Chapter 1
Measurement and Vectors

Conceptual Problems

1 • **[SSM]** Which of the following *is not* one of the base quantities in the SI system? (*a*) mass, (*b*) length, (*c*) energy, (*d*) time, (*e*) All of the above are base quantities.

Determine the Concept The base quantities in the SI system include mass, length, and time. Force is not a base quantity. $\boxed{(c)}$ is correct.

2 • In doing a calculation, you end up with m/s in the numerator and m/s^2 in the denominator. What are your final units? (*a*) m^2/s^3, (*b*) 1/s, (*c*) s^3/m^2, (*d*) s, (*e*) m/s.

Picture the Problem We can express and simplify the ratio of m/s to m/s^2 to determine the final units.

Express and simplify the ratio of m/s to m/s^2:

$$\frac{\frac{\text{m}}{\text{s}}}{\frac{\text{m}}{\text{s}^2}}=\frac{\text{m}\cdot\text{s}^2}{\text{m}\cdot\text{s}}=\text{s and }\boxed{(d)}\text{ is correct.}$$

3 • The prefix giga means (*a*) 10^3, (*b*) 10^6, (*c*) 10^9, (*d*) 10^{12}, (*e*) 10^{15}.

Determine the Concept Consulting Table 1-1 we note that the prefix giga means 10^9. $\boxed{(c)}$ is correct.

4 • The prefix mega means (*a*) 10^{-9}, (*b*) 10^{-6}, (*c*) 10^{-3}, (*d*) 10^6, (*e*) 10^9.

Determine the Concept Consulting Table 1-1 we note that the prefix mega means 10^6. $\boxed{(d)}$ is correct.

5 • **[SSM]** Show that there are 30.48 cm per foot. How many centimeters are there in one mile?

Picture the Problem We can use the facts that there are 2.540 centimeters in 1 inch and 12 inches in 1 foot to show that there are 30.48 cm per ft. We can then use the fact that there are 5280 feet in 1 mile to find the number of centimeters in one mile.

Multiply 2.540 cm/in by 12 in/ft to find the number of cm per ft:

$$\left(2.540\frac{\text{cm}}{\text{in}}\right)\left(12\frac{\text{in}}{\text{ft}}\right)=\boxed{30.48\,\text{cm/ft}}$$

Multiply 30.48 cm/ft by 5280 ft/mi to find the number of centimeters in one mile:

$$\left(30.48\frac{\text{cm}}{\text{ft}}\right)\left(5280\frac{\text{ft}}{\text{mi}}\right)=\boxed{1.609\times10^{5}\ \text{cm/mi}}$$

Remarks: Because there are exactly 2.54 cm in 1 in and exactly 12 inches in 1 ft, we are justified in reporting four significant figures in these results.

6 • The number 0.000 513 0 has ____ significant figures. (*a*) one, (*b*) three, (*c*) four, (*d*) seven, (*e*) eight.

Determine the Concept Counting from left to right and ignoring zeros to the left of the first nonzero digit, the last significant figure is the first digit that is in doubt. Applying this criterion, the three zeros after the decimal point are not significant figures, but the last zero is significant. Hence, there are four significant figures in this number. $\boxed{(c)}$ is correct.

7 • The number 23.0040 has ____ significant figures. (*a*) two, (*b*) three, (*c*) four, (*d*) five, (*e*) six.

Determine the Concept Counting from left to right, the last significant figure is the first digit that is in doubt. Applying this criterion, there are six significant figures in this number. $\boxed{(e)}$ is correct.

8 • Force has dimensions of mass times acceleration. Acceleration has dimensions of speed divided by time. Pressure is defined as force divided by area. What are the dimensions of pressure? Express pressure in terms of the SI base units kilogram, meter and second.

Determine the Concept We can use the definitions of force and pressure, together with the dimensions of mass, acceleration, and length, to find the dimensions of pressure. We can express pressure in terms of the SI base units by substituting the base units for mass, acceleration, and length in the definition of pressure.

Use the definition of pressure and the dimensions of force and area to obtain:

$$[P]=\frac{[F]}{[A]}=\frac{\frac{\mathrm{ML}}{\mathrm{T}^2}}{\mathrm{L}^2}=\boxed{\frac{\mathrm{M}}{\mathrm{LT}^2}}$$

Express pressure in terms of the SI base units to obtain:

$$\frac{\mathrm{N}}{\mathrm{m}^2}=\frac{\mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{m}^2}=\boxed{\frac{\mathrm{kg}}{\mathrm{m}\cdot\mathrm{s}^2}}$$

9 • True or false: Two quantities must have the same dimensions in order to be multiplied.

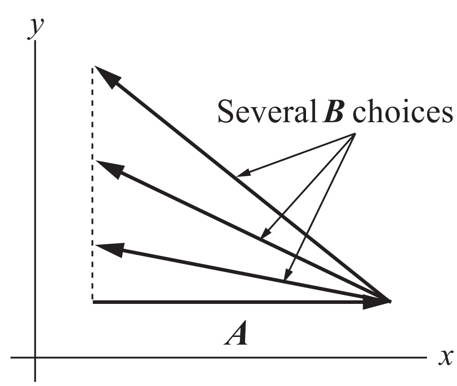
False. For example, the distance traveled by an object is the product of its speed (length/time) multiplied by its time of travel (time).

10 • A vector has a negative *x* component and a positive *y* component. Its angle measured counterclockwise from the positive *x* axis is (*a*) between zero and 90 degrees. (*b*) between 90 and 180 degrees. (*c*) More than 180 degrees.

Determine the Concept Because a vector with a negative *x*-component and a positive *y*-component is in the second quadrant, its angle is between 90 and 180 degrees. $\boxed{(b)\text{ is correct.}}$

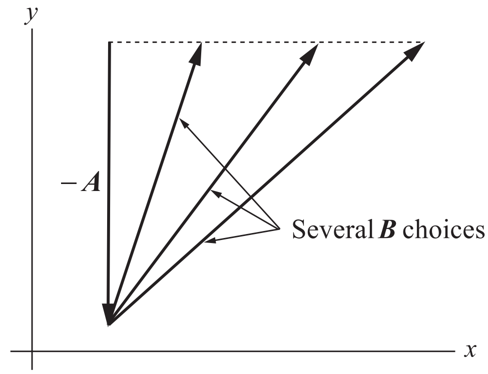
11 • **[SSM]** A vector $\vec{A}$ points in the +*x* direction. Show graphically at least three choices for a vector $\vec{B}$ such that $\vec{B}+\vec{A}$ points in the +*y* direction.

Determine the Concept The figure shows a vector $\vec{A}$ pointing in the positive *x* direction and three unlabeled possibilities for vector $\vec{B}$. Note that the choices for $\vec{B}$ start at the end of vector $\vec{A}$ rather than at its initial point. Note further that this configuration could be in any quadrant of the reference system shown.



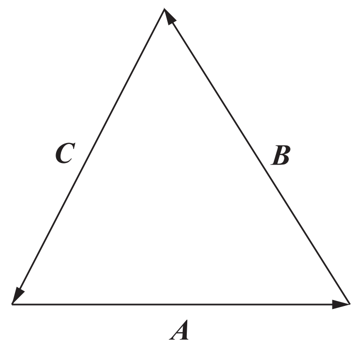
12 • A vector $\vec{A}$ points in the +*y* direction. Show graphically at least three choices for a vector $\vec{B}$ such that $\vec{B}-\vec{A}$ points in the + *x* direction.

Determine the Concept Let the $+x$ direction be to the right and the $+y$ direction be upward. The figure shows the vector $-\vec{A}$ pointing in the $-y$ direction and three unlabeled possibilities for vector $\vec{B}$. Note that the choices for $\vec{B}$ start at the end of vector $-\vec{A}$ rather than at its initial point.



13 • **[SSM]** Is it possible for three equal magnitude vectors to add to zero? If so, sketch a graphical answer. If not, explain why not.

Determine the Concept In order for the three equal magnitude vectors to add to zero, the sum of the three vectors must form a triangle. The equilateral triangle shown to the right satisfies this condition for the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ for which it is true that $A = B = C$, whereas $\vec{A}+\vec{B}+\vec{C}=0$.



Estimation and Approximation

14 • The angle subtended by the moon's diameter at a point on Earth is about 0.524° (Fig. 1-2). Use this and the fact that the moon is about 384 Mm away to find the diameter of the moon. *HINT: The angle can be determined from the diameter of the moon and the distance to the moon.*

Picture the Problem Let θ represent the angle subtended by the moon's diameter, D represent the diameter of the moon, and r_m the distance to the moon. Because θ is small, we can approximate it by $\theta \approx D/r_m$ where θ is in radian measure. We can solve this relationship for the diameter of the moon.

Express the moon's diameter D in terms of the angle it subtends at Earth θ and the Earth-moon distance r_m:

$$D = \theta r_m$$

Substitute numerical values and evaluate D:

$$D = \left(0.524° \times \frac{2\pi\ \text{rad}}{360°}\right)(384\ \text{Mm})$$

$$= \boxed{3.51\times10^{6}\ \text{m}}$$

15 • [SSM] Some good estimates about the human body can be made if it is assumed that we are made mostly of water. The mass of a water molecule is 29.9×10^{-27} kg. If the mass of a person is 60 kg, estimate the number of water molecules in that person.

Picture the Problem We can estimate the number of water molecules in a person whose mass is 60 kg by dividing this mass by the mass of a single water molecule.

Letting N represent the number of water molecules in a person of mass $m_{\text{human body}}$, express N in terms of $m_{\text{human body}}$ and the mass of a water molecule $m_{\text{water molecule}}$:

$$N = \frac{m_{\text{human body}}}{m_{\text{water molecule}}}$$

Substitute numerical values and evaluate N:

$$N = \frac{60\ \text{kg}}{29.9 \times 10^{-27}\ \dfrac{\text{kg}}{\text{molecule}}} = \boxed{2.0 \times 10^{27}\ \text{molecules}}$$

16 •• In 1989, IBM scientists figured out how to move atoms with a scanning tunneling microscope (STM). One of the first STM pictures seen by the general public was of the letters IBM spelled with xenon atoms on a nickel surface. The letters IBM were 15 xenon atoms across. If the space between the centers of adjacent xenon atoms is 5 nm (5×10^{-9} m), estimate how many times could ″IBM″ could be written across this 8.5 inch page.

Picture the Problem We can estimate the number of times N that ″IBM″ could be written across this 8.5-inch page by dividing the width w of the page by the distance d required by each writing of ″IBM.″

Express N in terms of the width w of the page and the distance d required by each writing of ″IBM″:

$$N = \frac{w}{d}$$

Express d in terms of the separation s of the centers of adjacent xenon atoms and the number n of xenon atoms in each writing of ″IBM″:

$$d = sn$$

Substitute for d in the expression for N to obtain:

$$N = \frac{w}{sn}$$

Substitute numerical values and evaluate N:

$$N=\frac{(8.5\text{ in})\left(2.540\,\frac{\text{cm}}{\text{in}}\right)}{\left(5\text{ nm}\times\frac{10^{-9}\text{ m}}{\text{nm}}\right)(15)}=\boxed{3\times10^{6}}$$

17 •• There is an environmental debate over the use of cloth versus disposable diapers. (*a*) If we assume that between birth and 2.5 y of age, a child uses 3 diapers per day, estimate the total number of disposable diapers used in the United States per year. (*b*) Estimate the total landfill volume due to these diapers, assuming that 1000 kg of waste fills about 1 m^3 of landfill volume. (*c*) How many square miles of landfill area at an average height of 10 m is needed for the disposal of diapers each year?

Picture the Problem We'll assume a population of 300 million and a life expectancy of 76 y. We'll also assume that a diaper has a volume of about half a liter. In (*c*) we'll assume the disposal site is a rectangular hole in the ground and use the formula for the volume of such an opening to estimate the surface area required.

(*a*) Express the total number N of disposable diapers used in the United States per year in terms of the number of children n in diapers and the number of diapers D used by each child per year:

$$N=nD \qquad (1)$$

Use the estimated daily consumption and the number of days in a year to estimate the number of diapers D required per child per year:

$$D=\frac{3\,\text{diapers}}{\text{child}\cdot\text{d}}\times\frac{365.24\,\text{d}}{\text{y}}$$
$$\approx1.1\times10^{3}\text{ diapers/child}\cdot\text{y}$$

Use the assumed life expectancy to estimate the number of children n in diapers yearly:

$$n=\left(\frac{2.5\,\text{y}}{76\,\text{y}}\right)\left(300\times10^{6}\text{ children}\right)$$
$$\approx10^{7}\text{ children}$$

Substitute numerical values in equation (1) to obtain:

$$N=\left(10^{7}\text{ children}\right)\left(3\times10^{3}\,\frac{\text{diapers}}{\text{y}}\right)$$
$$\approx\boxed{1.1\times10^{10}\text{ diapers}}$$

(*b*) Express the required landfill volume V in terms of the volume of diapers to be buried:

$$V = NV_{\text{one diaper}}$$

Substitute numerical values and evaluate V:

$$V = \left(1.1\times10^{10}\text{ diapers}\right)\left(\frac{0.5\,\text{L}}{\text{diaper}}\times\frac{10^{-3}\text{ m}^3}{\text{L}}\right)\approx\boxed{5.5\times10^{6}\text{ m}^3}$$

(*c*) Express the required volume in terms of the volume of a rectangular parallelepiped:

$$V = Ah \Rightarrow A = \frac{V}{h}$$

Substitute numerical values evaluate A:

$$A = \frac{5.5\times10^{6}\text{ m}^3}{10\,\text{m}} = 5.5\times10^{5}\text{ m}^2$$

Use a conversion factor (see Appendix A) to express this area in square miles:

$$A = 5.5\times10^{5}\text{ m}^2\times\frac{0.3861\text{ mi}^2}{1\,\text{km}^2}$$

$$\approx\boxed{0.2\,\text{mi}^2}$$

18 •• (*a*) Estimate the number of gallons of gasoline used per day by automobiles in the United States and the total amount of money spent on it. (*b*) If 19.4 gal of gasoline can be made from one barrel of crude oil, estimate the total number of barrels of oil imported into the United States per year to make gasoline. How many barrels per day is this?

Picture the Problem The population of the United States is roughly 3×10^8 people. Assuming that the average family has four people, with an average of two cars per family, there are about 1.5×10^8 cars in the United States. If we double that number to include trucks, cabs, etc., we have 3×10^8 vehicles. Let's assume that each vehicle uses, on average, 14 gallons of gasoline per week and that the United States imports half its oil.

(*a*) Find the daily consumption of gasoline G:

$$G = \left(3\times10^{8}\text{ vehicles}\right)\left(2\,\text{gal/d}\right)$$

$$= 6\times10^{8}\text{ gal/d}$$

Assuming a price per gallon P = \$3.00, find the daily cost C of gasoline:

$$C = GP = \left(6\times10^{8}\text{ gal/d}\right)\left(\$3.00/\text{gal}\right)$$

$$= \$18\times10^{8}/\text{d}$$

$$\approx\boxed{2\text{ billion dollars/d}}$$

(*b*) Relate the number of barrels N of crude oil imported annually to the yearly consumption of gasoline Y and the number of gallons of gasoline n that can be made from one barrel of crude oil:

$$N = \frac{fY}{n} = \frac{fG\Delta t}{n}$$

where f is the fraction of the oil that is imported.

Substitute numerical values and estimate N:

$$N = \frac{(0.5)\left(6\times10^{8}\,\frac{\text{gallons}}{\text{d}}\right)\left(365.24\,\frac{\text{d}}{\text{y}}\right)}{19.4\,\frac{\text{gallons}}{\text{barrel}}} \approx \boxed{6\times10^{9}\,\frac{\text{barrels}}{\text{y}}}$$

Convert barrels/y to barrels/d to obtain:

$$N = 6\times10^{9}\,\frac{\text{barrels}}{\text{y}}\times\frac{1\,\text{y}}{365.24\,\text{d}} \approx \boxed{2\times10^{7}\,\frac{\text{barrels}}{\text{d}}}$$

19 •• [SSM] A megabyte (MB) is a unit of computer memory storage. A CD has a storage capacity of 700 MB and can store approximately 70 min of high-quality music. (*a*) If a typical song is 5 min long, how many megabytes are required for each song? (*b*) If a page of printed text takes approximately 5 kilobytes, estimate the number of novels that could be saved on a CD.

Picture the Problem We can set up a proportion to relate the storage capacity of a CD to its playing time, the length of a typical song, and the storage capacity required for each song. In (*b*) we can relate the number of novels that can be stored on a CD to the number of megabytes required per novel and the storage capacity of the CD.

(*a*) Set up a proportion relating the ratio of the number of megabytes on a CD to its playing time to the ratio of the number of megabytes N required for each song:

$$\frac{700\,\text{MB}}{70\,\text{min}} = \frac{N}{5\,\text{min}}$$

Solve this proportion for N to obtain:

$$N = \left(\frac{700\,\text{MB}}{70\,\text{min}}\right)(5\,\text{min}) = \boxed{50\,\text{MB}}$$

(*b*) Letting *n* represent the number of megabytes per novel, express the number of novels N_{novels} that can be stored on a CD in terms of the storage capacity of the CD:

$$N_{\text{novels}} = \frac{700\ \text{MB}}{n}$$

Assuming that a typical page in a novel requires 5 kB of memory, express *n* in terms of the number of pages *p* in a typical novel:

$$n = \left(5\,\frac{\text{kB}}{\text{page}}\right)p$$

Substitute for *n* in the expression for N_{novels} to obtain:

$$N_{\text{novels}} = \frac{700\ \text{MB}}{\left(5\,\frac{\text{kB}}{\text{page}}\right)p}$$

Assuming that a typical novel has 200 pages:

$$N_{\text{novels}} = \frac{700\ \text{MB} \times \frac{10^3\ \text{kB}}{\text{MB}}}{\left(5\,\frac{\text{kB}}{\text{page}}\right)\left(200\,\frac{\text{pages}}{\text{novel}}\right)} = \boxed{7\times10^2\ \text{novels}}$$

Units

20 • Express the following quantities using the prefixes listed in Table 1-1 and the unit abbreviations listed in the table Abbreviations for Units. For example, 10,000 meters = 10 km. (*a*) 1,000,000 watts, (*b*) 0.002 gram, (*c*) 3 × 10^{-6} meter, (*d*) 30,000 seconds.

Picture the Problem We can use the metric prefixes listed in Table 1-1 and the abbreviations on page EP-1 to express each of these quantities.

(*a*)

$$1{,}000{,}000\ \text{watts} = 10^6\ \text{watts} = \boxed{1\,\text{MW}}$$

(*c*)

$$3\times10^{-6}\ \text{meter} = \boxed{3\ \mu\text{m}}$$

(*b*)

$$0.002\,\text{gram} = 2\times10^{-3}\ \text{g} = \boxed{2\,\text{mg}}$$

(*d*)

$$30{,}000\,\text{seconds} = 30\times10^{3}\ \text{s} = \boxed{30\,\text{ks}}$$

21 • Write each of the following without using prefixes: (*a*) 40 μW, (*b*) 4 ns, (*c*) 3 MW, (*d*) 25 km.

Picture the Problem We can use the definitions of the metric prefixes listed in Table 1-1 to express each of these quantities without prefixes.

(*a*)
$$40\ \mu\text{W} = 40 \times 10^{-6}\ \text{W} = \boxed{0.000040\ \text{W}}$$

(*b*)
$$4\ \text{ns} = 4 \times 10^{-9}\ \text{s} = \boxed{0.000000004\ \text{s}}$$

(*c*)
$$3\ \text{MW} = 3 \times 10^{6}\ \text{W} = \boxed{3{,}000{,}000\ \text{W}}$$

(*d*)
$$25\ \text{km} = 25 \times 10^{3}\ \text{m} = \boxed{25{,}000\ \text{m}}$$

22 • Write the following (which are not SI units) using prefixes (but not their abbreviations). For example, 10^3 meters = 1 kilometer: (*a*) 10^{-12} boo, (*b*) 10^9 low, (*c*) 10^{-6} phone, (*d*) 10^{-18} boy, (*e*) 10^6 phone, (*f*) 10^{-9} goat, (*g*) 10^{12} bull.

Picture the Problem We can use the definitions of the metric prefixes listed in Table 1-1 to express each of these quantities without abbreviations.

(*a*) $10^{-12}\ \text{boo} = \boxed{1\ \text{picoboo}}$

(*b*) $10^{9}\ \text{low} = \boxed{1\ \text{gigalow}}$

(*c*) $10^{-6}\ \text{phone} = \boxed{1\ \text{microphone}}$

(*d*) $10^{-18}\ \text{boy} = \boxed{1\ \text{attoboy}}$

(*e*) $10^{6}\ \text{phone} = \boxed{1\ \text{megaphone}}$

(*f*) $10^{-9}\ \text{goat} = \boxed{1\ \text{nanogoat}}$

(*g*) $10^{12}\ \text{bull} = \boxed{1\ \text{terabull}}$

23 •• **[SSM]** In the following equations, the distance x is in meters, the time t is in seconds, and the velocity v is in meters per second. What are the SI units of the constants C_1 and C_2? (*a*) $x = C_1 + C_2t$, (*b*) $x = \frac{1}{2}C_1t^2$, (*c*) $v^2 = 2C_1x$, (*d*) $x = C_1 \cos C_2t$, (*e*) $v^2 = 2C_1v - (C_2x)^2$.

Picture the Problem We can determine the SI units of each term on the right-hand side of the equations from the units of the physical quantity on the left-hand side.

(*a*) Because x is in meters, C_1 and C_2t must be in meters:

$$\boxed{C_1 \text{ is in m; } C_2 \text{ is in m/s}}$$

(*b*) Because x is in meters, $\frac{1}{2}C_1t^2$ must be in meters:

$$\boxed{C_1 \text{ is in m/s}^2}$$

(*c*) Because v^2 is in m^2/s^2, $2C_1x$ must be in m^2/s^2:

$$\boxed{C_1 \text{ is in m/s}^2}$$

(*d*) The argument of a trigonometric function must be dimensionless; i.e. without units. Therefore, because x is in meters:

$$\boxed{C_1 \text{ is in m}; C_2 \text{ is in s}^{-1}}$$

(*e*) All of the terms in the expression must have the same units. Therefore, because v is in m/s:

$$\boxed{C_1 \text{ is in m/s}; C_2 \text{ is in s}^{-1}}$$

24 •• If x is in feet, t is in milliseconds, and v is in feet per second, what are the units of the constants C_1 and C_2 in each part of Problem 23?

Picture the Problem We can determine the US customary units of each term on the right-hand side of the equations from the units of the physical quantity on the left-hand side.

(*a*) Because x is in feet, C_1 and C_2t must be in feet:

$$\boxed{C_1 \text{ is in ft}; C_2 \text{ is in ft/ms}}$$

(*b*) Because x is in feet, $\frac{1}{2}\boldsymbol{C_1t^2}$ must be in feet:

$$\boxed{C_1 \text{ is in ft}/(\text{ms})^2}$$

(*c*) Because v^2 is in $\text{ft}^2/(\text{ms})^2$, $2C_1x$ must be in ft^2/s^2:

$$\boxed{C_1 \text{ is in ft}/(\text{ms})^2}$$

(*d*) The argument of a trigonometric function must be dimensionless; that is, without units. Therefore, because x is in feet:

$$\boxed{C_1 \text{ is in ft}; C_2 \text{ is in } (\text{ms})^{-1}}$$

(*e*) The argument of an exponential function must be dimensionless; that is, without units. Therefore, because v is in ft/s:

$$\boxed{C_1 \text{ is in ft/ms}; C_2 \text{ is in } (\text{ms})^{-1}}$$

Conversion of Units

25 • From the original definition of the meter in terms of the distance along a meridian from the equator to the North Pole, find in meters (*a*) the circumference of Earth and (*b*) the radius of Earth. (*c*) Convert your answers for (*a*) and (*b*) from meters into miles.

Picture the Problem We can use the formula for the circumference of a circle to find the radius of Earth and the conversion factor 1 mi = 1.609 km to convert distances in meters into distances in miles.

(*a*) The Pole-Equator distance is one-fourth of the circumference:

$$C = 4D_{\text{pole-equator}} = \boxed{4\times 10^{7}\ \text{m}}$$

(*b*) The formula for the circumference of a circle is:

$$C = \pi D = 2\pi R \Rightarrow R = \frac{C}{2\pi}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{4\times 10^{7}\ \text{m}}{2\pi} = 6.37\times 10^{6}\ \text{m}$$
$$= \boxed{6\times 10^{6}\ \text{m}}$$

(*c*) Use the conversion factors 1 km = 1000 m and 1 mi = 1.609 km to express *C* in mi:

$$C = 4\times 10^{7}\ \text{m}\times\frac{1\,\text{km}}{10^{3}\ \text{m}}\times\frac{1\,\text{mi}}{1.609\,\text{km}}$$
$$= \boxed{2\times 10^{4}\ \text{mi}}$$

Use the conversion factors 1 km = 1000 m and 1 mi = 1.61 km to express *R* in mi:

$$R = 6.37\times 10^{6}\ \text{m}\times\frac{1\text{km}}{10^{3}\ \text{m}}\times\frac{1\text{mi}}{1.609\,\text{km}}$$
$$= \boxed{4\times 10^{3}\ \text{mi}}$$

26 • The speed of sound in air is 343 m/s at normal room temperature. What is the speed of a supersonic plane that travels at twice the speed of sound? Give your answer in kilometers per hour and miles per hour.

Picture the Problem We can use the conversion factor 1 mi = 1.61 km to convert speeds in km/h into mi/h.

Find the speed of the plane in km/h:

$$v = 2(343\,\text{m/s}) = 686\,\text{m/s}$$
$$= \left(686\frac{\text{m}}{\text{s}}\right)\left(\frac{1\text{km}}{10^{3}\ \text{m}}\right)\left(3600\frac{\text{s}}{\text{h}}\right)$$
$$= \boxed{2.47\times 10^{3}\ \text{km/h}}$$

Convert v into mi/h:

$$v = \left(2.47\times10^3\ \frac{\text{km}}{\text{h}}\right)\left(\frac{1\,\text{mi}}{1.609\,\text{km}}\right) = \boxed{1.53\times10^3\ \text{mi/h}}$$

27 • A basketball player is 6 ft $10\frac{1}{2}$ in tall. What is his height in centimeters?

Picture the Problem We'll first express his height in inches and then use the conversion factor 1 in = 2.540 cm.

Express the player's height in inches:

$$h = 6\,\text{ft}\times\frac{12\,\text{in}}{\text{ft}} + 10.5\,\text{in} = 82.5\,\text{in}$$

Convert h into cm:

$$h = 82.5\,\text{in}\times\frac{2.540\,\text{cm}}{\text{in}} = \boxed{210\,\text{cm}}$$

28 • Complete the following: (a) 100 km/h = ____ mi/h, (b) 60 cm = ____ in, (c) 100 yd = ____ m.

Picture the Problem We can use the conversion factors 1 mi = 1.609 km, 1 in = 2.540 cm, and 1 m = 1.094 yd to perform these conversions.

(a) Convert $100\frac{\text{km}}{\text{h}}$ to $\frac{\text{mi}}{\text{h}}$:

$$100\frac{\text{km}}{\text{h}} = 100\frac{\text{km}}{\text{h}}\times\frac{1\,\text{mi}}{1.609\,\text{km}} = \boxed{62.2\,\text{mi/h}}$$

(b) Convert 60 cm to in:

$$60\,\text{cm} = 60\,\text{cm}\times\frac{1\,\text{in}}{2.540\,\text{cm}} = 23.6\,\text{in} = \boxed{24\,\text{in}}$$

(c) Convert 100 yd to m:

$$100\,\text{yd} = 100\,\text{yd}\times\frac{1\,\text{m}}{1.094\,\text{yd}} = \boxed{91.4\,\text{m}}$$

29 • The main span of the Golden Gate Bridge is 4200 ft. Express this distance in kilometers.

Picture the Problem We can use the conversion factor 1.609 km = 5280 ft to convert the length of the main span of the Golden Gate Bridge into kilometers.

Convert 4200 ft into km by multiplying by 1 in the form $\frac{1.609\text{ km}}{5280\text{ ft}}$:

$$4200\,\text{ft} = 4200\,\text{ft}\times\frac{1.609\,\text{km}}{5280\,\text{ft}} = \boxed{1.28\,\text{km}}$$

30 • Find the conversion factor to convert from miles per hour into kilometers per hour.

Picture the Problem Let v be the speed of an object in mi/h. We can use the conversion factor 1 mi = 1.609 km to convert this speed to km/h.

Multiply v mi/h by 1.609 km/mi to convert v to km/h:

$$v\frac{\text{mi}}{\text{h}} = v\frac{\text{mi}}{\text{h}}\times\frac{1.609\,\text{km}}{\text{mi}} = \boxed{1.61v\,\text{km/h}}$$

31 • Complete the following: (*a*) 1.296×10^5 km/h^2 = ____ km/(h·s), (*b*) 1.296×10^5 km/h^2 = ____m/s^2, (*c*) 60 mi/h = ____ ft/s, (*d*) 60 mi/h = ____ m/s.

Picture the Problem Use the conversion factors 1 h = 3600 s, 1.609 km = 1 mi, and 1 mi = 5280 ft to make these conversions.

(*a*) $$1.296\times10^5\frac{\text{km}}{\text{h}^2} = \left(1.296\times10^5\frac{\text{km}}{\text{h}^2}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right) = \boxed{36.00\,\text{km/h}\cdot\text{s}}$$

(*b*) $$1.296\times10^5\frac{\text{km}}{\text{h}^2} = \left(1.296\times10^5\frac{\text{km}}{\text{h}^2}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)^2\left(\frac{10^3\,\text{m}}{\text{km}}\right) = \boxed{10.00\,\text{m/s}^2}$$

(*c*) $$60\frac{\text{mi}}{\text{h}} = \left(60\frac{\text{mi}}{\text{h}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mi}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right) = \boxed{88\,\text{ft/s}}$$

(*d*) $$60\frac{\text{mi}}{\text{h}} = \left(60\frac{\text{mi}}{\text{h}}\right)\left(\frac{1.609\,\text{km}}{1\,\text{mi}}\right)\left(\frac{10^3\,\text{m}}{\text{km}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right) = 26.8\,\text{m/s} = \boxed{27\,\text{m/s}}$$

32 • There are 640 acres in a square mile. How many square meters are there in one acre?

Picture the Problem We can use the conversion factor given in the problem statement and the fact that 1 mi = 1.609 km to express the number of square meters in one acre. Note that, because there are exactly 640 acres in a square mile, 640 acres has as many significant figures as we may wish to associate with it.

Multiply by 1 twice, properly chosen, to convert one acre into square miles, and then into square meters:

$$1\,\text{acre} = (1\,\text{acre})\left(\frac{1\,\text{mi}^2}{640\,\text{acres}}\right)\left(\frac{1609\,\text{m}}{\text{mi}}\right)^2$$
$$= \boxed{4045\,\text{m}^2}$$

33 •• [SSM] You are a delivery person for the Fresh Aqua Spring Water Company. Your truck carries 4 pallets. Each pallet carries 60 cases of water. Each case of water has 24 one-liter bottles. You are to deliver 10 cases of water to each convenience store along your route. The dolly you use to carry the water into the stores has a weight limit of 250 lb. (*a*) If a milliliter of water has a mass of 1 g, and a kilogram has a weight of 2.2 lb, what is the weight, in pounds, of all the water in your truck? (*b*) How many full cases of water can you carry on the cart?

Picture the Problem The weight of the water in the truck is the product of the volume of the water and its weight density of 2.2 lb/L.

(*a*) Relate the weight w of the water on the truck to its volume V and weight density (weight per unit volume) D:

$$w = DV$$

Find the volume V of the water:

$$V = (4\text{ pallets})\left(60\,\frac{\text{cases}}{\text{pallet}}\right)\left(24\,\frac{\text{L}}{\text{case}}\right)$$
$$= 5760\text{ L}$$

Substitute numerical values for D and L and evaluate w:

$$w = \left(2.2\,\frac{\text{lb}}{\text{L}}\right)(5760\text{ L}) = 1.267\times10^4\text{ lb}$$
$$= \boxed{1.3\times10^4\text{ lb}}$$

(*b*) Express the number of cases of water in terms of the weight limit of the cart and the weight of each case of water:

$$N = \frac{\text{weight limit of the cart}}{\text{weight of each case of water}}$$

Substitute numerical values and evaluate N:

$$N = \frac{250\text{ lb}}{\left(2.2\,\frac{\text{lb}}{\text{L}}\right)\left(24\,\frac{\text{L}}{\text{case}}\right)} = 4.7\text{ cases}$$

$$\boxed{\text{You can carry 4 cases.}}$$

34 •• A right circular cylinder has a diameter of 6.8 in and a height of 2.0 ft. What is the volume of the cylinder in (*a*) cubic feet, (*b*) cubic meters, (*c*) liters?

Picture the Problem The volume of a right circular cylinder is the area of its base multiplied by its height. Let *d* represent the diameter and *h* the height of the right circular cylinder; use conversion factors to express the volume *V* in the given units.

(*a*) Express the volume of the cylinder:

$$V = \tfrac{1}{4}\pi d^2 h$$

Substitute numerical values and evaluate *V*:

$$V = \tfrac{1}{4}\pi(6.8\,\text{in})^2(2.0\,\text{ft})$$
$$= \tfrac{1}{4}\pi(6.8\,\text{in})^2(2.0\,\text{ft})\left(\frac{1\,\text{ft}}{12\,\text{in}}\right)^2$$
$$= 0.504\,\text{ft}^3 = \boxed{0.50\,\text{ft}^3}$$

(*b*) Use the fact that 1 m = 3.281 ft to convert the volume in cubic feet into cubic meters:

$$V = \left(0.504\,\text{ft}^3\right)\left(\frac{1\,\text{m}}{3.281\,\text{ft}}\right)^3 = 0.0143\,\text{m}^3$$
$$= \boxed{0.014\,\text{m}^3}$$

(*c*) Because 1 L = 10^{-3} m^3:

$$V = \left(0.0143\,\text{m}^3\right)\left(\frac{1\,\text{L}}{10^{-3}\,\text{m}^3}\right) = \boxed{14\,\text{L}}$$

35 •• **[SSM]** In the following, *x* is in meters, *t* is in seconds, *v* is in meters per second, and the acceleration *a* is in meters per second squared. Find the SI units of each combination: (*a*) v^2/x, (*b*) $\sqrt{x/a}$, (*c*) $\tfrac{1}{2}at^2$.

Picture the Problem We can treat the SI units as though they are algebraic quantities to simplify each of these combinations of physical quantities and constants.

(*a*) Express and simplify the units of v^2/x:

$$\frac{(\text{m/s})^2}{\text{m}} = \frac{\text{m}^2}{\text{m}\cdot\text{s}^2} = \boxed{\frac{\text{m}}{\text{s}^2}}$$

(*b*) Express and simplify the units of $\sqrt{x/a}$:

$$\sqrt{\frac{\text{m}}{\text{m/s}^2}} = \sqrt{\text{s}^2} = \boxed{\text{s}}$$

(*c*) Noting that the constant factor $\frac{1}{2}$ has no units, express and simplify the units of $\frac{1}{2}at^2$:

$$\left(\frac{\text{m}}{\text{s}^2}\right)(\text{s})^2 = \boxed{\text{m}}$$

Dimensions of Physical Quantities

36 • What are the dimensions of the constants in each part of Problem 23?

Picture the Problem We can use the facts that each term in an equation must have the same dimensions and that the arguments of a trigonometric or exponential function must be dimensionless to determine the dimensions of the constants.

(*a*)

$$\begin{array}{ccccc} x & = & C_1 & + & C_2 \quad t \\ \text{L} & & \boxed{\text{L}} & & \boxed{\frac{\text{L}}{\text{T}}}\ \text{T} \end{array}$$

(*b*)

$$\begin{array}{cccc} x & = & \frac{1}{2} & C_1 \quad t^2 \\ \text{L} & & & \boxed{\frac{\text{L}}{\text{T}^2}}\ \text{T}^2 \end{array}$$

(*c*)

$$\begin{array}{ccc} v^2 & = & 2\ C_1 \quad x \\ \frac{\text{L}^2}{\text{T}^2} & & \boxed{\frac{\text{L}}{\text{T}^2}}\ \text{L} \end{array}$$

(*d*)

$$\begin{array}{ccc} x & = & C_1 \ \cos\ C_2 \quad t \\ \text{L} & & \boxed{\text{L}} \quad \boxed{\frac{1}{\text{T}}}\ \text{T} \end{array}$$

(*e*)

$$\begin{array}{ccccc} v^2 & = & 2\ C_1\ v & - & (C_2)^2\ x^2 \\ \left(\frac{\text{L}}{\text{T}}\right)^2 & & \boxed{\frac{\text{L}}{\text{T}}}\ \frac{\text{L}}{\text{T}} & & \boxed{\frac{1}{\text{T}}}\ \text{L}^2 \end{array}$$

37 • The law of radioactive decay is $N(t) = N_0 e^{-\lambda t}$, where N_0 is the number of radioactive nuclei at $t = 0$, $N(t)$ is the number remaining at time t, and λ is a quantity known as the decay constant. What is the dimension of λ?

Picture the Problem Because the exponent of the exponential function must be dimensionless, the dimension of λ must be $\boxed{\text{T}^{-1}.}$

38 •• The SI unit of force, the kilogram-meter per second squared (kg·m/s^2) is called the newton (N). Find the dimensions and the SI units of the constant G in Newton's law of gravitation $F = Gm_1m_2/r^2$.

Picture the Problem We can solve Newton's law of gravitation for G and substitute the dimensions of the variables. Treating them as algebraic quantities will allow us to express the dimensions in their simplest form. Finally, we can substitute the SI units for the dimensions to find the units of G.

Solve Newton's law of gravitation for G to obtain:

$$G = \frac{Fr^2}{m_1 m_2}$$

Substitute the dimensions of the variables:

$$[G] = \frac{\frac{\text{ML}}{\text{T}^2} \times \text{L}^2}{\text{M}^2} = \boxed{\frac{\text{L}^3}{\text{MT}^2}}$$

Use the SI units for L, M, and T to obtain:

$$\boxed{\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}}$$

39 •• The magnitude of the force (F) that a spring exerts when it is stretched a distance x from its unstressed length is governed by Hooke's law, $F = kx$. (a) What are the dimensions of the *force constant*, k? (b) What are the dimensions and SI units of the quantity kx^2?

Picture the Problem The dimensions of mass and velocity are M and L/T, respectively. We note from Table 1-2 that the dimensions of force are ML/T^2.

(a) We know, from Hooke's law, that:

$$k = \frac{F}{x}$$

Write the corresponding dimensional equation:

$$[k] = \frac{[F]}{[x]}$$

Substitute the dimensions of F and x and simplify to obtain:

$$[k] = \frac{\text{M}\frac{\text{L}}{\text{T}^2}}{\text{L}} = \boxed{\frac{\text{M}}{\text{T}^2}}$$

(b) Substitute the dimensions of k and x^2 and simplify to obtain:

$$[kx^2] = \frac{\text{M}}{\text{T}^2}\text{L}^2 = \boxed{\frac{\text{ML}^2}{\text{T}^2}}$$

Substitute the units of kx^2 to obtain:

$$\boxed{\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}}$$

40 •• Show that the product of mass, acceleration, and speed has the dimensions of power.

Picture the Problem We note from Table 1-2 that the dimensions of power are ML^2/T^3. The dimensions of mass, acceleration, and speed are M, L/T^2, and L/T respectively.

Express the dimensions of mav:

$$[mav] = \mathrm{M} \times \frac{\mathrm{L}}{\mathrm{T}^2} \times \frac{\mathrm{L}}{\mathrm{T}} = \frac{\mathrm{ML}^2}{\mathrm{T}^3}$$

From Table 1-2:

$$[P] = \frac{\mathrm{ML}^2}{\mathrm{T}^3}$$

Comparing these results, we see that the product of mass, acceleration, and speed has the dimensions of power.

41 •• **[SSM]** The momentum of an object is the product of its velocity and mass. Show that momentum has the dimensions of force multiplied by time.

Picture the Problem The dimensions of mass and velocity are M and L/T, respectively. We note from Table 1-2 that the dimensions of force are $\mathrm{ML/T^2}$.

Express the dimensions of momentum:

$$[mv] = \mathrm{M} \times \frac{\mathrm{L}}{\mathrm{T}} = \frac{\mathrm{ML}}{\mathrm{T}}$$

From Table 1-2:

$$[F] = \frac{\mathrm{ML}}{\mathrm{T}^2}$$

Express the dimensions of force multiplied by time:

$$[Ft] = \frac{\mathrm{ML}}{\mathrm{T}^2} \times \mathrm{T} = \frac{\mathrm{ML}}{\mathrm{T}}$$

Comparing these results, we see that momentum has the dimensions of force multiplied by time.

42 •• What combination of force and one other physical quantity has the dimensions of power?

Picture the Problem Let X represent the physical quantity of interest. Then we can express the dimensional relationship between F, X, and P and solve this relationship for the dimensions of X.

Express the relationship of X to force and power dimensionally:

$$[F][X]=[P]$$

Solve for $[X]$:

$$[X]=\frac{[P]}{[F]}$$

Substitute the dimensions of force and power and simplify to obtain:

$$[X]=\frac{\frac{\mathrm{ML}^2}{\mathrm{T}^3}}{\frac{\mathrm{ML}}{\mathrm{T}^2}}=\frac{\mathrm{L}}{\mathrm{T}}$$

Because the dimensions of velocity are L/T*,* we can conclude that:

$$[P]=\boxed{[F][v]}$$

Remarks: While it is true that *P* = *Fv*, dimensional analysis does not reveal the presence of dimensionless constants. For example, if $P=\pi Fv$, the analysis shown above would fail to establish the factor of π.

43 •• [SSM] When an object falls through air, there is a drag force that depends on the product of the cross sectional area of the object and the square of its velocity, that is, $F_{\text{air}} = CAv^2$, where C is a constant. Determine the dimensions of C.

Picture the Problem We can find the dimensions of C by solving the drag force equation for C and substituting the dimensions of force, area, and velocity.

Solve the drag force equation for the constant C:

$$C=\frac{F_{\text{air}}}{Av^2}$$

Express this equation dimensionally:

$$[C]=\frac{[F_{\text{air}}]}{[A][v]^2}$$

Substitute the dimensions of force, area, and velocity and simplify to obtain:

$$[C]=\frac{\frac{\mathrm{ML}}{\mathrm{T}^2}}{\mathrm{L}^2\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^2}=\boxed{\frac{\mathrm{M}}{\mathrm{L}^3}}$$

44 •• Kepler's third law relates the period of a planet to its orbital radius r, the constant G in Newton's law of gravitation ($F = Gm_1m_2/r^2$), and the mass of the sun M_s. What combination of these factors gives the correct dimensions for the period of a planet?

Picture the Problem We can express the period of a planet as the product of the factors *r, G,* and M_S (each raised to a power) and then perform dimensional analysis to determine the values of the exponents.

Express the period T of a planet as the product of $r^a, G^b, \text{and } M_S^c$:

$$T = Cr^aG^bM_S^c \qquad (1)$$

where C is a dimensionless constant.

Solve the law of gravitation for the constant G:

$$G = \frac{Fr^2}{m_1m_2}$$

Express this equation dimensionally:

$$[G] = \frac{[F][r]^2}{[m_1][m_2]}$$

Substitute the dimensions of F, r, and m:

$$[G] = \frac{\frac{\text{ML}}{\text{T}^2}\times(\text{L})^2}{\text{M}\times\text{M}} = \frac{\text{L}^3}{\text{MT}^2}$$

Noting that the dimension of time is represented by the same letter as is the period of a planet, substitute the dimensions in equation (1) to obtain:

$$T = (\text{L})^a\left(\frac{\text{L}^3}{\text{MT}^2}\right)^b(\text{M})^c$$

Introduce the product of M^0 and L^0 in the left hand side of the equation and simplify to obtain:

$$\text{M}^0\text{L}^0\text{T}^1 = \text{M}^{c-b}\text{L}^{a+3b}\text{T}^{-2b}$$

Equating the exponents on the two sides of the equation yields:

$0 = c - b$, $0 = a + 3b$, and $1 = -2b$

Solve these equations simultaneously to obtain:

$$a = \tfrac{3}{2}, b = -\tfrac{1}{2}, \text{and } c = -\tfrac{1}{2}$$

Substitute for a, b, and c in equation (1) and simplify to obtain:

$$T = Cr^{3/2}G^{-1/2}M_S^{-1/2} = \boxed{\frac{C}{\sqrt{GM_S}}r^{3/2}}$$

Scientific Notation and Significant Figures

45 • **[SSM]** Express as a decimal number without using powers of 10 notation: (*a*) 3×10^4, (*b*) 6.2×10^{-3}, (*c*) 4×10^{-6}, (*d*) 2.17×10^5.

Picture the Problem We can use the rules governing scientific notation to express each of these numbers as a decimal number.

(*a*) $3\times 10^{4} = \boxed{30{,}000}$

(*b*) $6.2\times 10^{-3} = \boxed{0.0062}$

(*c*) $4\times 10^{-6} = \boxed{0.000004}$

(*d*) $2.17\times 10^{5} = \boxed{217{,}000}$

46 • Write the following in scientific notation: (*a*) 1345100 m = ____ km, (*b*) 12340. kW = ____ MW, (*c*) 54.32 ps = ____ s, (*d*) 3.0 m = ____ mm

Picture the Problem We can use the rules governing scientific notation to express each of these numbers in scientific notation.

(*a*)

$$1345100\text{ m} = 1.3451\times 10^{6}\text{ m}$$
$$= \boxed{1.3451\times 10^{3}\text{ km}}$$

(*b*)

$$12340.\text{ kW} = 1.2340\times 10^{4}\text{ kW}$$
$$= 1.2340\times 10^{7}\text{ W}$$
$$= \boxed{1.2340\times 10^{1}\text{ MW}}$$

(*c*)

$$54.32\text{ ps} = 54.32\times 10^{-12}\text{ s}$$
$$= \boxed{5.432\times 10^{-11}\text{ s}}$$

(*d*)

$$3.0\text{ m} = 3.0\text{ m}\times\frac{10^{3}\text{ mm}}{\text{m}}$$
$$= \boxed{3.0\times 10^{3}\text{ mm}}$$

47 • **[SSM]** Calculate the following, round off to the correct number of significant figures, and express your result in scientific notation: (*a*) $(1.14)(9.99\times 10^{4})$, (*b*) $(2.78\times 10^{-8}) - (5.31\times 10^{-9})$, (*c*) $12\pi/(4.56\times 10^{-3})$, (*d*) $27.6 + (5.99\times 10^{2})$.

Picture the Problem Apply the general rules concerning the multiplication, division, addition, and subtraction of measurements to evaluate each of the given expressions.

(*a*) The number of significant figures in each factor is three; therefore the result has three significant figures:

$$(1.14)(9.99\times 10^{4}) = \boxed{1.14\times 10^{5}}$$

(*b*) Express both terms with the same power of 10. Because the first measurement has only two digits after the decimal point, the result can have only two digits after the decimal point:

$$\left(2.78\times10^{-8}\right)-\left(5.31\times10^{-9}\right)=\left(2.78-0.531\right)\times10^{-8}=\boxed{2.25\times10^{-8}}$$

(*c*) We'll assume that 12 is exact. Hence, the answer will have three significant figures:

$$\frac{12\pi}{4.56\times10^{-3}}=\boxed{8.27\times10^{3}}$$

(*d*) Proceed as in (*b*):

$$27.6+\left(5.99\times10^{2}\right)=27.6+599=627=\boxed{6.27\times10^{2}}$$

48 • Calculate the following, round off to the correct number of significant figures, and express your result in scientific notation: (*a*) (200.9)(569.3), (*b*) (0.000000513)(62.3 × 10^7), (*c*) 28 401 + (5.78 × 10^4), (*d*) 63.25/(4.17 × 10^{-3}).

Picture the Problem Apply the general rules concerning the multiplication, division, addition, and subtraction of measurements to evaluate each of the given expressions.

(*a*) Note that both factors have four significant figures.

$$\left(200.9\right)\left(569.3\right)=\boxed{1.144\times10^{5}}$$

(*b*) Express the first factor in scientific notation and note that both factors have three significant figures.

$$\left(0.000000513\right)\left(62.3\times10^{7}\right)=\left(5.13\times10^{-7}\right)\left(62.3\times10^{7}\right)=\boxed{3.20\times10^{2}}$$

(*c*) Express both terms in scientific notation and note that the second has only three significant figures. Hence the result will have only three significant figures.

$$28401+\left(5.78\times10^{4}\right)=\left(2.841\times10^{4}\right)+\left(5.78\times10^{4}\right)=\left(2.841+5.78\right)\times10^{4}=\boxed{8.62\times10^{4}}$$

(*d*) Because the divisor has three significant figures, the result will have three significant figures.

$$\frac{63.25}{4.17\times10^{-3}}=\boxed{1.52\times10^{4}}$$

49 • [SSM] A cell membrane has a thickness of 7.0 nm. How many cell membranes would it take to make a stack 1.0 in high?

Picture the Problem Let *N* represent the required number of membranes and express *N* in terms of the thickness of each cell membrane.

Express *N* in terms of the thickness of a single membrane:

$$N=\frac{1.0\,\text{in}}{7.0\,\text{nm}}$$

Convert the units into SI units and simplify to obtain:

$$N=\frac{1.0\,\text{in}}{7.0\,\text{nm}}\times\frac{2.540\,\text{cm}}{\text{in}}\times\frac{1\,\text{m}}{100\,\text{cm}}\times\frac{1\,\text{nm}}{10^{-9}\,\text{m}}=3.63\times10^{6}=\boxed{3.6\times10^{6}}$$

50 •• A circular hole of radius 8.470×10^{-1} cm must be cut into the front panel of a display unit. The *tolerance* is 1.0×10^{-3} cm, which means the actual hole cannot differ by more than this much from the desired radius. If the actual hole is larger than the desired radius by the allowed tolerance, what is the difference between the actual area and the desired area of the hole?

Picture the Problem Let r_0 represent the larger radius and r the desired radius of the hole. We can find the difference between the actual and the desired area of the hole by subtracting the smaller area from the larger area.

Express the difference between the two areas in terms of r and r_0:

$$\Delta A=\pi r_0^2-\pi r^2=\pi\left(r_0^2-r^2\right)$$

Factoring $r_0^2-r^2$ to obtain:

$$\Delta A=\pi\left(r_0-r\right)\left(r_0+r\right)$$

Substitute numerical values and evaluate ΔA:

$$\begin{aligned}\Delta A&=\pi\left(1.0\times10^{-3}\text{ cm}\right)\left[8.470\times10^{-1}\text{ cm}+\left(8.470\times10^{-1}\text{ cm}+1.0\times10^{-3}\text{ cm}\right)\right]\\&=\boxed{5.3\times10^{-3}\text{ cm}^2}\end{aligned}$$

51 •• [SSM] A square peg must be made to fit through a square hole. If you have a square peg that has an edge length of 42.9 mm, and the square hole has an edge length of 43.2 mm, (*a*) what is the area of the space available when

the peg is in the hole? (*b*) If the peg is made rectangular by removing 0.10 mm of material from one side, what is the area available now?

Picture the Problem Let s_h represent the side of the square hole and s_p the side of the square peg. We can find the area of the space available when the peg is in the hole by subtracting the area of the peg from the area of the hole.

(*a*) Express the difference between the two areas in terms of s_h and s_p:

$$\Delta A = s_h^2 - s_p^2$$

Substitute numerical values and evaluate ΔA:

$$\Delta A = (43.2\text{ mm})^2 - (42.9\text{ mm})^2 \approx \boxed{26\text{ mm}^2}$$

(*b*) Express the difference between the area of the square hole and the rectangular peg in terms of s_h, s_p and the new length of the peg ℓ_p:

$$\Delta A' = s_h^2 - s_p \ell_p$$

Substitute numerical values and evaluate $\Delta A'$:

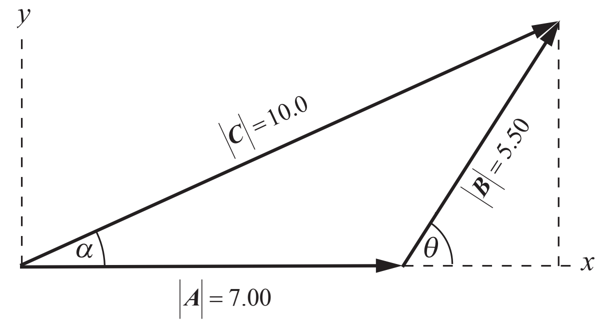
$$\Delta A' = (43.2\text{ mm})^2 - (42.9\text{ mm})(42.9\text{ mm} - 0.10\text{ mm}) \approx \boxed{30\text{ mm}^2}$$

Vectors and Their Properties

52 • A vector 7.00 units long and a vector 5.50 units long are added. Their sum is a vector 10.0 units long. (*a*) Show graphically at least one way that this can be accomplished. (*b*) Using your sketch in Part (*a*), determine the angle between the original two vectors.

Picture the Problem Let $\vec{A}$ be the vector whose length is 7.00 units and let $\vec{B}$ be the vector whose length is 5.50 units. $\vec{C} = \vec{A} + \vec{B}$ is the sum of $\vec{A}$ and $\vec{B}$. The fact that their sum is 10.0 long tells us that the vectors are not collinear. We can use the of the vectors to find the angle between $\vec{A}$ and $\vec{B}$.

(*a*) A graphical representation of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is shown below. θ is the angle between $\vec{A}$ and $\vec{B}$.



(*b*) The components of the vectors are related as follows:

$$A_x + B_x = C_x$$
and
$$A_y + B_y = C_y$$

Substituting for the components gives:

$$7.00 + (5.50)\cos\theta = (10.0)\cos\alpha$$
and
$$(5.50)\sin\theta = (10.0)\sin\alpha$$

Squaring and adding these equations yields:

$$(100)\sin^2\alpha + (100)\cos^2\alpha = (5.50)\sin^2\theta + [7.00 + (5.50)\cos\theta]^2$$

or

$$(100)(\sin^2\alpha + \cos^2\alpha) = (5.50)\sin^2\theta + [7.00 + (5.50)\cos\theta]^2$$

Because $\sin^2\alpha + \cos^2\alpha = 1$:

$$100 = (5.50)^2\sin^2\theta + [7.00 + (5.50)\cos\theta]^2$$

Solve this equation for θ (you can (1) use the same trigonometric identity used in the previous step to eliminate either $\sin^2\theta$ or $\cos^2\theta$ in favor of the other and then solve the resulting equation or (2) use your graphing calculator's SOLVER program) to obtain:

$$\theta = \boxed{74.4°}$$

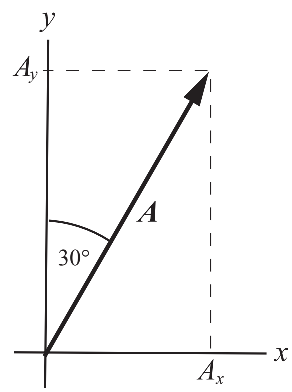
Remarks: You could also solve Part (*b*) of this problem by using the law of cosines.

53 • **[SSM]** Determine the *x* and *y* components of the following three vectors in the *xy* plane. (*a*) A 10-m displacement vector that makes an angle of 30° clockwise from the +*y* direction. (*b*) A 25-m/s velocity vector that makes an

angle of –40° counterclockwise from the –*x* direction. (*c*) A 40-lb force vector that makes an angle of 120° counterclockwise from the –*y* direction.

Picture the Problem The *x* and *y* components of these vectors are their projections onto the *x* and *y* axes. Note that the components are calculated using the angle each vector makes with the +*x* axis.

(*a*) Sketch the displacement vector (call it $\vec{A}$) and note that it makes an angle of 60° with the +*x* axis:



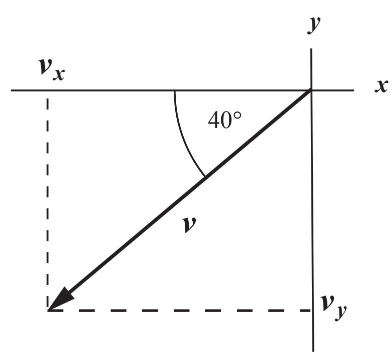
Find the *x* and *y* components of $\vec{A}$:

$$A_x = (10\text{ m})\cos 60° = \boxed{5.0\text{ m}}$$

and

$$A_y = (10\text{ m})\sin 60° = \boxed{8.7\text{ m}}$$

(*b*) Sketch the velocity vector (call it $\vec{v}$) and note that it makes an angle of 220° with the +*x* axis:



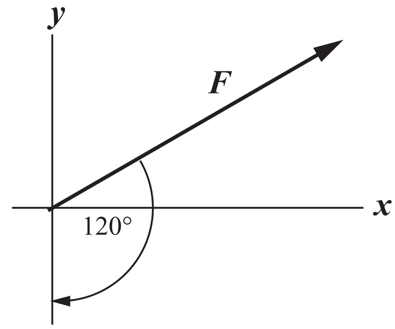
Find the *x* and *y* components of $\vec{v}$:

$$v_x = (25\text{ m/s})\cos 220° = \boxed{-19\text{ m/s}}$$

and

$$v_y = (25\text{ m/s})\sin 220° = \boxed{-16\text{ m/s}}$$

(c) Sketch the force vector (call it $\vec{F}$) and note that it makes an angle of 30° with the $+x$ axis:



Find the x and y components of $\vec{F}$:

$$F_x = (40\text{ lb})\cos 30° = \boxed{35\text{ lb}}$$

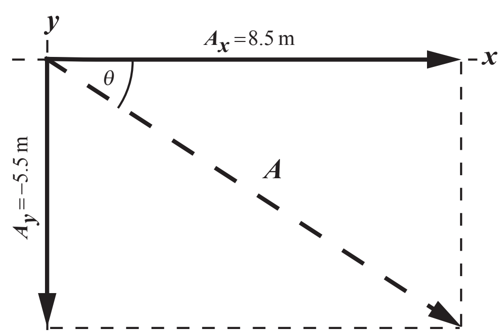
and

$$F_y = (40\text{ lb})\sin 30° = \boxed{20\text{ lb}}$$

54 • Rewrite the following vectors in terms of their magnitude and angle (counterclockwise from the $+x$ direction). (*a*) A displacement vector with an x component of +8.5 m and a y component of −5.5 m. (*b*) A velocity vector with an x component of −75 m/s and a y component of +35 m/s. (*c*) A force vector with a magnitude of 50 lb that is in the third quadrant with an x component whose magnitude is 40 lb.

Picture the Problem We can use the Pythagorean Theorem to find magnitudes of these vectors from their x and y components and trigonometry to find their direction angles.

(*a*) Sketch the components of the displacement vector (call it $\vec{A}$) and show their resultant:



Use the Pythagorean Theorem to find the magnitude of $\vec{A}$:

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(8.5\text{ m})^2 + (-5.5\text{ m})^2}$$

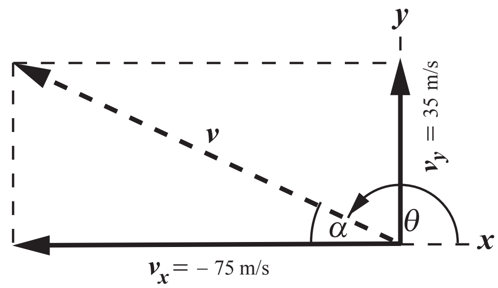
$$= \boxed{10\text{ m}}$$

Find the direction angle of $\vec{A}$:

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{-5.5\text{ m}}{8.5\text{ m}}\right)$$

$$= \boxed{-33°} = \boxed{327°}$$

(b) Sketch the components of the velocity vector (call it $\vec{v}$) and show their resultant:



Use the Pythagorean Theorem to find the magnitude of $\vec{v}$ and trigonometry to find its direction angle:

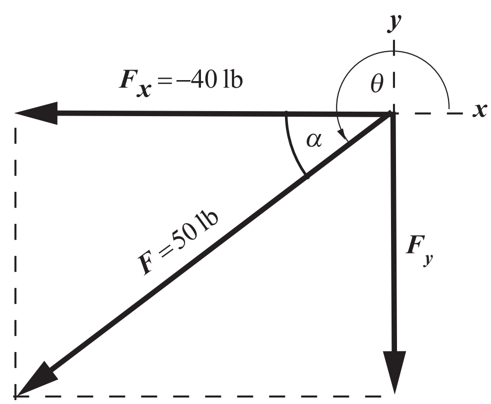
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-75\text{ m/s})^2 + (35\text{ m/s})^2}$$
$$= \boxed{83\text{ m/s}}$$

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{35\text{ m/s}}{75\text{ m/s}}\right) = 25°$$

and

$$\boldsymbol{\theta} = 180° - \boldsymbol{\alpha} = 180° - 25° = \boxed{155°}$$

(c) Sketch the force vector (call it $\vec{F}$) and its x-component and show F_y:



Use trigonometry to find α and, hence, θ:

$$\alpha = \cos^{-1}\left(\frac{F_x}{F}\right) = \cos^{-1}\left(\frac{40\text{ lb}}{50\text{ lb}}\right) = 37°$$

$$\theta = 180° + \alpha = 180° + 37° = \boxed{217°}$$

Use the relationship between a vector and its vertical component to find F_y:

$$F_y = F\sin\theta = (50\text{ lb})\sin 217°$$
$$= \boxed{-30\text{ lb}}$$

Remarks: In Part (*c*) we could have used the Pythagorean Theorem to find the magnitude of F_y.

55 • You walk 100 m in a straight line on a horizontal plane. If this walk took you 50 m east, what are your possible north or south movements? What are the possible angles that your walk made with respect to due east?

Picture the Problem There are two directions that you could have walked that are consistent with your having walked 50 m to the east. One possibility is walking in the north-east direction and the other is walking in the south-east direction. We can use the Pythagorean Theorem to find the distance you walked either north or south and then use trigonometry to find the two angles that correspond to your having walked in either of these directions.

Letting A represent the magnitude of your displacement on the walk, use the Pythagorean Theorem to relate A to its horizontal (A_x) and vertical (A_y) components:

$$A_y = \sqrt{A^2 - A_x^2}$$

Substitute numerical values and evaluate A_y:

$$A_y = \sqrt{(100\ \text{m})^2 - (50\ \text{m})^2} = \pm 87\ \text{m}$$

The plus-and-minus-signs mean that you could have gone 87 m north or 87 m south.

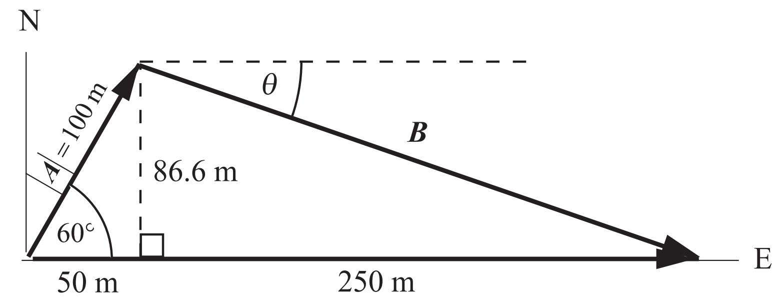
Use trigonometry to relate the directions you could have walked to the distance you walked and its easterly component:

$$\theta = \cos^{-1}\left(\frac{50\ \text{m}}{100\ \text{m}}\right) = \pm 60°$$

The plus-and-minus-signs mean that you could have walked 60° north of east or 60° south of east.

56 • The final destination of your journey is 300 m due east of your starting point. The first leg of this journey is the walk described in Problem 55, and the second leg is also a walk along a single straight-line path. Estimate graphically the length and heading for the second leg of your journey.

Picture the Problem Let $\vec{\boldsymbol{A}}$ be the vector whose length is 100 m and whose direction is 60° N of E. Let $\vec{\boldsymbol{B}}$ be the vector whose direction and magnitude we are to determine and assume that you initially walked in a direction north of east. The graphical representation that we can use to estimate these quantities is shown below. Knowing that the magnitude of $\vec{\boldsymbol{A}}$ is 100 m, use any convenient scale to determine the length of $\vec{\boldsymbol{B}}$ and a protractor to determine the value of θ.



The magnitude of the second leg of your journey is about 260 m at an angle of approximately 20° S of E. If you had initially walked into the southeast, the magnitude of the second leg of your journey would still be about 260 m but its direction would be approximately 20° N of E.

Remarks: If you use the Pythagorean Theorem and right-triangle trigonometry, you'll find that the length of the second leg of your journey is 265 m and that $\boldsymbol{\theta}$ = 19° S of E.

57 •• Given the following vectors: $\vec{\boldsymbol{A}} = 3.4\hat{\boldsymbol{i}} + 4.7\hat{\boldsymbol{j}}$, $\vec{\boldsymbol{B}} = (-7.7)\hat{\boldsymbol{i}} + 3.2\hat{\boldsymbol{j}}$, and $\vec{\boldsymbol{C}} = 5.4\hat{\boldsymbol{i}} + (-9.1)\hat{\boldsymbol{j}}$. (*a*) Find the vector $\vec{\boldsymbol{D}}$, in unit vector notation, such that $\vec{\boldsymbol{D}} + 2\vec{\boldsymbol{A}} - 3\vec{\boldsymbol{C}} + 4\vec{\boldsymbol{B}} = 0$. (*b*) Express your answer in Part (*a*) in terms of magnitude and angle with the +*x* direction.

Picture the Problem We can find the vector $\vec{\boldsymbol{D}}$ by solving the equation $\vec{\boldsymbol{D}} + 2\vec{\boldsymbol{A}} - 3\vec{\boldsymbol{C}} + 4\vec{\boldsymbol{B}} = 0$ for $\vec{\boldsymbol{D}}$ and then substituting for the vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$. In (*b*) we can use the components of $\vec{\boldsymbol{D}}$ to find its magnitude and direction.

(*a*) Solve the vector equation that gives the condition that must satisfied for $\vec{\boldsymbol{D}}$:

$$\vec{\boldsymbol{D}} = -2\vec{\boldsymbol{A}} + 3\vec{\boldsymbol{C}} - 4\vec{\boldsymbol{B}}$$

Substitute for $\vec{\boldsymbol{A}}, \vec{\boldsymbol{C}}$ and $\vec{\boldsymbol{B}}$ and simplify to obtain:

$$\begin{aligned}\vec{\boldsymbol{D}} &= -2\left(3.4\hat{\boldsymbol{i}} + 4.7\hat{\boldsymbol{j}}\right) + 3\left(5.4\hat{\boldsymbol{i}} - 9.1\hat{\boldsymbol{j}}\right) - 4\left(-7.7\hat{\boldsymbol{i}} + 3.2\hat{\boldsymbol{j}}\right)\\ &= (-6.8 + 16.2 + 30.8)\hat{\boldsymbol{i}} + (-9.4 - 27.3 - 12.8)\hat{\boldsymbol{j}}\\ &= 40.2\hat{\boldsymbol{i}} - 49.5\hat{\boldsymbol{j}} = \boxed{40\hat{\boldsymbol{i}} - 50\hat{\boldsymbol{j}}}\end{aligned}$$

(*b*) Use the Pythagorean Theorem to relate the magnitude of $\vec{\boldsymbol{D}}$ to its components D_x and D_y:

$$D = \sqrt{D_x^2 + D_y^2}$$

Substitute numerical values and evaluate D:

$$D = \sqrt{(40.2)^2 + (-49.5)^2} = \boxed{63.8}$$

Use trigonometry to express and evaluate the angle θ :

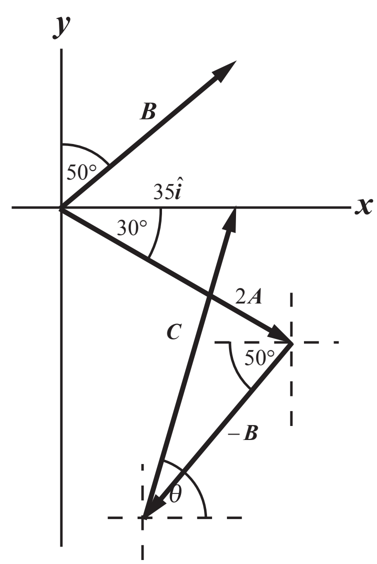
$$\theta = \tan^{-1}\left(\frac{D_y}{D_x}\right) = \tan^{-1}\left(\frac{-49.5}{40.2}\right)$$
$$= \boxed{-51°}$$

where the minus sign means that $\vec{D}$ is in the 4th quadrant.

58 •• Given the following force vectors: $\vec{A}$ is 25 lb at an angle of 30° clockwise from the +*x* axis, and $\vec{B}$ is 42 lb at an angle of 50° clockwise from the +*y* axis. (*a*) Make a sketch and visually estimate the magnitude and angle of the vector $\vec{C}$ such that $2\vec{A} + \vec{C} - \vec{B}$ results in a vector with a magnitude of 35 lb pointing in the +*x* direction. (*b*) Repeat the calculation in Part (*a*) using the method of components and compare your result to the estimate in (*a*).

Picture the Problem A diagram showing the condition that $2\vec{A} + \vec{C} - \vec{B} = 35\hat{\boldsymbol{i}}$ and from which one can scale the values of C and θ is shown below. In (*b*) we can use the two scalar equations corresponding to this vector equation to check our graphical results.

(*a*) A diagram showing the conditions imposed by $2\vec{A} + \vec{C} - \vec{B} = 35\hat{\boldsymbol{i}}$ approximately to scale is shown to the right.



The magnitude of $\vec{C}$ is approximately $\boxed{57\text{ lb}}$ and the angle θ is approximately $\boxed{68°.}$

(*b*) Express the condition relating the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$:

$$2\vec{A} + \vec{C} - \vec{B} = 35\hat{\boldsymbol{i}}$$

The corresponding scalar equations are:

$$2A_x + C_x - B_x = 35$$
and
$$2A_y + C_y - B_y = 0$$

Solve these equations for C_x and C_y to obtain:

$$C_x = 35 - 2A_x + B_x$$
and
$$C_y = -2A_y + B_y$$

Substitute for the x and y components of $\vec{A}$ and $\vec{B}$ to obtain:

$$\begin{aligned} C_x &= 35\text{ lb} - 2[(25\text{ lb})\cos 330°] \\ &\qquad + (42\text{ lb})\cos 40° \\ &= 23.9\text{ lb} \end{aligned}$$
and
$$\begin{aligned} C_y &= -2[(25\text{ lb})\sin 330°] + (42\text{ lb})\sin 40° \\ &= 52\text{ lb} \end{aligned}$$

Use the Pythagorean Theorem to relate the magnitude of $\vec{C}$ to its components:

$$C = \sqrt{C_x^2 + C_y^2}$$

Substitute numerical values and evaluate C:

$$C = \sqrt{(23.9\text{ lb})^2 + (52.0\text{ lb})^2} = \boxed{57\text{ lb}}$$

Use trigonometry to find the direction of $\vec{C}$:

$$\theta = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{52.0\text{ lb}}{23.9\text{ lb}}\right) = \boxed{65°}$$

Remarks: The analytical results for *C* and θ are in excellent agreement with the values determined graphically.

59 •• [SSM] Calculate the unit vector (in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$) in the direction opposite to the direction of each of the vectors in Problem 57.

Picture the Problem The unit vector in the direction opposite to the direction of a vector is found by taking the negative of the unit vector in the direction of the vector. The unit vector in the direction of a given vector is found by dividing the given vector by its magnitude.

The unit vector in the direction of $\vec{A}$ is given by:

$$\hat{\boldsymbol{A}} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{\sqrt{A_x^2 + A_y^2}}$$

Substitute for $\vec{A}$, A_x, and A_y and evaluate $\hat{\boldsymbol{A}}$:

$$\hat{\boldsymbol{A}} = \frac{3.4\hat{\boldsymbol{i}} + 4.7\hat{\boldsymbol{j}}}{\sqrt{(3.4)^2 + (4.7)^2}} = 0.59\hat{\boldsymbol{i}} + 0.81\hat{\boldsymbol{j}}$$

The unit vector in the direction opposite to that of $\vec{A}$ is given by:

$$-\hat{A} = \boxed{-0.59\hat{i} - 0.81\hat{j}}$$

The unit vector in the direction of $\vec{B}$ is given by:

$$\hat{B} = \frac{\vec{B}}{|\vec{B}|} = \frac{\vec{B}}{\sqrt{B_x^2 + B_y^2}}$$

Substitute for $\vec{B}$, B_x, and B_y and evaluate $\hat{B}$:

$$\hat{B} = \frac{-7.7\hat{i} + 3.2\hat{j}}{\sqrt{(-7.7)^2 + (3.2)^2}} = -0.92\hat{i} + 0.38\hat{j}$$

The unit vector in the direction opposite to that of $\vec{B}$ is given by:

$$-\hat{B} = \boxed{0.92\hat{i} - 0.38\hat{j}}$$

The unit vector in the direction of $\vec{C}$ is given by:

$$\hat{C} = \frac{\vec{C}}{|\vec{C}|} = \frac{\vec{C}}{\sqrt{C_x^2 + C_y^2}}$$

Substitute for $\vec{C}$, C_x, and C_y and evaluate $\hat{C}$:

$$\hat{C} = \frac{5.4\hat{i} - 9.1\hat{j}}{\sqrt{(5.4)^2 + (-9.1)^2}} = \boxed{0.51\hat{i} - 0.86\hat{j}}$$

The unit vector in the direction opposite that of $\vec{C}$ is given by:

$$-\hat{C} = \boxed{-0.51\hat{i} + 0.86\hat{j}}$$

60 •• Unit vectors $\hat{i}$ and $\hat{j}$ are directed east and north, respectively. Calculate the unit vector (in terms of $\hat{i}$ and $\hat{j}$) in the following directions. (*a*) northeast, (*b*) 70° clockwise from the $-y$ axis, (*c*) southwest.

Picture the Problem The unit vector in a given direction is a vector pointing in that direction whose magnitude is 1.

(*a*) The unit vector in the northeast direction is given by:

$$\hat{u}_{\text{NE}} = (1)\cos 45°\hat{i} + (1)\sin 45°\hat{j} = \boxed{0.707\hat{i} + 0.707\hat{j}}$$

(*b*) The unit vector 70° clockwise from the $-y$ axis is given by:

$$\hat{u} = (1)\cos 200°\hat{i} + (1)\sin 200°\hat{j} = \boxed{-0.940\hat{i} - 0.342\hat{j}}$$

(*c*) The unit vector in the southwest direction is given by:

$$\hat{\boldsymbol{u}}_{\mathrm{SW}} = (1)\cos 225°\hat{\boldsymbol{i}} + (1)\sin 225°\hat{\boldsymbol{j}}$$
$$= \boxed{-0.707\hat{\boldsymbol{i}} - 0.707\hat{\boldsymbol{j}}}$$

Remarks: One can confirm that a given vector is, in fact, a unit vector by checking its magnitude.

General Problems

61 • [SSM] The Apollo trips to the moon in the 1960's and 1970's typically took 3 days to travel the Earth-moon distance once they left Earth orbit. Estimate the spacecraft's average speed in kilometers per hour, miles per hour, and meters per second.

Picture the Problem Average speed is defined to be the distance traveled divided by the elapsed time. The Earth-moon distance and the distance and time conversion factors can be found on the inside-front cover of the text. We'll assume that 3 days means exactly three days.

Express the average speed of Apollo as it travels to the moon:

$$v_{\mathrm{av}} = \frac{\text{distance traveled}}{\text{elapsed time}}$$

Substitute numerical values to obtain:

$$v_{\mathrm{av}} = \frac{2.39\times10^{5}\ \mathrm{mi}}{3\ \mathrm{d}}$$

Use the fact that there are 24 h in 1 d to convert 3 d into hours:

$$v_{\mathrm{av}} = \frac{2.39\times10^{5}\ \mathrm{mi}}{3\ \mathrm{d}\times\dfrac{24\ \mathrm{h}}{\mathrm{d}}}$$
$$= 3.319\times10^{3}\ \mathrm{mi/h}$$
$$= \boxed{3.32\times10^{3}\ \mathrm{mi/h}}$$

Use the fact that 1 mi is equal to 1.609 km to convert the spacecraft's average speed to km/h:

$$v_{\mathrm{av}} = 3.319\times10^{3}\ \frac{\mathrm{mi}}{\mathrm{h}}\times1.609\,\frac{\mathrm{km}}{\mathrm{mi}} = 5.340\times10^{3}\ \frac{\mathrm{km}}{\mathrm{h}} = \boxed{5.34\times10^{3}\ \mathrm{km/h}}$$

Use the facts that there are 3600 s in 1 h and 1000 m in 1 km to convert the spacecraft's average speed to m/s:

$$v_{\text{av}} = 5.340\times10^{6}\,\frac{\text{km}}{\text{h}}\times\frac{10^{3}\text{ m}}{\text{km}}\times\frac{1\text{ h}}{3600\text{ s}} = 1.485\times10^{3}\text{ m/s} = \boxed{1.49\times10^{3}\text{ m/s}}$$

Remarks: An alternative to multiplying by 10^3 m/km in the last step is to replace the metric prefix ″k″ in ″km″by 10^3.

62 • On many of the roads in Canada the speed limit is 100 km/h. What is this speed limit in miles per hour?

Picture the Problem We can use the conversion factor 1 mi = 1.609 km to convert 100 km/h into mi/h.

Multiply 100 km/h by 1 mi/1.609 km to obtain:

$$100\,\frac{\text{km}}{\text{h}} = 100\,\frac{\text{km}}{\text{h}}\times\frac{1\,\text{mi}}{1.609\,\text{km}}$$
$$= \boxed{62.2\,\text{mi/h}}$$

63 • If you could count \$1.00 per second, how many years would it take to count 1.00 billion dollars?

Picture the Problem We can use a series of conversion factors to convert 1 billion seconds into years.

Multiply 1 billion seconds by the appropriate conversion factors to convert into years:

$$10^{9}\text{ s} = 10^{9}\text{ s}\times\frac{1\text{h}}{3600\text{s}}\times\frac{1\,\text{day}}{24\,\text{h}}\times\frac{1\,\text{y}}{365.24\,\text{days}} = \boxed{31.7\,\text{y}}$$

64 • (*a*) The speed of light in vacuum is 186,000 mi/s = 3.00×10^8 m/s. Use this fact to find the number of kilometers in a mile. (*b*) The weight of 1.00 ft^3 of water is 62.4 lb, and 1.00 ft = 30.5 cm. Use this and the fact that 1.00 cm^3 of water has a mass of 1.00 g to find the weight in pounds of a 1.00-kg mass.

Picture the Problem In both the examples cited we can equate expressions for the physical quantities, expressed in different units, and then divide both sides of the equation by one of the expressions to obtain the desired conversion factor.

(*a*) Divide both sides of the equation expressing the speed of light in the two systems of measurement by 186,000 mi/s to obtain:

$$1 = \frac{3.00\times10^{8}\ \text{m/s}}{1.86\times10^{5}\ \text{mi/h}} = 1.61\times10^{3}\ \text{m/mi}$$
$$= \left(1.61\times10^{3}\ \frac{\text{m}}{\text{mi}}\right)\left(\frac{1\,\text{km}}{10^{3}\ \text{m}}\right)$$
$$= \boxed{1.61\,\text{km/mi}}$$

(*b*) Find the volume of 1.00 kg of water:

Volume of 1.00 kg = 10^3 g is 10^3 cm^3

Express 10^3 cm^3 in ft^3:

$$(10\,\text{cm})^3\left(\frac{1.00\,\text{ft}}{30.5\ \text{cm}}\right)^3 = 0.03525\,\text{ft}^3$$

Relate the weight of 1.00 ft^3 of water to the volume occupied by 1.00 kg of water:

$$\frac{1.00\,\text{kg}}{0.03525\,\text{ft}^3} = 62.4\frac{\text{lb}}{\text{ft}^3}$$

Divide both sides of the equation by the left-hand side to obtain:

$$1 = \frac{62.4\dfrac{\text{lb}}{\text{ft}^3}}{\dfrac{1.00\,\text{kg}}{0.03525\,\text{ft}^3}} = \boxed{2.20\,\text{lb/kg}}$$

65 • The mass of one uranium atom is 4.0×10^{-26} kg. How many uranium atoms are there in 8.0 g of pure uranium?

Picture the Problem We can use the given information to equate the ratios of the number of uranium atoms in 8 g of pure uranium and of 1 atom to its mass.

Express the proportion relating the number of uranium atoms N_U in 8.0 g of pure uranium to the mass of 1 atom:

$$\frac{N_U}{8.0\,\text{g}} = \frac{1\,\text{atom}}{4.0\times10^{-26}\ \text{kg}}$$

Solve for and evaluate N_U:

$$N_U = \left(8.0\,\text{g}\times\frac{1\,\text{kg}}{10^3\ \text{g}}\right)\left(\frac{1\,\text{atom}}{4.0\times10^{-26}\ \text{kg}}\right)$$
$$= \boxed{2.0\times10^{23}}$$

66 •• During a thunderstorm, a total of 1.4 in of rain falls. How much water falls on one acre of land? (1 mi^2 = 640 acres.) Express your answer in (*a*) cubic inches, (*b*) cubic feet, (*c*) cubic meters, and (*d*) kilograms. Note that the density of water is 1000 kg/m^3.

Picture the Problem Assuming that the water is distributed uniformly over the one acre of land, its volume is the product of the area over which it is distributed and its depth. The mass of the water is the product of its density and volume. The required conversion factors can be found in the front material of the text.

(*a*) Express the volume V of water in terms of its depth h and the area A over which it falls:

$$V = Ah$$

Substitute numerical values and evaluate V:

$$V = (1\,\text{acre})\left(\frac{1\,\text{mi}^2}{640\,\text{acres}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)^2\left(\frac{12\,\text{in}}{\text{ft}}\right)^2(1.4\,\text{in}) = 8.782\times10^6\,\text{in}^3$$

$$= \boxed{8.8\times10^6\,\text{in}^3}$$

(*b*) Convert in^3 into ft^3:

$$V = 8.782\times10^6\,\text{in}^3\times\left(\frac{1\,\text{ft}}{12\,\text{in}}\right)^3$$

$$= 5.082\times10^3\,\text{ft}^3 = \boxed{5.1\times10^3\,\text{ft}^3}$$

(*c*) Convert ft^3 into m^3:

$$V = 5.082\times10^3\,\text{ft}^3\times\left(\frac{12\,\text{in}}{\text{ft}}\right)^3\times\left(\frac{2.540\,\text{cm}}{\text{in}}\right)^3\times\left(\frac{1\,\text{m}}{100\,\text{cm}}\right)^3 = 1.439\times102\,\text{m}^3$$

$$= \boxed{1.4\times10^2\,\text{m}^3}$$

(*d*) The mass of the water is the product of its density ρ and volume V:

$$m = \rho V$$

Substitute numerical values and evaluate m:

$$m = \left(10^3\,\frac{\text{kg}}{\text{m}^3}\right)(1.439\times10^2\,\text{m}^3)$$

$$= \boxed{1.4\times10^5\,\text{kg}}$$

67 •• An iron nucleus has a radius of 5.4×10^{-15} m and a mass of 9.3×10^{-26} kg. (*a*) What is its mass per unit volume in kg/ m^3? (*b*) If Earth had the same mass per unit volume, what would be its radius? (The mass of Earth is 5.98×10^{24} kg.)

Picture the Problem The mass per unit volume of an object is its density.

(*a*) The density ρ of an object is its mass m per unit volume V:

$$\rho = \frac{m}{V}$$

Assuming the iron nucleus to be spherical, its volume as a function of its radius r is given by:

$$V = \frac{4}{3}\pi r^3$$

Substitute for V and simplify to obtain:

$$\rho = \frac{m}{\frac{4}{3}\pi r^3} = \frac{3m}{4\pi r^3} \qquad (1)$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{3\left(9.3\times10^{-26}\ \text{kg}\right)}{4\pi\left(5.4\times10^{-15}\ \text{m}\right)^3} = 1.410\times10^{17}\ \text{kg/m}^3 = \boxed{1.4\times10^{17}\ \text{kg/m}^3}$$

(*b*) Solve equation (1) for r to obtain:

$$r = \sqrt[3]{\frac{3m}{4\pi\rho}}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\frac{3\left(5.98\times10^{24}\ \text{kg}\right)}{4\pi\left(1.410\times10^{17}\ \frac{\text{kg}}{\text{m}^3}\right)}} = \boxed{2.2\times10^{2}\ \text{m}}$$

or about 200 m!

68 •• The Canadian Norman Wells Oil Pipeline extends from Norman Wells, Northwest Territories, to Zama, Alberta. The 8.68×10^5-m-long pipeline has an inside diameter of 12 in and can be supplied with oil at 35 L/s. (*a*) What is the volume of oil in the pipeline if it is full at some instant in time? (*b*) How long would it take to fill the pipeline with oil if it is initially empty?

Picture the Problem The volume of a cylinder is the product of its cross-sectional area and its length. The time required to fill the pipeline with oil is the ratio of its volume to the flow rate R of the oil. We'll assume that the pipe has a diameter of exactly 12 in.

(*a*) Express the volume V of the cylindrical pipe in terms of its radius r and its length L:

$$V = \pi r^2 L$$

Substitute numerical values and evaluate V:

$$V = \pi\left(6\,\text{in} \times \frac{2.540\times10^{-2}\,\text{m}}{\text{in}}\right)^2 \left(8.68\times10^5\ \text{m}\right) = \boxed{6.3\times10^4\ \text{m}^3}$$

(*b*) Express the time Δt to fill the pipe in terms of its volume V and the flow rate R of the oil:

$$\Delta t = \frac{V}{R}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{6.3\times10^4\ \text{m}^3}{\left(35\,\frac{\text{L}}{\text{s}}\right)\left(\frac{10^{-3}\ \text{m}^3}{\text{L}}\right)} = \boxed{1.8\times10^6\ \text{s}}$$

or about 21 days!

69 •• The astronomical unit (AU) is defined as the mean center-to-center distance from Earth to the sun, namely 1.496×10^{11} m. The parsec is the radius of a circle for which a central angle of 1 s intercepts an arc of length 1 AU. The light-year is the distance that light travels in 1 y. (*a*) How many parsecs are there in one astronomical unit? (*b*) How many meters are in a parsec? (*c*) How many meters in a light-year? (*d*) How many astronomical units in a light-year? (*e*) How many light-years in a parsec?

Picture the Problem We can use the relationship between an angle θ, measured in radians, subtended at the center of a circle, the radius R of the circle, and the length L of the arc to answer these questions concerning the astronomical units of measure. We'll take the speed of light to be 2.998×10^8 m/s.

(*a*) Relate the angle θ subtended by an arc of length S to the distance R:

$$\theta = \frac{S}{R} \Rightarrow S = R\theta \qquad (1)$$

Substitute numerical values and evaluate S:

$$S = (1\,\text{parsec})(1\,\text{s})\left(\frac{1\,\text{min}}{60\,\text{s}}\right)\left(\frac{1°}{60\,\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{360°}\right) = \boxed{4.848\times 10^{-6}\,\text{parsec}}$$

(b) Solving equation (1) for R yields:

$$R = \frac{S}{\theta}$$

Substitute numerical values and evaluate R:

$$R = \frac{1.496\times 10^{11}\,\text{m}}{(1\,\text{s})\left(\frac{1\,\text{min}}{60\,\text{s}}\right)\left(\frac{1°}{60\,\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{360°}\right)} = \boxed{3.086\times 10^{16}\,\text{m}}$$

(c) The distance D light travels in a given interval of time Δt is given by:

$$D = c\Delta t$$

Substitute numerical values and evaluate D:

$$D = \left(2.998\times 10^{8}\,\frac{\text{m}}{\text{s}}\right)\left(365.24\frac{\text{d}}{\text{y}}\right)\left(24\frac{\text{h}}{\text{d}}\right)\left(60\frac{\text{min}}{\text{h}}\right)\left(60\frac{\text{s}}{\text{min}}\right) = \boxed{9.461\times 10^{15}\,\text{m}}$$

(d) Use the definition of 1 AU and the result from Part (c) to obtain:

$$1\,c\cdot\text{y} = \left(9.461\times 10^{15}\,\text{m}\right)\left(\frac{1\,\text{AU}}{1.496\times 10^{11}\,\text{m}}\right) = \boxed{6.324\times 10^{4}\,\text{AU}}$$

(e) Combine the results of Parts (b) and (c) to obtain:

$$1\,\text{parsec} = \left(3.086\times 10^{16}\,\text{m}\right)\left(\frac{1\,c\cdot\text{y}}{9.461\times 10^{15}\,\text{m}}\right) = \boxed{3.262\,c\cdot\text{y}}$$

70 •• If the average density of the universe is at least 6×10^{-27} kg/m^3, then the universe will eventually stop expanding and begin contracting. (a) How many electrons are needed in each cubic meter to produce the critical density? (b) How many protons per cubic meter would produce the critical density? ($m_e = 9.11 \times 10^{-31}$ kg; $m_p = 1.67 \times 10^{-27}$ kg.)

Picture the Problem Let N_e and N_p represent the number of electrons and the number of protons, respectively and ρ the critical average density of the universe. We can relate these quantities to the masses of the electron and proton using the definition of density.

(*a*) Using its definition, relate the required density ρ to the electron density N_e/V:

$$\rho = \frac{m}{V} = \frac{N_e m_e}{V} \Rightarrow \frac{N_e}{V} = \frac{\rho}{m_e} \quad (1)$$

Substitute numerical values and evaluate N_e/V:

$$\frac{N_e}{V} = \frac{6\times10^{-27}\ \text{kg/m}^3}{9.11\times10^{-31}\ \text{kg/electron}} = 6.586\times10^3\ \text{electrons/m}^3 \approx \boxed{7\times10^3\ \text{electrons/m}^3}$$

(*b*) Express and evaluate the ratio of the masses of an electron and a proton:

$$\frac{m_e}{m_p} = \frac{9.11\times10^{-31}\ \text{kg}}{1.67\times10^{-27}\ \text{kg}} = 5.455\times10^{-4}$$

Rewrite equation (1) in terms of protons:

$$\frac{N_p}{V} = \frac{\rho}{m_p} \quad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{\frac{N_p}{V}}{\frac{N_e}{V}} = \frac{m_e}{m_p} \text{ or } \frac{N_p}{V} = \frac{m_e}{m_p}\left(\frac{N_e}{V}\right)$$

Substitute numerical values and use the result from Part (*a*) to evaluate N_p/V:

$$\frac{N_p}{V} = \left(5.455\times10^{-4}\right)\left(6.586\times10^3\ \text{electrons/m}^3\right) \approx \boxed{4\,\text{protons/m}^3}$$

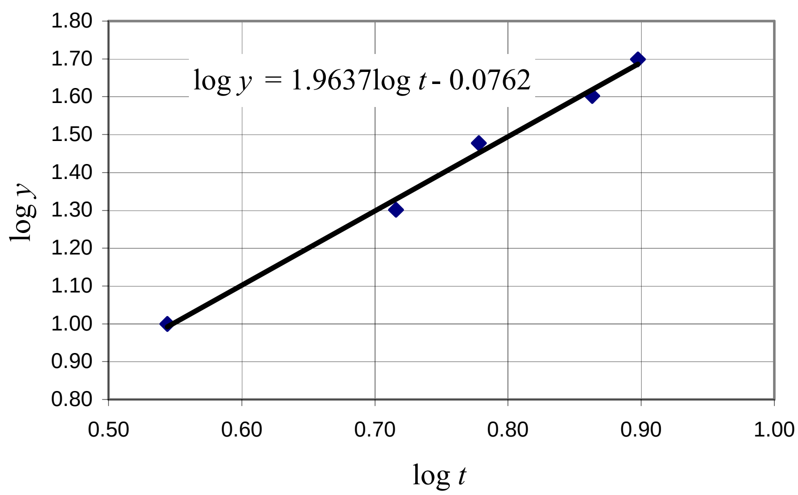
71 ••• You are an astronaut doing physics experiments on the moon. You are interested in the experimental relationship between distance fallen, y, and time elapsed, t, of falling objects dropped from rest. You have taken some data for a falling penny, which is represented in the table below. You expect that a general relationship between distance y and time t is $y = Bt^C$, where B and C are constants to be determined experimentally. To accomplish this, create a *log-log* plot of the data: (*a*) graph log(y) vs. log(t), with log(y) the ordinate variable and log(t) the abscissa variable. (*b*) Show that if you take the log of each side of your equation, you get log(y) = log(B) + Clog(t). (*c*) By comparing this linear relationship to the graph of the data, estimate the values of B and C. (*d*) If you

drop a penny, how long should it take to fall 1.0 m? (*e*) In the next chapter, we will show that the expected relationship between y and t is $y = \frac{1}{2}at^2$, where a is the acceleration of the object. What is the acceleration of objects dropped on the moon?

y (m)	10	20	30	40	50
t (s)	3.5	5.2	6.0	7.3	7.9

Picture the Problem We can plot log y versus log t and find the slope of the best-fit line to determine the exponent C. The value of B can be determined from the intercept of this graph. Once we know C and B, we can solve $y = Bt^C$ for t as a function of y and use this result to determine the time required for an object to fall a given distance on the surface of the moon.

(*a*) The following graph of log y versus log t was created using a spreadsheet program. The equation shown on the graph was obtained using Excel's ″Add Trendline″ function. (Excel's ″Add Trendline″ function uses regression analysis to generate the trendline.)



(*b*) Taking the logarithm of both sides of the equation $y = Bt^C$ yields:

$$\log y = \log\left(Bt^C\right) = \log B + \log t^C$$
$$= \boxed{\log B + C\log t}$$

(*c*) Note that this result is of the form:

$$Y = b + mX$$
where
$Y = \log y$, $b = \log B$, $m = C$, and
$X = \log t$

From the regression analysis (trendline) we have: log B = −0.076

Solving for B yields: $B = 10^{-0.076} = \boxed{0.84\ \text{m/s}^2}$

where we have inferred the units from th given for $y = Bt^C$.

Also, from the regression analysis we have: $C = 1.96 \approx \boxed{2.0}$

(d) Solve $y = Bt^C$ for t to obtain:

$$t = \left(\frac{y}{B}\right)^{\frac{1}{C}}$$

Substitute numerical values and evaluate t to determine how long it would take a penny to fall 1.0 m:

$$t = \left(\frac{1.0\ \text{m}}{0.84\ \dfrac{\text{m}}{\text{s}^2}}\right)^{\frac{1}{2}} \approx \boxed{1.1\ \text{s}}$$

(*e*) Substituting for *B* and *C* in $y = Bt^C$ yields:

$$y = \left(0.84\ \frac{\text{m}}{\text{s}^2}\right)t^2$$

Compare this equation to $y = \frac{1}{2}at^2$ to obtain:

$$\tfrac{1}{2}a = 0.84\ \frac{\text{m}}{\text{s}^2}$$

and

$$a = 2\left(0.84\ \frac{\text{m}}{\text{s}^2}\right) = \boxed{1.7\ \frac{\text{m}}{\text{s}^2}}$$

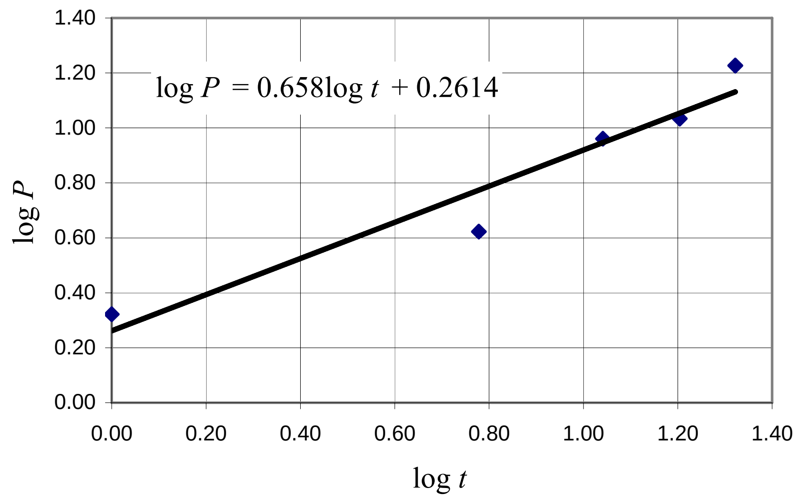
Remarks: One could use a graphing calculator to obtain the results in Parts (*a*) and (*c*).

72 **•••** A particular company's stock prices vary with the market and with the company's type of business, and can be very unpredictable, but people often try to look for mathematical patterns where they may not belong. Corning is a materials-engineering company located in upstate New York. Below is a table of the price of Corning stock on August 3, for every 5 years from 1981 to 2001. Assume that the price follows a power law: price (in \$) = Bt^C where *t* is expressed in years. (*a*) Evaluate the constants *B* and *C*. (*b*) According to the power law, what should the price of Corning stock have been on August 3, 2000? (It was actually \$82.83!)

Price (dollars)	2.10	4.19	9.14	10.82	16.85
Years since 1980	1	6	11	16	21

Picture the Problem We can plot log P versus log t and find the slope of the best-fit line to determine the exponent C. The value of B can be determined from the intercept of this graph. Once we know C and B, we can use $P = Bt^C$ to predict the price of Corning stock as a function of time.

(*a*) The following graph of log P versus log was created using a spreadsheet program. The equation shown on the graph was obtained using Excel's ″Add Trendline″ function. (Excel's ″Add Trendline″ function uses regression analysis to generate the trendline.)



Taking the logarithm of both sides of the equation $P = Bt^C$ yields:

$$\log P = \log\left(Bt^C\right) = \log B + \log t^C$$
$$= \log B + C\log t$$

Note that this result is of the form:

$Y = b + mX$
where
$Y = \log P$, $b = \log B$, $m = C$, and
$X = \log t$

From the regression analysis (trendline) we have:

$$\log B = 0.2614 \Rightarrow B = 10^{0.2614} = \boxed{\$1.83}$$

Also, from the regression analysis we have:

$$C = \boxed{0.658}$$

(*b*) Substituting for B and C in $P = Bt^C$ yields:

$$P = (\$1.83)t^{0.658}$$

Evaluate $P(20\text{ y})$ to obtain: $P(20\text{ y}) = (\$1.83)(20)^{0.658} = \boxed{\$13.14}$

Remarks: One could use a graphing calculator to obtain these results.

73 ••• [SSM] The Super-Kamiokande neutrino detector in Japan is a large transparent cylinder filled with ultra pure water. The height of the cylinder is 41.4 m and the diameter is 39.3 m. Calculate the mass of the water in the cylinder. Does this match the claim posted on the official Super-K Web site that the detector uses 50000 tons of water?

Picture the Problem We can use the definition of density to relate the mass of the water in the cylinder to its volume and the formula for the volume of a cylinder to express the volume of water used in the detector's cylinder. To convert our answer in kg to lb, we can use the fact that 1 kg weighs about 2.205 lb.

Relate the mass of water contained in the cylinder to its density and volume:

$$m = \rho V$$

Express the volume of a cylinder in terms of its diameter d and height h:

$$V = A_{\text{base}}h = \frac{\pi}{4}d^2h$$

Substitute in the expression for m to obtain:

$$m = \rho\frac{\pi}{4}d^2h$$

Substitute numerical values and evaluate m:

$$m = \left(10^3\text{ kg/m}^3\right)\left(\frac{\pi}{4}\right)(39.3\,\text{m})^2(41.4\,\text{m}) = 5.022\times10^7\text{ kg}$$

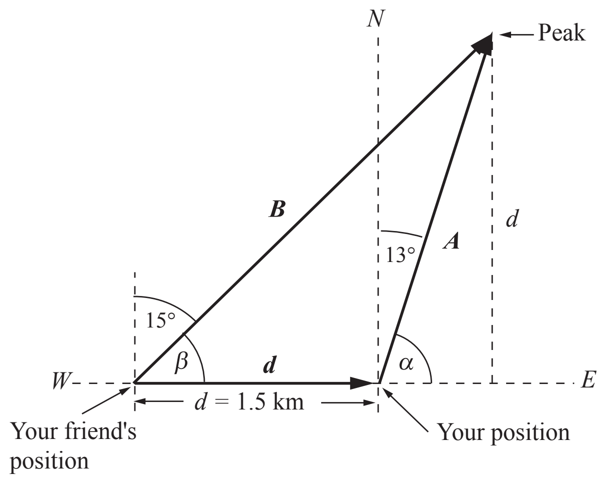
Convert 5.02×10^7 kg to tons:

$$m = 5.022\times10^7\text{ kg}\times\frac{2.205\,\text{lb}}{\text{kg}}\times\frac{1\,\text{ton}}{2000\,\text{lb}} = 55.4\times10^3\text{ ton}$$

The 50,000-ton claim is conservative. The actual weight is closer to 55,000 tons.

74 ••• You and a friend are out hiking across a large flat plain and decide to determine the height of a distant mountain peak, and also the horizontal distance from you to the peak Figure 1-19). In order to do this, you stand in one spot and determine that the sightline to the top of the peak is inclined at 7.5° above the horizontal. You also make note of the heading to the peak at that point: 13° east of north. You stand at the original position, and your friend hikes due west for 1.5 km. He then sights the peak and determines that its sightline has a heading of 15° east of north. How far is the mountain from your position, and how high is its summit above your position?

Picture the Problem Vector $\vec{A}$ lies in the plane of the plain and locates the base or the peak relative to you. Vector $\vec{B}$ also lies in the plane of the plain and locates the base of the peak relative to your friend when he/she has walked 1.5 km to the west. We can use the geometry of the diagram and the *E-W* components of the vectors $\vec{d}$, $\vec{B}$ and $\vec{A}$ to find the distance to the mountain from your position. Once we know the distance *A*, we can use a trigonometric relationship to find the height of the peak above your position.



Referring to the diagram, note that:

$$B\sin\beta = h$$
and
$$A\sin\alpha = h$$

Equating these expressions for *h* gives:

$$B\sin\beta = A\sin\alpha \Rightarrow B = \frac{\sin\alpha}{\sin\beta}A$$

Adding the *E-W* components of the vectors $\vec{d}$, $\vec{B}$ and $\vec{A}$ yields:

$$1.5\text{ km} = B\cos\beta - A\cos\alpha$$

Substitute for *B* and simplify to obtain:

$$1.5\text{ km} = A\frac{\sin\alpha}{\sin\beta}\cos\beta - A\cos\alpha$$
$$= A\left(\frac{\sin\alpha}{\tan\beta} - \cos\alpha\right)$$

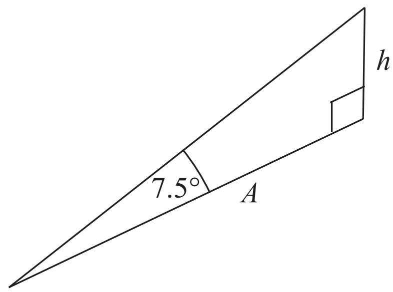
Solving for *A* yields:

$$A = \frac{1.5\text{ km}}{\dfrac{\sin\alpha}{\tan\beta} - \cos\alpha}$$

Substitute numerical values and evaluate A:

$$A = \frac{1.5\text{ km}}{\dfrac{\sin 77°}{\tan 75°} - \cos 77°} = 41.52\text{ km}$$
$$= \boxed{42\text{ km}}$$

Referring to the following diagram, we note that:



$$h = A\tan 7.5°$$
$$= (41.52\text{ km})\tan 7.5°$$
$$= \boxed{5.5\text{ km}}$$

Remarks: One can also solve this problem using the law of sines.

75 ••• The table below gives the periods T and orbit radii r for the motions of four satellites orbiting a dense, heavy asteroid. (a) These data can be fitted by the formula $T = Cr^n$. Find the values of the constants C and n. (b) A fifth satellite is discovered to have a period of 6.20 y. Find the radius for the orbit of this satellite, which fits the same formula.

Period T, y	0.44	1.61	3.88	7.89
Radius r, Gm	0.088	0.208	0.374	0.600

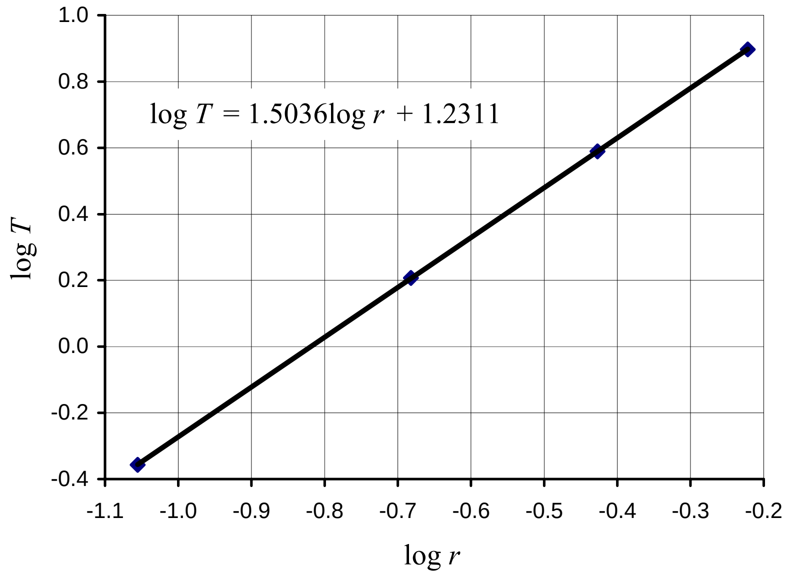
Picture the Problem We can plot log T versus log r and find the slope of the best-fit line to determine the exponent n. We can then use any of the ordered pairs to evaluate C. Once we know n and C, we can solve $T = Cr^n$ for r as a function of T.

(a) Take the logarithm (we'll arbitrarily use base 10) of both sides of $T = Cr^n$ and simplify to obtain:

$$\log(T) = \log(Cr^n) = \log C + \log r^n$$
$$= n\log r + \log C$$

Note that this equation is of the form $y = mx + b$. Hence a graph of log T vs. log r should be linear with a slope of n and a log T-intercept log C.

The following graph of log T versus log r was created using a spreadsheet program. The equation shown on the graph was obtained using Excel's ″Add Trendline″ function. (Excel's ″Add Trendline″ function uses regression analysis to generate the trendline.)



From the regression analysis we note that:

$$n = \boxed{1.50},$$

$$C = 10^{1.2311} = \boxed{17.0\,\text{y}/(\text{Gm})^{3/2}},$$

and

$$T = \boxed{\left(17.0\,\text{y}/(\text{Gm})^{3/2}\right)r^{1.50}} \qquad (1)$$

(*b*) Solve equation (1) for the radius of the planet's orbit:

$$r = \left(\frac{T}{17.0\,\text{y}/(\text{Gm})^{3/2}}\right)^{2/3}$$

Substitute numerical values and evaluate r:

$$r = \left(\frac{6.20\,\text{y}}{17.0\,\text{y}/(\text{Gm})^{3/2}}\right)^{2/3} = \boxed{0.510\text{ Gm}}$$

76 ••• The period T of a simple pendulum depends on the length L of the pendulum and the acceleration of gravity g (dimensions L/T^2). (*a*) Find a simple combination of L and g that has the dimensions of time. (*b*) Check the dependence of the period T on the length L by measuring the period (time for a complete swing back and forth) of a pendulum for two different values of L. (*c*) The correct formula relating T to L and g involves a constant that is a multiple of π, and cannot be obtained by the dimensional analysis of Part (*a*). It can be found by

experiment as in Part (*b*) if g is known. Using the value $g = 9.81$ m/s^2 and your experimental results from Part (*b*), find the formula relating T to L and g.

Picture the Problem We can express the relationship between the period T of the pendulum, its length L, and the acceleration of gravity g as $T = CL^a g^b$ and perform dimensional analysis to find the values of a and b and, hence, the function relating these variables. Once we've performed the experiment called for in Part (*b*), we can determine an experimental value for C.

(*a*) Express T as the product of L and g raised to powers a and b:

$$T = CL^a g^b \qquad (1)$$

where C is a dimensionless constant.

Write this equation in dimensional form:

$$[T] = [L]^a [g]^b$$

Substituting the dimensions of the physical quantities yields:

$$\mathrm{T} = \mathrm{L}^a \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^b$$

Because L does not appear on the left-hand side of the equation, we can write this equation as:

$$\mathrm{L}^0\mathrm{T}^1 = \mathrm{L}^{a+b}\mathrm{T}^{-2b}$$

Equate the exponents to obtain:

$$a + b = 0 \text{ and } -2b = 1$$

Solve these equations simultaneously to find a and b:

$$a = \tfrac{1}{2} \text{ and } b = -\tfrac{1}{2}$$

Substitute in equation (1) to obtain:

$$T = CL^{1/2} g^{-1/2} = \boxed{C\sqrt{\frac{L}{g}}} \qquad (2)$$

(*b*) If you use pendulums of lengths 1.0 m and 0.50 m; the periods should be about:

$$T(1.0\,\mathrm{m}) = \boxed{2.0\,\mathrm{s}}$$

and

$$T(0.50\,\mathrm{m}) = \boxed{1.4\,\mathrm{s}}$$

(*c*) Solving equation (2) for C yields:

$$C = T\sqrt{\frac{g}{L}}$$

Evaluate C with L = 1.0 m and T = 2.0 s:

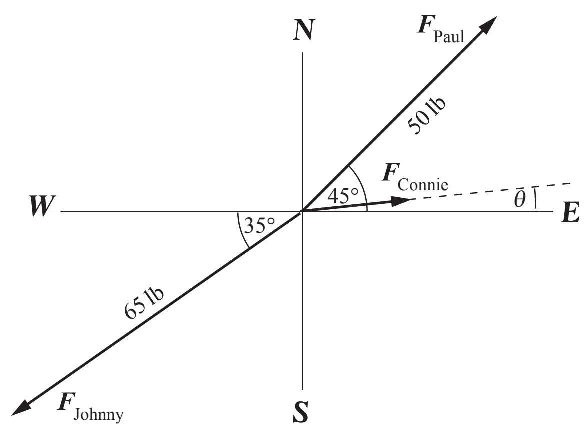
$$C=(2.0\,\text{s})\sqrt{\frac{9.81\,\text{m/s}^2}{1.0\,\text{m}}}=6.26\approx 2\pi$$

Substitute in equation (2) to obtain:

$$T=\boxed{2\pi\sqrt{\frac{L}{g}}}$$

77 ••• A sled at rest is suddenly pulled in three horizontal directions at the same time but it goes nowhere. Paul pulls to the northeast with a force of 50 lb. Johnny pulls at an angle of 35° south of due west with a force of 65 lb. Connie pulls with a force to be determined. (*a*) Express the boys' two forces in terms of the usual unit vectors (*b*) Determine the third force (from Connie), expressing it first in component form and then as a magnitude and angle (direction).

Picture the Problem A diagram showing the forces exerted by Paul, Johnny, and Connie is shown below. Once we've expressed the forces exerted by Paul and Johnny in vector form we can use them to find the force exerted by Connie.



(*a*) The force that Paul exerts is:

$$\vec{F}_{\text{Paul}}=[(50\,\text{lb})\cos 45°]\hat{i}+[(50\,\text{lb})\sin 45°]\hat{j}=(35.4\,\text{lb})\hat{i}+(35.4\,\text{lb})\hat{j}$$
$$=\boxed{(35\,\text{lb})\hat{i}+(35\,\text{lb})\hat{j}}$$

The force that Johnny exerts is:

$$\vec{F}_{\text{Johnny}}=[(65\,\text{lb})\cos 215°]\hat{i}+[(65\,\text{lb})\sin 215°]\hat{j}=(-53.2\,\text{lb})\hat{i}-(37.3\,\text{lb})\hat{j}$$
$$=\boxed{(-53\,\text{lb})\hat{i}+(-37\,\text{lb})\hat{j}}$$

The sum of the forces exerted by Paul and Johnny is:

$$\begin{aligned}\vec{F}_{\text{Paul}}+\vec{F}_{\text{Johnny}}&=(35.4\text{ lb})\hat{i}+(35.4\text{ lb})\hat{j}-(53.2\text{ lb})\hat{i}-(37.3\text{ lb})\hat{j}\\&=(-17.8\text{ lb})\hat{i}-(1.9\text{ lb})\hat{j}\end{aligned}$$

(*b*) The condition that the three forces must satisfy is:

$$\vec{F}_{\text{Paul}}+\vec{F}_{\text{Johnny}}+\vec{F}_{\text{Connie}}=0$$

Solving for $\vec{F}_{\text{Connie}}$ yields:

$$\vec{F}_{\text{Connie}}=-\left(\vec{F}_{\text{Paul}}+\vec{F}_{\text{Johnny}}\right)$$

Substitute for $\vec{F}_{\text{Paul}}+\vec{F}_{\text{Johnny}}$ to obtain:

$$\begin{aligned}\vec{F}_{\text{Connie}}&=-\left[(-17.8\text{ lb})\hat{i}-(1.9\text{ lb})\hat{j}\right]\\&=\boxed{(18\text{ lb})\hat{i}+(1.9\text{ lb})\hat{j}}\end{aligned}$$

The magnitude of $\vec{F}_{\text{Connie}}$ is:

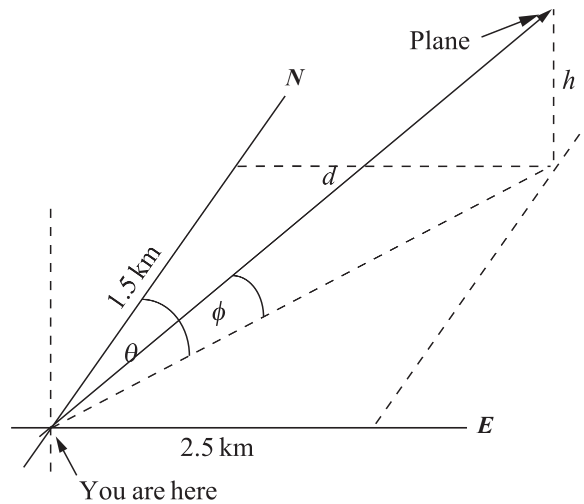
$$F_{\text{Connie}}=\sqrt{(17.8\text{ lb})^2+(1.9\text{ lb})^2}=\boxed{18\text{ lb}}$$

The direction that the force exerted by Connie acts is given by:

$$\theta=\tan^{-1}\left(\frac{1.9\text{ lb}}{17.8\text{ lb}}\right)=\boxed{6.1^\circ\text{ N of E}}$$

78 ••• You spot a plane that is 1.50 km North, 2.5 km East and at an altitude of 5.0 km above your position. (*a*) How far from you is the plane? (*b*) At what angle from due north (in the horizontal plane) are you looking? (*c*) Determine the plane's position vector (from your location) in terms of the unit vectors, letting $\hat{i}$ be toward the east direction, $\hat{j}$ be toward the north direction, and $\hat{k}$ be vertically upward. (*d*) At what elevation angle (above the horizontal plane of Earth) is the airplane?

Picture the Problem A diagram showing the given information is shown below. We can use the Pythagorean Theorem, trigonometry, and vector algebra to find the distance, angles, and expression called for in the problem statement.



(a) Use the Pythagorean Theorem to express d in terms of h and ℓ:

$$d = \sqrt{\ell^2 + h^2}$$

Substitute numerical values and evaluate d:

$$d = \sqrt{(2.5\text{ km})^2 + (1.5\text{ km})^2 + (5.0\text{ km})^2} = \boxed{5.8\text{ km}}$$

(b) Use trigonometry to evaluate the angle from due north at which you are looking at the plane:

$$\theta = \tan^{-1}\left(\frac{2.5\text{ km}}{1.5\text{ km}}\right) = \boxed{59^\circ\text{ E of N}}$$

(c) We can use the coordinates of the plane relative to your position to express the vector $\vec{\boldsymbol{d}}$:

$$\vec{\boldsymbol{d}} = \boxed{(2.5\text{ km})\hat{\boldsymbol{i}} + (1.5\text{ km})\hat{\boldsymbol{j}} + (5.0\text{ km})\hat{\boldsymbol{k}}}$$

(d) Express the elevation angle ϕ in terms of h and ℓ:

$$\phi = \tan^{-1}\left(\frac{h}{\ell}\right)$$

Substitute numerical values and evaluate ϕ:

$$\phi = \tan^{-1}\left(\frac{5.0\text{ km}}{\sqrt{(2.5\text{ km})^2 + (1.5\text{ km})^2}}\right) = \boxed{60^\circ\text{ above the horizon}}$$

Chapter 2
Motion in One Dimension

Conceptual Problems

1 • What is the average velocity over the ″round trip″ of an object that is launched straight up from the ground and falls straight back down to the ground?

Determine the Concept The "average velocity" is being requested as opposed to "average speed".

The average velocity is defined as the change in position or displacement divided by the change in time.

$$v_{av} = \frac{\Delta y}{\Delta t}$$

The change in position for any "round trip" is zero by definition. So the ***average velocity*** for any round trip must also be zero.

$$v_{av} = \frac{\Delta y}{\Delta t} = \frac{0}{\Delta t} = \boxed{0}$$

2 • An object thrown straight up falls back and is caught at the same place it is launched from. Its time of flight is T, its maximum height is H. Neglect air resistance. The correct expression for its average speed for the entire flight is (*a*) H/T, (*b*) 0, (*c*) $H/(2T)$, and (*d*) $2H/T$.

Determine the Concept The important concept here is that "average speed" is being requested as opposed to "average velocity".

Under all circumstances, including **constant acceleration**, the definition of the average speed is the ratio of the total distance traveled ($H + H$) to the elapsed time, in this case $2H/T$. $\boxed{(d)}$ is correct.

Remarks: **Because this motion involves a round trip, if the question asked for "average velocity," the answer would be zero.**

3 • Using the information in the previous question, what is its average speed just for the first half of the trip? What is its average velocity for the second half of the trip? (Answer in terms of H and T.)

Determine the Concept Under all circumstances, including ***constant acceleration***, the definition of the average speed is the ratio of the total distance traveled to the elapsed time. The average velocity, on the other hand, is the ratio of the displacement to the elapsed time.

The average speed for the first half of the trip is the height to which the object rises divided by one-half its time of flight:

$$v_{\text{av, 1st half}} = \frac{H}{\frac{1}{2}T} = \boxed{\frac{2H}{T}}$$

The average velocity for the second half of the trip is the distance the object falls divided by one-half its time of flight:

$$\text{vel}_{\text{av, 2nd half}} = \frac{-H}{\frac{1}{2}T} = \boxed{-\frac{2H}{T}}$$

Remarks: We could also say that the average velocity for the second half of the trip is –2*H*/*T*.

4 • Give an everyday example of one-dimensional motion where (*a*) the velocity is westward and the acceleration is eastward, and (*b*) the velocity is northward and the acceleration is northward.

Determine the Concept The important concept here is that $a = dv/dt$, where a is the acceleration and v is the velocity. Thus, the acceleration is positive if dv is positive; the acceleration is negative if dv is negative.

(*a*) An example of one-dimensional motion where the velocity is westward and acceleration is eastward is a car traveling westward and slowing down.

(*b*)An example of one-dimensional motion where the velocity is northward and the acceleration is northward is a car traveling northward and speeding up.

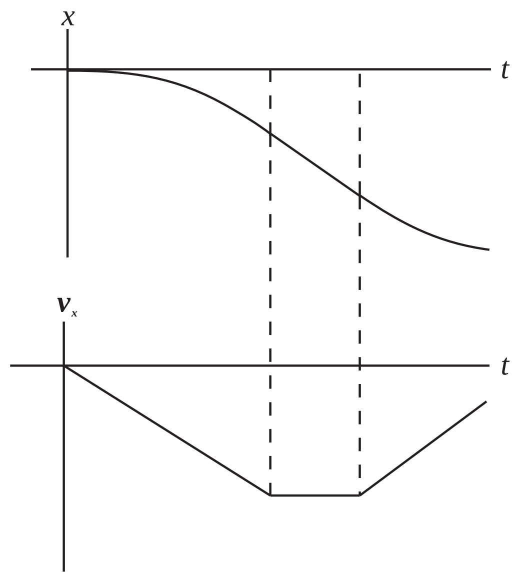
5 • **[SSM]** Stand in the center of a large room. Call the direction to your right ″positive,″ and the direction to your left ″negative.″ Walk across the room along a straight line, using a constant acceleration to quickly reach a steady speed along a straight line in the negative direction. After reaching this steady speed, keep your velocity negative but make your acceleration positive. (*a*) Describe how your speed varied as you walked. (*b*) Sketch a graph of x versus t for your motion. Assume you started at $x = 0$. (*c*) Directly under the graph of Part (*b*), sketch a graph of v_x versus t.

Determine the Concept The important concept is that when both the acceleration and the velocity are in the same direction, the speed increases. On the other hand, when the acceleration and the velocity are in opposite directions, the speed decreases.

(*a*) Your speed increased from zero, stayed constant for a while, and then decreased.

(*b*) A graph of your position as a function of time is shown to the right. Note that the slope starts out equal to zero, becomes more negative as the speed increases, remains constant while your speed is constant, and becomes less negative as your speed decreases.

(*c*) The graph of $v(t)$ consists of a straight line with negative slope (your acceleration is constant and negative) starting at (0,0), then a flat line for a while (your acceleration is zero), and finally an approximately straight line with a positive slope heading to $v = 0$.



6 • True/false: The displacement *always* equals the product of the average velocity and the time interval. Explain your choice.

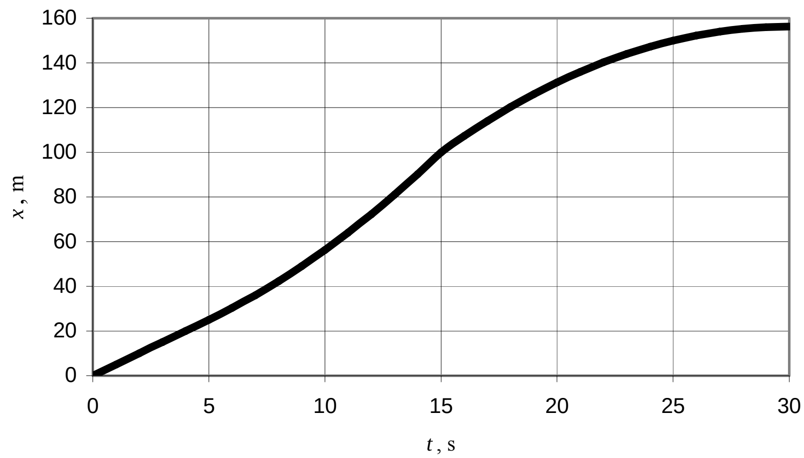
Determine the Concept True. We can use the definition of average velocity to express the displacement Δx as $\Delta x = v_{av}\Delta t$. Note that, if the acceleration is constant, the average velocity is also given by $v_{av} = (v_i + v_f)/2$.

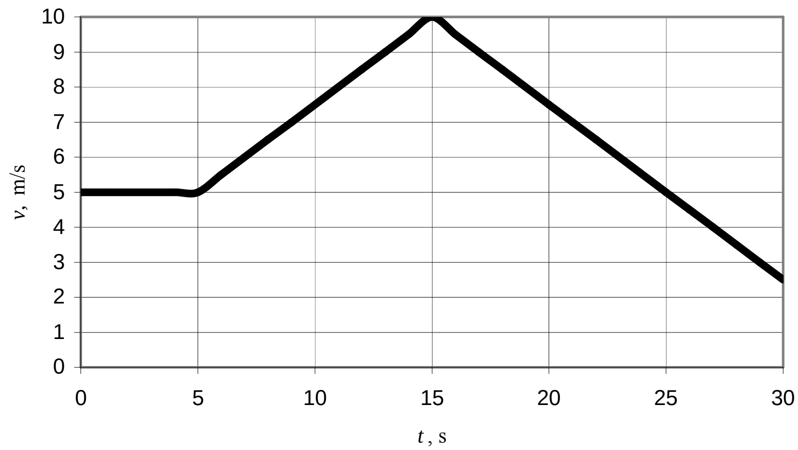
7 • Is the statement "for an object's velocity to remain constant, its acceleration must remain zero" true or false? Explain your choice.

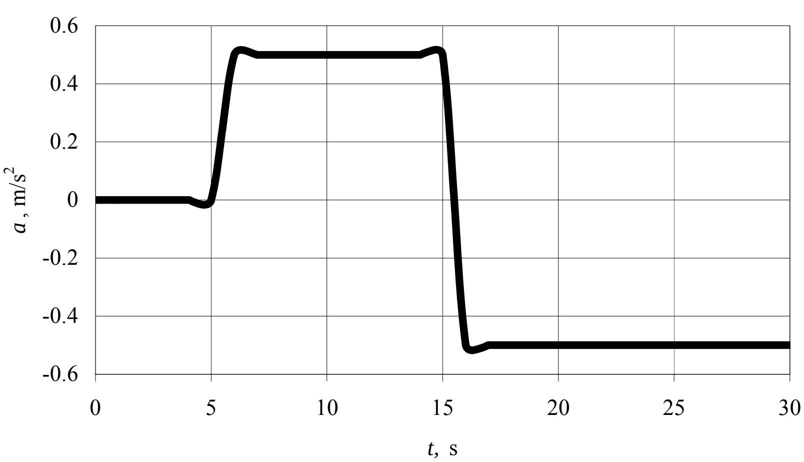
Determine the Concept True. Acceleration is the slope of the velocity versus time curve, $a = dv/dt$; while velocity is the slope of the position versus time curve, $v = dx/dt$. The speed of an object is the magnitude of its velocity. Zero acceleration implies that the velocity is constant. If the velocity is constant (including zero), the speed must also be constant.

8 • Draw careful graphs of the position and velocity and acceleration over the time period $0 \lesssim t \lesssim 30$ s for a cart that, in succession, has the following motion. The cart is moving at the constant speed of 5.0 m/s in the $+x$ direction. It passes by the origin at $t = 0.0$ s. It continues on at 5.0 m/s for 5.0 s, after which it gains speed at the constant rate of 0.50 m/s each second for 10.0 s. After gaining speed for 10.0 s, the cart loses speed at the constant rate of 0.50 m/s for the next 15.0 s.

Determine the Concept Velocity is the slope of the position versus time curve and acceleration is the slope of the velocity versus time curve. The following graphs were plotted using a spreadsheet program.


160
140
120
100
80
60
40
20
0
x , m
0
5
10
15
20
25
30
t , s


10
9
8
7
6
5
4
3
2
1
0
v, m/s
0
5
10
15
20
25
30
t , s


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
a , m/s2
0
5
10
15
20
25
30
t, s

9 • True/false; Average velocity *always* equals one-half the sum of the initial and final velocities. Explain your choice.

Determine the Concept False. The average velocity is defined (for any acceleration) as the change in position (the displacement) divided by the change in time $v_{av} = \Delta x/\Delta t$. It is always valid. If the acceleration remains constant the average velocity is also given by

$$v_{av} = \frac{v_i + v_f}{2}$$

Consider an engine piston moving up and down as an example of non-constant velocity. For one complete cycle, $v_f = v_i$ and $x_i = x_f$ so $v_{av} = \Delta x/\Delta t$ is zero. The formula involving the mean of v_f and v_i cannot be applied because the acceleration is not constant, and yields an incorrect nonzero value of v_i.

10 • Identical twin brothers standing on a horizontal bridge each throw a rock straight down into the water below. They throw rocks at exactly the same time, but one hits the water before the other. How can this be? Explain what they did differently. Ignore any effects due to air resistance.

Determine the Concept This can occur if the rocks have different initial speeds. Ignoring air resistance, the acceleration is constant. Choose a coordinate system in which the origin is at the point of release and upward is the positive direction. From the constant-acceleration equation $y = y_0 + v_0 t + \frac{1}{2}at^2$ we see that the only way two objects can have the same acceleration ($-g$ in this case) and cover the same distance, $\Delta y = y - y_0$, in different times would be if the initial velocities of the two rocks were different. Actually, the answer would be the same whether or not the acceleration is constant. It is just easier to see for the special case of constant acceleration.

11 •• [SSM] Dr. Josiah S. Carberry stands at the top of the Sears Tower in Chicago. Wanting to emulate Galileo, and ignoring the safety of the pedestrians below, he drops a bowling ball from the top of the tower. One second later, he drops a second bowling ball. While the balls are in the air, does their separation (*a*) increase over time, (*b*) decrease, (*c*) stay the same? Ignore any effects due to air resistance.

Determine the Concept Neglecting air resistance, the balls are in free fall, each with the same free-fall acceleration, which is a constant.

At the time the second ball is released, the first ball is already moving. Thus, during any time interval their velocities will increase by exactly the same amount. What can be said about the speeds of the two balls? *The first ball will always be moving faster than the second ball.* This being the case, what happens to the separation of the two balls while they are both falling? *Their separation increases.* $\boxed{(a)}$ is correct.

12 •• Which of the position-versus-time curves in Figure 2-28 best shows the motion of an object (*a*) with positive acceleration, (*b*) with constant positive velocity, (*c*) that is always at rest, and (*d*) with negative acceleration? (There may be more than one correct answer for each part of the problem.)

Determine the Concept The slope of an $x(t)$ curve at any point in time represents the speed at that instant. The way the slope changes as time increases gives the sign of the acceleration. If the slope becomes less negative or more positive as time increases (as you move to the right on the time axis), then the acceleration is positive. If the slope becomes less positive or more negative, then the acceleration is negative. The slope of the slope of an $x(t)$ curve at any point in time represents the acceleration at that instant.

(*a*) The correct answer is $\boxed{(d)}$. The slope of curve (*d*) is positive and increasing. Therefore the velocity and acceleration are positive. We would need more information to conclude that *a* is constant.

(*b)* The correct answer is $\boxed{(b)}$. The slope of curve (*b*) is positive and constant. Therefore the velocity is positive and constant.

(*c*) The correct answer is $\boxed{(e)}$. The slope of curve (*e*) is zero. Therefore, the velocity and acceleration are zero and the object remains at the same position.

(*d*) The correct answers are $\boxed{(a)\text{ and }(c)}$. The slope of curve (*a*) is negative and becomes more negative as time increases. Therefore the velocity is negative and the acceleration is negative. The slope of curve (*c*) is positive and decreasing. Therefore the velocity is positive and the acceleration is negative.

13 •• [SSM] Which of the velocity-versus-time curves in figure 2-29 best describes the motion of an object (*a*) with constant positive acceleration, (*b*) with positive acceleration that is decreasing with time, (*c*) with positive acceleration that is increasing with time, and (*d*) with no acceleration? (There may be more than one correct answer for each part of the problem.)

Determine the Concept The slope of a $v(t)$ curve at any point in time represents the acceleration at that instant.

(*a*) The correct answer is $\boxed{(b)}$. The slope of curve (*b*) is constant and positive. Therefore the acceleration is constant and positive.

(*b*) The correct answer is $\boxed{(c)}$. The slope of curve (*c*) is positive and decreasing with time. Therefore the acceleration is positive and decreasing with time.

(*c*) The correct answer is $\boxed{(d)}$. The slope of curve (*d*) is positive and increasing with time. Therefore the acceleration is positive and increasing with time.

(*d*) The correct answer is $\boxed{(e)}$. The slope of curve (*e*) is zero. Therefore the velocity is constant and the acceleration is zero.

14 •• The diagram in Figure 2-30 tracks the location of an object moving in a straight line along the *x* axis. Assume that the object is at the origin at $t = 0$. Of the five times shown, which time (or times) represents when the object is (*a*) farthest from the origin, (*b*) at rest for an instant, (*c*) in the midst of being at rest for awhile, and (*d*) moving away from the origin?

Determine the Concept Because this graph is of distance-versus-time we can use its displacement from the time axis to draw conclusions about how far the object is from the origin. We can also use the instantaneous slope of the graph to decide whether the object is at rest and whether it is moving toward or away from the origin.

(*a*) The correct answer is $\boxed{\text{B}.}$ Because the object's initial position is at $x = 0$, point B represents the instant that the object is farthest from $x = 0$.

(*b*) The correct answers are $\boxed{\text{B and D.}}$ Because the slope of the graph is zero at points B and D, the velocity of the object is zero and it is momentarily at rest at these points.

(*c*) The correct answer is $\boxed{E}$. Because the graph is a horizontal line with zero slope, the object remains at rest at the same position (its velocity is zero).

(*d*) The correct answer is $\boxed{\text{A.}}$ Because the slope of the graph is positive at point A, the velocity of the object is positive and it is moving away from the origin.

15 •• **[SSM]** An object moves along a straight line. Its position versus time graph is shown in Figure 2-30. At which time or times is its (*a*) speed at a minimum, (*b*) acceleration positive, and (*c*) velocity negative?

Determine the Concept Because this graph is of distance-versus-time we can use its instantaneous slope to describe the object's speed, velocity, and acceleration.

(*a*) The minimum speed is at $\boxed{\text{B, D, and E}}$, where it is zero. In the one-dimensional motion shown in the figure, the velocity is a minimum when the slope of a position-versus-time plot goes to zero (i.e., the curve becomes horizontal). At these points the velocity is zero and, therefore, the speed is zero.

(*b*) The acceleration is positive at points $\boxed{\text{A and D.}}$ Because the slope of the graph is increasing at these points, the velocity of the object is increasing and its acceleration is positive.

(*c*) The velocity is negative at point $\boxed{\text{C.}}$ Because the slope of the graph is negative at point C, the velocity of the object is negative.

16 •• For each of the four graphs of x versus t in Figure 2-31 answer the following questions. (*a*) Is the velocity at time t_2 greater than, less than, or equal to the velocity at time t_1? (*b*) Is the speed at time t_2 greater than, less than, or equal to the speed at time t_1?

Determine the Concept In one-dimensional motion, the velocity is the slope of a position-versus-time plot and can be either positive or negative. On the other hand, the speed is the magnitude of the velocity and can only be positive. We'll use v to denote velocity and the word "speed" for how fast the object is moving.

(*a*)
curve a: $v(t_2) < v(t_1)$
curve b: $v(t_2) = v(t_1)$
curve c: $v(t_2) > v(t_1)$
curve d: $v(t_2) < v(t_1)$

(*b*)
curve a: $\text{speed}(t_2) < \text{speed}(t_1)$
curve b: $\text{speed}(t_2) = \text{speed}(t_1)$
curve c: $\text{speed}(t_2) < \text{speed}(t_1)$
curve d: $\text{speed}(t_2) > \text{speed}(t_1)$

17 •• True/false: Explain your reasoning for each answer. If the answer is true, give an example.

(*a*) If the acceleration of an object is always zero, then it cannot be moving.
(*b*) If the acceleration of an object is always zero, then its *x*-versus-*t* curve must be a straight line.
(*c*) If the acceleration of an object is nonzero at an instant, it may be momentarily at rest at that instant.

Explain your reasoning for each answer. If an answer is true, give an example.

(*a*) False. An object moving in a straight line with constant speed has zero acceleration.

(*b*) True. If the acceleration of the object is zero, then its speed must be constant. The graph of *x*-versus-*t* for an object moving with constant speed is a straight line.

(c) True. A ball thrown upward is momentarily at rest when it is at the top of its trajectory. Its acceleration, however, is non-zero at this instant. Its value is the same as it was just before it came to rest and after it has started its descent.

18 •• A hard-thrown tennis ball is moving horizontally when it bangs into a vertical concrete wall at perpendicular incidence. The ball rebounds straight back off the wall. Neglect any effects due to gravity for the small time interval described here. Assume that towards the wall is the $+x$ direction. What are the directions of its velocity and acceleration (a) just before hitting the wall, (b) at maximum impact, and (c) just after leaving the wall.

Determine the Concept The tennis ball will be moving with constant velocity immediately before and after its collision with the concrete wall. It will be accelerated during the duration of its collision with the wall.

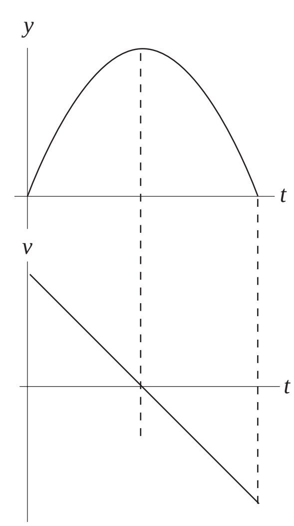
(a) Just before hitting the wall the velocity of the ball is in the $+x$ direction and, because its velocity is constant, its acceleration is zero.

(b) At maximum impact, the ball is reversing direction and its velocity is zero. Its acceleration is in the $-x$ direction.

(c) Just after leaving the wall, the velocity of the ball is in the $-x$ direction and constant. Because its velocity is constant, its acceleration is zero.

19 •• **[SSM]** A ball is thrown straight up. Neglect any effects due to air resistance. (a) What is the velocity of the ball at the top of its flight? (b) What is its acceleration at that point? (c) What is different about the velocity and acceleration at the top of the flight if instead the ball impacts a horizontal ceiling very hard and then returns.

Determine the Concept In the absence of air resistance, the ball will experience a constant acceleration and the graph of its position as a function of time will be parabolic. In the graphs to the right, a coordinate system was chosen in which the origin is at the point of release and the upward direction is the $+y$ direction. The top graph shows the position of the ball as a function of time and the bottom graph shows the velocity of a ball as a function of time.



(a) $v_{\text{top of flight}} = \boxed{0}$

(b) The acceleration of the ball is the same at every point of its trajectory, including the point at which $v = 0$ (at the top of its flight). Hence $a_{\substack{\text{top of}\\ \text{flight}}} = \boxed{-g}$.

(*c*) If the ball impacts a horizontal ceiling very hard and then returns, its velocity at the top of its flight is still zero and its acceleration is still downward but greater than g in magnitude.

20 •• An object that is launched straight up from the ground, reaches a maximum height H, and falls straight back down to the ground, hitting it T seconds after launch. Neglect any effects due to air resistance. (*a*) Express the average speed for the entire trip as a function of H and T. (*b*) Express the average speed for the same interval of time as a function of the initial launch speed v_0.

Picture the Problem The average speed is being requested as opposed to average velocity. We can use the definition of average speed as distance traveled divided by the elapsed time and the expression for the average speed of an object when it is experiencing constant acceleration to express v_{av} in terms of v_0.

(*a*) The average speed is defined as the total distance traveled divided by the change in time:

$$v_{av} = \frac{\text{total distance traveled}}{\text{total time}}$$

Substitute for the total distance traveled and the total time and simplify to obtain:

$$v_{av} = \frac{H+H}{T} = \boxed{\frac{2H}{T}}$$

(*b*) The average speed for the upward flight of the object is given by:

$$v_{av,\,up} = \frac{v_0+0}{2} = \frac{H}{\frac{1}{2}T} \Rightarrow \frac{H}{T} = \tfrac{1}{4}v_0$$

The average speed for the same interval of time as a function of the initial launch speed v_0 is twice the average speed during the upward portion of the flight:

$$v_{av} = 2v_{av,\,up} = 2\left(\tfrac{1}{4}v_0\right) = \boxed{\tfrac{1}{2}v_0}$$

Because $v_0 \neq 0$, the average speed is not zero.

Remarks: 1) Because this motion involves a roundtrip, if the question asked for "average velocity", the answer would be zero. 2) Another easy way to obtain this result is take the absolute value of the velocity of the object to obtain a graph of its speed as a function of time. A simple geometric argument leads to the result we obtained above.

21 •• A small lead ball is thrown directly upward. Neglect any effects due to air resistance. True or false: (*a*) The magnitude of its acceleration decreases on the way up. (*b*) The direction of its acceleration on its way down is opposite to the direction of its acceleration on its way up. (*c*) The direction of its velocity on its way down is opposite to the direction of its velocity on its way up.

Determine the Concept For free fall, the acceleration is the same (g) throughout the entire flight.

(*a*) False. The velocity of the ball decreases at a steady rate. This means that the acceleration of the ball is constant.

(*b*) False. The velocity of the ball decreases at a steady rate (g) throughout its entire flight.

(*c*) True. On the way up the velocity vector points upward and on the way down it points downward.

22 •• At $t = 0$, object A is dropped from the roof of a building. At the same instant, object B is dropped from a window 10 m below the roof. Air resistance is negligible. During the descent of B to the ground, the distance between the two objects (*a*) is proportional to t, (*b*) is proportional to t^2, (*c*) decreases, (*d*) remains 10 m throughout.

Determine the Concept Both objects experience the same constant acceleration. Choose a coordinate system in which downward is the positive direction and use a constant-acceleration equation to express the position of each object as a function of time.

Using constant-acceleration equations, express the positions of both objects as functions of time:

$$x_A = x_{0,A} + v_0 t + \tfrac{1}{2} g t^2$$

and

$$x_B = x_{0,B} + v_0 t + \tfrac{1}{2} g t^2$$

where $v_0 = 0$.

Express the separation of the two objects by evaluating $x_B - x_A$:

$$x_B - x_A = x_{0,B} - x_{0,A} = 10\,\text{m}$$

$\boxed{(d)}$ is correct.

23 •• You are driving a Porsche that accelerates uniformly from 80.5 km/h (50 mi/h) at $t = 0.00$ to 113 km/h (70 mi/h) at $t = 9.00$ s. (*a*) Which graph in Figure 2-32 best describes the velocity of your car? (*b*) Sketch a position-versus-time graph showing the location of your car during these nine seconds, assuming we let its position x be zero at $t = 0$.

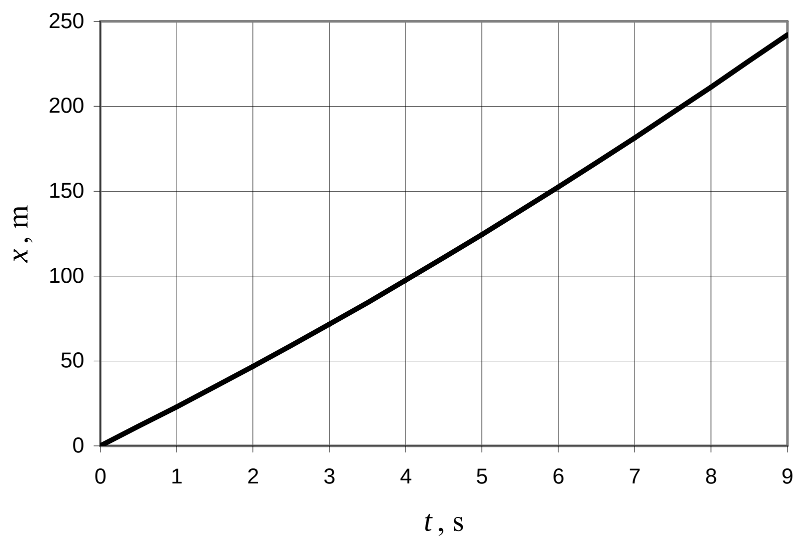
Determine the Concept Because the Porsche accelerates uniformly, we need to look for a graph that represents constant acceleration.

(*a*) Because the Porsche has a constant acceleration that is positive (the velocity is increasing), we must look for a velocity-versus-time curve with a positive constant slope and a nonzero intercept. Such a graph is shown in $\boxed{(c).}$

(*b*) Use the data given in the problem statement to determine that the acceleration of the Porsche is 1.00 m/s^2 and that its initial speed is 22.4 m/s. The equation describing the position of the car as a function of time is

$$x = \left(22.4\ \text{m/s}^2\right)t + \tfrac{1}{2}\left(1.00\ \text{m/s}^2\right)t^2.$$

The following graph of this equation was plotted using a spreadsheet program.



24 •• A small heavy object is dropped from rest and falls a distance *D* in a time *T*. After it has fallen for a time 2*T*, what will be its (*a*) fall distance from its initial location in terms of *D*, (*b*) its speed in terms of *D* and *t*, and (*c*) its acceleration? (Neglect air resistance.)

Picture the Problem In the absence of air resistance, the object experiences constant acceleration. Choose a coordinate system in which the downward direction is positive and use the constant-acceleration equation to describe its motion.

(*a*) Relate the distance *D* that the object, released from rest, falls in time *t*:

$$x(t) = D = \tfrac{1}{2}gt^2 \qquad (1)$$

Evaluate *x*(2*t*) to obtain:

$$x(2t) = \tfrac{1}{2}g(2t)^2 = 2gt^2 \qquad (2)$$

Dividing equation (2) by equation (1) and simplifying yields:

$$\frac{x(2t)}{D} = \frac{2gt^2}{\frac{1}{2}gt^2} = 4 \Rightarrow x(2t) = \boxed{4D}$$

(*b*) Express the speed of the object as a function of time:

$$v = v_0 + gt$$

or, because $v_0 = 0$,

$$v = gt \qquad (3)$$

Solving equation (1) for g yields:

$$g = \frac{2x}{t^2}$$

Substitute for g in equation (3) to obtain:

$$v = \frac{2x}{t^2}t = \frac{2x}{t} \qquad (4)$$

Evaluating equation (4) at time $2t$ and simplifying yields:

$$v(2t) = \frac{2x(2t)}{2t} = \boxed{\frac{4D}{t}}$$

(*c*) The acceleration of the object is independent of time (that is, it is constant) and is equal to $\boxed{g.}$

25 •• In a race, at an instant when two horses are running right next to each other and in the same direction (the $+x$ direction), horse A's instantaneous velocity and acceleration are $+10$ m/s and $+2.0$ m/s^2 respectively, and horse B's instantaneous velocity and acceleration are $+12$ m/s and -1.0 m/s^2 respectively. Which horse is passing the other at this instant? Explain.

Determine the Concept The information about the horses' accelerations is irrelevant to the determination of which horse is passing the other at this instant. The horse running with the greater instantaneous velocity will by passing the slower horse. Hence $\boxed{\text{B is passing A.}}$ The accelerations are relevant to the determination of which horse will be in the lead at some later time.

26 •• True or false: (*a*) The equation $x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$ is always valid for all particle motion in one dimension. (*b*) If the velocity at a given instant is zero, the acceleration at that instant must also be zero. (*c*) The equation $\Delta x = v_{av}\Delta t$ holds for all particle motion in one dimension.

Determine the Concept As long as the acceleration remains constant the following constant-acceleration equations hold. If the acceleration is not constant, they do not, in general, give correct results except by coincidence.

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \qquad v = v_0 + at \qquad v^2 = v_0^2 + 2a\Delta x \qquad v_{av} = \frac{v_i + v_f}{2}$$

(*a*) False. This statement is true if and only if the acceleration is constant.

(*b*) False. Consider a rock thrown straight up into the air. At the "top" of its flight, the velocity is zero but it is changing (otherwise the velocity would remain zero and the rock would hover); therefore the acceleration is not zero.

(*c*) True. The definition of average velocity, $v_{\text{av}} = \Delta x/\Delta t$, requires that this always be true.

27 •• If an object is moving in a straight line at constant acceleration, its instantaneous velocity halfway through any time interval is (*a*) greater than its average velocity, (*b*) less than its average velocity, (*c*) equal to its average velocity, (*d*) half its average velocity, (*e*) twice its average velocity.

Determine the Concept Because the acceleration of the object is constant, the constant-acceleration equations can be used to describe its motion. The special expression for average velocity for constant acceleration is $v_{\text{av}} = \dfrac{v_{\text{i}} + v_{\text{f}}}{2}$. $\boxed{(c)}$ is correct.

28 •• A turtle, seeing his owner put some fresh lettuce on the opposite side of his terrarium, begins to accelerate (at a constant rate) from rest at time $t = 0$, heading directly toward the food. Let t_1 be the time at which the turtle has covered half the distance to his lunch. Derive an expression for the ratio of t_2 to t_1, where t_2 is the time at which the turtle reaches the lettuce.

Picture the Problem We are asked, essentially, to determine the time t_2, at which a displacement, Δx, is twice what it was at an earlier time, t_1. The turtle is crawling with constant acceleration so we can use the constant-acceleration equation $\Delta x = v_{0x}\Delta t + \frac{1}{2}a_x(\Delta t)^2$ to describe the turtle's displacement as a function of time.

Express the displacement Δt of the turtle at the end of a time interval Δt:	$\Delta x = v_{0x}\Delta t + \frac{1}{2}a_x(\Delta t)^2$ or, because $v_{0x} = 0$, $\Delta x = \frac{1}{2}a_x(\Delta t)^2$
For the two time intervals:	$\Delta x_1 = \frac{1}{2}a_x t_1^2$ and $\Delta x_2 = \frac{1}{2}a_x t_2^2$
Express the ratio of Δx_2 to Δx_1 to obtain:	$\dfrac{\Delta x_2}{\Delta x_1} = \dfrac{\frac{1}{2}a_x t_2^2}{\frac{1}{2}a_x t_1^2} = \dfrac{t_2^2}{t_1^2}$ (1)
We're given that:	$\dfrac{\Delta x_2}{\Delta x_1} = 2$

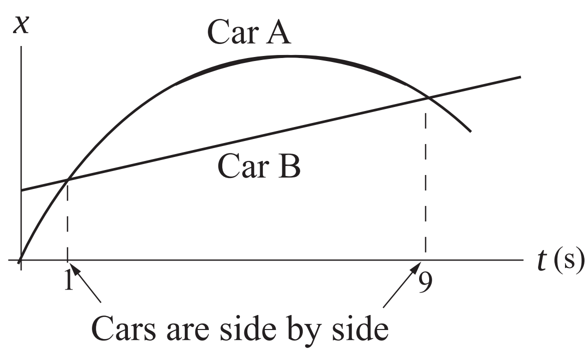
Substitute in equation (1) and simplify to obtain:

$$\frac{t_2^2}{t_1^2}=2\Rightarrow\frac{t_2}{t_1}=\boxed{\sqrt{2}}$$

29 •• [SSM] The positions of two cars in parallel lanes of a straight stretch of highway are plotted as functions of time in the Figure 2-33.Take positive values of *x* as being to the right of the origin. Qualitatively answer the following: (*a*) Are the two cars ever side by side? If so, indicate that time (those times) on the axis. (*b*) Are they always traveling in the same direction, or are they moving in opposite directions for some of the time? If so, when? (*c*) Are they ever traveling at the same velocity? If so, when? (*d*) When are the two cars the farthest apart? (*e*) Sketch (no numbers) the velocity versus time curve for each car.

Determine the Concept Given the positions of the two cars as a function of time, we can use the intersections of the curves and their slopes to answer these questions.

(*a*) The positions of cars A and B are the same at two places where the graphs cross.

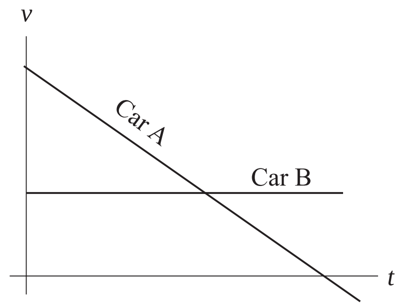


(*b*) When the slopes of the curves have opposite signs, the velocities of the cars are oppositely directed. Thus, after approximately 7 s, car A is moving leftward while car B is moving rightward. Before t = 7 s, the two cars are traveling in the same direction.

(*c*) The two cars have the same velocity when their curves have the same slopes. This occurs at about 6 s.

(*d*) The time at which the two cars are farthest apart is roughly 6 s as indicated by the place at which, vertically, the two curves are farthest part.

(*e*)



30 •• A car driving at constant velocity passes the origin at time $t = 0$. At that instant, a truck, at rest at the origin, begins to accelerate uniformly from rest. Figure 2-34 shows a qualitative plot of the velocities of truck and car as functions of time. Compare their displacements (from the origin), velocities, and accelerations at the instant that their curves intersect.

Determine the Concept The graph is a plot of velocity versus time. Thus, where the two curves cross, the truck and car are, at that instant, moving with equal velocities. The slope of a velocity versus time curve is equal to the instantaneous acceleration – thus, since the curve that represents the truck's velocity has a positive slope, and the car's curve has zero slope, the truck is accelerating at a higher rate than the car. Finally, the displacements of the two cars are determined by calculating the areas under the curves. In this instance, the curve representing the truck's velocity as a function of time encloses a triangular area that is exactly half that of the curve representing the car's velocity. Thus, at the instant represented by the point where the curves intersect, the truck has gone half as far as has the car.

31 •• Reginald is out for a morning jog, and during the course of his run on a straight track, has a velocity that depends upon time as shown in Figure 2-35. That is, he begins at rest, and ends at rest, peaking at a maximum velocity v_{max} at an arbitrary time t_{max}. A second runner, Josie, runs throughout the time interval $t = 0$ to $t = t_f$ at a constant speed v_R, so that each has the same displacement during the time interval. Note: t_f is NOT twice t_{max}, but represents an arbitrary time. What is the relationship between v_J and v_{max}?

Determine the Concept In this problem we are presented with curves representing the velocity as a function of time for both runners. The area under each curve represents the displacement for each runner and we are told that Josie and Reginald each have the same displacement during the time interval of length t_f. Since this is the case, we can find the relationship between v_R and v_{max} by equating the areas under the two curves.

Express the condition on the displacement of the two runners:

$$\Delta x_R = \Delta x_J \quad (1)$$

Josie runs at a constant velocity v for the whole of the time interval. Express her displacement Δx_J:

$$\Delta x_J = v_J t_f$$

Reginald has a different velocity profile, one which results in a triangle of height v_{max} and length t_f. Express his displacement Δx_J:

$$\Delta x_R = \tfrac{1}{2} v_{max} t_f$$

Substitute for Δx_R and Δx_J in equation (1) and simplify to obtain:

$$\tfrac{1}{2} v_{max} t_f = v_J t_f \Rightarrow v_J = \boxed{\tfrac{1}{2} v_{max}}$$

32 •• Which graph (or graphs), if any, of v versus t in Figure 2-36 best describes the motion of a particle with (*a*) positive velocity and increasing speed, (*b*) positive velocity and zero acceleration, (*c*) constant non-zero acceleration, and (*d*) a speed decrease?

Determine the Concept The velocity of the particle is positive if the curve is above the $v = 0$ line (the t axis), and the acceleration is positive if the curve has a positive slope. The speed of the particle is the magnitude of its velocity.

(*a*) Graph $\boxed{(c)}$ describes the motion of a particle with positive velocity and increasing speed because $v(t)$ is above the t axis and has a positive slope.

(*b*) Graph $\boxed{(a)}$ describes the motion of a particle with positive velocity and zero acceleration because $v(t)$ is above the t axis and its slope is zero.

(*c*) Graphs $\boxed{(c), (d) \text{ and } (e)}$ describe the motion of a particle with constant non-zero acceleration because $v(t)$ is linear and has a non-zero slope.

(*d*) Graph $\boxed{(e)}$ describes the motion of a particle with a speed decrease because it shows the speed of the particle decreasing with time.

33 •• Which graph (or graphs), if any, of v_x versus t in Figure 2-36 best describes the motion of a particle with (*a*) negative velocity and increasing speed, (*b*) negative velocity and zero acceleration, (*c*) variable acceleration, and (*d*) increasing speed?

Determine the Concept The velocity of the particle is positive if the curve is above the $v = 0$ line (the t axis), and the acceleration is positive if the curve has a positive slope. The speed of the particle is the magnitude of its velocity.

(*a*) Graph $\boxed{(d)}$ describes the motion of a particle with negative velocity and increasing speed because *v*(*t*) is below the *t* axis and has a negative slope.

(*b*) Graph $\boxed{(b)}$ describes the motion of a particle with negative velocity and zero acceleration because *v*(*t*) is below the *t* axis and its slope is zero.

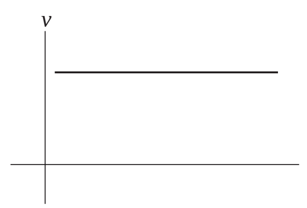
(*c*) None of these graphs describe the motion of a particle with a variable acceleration because *v*(*t*) is linear.

(*d*) Graphs $\boxed{(c)\text{ and }(d)}$ describe the motion of a particle with an increasing speed because they show the speed of the particle increasing with time.

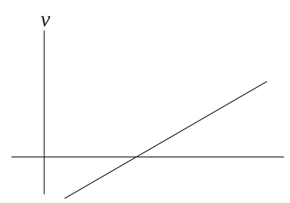
34 •• Sketch a *v*-versus-*t* curve for each of the following conditions: (*a*) Acceleration is zero and constant while velocity is not zero. (*b*) Acceleration is constant but not zero. (*c*) Velocity and acceleration are both positive. (*d*) Velocity and acceleration are both negative. (*e*) Velocity is positive and acceleration is negative. (*f*) Velocity is negative and acceleration is positive. (*g*) Velocity is momentarily zero but the acceleration is not zero.

Determine the Concept Acceleration is the slope of a velocity-versus-time curve.

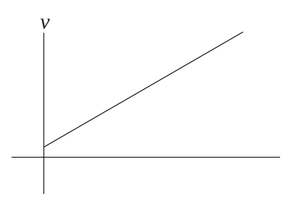
(*a*) Acceleration is zero and constant while velocity is not zero.



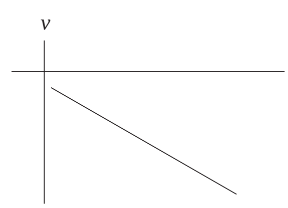
(*b*) Acceleration is constant but not zero.



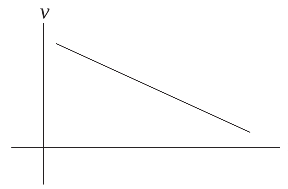
(*c*) Velocity and acceleration are both positive.



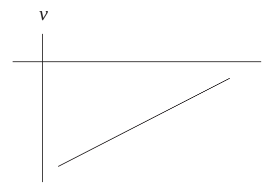
(*d*) Velocity and acceleration are both negative.



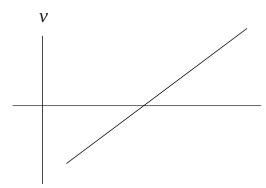
(*e*) Velocity is positive and acceleration is negative.



(*f*) Velocity is negative and acceleration is positive.



(*g*) Velocity is momentarily zero but the acceleration is not zero.



35 •• Figure 2-37 shows nine graphs of position, velocity, and acceleration for objects in motion along a straight line. Indicate the graphs that meet the following conditions: (*a*) Velocity is constant, (*b*) velocity reverses its direction, (*c*) acceleration is constant, and (*d*) acceleration is not constant. (*e*) Which graphs of position, velocity, and acceleration are mutually consistent?

Determine the Concept Velocity is the slope and acceleration is the slope of the slope of a position-versus-time curve. Acceleration is the slope of a velocity-versus-time curve.

(*a*) Graphs $\boxed{(a), (f), \text{and } (i)}$ describe motion at constant velocity. For constant velocity, *x* versus *t* must be a straight line; *v*-versus-*t* must be a horizontal straight line; and *a* versus *t* must be a straight horizontal line at $a = 0$.

(*b*) Graphs $\boxed{(c) \text{ and } (d)}$ describe motion in which the velocity reverses its direction. For velocity to reverse its direction *x*-versus-*t* must have a slope that changes sign and *v* versus *t* must cross the time axis. The acceleration cannot remain zero at all times.

(*c*) Graphs $\boxed{(a), (d), (e), (f), (h), \text{and } (i)}$ describe motion with constant acceleration. For constant acceleration, *x* versus *t* must be a straight horizontal line or a parabola, *v* versus *t* must be a straight line, and *a* versus *t* must be a horizontal straight line.

(*d*) Graphs $\boxed{(b),(c),\text{ and }(g)}$ describe motion with non-constant acceleration. For non-constant acceleration, *x* versus *t* must not be a straight line or a parabola; *v* versus *t* must not be a straight line, or *a* versus *t* must not be a horizontal straight line.

(*e*) The following pairs of graphs are mutually consistent: $\boxed{(a)\text{ and }(i),}$ $\boxed{(d)\text{ and }(h),}$ and $\boxed{(f)\text{ and }(i).}$ For two graphs to be mutually consistent, the curves must be consistent with the definitions of velocity and acceleration.

Estimation and Approximation

36 • While engrossed in thought about the scintillating lecture just delivered by your physics professor you mistakenly walk directly into the wall (rather than through the open lecture hall door). Estimate the magnitude of your average acceleration as you rapidly come to a halt.

Picture the Problem The speed of one's walk varies from person to person, but 1.0 m/s is reasonable. We also need to estimate a distance within which you would stop in such a case. We'll assume a fairly short stopping distance of 1.5 cm. We'll also assume (unrealistically) that you experience constant acceleration and choose a coordinate system in which the direction you are walking is the +*x* direction.

Using a constant-acceleration equation, relate your final speed to your initial speed, acceleration, and displacement while stopping:

$$v_{\text{f}}^2 = v_{\text{i}}^2 + 2a_x\Delta x \Rightarrow a_x = \frac{v_{\text{f}}^2 - v_{\text{i}}^2}{2\Delta x}$$

Substitute numerical values and evaluate the magnitude of your acceleration:

$$a_x = \left|\frac{(0)^2 - \left(1.0\,\frac{\text{m}}{\text{s}}\right)^2}{2\left(1.5\times10^{-2}\,\text{m}\right)}\right| = \boxed{33\ \text{m/s}^2}$$

37 • **[SSM]** Occasionally, people can survive falling large distances if the surface they land on is soft enough. During a traverse of the Eiger's infamous Nordvand, mountaineer Carlos Ragone's rock anchor gave way and he plummeted 500 feet to land in snow. Amazingly, he suffered only a few bruises and a wrenched shoulder. Assuming that his impact left a hole in the snow 4.0 ft deep, estimate his average acceleration as he slowed to a stop (that is, while he was impacting the snow).

Picture the Problem In the absence of air resistance, Carlos' acceleration is constant. Because all the motion is downward, let's use a coordinate system in which downward is the positive direction and the origin is at the point at which the fall began.

Using a constant-acceleration equation, relate Carlos' final velocity v_2 to his velocity v_1 just before his impact, his stopping acceleration a_s upon impact, and his stopping distance Δy:

$$v_2^2 = v_1^2 + 2a_s\Delta y \Rightarrow a_s = \frac{v_2^2 - v_1^2}{2\Delta y}$$

or, because $v_2 = 0$,

$$a_s = -\frac{v_1^2}{2\Delta y} \qquad (1)$$

Using a constant-acceleration equation, relate Carlos' speed just before impact to his acceleration during free-fall and the distance he fell h:

$$v_1^2 = v_0^2 + 2a_{\text{free-fall}}h$$

or, because $v_0 = 0$ and $a_{\text{free-fall}} = g$,

$$v_1^2 = 2gh$$

Substituting for v_1^2 in equation (1) yields:

$$a_s = -\frac{2gh}{2\Delta y}$$

Substitute numerical values and evaluate a_s:

$$a = -\frac{2(9.81\,\text{m/s}^2)(500\,\text{ft})}{2(4.0\,\text{ft})} = \boxed{-1.2\times10^3\,\text{m/s}^2}$$

Remarks: The magnitude of this acceleration is about 125*g*!

38 •• When we solve free-fall problems near Earth, it's important to remember that air resistance may play a significant role. If its effects are significant, we may get answers that are wrong by orders of magnitude if we ignore it. How can we tell when it is valid to ignore the affects of air resistance? One way is to realize that air resistance increases with increasing speed. Thus, as an object falls and its speed *increases,* its downward acceleration *decreases.* Under these circumstances, the object's speed will approach, as a limit, a value called its *terminal speed.* This terminal speed depends upon such things as the mass and cross-sectional area of the body. Upon reaching its terminal speed, its acceleration is zero. For a "typical" skydiver falling through the air, a typical the terminal speed is about 50 m/s (roughly 120 mph). At half its terminal speed, the skydiver's acceleration will be about $\frac{3}{4}g$. Let's take half the terminal speed as a reasonable "upper bound" beyond which we shouldn't use our constant acceleration free-fall relationships. Assuming the skydiver started from rest, (*a*) estimate how far, and for how long, the skydiver falls before we can no longer neglect air resistance. (*b*) Repeat the analysis for a ping-pong ball, which has a terminal speed of about 5.0 m/s. (*c*) What can you conclude by comparing your answers for Parts (*a*) and (*b*)?

Picture the Problem Because we're assuming that the accelerations of the skydiver and a ping-pong ball are constant to one-half their terminal velocities, we can use constant-acceleration equations to find the times required for them to reach their ″upper-bound″ velocities and their distances of fall. Let's use a coordinate system in which downward is the +*y* direction.

(*a*) Using a constant-acceleration equation, relate the upper-bound velocity to the free-fall acceleration and the time required to reach this velocity:

$$v_{\text{upper bound}} = v_0 + g\Delta t$$

or, because $v_0 = 0$,

$$v_{\text{upper bound}} = g\Delta t \Rightarrow \Delta t = \frac{v_{\text{upper bound}}}{g}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{25\,\text{m/s}}{9.81\,\text{m/s}^2} = 2.55\text{ s} \approx \boxed{2.6\,\text{s}}$$

Using a constant-acceleration equation, relate the skydiver's terminal speed to his/her acceleration and distance of fall:

$$\left(\tfrac{1}{2}v_t\right)^2 = v_0^2 + 2g\Delta y$$

or, because $v_0 = 0$,

$$\left(\tfrac{1}{2}v_t\right)^2 = 2g\Delta y \Rightarrow \Delta y = \frac{\left(\tfrac{1}{2}v_t\right)^2}{2g}$$

Substitute numerical values and evaluate Δy:

$$\Delta y = \frac{\left[\tfrac{1}{2}(50\text{ m/s})\right]^2}{2\left(9.81\text{ m/s}^2\right)} \approx \boxed{32\text{ m}}$$

(*b*) Proceed as in (*a*) with $v_{\text{upper bound}} = 5.0\text{ m/s}$ to obtain:

$$\Delta t = \frac{\tfrac{1}{2}(5.0\,\text{m/s})}{9.81\,\text{m/s}^2} = 0.255\text{ s} \approx \boxed{0.26\,\text{s}}$$

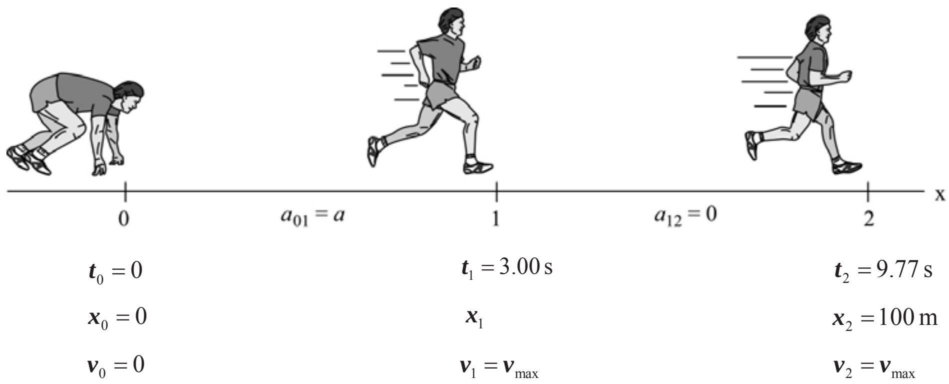
and

$$\Delta y = \frac{\left[\tfrac{1}{2}(5\text{ m/s})\right]^2}{2\left(9.81\text{ m/s}^2\right)} \approx \boxed{32\text{ cm}}$$

(*c*) The analysis of the motion of a ping-pong ball requires the inclusion of air resistance for almost any situation, whereas the analysis of the motion of the sky diver doesn't require it until the fall distances and times are considerably longer.

39 •• On June 14, 2005 Asafa Powell of the Jamaica set a world's record for the 100-m dash with a time $t = 9.77$ s. Assuming he reached his maximum speed in 3.00 s, and then maintained that speed until the finish, estimate his acceleration during the first 3.00 s.

Picture the Problem This is a constant-acceleration problem. Choose a coordinate system in which the direction Powell is running is the +*x* direction. During the first 3 s of the race his acceleration is positive and during the rest of the race it is zero. The pictorial representation summarizes what we know about Powell's race.



Express the total distance covered by Powell in terms of the distances covered in the two phases of his race:

$$100\,\text{m} = \Delta x_{01} + \Delta x_{12} \qquad (1)$$

Express the distance he runs getting to his maximum velocity:

$$\Delta x_{01} = v_0 \Delta t_{01} + \tfrac{1}{2} a_{01} (\Delta t_{01})^2 = \tfrac{1}{2} a (3\,\text{s})^2$$

The distance covered during the rest of the race at the constant maximum velocity is given by:

$$\begin{aligned}\Delta x_{12} &= v_{max} \Delta t_{12} + \tfrac{1}{2} a_{12} (\Delta t_{12})^2 \\ &= (a \Delta t_{01}) \Delta t_{12} \\ &= a(3.00\,\text{s})(6.77\,\text{s})\end{aligned}$$

Substitute for these displacements in equation (1) to obtain:

$$100\,\text{m} = \tfrac{1}{2} a (3.00\,\text{s})^2 + a(3.00\,\text{s})(6.77\,\text{s})$$

Solving for a yields:

$$a = \frac{100\,\text{m}}{\tfrac{1}{2}(3.00\,\text{s})^2 + (3.00\,\text{s})(6.77\,\text{s})} = \boxed{4.03\,\text{m/s}^2}$$

40 •• The photograph in Figure 2-38 is a short-time exposure (1/30 s) of a juggler with two tennis balls in the air. (*a*) The tennis ball near the top of its trajectory is less blurred than the lower one. Why is that? (*b*) Estimate the speed of the ball that he is just releasing from his right hand. (*c*) Determine how high the ball should have gone above the launch point and compare it to an estimate from the picture. (*Hint: You have a built-in distance scale if you assume some reasonable value for the height of the juggler*.)

Determine the Concept This is a constant-acceleration problem with $a = -g$ if we take upward to be the positive direction. At the maximum height the ball will reach, its speed will be near zero and when the ball has just been tossed in the air its speed is near its maximum value.

(*a*) Because the ball is moving slowly its blur is relatively short (i.e., there is less blurring).

(*b*) The average speed of the ball is given by:

$$v_{av} = \frac{\text{distance traveled}}{\text{elapsed time}}$$

Estimating how far the ball has traveled in 1/30 s yields:

$$v_{av} = \frac{2\text{ ball diameters}}{\frac{1}{30}\text{ s}}$$

The diameter of a tennis ball is 6.5 cm:

$$v_{av} = \frac{2(6.5\text{ cm})}{\frac{1}{30}\text{ s}} \approx \boxed{3.9\text{ m/s}}$$

(*c*) Use a constant-acceleration equation to relate the initial and final speeds of the ball to its maximum height *h*:

$$v^2 = v_0^2 + 2a_y h \Rightarrow h = \frac{v^2 - v_0^2}{2a_y}$$

or, because $v = 0$ and $a_y = g$,

$$h = -\frac{v_0^2}{2g}$$

Substitute numerical values and evaluate *h*:

$$h = -\frac{(3.9\text{ m/s})^2}{2(-9.81\text{ m/s}^2)} \approx \boxed{78\text{ cm}}$$

If we assume that the juggler is approximately 6 ft (1.8 m) tall, then our calculated value for *h* seems to be a good approximation to the height shown in the photograph.

41 •• A rough rule of thumb for determining the distance between you and a lightning strike is to start counting the seconds that elapse ("one-Mississippi, two-Mississippi, …") until you hear the thunder (sound emitted by the lightning as it rapidly heats the air around it). Assuming the speed of sound is about 750 mi/h, (*a*) estimate how far away is a lightning strike if you counted about 5 s until you heard the thunder. (*b*) Estimate the uncertainty in the distance to the strike in Part (*a*). Be sure to explain your assumptions and reasoning. (*Hint: The speed of sound depends on the air temperature and your counting is far from exact!*)

Picture the Problem We can use the relationship between distance, speed, and time to estimate the distance to the lightning strike.

(*a*) Relate the distance Δd to the lightning strike to the speed of sound in air *v* and the elapsed time Δt:

$$\Delta d = v\Delta t$$

Substitute numerical values and evaluate Δd:

$$\Delta d = \left(750\,\frac{\text{mi}}{\text{h}}\right)\left(\frac{0.3048\,\frac{\text{m}}{\text{s}}}{0.6818\,\frac{\text{mi}}{\text{h}}}\right)(5\,\text{s})$$

$$\approx \boxed{1.7\,\text{km}} \approx \boxed{1\,\text{mi}}$$

(*b*) You are probably lucky if the uncertainty in your time estimate is less than 1 s (±20%), so the uncertainty in the distance estimate is about 20% of 1.7 km or approximately 300 m. This is probably much greater than the error made by assuming *v* is constant.

Speed, Displacement, and Velocity

42 • (*a*) An electron in a television tube travels the 16-cm distance from the grid to the screen at an average speed of 4.0×10^7 m/s. How long does the trip take? (*b*) An electron in a current-carrying wire travels at an average speed of 4.0×10^{-5} m/s. How long does it take to travel 16 cm?

Picture the Problem Think of the electron as traveling in a straight line at constant speed and use the definition of average speed.

(*a*) Using its definition, express the average speed of the electron:

$$\text{Average speed} = \frac{\text{distance traveled}}{\text{time of flight}} = \frac{\Delta s}{\Delta t}$$

Solve for and evaluate the time of flight:

$$\Delta t = \frac{\Delta s}{\text{Average speed}} = \frac{0.16\ \text{m}}{4.0 \times 10^7\ \text{m/s}}$$

$$= 4.0 \times 10^{-9}\,\text{s} = \boxed{4.0\ \text{ns}}$$

(*b*) Calculate the time of flight for an electron in a 16-cm long current carrying wire similarly.

$$\Delta t = \frac{\Delta s}{\text{Average speed}} = \frac{0.16\ \text{m}}{4.0 \times 10^{-5}\ \text{m/s}}$$

$$= 4.0 \times 10^3\,\text{s} \times \frac{1\,\text{min}}{60\,\text{s}} = \boxed{67\ \text{min}}$$

43 • [SSM] A runner runs 2.5 km, in a straight line, in 9.0 min and then takes 30 min to walk back to the starting point. (*a*) What is the runner's average velocity for the first 9.0 min? (*b*) What is the average velocity for the time spent walking? (*c*) What is the average velocity for the whole trip? (*d*) What is the average speed for the whole trip?

Picture the Problem In this problem the runner is traveling in a straight line but not at constant speed - first she runs, then she walks. Let's choose a coordinate system in which her initial direction of motion is taken as the +*x* direction.

(*a*) Using the definition of average velocity, calculate the average velocity for the first 9 min:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{2.5\ \text{km}}{9.0\ \text{min}} = \boxed{0.28\ \text{km/min}}$$

(*b*) Using the definition of average velocity, calculate her average velocity for the 30 min spent walking:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{-2.5\ \text{km}}{30\ \text{min}}$$
$$= \boxed{-83\ \text{m/min}}$$

(*c*) Express her average velocity for the whole trip:

$$v_{\text{av}} = \frac{\Delta x_{\text{round trip}}}{\Delta t} = \frac{0}{\Delta t} = \boxed{0}$$

(*d*) Finally, express her average speed for the whole trip:

$$\text{speed}_{\text{av}} = \frac{\text{distance traveled}}{\text{elapsed time}}$$
$$= \frac{2(2.5\ \text{km})}{30\ \text{min} + 9.0\ \text{min}}$$
$$= \boxed{0.13\ \text{km/min}}$$

44 • A car travels in a straight line with an average velocity of 80 km/h for 2.5 h and then with an average velocity of 40 km/h for 1.5 h. (*a*) What is the total displacement for the 4.0-h trip? (*b*) What is the average velocity for the total trip?

Picture the Problem The car is traveling in a straight line but not at constant speed. Let the direction of motion be the +*x* direction.

(*a*) The total displacement of the car for the entire trip is the sum of the displacements for the two legs of the trip:

$$\Delta x_{\text{total}} = \Delta x_1 + \Delta x_2$$

Find the displacement for each leg of the trip:

$$\Delta x_1 = v_{av,1}\Delta t_1 = (80\ \text{km/h})(2.5\ \text{h})$$
$$= 200\,\text{km}$$

and

$$\Delta x_2 = v_{av,2}\Delta t_2 = (40\ \text{km/h})(1.5\ \text{h})$$
$$= 60.0\,\text{km}$$

Add the individual displacements to get the total displacement:

$$\Delta x_{\text{total}} = \Delta x_1 + \Delta x_2 = 200\,\text{km} + 60.0\,\text{km}$$
$$= \boxed{2.6\times10^5\,\text{m}}$$

(*b*) As long as the car continues to move in the same direction, the average velocity for the total trip is given by:

$$v_{av} = \frac{\Delta x_{total}}{\Delta t_{total}}$$

Substitute numerical values and evaluate v_{av}:

$$v_{av} = \frac{2.6\times10^5\ \text{m}}{2.5\ \text{h} + 1.5\ \text{h}} = \boxed{65\,\text{km/h}}$$

45 • One busy air route across the Atlantic Ocean is about 5500 km. The now-retired Concord, a supersonic jet capable of flying at twice the speed of sound was used to travel such routes. (*a*) Roughly how long did it take for a one-way flight? (Use 343 m/s for the speed of sound.) (*b*) Compare this time to the time taken by a subsonic jet flying at 0.90 times the speed of sound.

Picture the Problem However unlikely it may seem, imagine that both jets are flying in a straight line at constant speed and use the definition of average speed to find the flight times.

(*a*) The time of flight is the ratio of the distance traveled to the speed of the supersonic jet.

$$t_{supersonic} = \frac{s_{Atlantic}}{v_{supersonic}} = \frac{5500\ \text{km}}{2(343\,\text{m/s})(3600\,\text{s/h})} = 2.23\,\text{h} = \boxed{2.2\ \text{h}}$$

(*b*) Express the ratio of the time for the trip at supersonic speed to the time for the trip at subsonic speed and simplify to obtain:

$$\frac{t_{supersonic}}{t_{subsonic}} = \frac{\dfrac{s_{Atlantic}}{v_{supersonic}}}{\dfrac{s_{Atlantic}}{v_{subsonic}}} = \frac{v_{subsonic}}{v_{supersonic}}$$

Substitute numerical values and evaluate the ratio of the flight times:

$$\frac{t_{supersonic}}{t_{subsonic}} = \frac{(0.90)(343\,\text{m/s})}{(2)(343\,\text{m/s})} = \boxed{0.45}$$

46 •• The speed of light, designated by the universally recognized symbol *c*, has a value, to two significant figures, of 3.0×10^8 m/s. (*a*) How long does it take for light to travel from the Sun to Earth, a distance of 1.5×10^{11} m? (*b*) How long does it take light to travel from the Moon to Earth, a distance of 3.8×10^8 m?

Picture the Problem In free space, light travels in a straight line at constant speed, *c*.

(*a*) Using the definition of average speed, express the time Δt required for light to travel from the Sun to Earth:

$$\Delta t = \frac{\Delta s}{\text{average speed}}$$

where Δs is the distance from the Sun to Earth.

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1.5\times10^{11}\ \text{m}}{3.0\times10^{8}\ \text{m/s}} = 5.0\times10^{2}\ \text{s}$$

$$\approx \boxed{8.3\,\text{min}}$$

(*b*) Proceed as in (*a*) this time using the Moon-Earth distance:

$$\Delta t = \frac{3.8\times10^{8}\ \text{m}}{3.0\times10^{8}\ \text{m/s}} \approx \boxed{1.3\,\text{s}}$$

47 • [SSM] Proxima Centauri, the closest star to us besides our own sun, is 4.1×10^{13} km from Earth. From Zorg, a planet orbiting this star, a Gregor places an order at Tony's Pizza in Hoboken, New Jersey, communicating via light signals. Tony's fastest delivery craft travels at $1.00 \times 10^{-4}c$ (see Problem 46). (*a*) How long does it take Gregor's order to reach Tony's Pizza? (*b*) How long does Gregor wait between sending the signal and receiving the pizza? If Tony's has a "1000-years-or-it's-free" delivery policy, does Gregor have to pay for the pizza?

Picture the Problem In free space, light travels in a straight line at constant speed, *c*. We can use the definition of average speed to find the elapsed times called for in this problem.

(*a*) Using the definition of average speed (equal here to the assumed constant speed of light), solve for the time Δt required to travel the distance to Proxima Centauri:

$$\Delta t = \frac{\text{distance traveled}}{\text{speed of light}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{4.1\times10^{16}\ \text{m}}{3.0\times10^{8}\ \text{m/s}} = 1.37\times10^{8}\ \text{s}$$

$$= \boxed{4.3\,\text{y}}$$

(b) The delivery time (Δt_{total}) is the sum of the time for the order to reach Hoboken and the travel time for the delivery craft to travel to Proxima Centauri:

$$\begin{aligned}\Delta t_{\text{total}} &= \Delta t_{\substack{\text{order to be}\\ \text{sent to Hoboken}}} + \Delta t_{\substack{\text{order to}\\ \text{be delivered}}}\\ &= 4.33\,y + \frac{4.1\times10^{13}\ \text{km}}{(1.00\times10^{-4})(3.0\times10^{8}\ \text{m/s})}\left(\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}}\right)\\ &= 4.3\ \text{y} + 4.3\times10^{4}\ \text{y} \approx 4.3\times10^{4}\ \text{y}\end{aligned}$$

Because 4.3×10^4 y >> 1000 y, Gregor does not have to pay.

48 • A car making a 100-km journey travels 40 km/h for the first 50 km. How fast must it go during the second 50 km to average 50 km/h?

Picture the Problem The time for the second 50 km is equal to the time for the entire journey less the time for the first 50 km. We can use this time to determine the average speed for the second 50 km interval from the definition of average speed.

Using the definition of average speed, find the time required for the total journey:

$$\Delta t_{\text{total}} = \frac{\Delta x}{v_{\text{av}}} = \frac{100\,\text{km}}{50\,\text{km/h}} = 2.0\,\text{h}$$

Find the time required for the first 50 km:

$$\Delta t_{\text{1st 50 km}} = \frac{50\,\text{km}}{40\,\text{km/h}} = 1.25\,\text{h}$$

Find the time remaining to travel the last 50 km:

$$\begin{aligned}\Delta t_{\text{2nd 50 km}} &= t_{\text{total}} - t_{\text{1st 50 km}} = 2.0\,\text{h} - 1.25\,\text{h}\\ &= 0.75\,\text{h}\end{aligned}$$

Finally, use the time remaining to travel the last 50 km to determine the average speed over this distance:

$$\begin{aligned}v_{\text{av, 2nd 50 km}} &= \frac{\Delta x_{\text{2nd 50 km}}}{\Delta t_{\text{2nd 50 km}}} = \frac{50\,\text{km}}{0.75\,\text{h}}\\ &= \boxed{67\,\text{km/h}}\end{aligned}$$

49 •• Late in ice hockey games, the team that is losing sometimes "pulls" their goalkeeper off the ice to add an additional offensive player and increase their chances of scoring. In such cases, the goalie on the opposing team might have an opportunity to score into the unguarded net that is 55.0 m away. Suppose you are the goaltender for your university team and are in just such a situation. You launch a shot (in hopes of getting your first career goal) on the frictionless ice. You hear a disappointing "clang" as the puck strikes a goalpost (instead of going in!) exactly 2.50 s later. In this case, how fast did the puck travel? You should assume 343 m/s for the speed of sound.

Picture the Problem The distance over which both the puck and the sound from the puck hitting the goalpost must travel is 55.0 m. The time between the shot being released and the sound reaching the goalie's ear can be found by expressing the total elapsed time as the sum of the elapsed times for the shot to travel 55.0 m and for the sound to travel back to you.

The total elapsed time as the sum of the elapsed times for the shot and for the sound to travel back to you :

$$\Delta t_{total} = \Delta t_{shot} + \Delta t_{sound}$$

Express the time for the shot to travel to the other net a distance Δx away:

$$\Delta t_{shot} = \frac{\Delta x}{v_{shot}}$$

Express the time for the sound to travel a distance Δx back to you:

$$\Delta t_{sound} = \frac{\Delta x}{v_{sound}}$$

Substitute in the expression for Δt_{total} to obtain:

$$\Delta t_{total} = \frac{\Delta x}{v_{shot}} + \frac{\Delta x}{v_{sound}}$$

Solving this equation for v_{shot} yields:

$$v_{shot} = \frac{v_{sound}\Delta x}{v_{sound}\Delta t_{total} - \Delta x}$$

Substitute numerical values and evaluate v_{shot} :

$$v_{shot} = \frac{(343\text{ m/s})(55.0\text{ m})}{(343\text{ m/s})(2.50\text{ s}) - 55.0\text{ m}} = \boxed{23.5\text{ m/s}}$$

50 •• Cosmonaut Andrei, your co-worker at the International Space Station, tosses a banana at you with a speed of 15 m/s. At exactly the same instant, you fling a scoop of ice cream at Andrei along exactly the same path. The collision between banana and ice cream produces a banana split 7.2 m from your location 1.2 s after the banana and ice cream were launched. (*a*) How fast did you toss the ice cream? (*b*) How far were you from Andrei when you tossed the ice cream? (Neglect any effects due to gravity.)

Picture the Problem Let the subscript b refer to the banana and the subscript ic refer to the ice cream. Then the distance covered by the ice cream before collision is given by $\Delta x_{ic} = v_{ic}\Delta t$ and the distance covered by the banana is $\Delta x_b = v_b\Delta t$. The distance between you and Andrei is then the sum of these distances: $\Delta x_{tot} = \Delta x_{ic} + \Delta x_b$.

(*a*) The speed of the ice cream is given by:

$$v_{ic} = \frac{\Delta x_{ic}}{\Delta t}$$

where Δt is the time-to-collision.

Substitute numerical values and evaluate v_{ic}:

$$v_{ic} = \frac{7.2\text{ m}}{1.2\text{ s}} = \boxed{6.0\text{ m/s}}$$

(*b*) Express the distance between yourself and Andrei as the sum of the distances the ice cream and the banana travel:

$$\Delta \boldsymbol{x}_{\text{total}} = \Delta \boldsymbol{x}_{\text{ic}} + \Delta \boldsymbol{x}_{\text{b}}$$

Because $\Delta x_{\text{b}} = v_{\text{b}} \Delta t$:

$$\Delta x_{\text{total}} = \Delta x_{\text{ic}} + v_{\text{b}} \Delta t$$

Substitute numerical values and evaluate Δx_{total}:

$$\Delta x_{\text{total}} = 7.2\text{ m} + (15\text{ m/s})(1.2\text{ s}) = \boxed{25\text{ m}}$$

51 •• Figure 2-39 shows the position of a particle as a function of time. Find the average velocities for the time intervals *a*, *b*, *c*, and *d* indicated in the figure.

Picture the Problem The average velocity in a time interval is defined as the displacement divided by the time elapsed; that is $v_{\text{av}} = \Delta x / \Delta t$.

(*a*) $\Delta x_a = 0$

$$v_{\text{av}} = \boxed{0}$$

(*b*) $\Delta x_b = 1$ m and $\Delta t_b = 3$ s

$$v_{\text{av}} = \boxed{0.3\text{ m/s}}$$

(*c*) $\Delta x_c = -6$ m and $\Delta t_c = 3$ s

$$v_{\text{av}} = \boxed{-2\text{ m/s}}$$

(*d*) $\Delta x_d = 3$ m and $\Delta t_d = 3$ s

$$v_{\text{av}} = \boxed{1\text{ m/s}}$$

52 •• It has been found that, on average, galaxies are moving away from Earth at a speed that is proportional to their distance from Earth. This discovery is known as Hubble's law, named for its discoverer, astrophysicist Sir Edwin Hubble. He found that the recessional speed v of a galaxy a distance r from Earth is given by $v = Hr$, where $H = 1.58 \times 10^{-18}\text{ s}^{-1}$ is called the Hubble constant. What are the expected recessional speeds of galaxies (*a*) 5.00×10^{22} m from Earth and (*b*) 2.00×10^{25} m from Earth? (*c*) If the galaxies at each of these distances had traveled at their expected recessional speeds, how long ago would they have been at our location?

Picture the Problem In free space, light travels in a straight line at constant speed *c*. We can use Hubble's law to find the speed of the two planets.

(*a*) Using Hubble's law, calculate the speed of the first galaxy:

$$v_{\text{a}} = (5.00 \times 10^{22}\text{ m})(1.58 \times 10^{-18}\text{ s}^{-1}) = \boxed{7.90 \times 10^{4}\text{ m/s}}$$

(*b*) Using Hubble's law, calculate the speed of the second galaxy:

$$v_b = (2.00\times10^{25}\text{ m})(1.58\times10^{-18}\text{ s}^{-1}) = \boxed{3.16\times10^{7}\text{ m/s}}$$

(*c*) Using the relationship between distance, speed, and time for both galaxies, express how long ago Δt they were both located at the same place as Earth:

$$\Delta t = \frac{r}{v} = \frac{r}{rH} = \frac{1}{H}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = 6.33\times10^{17}\text{ s} \approx \boxed{20\times10^{9}\text{ y}}$$

53 •• [SSM] The cheetah can run as fast as 113 km/h, the falcon can fly as fast as 161 km/h, and the sailfish can swim as fast as 105 km/h. The three of them run a relay with each covering a distance *L* at maximum speed. What is the average speed of this relay team for the entire relay? Compare this average speed with the numerical average of the three individual speeds. Explain carefully why the average speed of the relay team is *not* equal to the numerical average of the three individual speeds.

Picture the Problem We can find the average speed of the relay team from the definition of average speed.

Using its definition, relate the average speed to the total distance traveled and the elapsed time:

$$v_{av} = \frac{\text{distance traveled}}{\text{elapsed time}}$$

Express the time required for each animal to travel a distance L:

$$t_{cheetah} = \frac{L}{v_{cheetah}},\ t_{falcon} = \frac{L}{v_{falcon}}$$

and

$$t_{sailfish} = \frac{L}{v_{sailfish}}$$

Express the total time Δt:

$$\Delta t = L\left(\frac{1}{v_{cheetah}} + \frac{1}{v_{falcon}} + \frac{1}{v_{sailfish}}\right)$$

Use the total distance traveled by the relay team and the elapsed time to calculate the average speed:

$$v_{av} = \frac{3L}{L\left(\frac{1}{113\,\text{km/h}} + \frac{1}{161\,\text{km/h}} + \frac{1}{105\,\text{km/h}}\right)} = 122.03\,\text{km/h} = \boxed{122\,\text{km/h}}$$

Calculating the average of the three speeds yields:

$$\text{Average}_{\text{three speeds}} = \frac{113\,\text{km/h} + 161\,\text{km/h} + 105\,\text{km/h}}{3} = 126.33\,\text{km/h} = 126\,\text{km/h}$$
$$= \boxed{1.04 v_{\text{av}}}$$

The average speed of the relay team is *not* equal to the numerical average of the three individual speeds because the ″runners″ did not run for the same interval of time. The average speed would be equal to one-third the sum of the three speeds if the three speeds were each maintained for the same length of time instead of for the same distance.

54 •• Two cars are traveling along a straight road. Car A maintains a constant speed of 80 km/h and car B maintains a constant speed of 110 km/h. At $t = 0$, car B is 45 km behind car A. (*a*) How much farther will car A travel before car B overtakes it? (*b*) How much ahead of A will B be 30 s after it overtakes A?

Picture the Problem Let the position of car A at $t = 0$ be the origin of our coordinate system. Then we can use a constant-acceleration equation to express the positions of both cars as functions of time and equate these expressions to determine the time at which car A is overtaken by car B.

(*a*) Car B overtakes car A when their *x* coordinates are the same:

$$x_A(t) = x_B(t) \qquad (1)$$

Using a constant-acceleration equation with $a = 0$, express the position of car A as a function of time:

$$x_A(t) = x_{0A} + v_A t$$

where x_{0A} is the position of car A at $t = 0$.

Because we've let car A be at the origin at $t = 0$:

$$x_A(t) = v_A t \qquad (2)$$

Using a constant-acceleration equation, express the position of car B as a function of time:

$$x_B(t) = x_{0B} + v_B t$$

where x_{0B} is the position of car B at $t = 0$.

Substitute for $x_A(t)$ and $x_B(t)$ in equation (1) to obtain:

$$v_A t = x_{0B} + v_B t \Rightarrow t = \frac{x_{0B}}{v_A - v_B}$$

Substitute numerical values and evaluate the time *t* at which car B overtakes car A:

$$t = \frac{-45\,\text{km}}{80\,\text{km/h} - 110\,\text{km/h}} = 1.50\,\text{h}$$

Now we can evaluate equation (2) at $t = 1.50$ h to obtain:

$$x_A(1.50\text{ h}) = \left(80\,\frac{\text{km}}{\text{h}}\right)(1.50\text{ h}) = 120\text{ km}$$
$$= \boxed{1.2\times10^5\text{ m}}$$

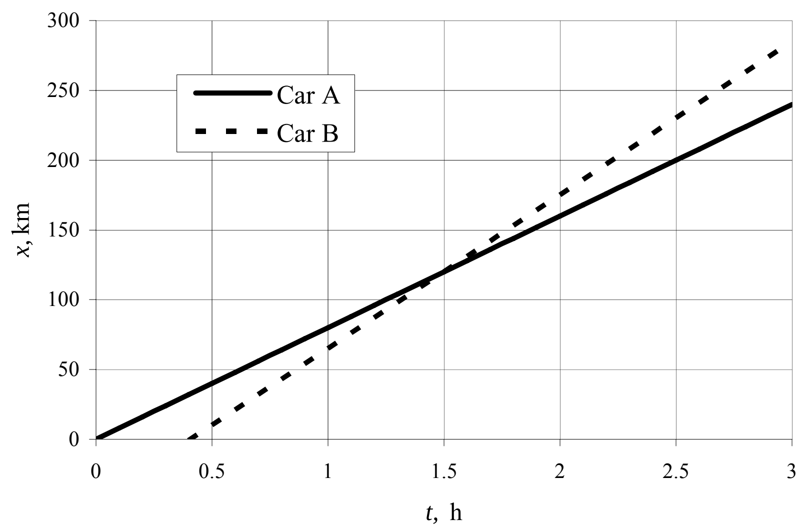
(*b*) The separation of the cars as a function of time is given by:

$$\Delta x(t) = x_B(t) - x_A(t) = x_{0B} - v_B t - v_A t$$

Substitute numerical values and evaluate $\Delta x(1.50\text{ h} + 30\text{ s}) = \Delta x\left(1.50\text{ h} + \tfrac{1}{120}\text{ h}\right)$ to obtain:

$$\Delta x\left(1.50\text{ h} + \tfrac{1}{120}\text{ h}\right) = -45\text{ km} + \left(110\,\frac{\text{km}}{\text{h}} - 80\,\frac{\text{km}}{\text{h}}\right)\left(1.50\text{ h} + \tfrac{1}{120}\text{ h}\right) = \boxed{0.25\text{ km}}$$

Remarks: One can use a graphing calculator or a spreadsheet program to solve this problem. A spreadsheet program was used to plot the following graph:



Note that this graph confirms our result that the cars are at the same location at t = 1.5 h.

55 •• **[SSM]** A car traveling at a constant speed of 20 m/s passes an intersection at time $t = 0$. A second car traveling at a constant speed of 30 m/s in the same direction passes the same intersection 5.0 s later. (*a*) Sketch the position functions $x_1(t)$ and $x_2(t)$ for the two cars for the interval $0 \le t \le 20$ s. (*b*) Determine when the second car will overtake the first. (*c*) How far from the intersection will the two cars be when they pull even? (*d*) Where is the first car when the second car passes the intersection?

Picture the Problem One way to solve this problem is by using a graphing calculator to plot the positions of each car as a function of time. Plotting these positions as functions of time allows us to visualize the motion of the two cars

relative to the (fixed) ground. More importantly, it allows us to see the motion of the two cars relative to each other. We can, for example, tell how far apart the cars are at any given time by determining the length of a vertical line segment from one curve to the other.

(*a*) Letting the origin of our coordinate system be at the intersection, the position of the slower car, $x_1(t)$, is given by:

$x_1(t) = 20t$
where x_1 is in meters if t is in seconds.

Because the faster car is also moving at a constant speed, we know that the position of this car is given by a function of the form:

$x_2(t) = 30t + b$

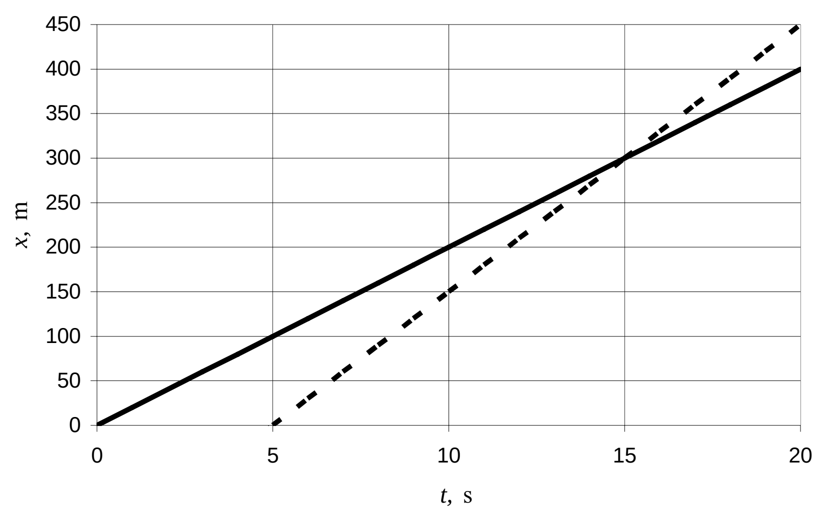
We know that when $t = 5.0$ s, this second car is at the intersection (that is, $x_2(5.0\text{ s}) = 0$). Using this information, you can convince yourself that:

$b = -150$ m

Thus, the position of the faster car is given by:

$x_2(t) = 30t - 150$

One can use a graphing calculator, graphing paper, or a spreadsheet to obtain the following graphs of $x_1(t)$ (the solid line) and $x_2(t)$ (the dashed line):



(*b*) Use the time coordinate of the intersection of the two lines to determine the time at which the second car overtakes the first:

From the intersection of the two lines, one can see that the second car will "overtake" (catch up to) the first car at $\boxed{t = 15\text{ s.}}$

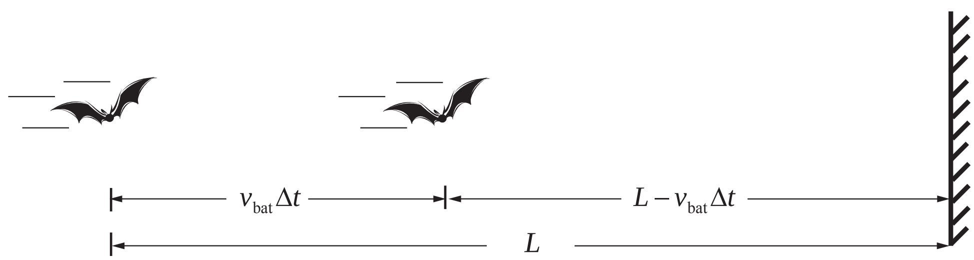
(*c*) Use the position coordinate of the intersection of the two lines to determine the distance from the intersection at which the second car catches up to the first car:

From the intersection of the two lines, one can see that the distance from the intersection is $\boxed{300\text{ m.}}$

(*d*) Draw a vertical line from $t = 5$ s to the solid line and then read the position coordinate of the intersection of the vertical line and the solid line to determine the position of the first car when the second car went through the intersection. From the graph, when the second car passes the intersection, the first car was $\boxed{100\text{ m ahead.}}$

56 •• Bats use echolocation to determine their distance from objects they cannot easily see in the dark. The time between the emission of high-frequency sound pulse (a click) and the detection of its echo is used to determine such distances. A bat, flying at a constant speed of 19.5 m/s in a straight line toward a vertical cave wall, makes a single clicking noise and hears the echo 0.15 s later. Assuming that she continued flying at her original speed, how close was she to the wall when she received the echo? Assume a speed of 343 m/s for the speed of sound.

Picture the Problem The sound emitted by the bat travels at v_{sound} and during the time interval Δt during which the sound travels to the wall (a distance L from the bat's initial position) and back to the bat, the bat travels a distance of $\Delta x_{\text{bat}} = v_{\text{bat}}\Delta t$, where v_{bat} is the bat's flying speed. The distance of the bat from the wall when she received the echo of her click is $\Delta x_{\text{away}} = L - v_{\text{bat}}\Delta t$ where $\Delta t = \Delta t_{\text{to wall}} + \Delta t_{\substack{\text{back}\\\text{to bat}}}$.



Express the distance of the bat from the wall of the cave when it hears the echo of its click:

$$\Delta x_{\text{away}} = L - v_{\text{bat}}\Delta t \qquad (1)$$

The elapsed time between the bat clicking and hearing the sound is:

$$\Delta t = \Delta t_{\text{to wall}} + \Delta t_{\substack{\text{back}\\\text{to bat}}}$$

Substituting for $\Delta t_{\text{to wall}}$ and $\Delta t_{\substack{\text{back}\\ \text{to bat}}}$ gives:

$$\Delta t = \frac{L}{v_{\text{sound}}} + \frac{L - v_{\text{bat}}\Delta t}{v_{\text{sound}}}$$

Solving for L yields:

$$L = \tfrac{1}{2}\left(v_{\text{sound}} + v_{\text{bat}}\right)\Delta t$$

Substitute for L in equation (1) and simplify to obtain:

$$\Delta x_{\text{away}} = \tfrac{1}{2}\left(v_{\text{sound}} + v_{\text{bat}}\right)\Delta t - v_{\text{bat}}\Delta t = \tfrac{1}{2}\left(v_{\text{sound}} - v_{\text{bat}}\right)\Delta t$$

Substitute numerical values and evaluate Δx_{away}:

$$\Delta x_{\text{away}} = \tfrac{1}{2}\left(343\ \text{m/s} - 19.5\ \text{m/s}\right)\left(0.15\ \text{s}\right) = \boxed{24\ \text{m}}$$

57 ••• A submarine can use *sonar* (sound traveling through water) to determine its distance from other objects. The time between the emission of a sound pulse (a ″ping″) and the detection of its echo can be used to determine such distances. Alternatively, by measuring the time between *successive* echo receptions of a *regularly timed set* of pings, the submarine's *speed* may be determined by comparing the time between echoes to the time between pings. Assume you are the sonar operator in a submarine traveling at a constant velocity underwater. Your boat is in the eastern Mediterranean Sea, where the speed of sound is known to be 1522 m/s. If you send out pings every 2.000 s, and your apparatus receives echoes reflected from an undersea cliff every 1.980 s, how fast is your submarine traveling?

Picture the Problem Both the pulses sent out by the submarine and the pulses returning from the sea-wall are traveling at 1522 m/s. Consequently, we can determine the distance in water between two successive echo (or emitted) pulses of sound which were emitted with a time interval $\Delta t_{\text{emitted}}$ between them. The actual distance in the seawater between the echoed pulses is given by $\Delta x = v_{\text{sound}}\Delta t_{\text{emitted}}$. We need to find the time $\Delta t_{\text{received}}$ between successive pulses received by the submarine. We start our ″clock″, as it were, when the submarine passes one of two successive pulses that approach it, separated by the distance Δx. After passing the first pulse, the next sound pulse moves toward the submarine at v_{sound} and the submarine moves toward the pulse at speed v_{sub}. The distance between successive pulses Δx may be divided into Δx_{sub} and Δx_{sound}, which are equal to $v_{\text{sub}}\Delta t_{\text{received}}$ and $v_{\text{sub}}\Delta t_{\text{received}}$, respectively.

The distance between successive pulses is given by:

$$\Delta x = \Delta x_{\text{sub}} + \Delta x_{\text{received}}$$

Substituting for all three terms in this equation yields:

$$v_{\text{sound}}\Delta t_{\text{emitted}} = v_{\text{sub}}\Delta t_{\text{received}} + v_{\text{sound}}\Delta t_{\text{received}}$$

Solve for v_{sub} to obtain:

$$v_{sub} = \frac{v_{sound}\left(\Delta t_{emitted} - \Delta t_{received}\right)}{\Delta t_{received}}$$

Substitute numerical values and evaluate v_{sub}:

$$v_{sub} = \frac{(1522\text{ m/s})(2.000\text{ s} - 1.980\text{ s})}{2.000\text{ s}} = \boxed{15\text{ m/s}}$$

Acceleration

58 • A sports car accelerates in third gear from 48.3 km/h (about 30 mi/h) to 80.5 km/h (about 50 mi/h) in 3.70 s. (*a*) What is the average acceleration of this car in m/s^2? (*b*) If the car maintained this acceleration, how fast would it be moving one second later?

Picture the Problem In Part (*a*), we can apply the definition of average acceleration to find a_{av}. In Part (*b*), we can find the change in the car's velocity in one second and add this change to its velocity at the beginning of the interval to find its speed one second later.

(*a*) The definition of average acceleration is:

$$a_{av} = \frac{\Delta v}{\Delta t}$$

Substitute numerical values and evaluate a_{av}:

$$a_{av} = \frac{80.5\text{ km/h} - 48.3\text{ km/h}}{3.70\text{ s}} = 8.70\frac{\text{km}}{\text{h}\cdot\text{s}}$$

Convert a_{av} to m/s^2:

$$a_{av} = \left(8.70\times 10^3\frac{\text{m}}{\text{h}\cdot\text{s}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = \boxed{2.42\text{ m/s}^2}$$

(*b*) Express the speed of the car at the end of 4.7 s:

$$v(4.7\text{ s}) = v(3.70\text{ s}) + \Delta v_{1\text{s}} = 80.5\text{ km/h} + \Delta v_{1\text{s}}$$

Find the change in the speed of the car in 1.00 s:

$$\Delta v = a_{av}\Delta t = \left(8.70\frac{\text{km}}{\text{h}\cdot\text{s}}\right)(1.00\text{ s}) = 8.70\text{ km/h}$$

Substitute and evaluate $v(4.7\text{ s})$:

$$v(4.7\text{ s}) = 80.5\text{ km/h} + 8.7\text{ km/h} = \boxed{89.2\text{ km/h}}$$

59 • **[SSM]** An object is moving along the x axis. At $t = 5.0$ s, the object is at $x = +3.0$ m and has a velocity of +5.0 m/s. At $t = 8.0$ s, it is at $x = +9.0$ m and its velocity is –1.0 m/s. Find its average acceleration during the time interval $5.0\text{ s} \le t \le 8.0\text{ s}$.

Picture the Problem We can find the change in velocity and the elapsed time from the given information and then use the definition of average acceleration.

The average acceleration is defined as the change in velocity divided by the change in time:

$$a_{\text{av}} = \frac{\Delta v}{\Delta t}$$

Substitute numerical values and evaluate a_{av}:

$$a_{\text{av}} = \frac{(-1.0\text{ m/s}) - (5.0\text{ m/s})}{(8.0\text{ s}) - (5.0\text{ s})} = \boxed{-2.0\text{ m/s}^2}$$

60 •• A particle moves along the x axis with velocity $v_x = (8.0\text{ m/s}^2)\,t - 7.0\text{ m/s}$. (*a*) Find the average acceleration for two different one-second intervals, one beginning at $t = 3.0$ s and the other beginning at $t = 4.0$ s. (*b*) Sketch v_x versus t over the interval $0 < t < 10$ s. (*c*) How do the instantaneous accelerations at the middle of each of the two time intervals specified in Part (*a*) compare to the average accelerations found in Part (*a*)? Explain.

Picture the Problem The important concept here is the difference between average acceleration and instantaneous acceleration.

(*a*) The average acceleration is defined as the change in velocity divided by the change in time:

$$a_{\text{av}} = \frac{\Delta v}{\Delta t}$$

Determine v_x at $t = 3.0$ s, $t = 4.0$ s, and $t = 5.0$ s:

$$v_x(3.0\text{ s}) = (8.0\text{ m/s}^2)(3.0\text{ s}) - 7.0\text{ m/s} = 17\text{ m/s}$$

$$v_x(4.0\text{ s}) = (8.0\text{ m/s}^2)(4.0\text{ s}) - 7.0\text{ m/s} = 25\text{ m/s}$$

$$v_x(5.0\text{ s}) = (8.0\text{ m/s}^2)(5.0\text{ s}) - 7.0\text{ m/s} = 33\text{ m/s}$$

Find a_{av} for the two 1-s intervals:

$$a_{\text{av, 3.0 s to 4.0 s}} = \frac{25\text{ m/s} - 17\text{ m/s}}{1.0\text{ s}} = 8.0\text{ m/s}^2$$

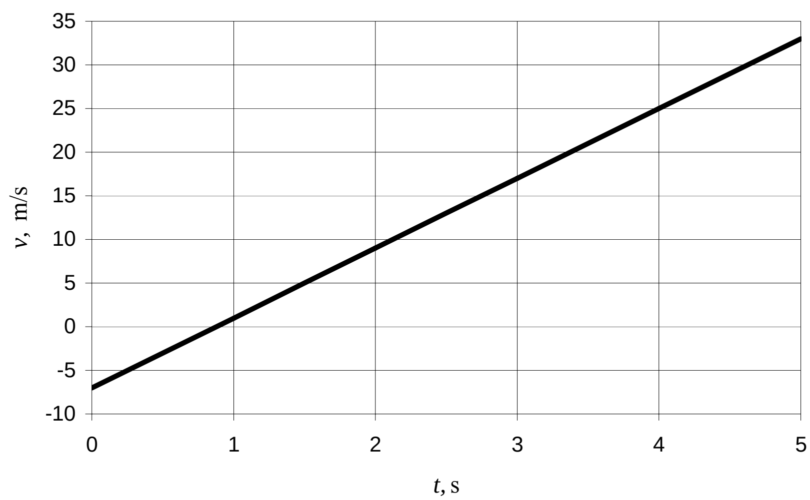
and

$$a_{\text{av, 4.0 s to 5.0 s}} = \frac{33\text{ m/s} - 25\text{ m/s}}{1.0\text{ s}} = 8.0\text{ m/s}^2$$

The instantaneous acceleration is defined as the time derivative of the velocity or the slope of the velocity- versus-time curve:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left[\left(8.0\text{ m/s}^2\right)t - 7.0\text{ m/s}\right] = \boxed{8.0\text{ m/s}^2}$$

(b) The given function and a spreadsheet program were used to plot the following graph of v-versus-t:



(c) Because the particle's speed varies linearly with time, these accelerations are the same.

61 •• [SSM] The position of a certain particle depends on time according to the equation $x(t) = t^2 - 5.0t + 1.0$, where x is in meters if t is in seconds. (a) Find the displacement and average velocity for the interval $3.0\text{ s} \le t \le 4.0\text{ s}$. ($b$) Find the general formula for the displacement for the time interval from t to $t + \Delta t$. (c) Use the limiting process to obtain the instantaneous velocity for any time t.

Picture the Problem We can closely approximate the instantaneous velocity by the average velocity in the limit as the time interval of the average becomes small. This is important because all we can ever obtain from any measurement is the average velocity, v_{av}, which we use to approximate the instantaneous velocity v.

(*a*) The displacement of the particle during the interval 3.0 s ≤ t ≤ 4.0 s is given by:

$$\Delta x = x(4.0\ \text{s}) - x(3.0\ \text{s}) \qquad (1)$$

The average velocity is given by:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} \qquad (2)$$

Find $x(4.0\ \text{s})$ and $x(3.0\ \text{s})$:

$x(4.0\ \text{s}) = (4.0)^2 - 5(4.0) + 1 = -3.0\ \text{m}$
and
$x(3.0\ \text{s}) = (3.0)^2 - 5(3.0) + 1 = -5.0\ \text{m}$

Substitute numerical values in equation (1) and evaluate Δx:

$$\Delta x = (-3.0\ \text{m}) - (-5.0\ \text{m}) = \boxed{2.0\ \text{m}}$$

Substitute numerical values in equation (2) and evaluate v_{av}:

$$v_{\text{av}} = \frac{2.0\ \text{m}}{1.0\ \text{s}} = \boxed{2.0\ \text{m/s}}$$

(*b*) Find $x(t + \Delta t)$:

$$\begin{aligned} x(t+\Delta t) &= (t+\Delta t)^2 - 5(t+\Delta t) + 1 \\ &= (t^2 + 2t\Delta t + (\Delta t)^2) \\ &\quad - 5(t+\Delta t) + 1 \end{aligned}$$

Express $x(t + \Delta t) - x(t) = \Delta x$:

$$\Delta x = \boxed{(2t-5)\Delta t + (\Delta t)^2}$$

where Δx is in meters if t is in seconds.

(*c*) From (*b*) find $\Delta x/\Delta t$ as $\Delta t \to 0$:

$$\begin{aligned} \frac{\Delta x}{\Delta t} &= \frac{(2t-5)\Delta t + (\Delta t)^2}{\Delta t} \\ &= 2t - 5 + \Delta t \end{aligned}$$

and

$$v = \lim_{\Delta t \to 0}(\Delta x/\Delta t) = \boxed{2t-5}$$

where v is in m/s if t is in seconds.

Alternatively, we can take the derivative of $x(t)$ with respect to time to obtain the instantaneous velocity.

$$\begin{aligned} v(t) &= \frac{dx(t)}{dt} = \frac{d}{dt}\left(at^2 + bt + 1\right) \\ &= 2at + b \\ &= \boxed{2t - 5} \end{aligned}$$

62 •• The position of an object as a function of time is given by $x = At^2 - Bt + C$, where $A = 8.0\ \text{m/s}^2$, $B = 6.0\ \text{m/s}$, and $C = 4.0\ \text{m}$. Find the instantaneous velocity and acceleration as functions of time.

Picture the Problem The instantaneous velocity is dx/dt and the acceleration is dv/dt.

Using the definitions of instantaneous velocity and acceleration, determine v and a:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left[At^2 - Bt + C\right] = 2At - B$$

and

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[2At - B\right] = 2A$$

Substitute numerical values for A and B and evaluate v and a:

$$v = 2\left(8.0\ \text{m/s}^2\right)t - 6.0\ \text{m/s}$$
$$= \boxed{\left(16\ \text{m/s}^2\right)t - 6.0\ \text{m/s}}$$

and

$$a = 2\left(8.0\ \text{m/s}^2\right) = \boxed{16\ \text{m/s}^2}$$

63 ••• The one-dimensional motion of a particle is plotted in Figure 2-40. (*a*) What is the average acceleration in each of the intervals *AB*, *BC*, and *CE*? (*b*) How far is the particle from its starting point after 10 s? (*c*) Sketch the displacement of the particle as a function of time; label the instants A, B, C, D, and E on your graph. (*d*) At what time is the particle traveling most slowly?

Picture the Problem We can use the definition of average acceleration ($a_{av} = \Delta v/\Delta t$) to find a_{av} for the three intervals of constant acceleration shown on the graph.

(*a*) Using the definition of average acceleration, find a_{av} for the interval AB:

$$a_{\text{av, AB}} = \frac{15.0\ \text{m/s} - 5.0\ \text{m/s}}{3.0\ \text{s}} = \boxed{3.3\ \text{m/s}^2}$$

Find a_{av} for the interval BC:

$$a_{\text{av, BC}} = \frac{15.0\ \text{m/s} - 15.0\ \text{m/s}}{3.0\ \text{s}} = \boxed{0}$$

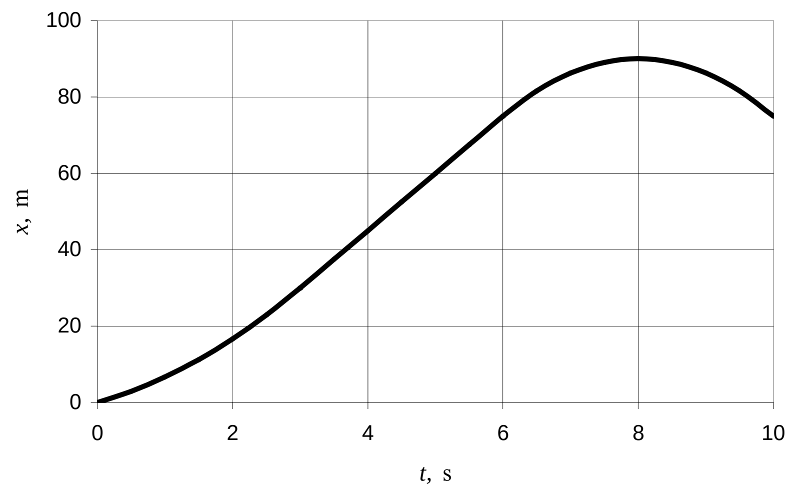
Find a_{av} for the interval CE:

$$a_{\text{av, CE}} = \frac{-15.0\ \text{m/s} - 15.0\ \text{m/s}}{4.0\ \text{s}}$$
$$= \boxed{-7.5\ \text{m/s}^2}$$

(*b*) Use the formulas for the areas of trapezoids and triangles to find the area under the graph of v as a function of t.

$$\begin{aligned}\Delta x &= (\Delta x)_{\text{A}\to\text{B}} + (\Delta x)_{\text{B}\to\text{C}} + (\Delta x)_{\text{C}\to\text{D}} + (\Delta x)_{\text{D}\to\text{E}}\\ &= \tfrac{1}{2}(5.0\ \text{m/s} + 15.0\ \text{m/s})(3.0\ \text{s}) + (15.0\ \text{m/s})(3.0\ \text{s}) + \tfrac{1}{2}(15.0\ \text{m/s})(2.0\ \text{s})\\ &\qquad + \tfrac{1}{2}(-15.0\ \text{m/s})(2.0\ \text{s})\\ &= \boxed{75\ \text{m}}\end{aligned}$$

(*c*) The graph of displacement, *x*, as a function of time, *t*, is shown in the following figure. In the region from B to C the velocity is constant so the *x*- versus-*t* curve is a straight line.



(*d*) Reading directly from the figure, we can find the time when the particle is moving the slowest. At point D, $t = 8$ s, the graph crosses the time axis; therefore $v = 0$.

Constant Acceleration and Free-Fall

64 • An object projected vertically upward with initial speed v_0 attains a maximum height h above its launch point. Another object projected up with initial speed $2v_0$ from the same height will attain a maximum height of (*a*) $4h$, (*b*) $3h$, (*c*) $2h$, (*d*) h. (Air resistance is negligible.)

Picture the Problem Because the acceleration is constant ($-g$) we can use a constant-acceleration equation to find the height of the projectile.

Using a constant-acceleration equation, express the height of the object as a function of its initial speed, the acceleration due to gravity, and its displacement:

$$v^2 = v_0^2 - 2g\Delta y$$

or, because $v(h) = 0$,

$$0 = v_0^2 - 2gh \Rightarrow h = \frac{v_0^2}{2g}$$

Express the ratio of the maximum height of the second object to that of the first object and simplify to obtain:

$$\frac{h_{\text{2nd object}}}{h_{\text{1st object}}} = \frac{\dfrac{(2v_0)^2}{2g}}{\dfrac{(v_0)^2}{2g}} = 4$$

Solving for $h_{\text{2nd object}}$ yields: $h_{\text{2nd object}} = 4h \Rightarrow (a)$ is correct.

65 •• A car traveling along the x axis starts from rest at x = 50 m and accelerates at a constant rate of 8.0 m/s^2. (a) How fast is it going after 10 s? (b) How far has it gone after 10 s? (c) What is its average velocity for the interval $0 \le t \le 10$ s?

Picture the Problem Because the acceleration of the car is constant we can use constant-acceleration equations to describe its motion.

(a) Using a constant-acceleration equation, relate the velocity to the acceleration and the time:

$$v = v_0 + at$$

Substitute numerical values and evaluate v:

$$v = 0 + \left(8.0\,\frac{\text{m}}{\text{s}^2}\right)(10\,\text{s}) = \boxed{80\,\text{m/s}}$$

(b) Using a constant-acceleration equation, relate the displacement to the acceleration and the time:

$$\Delta x = x - x_0 = v_0 t + \frac{a}{2}t^2$$

Substitute numerical values and evaluate Δx:

$$\Delta \boldsymbol{x} = \frac{1}{2}\left(8.0\,\frac{\text{m}}{\text{s}^2}\right)(10\,\text{s})^2 = \boxed{0.40\,\text{km}}$$

(c) Use the definition of v_{av} :

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{400\,\text{m}}{10\,\text{s}} = \boxed{40\,\text{m/s}}$$

Remarks: Because the area under a velocity-versus-time graph is the displacement of the object, we could solve this problem graphically.

66 • An object traveling along the x axis with an initial velocity of +5.0 m/s has a constant acceleration of +2.0 m/s^2. When its speed is 15 m/s, how far has it traveled?

Picture the Problem Because the acceleration of the object is constant we can use constant-acceleration equations to describe its motion.

Using a constant-acceleration equation, relate the speed of the object to its acceleration and displacement:

$$v^2 = v_0^2 + 2a\,\Delta x \Rightarrow \Delta x = \frac{v^2 - v_0^2}{2a}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{\left(15^2 - 5.0^2\right)\text{m}^2/\text{s}^2}{2\left(2.0\,\text{m/s}^2\right)} = \boxed{50\,\text{m}}$$

67 • **[SSM]** An object traveling along the x axis at constant acceleration has a velocity of +10 m/s when it is at $x = 6.0$ m and of +15 m/s when it is at $x = 10$ m. What is its acceleration?

Picture the Problem Because the acceleration of the object is constant we can use constant-acceleration equations to describe its motion.

Using a constant-acceleration equation, relate the velocity to the acceleration and the displacement:

$$v^2 = v_0^2 + 2a\Delta x \Rightarrow a = \frac{v^2 - v_0^2}{2\Delta x}$$

Substitute numerical values and evaluate a:

$$a = \frac{\left(15^2 - 10^2\right)\mathrm{m^2/s^2}}{2\left(10\,\mathrm{m} - 6.0\,\mathrm{m}\right)} = \boxed{16\,\mathrm{m/s^2}}$$

68 • The speed of an object traveling along the x axis increases at the constant rate of +4.0 m/s each second. At $t = 0.0$ s, its velocity is +1.0 m/s and its position is +7.0 m. How fast is it moving when its position is +8.0 m, and how much time has elapsed from the start at $t = 0.0$ s?

Picture the Problem Because the acceleration of the object is constant we can use constant-acceleration equations to describe its motion.

Using a constant-acceleration equation, relate the velocity to the acceleration and the displacement:

$$v^2 = v_0^2 + 2a\Delta x \Rightarrow v = \sqrt{v_0^2 + 2a\Delta x}$$

Substitute numerical values and evaluate v to obtain:

$$v = \sqrt{\left(1.0\,\mathrm{m/s}\right)^2 + 2\left(4.0\,\mathrm{m/s^2}\right)\left(8.0\,\mathrm{m} - 7.0\,\mathrm{m}\right)} = \boxed{3.0\,\mathrm{m/s}}$$

From the definition of average acceleration we have:

$$\Delta t = \frac{\Delta v}{a_{\mathrm{av}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{3.0\,\mathrm{m/s} - 1.0\,\mathrm{m/s}}{4.0\,\mathrm{m/s^2}} = \boxed{0.50\,\mathrm{s}}$$

69 •• A ball is launched directly upward from ground level with an initial speed of 20 m/s. (Air resistance is negligible.) (*a*) How long is the ball in the air? (*b*) What is the greatest height reached by the ball? (*c*) How many seconds after launch is the ball 15 m above the release point?

Picture the Problem In the absence of air resistance, the ball experiences constant acceleration. Choose a coordinate system with the origin at the point of release and the positive direction upward.

(*a*) Using a constant-acceleration equation, relate the displacement of the ball to the acceleration and the time:

$$\Delta y = v_0 t + \tfrac{1}{2}at^2$$

Setting $\Delta y = 0$ (the displacement for a round trip), solve for the time required for the ball to return to its starting position:

$$0 = v_0 t_{\text{roundtrip}} + \tfrac{1}{2}at^2_{\substack{\text{round}\\\text{trip}}}$$

and

$$t_{\substack{\text{round}\\\text{trip}}} = \frac{2v_0}{g}$$

Substitute numerical values and evaluate $t_{\substack{\text{round}\\\text{trip}}}$:

$$t_{\substack{\text{round}\\\text{trip}}} = \frac{2(20\,\text{m/s})}{9.81\,\text{m/s}^2} = \boxed{4.1\,\text{s}}$$

(*b*) Using a constant-acceleration equation, relate the final speed of the ball to its initial speed, the acceleration, and its displacement:

$$v_{\text{top}}^2 = v_0^2 + 2a\Delta y$$

or, because $v_{\text{top}} = 0$ and $a = -g$,

$$0 = v_0^2 + 2(-g)H \Rightarrow H = \frac{v_0^2}{2g}$$

Substitute numerical values and evaluate *H*:

$$H = \frac{(20\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)} = \boxed{20\,\text{m}}$$

(*c*) Using the same constant-acceleration equation with which we began part (*a*), express the displacement as a function of time:

$$\Delta y = v_0 t + \tfrac{1}{2}at^2$$

Substitute numerical values to obtain:

$$15\,\text{m} = (20\,\text{m/s})t - \left(\frac{9.81\,\text{m/s}^2}{2}\right)t^2$$

Use your graphing calculator or the quadratic formula to solve this equation for the times at which the displacement of the ball is 15 m:

The solutions are $t = \boxed{0.99\,\text{s}}$ (this corresponds to passing 15 m on the way up) and $t = \boxed{3.1\,\text{s}}$ (this corresponds to passing 15 m on the way down).

70 •• In the Blackhawk landslide in California, a mass of rock and mud fell 460 m down a mountain and then traveled 8.00 km across a level plain. It has been theorized that the rock and mud moved on a cushion of water vapor. Assume that the mass dropped with the free-fall acceleration and then slid horizontally, losing speed at a constant rate. (*a*) How long did the mud take to drop the 460 m? (*b*) How fast was it traveling when it reached the bottom? (*c*) How long did the mud take to slide the 8.00 km horizontally?

Picture the Problem This is a multipart constant-acceleration problem using two different constant accelerations. We'll choose a coordinate system in which downward is the positive direction and apply constant-acceleration equations to find the required times.

(*a*) Using a constant-acceleration equation, relate the time for the slide to the distance of fall and the acceleration:

$$\Delta y = y - y_0 = h - 0 = v_0 t_1 + \tfrac{1}{2} a t_1^2$$

or, because $v_0 = 0$,

$$h = \tfrac{1}{2} a t_1^2 \Rightarrow t_1 = \sqrt{\frac{2h}{g}}$$

Substitute numerical values and evaluate t_1:

$$t_1 = \sqrt{\frac{2(460\,\text{m})}{9.81\,\text{m/s}^2}} = 9.684\,\text{s} = \boxed{9.68\,\text{s}}$$

(*b*) Using a constant-acceleration equation, relate the velocity at the bottom of the mountain to the acceleration and time:

$$v_1 = v_0 + a_1 t_1$$

or, because $v_0 = 0$ and $a_1 = g$,

$$v_1 = g t_1$$

Substitute numerical values and evaluate v_1:

$$v_1 = (9.81\,\text{m/s}^2)(9.684\,\text{s}) = \boxed{95.0\,\text{m/s}}$$

(*c*) Using a constant-acceleration equation, relate the time required to stop the mass of rock and mud to its average speed and the distance it slides:

$$\Delta t = \frac{\Delta x}{v_{\text{av}}}$$

Because the acceleration is constant:

$$v_{\text{av}} = \frac{v_1 + v_{\text{f}}}{2} = \frac{v_1 + 0}{2} = \frac{v_1}{2}$$

Substitute for v_{av} to obtain:

$$\Delta t = \frac{2\Delta x}{v_1}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(8000\,\text{m})}{95.0\,\text{m/s}} = \boxed{168\,\text{s}}$$

71 •• [SSM] A load of bricks is lifted by a crane at a steady velocity of 5.0 m/s when one brick falls off 6.0 m above the ground. (*a*) Sketch the position of the brick $y(t)$ versus time from the moment it leaves the pallet until it hits the ground. (*b*) What is the greatest height the brick reaches above the ground? (*c*) How long does it take to reach the ground? (*d*) What is its speed just before it hits the ground?

Picture the Problem In the absence of air resistance, the brick experiences constant acceleration and we can use constant-acceleration equations to describe its motion. Constant acceleration implies a parabolic position-versus-time curve.

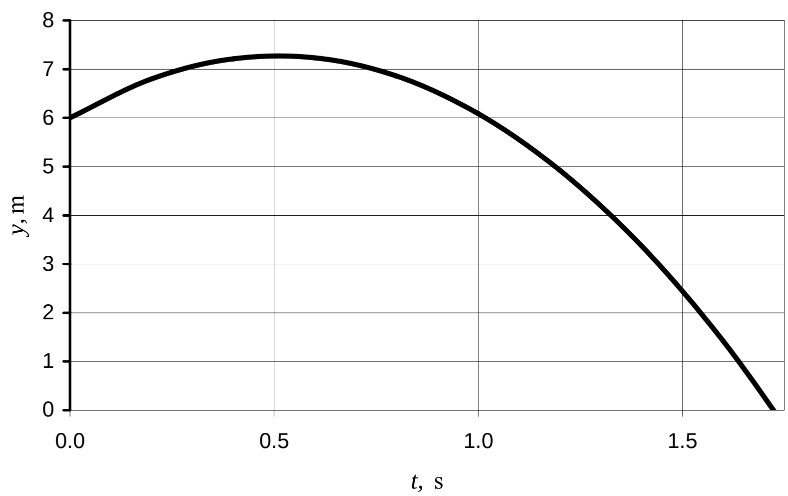
(*a*) Using a constant-acceleration equation, relate the position of the brick to its initial position, initial velocity, acceleration, and time into its fall:

$$y = y_0 + v_0 t + \tfrac{1}{2}(-g)t^2$$

Substitute numerical values to obtain:

$$y = 6.0\,\text{m} + (5.0\,\text{m/s})t - (4.91\,\text{m/s}^2)t^2 \qquad (1)$$

The following graph of $y = 6.0\,\text{m} + (5.0\,\text{m/s})t - (4.91\,\text{m/s}^2)t^2$ was plotted using a spreadsheet program:



(*b*) Relate the greatest height reached by the brick to its height when it falls off the load and the additional height it rises Δy_{max}:

$$h = y_0 + \Delta y_{max} \qquad (2)$$

Using a constant-acceleration equation, relate the height reached by the brick to its acceleration and initial velocity:

$$v_{top}^2 = v_0^2 + 2(-g)\Delta y_{max}$$

or, because $v_{top} = 0$,

$$0 = v_0^2 + 2(-g)\Delta y_{max} \Rightarrow \Delta y_{max} = \frac{v_0^2}{2g}$$

Substitute numerical values and evaluate Δy_{max}:

$$\Delta y_{max} = \frac{(5.0\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)} = 1.3\,\text{m}$$

Substitute numerical values in equation (2) and evaluate *h*:

$$h = 6.0\,\text{m} + 1.3\,\text{m} = \boxed{7.3\,\text{m}}$$

The graph shown above confirms this result.

(*c*) Setting $y = 0$ in equation (1) yields:

$$0 = 6.0\,\text{m} + (5.0\,\text{m/s})t - (4.91\,\text{m/s}^2)t^2$$

Use the quadratic equation or your graphing calculator to obtain:

$t = \boxed{1.7\,\text{s}}$ and $t = -0.71$ s. The negative value for *t* yields (from $v = v_0 - gt$) $v = -12$ m/s and so has no physical meaning.

(*d*) Using a constant-acceleration equation, relate the speed of the brick on impact to its acceleration and displacement:

$$v^2 = v_0^2 + 2gh$$

or, because $v_0 = 0$,

$$v^2 = 2gh \Rightarrow v = \sqrt{2gh}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{2(9.81\,\text{m/s}^2)(7.3\,\text{m})} = \boxed{12\,\text{m/s}}$$

72 •• A bolt comes loose from underneath an elevator that is moving upward at a speed of 6.0 m/s. The bolt reaches the bottom of the elevator shaft in 3.0 s. (*a*) How high above the bottom of the shaft was the elevator when the bolt came loose? (*b*) What is the speed of the bolt when it hits the bottom of the shaft?

Picture the Problem In the absence of air resistance, the acceleration of the bolt is constant. Choose a coordinate system in which upward is positive and the origin is at the bottom of the shaft ($y = 0$).

(*a*) Using a constant-acceleration equation, relate the position of the bolt to its initial position, initial velocity, and fall time:

$$y_{\text{bottom}} = 0 = y_0 + v_0 t + \tfrac{1}{2}(-g)t^2$$

Solve for the position of the bolt when it came loose:

$$y_0 = -v_0 t + \tfrac{1}{2}gt^2$$

Substitute numerical values and evaluate y_0:

$$y_0 = -\left(6.0\,\frac{\text{m}}{\text{s}}\right)(3.0\,\text{s}) + \tfrac{1}{2}\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(3.0\,\text{s})^2 = \boxed{26\,\text{m}}$$

(*b*) Using a constant-acceleration equation, relate the speed of the bolt to its initial speed, acceleration, and fall time:

$$v = v_0 + at$$

Substitute numerical values and evaluate $|v|$:

$$v = 6.0\,\frac{\text{m}}{\text{s}} - \left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(3.0\text{ s}) = -23\text{ m/s}$$

and

$$|v| = \boxed{23\text{ m/s}}$$

73 •• An object is dropped from rest at a height of 120 m. Find the distance it falls during its final second in the air.

Picture the Problem In the absence of air resistance, the object's acceleration is constant. Choose a coordinate system in which downward is the +*y* direction and the origin is at the point of release. In this coordinate system, $a = g$ and $y = 120$ m at the bottom of the fall.

Express the distance fallen in the last second in terms of the object's position at impact and its position 1 s before impact:

$$\Delta y_{\substack{\text{last}\\\text{second}}} = y - \tfrac{1}{2}gt^2 \qquad (1)$$

where *t* is the time the object has been falling.

Using a constant-acceleration equation, relate the object's position upon impact to its initial position, initial velocity, and fall time:

$$y = y_0 + v_0 t + \tfrac{1}{2}gt^2$$

or, because $y_0 = 0$ and $v_0 = 0$,

$$y = \tfrac{1}{2}gt_{\text{fall}}^2 \Rightarrow t_{\text{fall}} = \sqrt{\frac{2y}{g}}$$

Substitute numerical values and evaluate t_{fall}:

$$t_{\text{fall}} = \sqrt{\frac{2(120\text{ m})}{9.81\text{ m/s}^2}} = 4.95\text{ s}$$

Evaluating equation (1) for $t = 3.95$ s gives:

$$\Delta y_{\substack{\text{last}\\\text{second}}} = 120\text{ m} - \tfrac{1}{2}(9.81\text{ m/s}^2)(3.95\text{ s})^2 = \boxed{44\text{ m}}$$

74 •• An object is released from rest at a height *h*. During the final second of its fall, it traverses a distance of 38 m. Determine *h*.

Picture the Problem In the absence of air resistance, the acceleration of the object is constant. Choose a coordinate system with the origin at the point of release and downward as the positive direction.

Using a constant-acceleration equation, relate the height h of the object to its initial and final velocities and acceleration:

$$v_{\rm f}^2 = v_0^2 + 2a\Delta y$$

or, because $v_0 = 0$, $a = g$, and $\Delta y = h$,

$$v_{\rm f}^2 = 2gh \Rightarrow h = \frac{v_{\rm f}^2}{2g} \qquad (1)$$

Using the definition of average velocity, find the average velocity of the object during its final second of fall:

$$v_{\rm av} = \frac{v_{\rm f\text{-}1s} + v_{\rm f}}{2} = \frac{\Delta y}{\Delta t} = \frac{38\,{\rm m}}{1\,{\rm s}} = 38\,{\rm m/s}$$

Express the sum of the final velocity and the velocity 1 s before impact:

$$v_{\rm f\text{-}1s} + v_{\rm f} = 2(38\,{\rm m/s}) = 76\,{\rm m/s}$$

From the definition of acceleration, we know that the change in velocity of the object, during 1 s of fall, is 9.81 m/s:

$$\Delta v = v_{\rm f} - v_{\rm f\text{-}1s} = 9.81\ {\rm m/s}$$

Add the equations that express the sum and difference of $v_{\rm f\text{-}1s}$ and $v_{\rm f}$ to obtain:

$$v_{\rm f\text{-}1s} + v_{\rm f} + v_{\rm f} - v_{\rm f\text{-}1s} = 76\,{\rm m/s} + 9.81\ {\rm m/s}$$

Solving for $v_{\rm f}$ yields:

$$v_{\rm f} = \frac{76\,{\rm m/s} + 9.81\,{\rm m/s}}{2} = 43\,{\rm m/s}$$

Substitute numerical values in equation (1) and evaluate h:

$$h = \frac{(43\,{\rm m/s})^2}{2(9.81\,{\rm m/s^2})} = \boxed{94\,{\rm m}}$$

75 •• [SSM] A stone is thrown vertically downward from the top of a 200-m cliff. During the last half second of its flight, the stone travels a distance of 45 m. Find the initial speed of the stone.

Picture the Problem In the absence of air resistance, the acceleration of the stone is constant. Choose a coordinate system with the origin at the bottom of the trajectory and the upward direction positive. Let $v_{\rm f\text{-}1/2}$ be the speed one-half second before impact and $v_{\rm f}$ the speed at impact.

Using a constant-acceleration equation, express the final speed of the stone in terms of its initial speed, acceleration, and displacement:

$$v_{\rm f}^2 = v_0^2 + 2a\Delta y \Rightarrow v_0 = \sqrt{v_{\rm f}^2 + 2g\Delta y} \quad (1)$$

Find the average speed in the last half second:

$$v_{\text{av}} = \frac{v_{\text{f-1/2}} + v_{\text{f}}}{2} = \frac{\Delta x_{\text{last half second}}}{\Delta t} = \frac{45\,\text{m}}{0.5\,\text{s}}$$
$$= 90\,\text{m/s}$$

and

$$v_{\text{f-1/2}} + v_{\text{f}} = 2(90\,\text{m/s}) = 180\,\text{m/s}$$

Using a constant-acceleration equation, express the change in speed of the stone in the last half second in terms of the acceleration and the elapsed time and solve for the change in its speed:

$$\Delta v = v_{\text{f}} - v_{\text{f-1/2}} = g\Delta t$$
$$= (9.81\,\text{m/s}^2)(0.5\,\text{s})$$
$$= 4.91\,\text{m/s}$$

Add the equations that express the sum and difference of $v_{\text{f}-½}$ and v_{f} and solve for v_{f}:

$$v_{\text{f}} = \frac{180\,\text{m/s} + 4.91\,\text{m/s}}{2} = 92.5\,\text{m/s}$$

Substitute numerical values in equation (1) and evaluate v_0:

$$v_0 = \sqrt{(92.5\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(-200\text{m})}$$
$$= \boxed{68\,\text{m/s}}$$

Remarks: The stone may be thrown either up or down from the cliff and the results after it passes the cliff on the way down are the same.

76 •• An object is released from rest at a height *h*. It travels 0.4*h* during the first second of its descent. Determine the average velocity of the object during its entire descent.

Picture the Problem In the absence of air resistance, the acceleration of the object is constant. Choose a coordinate system in which downward is the positive direction and the object starts from rest. Apply constant-acceleration equations to find the average velocity of the object during its descent.

Express the average velocity of the falling object in terms of its initial and final velocities:

$$v_{\text{av}} = \frac{v_0 + v_{\text{f}}}{2}$$

Using a constant-acceleration equation, express the displacement of the object during the 1st second in terms of its acceleration and the elapsed time:

$$\Delta y_{\text{1st second}} = \frac{gt^2}{2} = 4.91\,\text{m} = 0.4\,h$$

Solving for the displacement *h* gives: $h = 12.3$ m

Using a constant-acceleration equation, express the final velocity of the object in terms of its initial velocity, acceleration, and displacement:

$$v_f^2 = v_0^2 + 2g\Delta y$$

or, because $v_0 = 0$,

$$v_f = \sqrt{2g\Delta y}$$

Substitute numerical values and evaluate the final velocity of the object:

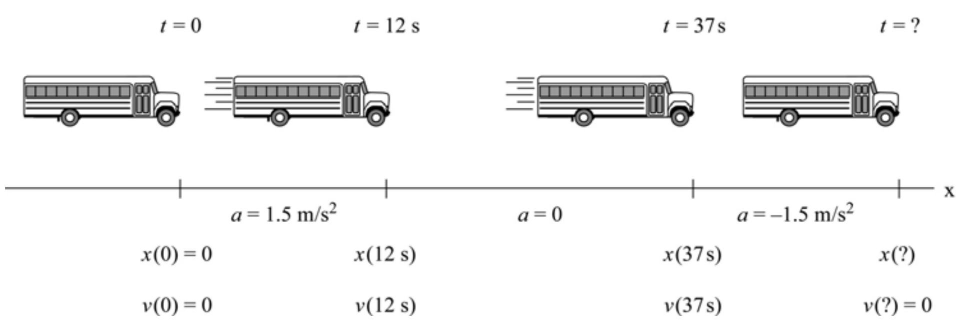
$$v_f = \sqrt{2\left(9.81\,\text{m/s}^2\right)\left(12.3\,\text{m}\right)} = 15.5\,\text{m/s}$$

Substitute in the equation for the average velocity to obtain:

$$v_{av} = \frac{0 + 15.5\,\text{m/s}}{2} = \boxed{7.77\,\text{m/s}}$$

77 •• A bus accelerates from rest at 1.5 m/s^2 for 12 s. It then travels at constant velocity for 25 s, after which it slows to a stop with an acceleration of 1.5 m/s^2. (*a*) What is the total distance that the bus travels? (*b*) What is its average velocity?

Picture the Problem This is a three-part constant-acceleration problem. The bus starts from rest and accelerates for a given period of time, and then it travels at a constant velocity for another period of time, and, finally, decelerates uniformly to a stop. The pictorial representation will help us organize the information in the problem and develop our solution strategy.



(*a*) Express the total displacement of the bus during the three intervals of time:

$$\Delta x_{total} = \Delta x(0 \to 12\,\text{s}) + \Delta x(12\,\text{s} \to 37\,\text{s}) + \Delta x(37\,\text{s} \to \text{end})$$

Using a constant-acceleration equation, express the displacement of the bus during its first 12 s of motion in terms of its initial velocity, acceleration, and the elapsed time; solve for its displacement:

$$\Delta x(0 \to 12\,\text{s}) = v_0 t + \tfrac{1}{2}at^2$$

or, because $v_0 = 0$,

$$\Delta x(0 \to 12\,\text{s}) = \tfrac{1}{2}at^2 = 108\,\text{m}$$

Using a constant-acceleration equation, express the velocity of the bus after 12 seconds in terms of its initial velocity, acceleration, and the elapsed time; solve for its velocity at the end of 12 s:

$$v_{12\,\text{s}} = v_0 + a_{0\to 12\,\text{s}}\Delta t = \left(1.5\,\text{m/s}^2\right)\left(12\,\text{s}\right) = 18\,\text{m/s}$$

During the next 25 s, the bus moves with a constant velocity. Using the definition of average velocity, express the displacement of the bus during this interval in terms of its average (constant) velocity and the elapsed time:

$$\Delta x\left(12\,\text{s} \to 37\,\text{s}\right) = v_{12\,\text{s}}\Delta t = \left(18\,\text{m/s}\right)\left(25\,\text{s}\right) = 450\,\text{m}$$

Because the bus slows down at the same rate that its velocity increased during the first 12 s of motion, we can conclude that its displacement during this braking period is the same as during its acceleration period and the time to brake to a stop is equal to the time that was required for the bus to accelerate to its cruising speed of 18 m/s. Hence:

$$\Delta x\left(37\,\text{s} \to 49\text{s}\right) = 108\,\text{m}$$

Add the displacements to find the distance the bus traveled:

$$\Delta x_{\text{total}} = 108\text{ m} + 450\text{ m} + 108\text{ m} = 666\text{ m} = \boxed{0.67\text{ km}}$$

(*b*) Use the definition of average velocity to calculate the average velocity of the bus during this trip:

$$v_{\text{av}} = \frac{\Delta x_{\text{total}}}{\Delta t} = \frac{666\,\text{m}}{49\,\text{s}} = \boxed{14\,\text{m/s}}$$

Remarks: One can also solve this problem graphically. Recall that the area under a velocity as a function-of-time graph equals the displacement of the moving object.

78 •• Al and Bert are jogging side-by-side on a trail in the woods at a speed of 0.75 m/s. Suddenly Al sees the end of the trail 35 m ahead and decides to speed up to reach it. He accelerates at a constant rate of 0.50 m/s^2, while Bert continues on at a constant speed. (*a*) How long does it take Al to reach the end of the trail? (*b*) Once he reaches the end of the trail, he immediately turns around and heads back along the trail with a constant speed of 0.85 m/s. How long does it take him

to meet up with Bert? (*c*) How far are they from the end of the trail when they meet?

Picture the Problem Because the accelerations of both Al and Bert are constant, constant-acceleration equations can be used to describe their motions. Choose the origin of the coordinate system to be where Al decides to begin his sprint.

(*a*) Using a constant-acceleration equation, relate Al's initial velocity, his acceleration, and the time to reach the end of the trail to his displacement in reaching the end of the trail:

$$\Delta x = v_0 t + \tfrac{1}{2}at^2$$

Substitute numerical values to obtain:

$$35\,\text{m} = (0.75\,\text{m/s})t + \tfrac{1}{2}(0.50\,\text{m/s}^2)t^2$$

Use your graphing calculator or the quadratic formula to solve for the time required for Al to reach the end of the trail:

$$t = 10.43\,\text{s} = \boxed{10\,\text{s}}$$

(*b*) Using constant-acceleration equations, express the positions of Bert and Al as functions of time. At the instant Al turns around at the end of the trail, $t = 0$. Also, $x = 0$ at a point 35 m from the end of the trail:

$$\boldsymbol{x}_{\text{Bert}} = \boldsymbol{x}_{\text{Bert,0}} + (0.75\ \text{m/s})\boldsymbol{t}$$

and

$$x_{\text{Al}} = x_{\text{Al,0}} - (0.85\,\text{m/s})t = 35\,\text{m} - (0.85\,\text{m/s})t$$

Calculate Bert's position at $t = 0$. At that time he has been running for 10.4 s:

$$\boldsymbol{x}_{\text{Bert,0}} = (0.75\,\text{m/s})(10.43\,\text{s}) = 7.823\,\text{m}$$

Because Bert and Al will be at the same location when they meet, equate their position functions and solve for t:

$$7.823\,\text{m} + (0.75\,\text{m/s})t = 35\,\text{m} - (0.85\,\text{m/s})t$$

and

$$t = 16.99\ \text{s}$$

To determine the elapsed time from when Al began his accelerated run, we need to add 10.43 s to this time:

$$t_{\text{start}} = 16.99\,\text{s} + 10.43\,\text{s} = 27.42\,\text{s} = \boxed{27\,\text{s}}$$

(*c*) Express Bert's distance from the end of the trail when he and Al meet:

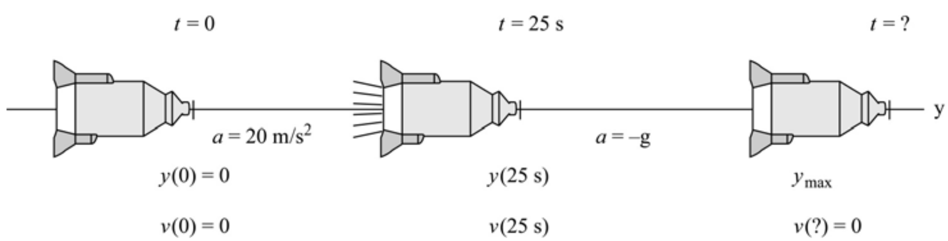
$$d_{\substack{\text{end of}\\ \text{trail}}} = 35\,\text{m} - x_{\text{Bert,0}} - d_{\substack{\text{Bert runs until}\\ \text{he meets Al}}}$$

Substitute numerical values and evaluate $d_{\substack{\text{end of}\\\text{trail}}}$:

$$d_{\substack{\text{end of}\\\text{trail}}} = 35\,\text{m} - 7.823\,\text{m} - (16.99\,\text{s})(0.75\,\text{m/s}) = 14.43\,\text{m} = \boxed{14\,\text{m}}$$

79 •• You have designed a rocket to be used to sample the local atmosphere for pollution. It is fired vertically with a constant upward acceleration of 20 m/s^2. After 25 s, the engine shuts off and the rocket continues rising (in freefall) for a while. (Neglect any effects due to air resistance.) The rocket eventually stops rising and then falls back to the ground. You want to get a sample of air that is 20 km above the ground. (*a*) Did you reach your height goal? If not, what would you change so that the rocket reaches 20 km? (*b*) Determine the total time the rocket is in the air. (*c*) Find the speed of the rocket just before it hits the ground.

Picture the Problem This is a two-part constant-acceleration problem. Choose a coordinate system in which the upward direction is positive and to the right. The pictorial representation will help us organize the information in the problem and develop our solution strategy.



(*a*) Express the highest point the rocket reaches, *h*, as the sum of its displacements during the first two stages of its flight:

$$h = \Delta y_{\text{1st stage}} + \Delta y_{\text{2nd stage}} \qquad (1)$$

Using a constant-acceleration equation, express the altitude reached in the first stage in terms of the rocket's initial velocity, acceleration, and burn time; solve for the first stage altitude:

$$\Delta y_{\text{1st stage}} = v_{0y}t + \tfrac{1}{2}a_{\text{1st stage}}t_{\text{1st stage}}^2$$

or, because $y_{0y} = 0$,

$$\Delta y_{\text{1st stage}} = \tfrac{1}{2}a_{\text{1st stage}}t_{\text{1st stage}}^2$$

Using a constant-acceleration equation, express the speed of the rocket at the end of its first stage in terms of its initial speed, acceleration, and displacement:

$$v_{\text{1st stage}} = v_{0y} + a_{\text{1st stage}}t_{\text{1st stage}}$$

or, because $y_{0y} = 0$,

$$v_{\text{1st stage}} = a_{\text{1st stage}}t_{\text{1st stage}}$$

Using a constant-acceleration equation, express the final speed of the rocket during the remainder of its climb in terms of its shut-off speed, free-fall acceleration, and displacement:

$$v_{\substack{\text{highest}\\\text{point}}}^2 = v_{\text{shutoff}}^2 + 2a_{\text{2nd stage}}\Delta y_{\text{2nd stage}}$$

or, because $v_{\substack{\text{highest}\\\text{point}}} = 0$ and $a_{\text{2nd stage}} = -g$,

$$0 = v_{\text{shutoff}}^2 - 2g\Delta y_{\text{2nd stage}}$$

Solving for $\Delta y_{\text{2nd stage}}$ yields:

$$\Delta y_{\text{2nd stage}} = \frac{v_{\text{shutoff}}^2}{2g}$$

Using a constant-acceleration equation, express the speed of the rocket at the end of its first stage in terms is its initial speed, acceleration, and displacement:

$$v_{\text{1st stage}} = v_0 + a_{\text{1st stage}}t_{\text{1st stage}}$$

or, because $v_0 = 0$ and $v_{\text{1st stage}} = v_{\text{shutoff}}$,

$$v_{\text{shutoff}} = a_{\text{1st stage}}t_{\text{1st stage}}$$

Substituting for v_{shutoff} yields:

$$\Delta y_{\text{2nd stage}} = \frac{(a_{\text{1st stage}}t_{\text{1st stage}})^2}{2g}$$

Substitute for $\Delta y_{\text{1st stage}}$ and $\Delta y_{\text{2nd stage}}$ in equation (1) and simplify to obtain:

$$h = \tfrac{1}{2}a_{\text{1st stage}}t_{\text{1st stage}}^2 + \frac{(a_{\text{1st stage}}t_{\text{1st stage}})^2}{2g}$$

$$= \left(\frac{1}{2} + \frac{a_{\text{1st stage}}}{2g}\right)a_{\text{1st stage}}t_{\text{1st stage}}^2$$

Substitute numerical values and evaluate *h*:

$$h = \left(\frac{1}{2} + \frac{20\,\dfrac{\text{m}}{\text{s}^2}}{2\left(9.81\,\dfrac{\text{m}}{\text{s}^2}\right)}\right)\left(20\,\frac{\text{m}}{\text{s}^2}\right)(25\text{ s})^2$$

$$\approx 19\text{ km}$$

You did not achieve your goal. To go higher, you can increase the acceleration value or the time of acceleration.

(*b*) Express the total time the rocket is in the air in terms of the three segments of its flight:

$$\Delta t_{\text{total}} = \Delta t_{\substack{\text{powered}\\\text{climb}}} + \Delta t_{\substack{\text{2nd}\\\text{segment}}} + \Delta t_{\text{descent}}$$

$$= 25\text{ s} + \Delta t_{\substack{\text{2nd}\\\text{segment}}} + \Delta t_{\text{descent}}$$

Express $\Delta t_{\text{2nd segment}}$ in terms of the rocket's displacement and average speed during the second segment:

$$\Delta t_{\text{2nd segment}} = \frac{\Delta y_{\substack{\text{2nd}\\\text{segment}}}}{v_{\text{av, 2nd segment}}}$$

Substituting for $\Delta y_{\text{2nd segment}}$ and simplifying yields:

$$\Delta t_{\text{2nd segment}} = \frac{\dfrac{\left(a_{\text{1st stage}} t_{\text{1st stage}}\right)^2}{2g}}{v_{\text{av, 2nd segment}}} = \frac{\left(a_{\text{1st stage}} t_{\text{1st stage}}\right)^2}{2g v_{\text{av, 2nd segment}}}$$

Using a constant-acceleration equation, relate the fall distance to the descent time:

$$\Delta y_{\text{descent}} = v_0 t + \tfrac{1}{2} g \left(\Delta t_{\text{descent}}\right)^2$$

or, because $v_0 = 0$,

$$\Delta y_{\text{descent}} = \tfrac{1}{2} g \left(\Delta t_{\text{descent}}\right)^2$$

Solving for $\Delta t_{\text{descent}}$ yields:

$$\Delta t_{\text{descent}} = \sqrt{\frac{2\Delta y_{\text{descent}}}{g}}$$

Substitute for $\Delta t_{\text{2nd segment}}$ and $\Delta t_{\text{descent}}$ in the expression for Δt_{total} to obtain:

$$\Delta t_{\text{total}} = 25\,\text{s} + \frac{\left(a_{\text{1st stage}} \Delta t_{\text{1st stage}}\right)^2}{2g v_{\text{av, 2nd segment}}} + \sqrt{\frac{2\Delta y_{\text{descent}}}{g}}$$

Noting that, because the acceleration is constant, $v_{\text{av, 2nd segment}}$ is the average of the initial and final speeds during the second stage, substitute numerical values and evaluate Δt_{total}:

$$\Delta t_{\text{total}} = 25\,\text{s} + \frac{\left(\left(20\,\dfrac{\text{m}}{\text{s}^2}\right)(25\,\text{s})\right)^2}{2\left(9.81\,\dfrac{\text{m}}{\text{s}^2}\right)\left(\dfrac{0 + 500\,\dfrac{\text{m}}{\text{s}}}{2}\right)} + \sqrt{\frac{2\left(19\times10^3\ \text{m}\right)}{9.81\,\dfrac{\text{m}}{\text{s}^2}}} \approx \boxed{1.4\times10^2\ \text{s}}$$

(*c*) Using a constant-acceleration equation, express the impact velocity of the rocket in terms of its initial downward velocity, acceleration under free-fall, and time of descent; solve for its impact velocity:

$$v_{\text{impact}} = v_0 + g\Delta t_{\text{descent}}$$

and, because $v_0 = 0$,

$$v_{\text{impact}} = g\Delta t_{\text{descent}}$$

Substituting for $\Delta t_{\text{descent}}$ yields:

$$v_{\text{impact}} = g\sqrt{\frac{2\Delta y_{\text{descent}}}{g}} = \sqrt{2g\Delta y_{\text{descent}}}$$

Substitute numerical values and evaluate v_{impact}:

$$v_{\text{impact}} = \sqrt{2\left(9.81\,\text{m/s}^2\right)\left(19\,\text{km}\right)} = \boxed{6.1\times10^2\,\text{m/s}}$$

80 •• A flowerpot falls from a windowsill of an apartment that is on the tenth floor of an apartment building. A person in an apartment below, coincidentally in possession of a high-speed high-precision timing system, notices that it takes 0.20 s for the pot to fall past his window, which is 4.0-m from top to bottom. How far above the top of the window is the windowsill from which the pot fell? (Air resistance is negligible.)

Picture the Problem In the absence of air resistance, the acceleration of the flowerpot is constant. Choose a coordinate system in which downward is positive and the origin is at the point from which the flowerpot fell. Let t = time when the pot is at the top of the window, and $t + \Delta t$ the time when the pot is at the bottom of the window. To find the distance from the ledge to the top of the window, first find the time t_{top} that it takes the pot to fall to the top of the window.

Using a constant-acceleration equation, express the distance y below the ledge from which the pot fell as a function of time:

$$y = y_0 + v_0 t + \tfrac{1}{2}at^2$$

or, because $a = g$ and $v_0 = y_0 = 0$,

$$y = \tfrac{1}{2}gt^2$$

Express the position of the pot as it reaches the top of the window:

$$y_{\text{top}} = \tfrac{1}{2}gt_{\text{top}}^2 \qquad (1)$$

Express the position of the pot as it reaches the bottom of the window:

$$y_{\text{bottom}} = \tfrac{1}{2}g\left(t_{\text{top}} + \Delta t_{\text{window}}\right)^2$$

where $\Delta t_{\text{window}} = t_{\text{top}} - t_{\text{bottom}}$

Subtract y_{bottom} from y_{top} to obtain an expression for the displacement Δy_{window} of the pot as it passes the window:

$$\Delta y_{\text{window}} = \tfrac{1}{2}g\left[\left(t_{\text{top}} + \Delta t_{\text{window}}\right)^2 - t_{\text{top}}^2\right] = \tfrac{1}{2}g\left[2t_{\text{top}}\Delta t_{\text{window}} + \left(\Delta t_{\text{window}}\right)^2\right]$$

Solving for t_{top} yields:

$$t_{\text{top}} = \frac{\dfrac{2\Delta y_{\text{window}}}{g} - \left(\Delta t_{\text{window}}\right)^2}{2\Delta t_{\text{window}}}$$

Substitute for t_{top} in equation (1) to obtain:

$$y_{\text{top}} = \tfrac{1}{2}g\left(\frac{\dfrac{2\Delta y_{\text{window}}}{\text{g}} - \left(\Delta t_{\text{window}}\right)^2}{2\Delta t_{\text{window}}}\right)^2$$

Substitute numerical values and evaluate y_{top}:

$$y_{\text{top}} = \tfrac{1}{2}\left(9.81\ \text{m/s}^2\right)\left(\frac{\dfrac{2(4.0\ \text{m})}{9.81\ \text{m/s}^2} - (0.20\ \text{s})^2}{2(0.20\ \text{s})}\right)^2 = \boxed{18\ \text{m}}$$

81 •• **[SSM]** In a classroom demonstration, a glider moves along an inclined air track with constant acceleration. It is projected from the low end of the track with an initial velocity. After 8.00 s have elapsed, it is 100 cm from the low end and is moving along the track at a velocity of –15 cm/s. Find the initial velocity and the acceleration.

Picture the Problem The acceleration of the glider on the air track is constant. Its average acceleration is equal to the instantaneous (constant) acceleration. Choose a coordinate system in which the initial direction of the glider's motion is the positive direction.

Using the definition of acceleration, express the average acceleration of the glider in terms of the glider's velocity change and the elapsed time:

$$a = a_{\text{av}} = \frac{\Delta v}{\Delta t}$$

Using a constant-acceleration equation, express the average velocity of the glider in terms of the displacement of the glider and the elapsed time:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{v_0 + v}{2} \Rightarrow v_0 = \frac{2\Delta x}{\Delta t} - v$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{2(100\ \text{cm})}{8.00\ \text{s}} - (-15\ \text{cm/s}) = \boxed{40\ \text{cm/s}}$$

The average acceleration of the glider is:

$$a = \frac{-15\ \text{cm/s} - (40\ \text{cm/s})}{8.00\ \text{s}} = \boxed{-6.9\ \text{cm/s}^2}$$

82 •• A rock dropped from a cliff covers one-third of its total distance to the ground in the last second of its fall. Air resistance is negligible. How high is the cliff?

Picture the Problem In the absence of air resistance, the acceleration of the rock is constant and its motion can be described using the constant-acceleration equations. Choose a coordinate system in which the downward direction is positive and let the height of the cliff, which equals the displacement of the rock, be represented by *h*.

Using a constant-acceleration equation, express the height *h* of the cliff in terms of the initial velocity of the rock, acceleration, and time of fall:

$$\Delta y = v_0 t + \tfrac{1}{2}at^2$$

or, because $v_0 = 0$, $a = g$, and $\Delta y = h$,

$$h = \tfrac{1}{2}gt^2$$

Using this equation, express the displacement of the rock during the first two-thirds of its fall:

$$\tfrac{2}{3}h = \tfrac{1}{2}gt^2 \qquad (1)$$

Using the same equation, express the displacement of the rock during its complete fall in terms of the time required for it to fall this distance:

$$h = \tfrac{1}{2}g(t + 1\,\text{s})^2 \qquad (2)$$

Substitute equation (2) in equation (1) to obtain:

$$t^2 - (4\text{ s})t - 2\text{ s}^2 = 0$$

Use the quadratic formula or your graphing calculator to solve for the positive root:

$$t = 4.45\,\text{s}$$

Evaluate $\Delta t = t + 1$ s:

$$\Delta t = 4.45\text{ s} + 1\text{ s} = 5.45\text{ s}$$

Substitute numerical values in equation (2) and evaluate *h*:

$$h = \tfrac{1}{2}\left(9.81\,\text{m/s}^2\right)(5.45\,\text{s})^2 = \boxed{146\,\text{m}}$$

83 •• [SSM] A typical automobile under hard braking loses speed at a rate of about 7.0 m/s^2; the typical reaction time to engage the brakes is 0.50 s. A local school board sets the speed limit in a school zone such that all cars should be able to stop in 4.0 m. (*a*) What maximum speed does this imply for an automobile in this zone? (*b*) What fraction of the 4.0 m is due to the reaction time?

Picture the Problem Assume that the acceleration of the car is constant. The total distance the car travels while stopping is the sum of the distances it travels during the driver's reaction time and the time it travels while braking. Choose a coordinate system in which the positive direction is the direction of motion of the automobile and apply a constant-acceleration equation to obtain a quadratic equation in the car's initial speed v_0.

(*a*) Using a constant-acceleration equation, relate the speed of the car to its initial speed, acceleration, and displacement during braking:

$$v^2 = v_0^2 + 2a\Delta x_{\text{brk}}$$

or, because the final speed of the car is zero,

$$0 = v_0^2 + 2a\Delta x_{\text{brk}} \Rightarrow \Delta x_{\text{brk}} = -\frac{v_0^2}{2a}$$

Express the total distance traveled by the car as the sum of the distance traveled during the reaction time and the distance traveled while slowing down:

$$\Delta x_{\text{tot}} = \Delta x_{\text{react}} + \Delta x_{\text{brk}} = v_0\Delta t_{\text{react}} - \frac{v_0^2}{2a}$$

Rearrange this quadratic equation to obtain:

$$v_0^2 - 2a\Delta t_{\text{react}}v_0 + 2a\Delta x_{\text{tot}} = 0$$

Substitute numerical values and simplify to obtain:

$$v_0^2 + (7.0\,\text{m/s})v_0 - 56\,\text{m}^2/\text{s}^2 = 0$$

Use your graphing calculator or the quadratic formula to solve the quadratic equation for its positive root:

$$v_0 = 4.76\,\text{m/s}$$

Convert this speed to mi/h:

$$v_0 = (4.76\,\text{m/s})\left(\frac{1\,\text{mi/h}}{0.4470\,\text{m/s}}\right) = \boxed{11\,\text{mi/h}}$$

(*b*) Find the reaction-time distance:

$$\Delta x_{\text{react}} = v_0\Delta t_{\text{react}} = (4.76\,\text{m/s})(0.50\,\text{s}) = 2.38\,\text{m}$$

Express and evaluate the ratio of the reaction distance to the total distance:

$$\frac{\Delta x_{\text{react}}}{\Delta x_{\text{tot}}} = \frac{2.38\,\text{m}}{4.0\,\text{m}} = \boxed{0.60}$$

84 •• Two trains face each other on adjacent tracks. They are initially at rest, and their front ends are 40 m apart. The train on the left accelerates rightward at 1.0 m/s^2. The train on the right accelerates leftward at 1.3 m/s^2. (*a*) How far does the train on the left travel before the front ends of the trains pass? (*b*) If the trains are each 150 m in length, how long after the start are they completely past one another, assuming their accelerations are constant?

Picture the Problem Assume that the accelerations of the trains are constant. Choose a coordinate system in which the direction of the motion of the train on the left is the positive direction. Take $x_0 = 0$ as the position of the train on the left at $t = 0$ and note that the acceleration of the train on the right is negative.

(*a*) Using a constant-acceleration equation, express the position of the front of the train on the left as a function of time:

$$x_L = \tfrac{1}{2}a_L t^2 \quad (1)$$

Using a constant-acceleration equation, express the position of the front of the train on the right as a function of time:

$$x_R = 40\,\text{m} + \tfrac{1}{2}a_R t^2$$

Equate x_L and x_R to obtain:

$$\tfrac{1}{2}a_L t^2 = 40\,\text{m} + \tfrac{1}{2}a_R t^2 \Rightarrow t = \sqrt{\frac{80\,\text{m}}{a_L - a_R}}$$

Substituting for t in equation (1) gives:

$$x_L = \tfrac{1}{2}a_L\left(\frac{80\,\text{m}}{a_L - a_R}\right) = \tfrac{1}{2}\left(\frac{80\,\text{m}}{1 - \dfrac{a_R}{a_L}}\right)$$

The acceleration of the train on the left is 1.0 m/s^2 and the acceleration of the train on the right is −1.3 m/s^2. Substitute numerical values and evaluate x_L:

$$x_L = \tfrac{1}{2}\left(\frac{80\,\text{m}}{1 - \dfrac{-1.3\,\text{m/s}^2}{1.0\,\text{m/s}^2}}\right) = \boxed{17\,\text{m}}$$

(*b*) Let the rear of the left train be at the origin at $t = 0$. Then the initial location of the rear end of the train on the right is at $x = 340$ m (150 m + 40 m + 150 m).

Using a constant-acceleration equation, express the position of the rear of the train on the left as a function of time:

$$x_L = \tfrac{1}{2}a_L t^2$$

Using a constant-acceleration equation, express the position of the rear of the train on the right as a function of time:

$$x_R = 340\,\text{m} + \tfrac{1}{2}a_R t^2$$

The trains will be completely past each other when:

$$x_L = x_R \Rightarrow \tfrac{1}{2}a_L t^2 = 340\,\text{m} + \tfrac{1}{2}a_R\, t^2$$

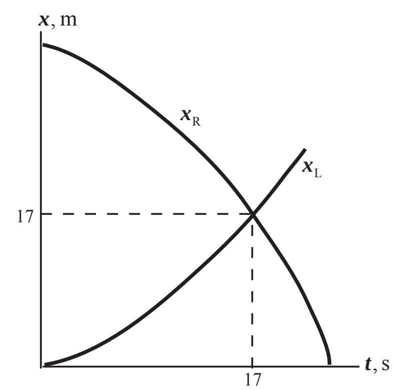
Solving for t yields:

$$t = \sqrt{\frac{680\text{ m}}{a_L - a_R}}$$

Substitute numerical values and evaluate t:

$$t = \sqrt{\frac{680\text{ m}}{1.0\text{ m/s}^2 - \left(-1.3\text{ m/s}^2\right)}} = \boxed{17\text{ s}}$$

Remarks: One can also solve this problem by graphing x_L and x_R as functions of *t*. The coordinates of the intersection of the two curves give one the time-to-passing and the distance traveled by the train on the left for Part (*a*).



85 •• Two stones are dropped from the edge of a 60-m cliff, the second stone 1.6 s after the first. How far below the top of the cliff is the second stone when the separation between the two stones is 36 m?

Picture the Problem In the absence of air resistance, the acceleration of the stones is constant. Choose a coordinate system in which the downward direction is positive and the origin is at the point of release of the stones.

Using constant-acceleration equations, relate the positions of the two stones to their initial positions, accelerations, and time-of-fall:

$$x_1 = \tfrac{1}{2}gt^2$$

and

$$x_2 = \tfrac{1}{2}g(t - 1.6\text{ s})^2$$

Express the difference between x_1 and x_2:

$$x_1 - x_2 = 36\text{ m}$$

Substitute for x_1 and x_2 to obtain:

$$36\text{ m} = \tfrac{1}{2}gt^2 - \tfrac{1}{2}g\left(t - 1.6\text{ s}\right)^2$$

Solve this equation for the time t at which the stones will be separated by 36 m:

$$t = 3.09\text{ s}$$

Substitute this result in the expression for x_2 and solve for x_2:

$$x_2 = \tfrac{1}{2}\left(9.81\text{ m/s}^2\right)\left(3.09\text{ s} - 1.6\text{ s}\right)^2 = \boxed{11\text{ m}}$$

86 •• A motorcycle officer hidden at an intersection observes a car driven by an oblivious driver who ignores a stop sign and continues through the intersection at constant speed. The police officer takes off in pursuit 2.0 s after the car has passed the stop sign. She accelerates at 4.2 m/s^2 until her speed is 110 km/h, and then continues at this speed until she catches the car. At that instant, the car is 1.4 km from the intersection. (*a*) How long did it take for the officer to catch up to the car? (*b*) How fast was the car traveling?

Picture the Problem The acceleration of the police officer's car is positive and constant and the acceleration of the speeder's car is zero. Choose a coordinate system such that the direction of motion of the two vehicles is the positive direction and the origin is at the stop sign.

(*a*) The time traveled by the car is given by:

$$t_{car} = 2.0\text{ s} + t_1 + t_2 \qquad (1)$$

where t_1 is the time during which the motorcycle was accelerating and t_2 is the time during which the motorcycle moved with constant speed.

Convert 110 km/h into m/s:

$$v_1 = 110\,\frac{\text{km}}{\text{h}} \times \frac{10^3\text{ m}}{\text{km}} \times \frac{1\text{ h}}{3600\text{ s}} = 30.56\,\frac{\text{m}}{\text{s}}$$

Express and evaluate t_1:

$$t_1 = \frac{v_1}{a_{motorcycle}} = \frac{30.56\text{ m/s}}{4.2\text{ m/s}^2} = 7.276\text{ s}$$

Express and evaluate d_1:

$$d_1 = \tfrac{1}{2}v_1 t_1 = \tfrac{1}{2}(30.56\text{ m/s})(7.276\text{ s}) = 111.2\text{ m}$$

Determine d_2:

$$d_2 = d_{caught} - d_1 = 1400\text{ m} - 111\text{ m} = 1289\text{ m}$$

Express and evaluate t_2:

$$t_2 = \frac{d_2}{v_1} = \frac{1289\text{ m}}{30.56\text{ m/s}} = 42.18\text{ s}$$

Substitute in equation (1) and evaluate t_{car}:

$$t_{car} = 2.0\text{ s} + 7.3\text{ s} + 42.2\text{ s} = 51.5\text{ s} = \boxed{52\text{ s}}$$

(*b*) The speed of the car when it was overtaken is the ratio of the distance it traveled to the elapsed time:

$$v_{car} = \frac{d_{caught}}{t_{car}}$$

Substitute numerical values and evaluate v_{car}:

$$v_{car} = \left(\frac{1400\text{ m}}{51.5\text{ s}}\right)\left(\frac{1\text{ mi/h}}{0.447\text{ m/s}}\right) = \boxed{61\text{ mi/h}}$$

87 •• At $t = 0$, a stone is dropped from the top of a cliff above a lake. Another stone is thrown downward 1.6 s later from the same point with an initial speed of 32 m/s. Both stones hit the water at the same instant. Find the height of the cliff.

Picture the Problem In the absence of air resistance, the acceleration of the stone is constant. Choose a coordinate system in which downward is positive and the origin is at the point of release of the stone and apply constant-acceleration equations.

Using a constant-acceleration equation, express the height of the cliff in terms of the initial position of the stones, acceleration due to gravity, and time for the first stone to hit the water:

$$h = \tfrac{1}{2}gt_1^2$$

Express the displacement of the second stone when it hits the water in terms of its initial velocity, acceleration, and time required for it to hit the water.

$$d_2 = v_{02}t_2 + \tfrac{1}{2}gt_2^2$$

where $t_2 = t_1 - 1.6$ s.

Because the stones will travel the same distances before hitting the water, equate h and d_2 to obtain:

$$\tfrac{1}{2}gt_1^2 = v_{02}t_2 + \tfrac{1}{2}gt_2^2$$

or, because $t_2 = t_1 - 1.6$ s,

$$\tfrac{1}{2}gt_1^2 = v_{02}t_2 + \tfrac{1}{2}g(t_1 - 1.6\text{ s})^2$$

Substitute numerical values to obtain:

$$\tfrac{1}{2}(9.81\text{ m/s}^2)t_1^2 = (32\text{ m/s})(t_1 - 1.6\text{ s}) + \tfrac{1}{2}(9.81\text{ m/s}^2)(t_1 - 1.6\text{ s})^2$$

Solving this quadratic equation for t_1 yields:

$$t_1 = 2.370\text{ s}$$

Substitute for t_1 and evaluate h:

$$h = \tfrac{1}{2}(9.81\text{ m/s}^2)(2.370\text{ s})^2 = \boxed{28\text{ m}}$$

88 •• A passenger train is traveling at 29 m/s when the engineer sees a freight train 360 m ahead of his train traveling in the same direction on the same track. The freight train is moving at a speed of 6.0 m/s. (*a*) If the reaction time of the engineer is 0.40 s, what is the minimum (constant) rate at which the passenger train must lose speed if a collision is to be avoided? (*b*) If the engineer's reaction

time is 0.80 s and the train loses speed at the minimum rate described in Part (*a*), at what rate is the passenger train approaching the freight train when the two collide? (*c*) How far will the passenger train have traveled in the time between the sighting of the freight train and the collision?

Picture the Problem Assume that the acceleration of the passenger train is constant. Let $x_p = 0$ be the location of the passenger train engine at the moment of sighting the freight train's end; let $t = 0$ be the instant the passenger train begins to slow (0.40 s after the passenger train engineer sees the freight train ahead). Choose a coordinate system in which the direction of motion of the trains is the positive direction and use constant-acceleration equations to express the positions of the trains in terms of their initial positions, speeds, accelerations, and elapsed time.

(*a*) Using constant-acceleration equations, write expressions for the positions of the front of the passenger train and the rear of the freight train, x_p and x_f, respectively:

$$x_p = (29\,\text{m/s})(t + 0.40\,\text{s}) - \tfrac{1}{2}at^2 \quad (1)$$

and

$$\boldsymbol{x}_f = (360\,\text{m}) + (6.0\,\text{m/s})(\boldsymbol{t} + 0.40\,\text{s})$$

where x_p and x_f are in meters if t is in seconds.

Equate x_f and x_p to obtain an equation for t:

$$\tfrac{1}{2}at^2 - (23\,\text{m/s})t + 350.8\,\text{m} = 0$$

Find the discriminant $(D = \text{B}^2 - 4\text{AC})$ of this equation:

$$D = (23\,\text{m/s})^2 - 4\left(\frac{a}{2}\right)(350.8\,\text{m})$$

The equation must have real roots if it is to describe a collision. The necessary condition for real roots is that the discriminant be greater than or equal to zero:

If $(23\text{ m/s})^2 - a\,(701.6\text{ m}) \geq 0$, then $a \leq \boxed{0.75\text{ m/s}^2}$

(*b*) Differentiate equation (1) to express the speed of the passenger train as a function of time:

$$v_p(t) = \frac{dx_p(t)}{dt} = 29\,\text{m/s} - at \qquad (2)$$

Repeat the previous steps with $a = 0.75\text{ m/s}^2$ and a 0.80 s reaction time. The quadratic equation that guarantees real roots with the longer reaction time is:

$$(0.375\,\text{m/s}^2)t^2 - (23\,\text{m/s})t + 341.6\text{m} = 0$$

Use the quadratic formula or your graphing calculator to find the collision times:

$t = 25.23$ s and $t = 36.10$ s

Because, when $t = 36.10$ s, the trains have already collided, this root is not a meaningful solution to our problem.

Note: In the graph shown below, you will see why we keep only the smaller of the two solutions.

Evaluate equation (2) for $t = 25.23$ s and $a = 0.75$ m/s^2:

$$v_p(t) = 29\ \text{m/s} - \left(0.75\ \text{m/s}^2\right)(25.23\ \text{s})$$
$$= \boxed{10\ \text{m/s}}$$

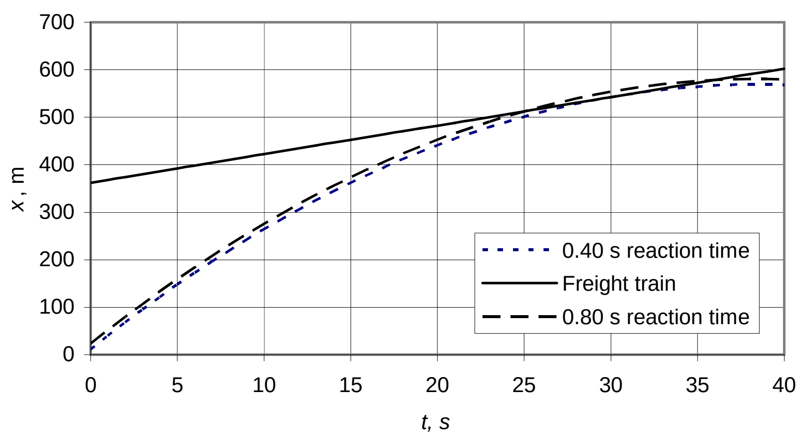
(*c*) The position of the passenger train as a function of time is given by:

$$x_p(t) = (29\ \text{m/s})(t + 0.80\ \text{s}) - \tfrac{1}{2}\left(0.75\ \text{m/s}^2\right)t^2$$

Evaluate $\boldsymbol{x}_p(25.23\ \text{s})$ to obtain:

$$\boldsymbol{x}_p(26\ \text{s}) = (29\ \text{m/s})(25.23\ \text{s} + 0.80\ \text{s}) - \tfrac{1}{2}\left(0.75\ \text{m/s}^2\right)(25.23\ \text{s})^2 = \boxed{0.52\ \text{km}}$$

The following graph shows the positions of the trains as a function of time. The solid straight line is for the constant velocity freight train; the dashed curves are for the passenger train, with reaction times of 0.40 s for the lower curve and 0.80 s for the upper curve.



Remarks: A collision occurs the first time the curve for the passenger train crosses the curve for the freight train. The smaller of the two solutions will always give the time of the collision.

89 •• The click beetle can project itself vertically with an acceleration of about 400*g* (an order of magnitude more than a human could survive!). The beetle jumps by "unfolding" its 0.60-cm long legs. (*a*) How high can the click beetle jump? (*b*) How long is the beetle in the air? (Assume constant acceleration while in contact with the ground and neglect air resistance.)

Picture the Problem Choose a coordinate system in which the upward direction is positive. We can use a constant-acceleration equation to find the beetle's speed as its feet lose contact with the ground and then use this speed to calculate the height of its jump.

(*a*) Using a constant-acceleration equation, relate the beetle's maximum height to its launch speed, speed at the top of its trajectory, and acceleration once it is airborne; solve for its maximum height:

$$v_{\substack{\text{highest}\\\text{point}}}^2 = v_{\text{launch}}^2 + 2a\Delta y_{\text{free fall}} = v_{\text{launch}}^2 + 2(-g)h$$

or, because $v_{\substack{\text{highest}\\\text{point}}} = 0$,

$$0 = v_{\text{launch}}^2 + 2(-g)h \Rightarrow h = \frac{v_{\text{launch}}^2}{2g} \quad (1)$$

Now, in order to determine the beetle's launch speedy, relate its time of contact with the ground to its acceleration and push-off distance:

$$v_{\text{launch}}^2 = v_0^2 + 2a\Delta y_{\text{launch}}$$

or, because $v_0 = 0$,

$$v_{\text{launch}}^2 = 2a\Delta y_{\text{launch}}$$

Substituting for v_{launch}^2 in equation (1) yields:

$$h = \frac{2a\Delta y_{\text{launch}}}{2g}$$

Substitute numerical values in equation (1) to find the height to which the beetle can jump:

$$h = \frac{2(400)(9.81\,\text{m/s}^2)(0.60\times10^{-2}\,\text{m})}{2(9.81\,\text{m/s}^2)} = \boxed{2.4\,\text{m}}$$

(*b*) Using a constant-acceleration equation, relate the speed of the beetle at its maximum height to its launch speed, free-fall acceleration while in the air, and time-to-maximum height:

$$v = v_0 + at$$

or

$$v_{\substack{\text{max.}\\\text{height}}} = v_{\text{launch}} - gt_{\substack{\text{max}\\\text{height}}}$$

and, because $v_{\substack{\text{max}\\\text{height}}} = 0$,

$$0 = v_{\text{launch}} - gt_{\substack{\text{max}\\\text{height}}} \Rightarrow t_{\substack{\text{max}\\\text{height}}} = \frac{v_{\text{launch}}}{g}$$

For zero displacement and constant acceleration, the time-of-flight is twice the time-to-maximum height:

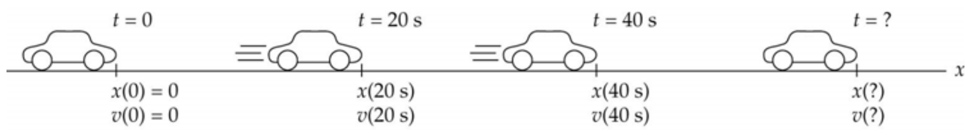
$$t_{\text{flight}} = 2t_{\substack{\text{max}\\\text{height}}} = \frac{2v_{\text{launch}}}{g}$$

Substitute numerical values and evaluate t_{flight}:

$$t_{\text{flight}} = \frac{2(6.9\,\text{m/s})}{9.81\,\text{m/s}^2} = \boxed{1.4\,\text{s}}$$

90 •• An automobile accelerates from rest at 2.0 m/s^2 for 20 s. The speed is then held constant for 20 s, after which there is an acceleration of –3.0 m/s^2 until the automobile stops. What is the total distance traveled?

Picture the Problem This is a multipart constant-acceleration problem using three different constant accelerations (+2.0 m/s^2 for 20 s, then zero for 20 s, and then –3.0 m/s^2 until the automobile stops). The final speed is zero. The pictorial representation will help us organize the information in the problem and develop a solution strategy.



Add up all the displacements to get the total:

$$\Delta x_{03} = \Delta x_{01} + \Delta x_{12} + \Delta x_{23} \qquad (1)$$

Using constant-acceleration formulas, find the first displacement:

$$\Delta x_{01} = v_0 t_1 + \tfrac{1}{2} a_{01} t_1^2$$

or, because $v_0 = 0$,

$$\Delta x_{01} = \tfrac{1}{2} a_{01} t_1^2$$

The speed is constant for the second displacement. The second displacement is given by:

$$\Delta x_{12} = v_1 (t_2 - t_1)$$

or, because $v_1 = v_0 + a_{01} t_1 = 0 + a_{01} t_1$,

$$\Delta x_{12} = a_{01} t_1 (t_2 - t_1)$$

The displacement during the braking interval is related to v_3, v_2, and a_{23}:

$$v_3^2 = v_2^2 + 2a_{23} \Delta x_{23}$$

and, because $v_2 = v_1 = a_{01} t_1$ and $v_3 = 0$,

$$\Delta x_{23} = \frac{-(a_{01} t_1)^2}{2a_{23}}$$

Substituting for $\Delta x_{01}, \Delta x_{12}$, and Δx_{23} in equation (1) yields:

$$\Delta x_{03} = \tfrac{1}{2} a_{01} t_1^2 + a_{01} t_1 (t_2 - t_1) - \frac{(a_{01} t_1)^2}{2a_{23}}$$

Substitute numerical values and evaluate Δx_{03}:

$$\Delta x_{03} = \tfrac{1}{2}(2.0\ \text{m/s})(20\ \text{s})^2 + (2.0\ \text{m/s})(20\ \text{s})(40\ \text{s} - 20\ \text{s}) - \frac{[(2.0\ \text{m/s})(20\ \text{s})]^2}{2(-3.0\ \text{m/s}^2)}$$

$$= \boxed{1.5\ \text{km}}$$

Remarks: Because the area under the curve of a velocity-versus-time graph equals the displacement of the object experiencing the acceleration, we could

solve this problem by plotting the velocity as a function of time and finding the area bounded by it and the time axis.

91 •• [SSM] Consider measuring the free-fall motion of a particle (neglect air resistance). Before the advent of computer-driven data-logging software, these experiments typically employed a wax-coated tape placed vertically next to the path of a dropped electrically conductive object. A spark generator would cause an arc to jump between two vertical wires through the falling object and through the tape, thereby marking the tape at fixed time intervals Δt. Show that the change in height during successive time intervals for an object falling from rest follows *Galileo's Rule of Odd Numbers*: $\Delta y_{21} = 3\Delta y_{10}$, $\Delta y_{32} = 5\Delta y_{10}$, . . . , where Δy_{10} is the change in y during the first interval of duration Δt, Δy_{21} is the change in y during the second interval of duration Δt, etc.

Picture the Problem In the absence of air resistance, the particle experiences constant acceleration and we can use constant-acceleration equations to describe its position as a function of time. Choose a coordinate system in which downward is positive, the particle starts from rest ($v_o = 0$), and the starting height is zero ($y_0 = 0$).

Using a constant-acceleration equation, relate the position of the falling particle to the acceleration and the time. Evaluate the y-position at successive equal time intervals Δt, $2\Delta t$, $3\Delta t$, etc:

$$y_1 = -\frac{1}{2}g(\Delta t)^2$$

$$y_2 = -\frac{1}{2}g(2\Delta t)^2 = -\frac{4}{2}g(\Delta t)^2$$

$$y_3 = -\frac{1}{2}(3\Delta t)^2 = -\frac{9}{2}g(\Delta t)^2$$

$$y_4 = -\frac{1}{2}g(4\Delta t)^2 = -\frac{16}{2}g(\Delta t)^2$$

etc.

Evaluate the changes in those positions in each time interval:

$$\Delta y_{10} = y_1 - 0 = \left(-\frac{1}{2}g\right)(\Delta t)^2$$

$$\begin{aligned}\Delta y_{21} = y_2 - y_1 &= -\frac{4}{2}g(\Delta t)^2 + \frac{1}{2}g(\Delta t)^2 \\ &= -\frac{3}{2}g(\Delta t)^2 = 3\left[-\frac{1}{2}g(\Delta t)^2\right] \\ &= 3\Delta y_{10}\end{aligned}$$

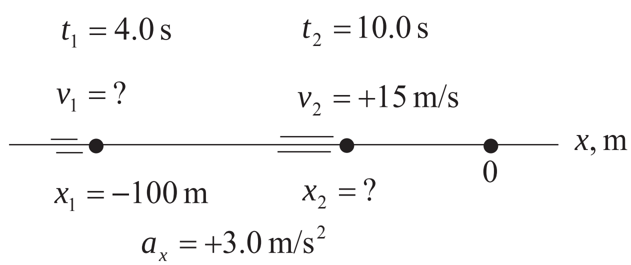
$$\Delta y_{32} = y_3 - y_2 = -\frac{9}{2}g(\Delta t)^2 + \frac{4}{2}g(\Delta t)^2$$
$$= -\frac{5}{2}g(\Delta t)^2 = 5\left[-\frac{1}{2}g(\Delta t)^2\right]$$
$$= 5\Delta y_{10}$$

$$\Delta y_{43} = y_4 - y_3 = -\frac{16}{2}g(\Delta t)^2 + \frac{9}{2}g(\Delta t)^2$$
$$= -\frac{7}{2}g(\Delta t)^2 = 7\left[-\frac{1}{2}g(\Delta t)^2\right]$$
$$= 7\Delta y_{10}$$

etc.

92 •• A particle travels along the x axis with a constant acceleration of +3.0 m/s^2. It is at $x = -100$ m at time $t = 4.0$ s. In addition, it has a velocity of +15 m/s at a time 6.0 s later. Find its position at this later time.

Picture the Problem Because the particle moves with a constant acceleration we can use constant-acceleration equations to relate its positions and velocities to the time into its motion. A pictorial representation will help us organize the information in the problem and develop our solution strategy.



Using a constant-acceleration equation, relate the velocity of the particle at time t_2 to its velocity at time t_1:

$$v_2 - v_1 = a_x(t_2 - t_1)$$

Using a constant-acceleration equation, relate the position of the particle at time t_2 to its position at time t_1:

$$x_2 - x_1 = v_1(t_2 - t_1) + \tfrac{1}{2}a_x(t_2 - t_1)^2$$

Eliminating v_1 and solving for x_2 gives:

$$x_2 = x_1 + v_2(t_2 - t_1) - \tfrac{1}{2}a_x(t_2 - t_1)^2$$

Substitute numerical values and evaluate x_2:

$$x_2 = -100\,\text{m} + (+15\,\text{m/s})(10.0\,\text{s} - 4.0\,\text{s}) - \tfrac{1}{2}(+3.0\,\text{m/s}^2)(10.0\,\text{s} - 4.0\,\text{s})^2 = \boxed{-64\,\text{m}}$$

93 •• [SSM] If it were possible for a spacecraft to maintain a constant acceleration indefinitely, trips to the planets of the Solar System could be undertaken in days or weeks, while voyages to the nearer stars would only take a few years. (*a*) Using data from the tables at the back of the book, find the time it would take for a one-way trip from Earth to Mars (at Mars' closest approach to Earth). Assume that the spacecraft starts from rest, travels along a straight line, accelerates halfway at 1*g*, flips around, and decelerates at 1*g* for the rest of the trip. (*b*) Repeat the calculation for a 4.1×10^{13}-km trip to Proxima Centauri, our nearest stellar neighbor outside of the sun. (See Problem 47.)

Picture the Problem Note: No material body can travel at speeds faster than light. When one is dealing with problems of this sort, the kinematic formulae for displacement, velocity and acceleration are no longer valid, and one must invoke the special theory of relativity to answer questions such as these. For now, ignore such subtleties. Although the formulas you are using (i.e., the constant-acceleration equations) are not quite correct, your answer to Part (*b*) will be wrong by about 1%.

(*a*) Let $t_{1/2}$ represent the time it takes to reach the halfway point. Then the total trip time is:

$$t = 2\,t_{1/2} \qquad (1)$$

Use a constant- acceleration equation to relate the half-distance to Mars Δx to the initial speed, acceleration, and half-trip time $t_{1/2}$:

$$\Delta x = v_0 t + \tfrac{1}{2} a t_{1/2}^2$$

Because $v_0 = 0$ and $a = g$:

$$t_{1/2} = \sqrt{\frac{2\Delta x}{a}}$$

Substitute for $t_{1/2}$ in equation (1) to obtain:

$$t = 2\sqrt{\frac{2\Delta x}{a}} \qquad (2)$$

The distance from Earth to Mars at closest approach is 7.8×10^{10} m. Substitute numerical values and evaluate t :

$$t_{\text{round trip}} = 2\sqrt{\frac{2(3.9\times10^{10}\ \text{m})}{9.81\,\text{m/s}^2}} = 18\times10^4\ \text{s}$$

$$\approx \boxed{2.1\,\text{d}}$$

(*b*) From Problem 47 we have:

$$d_{\substack{\text{Proxima}\\ \text{Centauri}}} = 4.1\times10^{13}\ \text{km}$$

Substitute numerical values in equation (2) to obtain:

$$t_{\text{round trip}} = 2\sqrt{\frac{2\left(4.1\times10^{13}\text{ km}\right)}{9.81\,\text{m/s}^2}} = 18\times10^{7}\text{ s}$$

$$\approx \boxed{5.8\text{ y}}$$

Remarks: Our result in Part (*a*) seems remarkably short, considering how far Mars is and how low the acceleration is.

94 • The Stratosphere Tower in Las Vegas is 1137 ft high. It takes 1 min, 20 s to ascend from the ground floor to the top of the tower using the high-speed elevator. The elevator starts and ends at rest. Assume that it maintains a constant upward acceleration until it reaches its maximum speed, and then maintains a constant acceleration of equal magnitude until it comes to a stop. Find the magnitude of the acceleration of the elevator. Express this acceleration magnitude as a multiple of g (the acceleration due to gravity).

Picture the Problem Because the elevator accelerates uniformly for half the distance and uniformly decelerates for the second half, we can use constant-acceleration equations to describe its motion

Let $t_{1/2}$ = 40 s be the time it takes to reach the halfway mark. Use the constant-acceleration equation that relates the acceleration to the known variables to obtain:

$$\Delta y = v_0 t + \tfrac{1}{2}at^2$$

or, because $v_0 = 0$,

$$\Delta y = \tfrac{1}{2}at^2 \Rightarrow a = \frac{2\,\Delta y}{t_{1/2}^2}$$

Substitute numerical values and evaluate a:

$$a = \frac{2\left(\frac{1}{2}\right)\left(1137\,\text{ft}\right)\left(1\,\text{m}/3.281\,\text{ft}\right)}{\left(40\,\text{s}\right)^2} = 0.22\,\text{m/s}^2$$

$$= \boxed{0.022g}$$

95 •• A train pulls away from a station with a constant acceleration of 0.40 m/s^2. A passenger arrives at a point next to the track 6.0 s after the end of the train has passed the very same point. What is the slowest constant speed at which she can run and still catch the train? On a single graph plot the position versus time curves for both the train and the passenger.

Picture the Problem Because the acceleration is constant, we can describe the motions of the train using constant-acceleration equations. Find expressions for the distances traveled, separately, by the train and the passenger. When are they equal? Note that the train is accelerating and the passenger runs at a constant minimum speed (zero acceleration) such that she can just catch the train.

1. Using the subscripts "train" and "p" to refer to the train and the passenger and the subscript "c" to identify "critical" conditions, express the position of the train and the passenger:

$$x_{\text{train c}}(t_c) = \frac{a_{\text{train}}}{2}t_c^2$$

and

$$x_{\text{p,c}}(t_c) = v_{\text{p,c}}(t_c - \Delta t)$$

Express the critical conditions that must be satisfied if the passenger is to catch the train:

$$v_{\text{train c}} = v_{\text{p,c}}$$

and

$$x_{\text{train c}} = x_{\text{p,c}}$$

2. Express the train's average speed:

$$v_{\text{av}}(0 \text{ to } t_c) = \frac{0 + v_{\text{train c}}}{2} = \frac{v_{\text{train c}}}{2}$$

3. Using the definition of average speed, express v_{av} in terms of $x_{\text{p,c}}$ and t_c.

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{0 + x_{\text{p,c}}}{0 + t_c} = \frac{x_{\text{p,c}}}{t_c}$$

4. Combine steps 2 and 3 and solve for $x_{\text{p,c}}$.

$$x_{\text{p,c}} = \frac{v_{\text{train c}}t_c}{2}$$

5. Combine steps 1 and 4 and solve for t_c.

$$v_{\text{p,c}}(t_c - \Delta t) = \frac{v_{\text{train c}}t_c}{2}$$

or

$$t_c - \Delta t = \frac{t_c}{2}$$

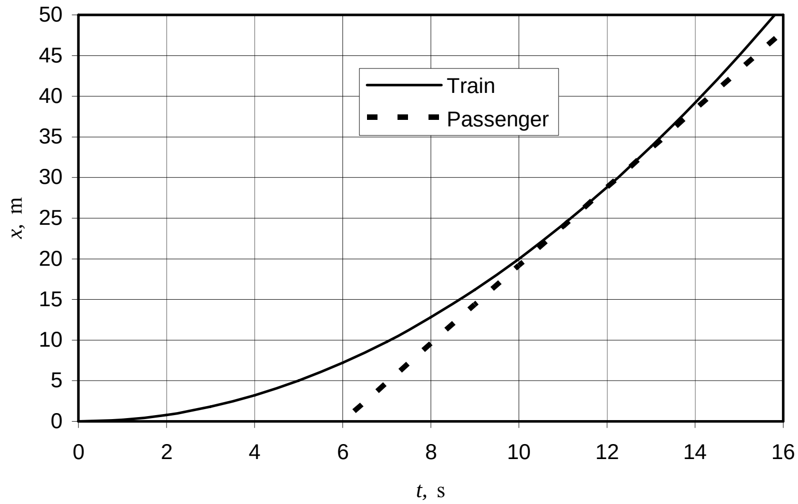
and

$t_c = 2\ \Delta t = 2\ (6\text{ s}) = 12\text{ s}$

6. Finally, combine steps 1 and 5 and solve for $v_{\text{train c}}$.

$$v_{\text{p,c}} = v_{\text{train c}} = a_{\text{train}}t_c = (0.40\,\text{m/s}^2)(12\,\text{s})$$
$$= \boxed{4.8\,\text{m/s}}$$

The following graph shows the location of both the passenger and the train as a function of time. The parabolic solid curve is the graph of $x_{\text{train}}(t)$ for the accelerating train. The straight dashed line is passenger's position $x_p(t)$ if she arrives at $\Delta t = 6.0$ s after the train departs. When the passenger catches the train, our graph shows that her speed and that of the train must be equal ($v_{\text{train c}} = v_{\text{p,c}}$). Do you see why?



96 ••• Ball A is dropped from the top of a building of height h at the same instant that ball B is thrown vertically upward from the ground. When the balls collide, they are moving in opposite directions, and the speed of A is twice the speed of B. At what fraction of the height of the building does the collision occur?

Picture the Problem Both balls experience constant acceleration once they are in flight. Choose a coordinate system with the origin at the ground and the upward direction positive. When the balls collide they are at the same height above the ground.

Using constant-acceleration equations, express the positions of both balls as functions of time. At the ground $y = 0$.

$$y_A = h - \tfrac{1}{2}gt^2$$
and
$$y_B = v_0 t - \tfrac{1}{2}gt^2$$

The conditions at collision are that the heights are equal and the speeds are related:

$$y_A = y_B$$
and
$$v_A = -2v_B$$

Express the speeds of both balls as functions of time:

$$v_A = -gt \quad \text{and} \quad v_B = v_0 - gt$$

Substituting the position and speed functions into the conditions at collision gives:

$$h - \tfrac{1}{2}gt_c^2 = v_0 t_c - \tfrac{1}{2}gt_c^2$$
and
$$-gt_c = -2(v_0 - gt_c)$$
where t_c is the time of collision.

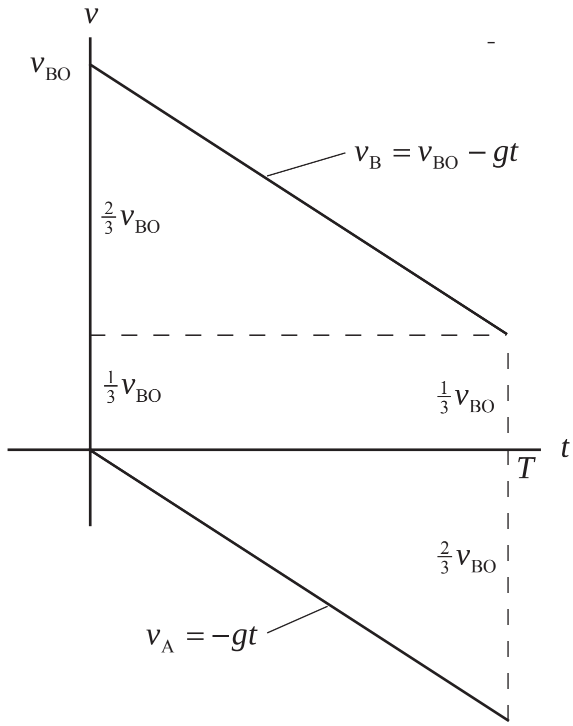
We now have two equations and two unknowns, t_c and v_0. Solving the equations for the unknowns gives:

$$t_c = \sqrt{\frac{2h}{3g}} \quad \text{and} \quad v_0 = \sqrt{\frac{3gh}{2}}$$

Substitute the expression for t_c into the equation for y_A to obtain the height at collision:

$$y_A = h - \tfrac{1}{2}g\left(\frac{2h}{3g}\right) = \boxed{\frac{2h}{3}}$$

Remarks: We can also solve this problem graphically by plotting velocity-versus-time for both balls. Because ball A starts from rest its velocity is given by $v_A = -gt$. Ball B initially moves with an unknown velocity v_{B0} and its velocity is given by $v_B = v_{B0} - gt$. The graphs of these equations are shown below with T representing the time at which they collide.



The height of the building is the sum of the sum of the distances traveled by the balls. Each of these distances is equal to the magnitude of the area "under" the corresponding v-versus-t curve. Thus, the height of the building equals the area of the parallelogram, which is $v_{B0}T$. The distance that A falls is the area of the lower triangle, which is $\tfrac{1}{3}v_{BO}T$. Therefore, the ratio of the distance fallen by A to the height of the building is 1/3, so the collision takes place at 2/3 the height of the building.

97 ••• Solve Problem 96 if the collision occurs when the balls are moving in the same direction and the speed of A is 4 times that of B.

Picture the Problem Both balls are moving with constant acceleration. Take the origin of the coordinate system to be at the ground and the upward direction to be positive. When the balls collide they are at the same height above the ground. The speeds at collision are related by $v_A = 4v_B$.

Using constant-acceleration equations, express the positions of both balls as functions of time:

$$y_A = h - \tfrac{1}{2}gt^2 \text{ and } y_B = v_0 t - \tfrac{1}{2}gt^2$$

The conditions at collision are that the heights are equal and the speeds are related:

$$y_A = y_B \text{ and } v_A = 4v_B$$

Express the speeds of both balls as functions of time:

$$v_A = -gt \text{ and } v_B = v_0 - gt$$

Substitute the position and speed functions into the conditions at collision to obtain:

$$h - \tfrac{1}{2}gt_c^2 = v_0 t_c - \tfrac{1}{2}gt_c^2$$

and

$$-gt_c = 4(v_0 - gt_c)$$

where t_c is the time of collision.

We now have two equations and two unknowns, t_c and v_0. Solving the equations for the unknowns gives:

$$t_c = \sqrt{\frac{4h}{3g}} \text{ and } v_0 = \sqrt{\frac{3gh}{4}}$$

Substitute the expression for t_c into the equation for y_A to obtain the height at collision:

$$y_A = h - \tfrac{1}{2}g\left(\frac{4h}{3g}\right) = \boxed{\frac{h}{3}}$$

98 ••• Starting at one station, a subway train accelerates from rest at a constant rate of 1.00 m/s^2 for half the distance to the next station, then slows down at the same rate for the second half of the journey. The total distance between stations is 900 m. (*a*) Sketch a graph of the velocity v_x as a function of time over the full journey. (*b*) Sketch a graph of the position as a function of time over the full journey. Place appropriate numerical values on both axes.

Picture the Problem The problem describes two intervals of constant acceleration; one when the train's speed is increasing, and a second when it is decreasing.

(*a*) Using a constant-acceleration equation, relate the half-distance Δx between stations to the initial speed v_0, the acceleration a of the train, and the time-to-midpoint Δt:

$$\Delta x = v_0 \Delta t + \tfrac{1}{2}a(\Delta t)^2$$

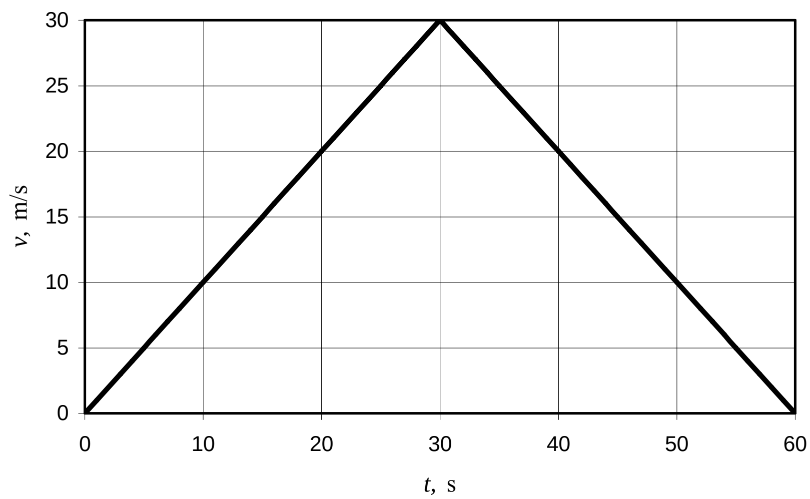
or, because $v_0 = 0$,

$$\Delta x = \tfrac{1}{2}a(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta x}{a}}$$

Substitute numerical values and evaluate the time-to-midpoint Δt:

$$\Delta t = \sqrt{\frac{2(450\,\text{m})}{1.00\,\text{m/s}^2}} = 30.0\,\text{s}$$

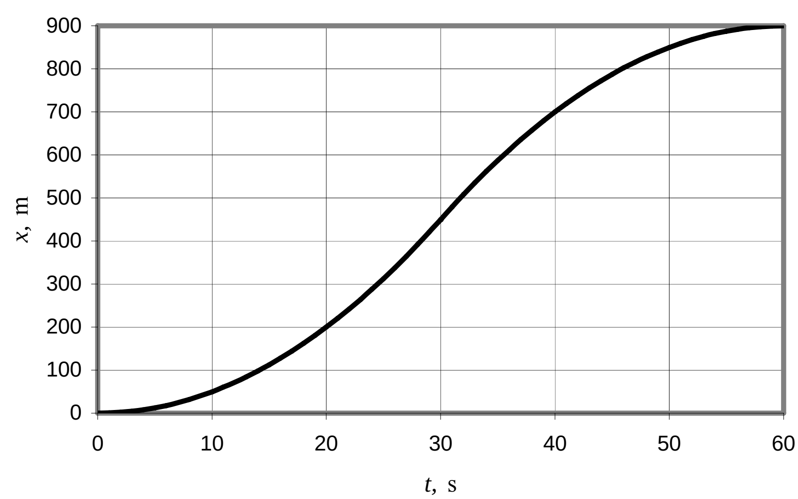
Because the train accelerates uniformly and from rest, the first part of its velocity graph will be linear, pass through the origin, and last for 30 s. Because it slows down uniformly and at the same rate for the second half of its journey, this part of its graph will also be linear but with a negative slope. A graph of v as a function of t follows.



(b) The graph of x as a function of t is obtained from the graph of v as a function of t by finding the area under the velocity curve. Looking at the velocity graph, note that when the train has been in motion for 10 s, it will have traveled a distance of

$$\tfrac{1}{2}(10\,\mathrm{s})(10\,\mathrm{m/s}) = 50\,\mathrm{m}$$

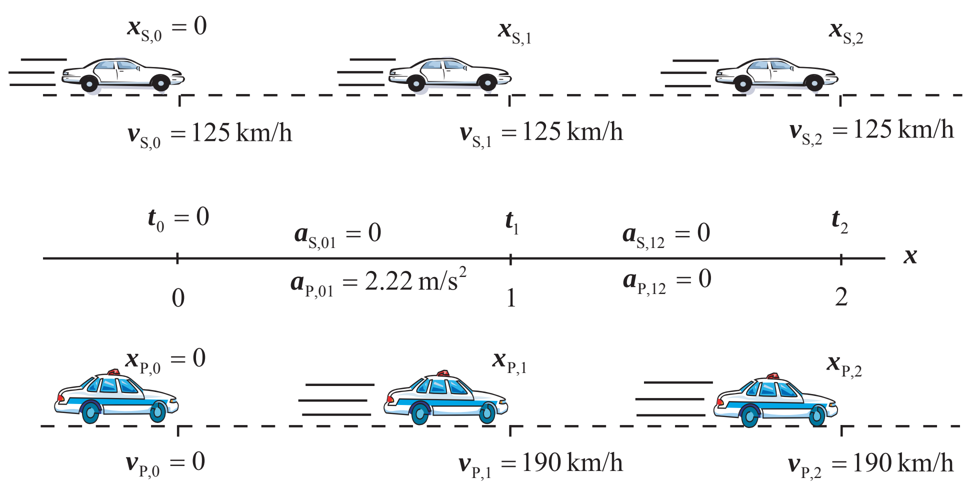
and that this distance is plotted above 10 s on the following graph.



Selecting additional points from the velocity graph and calculating the areas under the curve will confirm the graph of x as a function of t.

99 ••• **[SSM]** A speeder traveling at a constant speed of 125 km/h races past a billboard. A patrol car pursues from rest with constant acceleration of (8.0 km/h)/s until it reaches its maximum speed of 190 km/h, which it maintains until it catches up with the speeder. (*a*) How long does it take the patrol car to catch the speeder if it starts moving just as the speeder passes? (*b*) How far does each car travel? (*c*) Sketch *x*(*t*) for each car.

Picture the Problem This is a two-stage constant-acceleration problem. Choose a coordinate system in which the direction of the motion of the cars is the positive direction. The pictorial representation summarizes what we know about the motion of the speeder's car and the patrol car.



Convert the speeds of the vehicles and the acceleration of the police car into SI units:

$$8.0\frac{\text{km}}{\text{h}\cdot\text{s}} = 8.0\frac{\text{km}}{\text{h}\cdot\text{s}}\times\frac{1\text{ h}}{3600\text{ s}} = 2.2\text{ m/s}^2,$$

$$125\frac{\text{km}}{\text{h}} = 125\frac{\text{km}}{\text{h}}\times\frac{1\text{ h}}{3600\text{ s}} = 34.7\text{ m/s},$$

and

$$190\frac{\text{km}}{\text{h}} = 190\frac{\text{km}}{\text{h}}\times\frac{1\text{ h}}{3600\text{ s}} = 52.8\text{ m/s}$$

(*a*) Express the condition that determines when the police car catches the speeder; that is, that their displacements will be the same:

$$\Delta x_{P,02} = \Delta x_{S,02}$$

Using a constant-acceleration equation, relate the displacement of the patrol car to its displacement while accelerating and its displacement once it reaches its maximum velocity:

$$\begin{aligned}\Delta x_{P,02} &= \Delta x_{P,01} + \Delta x_{P,12}\\ &= \Delta x_{P,01} + v_{P,1}(t_2 - t_1)\end{aligned}$$

Using a constant-acceleration equation, relate the displacement of the speeder to its constant velocity and the time it takes the patrol car to catch it:

$$\Delta x_{\mathrm{S},02} = v_{\mathrm{S},02}\Delta t_{02} = (34.7\ \mathrm{m/s})t_2$$

Calculate the time during which the police car is speeding up:

$$\Delta t_{\mathrm{P},01} = \frac{\Delta v_{\mathrm{P},01}}{a_{\mathrm{P},01}} = \frac{v_{\mathrm{P},1} - v_{\mathrm{P},0}}{a_{\mathrm{P},01}} = \frac{52.8\ \mathrm{m/s} - 0}{2.2\ \mathrm{m/s^2}} = 24\ \mathrm{s}$$

Express the displacement of the patrol car:

$$\Delta x_{\mathrm{P},01} = v_{\mathrm{P},0}\Delta t_{\mathrm{P},01} + \tfrac{1}{2}a_{\mathrm{P},01}\Delta t_{\mathrm{P},01}^2 = 0 + \tfrac{1}{2}(2.2\ \mathrm{m/s^2})(24\ \mathrm{s})^2 = 630\ \mathrm{m}$$

Equate the displacements of the two vehicles:

$$\Delta x_{\mathrm{P},02} = \Delta x_{\mathrm{P},01} + \Delta x_{\mathrm{P},12} = \Delta x_{\mathrm{P},01} + v_{\mathrm{P},1}(t_2 - t_1) = 630\ \mathrm{m} + (52.8\ \mathrm{m/s})(t_2 - 24\ \mathrm{s})$$

Substitute for $\Delta x_{\mathrm{P},02}$ to obtain:

$$(34.7\ \mathrm{m/s})\ t_2 = 630\ \mathrm{m} + (52.8\ \mathrm{m/s})(t_2 - 24\ \mathrm{s})$$

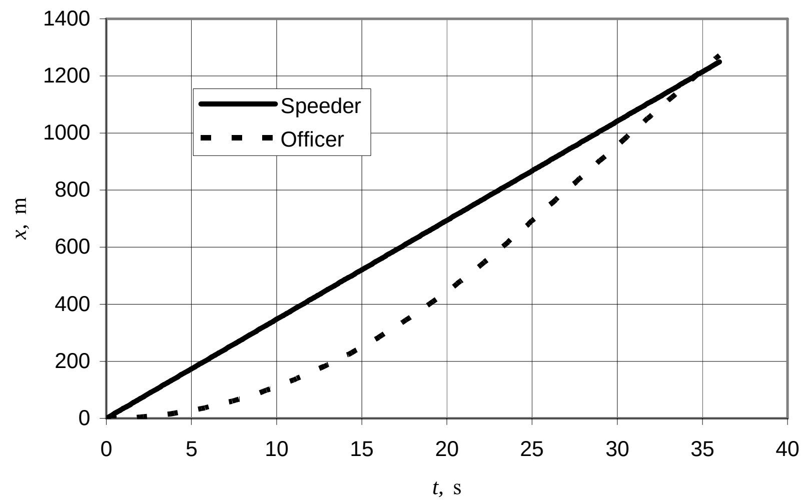
Solving for the time to catch up yields:

$$t_2 = 35.19\ \mathrm{s} = \boxed{35\ \mathrm{s}}$$

(*b*) The distance traveled is the displacement, $\Delta x_{02,\mathrm{S}}$, of the speeder during the catch:

$$\Delta x_{\mathrm{S},02} = v_{\mathrm{S},02}\Delta t_{02} = (35\ \mathrm{m/s})(35.19\ \mathrm{s}) = \boxed{1.2\ \mathrm{km}}$$

(*c*) The graphs of x_S and x_P follow. The straight line (solid) represents $x_S(t)$ and the parabola (dashed) represents $x_P(t)$.



100 ••• When the patrol car in Problem 99 (traveling at 190 km/h), is 100 m behind the speeder (traveling at 125 km/h), the speeder sees the police car and slams on his brakes, locking the wheels. (*a*) Assuming that each car can brake at 6.0 m/s^2 and that the driver of the police car brakes instantly as she sees the brake lights of the speeder (reaction time = 0.0 s), show that the cars collide. (*b*) At what time after the speeder applies his brakes do the two cars collide? (*c*) Discuss how reaction time affects this problem.

Picture the Problem The accelerations of both cars are constant and we can use constant-acceleration equations to describe their motions. Choose a coordinate system in which the direction of motion of the cars is the positive direction, and the origin is at the initial position of the police car.

(*a*) The collision will ***not*** occur if, during braking, the displacements of the two cars differ by less than 100 m:

$$\Delta x_P - \Delta x_S < 100\,\text{m}$$

Using a constant-acceleration equation, relate the speeder's initial and final speeds to its displacement and acceleration and solve for the displacement:

$$v_s^2 = v_{0,s}^2 + 2a_s\Delta x_s$$

or, because $v_S = 0$,

$$\Delta x_s = \frac{-v_{0,s}^2}{2a_s}$$

Substitute numerical values and evaluate Δx_S:

$$\Delta x_S = \frac{-(34.7\,\text{m/s})^2}{2(-6.0\,\text{m/s}^2)} = 100\,\text{m}$$

Using a constant-acceleration equation, relate the patrol car's initial and final speeds to its displacement and acceleration and solve for the displacement:

$v_p^2 = v_{0,p}^2 + 2a_p\Delta x_p$
or, assuming $v_p = 0$,
$\Delta x_p = \dfrac{-v_{0,p}^2}{2a_p}$

Substitute numerical values and evaluate Δx_P:

$$\Delta x_P = \frac{-(52.8\,\text{m/s})^2}{2(-6.0\,\text{m/s}^2)} = 232\,\text{m}$$

Finally, substitute these displacements into the inequality that determines whether a collision occurs:

232 m – 100 m = 132 m
Because this difference is greater than 100 m, $\boxed{\text{the cars collide}}$.

(*b*) Using constant-acceleration equations, relate the positions of both vehicles to their initial positions, initial speeds, accelerations, and time in motion:

$x_S = 100\,\text{m} + (34.7\,\text{m/s})t - (3.0\,\text{m/s}^2)t^2$
and
$x_P = (52.8\,\text{m/s})t - (3.0\,\text{m/s}^2)t^2$

Equate x_S and x_P to obtain:

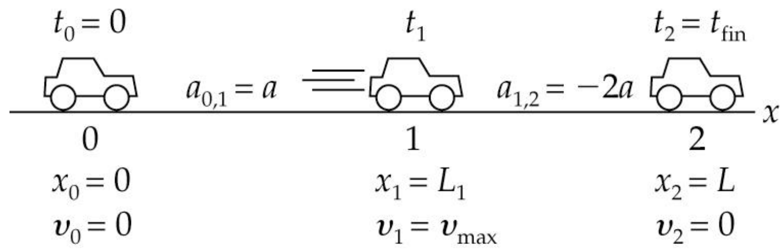
$$100\,\text{m} + (34.7\,\text{m/s})t - (3.0\,\text{m/s}^2)t^2 = (52.8\,\text{m/s})t - (3.0\,\text{m/s}^2)t^2$$

Solving this equation for t yields: $t = \boxed{5.5\,\text{s}}$

(*c*) If you take the reaction time into account, the collision will occur sooner and be more severe.

101 ••• Leadfoot Lou enters the "Rest-to-Rest" auto competition, in which each contestant's car begins and ends at rest, covering a fixed distance L in as short a time as possible. The intention is to demonstrate driving skills, and to find which car is the best at the *total combination* of speeding up and slowing down. The course is designed so that maximum speeds of the cars are never reached. (*a*) If Lou's car maintains an acceleration (magnitude) of a during speedup, and maintains a deceleration (magnitude) of $2a$ during braking, at what fraction of L should Lou move his foot from the gas pedal to the brake? (*b*) What fraction of the total time for the trip has elapsed at that point? (*c*) What is the fastest speed Lou's car ever reaches? Neglect Lou's reaction time, and answer in terms of a and L.

Picture the Problem Lou's acceleration is constant during both parts of his trip. Let t_1 be the time when the brake is applied; L_1 the distance traveled from $t = 0$ to $t = t_1$. Let t_{fin} be the time when Lou's car comes to rest at a distance L from the starting line. A pictorial representation will help organize the given information and plan the solution.



(*a*) Express the total length, L, of the course in terms of the distance over which Lou will be accelerating, Δx_{01}, and the distance over which he will be braking, Δx_{12}:

$$L = \Delta x_{01} + \Delta x_{12} \qquad (1)$$

Express the final speed over the first portion of the course in terms of the initial speed, acceleration, and displacement; solve for the displacement:

$$v_1^2 = v_0^2 + 2a_{01}\Delta x_{01}$$

or, because $v_0 = 0$, $\Delta x_{01} = L_1$, and $a_{01} = a$,

$$\Delta x_{01} = L_1 = \frac{v_1^2}{2a} = \frac{v_{\max}^2}{2a}$$

Express the final speed over the second portion of the course in terms of the initial speed, acceleration, and displacement; solve for the displacement:

$$v_2^2 = v_1^2 + 2a_{12}\Delta x_{12}$$

or, because $v_2 = 0$ and $a_{12} = -2a$,

$$\Delta x_{12} = \frac{v_1^2}{4a} = \frac{L_1}{2}$$

Substitute for Δx_{01} and Δx_{12} to obtain:

$$L = \Delta x_{01} + \Delta x_{12} = L_1 + \tfrac{1}{2}L_1 = \tfrac{3}{2}L_1$$

and

$$L_1 = \boxed{\tfrac{2}{3}L}$$

(*b*) Using the fact that the acceleration was constant during both legs of the trip, express Lou's average speed over each leg:

$$v_{\text{av},01} = v_{\text{av},12} = \frac{v_{\max}}{2}$$

Express the time for Lou to reach his maximum speed as a function of L_1 and his maximum speed:

$$\Delta t_{01} = \frac{\Delta x_{01}}{v_{\text{av},01}} = \frac{2L_1}{v_{\max}}$$

and

$$\Delta t_{01} \propto L_1 = \tfrac{2}{3}L$$

Having just shown that the time required for the first segment of the trip is proportional to the length of the segment, use this result to express Δt_{01} (= t_1) in terms t_{fin}:

$$t_1 = \boxed{\tfrac{2}{3}t_{\text{fin}}}$$

(*c*) Express Lou's displacement during the speeding up portion of his trip:

$$\Delta x_{01} = \tfrac{1}{2}at_1^2$$

The time required for Lou to make this speeding portion of his trip is given by:

$$t_1 = \frac{v_1}{a}$$

Express Lou's displacement during the slowing down portion of his trip:

$$\Delta x_{12} = v_1t_1 - \tfrac{1}{2}(2a)t_2^2 = v_1t_1 - at_2^2$$

where t_2 is his slowing down time.

Substitute in equation (1) to obtain:

$$L = \tfrac{1}{2}at_1^2 + v_1t_1 - at_2^2 \qquad (2)$$

Using a constant-acceleration equation, relate Lou's slowing down time for this portion of his trip to his initial and final speeds:

$$0 = v_1 - 2at_2 \Rightarrow t_2 = \frac{v_1}{2a}$$

Substituting for t_1 and t_2 in equation (2) yields:

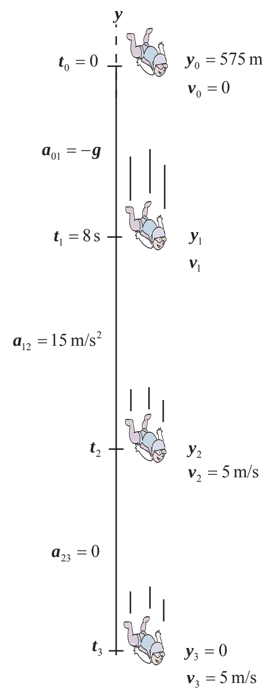
$$L = \tfrac{1}{2}a\left(\frac{v_1}{a}\right)^2 + v_1\left(\frac{v_1}{a}\right) - a\left(\frac{v_1}{2a}\right)^2$$

Solve for v_1 to obtain:

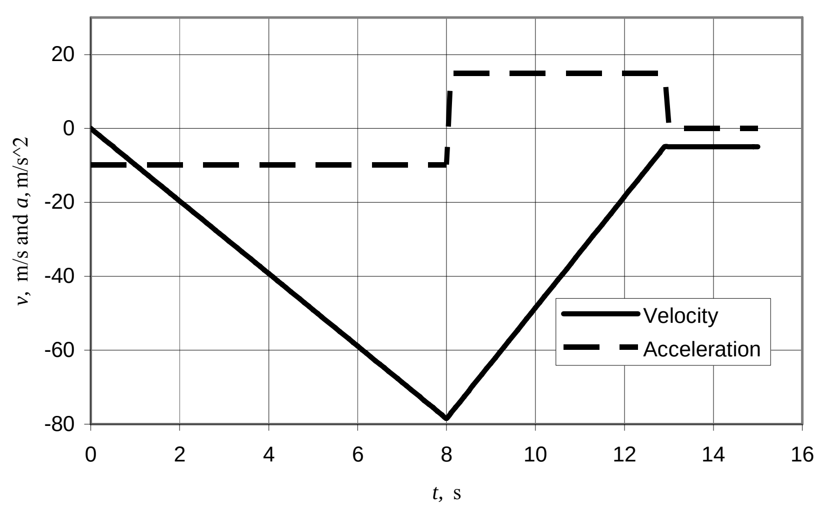
$$v_1 = \boxed{\sqrt{\frac{4aL}{5}}}$$

102 ••• A physics professor, equipped with a rocket backpack, steps out of a helicopter at an altitude of 575 m with zero initial velocity. (Neglect air resistance.) For 8.0 s, she falls freely. At that time she fires her rockets and slows her rate of descent at 15 m/s^2 until her rate of descent reaches 5.0 m/s. At this point she adjusts her rocket engine controls to maintain that rate of descent until she reaches the ground. (*a*) On a single graph, sketch her acceleration and velocity as functions of time. (Take upward to be positive.) (*b*) What is her speed at the end of the first 8.0 s? (*c*) What is the duration of her slowing down period? (*d*) How far does she travel while slowing down? (*e*) How much time is required for the entire trip from the helicopter to the ground? (*f*) What is her average velocity for the entire trip?

Picture the Problem There are three intervals of constant acceleration described in this problem. Choose a coordinate system in which the upward direction is positive. The pictorial representation summarizes the information we are given. We can apply constant-acceleration equations to find her speeds and distances traveled at the end of each phase of her descent.



(a) The graphs of $a(t)$ (dashed lines) and $v(t)$ (solid lines) follow.



(b) Use a constant- acceleration equation to express her speed in terms of her acceleration and the elapsed time:

$$v_1 = v_0 + a_{01}t_1$$

or, because $a_{01} = -g$ and $v_0 = 0$,

$$v_1 = -gt_1$$

Substitute numerical values and evaluate v_1:

$$v_1 = \left(-9.81\ \text{m/s}^2\right)\left(8.0\ \text{s}\right) = -78.5\ \text{m/s}$$

and her speed is $\boxed{79\ \text{m/s}}$

(*c*) As in (*b*), use a constant-acceleration equation to express her speed in terms of her acceleration and the elapsed time:

$$v_2 = v_1 + a_{12}\Delta t_{12} \Rightarrow \Delta t_{12} = \frac{v_2 - v_1}{a_{12}}$$

Substitute numerical values and evaluate Δt_{12}:

$$\Delta t_{12} = \frac{-5.0\,\text{m/s} - (-78.5\,\text{m/s})}{15\,\text{m/s}^2} = \boxed{4.9\,\text{s}}$$

(*d*) Find her average speed as she slows from 78.5 m/s to 5 m/s:

$$v_{\text{av}} = \frac{v_1 + v_2}{2} = \frac{78.5\,\text{m/s} + 5.0\,\text{m/s}}{2} = 41.7\,\text{m/s}$$

Use this value to calculate how far she travels in 4.90 s:

$$\Delta y_{12} = v_{\text{av}}\Delta t_{12} = (41.7\ \text{m/s})(4.90\,\text{s}) = 204.3\,\text{m}$$

She travels approximately 204 m while slowing down.

(*e*) Express the total time in terms of the times for each segment of her descent:

$$\Delta t_{\text{total}} = \Delta t_{01} + \Delta t_{12} + \Delta t_{23}$$

We know the times for the intervals from 0 to 1 and 1 to 2 so we only need to determine the time for the interval from 2 to 3. We can calculate Δt_{23} from her displacement and constant speed during that segment of her descent.

$$\Delta y_{23} = \Delta y_{\text{total}} - \Delta y_{01} - \Delta y_{12} = 575\,\text{m} - \left(\frac{78.5\,\text{m/s}}{2}\right)(8.0\,\text{s}) - 204.3\,\text{m} = 56.7\,\text{m}$$

Add the times to get the total time:

$$\Delta t_{\text{total}} = \Delta t_{01} + \Delta t_{12} + \Delta t_{23} = 8.0\,\text{s} + 4.9\,\text{s} + \frac{56.7\,\text{m}}{5.0\,\text{m/s}} = 24.24\,\text{s} = \boxed{24\,\text{s}}$$

(*f*) Using its definition, calculate her average velocity:

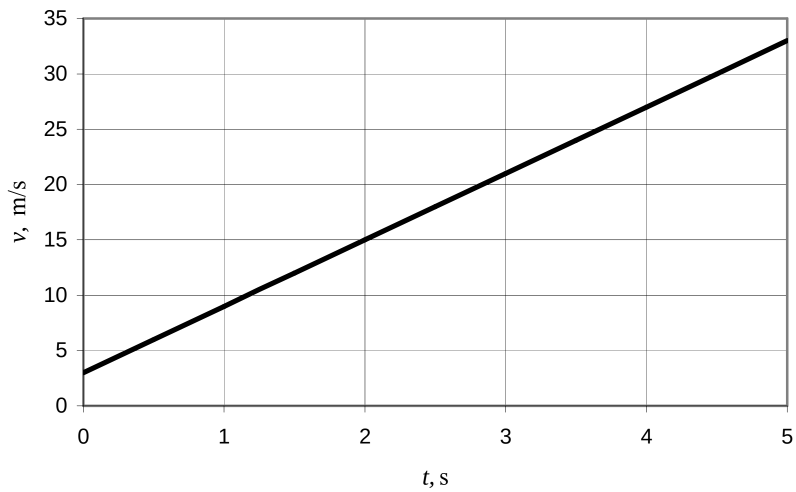
$$v_{\text{av}} = \frac{\Delta y}{\Delta t_{\text{total}}} = \frac{-575\,\text{m}}{24.24\,\text{s}} = \boxed{-24\,\text{m/s}}$$

Integration of the Equations of Motion

103 • [SSM] The velocity of a particle is given by $v_x(t) = (6.0\text{ m/s}^2)t + (3.0\text{ m/s})$. (*a*) Sketch *v* versus *t* and find the area under the curve for the interval $t = 0$ to $t = 5.0$ s. (*b*) Find the position function $x(t)$. Use it to calculate the displacement during the interval $t = 0$ to $t = 5.0$ s.

Picture the Problem The integral of a function is equal to the "area" between the curve for that function and the independent-variable axis.

(*a*) The following graph was plotted using a spreadsheet program:



The distance is found by determining the area under the curve. There are approximately 18 blocks each having an area of

$(5.0\text{ m/s})(1.0\text{ s}) = 5.0\text{ m}.$

$$A_{\substack{\text{under}\\\text{curve}}} = (18\text{ blocks})(5.0\text{ m/block}) = \boxed{90\text{ m}}$$

You can confirm this result by using the formula for the area of a trapezoid:

$$A = \left(\frac{33\text{ m/s} + 3\text{ m/s}}{2}\right)(5.0\text{ s} - 0\text{ s}) = 90\text{ m}$$

(*b*) To find the position function $x(t)$, we integrate the velocity function $v(t)$ over the time interval in question:

$$x(t) = \int_0^t v(t)\,\mathrm{d}t = \int_0^t \left[(6.0\text{ m/s}^2)t + (3.0\text{ m/s})\right] dt$$

and

$$x(t) = \boxed{(3.0\text{ m/s}^2)t^2 + (3.0\text{ m/s})\,t}$$

Now evaluate $x(t)$ at 0 s and 5.0 s respectively and subtract to obtain Δx:

$$\Delta x = x(5.0\,\text{s}) - x(0\,\text{s}) = 90\text{ m} - 0\text{ m}$$
$$= \boxed{90\text{ m}}$$

104 • Figure 2-41 shows the velocity of a particle versus time. (*a*) What is the magnitude in meters represented by the area of the shaded box? (*b*) Estimate the displacement of the particle for the two 1-s intervals, one beginning at $t = 1.0$ s and the other at $t = 2.0$ s. (*c*) Estimate the average velocity for the interval $1.0\text{ s} \le t \le 3.0\text{ s}$. (*d*) The equation of the curve is $v_x = (0.50\text{ m/s}^3)t^2$. Find the displacement of the particle for the interval $1.0\text{ s} \le t \le 3.0\text{ s}$ by integration and compare this answer with your answer for Part (*b*). Is the average velocity equal to the mean of the initial and final velocities for this case?

Picture the Problem The integral of $v(t)$ over a time interval is the displacement (change in position) during that time interval. The integral of a function is equivalent to the "area" between the curve for that function and the independent-variable axis. Count the grid boxes.

(*a*) Find the area of the shaded grid box:

$$\text{Area} = (1\text{ m/s})(1\text{ s}) = \boxed{1\text{ m per box}}$$

(*b*) Find the approximate area under the curve for $1.0\text{ s} \le t \le 2.0\text{ s}$:

$$\Delta x_{1.0\text{ s to }2.0\text{ s}} = \boxed{1.2\text{ m}}$$

Find the approximate area under the curve for $2.0\text{ s} \le t \le 3.0\text{ s}$:

$$\Delta x_{2.0\text{ s to }3.0\text{ s}} = \boxed{3.2\text{ m}}$$

(*c*) Sum the displacements to obtain the total in the interval $1.0\text{ s} \le t \le 3.0\text{ s}$:

$$\Delta x_{1.0\text{ s to }3.0\text{ s}} = 1.2\text{ m} + 3.2\text{ m}$$
$$= 4.4\text{ m}$$

Using its definition, express and evaluate v_{av}:

$$v_{av} = \frac{\Delta x_{1.0\text{ s to }3.0\text{ s}}}{\Delta t_{1.0\text{ s to }3.0\text{ s}}} = \frac{4.4\text{ m}}{2.0\text{ s}} = \boxed{2.2\text{ m/s}}$$

(*d*) Because the velocity of the particle is dx/dt, separate the variables and integrate over the interval 1.0 s ≤ t ≤ 3.0 s to determine the displacement in this time interval:

$$dx = \left(0.50\ \text{m/s}^3\right)t^2 dt$$

so

$$\Delta x_{1.0\,\text{s}\to 3.0\,\text{s}} = \int_{x_0}^{x} dx = \left(0.50\ \text{m/s}^3\right)\int_{1.0\ \text{s}}^{3.0\ \text{s}} t^2 dt$$

$$= \left(0.50\,\text{m/s}^3\right)\left[\frac{t^3}{3}\right]_{1.0\text{s}}^{3.0\text{s}} = 4.33\ \text{m}$$

$$= \boxed{4.3\ \text{m}}$$

This result is a little smaller than the sum of the displacements found in Part (*b*).

Calculate the average velocity over the 2-s interval from 1.0 s to 3.0 s:

$$v_{\text{av}\,(1.0\text{s}-3.0\text{s})} = \frac{\Delta x_{1.0\text{s}-3.0\text{s}}}{\Delta t_{1.0\text{s}-3.0\text{s}}} = \frac{4.33\ \text{m}}{2.0\ \text{s}} = 2.2\ \text{m/s}$$

Calculate the initial and final velocities of the particle over the same interval:

$$v(1.0\,\text{s}) = \left(0.50\,\text{m/s}^3\right)(1.0\,\text{s})^2 = 0.50\ \text{m/s}$$

$$v(3.0\,\text{s}) = \left(0.50\,\text{m/s}^3\right)(3.0\,\text{s})^2 = 4.5\ \text{m/s}$$

Finally, calculate the average value of the velocities at t = 1.0 s and t = 3.0 s:

$$\frac{v(1.0\,\text{s}) + v(3.0\,\text{s})}{2} = \frac{0.50\ \text{m/s} + 4.5\ \text{m/s}}{2}$$

$$= 2.5\ \text{m/s}$$

This average of 2.5 m/s is not equal to the average velocity calculated above because this is not an example of constant acceleration.

105 •• The velocity of a particle is given by $v_x = (7.0\ \text{m/s}^3)t^2 - 5.0\ \text{m/s}$, where t is in seconds and v is in meters per second. If the particle is at the origin at $t_0 = 0$, find the position function $x(t)$.

Picture the Problem Because the velocity of the particle varies with the square of the time, the acceleration is not constant. The position of the particle is found by integrating the velocity function.

Express the velocity of a particle as the derivative of its position function:

$$v(t) = \frac{dx(t)}{dt}$$

Separate the variables to obtain:

$$dx(t) = v(t)dt$$

Express the integral of x from $x_o = 0$ to x and t from $t_0 = 0$ to t:

$$x(t) = \int_{t_0=0}^{x(t)} dx = \int_{t_0=0}^{t} v(t)dt$$

Substitute for $v(t)$ and carry out the integration to obtain:

$$x(t) = \int_{t_0=0}^{t} \left[\left(7.0\,\text{m/s}^3\right)t^2 - \left(5.0\,\text{m/s}\right)\right]dt$$
$$= \boxed{\left(2.3\,\text{m/s}^3\right)t^3 - \left(5.0\,\text{m/s}\right)t}$$

106 •• Consider the velocity graph in Figure 2-42. Assuming $x = 0$ at $t = 0$, write correct algebraic expressions for $x(t)$, $v_x(t)$, and $a_x(t)$ with appropriate numerical values inserted for all constants.

Picture the Problem The graph is one of constant negative acceleration. Because $v_x = v(t)$ is a linear function of t, we can make use of the slope-intercept form of the equation of a straight line to find the relationship between these variables. We can then differentiate $v(t)$ to obtain $a(t)$ and integrate $v(t)$ to obtain $x(t)$.

Find the acceleration (the slope of the graph) and the velocity at time 0 (the v-intercept) and use the slope-intercept form of the equation of a straight line to express $v_x(t)$:

$$a_x = \boxed{-10\,\text{m/s}^2}$$
and
$$v_x(t) = \boxed{50\,\text{m/s} + (-10\,\text{m/s}^2)t}$$

Find $x(t)$ by integrating $v(t)$:

$$x(t) = \int \left[\left(-10\,\text{m/s}^2\right)t + 50\,\text{m/s}\right]dt$$
$$= \left(50\,\text{m/s}\right)t - \left(5.0\,\text{m/s}^2\right)t^2 + \text{C}$$

Using the fact that $x = 0$ when $t = 0$, evaluate C:

$$0 = \left(50\,\text{m/s}\right)(0) - \left(5.0\,\text{m/s}^2\right)(0)^2 + \text{C}$$
and
C = 0

Substitute to obtain:

$$x(t) = \boxed{\left(50\,\text{m/s}\right)t - \left(5.0\,\text{m/s}^2\right)t^2}$$

Note that this expression is quadratic in t and that the coefficient of t^2 is negative and equal in magnitude to half the constant acceleration.

Remarks: We can check our result for $x(t)$ by evaluating it over the 10-s interval shown and comparing this result with the area bounded by this curve and the time axis.

107 ••• Figure 2-43 shows the acceleration of a particle versus time. (*a*) What is the magnitude, in m/s, of the area of the shaded box? (*b*) The particle starts from rest at $t = 0$. Estimate the velocity at $t = 1.0$ s, 2.0 s, and 3.0 s by counting the boxes under the curve. (*c*) Sketch the curve v versus t from your results for Part (*b*), then estimate how far the particle travels in the interval $t = 0$ to $t = 3.0$ s using your curve.

Picture the Problem During any time interval, the integral of $a(t)$ is the change in velocity and the integral of $v(t)$ is the displacement. The integral of a function equals the "area" between the curve for that function and the independent-variable axis.

(*a*) Find the area of the shaded grid box in Figure 2-44:

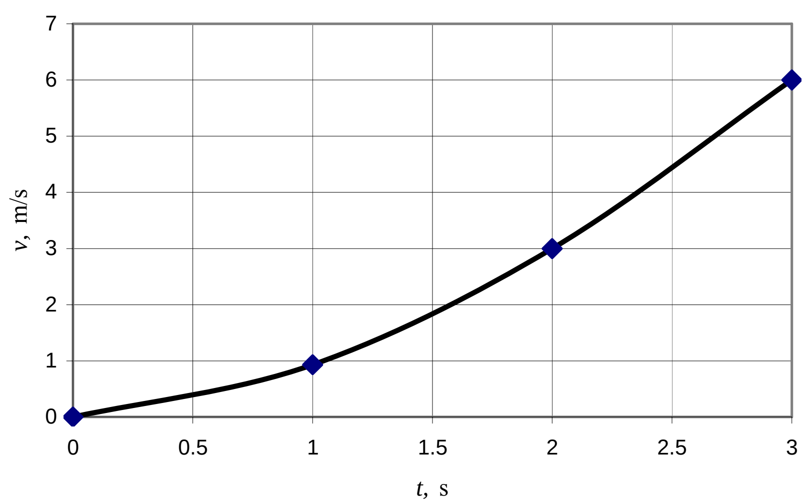
$$Area = \left(0.50\,\frac{\text{m}}{\text{s}^2}\right)(0.50\text{ s})$$

$$= \boxed{0.25\text{ m/s per box}}$$

(*b*) We start from rest ($v_0 = 0$) at $t = 0$. For the velocities at the other times, count boxes and multiply by the 0.25 m/s per box that we found in Part (*a*):

t	# of boxes	$v(t)$
(s)		(m/s)
1.0	3.7	$\boxed{0.93\text{ m/s}}$
2.0	12	$\boxed{3.0\text{ m/s}}$
3.0	24	$\boxed{6.0\text{ m/s}}$

(*c*) The graph of v as a function of t follows:



$$Area = hw = (1.0\text{ m/s})(0.50\text{ s}) = 0.50\text{ m per box}$$

Count the boxes under the $v(t)$ curve to find the distance traveled:

$$x(3.0\,\text{s}) = \Delta x(0 \to 3.0\,\text{s}) = (13\,\text{boxes})[(0.50\,\text{m})/\text{box}] = \boxed{6.5\,\text{m}}$$

108 ••• Figure 2-44 is a graph of v_x versus t for a particle moving along a straight line. The position of the particle at time $t = 0$ is $x_0 = 5.0$ m. (*a*) Find x for various times t by counting boxes, and sketch x as a function of t. (*b*) Sketch a graph of the acceleration a as a function of the time t. (*c*) Determine the displacement of the particle between $t = 3.0$ s and 7.0 s.

Picture the Problem The integral of $v(t)$ over a time interval is the displacement (change in position) during that time interval. The integral of a function equals the "area" between the curve for that function and the independent-variable axis. We can estimate this value by counting the number of squares under the curve between given limits and multiplying this count by the "area" of each square. Because acceleration is the slope of a velocity versus time curve, this is a non-constant-acceleration problem. The derivative of a function is equal to the "slope" of the function at that value of the independent variable. We can approximate the slope of any graph by drawing tangent lines and estimating their slopes.

(*a*) To obtain the data for $x(t)$, we must estimate the accumulated area under the $v(t)$ curve at each time interval:

Find the area of a shaded grid box in Figure 2-44:

$$\text{Area} = hw = (1.0\text{ m/s})(0.50\text{ s}) = 0.50\text{ m per box}$$

We start from rest ($v_o = 0$) at $t_o = 0$. For the position at the other times, count boxes and multiply by the 0.50 m per box that we found above. Remember to add the offset from the origin, $x_o = 5.0$ m, and that boxes below the $v = 0$ line are counted as negative:

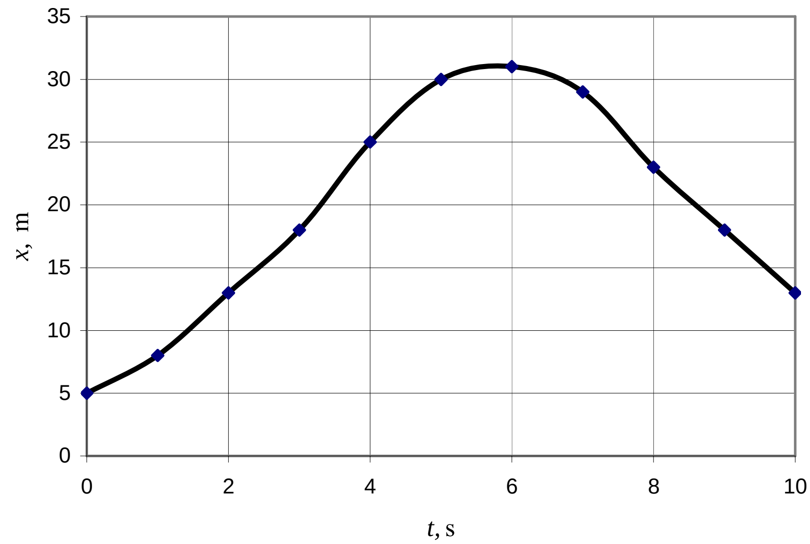
Examples:

$$x(2\,\text{s}) = (15.5\,\text{boxes})\left(\frac{0.50\text{ m}}{\text{box}}\right) + 5.0\,\text{m} = 13\text{ m}$$

$$x(5\,\text{s}) = (49\text{ boxes})\left(\frac{0.50\text{ m}}{\text{box}}\right) + 5.0\text{ m} = 30\text{ m}$$

$$x(10\,\text{s}) = (51\,\text{boxes})\left(\frac{0.50\text{ m}}{\text{box}}\right) - (36\,\text{boxes})\left(\frac{0.50\text{ m}}{\text{box}}\right) + 5.0\text{ m} = 13\text{ m}$$

A graph of x as a function of t follows:



(b) To obtain the data for $a(t)$, we must estimate the slope ($\Delta v/\Delta t$) of the $v(t)$ curve at several times. A good way to get reasonably reliable readings from the graph is to enlarge it several fold and then estimate the slope of the tangent line at selected points on the graph:

Examples:

$$a(2\text{ s}) = \frac{8.0\text{ m/s} - 4.1\text{ m/s}}{4.9\text{ s}} = 0.8\text{ m/s}^2$$

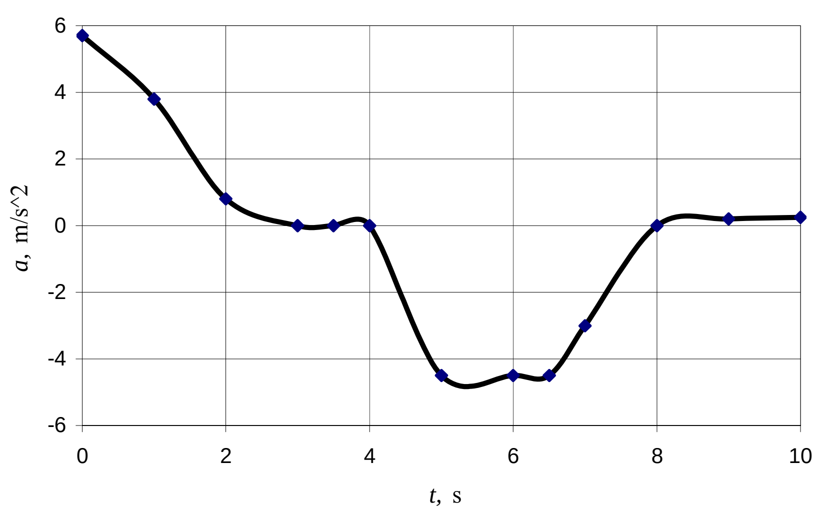
$$a(3\text{ s}) = 0$$

$$a(5\text{ s}) = \frac{-6.0\text{ m/s} - 8.0\text{ m/s}}{7.1\text{ s} - 4.0\text{ s}} = -4.5\text{ m/s}^2$$

$$a(7\text{ s}) = \frac{-6.0\text{ m/s} - 1.4\text{ m/s}}{7.6\text{ s} - 5.0\text{ s}} = -3.0\text{ m/s}^2$$

$$a(8\text{ s}) = 0$$

A graph of a as a function of t follows:



(*c*) The displacement of the particle between t = 3.0 s and 7.0 s is given by:

$$\Delta x_{3.0\,\text{s}\to 7.0\,\text{s}} = x(7.0\ \text{s}) - x(3.0\ \text{s})$$

Use the graph in (*a*) to obtain:

$$x(7.0\ \text{s}) = 29\ \text{m and } x(3.0\ \text{s}) = 18\ \text{m}$$

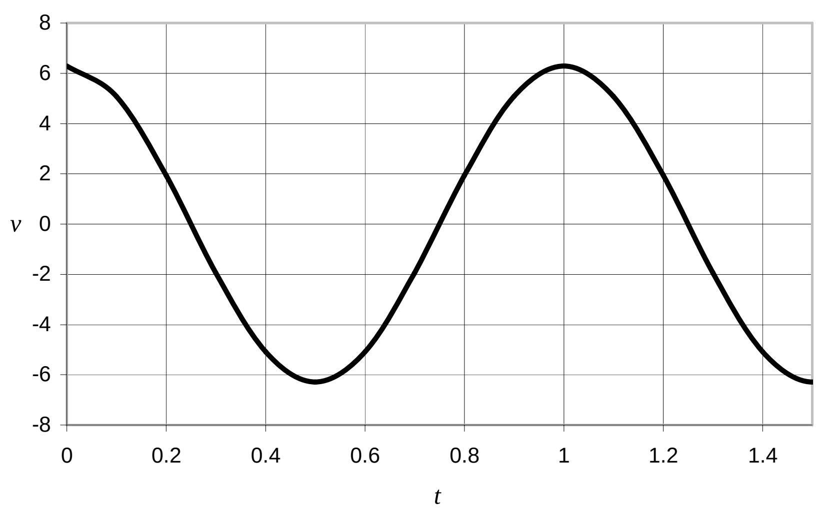
Substitute for Δx(7.0 s) and Δx(3.0 s) to obtain:

$$\Delta x_{3.0\,\text{s}\to 7.0\,\text{s}} = 29\ \text{m} - 18\ \text{m} = \boxed{11\ \text{m}}$$

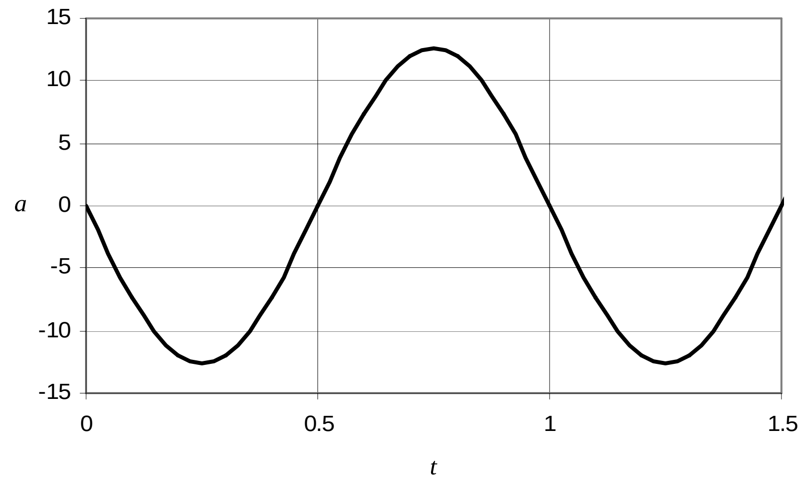
109 ••• [SSM] Figure 2-45 shows a plot of x versus t for a body moving along a straight line. For this motion, sketch graphs (using the same t axis) of (*a*) v_x as a function of t, and (*b*) a_x as a function of t. (*c*) Use your sketches to qualitatively compare the time(s) when the object is at its largest distance from the origin to the time(s) when its speed is greatest. Explain why the times are *not* the same. (*d*) Use your sketches to qualitatively compare the time(s) when the object is moving fastest to the time(s) when its acceleration is the largest. Explain why the times are *not* the same.

Picture the Problem Because the position of the body is not described by a parabolic function, the acceleration is not constant.

(*a*) Select a series of points on the graph of $x(t)$ (e.g., at the extreme values and where the graph crosses the t axis), draw tangent lines at those points, and measure their slopes. In doing this, you are evaluating $v = dx/dt$ at these points. Plot these slopes above the times at which you measured the slopes. Your graph should closely resemble the following graph.



(*b*) Select a series of points on the graph of $v(t)$ (e.g., at the extreme values and where the graph crosses the t axis), draw tangent lines at those points, and measure their slopes. In doing this, you are evaluating $a = dv/dt$ at these points. Plot these slopes above the times at which you measured the slopes. Your graph should closely resemble the following graph.



(c) The points at the greatest distances from the time axis correspond to turn-around points. The velocity of the body is zero at these points.

(d) The body is moving fastest as it goes through the origin. At these times the velocity is not changing and hence the acceleration is zero. The maximum acceleration occurs at the maximum distances where the velocity is zero but changing direction rapidly.

110 ••• The acceleration of a certain rocket is given by $a = bt$, where b is a positive constant. (a) Find the position function $x(t)$ if $x = x_0$ and $v_x = v_{0x}$ at $t = 0$. (b) Find the position and velocity at $t = 5.0$ s if $x_0 = 0$, $v_{0x} = 0$ and $b = 3.0$ m/s^3. (c) Compute the average velocity of the rocket between $t = 4.5$ s and 5.5 s at $t = 5.0$ s if $x_0 = 0$, $v_{0x} = 0$ and $b = 3.0$ m/s^3. Compare this average velocity with the instantaneous velocity at $t = 5.0$ s.

Picture the Problem Because the acceleration of the rocket varies with time, it is not constant and integration of this function is required to determine the rocket's velocity and position as functions of time. The conditions on x and v at $t = 0$ are known as ***initial conditions***.

(a) Integrate $a(t)$ to find $v(t)$:

$$v(t) = \int a(t)\,dt = b\int t\,dt = \tfrac{1}{2}bt^2 + \mathrm{C}$$

where C, the constant of integration, can be determined from the initial conditions.

Integrate $v(t)$ to find $x(t)$:

$$x(t) = \int v(t)dt = \int\left[\tfrac{1}{2}bt^2 + \mathrm{C}\right]dt$$

$$= \tfrac{1}{6}bt^3 + \mathrm{C}t + \mathrm{D}$$

where D is a second constant of integration.

Using the initial conditions, find the constants C and D:

$v_x(0) = v_{0x} \Rightarrow C = v_{0x}$
and
$x(0) = x_0 \Rightarrow D = x_0$
and so
$x(t) = \boxed{\tfrac{1}{6}bt^3 + v_{0x}t + x_0}$

(*b*) Evaluate $v(5.0\text{ s})$ and $x(5.0\text{ s})$ with C = D = 0 and $b = 3.0\text{ m/s}^3$:

$v(5.0\,\text{s}) = \tfrac{1}{2}(3.0\text{ m/s}^3)(5.0\text{ s})^2 = \boxed{38\,\text{m/s}}$
and
$x(5.0\,\text{s}) = \tfrac{1}{6}(3.0\text{ m/s}^3)(5.0\text{ s})^3 = \boxed{63\,\text{m}}$

(*c*) The average value of $v(t)$ over the interval Δt is given by:

$$\bar{v} = \frac{1}{\Delta t}\int_{t_1}^{t_2} v(t)dt$$

Substitute for Δt and $v(t)$ and evaluate the integral to obtain:

$$\bar{v} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \tfrac{1}{2}bt^2dt = \frac{b}{6\Delta t}t^3\Big|_{t_1}^{t_2} = \frac{b(t_2^3 - t_1^3)}{6\Delta t}$$

Substitute numerical values and evaluate $\bar{v}$:

$$\bar{v} = \frac{(3.0\text{ m/s}^3)\left[(5.5\text{ s})^3 - (4.5\text{ s})^3\right]}{6(5.5\text{ s} - 4.5\text{ s})}$$

$= \boxed{38\text{ m/s}}$ ... a result in good agreement with the value found in (*b*).

111 ••• [SSM] In the time interval from 0.0 s to 10.0 s, the acceleration of a particle traveling in a straight line is given by $a_x = (0.20\text{ m/s}^3)t$. Let to the right be the $+x$ direction. The particle initially has a velocity to the right of 9.5 m/s and is located 5.0 m to the left of the origin. (*a*) Determine the velocity as a function of time during the interval, (*b*) determine the position as a function of time during the interval, (*c*) determine the average velocity between $t = 0.0$ s and10.0 s, and compare it to the average of the instantaneous velocities at the start and ending times. Are these two averages equal? Explain.

Picture the Problem The acceleration is a function of time; therefore it is not constant. The instantaneous velocity can be determined by integration of the acceleration function and the average velocity from the general expression for the average value of a non-linear function.

(*a*) The ***instantaneous velocity*** function $v(t)$ is the time-integral of the acceleration function:

$$v(t) = \int a_x dt = \int bt\,dt = \frac{b}{2}t^2 + C_1$$

where $b = 0.20\text{ m/s}^3$

The initial conditions are: 1) $x(0) = -5.0\text{ m}$ and 2) $v(0) = 9.5\text{ m/s}$

Use initial condition 2) to obtain: $v(0) = 9.5\text{ m/s} = C_1$

Substituting in $v(t)$ for b and C_1 yields: $v(t) = \boxed{\left(0.10\text{ m/s}^3\right)t^2 + 9.5\text{ m/s}}$ (1)

(*b*) The ***instantaneous position*** function $x(t)$ is the time-integral of the velocity function:

$$x(t) = \int v(t)dt = \int\left(ct^2 + C_1\right)dt$$

$$= \frac{c}{3}t^3 + C_1 t + C_2$$

where $c = 0.10\text{ m/s}^3$.

Using initial condition 1) yields: $x(0) = -5.0\text{ m} = C_2$

Substituting in $x(t)$ for C_1 and C_2 yields:

$$x(t) = \boxed{\tfrac{1}{3}\left(0.20\text{ m/s}^3\right)t^3 + (9.5\text{ m/s})t - 5.0\text{ m/s}}$$

(*c*) The average value of $v(t)$ over the interval Δt is given by:

$$\bar{v} = \frac{1}{\Delta t}\int_{t_1}^{t_2} v(t)dt$$

Substitute for $v(t)$ and evaluate the integral to obtain:

$$\bar{v} = \frac{1}{\Delta t}\int_{t_1}^{t_2}\left(\frac{b}{2}t^2 + C_1\right)dt = \frac{1}{\Delta t}\left[\frac{b}{6}t^3 + C_1 t\right]_{t_1}^{t_2} = \frac{1}{\Delta t}\left[\frac{b}{6}t_2^3 + C_1 t_2 - \left(\frac{b}{6}t_1^3 + C_1 t_1\right)\right]$$

Simplifying this expression yields:

$$\bar{v} = \frac{1}{\Delta t}\left[\frac{b}{6}\left(t_2^3 - t_1^3\right) + C_1\left(t_2 - t_1\right)\right]$$

Because $t_1 = 0$:

$$\bar{v} = \frac{1}{\Delta t}\left[\frac{b}{6}t_2^3 + C_1 t_2\right]$$

Substitute numerical values and simplify to obtain:

$$\bar{v} = \frac{1}{10.0\text{ s}}\left[\left(\frac{0.20\text{ m/s}^3}{6}\right)(10.0\text{ s})^3 + (9.5\text{ m/s})(10.0\text{ s})\right] = \boxed{13\text{ m/s}}$$

The average of the initial instantaneous and final instantaneous velocities is given by:

$$v_{\text{av}} = \frac{v(0) + v(10.0\text{ s})}{2} \qquad (2)$$

Using equation (1), evaluate $v(0)$ and $v(10\text{ s})$:

$$v(0) = 9.5\text{ m/s}$$

and

$$v(10\text{ s}) = (0.10\text{ m/s}^3)(10.0\text{ s})^2 + 9.5\text{ m/s}$$
$$= 19.5\text{ m/s}$$

Substitute in equation (2) to obtain:

$$v_{\text{av}} = \frac{9.5\text{ m/s} + 19.5\text{ m/s}}{2} = \boxed{15\text{ m/s}}$$

v_{av} is not the same as $\bar{v}$ because the velocity does not change linearly with time. The velocity does not change linearly with time because the acceleration is not constant.

112 ••• Consider the motion of a particle that experiences a variable acceleration given by $a_x = a_{0x} + bt$, where a_{0x} and b are constants and $x = x_0$ and $v_x = v_{0x}$ at $t = 0$. (*a*) Find the instantaneous velocity as a function of time. (*b*) Find the position as a function of time. (*c*) Find the average velocity for the time interval with an initial time of zero and arbitrary final time t. (*d*) Compare the average of the initial and final velocities to your answer to Part (*c*). Are these two averages equal? Explain.

Determine the Concept Because the acceleration is a function of time, it is not constant. Hence we'll need to integrate the acceleration function to find the velocity of the particle as a function of time and integrate the velocity function to find the position of the particle as a function of time. We can use the general expression for the average value of a non-linear function to find the average velocity over an arbitrary interval of time.

(*a*) From the definition of acceleration we have:

$$\int_{v_{0x}}^{v_x} dv = \int_0^t a\,dt$$

Integrating the left-hand side of the equation and substituting for a on the right-hand side yields:

$$v_x - v_{0x} = \int_0^t (a_0 + bt)\,dt$$

Integrate the right-hand side to obtain:

$$v_x - v_{0x} = a_0 t + \tfrac{1}{2}bt^2$$

Solving this equation for v_x yields:

$$v_x = \boxed{v_{0x} + a_0 t + \tfrac{1}{2}bt^2} \qquad (1)$$

(*b*) The position of the particle as a function of time is the integral of the velocity function:

$$\int_{x_0}^{x} dx = \int_0^t v_x(t)dt$$

Integrating the left-hand side of the equation and substituting for *a* on the right-hand side yields:

$$x - x_0 = \int_0^t \left(v_{0x} + a_0 t + \frac{b}{2}t^2 \right) dt$$

Integrate the right-hand side to obtain:

$$x - x_0 = v_{0x}t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3$$

Solving this equation for *x* yields:

$$x = \boxed{x_{0x} + v_0 t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3}$$

(*c*) The average value of $v(t)$ over an interval Δt is given by:

$$\bar{v}_x = \frac{1}{\Delta t}\int_{t_1}^{t_2} v_x(t)dt$$

or, because $t_1 = 0$ and $t_2 = t$,

$$\bar{v}_x = \frac{1}{t}\int_0^t v_x(t)dt$$

Substituting for $v_x(t)$ yields:

$$\bar{v}_x = \frac{1}{t}\int_0^t \left(v_{0x} + a_0 t + \tfrac{1}{2}bt^2 \right) dt$$

Carry out the details of the integration to obtain:

$$\bar{v}_x = \frac{1}{t}\left[v_{0x}t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3 \right]_0^t$$

$$= \frac{1}{t}\left[v_{0x}t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3 \right]$$

$$= \boxed{v_{0x} + \tfrac{1}{2}a_0 t + \tfrac{1}{6}bt^2}$$

(*d*) The average of the initial instantaneous and final instantaneous velocities is given by:

$$v_{\text{av},x} = \frac{v_{0x} + v_x}{2}$$

Using equation (1), substitute to obtain:

$$v_{\text{av},x} = \frac{v_{0x} + v_{0x} + a_0 t + \tfrac{1}{2}bt^2}{2}$$

$$= \frac{2v_{0x} + a_0 t + \tfrac{1}{2}bt^2}{2}$$

$$= \boxed{v_{0x} + \tfrac{1}{2}a_0 t + \tfrac{1}{4}bt^2}$$

$v_{\text{av},x}$ is not the same as $\bar{v}_x$ because the acceleration is not constant.

General Problems

113 ••• You are a student in a science class that is using the following apparatus to determine the value of g. Two photogates are used. (Note: You may be familiar with photogates in everyday living. You see them in the doorways of some stores. They are designed to ring a bell when someone interrupts the beam while walking through the door.) One photogate is located at the edge of a table that is 1.00 m above the floor, and the second photogate is located directly below the first, at a height 0.500 m above the floor. You are told to drop a marble through these gates, releasing it from rest a negligible distance above the upper gate. The upper gate starts a timer as the marble passes through its beam. The second photogate stops the timer when the marble passes through its beam. (*a*) Prove that the experimental magnitude of free-fall acceleration is given by $g_{exp} = (2\Delta y)/(\Delta t)^2$, where Δy is the vertical distance between the photogates and Δt is the fall time. (*b*) For your setup, what value of Δt would you expect to measure, assuming g_{exp} is the value (9.81 m/s^2)? (*c*) During the experiment, a slight error is made. Instead of locating the first photogate even with the top of the table, your not-so-careful lab partner locates it 0.50 cm lower than the top of the table. However, she releases the marble from the same height that it was released from when the photogate was 1.00 m above the floor. What value of g_{exp} will you and your partner determine? What percentage difference does this represent from the standard value at sea level?

Picture the Problem The acceleration of the marble is constant. Because the motion is downward, choose a coordinate system with downward as the positive direction and use constant-acceleration equations to describe the motion of the marble.

(*a*) Use a constant-acceleration equation to relate the vertical distance the marble falls to its time of fall and its free-fall acceleration:

$$\Delta y = v_0\Delta t + \tfrac{1}{2}g_{exp}(\Delta t)^2$$

or, because $v_0 = 0$,

$$\Delta y = \tfrac{1}{2}g_{exp}(\Delta t)^2 \Rightarrow g_{exp} = \boxed{\frac{2\Delta y}{(\Delta t)^2}} \quad (1)$$

(*b*) Solve equation (1) for Δt to obtain:

$$\Delta t = \sqrt{\frac{2\Delta y}{g_{exp}}}$$

Substitute numerical values and evaluate Δt

$$\Delta t = \sqrt{\frac{2(1.00\text{ m})}{9.81\text{ m/s}^2}} = \boxed{0.452\text{ s}}$$

(*c*) Using a constant-acceleration equation, express the speed of the marble in terms of its initial speed, acceleration, and displacement:

$$v_f^2 = v_0^2 + 2a\Delta y$$

or, because $v_0 = 0$ and $a = g$,

$$v_f^2 = 2g\Delta y \Rightarrow v_f = \sqrt{2g\Delta y}$$

Let v_1 be the speed the ball has reached when it has fallen 0.50 cm, and v_2 be the speed the ball has reached when it has fallen 0.500 m to obtain.

$$v_1 = \sqrt{2(9.81\,\text{m/s}^2)(0.0050\,\text{m})}\ \text{and}$$
$$= 0.313\,\text{m/s}$$
$$v_2 = \sqrt{2(9.81\,\text{m/s}^2)(0.500\,\text{m})}$$
$$= 3.13\,\text{m/s}$$

Using a constant-acceleration equation, express v_2 in terms of v_1, g and Δt:

$$v_2 = v_1 + g\Delta t \Rightarrow \Delta t = \frac{v_2 - v_1}{g}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{3.13\,\text{m/s} - 0.313\,\text{m/s}}{9.81\,\text{m/s}^2} = 0.2872\,\text{s}$$

The distance the marble falls during this interval is:

$$\Delta y = 1.00\,\text{m} - 0.5000\,\text{m} - 0.0050\,\text{m}$$
$$= 0.50\,\text{m}$$

Use equation (1) to calculate the experimental value of the acceleration due to gravity:

$$g_{\text{exp}} = \frac{2(0.50\,\text{m})}{(0.2872\,\text{s})^2} = \boxed{12\,\text{m/s}^2}$$

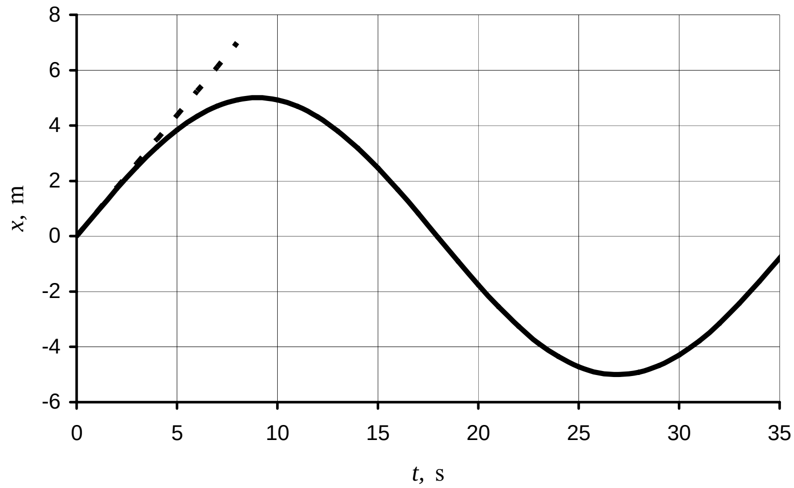
Finally, calculate the percent difference between this experimental result and the value accepted for g at sea level.

$$\%\ \text{difference} = \frac{\left|9.81\,\text{m/s}^2 - 12\,\text{m/s}^2\right|}{9.81\,\text{m/s}^2}$$
$$= \boxed{22\%}$$

114 ••• The position of a body oscillating on a spring is given by $x = A \sin \omega t$, where A and ω are constants, A = 5.0 cm. and $\omega = 0.175\ \text{s}^{-1}$. (*a*) Plot x as a function of t for $0 \le t \le 36$ s. (*b*) Measure the slope of your graph at $t = 0$ to find the velocity at this time. (*c*) Calculate the average velocity for a series of intervals, beginning at $t = 0$ and ending at t = 6.0, 3.0, 2.0, 1.0, 0.50, and 0.25 s. (*d*) Compute dx/dt to find the velocity at time $t = 0$. (*e*) Compare your results in Parts (*c*) and (*d*) and explain why your Part (*c*) results approach your Part (*d*) results.

Picture the Problem We can obtain an average velocity, $v_{\text{av}} = \Delta x/\Delta t$, over fixed time intervals. The instantaneous velocity, $v = dx/dt$ can only be obtained by differentiation.

(*a*) The following graph of *x* versus *t* was plotted using a spreadsheet program:



(*b*) Draw a tangent line at the origin and measure its rise and run. Use this ratio to obtain an approximate value for the slope at the origin:

The tangent line appears to, at least approximately, pass through the point (5, 4). Using the origin as the second point,

$$\Delta x = 4\text{ cm} - 0 = 4\text{ cm}$$

and

$$\Delta t = 5\text{ s} - 0 = 5\text{ s}$$

Therefore, the slope of the tangent line and the velocity of the body as it passes through the origin is approximately:

$$v(0) = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{4\,\text{cm}}{5\,\text{s}} = \boxed{0.8\,\text{cm/s}}$$

(*c*) Calculate the average velocity for the series of time intervals given by completing the table shown below:

t_0	t	Δt	x_0	x	Δx	$v_{av}=\Delta x/\Delta t$
(s)	(s)	(s)	(cm)	(cm)	(cm)	(m/s)
0	6	6	0	4.34	4.34	0.723
0	3	3	0	2.51	2.51	0.835
0	2	2	0	1.71	1.71	0.857
0	1	1	0	0.871	0.871	0.871
0	0.5	0.5	0	0.437	0.437	0.874
0	0.25	0.25	0	0.219	0.219	0.875

(*d*) Express the time derivative of the position function:

$$\frac{dx}{dt} = A\omega\cos\omega t$$

Substitute numerical values and evaluate $\frac{dx}{dt}$ at $t = 0$:

$$\frac{dx}{dt} = A\omega\cos 0 = A\omega$$
$$= (0.050\text{ m})(0.175\text{ s}^{-1})$$
$$= \boxed{0.88\text{ cm/s}}$$

(*e*) Compare the average velocities from Part (*c*) with the instantaneous velocity from Part (*d*):

As Δt, and thus Δx, becomes small, the value for the average velocity approaches that for the instantaneous velocity obtained in Part (*d*). For $\Delta t = 0.25$ s, they agree to three significant figures.

115 ••• [SSM] Consider an object that is attached to a horizontally oscillating piston. The object moves with a velocity given by $v = B\sin(\omega t)$, where B and ω are constants and ω is in s^{-1}. (*a*) Explain why B is equal to the maximum speed v_{max}. (*b*) Determine the acceleration of the object as a function of time. Is the acceleration constant? (*c*) What is the maximum acceleration (magnitude) in terms of ω and v_{max}. (*d*) At $t = 0$, the object's position is known to be x_0. Determine the position as a function of time in terms of in terms of t, ω, x_0 and v_{max}.

Determine the Concept Because the velocity varies nonlinearly with time, the acceleration of the object is not constant. We can find the acceleration of the object by differentiating its velocity with respect to time and its position function by integrating the velocity function.

(*a*) The maximum value of the sine function (as in $v = v_{max}\sin(\omega t)$) is 1. Hence the coefficient B represents the maximum possible speed v_{max}.

(*b*) The acceleration of the object is the derivative of its velocity with respect to time:

$$a = \frac{dv}{dt} = \frac{d}{dt}[v_{max}\sin(\omega t)]$$
$$= \boxed{\omega v_{max}\cos(\omega t)}$$

Because *a* varies sinusoidally with time it is *not* constant.

(*c*) Examination of the coefficient of the cosine function in the expression for *a* leads one to the conclusion that $|a_{max}| = \boxed{\omega v_{max}\,.}$

(*d*) The position of the object as a function of time is the integral of the velocity function:

$$\int dx = \int v(t)\,dt$$

Integrating the left-hand side of the equation and substituting for v on the right-hand side yields:

$$x = \int v_{max}\sin(\omega t)dt + C$$

Integrate the right-hand side to obtain:

$$x = \frac{-v_{max}}{\omega}\cos(\omega t) + C \qquad (1)$$

Use the initial condition $x(0) = x_0$ to obtain:

$$x_0 = \frac{-v_{max}}{\omega} + C$$

Solving for C yields:

$$C = x_0 + \frac{v_{max}}{\omega}$$

Substitute for C in equation (1) to obtain:

$$x = \frac{-v_{max}}{\omega}\cos(\omega t) + x_0 + \frac{v_{max}}{\omega}$$

Solving this equation for x yields:

$$x = \boxed{x_0 + \frac{v_{max}}{\omega}[1 - \cos(\omega t)]}$$

116 ••• Suppose the acceleration of a particle is a function of x, where $a_x(x) = (2.0\text{ s}^{-2})x$. (*a*) If the velocity is zero when $x = 1.0$ m, what is the speed when $x = 3.0$ m? (*b*) How long does it take the particle to travel from $x = 1.0$ m to $x = 3.0$ m?

Picture the Problem Because the acceleration of the particle is a function of its position, it is not constant. Changing the variable of integration in the definition of acceleration will allow us to determine its velocity and position as functions of position.

(*a*) The acceleration of the particle is given by:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Separating variables yields:

$$vdv = adx$$

Substitute for a to obtain:

$$vdv = bxdx$$

where $b = 2.0\text{ s}^{-2}$

The limits of integration are $v_o = 0$ and v on the left-hand side and x_o and x on the right-hand:

$$\int_{v_0=0}^{v} vdv = \int_{x_0}^{x} bxdx$$

Integrating both sides of the equation yields:

$$v^2 - v_0^2 = b\left(x^2 - x_0^2\right)$$

Solve for v to obtain:

$$v = \sqrt{v_0^2 + b\left(x^2 - x_0^2\right)}$$

Now set $v_o = 0$, $x_o = 1.0$ m, $x = 3$ m, and $b = 2.0\ \text{s}^{-2}$ and evaluate the speed:

$$\left|v(3.0\ \text{m})\right| = \sqrt{\left(2.0\,\text{s}^{-2}\right)\left[(3.0\,\text{m})^2 - (1.0\,\text{m})^2\right]} = \boxed{4.0\ \text{m/s}}$$

(b) From the definition of instantaneous velocity we have:

$$t = \int_0^t dt = \int_{x_0}^{x} \frac{dx}{v(x)}$$

Substitute the expression for $v(x)$ derived in (a) to obtain:

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{b\left(x^2 - x_0^2\right)}}$$

Evaluate the integral using the formula found in standard integral tables to obtain:

$$t = \frac{1}{\sqrt{b}} \int_{x_0}^{x} \frac{dx}{\sqrt{x^2 - x_0^2}}$$

$$= \frac{1}{\sqrt{b}} \ln\left(\frac{x + \sqrt{x^2 - x_0^2}}{x_0}\right)$$

Substitute numerical values and evaluate t:

$$t = \frac{1}{\sqrt{2.0\,\text{s}^{-2}}} \ln\left(\frac{3.0\,\text{m} + \sqrt{(3.0\,\text{m})^2 - (1.0\,\text{m})^2}}{1.0\,\text{m}}\right) = \boxed{1.2\,\text{s}}$$

117 ••• [SSM] A rock falls through water with a continuously decreasing acceleration. Assume that the rock's acceleration as a function of *velocity* has the form $a_y = g - bv_y$ where b is a positive constant. (The $+y$ direction is directly downward.) (a) What are the SI units of b? (b) Prove mathematically that if the rock is released from rest at time $t = 0$, the acceleration will depend exponentially on *time* according to $a_y(t) = ge^{-bt}$. (c) What is the terminal speed for the rock in terms of g and b? (See Problem 38 for an explanation of the phenomenon of *terminal speed*.)

Picture the Problem Because the acceleration of the rock is a function of its velocity, it is not constant and we will have to integrate the acceleration function in order to find the velocity function. Choose a coordinate system in which downward is positive and the origin is at the point of release of the rock.

(a) All three terms in $a_y = g - bv_y$ must have the same units in order for the equation to be valid. Hence the units of bv_y must be acceleration units. Because the SI units of v_y are m/s, b must have units of $\boxed{\text{s}^{-1}.}$

(*b*) Rewrite $a_y = g - bv_y$ explicitly as a differential equation:

$$\frac{dv_y}{dt} = g - bv_y$$

Separate the variables, v_y on the left, t on the right:

$$\frac{dv_y}{g - bv_y} = dt$$

Integrate the left-hand side of this equation from 0 to v_y and the right-hand side from 0 to t:

$$\int_0^{v_y} \frac{dv_y}{g - bv_y} = \int_0^t dt$$

Integrating this equation yields:

$$-\frac{1}{b}\ln\left(\frac{g - bv_y}{g}\right) = t$$

Solve this expression for v_y to obtain:

$$v_y = \frac{g}{b}\left(1 - e^{-bt}\right) \qquad (1)$$

Differentiate this expression with respect to time to obtain an expression for the acceleration and complete the proof:

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left(\frac{g}{b}\left(1 - e^{-bt}\right)\right) = \boxed{ge^{-bt}}$$

(*c*) Take the limit, as $t \to \infty$, of both sides of equation (1):

$$\lim_{t\to\infty} v_y = \lim_{t\to\infty}\left[\frac{g}{b}\left(1 - e^{-bt}\right)\right]$$

and

$$v_t = \boxed{\frac{g}{b}}$$

Notice that this result depends only on *b* (inversely so). Thus *b* must include geometric factors like the shape and cross-sectional area of the falling object, as well as properties of the liquid such as density and temperature.

118 ••• A small rock sinking through water (see Problem 117) experiences an exponentially decreasing acceleration given by $a_y(t) = ge^{-bt}$, where *b* is a positive constant that depends on the shape and size of the rock and the physical properties of the water. Based upon this, find expressions for the velocity and position of the rock as functions of time. Assume that its initial position and velocity are both zero and that the $+y$ direction is directly downward.

Picture the Problem Because the acceleration of the rock is a function of time, it is not constant and we will have to integrate the acceleration function in order to find the velocity of the rock as a function of time. Similarly, we will have to integrate the velocity function in order to find the position of the rock as a function of time. Choose a coordinate system in which downward is positive and the origin at the point of release of the rock.

Separate variables in $a_y(t) = dv_y/dt = ge^{-bt}$ to obtain:

$$dv_y = ge^{-bt}dt$$

Integrating from $t_0 = 0$, $v_{0y} = 0$ to some later time t and velocity v_y yields:

$$v = \int_0^v dv = \int_0^t ge^{-bt}dt = \frac{g}{-b}\left[e^{-bt}\right]_0^t = \frac{g}{b}\left(1-e^{-bt}\right) = \boxed{v_\mathrm{t}\left(1-e^{-bt}\right)}$$

where $v_\mathrm{t} = \dfrac{g}{b}$

Separate variables in $v = \dfrac{dy}{dt} = v_\mathrm{t}\left(1-e^{-bt}\right)$ to obtain:

$$dy = v_\mathrm{t}\left(1-e^{-bt}\right)dt$$

Integrate from $t_\mathrm{o} = 0$, $y_\mathrm{o} = 0$ to some later time t and position y:

$$y = \int_0^y dy = \int_0^t v_\mathrm{t}\left(1-e^{-bt}\right)dt = v_\mathrm{term}\left[t + \frac{1}{b}e^{-bt}\right]_0^t = \boxed{v_\mathrm{t}t - \frac{v_\mathrm{t}}{b}\left(1-e^{-bt}\right)}$$

Remarks: This last result is very interesting. It says that throughout its free-fall, the object experiences drag; therefore it has not fallen as far at any given time as it would have if it were falling at the constant velocity, v_t.

119 ••• The acceleration of a skydiver jumping from an airplane is given by $a_y = g - bv_y^2$ where b is a positive constant depending on the skydiver's cross-sectional area and the density of the surrounding atmosphere she is diving through. The +y direction is directly downward. (a) If her initial speed is zero when stepping from a hovering helicopter, show that her speed as a function of time is given by $v_y(t) = v_\mathrm{t}\tanh(t/T)$ where v_t is the terminal speed (see Problem 38) given by $v_\mathrm{t} = \sqrt{g/b}$ and $T = v_\mathrm{t}/g$ is a time-scale parameter (b) What fraction of the terminal speed is the speed at $t = T$? (c) Use a **spreadsheet** program to graph $v_y(t)$ as a function of time, using a terminal speed of 56 m/s (use this value to calculate b and T). Does the resulting curve make sense?

Picture the Problem The skydiver's acceleration is a function of her speed; therefore it is not constant. Expressing her acceleration as the derivative of her speed, separating the variables, and then integrating will give her speed as a function of time.

(*a*) Rewrite $a_y = g - bv_y^2$ explicitly as a differential equation:

$$\frac{dv_y}{dt} = g - bv_y^2$$

Separate the variables, with v_y on the left, and t on the right:

$$\frac{dv_y}{g - bv_y^2} = dt$$

Eliminate b by using $b = \frac{g}{v_t^2}$:

$$\frac{dv_y}{g - \frac{g}{v_t^2}v_y^2} = \frac{dv_y}{g\left[1 - \left(\frac{v_y}{v_t}\right)^2\right]} = dt$$

or, separating variables,

$$\frac{dv_y}{1 - \left(\frac{v_y}{v_t}\right)^2} = gdt$$

Integrate the left-hand side of this equation from 0 to v_y and the right-hand side from 0 to t:

$$\int_0^{v_y} \frac{dv_y}{1 - \left(\frac{v_y}{v_t}\right)^2} = g\int_0^t dt = gt$$

The integral on the left-hand side can be found in integral tables:

$$v_t \tanh^{-1}\left(\frac{v_y}{v_t}\right) = gt$$

Solving this equation for v_y yields:

$$v_y = v_t \tanh\left(\frac{g}{v_t}\right)t$$

Because c has units of m^{-1}, and g has units of m/s^2, $(bg)^{-1/2}$ will have units of time. Let's represent this expression with the time-scale factor T:

$$T = (bg)^{-1/2}$$

The skydiver falls with her terminal speed when $a = 0$. Using this definition, relate her terminal speed to the acceleration due to gravity and the constant b in the acceleration equation:

$$0 = g - bv_t^2 \Rightarrow v_t = \sqrt{\frac{g}{b}}$$

Convince yourself that T is also equal to v_t/g and use this relationship to eliminate g and v_t in the solution to the differential equation:

$$v_y(t) = \boxed{v_t \tanh\left(\frac{t}{T}\right)}$$

(*b*) The ratio of the speed at any time to the terminal speed is given by:

$$\frac{v_y(t)}{v_t} = \frac{v_t \tanh\left(\frac{t}{T}\right)}{v_t} = \tanh\left(\frac{t}{T}\right)$$

Evaluate this ratio for $t = T$ to obtain:

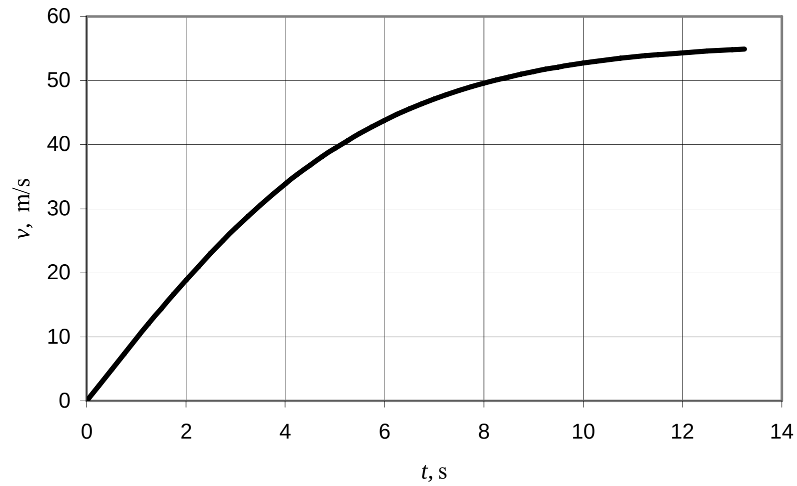
$$\frac{v_y(T)}{v_t} = \tanh\left(\frac{T}{T}\right) = \tanh(1) = \boxed{0.762}$$

(*c*) The following table was generated using a spreadsheet and the equation we derived in Part (*a*) for $v(t)$. The cell formulas and their algebraic forms are:

Cell	Content/Formula	Algebraic Form
D1	56	v_T
D2	5.71	T
B7	B6 + 0.25	$t + 0.25$
C7	$B$1*TANH(B7/$B$2)	$v_T \tanh\left(\frac{t}{T}\right)$

	A	B	C
1	v_T =	56	m/s
2	T=	5.71	s
3			
4			
5		t (s)	v (m/s)
6		0.00	0.00
7		0.25	2.45
8		0.50	4.89
9		0.75	7.32
56		12.50	54.61
57		12.75	54.73
58		13.00	54.83
59		13.25	54.93

A graph of v as a function of t follows:



Note that the speed increases linearly over time (that is, with constant acceleration) for about time T, but then it approaches the terminal speed as the acceleration decreases.

120 ••• Imagine that you are standing at a wishing well, wishing that you knew how deep the surface of the water was. Cleverly, you make your wish. Then, you take a penny from your pocket and drop it into the well. Exactly three seconds after you dropped the penny, you hear the sound it made when it struck the water. If the speed of sound is 343 m/s, how deep is the well? Neglect any effects due to air resistance.

Picture the Problem We know that the sound was heard exactly 3.00 s after the penny was dropped. This total time may be broken up into the time required for the penny to drop from your hand to the water's surface, and the time required for the sound to bounce back up to your ears. The time required for the penny to drop is related to the depth of the well. The use of a constant-acceleration equation in expressing the total fall time will lead to a quadratic equation one of whose roots will be the depth of the well

Express the time required for the sound to reach your ear as the sum of the drop time for the penny and the time for the sound to travel to your ear from the bottom of the well:

$$\Delta t_{\text{tot}} = \Delta t_{\text{drop}} + \Delta t_{\text{sound}} \qquad (1)$$

Use a constant-acceleration equation to express the depth of the well:

$$\Delta y_{\text{well}} = \tfrac{1}{2} g (\Delta t_{\text{drop}})^2 \Rightarrow \Delta t_{\text{drop}} = \sqrt{\frac{2\Delta y_{\text{well}}}{g}}$$

From the definition of average speed, the time Δt_{sound} required for the sound to travel from the bottom of the well to your ear is given by:

$$\Delta t_{\text{sound}} = \frac{\Delta y_{\text{well}}}{v_{\text{sound}}}$$

Substituting for Δt_{sound} and Δt_{drop} in equation (1) yields:

$$\Delta t_{\text{tot}} = \sqrt{\frac{2\Delta y_{\text{well}}}{g}} + \frac{\Delta y_{\text{well}}}{v_{\text{sound}}}$$

Isolate the radical term and square both sides of the equation to obtain:

$$\left(\Delta t_{\text{tot}} - \frac{\Delta y_{\text{well}}}{v_{\text{sound}}}\right)^2 = \frac{2\Delta y_{\text{well}}}{g}$$

Expanding the left side of this equation yields:

$$(\Delta t_{\text{tot}})^2 - 2\Delta t_{\text{tot}}\frac{\Delta y_{\text{well}}}{v_{\text{sound}}} + \left(\frac{\Delta y_{\text{well}}}{v_{\text{sound}}}\right)^2 = \frac{2\Delta y_{\text{well}}}{g}$$

Rewrite this equation explicitly as a quadratic equation in Δy_{well}:

$$\left(\frac{1}{v_{\text{sound}}}\right)^2 (\Delta y_{\text{well}})^2 - 2\left(\frac{1}{g} + \frac{\Delta t_{\text{tot}}}{v_{\text{sound}}}\right)\Delta y_{\text{well}} + (\Delta t_{\text{tot}})^2 = 0$$

Simplify further to obtain:

$$(\Delta y_{\text{well}})^2 - 2\left(\frac{v_{\text{sound}}^2}{g} + v_{\text{sound}}\Delta t_{\text{tot}}\right)\Delta y_{\text{well}} + (v_{\text{sound}}\Delta t_{\text{tot}})^2 = 0$$

Substituting numerical values yields:

$$(\Delta y_{\text{well}})^2 - 2\left(\frac{(343\text{ m/s})^2}{9.81\text{ m/s}^2} + (343\text{ m/s})(3.00\text{ s})\right)\Delta y_{\text{well}} + [(343\text{ m/s})(3.00\text{ s})]^2 = 0$$

or

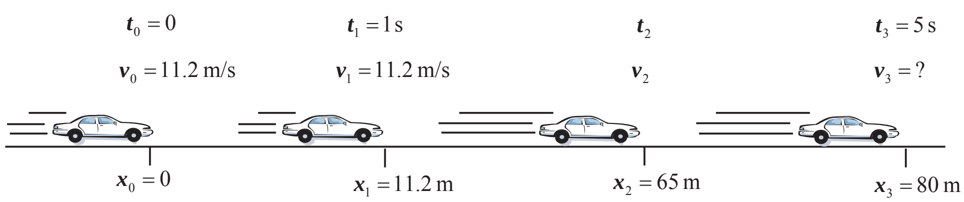
$$(\Delta y_{\text{well}})^2 - (2.604\times10^4\text{ m})\Delta y_{\text{well}} + 1.059\times10^6\text{ m}^2 = 0$$

Use the quadratic formula or your graphing calculator to solve for the smaller root of this equation (the larger root is approximately 26 km and is too large to be physically meaningful):

$$\Delta y_{\text{well}} = 40.73\text{ m} = \boxed{41\text{ m}}$$

121 ••• You are driving a car at the speed limit of 25-mi/h speed limit when you observe the light at the intersection 65 m in front of you turn yellow. You know that at that particular intersection the light remains yellow for exactly 5.0 s before turning red. After you think for 1.0 s, you then accelerate the car at a constant rate. You somehow manage to pass your 4.5-m-long car completely through the 15.0-m-wide intersection just as the light turns red, thus narrowly avoiding a ticket for being in an intersection when the light is red. Immediately after passing through the intersection, you take your foot off the accelerator, relieved. However, down the road, you are pulled over for speeding. You assume that you were ticketed for the speed of your car as it exited the intersection. Determine this speed and decide whether you should fight this ticket in court. Explain.

Picture the Problem First, we should find the total distance covered by the car between the time that the light turned yellow (and you began your acceleration) and the time that the back end of the car left the intersection. This distance is Δx_{tot} = 65.0 m + 15.0 m + 4.5 m = 84.5 m. Part of this distance, Δx_c is covered at constant speed, as you think about trying to make it through the intersection in time – this can be subtracted from the full distance Δx_{tot} above to yield Δx_{acc}, the displacement during the constant acceleration. With this information, we then can utilize a constant-acceleration equation to obtain an expression for your velocity as you exited the intersection. The following pictorial representation will help organize the information given in this problem.



Using a constant-acceleration equation relate the distance over which you accelerated to your initial velocity v_1, your acceleration a, and the time during which you accelerated a_{acc}:

$$\Delta x_{acc} = \Delta x_{tot} - \Delta x_c = v_1 \Delta t_{acc} + \tfrac{1}{2} a (\Delta t_{acc})^2$$

Relate the final velocity v_3 of the car to its velocity v_1 when it begins to accelerate, its acceleration, and the time during which it accelerates:

$$v_3 = v_1 + a\Delta t_{acc} \Rightarrow a = \frac{v_3 - v_1}{\Delta t_{acc}}$$

Substitute for a in the expression for Δx_{acc} to obtain:

$$\Delta x_{tot} - \Delta x_c = v_1 \Delta t_{acc} + \tfrac{1}{2}\left(\frac{v_3 - v_1}{\Delta t_{acc}}\right)(\Delta t_{acc})^2 = v_1 \Delta t_{acc} + \tfrac{1}{2}(v_3 - v_1)(\Delta t_{acc})$$

Solving for v_3 yields:

$$v_3 = \frac{2(\Delta x_{\text{tot}} - \Delta x_{\text{c}})}{\Delta t_{\text{acc}}} - v_1$$

Substitute numerical values and evaluate v_3:

$$v_3 = \frac{2(84.5\ \text{m} - 11.2\ \text{m})}{4.0\ \text{s}} - 11.2\ \text{m/s} = \boxed{25\ \text{m/s}}$$

Because 25 m/s is approximately 57 mi/h, you were approximately 32 mi/h over the speed limit! You would be foolish to contest your ticket.

122 ••• For a spherical celestial object of radius R, the acceleration due to gravity g at a distance x from the *center* of the object is $g = g_0 R^2/x^2$, where g_0 is the acceleration due to gravity at the object's surface and $x > R$. For the moon, take $g_0 = 1.63\ \text{m/s}^2$ and $R = 3200$ km. If a rock is released from rest at a height of $4R$ above the lunar surface, with what speed does the rock impact the moon? *Hint: Its acceleration is a function of position and increases as the object falls. So do not use constant acceleration free-fall equations, but go back to basics.*

Picture the Problem Let the origin be at the center of the moon and the $+x$ direction be radially outward. Because the acceleration of the rock is a function of its distance from the center of the moon, we'll need to change the variables of integration in the definition of acceleration to v and x in order to relate the rock's acceleration to its speed. Separating variables and integrating will yield an expression for the speed of the rock as a function of its distance from the center of the moon.

The acceleration of the object is given by:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Separating variables yields:

$$vdv = adx$$

Because $a = -g$:

$$vdv = -\frac{g_0 R^2}{x^2}dx$$

Expressing the integral of v from 0 to v and x from $4R$ to R yields:

$$\int_0^v vdv = \int_{4R}^{R} -\frac{g_0 R^2}{x^2}dx = -g_0 R^2 \int_{4R}^{R} \frac{dx}{x^2}$$

Carry out the integration to obtain:

$$\frac{v^2}{2} = g_0 R^2 \left[\left.\frac{1}{x}\right|_{4R}^{R}\right] = \frac{3}{4} g_0 R$$

Solve for v to obtain an expression for the speed of the object upon impact:

$$v = \sqrt{\frac{3}{2} g_0 R}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{3}{2}\left(1.63\ \text{m/s}^2\right)\left(1.738\times 10^6\ \text{m}\right)}$$

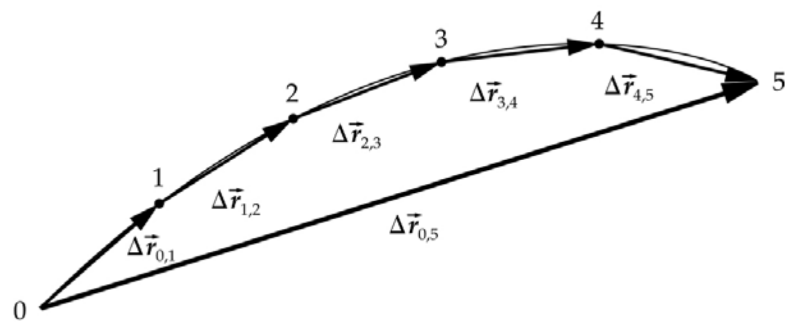
$$= \boxed{2.06\ \text{km/s}}$$

Chapter 3
Motion in Two and Three Dimensions

Conceptual Problems

1 • [SSM] Can the magnitude of the displacement of a particle be less than the distance traveled by the particle along its path? Can its magnitude be more than the distance traveled? Explain.

Determine the Concept The distance traveled along a path can be represented as a sequence of displacements.



Suppose we take a trip along some path and consider the trip as a sequence of many very small displacements. The net displacement is the vector sum of the very small displacements, and the total distance traveled is the sum of the magnitudes of the very small displacements. That is,

$$\textit{total distance} = \left|\Delta\vec{r}_{0,1}\right| + \left|\Delta\vec{r}_{1,2}\right| + \left|\Delta\vec{r}_{2,3}\right| + \ldots + \left|\Delta\vec{r}_{N-1,N}\right|$$

where N is the number of very small displacements. (For this to be exactly true we have to take the limit as N goes to infinity and each displacement magnitude goes to zero.) Now, using ″the shortest distance between two points is a straight line,″ we have

$$\left|\Delta\vec{r}_{0,N}\right| \le \left|\Delta\vec{r}_{0,1}\right| + \left|\Delta\vec{r}_{1,2}\right| + \left|\Delta\vec{r}_{2,3}\right| + \ldots + \left|\Delta\vec{r}_{N-1,N}\right|,$$

where $\left|\Delta\vec{r}_{0,N}\right|$ is the magnitude of the net displacement.

Hence, we have shown that the magnitude of the displacement of a particle is less than or equal to the distance it travels along its path.

2 • Give an example in which the distance traveled is a significant amount, yet the corresponding displacement is zero. Can the reverse be true? If so, give an example.

Determine the Concept The displacement of an object is its final position vector minus its initial position vector ($\Delta\vec{r} = \vec{r}_f - \vec{r}_i$). The displacement can be less but never more than the distance traveled. Suppose the path is one complete trip around Earth at the equator. Then, the displacement is 0 but the distance traveled is $2\pi R_e$. No, the reverse cannot be true.

3 • What is the average velocity of a batter who hits a home run (from when he hits the ball to when he touches home plate after rounding the bases)?

Determine the Concept The important distinction here is that ***average velocity*** is being requested, as opposed to ***average speed***.

The average velocity is defined as the displacement divided by the elapsed time.

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{0}{\Delta t} = 0$$

The displacement for any trip around the bases is zero. Thus we see that no matter how fast the runner travels, the average velocity is always zero at the end of each complete circuit of the bases.

What is the correct answer if we were asked for *average speed*?

The average speed is defined as the distance traveled divided by the elapsed time.

$$v_{av} = \frac{\text{total distance}}{\Delta t}$$

For one complete circuit of any track, the total distance traveled will be greater than zero and so the average speed is not zero.

4 • A baseball is hit so its initial velocity upon leaving the bat makes an angle of 30° above the horizontal. It leaves that bat at a height of 1.0 m above the ground and lands untouched for a single. During its flight, from just after it leaves the bat to just before it hits the ground, describe how the angle between its velocity and acceleration vectors changes. Neglect any effects due to air resistance.

Determine the Concept The angle between its velocity and acceleration vectors starts at 30° + 90° or 120° because the acceleration of the ball is straight down. At the peak of the flight of the ball the angle reduces to 90° because the ball's velocity vector is horizontal. When the ball reaches the same elevation that it started from the angle is 90° − 30° or 60°.

5 • If an object is moving toward the west at some instant, in what direction is its acceleration? (*a*) north, (*b*) east, (*c*) west, (*d*) south, (*e*) may be any direction.

Determine the Concept The *instantaneous acceleration* is the limiting value, as Δt approaches zero, of $\Delta\vec{v}/\Delta t$ and is in the same direction as $\Delta\vec{v}$.

Other than through the definition of $\vec{a}$, the instantaneous velocity and acceleration vectors are unrelated. Knowing the direction of the velocity at one instant tells one nothing about how the velocity is changing at that instant. $\boxed{(e)}$ is correct.

6 • Two astronauts are working on the lunar surface to install a new telescope. The acceleration due to gravity on the Moon is only 1.64 m/s^2. One astronaut tosses a wrench to the other astronaut but the speed of throw is excessive and the wrench goes over her colleague's head. When the wrench is at the highest point of its trajectory (*a*) its velocity and acceleration are both zero, (*b*) its velocity is zero but its acceleration is nonzero, (*c*) its velocity is nonzero but its acceleration is zero, (*d*) its velocity and acceleration are both nonzero, (*e*) insufficient information is given to choose between any of the previous choices.

Determine the Concept When the wrench reaches its maximum height, it is still moving horizontally but its acceleration is downward. $\boxed{(d)}$ is correct.

7 • The velocity of a particle is directed toward the east while the acceleration is directed toward the northwest as shown in Figure 3-27. The particle is (*a*) speeding up and turning toward the north, (*b*) speeding up and turning toward the south, (*c*) slowing down and turning toward the north, (*d*) slowing down and turning toward the south, (*e*) maintaining constant speed and turning toward the south.

Determine the Concept The change in the velocity is in the same direction as the acceleration. Choose an *x*-*y* coordinate system with east being the positive *x* direction and north the positive *y* direction. Given our choice of coordinate system, the *x* component of $\vec{a}$ is negative and so $\vec{v}$ will decrease. The *y* component of $\vec{a}$ is positive and so $\vec{v}$ will increase toward the north. $\boxed{(c)}$ is correct.

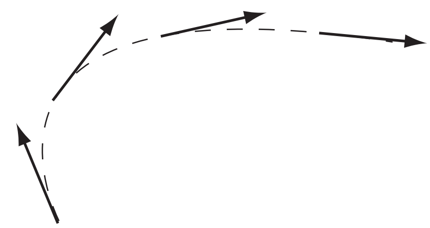
8 • Assume you know the position vectors of a particle at two points on its path, one earlier and one later. You also know the time it took the particle to move from one point to the other. Then you can then compute the particle's (*a*) average velocity, (*b*) average acceleration, (*c*) instantaneous velocity, (*d*) instantaneous acceleration.

Determine the Concept All you can compute is the average velocity, since no instantaneous quantities can be computed and you need two instantaneous velocities to compute the average acceleration. $\boxed{(a)}$ is correct.

9 • Consider the path of a moving particle. (*a*) How is the velocity vector related geometrically to the path of the particle? (*b*) Sketch a curved path and draw the velocity vector for the particle for several positions along the path.

Determine the Concept (*a*) The velocity vector, as a consequence of always being in the direction of motion, is tangent to the path.

(*b*) A sketch showing two velocity vectors for a particle moving along a curved path is shown to the right.



10 • The acceleration of a car is zero when it is (*a*) turning right at a constant speed, (*b*) driving up a long straight incline at constant speed, (*c*) traveling over the crest of a hill at constant speed, (*d*) bottoming out at the lowest point of a valley at constant speed, (*e*) speeding up as it descends a long straight decline.

Determine the Concept An object experiences acceleration whenever either its speed changes or it changes direction.

The acceleration of a car moving in a straight path at constant speed is zero. In the other examples, either the magnitude or the direction of the velocity vector is changing and, hence, the car is accelerated. $\boxed{(b)}$ is correct.

11 • **[SSM]** Give examples of motion in which the directions of the velocity and acceleration vectors are (*a*) opposite, (*b*) the same, and (*c*) mutually perpendicular.

Determine the Concept The velocity vector is defined by $\vec{v} = d\vec{r}/dt$, while the acceleration vector is defined by $\vec{a} = d\vec{v}/dt$.

(*a*) A car moving along a straight road while braking.

(*b*) A car moving along a straight road while speeding up.

(*c*) A particle moving around a circular track at constant speed.

12 • How is it possible for a particle moving at constant speed to be accelerating? Can a particle with constant velocity be accelerating at the same time?

Determine the Concept A particle experiences accelerated motion when either its speed or direction of motion changes.

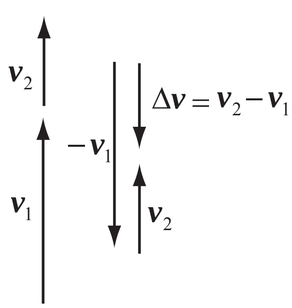
A particle moving at constant speed in a circular path is accelerating because the direction of its velocity vector is changing.

If a particle is moving at constant velocity, it is not accelerating.

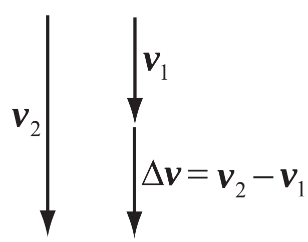
13 • **[SSM]** Imagine throwing a dart straight upward so that it sticks into the ceiling. After it leaves your hand, it steadily slows down as it rises before it sticks. (*a*) Draw the dart's velocity vector at times t_1 and t_2, where t_1 and t_2 occur after it leaves your hand but before it impacts the ceiling, and $\Delta t = t_2 - t_1$ is small. From your drawing find the direction of the change in velocity $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, and thus the direction of the acceleration vector. (*b*) After it has stuck in the ceiling for a few seconds, the dart falls down to the floor. As it falls it speeds up, of course, until it hits the floor. Repeat Part (*a*) to find the direction of its acceleration vector as it falls. (*c*) Now imagine tossing the dart horizontally. What is the direction of its acceleration vector after it leaves your hand, but before it strikes the floor?

Determine the Concept The acceleration vector is in the same direction as the *change in velocity vector,* $\Delta\vec{v}$.

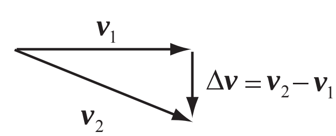
(*a*) The sketch for the dart thrown upward is shown below. The acceleration vector is in the direction of the *change* in the velocity vector $\Delta\vec{v}$.



(*b*) The sketch for the falling dart is shown below. Again, the acceleration vector is in the direction of the *change* in the velocity vector $\Delta\vec{v}$.



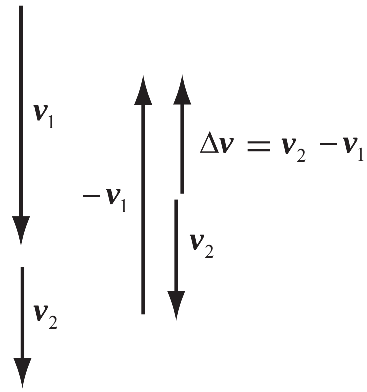
(*c*) The acceleration vector is in the direction of the *change* in the velocity vector … and hence is downward as shown below:



14 • As a bungee jumper approaches the lowest point in her descent, the rubber cord holding her stretches and she loses speed as she continues to move downward. Assuming that she is dropping straight down, make a motion diagram to find the direction of her acceleration vector as she slows down by drawing her velocity vectors at times t_1 and t_2, where t_1 and t_2 are two instants during the portion of her descent that she is losing speed and $t_2 - t_1$ is small. From your

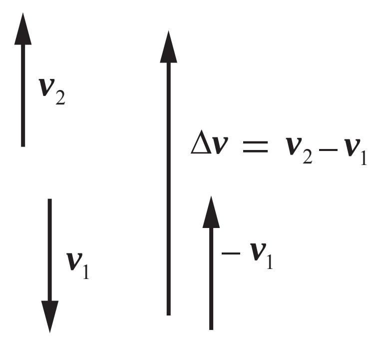
drawing find the direction of the change in velocity $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, and thus the direction of the acceleration vector.

Determine the Concept The acceleration vector is in the same direction as the *change in velocity vector*, $\Delta\vec{v}$. The drawing is shown to the right.



15 • After reaching the lowest point in her jump at time t_{low}, a bungee jumper moves upward, gaining speed for a short time until gravity again dominates her motion. Draw her velocity vectors at times t_1 and t_2, where $\Delta t = t_2 - t_1$ is small and $t_1 < t_{low} < t_2$. From your drawing find the direction of the change in velocity $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, and thus the direction of the acceleration vector at time t_{low}.

Determine the Concept The acceleration vector is in the same direction as the *change in the velocity vector,* $\Delta\vec{v}$. The drawing is shown to the right.



16 • A river is 0.76 km wide. The banks are straight and parallel (Figure 3-28). The current is 4.0 km/h and is parallel to the banks. A boat has a maximum speed of 4.0 km/h in still water. The pilot of the boat wishes to go on a straight line from A to B, where the line AB is perpendicular to the banks. The pilot should (*a*) head directly across the river, (*b*) head 53° upstream from the line AB, (*c*) head 37° upstream from the line AB, (*d*) give up—the trip from A to B is not possible with a boat of this limited speed, (*e*) do none of the above.

Determine the Concept We can decide what the pilot should do by considering the speeds of the boat and of the current. The speed of the stream is equal to the maximum speed of the boat in still water. The best the boat can do is, while facing directly upstream, maintain its position relative to the bank. So, the pilot should give up. $\boxed{(d)}$ is correct.

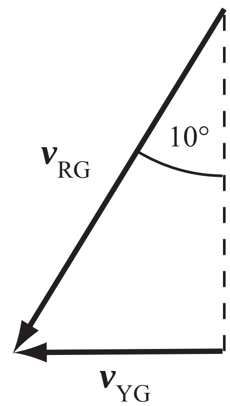
17 • **[SSM]** During a heavy rain, the drops are falling at a constant velocity and at an angle of 10° west of the vertical. You are walking in the rain and notice that only the top surfaces of your clothes are getting wet. In what direction are you walking? Explain.

Determine the Concept You must be walking west to make it appear to you that the rain is exactly vertical.

18 • In Problem 17, what is your walking speed if the speed of the drops relative to the ground is 5.2 m/s?

Determine the Concept Let $\vec{v}_{YG}$ represent your velocity relative to the ground and $\vec{v}_{RG}$ represent the velocity of the rain relative to the ground. Then your speed relative to the ground is given by

$$v_{YG} = (5.2\text{ m/s})\sin 10° = \boxed{0.90\text{ m/s}}$$



19 • True or false (Ignore any effects due to air resistance):

(*a*) When a projectile is fired horizontally, it takes the same amount of time to reach the ground as an identical projectile dropped from rest from the same height.
(*b*) When a projectile is fired from a certain height at an upward angle, it takes longer to reach the ground than does an identical projectile dropped from rest from the same height.
(*c*) When a projectile is fired horizontally from a certain height, it has a higher speed upon reaching the ground than does an identical projectile dropped from rest from the same height.

(*a*) True. In the absence of air resistance, both projectiles experience the same downward acceleration. Because both projectiles have initial vertical velocities of zero, their vertical motions must be identical.

(*b*) True. When a projectile is fired from a certain height at an upward angle, its time in the air is larger because it first rises to a greater height before beginning to fall with zero initial vertical velocity.

(*c*) True. When a projectile is fired horizontally, its velocity upon reaching the ground has a horizontal component in addition to the vertical component it has when it is dropped from rest. The magnitude of this velocity is related to its horizontal and vertical components through the Pythagorean Theorem.

20 • A projectile is fired at 35º above the horizontal. Any effects due to air resistance are negligible. At the highest point in its trajectory, its speed is 20 m/s. The initial velocity had a horizontal component of (*a*) 0, (*b*) (20 m/s) cos 35º, (*c*) (20 m/s) sin 35º, (*d*) (20 m/s)/cos 35º, (*e*) 20 m/s.

Determine the Concept In the absence of air resistance, the horizontal component of the projectile's velocity is constant for the duration of its flight. At the highest point, the speed is the horizontal component of the initial velocity. The vertical component is zero at the highest point. $\boxed{(e)}$ is correct.

21 • **[SSM]** A projectile is fired at 35º above the horizontal. Any effects due to air resistance are negligible. The initial velocity of the projectile in Problem 20 has a vertical component that is (*a*) less than 20 m/s, (*b*) greater than 20 m/s, (*c*) equal to 20 m/s, (*d*) cannot be determined from the data given.

Determine the Concept $\boxed{(a)}$ is correct. Because the initial horizontal velocity is 20 m/s and the launch angle is less than 45 degrees, the initial vertical velocity must be less than 20 m/s.

22 • A projectile is fired at 35º above the horizontal. Any effects due to air resistance are negligible. The projectile lands at the same elevation of launch, so the vertical component of the impact velocity is (*a*) the same as the vertical component of its initial velocity in both magnitude and direction, (*b*) the same as the vertical component of its initial velocity, (*c*) less than the vertical component of its initial velocity, (*d*) less than the vertical component of its initial velocity.

Determine the Concept $\boxed{(b)}$ is correct. The landing speed is the same as the launch speed. Because the horizontal component of its initial velocity does not change, the vertical component of the velocity at landing must be the same magnitude but oppositely directed its vertical component at launch.

23 • Figure 3-29 represents the trajectory of a projectile going from A to E. Air resistance is negligible. What is the direction of the acceleration at point B? (*a*) up and to the right, (*b*) down and to the left, (*c*) straight up, (*d*) straight down, (*e*) The acceleration of the ball is zero.

Determine the Concept $\boxed{(d)}$ is correct. In the absence of air resistance, the acceleration of the ball depends only on the *change in its velocity* and is independent of its velocity. As the ball moves along its trajectory between points A and C, the vertical component of its velocity decreases and the *change* in its velocity is a downward pointing vector. Between points C and E, the vertical component of its velocity increases and the *change* in its velocity is also a downward pointing vector. There is no change in the horizontal component of the velocity.

24 • Figure 3-29 represents the trajectory of a projectile going from A to E. Air resistance is negligible. (*a*) At which point(s) is the speed the greatest? (*b*) At which point(s) is the speed the least? (*c*) At which two points is the speed the same? Is the velocity also the same at these points?

Determine the Concept In the absence of air resistance, the horizontal component of the velocity remains constant throughout the flight. The vertical component has its maximum values at launch and impact.

(*a*) The speed is greatest at A and E.

(*b*) The speed is least at point C.

(*c*) The speed is the same at A and E. No. The horizontal components are equal at these points but the vertical components are oppositely directed, so the velocity is not the same at A and E.

25 • **[SSM]** True or false:

(*a*) If an object's speed is constant, then its acceleration must be zero.
(*b*) If an object's acceleration is zero, then its speed must be constant.
(*c*) If an object's acceleration is zero, its velocity must be constant.
(*d*) If an object's speed is constant, then its velocity must be constant.
(*e*) If an object's velocity is constant, then its speed must be constant.

Determine the Concept Speed is a scalar quantity, whereas acceleration, equal to the *rate of change* of velocity, is a vector quantity.

(*a*) False. Consider a ball moving in a horizontal circle on the end of a string. The ball can move with constant *speed* (a scalar) even though its *acceleration* (a vector) is always changing direction.

(*b*) True. From its definition, if the acceleration is zero, the velocity must be constant and so, therefore, must be the speed.

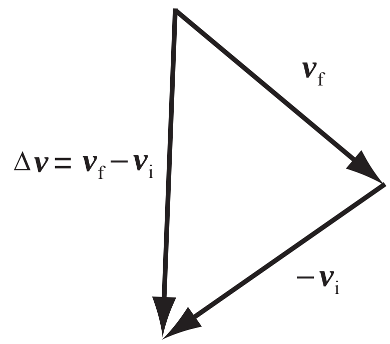
(*c*) True. An object’s velocity must change in order for the object to have acceleration other than zero.

(*d*) False. Consider an object moving at constant speed along a circular path. Its velocity changes continuously along such a path.

(*e*) True. If the velocity of an object is constant, then both its direction and magnitude (speed) must be constant.

26 • The initial and final velocities of a particle are as shown in Figure 3-30. Find the direction of the average acceleration.

Determine the Concept The average acceleration vector is defined by $\vec{a}_{av} = \Delta\vec{v}/\Delta t$. The direction of $\vec{a}_{av}$ is that of $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$, as shown to the right.



27 •• The automobile path shown in Figure 3-31 is made up of straight lines and arcs of circles. The automobile starts from rest at point A. After it reaches point B, it travels at constant speed until it reaches point E. It comes to rest at point F. (*a*) At the middle of each segment (AB, BC, CD, DE, and EF), what is the direction of the velocity vector? (*b*) At which of these points does the automobile have a nonzero acceleration? In those cases, what is the direction of the acceleration? (*c*) How do the magnitudes of the acceleration compare for segments BC and DE?

Determine the Concept The velocity vector is in the same direction as *the change in the position vector* while the acceleration vector is in the same direction as *the change in the velocity vector*. Choose a coordinate system in which the y direction is north and the x direction is east.

(*a*)

Path	Direction of velocity vector
AB	north
BC	northeast
CD	east
DE	southeast
EF	south

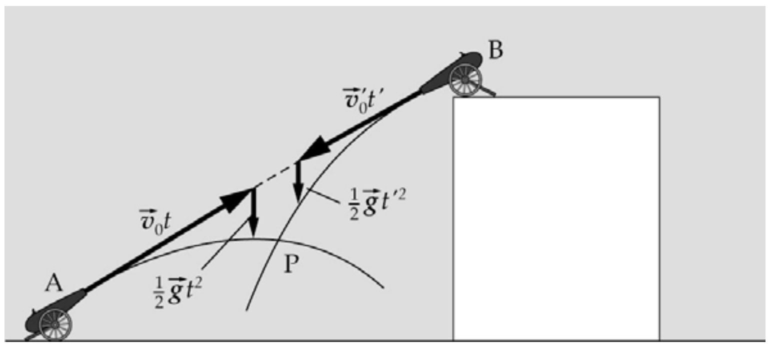
(*b*)

Path	Direction of acceleration vector
AB	north
BC	southeast
CD	0
DE	southwest
EF	north

(*c*) The magnitudes are comparable, but larger for DE because the radius of the path is smaller there.

28 •• Two cannons are pointed directly toward each other as shown in Figure 3-32. When fired, the cannonballs will follow the trajectories shown—P is the point where the trajectories cross each other. If we want the cannonballs to hit each other, should the gun crews fire cannon A first, cannon B first, or should they fire simultaneously? Ignore any effects due to air resistance.

Determine the Concept We'll assume that the cannons are identical and use a constant-acceleration equation to express the displacement of each cannonball as a function of time. Having done so, we can then establish the condition under which they will have the same vertical position at a given time and, hence, collide. The modified diagram shown below shows the displacements of both cannonballs.



Express the displacement of the cannonball from cannon A at any time t after being fired and before any collision:

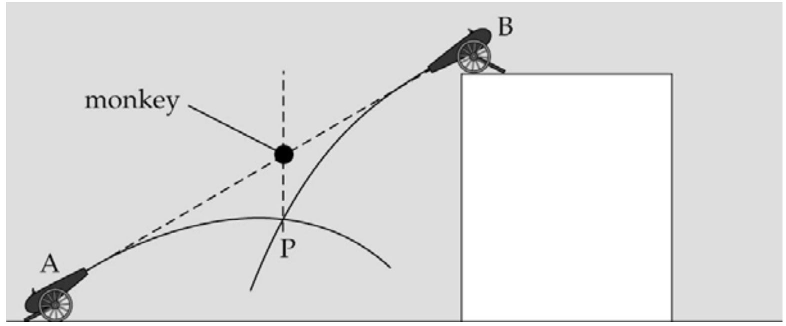
$$\Delta\vec{r} = \vec{v}_0t + \tfrac{1}{2}\vec{g}t^2$$

Express the displacement of the cannonball from cannon A at any time t' after being fired and before any collision:

$$\Delta\vec{r}\,' = \vec{v}_0't' + \tfrac{1}{2}\vec{g}t'^2$$

If the guns are fired simultaneously, $t = t'$ and the balls are the same distance $\frac{1}{2}gt^2$ below the line of sight at all times. Also, because the cannons are identical, the cannonballs have the same horizontal component of velocity and will reach the horizontal midpoint at the same time. Therefore, they should fire the guns simultaneously.

Remarks: This is the ″monkey and hunter″ problem in disguise. If you imagine a monkey in the position shown below, and the two guns are fired simultaneously, and the monkey begins to fall when the guns are fired, then the monkey and the two cannonballs will all reach point P at the same time.

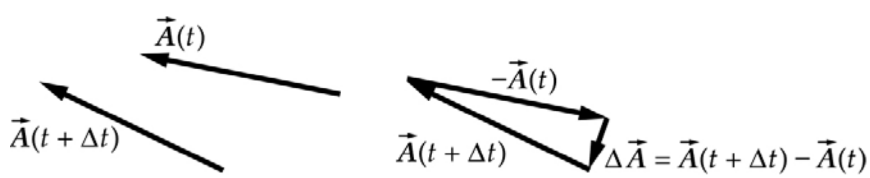


29 •• Galileo wrote the following in his ***Dialogue concerning the two world systems****:* "Shut yourself up . . . in the main cabin below decks on some large ship, and . . . hang up a bottle that empties drop by drop into a wide vessel beneath it. When you have observed [this] carefully . . . have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the drops are in the air the ship runs many spans." Explain this quotation.

Determine the Concept The droplet leaving the bottle has the same horizontal velocity as the ship. During the time the droplet is in the air, it is also moving horizontally with the same velocity as the rest of the ship. Because of this, it falls into the vessel, which has the same horizontal velocity. Because you have the same horizontal velocity as the ship does, you see the same thing as if the ship were standing still.

30 • A man swings a stone attached to a rope in a horizontal circle at constant speed. Figure 3-33 represents the path of the rock looking down from above. (*a*) Which of the vectors could represent the velocity of the stone? (*b*) Which could represent the acceleration?

Determine the Concept (*a*) Because $\vec{A}$ and $\vec{D}$ are tangent to the path of the stone, either of them could represent the velocity of the stone.



(*b*) Let the vectors $\vec{A}(t)$ and $\vec{A}(t+\Delta t)$ be of equal length but point in slightly different directions as the stone moves around the circle. These two vectors and $\Delta\vec{A}$ are shown in the diagram above. Note that $\Delta\vec{A}$ is nearly perpendicular to $\vec{A}(t)$. For very small time intervals, $\Delta\vec{A}$ and $\vec{A}(t)$ are perpendicular to one another. Therefore, $d\vec{A}/dt$ is perpendicular to $\vec{A}$ and only the vector $\vec{E}$ could represent the acceleration of the stone.

31 •• True or false:

(*a*) An object cannot move in a circle unless it has centripetal acceleration.
(*b*) An object cannot move in a circle unless it has tangential acceleration.
(*c*) An object moving in a circle cannot have a variable speed.
(*d*) An object moving in a circle cannot have a constant velocity.

(*a*) True. An object accelerates when its velocity changes; that is, when either its speed or its direction changes. When an object moves in a circle the direction of its motion is continually changing.

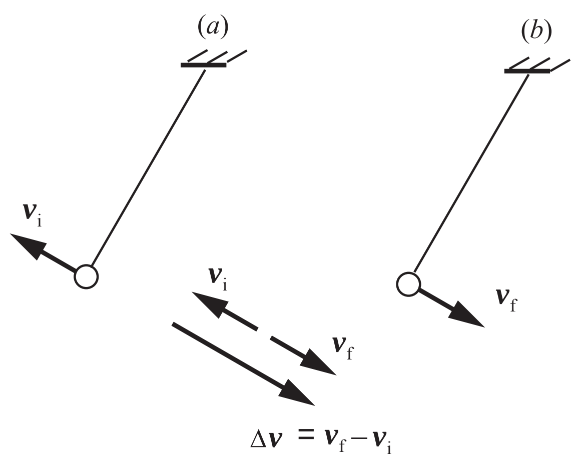
(*b*) False. An object moving in a circular path at constant speed has a tangential acceleration of zero.

(*c*) False. A good example is a rock wedged in the tread of an automobile tire. Its speed changes whenever the car's speed changes.

(*d*) True. The velocity vector of any object moving in a circle is continually changing direction.

32 •• Using a motion diagram, find the direction of the acceleration of the bob of a pendulum when the bob is at a point where it is just reversing its direction.

Picture the Problem In the diagram, (*a*) shows the pendulum just before it reverses direction and (*b*) shows the pendulum just after it has reversed its direction. The acceleration of the bob is in the direction of the *change* in the velocity $\Delta\vec{v} = \vec{v}_{\mathrm{f}} - \vec{v}_{\mathrm{i}}$ and is tangent to the pendulum trajectory at the point of reversal of direction. This makes sense because, at an extremum of motion, $v = 0$, so there is no centripetal acceleration. However, because the velocity is reversing direction, the tangential acceleration is nonzero.

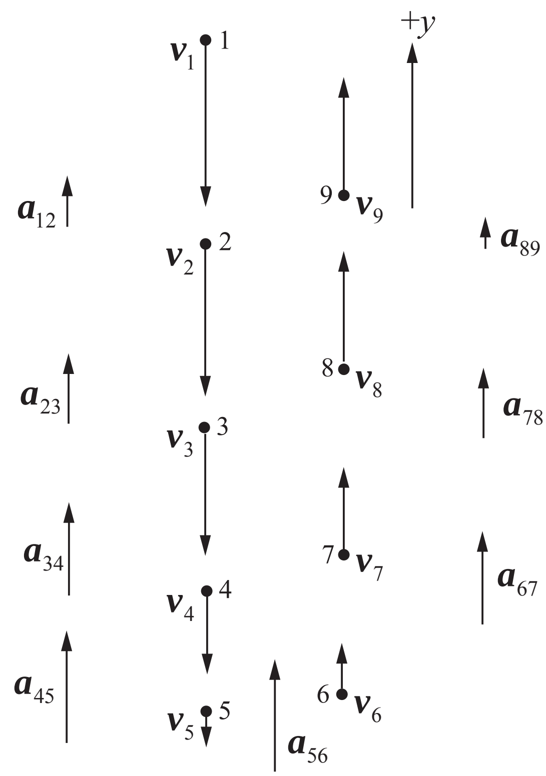


33 •• **[SSM]** During your rookie bungee jump, your friend records your fall using a camcorder. By analyzing it frame by frame, he finds that the *y*-component of your velocity is (recorded every 1/20th of a second) as follows:

t (s)	12.05	12.10	12.15	12.20	12.25	12.30	12.35	12.40	12.45
v_y (m/s)	–0.78	–0.69	–0.55	–0.35	–0.10	0.15	0.35	0.49	0.53

(*a*) Draw a motion diagram. Use it to find the direction and relative magnitude of your average acceleration for each of the eight successive 0.050 s time intervals in the table. (*b*) Comment on how the *y* component of your acceleration does or does not vary in sign and magnitude as you reverse your direction of motion.

Determine the Concept (*a*) The motion diagram shown below was constructed using the data in the table shown below the motion diagram. The column for Δv in the table was calculated using $\Delta v = v_i - v_{i-1}$ and the column for a was calculated using $a = (v_i - v_{i-1})/\Delta t$.



i	v	Δv	a_{ave}
	(m/s)	(m/s)	(m/s^2)
1	–0.78		
2	–0.69	0.09	1.8
3	–0.55	0.14	2.8
4	–0.35	0.20	4.0
5	–0.10	0.25	5.0
6	0.15	0.25	5.0

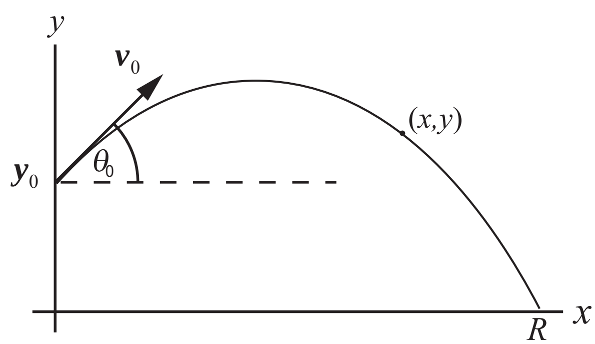
7	0.35	0.20	4.0
8	0.49	0.14	2.8
9	0.53	0.04	0.8

(*b*) The acceleration vector always points upward and so the sign of its *y* component does not change. The magnitude of the acceleration vector is greatest when the bungee cord has its maximum extension (your speed, the magnitude of your velocity, is least at this time and times near it) and is less than this maximum value when the bungee cord has less extension.

Estimation and Approximation

34 •• Estimate the speed in mph with which water comes out of a garden hose using your past observations of water coming out of garden hoses and your knowledge of projectile motion.

Picture the Problem Based on your experience with garden hoses, you probably know that the maximum range of the water is achieved when the hose is inclined at about 45° with the vertical. A reasonable estimate of the range of such a stream is about 4.0 m when the initial height of the stream is 1.0 m. Use constant-acceleration equations to obtain expressions for the *x* and *y* coordinates of a droplet of water in the stream and then eliminate time between these equations to obtain an equation that you can solve for v_0.



Use constant-acceleration equations to express the *x* and *y* components of a molecule of water in the stream:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$
and
$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

Because $x_0 = 0$, $x = R$, $a_y = -g$, and $a_x = 0$:

$$x = v_{0x}t \qquad (1)$$
and
$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2 \qquad (2)$$

Express v_{0x} and v_{0y} in terms of v_0 and θ:

$v_{0x} = v_0 \cos\theta_0$ and $v_{0y} = v_0 \sin\theta_0$

where θ_0 is the angle the stream makes with the horizontal.

Substitute in equations (1) and (2) to obtain:

$$x = (v_0 \cos\theta_0)t \qquad (3)$$

and

$$y = y_0 + (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \qquad (4)$$

Eliminating t from equations (3) and (4) yields:

$$y = y_0 + (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

When the stream of water hits the ground, $y = 0$ and $x = R$:

$$0 = y_0 + (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Solving this equation for v_0 gives:

$$v_0 = R\sqrt{\frac{g}{2\cos^2\theta\left[R\tan\theta + y_0\right]}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = (4.0\text{ m})\sqrt{\frac{9.81\text{ m/s}^2}{2\cos^2 45°\left[(4.0\text{ m})\tan 45° + 1.0\text{ m}\right]}} = 5.603\text{ m/s}$$

Use a conversion factor, found in Appendix A, to convert m/s to mi/h:

$$v_0 = 5.603\text{ m/s} \times \frac{1\text{ mi/h}}{0.4470\text{ m/s}} \approx \boxed{13\text{ mi/h}}$$

35 •• [SSM] You won a contest to spend a day with all team during their spring training camp. You are allowed to try to hit some balls thrown by a pitcher. Estimate the acceleration during the hit of a fastball thrown by a major league pitcher when you hit the ball squarely-straight back at the pitcher. You will need to make reasonable choices for ball speeds, both just before and just after the ball is hit, and of the contact time the ball has with the bat.

Determine the Concept The magnitude of the acceleration of the ball is given by $a = |\vec{a}| = \left|\frac{\Delta\vec{v}}{\Delta t}\right|$. Let $\vec{v}_{\text{after}}$ represent the velocity of the ball just after its collision with the bat and $\vec{v}_{\text{before}}$ its velocity just before this collision. Most major league pitchers can throw a fastball at least 90 mi/h and some occasionally throw as fast as 100 mi/h. Let's assume that the pitcher throws you an 80 mph fastball.

The magnitude of the acceleration of the ball is:

$$a=\left|\frac{\vec{v}_{\text{after}}-\vec{v}_{\text{before}}}{\Delta t}\right|.$$

Assuming that v_{after} and v_{before} are both 80 mi/h and that the ball is in contact with the bat for 1 ms:

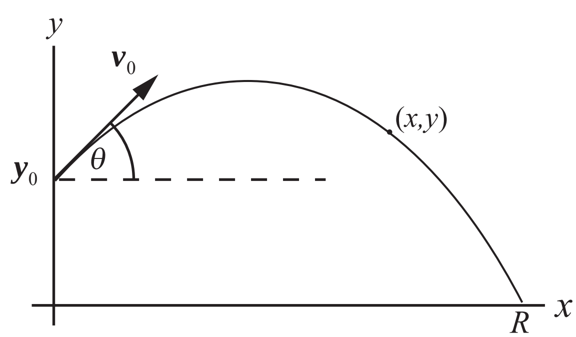
$$a=\frac{80\ \text{mi/h}-(-80\ \text{mi/h})}{1\ \text{ms}}=\frac{160\ \text{mi/h}}{1\ \text{ms}}$$

Converting a to m/s^2 yields:

$$a=\frac{160\ \text{mi/h}}{1\ \text{ms}}\times\frac{0.447\ \text{m/s}}{1\ \text{mi/h}}\approx\boxed{7\times10^4\ \text{m/s}^2}$$

36 •• Estimate how far you can throw a ball if you throw it (*a*) horizontally while standing on level ground, (*b*) at $\theta = 45°$ above horizontal while standing on level ground, (*c*) horizontally from the top of a building 12 m high, (*d*) at $\theta = 45°$ above horizontal from the top of a building 12 m high. Ignore any effects due to air resistance.

Picture the Problem During the flight of the ball the acceleration is constant and equal to 9.81 m/s^2 directed downward. We can find the flight time from the vertical part of the motion, and then use the horizontal part of the motion to find the horizontal distance. We'll assume that the release point of the ball is 2.0 m above your feet. A sketch of the motion that includes coordinate axes, the initial and final positions of the ball, the launch angle, and the initial velocity follows.



Obviously, how far you throw the ball will depend on how fast you can throw it. A major league baseball pitcher can throw a fastball at 90 mi/h or so. Assume that you can throw a ball at two-thirds that speed to obtain:

$$v_0=60\ \text{mi/h}\times\frac{0.447\ \text{m/s}}{1\ \text{mi/h}}\approx 27\ \text{m/s}$$

There is no acceleration in the x direction, so the horizontal motion is one of constant velocity. Express the horizontal position of the ball as a function of time:

$$x = v_{0x}t = (v_0 \cos\theta_0)t \qquad (1)$$

Assuming that the release point of the ball is a distance y_0 above the ground, express the vertical position of the ball as a function of time:

$$\begin{aligned} y &= y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= y_0 + (v_0 \sin\theta_0)t + \tfrac{1}{2}a_y t^2 \end{aligned} \qquad (2)$$

Eliminating t between equations (1) and (2) yields:

$$y = y_0 + (\tan\theta_0)x + \left(\frac{a_y}{2v_0^2\cos^2\theta_0}\right)x^2$$

(a) If you throw the ball horizontally our equation becomes:

$$y = y_0 + \left(\frac{a_y}{2v_0^2}\right)x^2$$

Substitute numerical values to obtain:

$$y = 2.0\text{ m} + \left(\frac{-9.81\text{ m/s}^2}{2(27\text{ m/s})^2}\right)x^2$$

At impact, $y = 0$ and $x = R$:

$$0 = 2.0\text{ m} + \left(\frac{-9.81\text{ m/s}^2}{2(27\text{ m/s})^2}\right)R^2$$

Solving for R yields:

$$R \approx \boxed{17\text{ m}}$$

(b) For $\theta = 45°$ we have:

$$y = 2.0\text{ m} + (\tan 45°)x + \left(\frac{-9.81\text{ m/s}^2}{2(27\text{ m/s})^2\cos^2 45°}\right)x^2$$

At impact, $y = 0$ and $x = R$:

$$0 = 2.0\text{ m} + (\tan 45°)R + \left(\frac{-9.81\text{ m/s}^2}{2(27\text{ m/s})^2\cos^2 45°}\right)R^2$$

Use the quadratic formula or your graphing calculator to obtain:

$$R = \boxed{75\text{ m}}$$

(c) If you throw the ball horizontally from the top of a building that is 12 m high our equation becomes:

$$y = 14.0\text{ m} + \left(\frac{-9.81\text{ m/s}^2}{2(27\text{ m/s})^2}\right)x^2$$

At impact, $y = 0$ and $x = R$:

$$0 = 14.0\,\text{m} + \left(\frac{-9.81\,\text{m/s}^2}{2(27\,\text{m/s})^2}\right)R^2$$

Solve for R to obtain:

$$R = \boxed{45\,\text{m}}$$

(*d*) If you throw the ball at an angle of 45° from the top of a building that is 12 m high our equation becomes:

$$y = 14\,\text{m} + (\tan 45°)x + \left(\frac{-9.81\,\text{m/s}^2}{2(27\,\text{m/s})^2\cos^2 45°}\right)x^2$$

At impact, $y = 0$ and $x = R$:

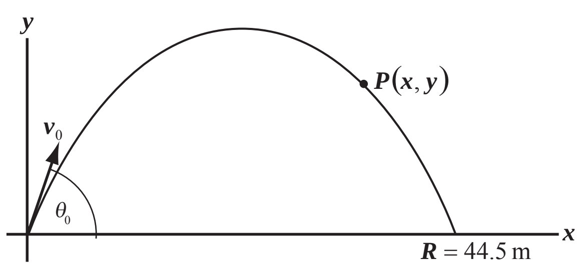
$$0 = 14\,\text{m} + (\tan 45°)R + \left(\frac{-9.81\,\text{m/s}^2}{2(27\,\text{m/s})^2\cos^2 45°}\right)R^2$$

Use the quadratic formula or your graphing calculator to obtain:

$$R = \boxed{85\,\text{m}}$$

37 •• In 1978, Geoff Capes of Great Britain threw a heavy brick a horizontal distance of 44.5 m. Find the approximate speed of the brick at the highest point of its flight, neglecting any effects due to air resistance. Assume the brick landed at the same height it was launched.

Picture the Problem We'll ignore the height of Geoff's release point above the ground and assume that he launched the brick at an angle of 45°. Because the velocity of the brick at the highest point of its flight is equal to the horizontal component of its initial velocity, we can use constant-acceleration equations to relate this velocity to the brick's x and y coordinates at impact. The diagram shows an appropriate coordinate system and the brick when it is at point P with coordinates (x, y).



Using a constant-acceleration equation, express the x and y coordinates of the brick as a function of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

and

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

Because $x_0 = 0$, $y_0 = 0$, $a_y = -g$, and $a_x = 0$:

$$x = v_{0x}t$$
and
$$y = v_{0y}t - \tfrac{1}{2}gt^2$$

Eliminate t between these equations to obtain:

$$y = (\tan\theta_0)x - \frac{g}{2v_{0x}^2}x^2$$

where we have used $\tan\theta_0 = \dfrac{v_{0y}}{v_{0x}}$.

When the brick strikes the ground $y = 0$ and $x = R$:

$$0 = (\tan\theta_0)R - \frac{g}{2v_{0x}^2}R^2$$

where R is the range of the brick.

Solve for v_{0x} to obtain:

$$v_{0x} = \sqrt{\frac{gR}{2\tan\theta_0}}$$

Substitute numerical values and evaluate v_{0x}:

$$v_{0x} = \sqrt{\frac{(9.81\,\text{m/s}^2)(44.5\,\text{m})}{2\tan 45°}} \approx \boxed{15\,\text{m/s}}$$

Note that, at the brick's highest point, $v_y = 0$.

Position, Displacement, Velocity and Acceleration Vectors

38 • A wall clock has a minute hand with a length of 0.50 m and an hour hand with a length of 0.25 m. Take the center of the clock as the origin, and use a Cartesian coordinate system with the positive x axis pointing to 3 o'clock and the positive y axis pointing to 12 o'clock. Using unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, express the position vectors of the tip of the hour hand ($\vec{\boldsymbol{A}}$) and the tip of the minute hand ($\vec{\boldsymbol{B}}$) when the clock reads (*a*) 12:00, (*b*) 3:00, (*c*) 6:00, (*d*) 9:00.

Picture the Problem Let the +y direction be straight up and the +x direction be to the right.

(*a*) At 12:00, both hands are positioned along the +y axis.

The position vector for the tip of the hour hand at12:00 is:

$$\vec{\boldsymbol{A}} = \boxed{(0.25\,\text{m})\hat{\boldsymbol{j}}}$$

The position vector for the tip of the minute hand at 12:00 is:

$$\vec{\boldsymbol{B}} = \boxed{(0.50\,\text{m})\hat{\boldsymbol{j}}}$$

(*b*) At 3:00, the minute hand is positioned along the +*y* axis, while the hour hand is positioned along the +*x* axis.

The position vector for the tip of the hour hand at 3:00 is: $\vec{A} = \boxed{(0.25\,\text{m})\hat{i}}$

The position vector for the tip of the minute hand at 3:00 is: $\vec{B} = \boxed{(0.50\,\text{m})\hat{j}}$

(*c*) At 6:00, the minute hand is positioned along the −*y* axis, while the hour hand is positioned along the +*y* axis.

The position vector for the tip of the hour hand at 6:00 is: $\vec{A} = \boxed{-(0.25\,\text{m})\hat{j}}$

The position vector for the tip of the minute hand at 6:00 is: $\vec{B} = \boxed{(0.50\,\text{m})\hat{j}}$

(*d*) At 9:00, the minute hand is positioned along the +*y* axis, while the hour hand is positioned along the −*x* axis.

The position vector for the tip of the hour hand at 9:00 is: $\vec{A} = \boxed{-(0.25\,\text{m})\hat{i}}$

The position vector for the tip of the minute hand at 9:00 is: $\vec{B} = \boxed{(0.50\,\text{m})\hat{j}}$

39 • [SSM] In Problem 38, find the displacements of the tip of each hand (that is, $\Delta\vec{A}$ and $\Delta\vec{B}$) when the time advances from 3:00 P.M. to 6:00 P.M.

Picture the Problem Let the +*y* direction be straight up, the +*x* direction be to the right, and use the vectors describing the ends of the hour and minute hands in Problem 38 to find the displacements $\Delta\vec{A}$ and $\Delta\vec{B}$.

The displacement of the minute hand as time advances from 3:00 P.M. to 6:00 P.M. is given by: $\Delta\vec{B} = \vec{B}_6 - \vec{B}_3$

From Problem 38: $\vec{B}_6 = (0.50\,\text{m})\hat{j}$ and $\vec{B}_3 = (0.50\,\text{m})\hat{j}$

Substitute and simplify to obtain: $\Delta\vec{B} = (0.50\,\text{m})\hat{j} - (0.50\,\text{m})\hat{j} = \boxed{0}$

The displacement of the hour hand as time advances from 3:00 P.M. to 6:00 P.M. is given by:

$$\Delta\vec{A} = \vec{A}_6 - \vec{A}_3$$

From Problem 38:

$$\vec{A}_6 = -(0.25\,\text{m})\hat{j} \text{ and } \vec{A}_3 = (0.25\,\text{m})\hat{i}$$

Substitute and simplify to obtain:

$$\Delta\vec{A} = \boxed{-(0.25\,\text{m})\hat{j} - (0.25\,\text{m})\hat{i}}$$

40 • In Problem 38, write the vector that describes the displacement of a fly if it quickly goes from the tip of the minute hand to the tip of the hour hand at 3:00 P.M.

Picture the Problem Let the positive y direction be straight up, the positive x direction be to the right, and use the vectors describing the ends of the hour and minute hands in Problem 38 to find the displacement $\vec{D}$ of the fly as it goes from the tip of the minute hand to the tip of the hour hand at 3:00 P.M.

The displacement of the fly is given by:

$$\vec{D} = \vec{A}_3 - \vec{B}_3$$

From Problem 38:

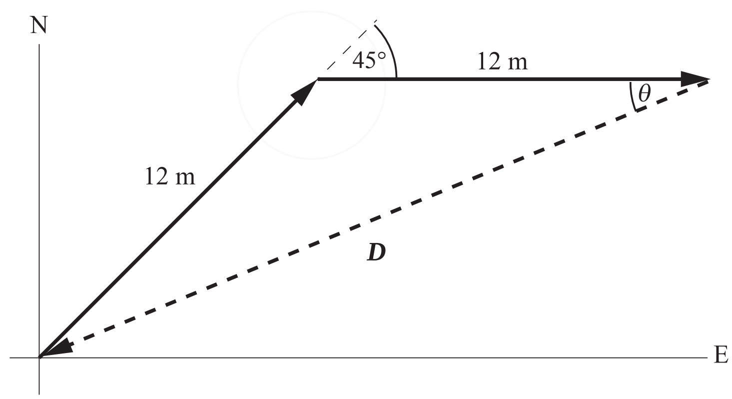
$$\vec{A}_3 = (0.25\,\text{m})\hat{i} \text{ and } \vec{B}_3 = (0.50\,\text{m})\hat{j}$$

Substitute for $\vec{A}_3$ and $\vec{B}_3$ to obtain:

$$\vec{D} = \boxed{(0.25\,\text{m})\,\hat{i} - (0.50\,\text{m})\hat{j}}$$

41 • A bear, awakening from winter hibernation, staggers directly northeast for 12 m and then due east for 12 m. Show each displacement graphically and graphically determine the single displacement that will take the bear back to her cave, to continue her hibernation.

Picture the Problem The single displacement for the bear to make it back to its cave is the vector $\vec{D}$. Its magnitude D and direction θ can be determined by drawing the bear's displacement vectors to scale.

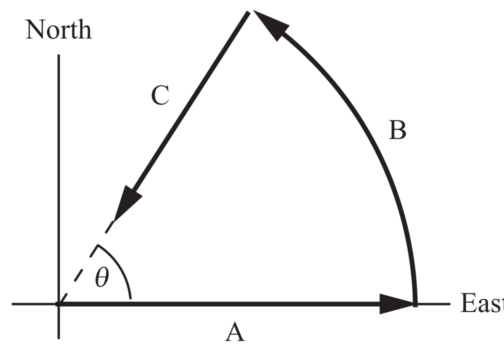


$$D \approx \boxed{22\,\text{m}} \text{ and } \boldsymbol{\theta} \approx \boxed{23°}$$

Remarks: The direction of $\vec{D}$ is 180° + θ ≈ 203°

42 • A scout walks 2.4 km due East from camp, then turns left and walks 2.4 km along the arc of a circle centered at the campsite, and finally walks 1.5 km directly toward the camp. (*a*) How far is the scout from camp at the end of his walk? (*b*) In what direction is the scout's position relative to the campsite? (*c*) What is the ratio of the final magnitude of the displacement to the total distance walked?

Picture the Problem The figure shows the paths walked by the Scout. The length of path A is 2.4 km; the length of path B is 2.4 km; and the length of path C is 1.5 km:



(*a*) Express the distance from the campsite to the end of path C:

$$2.4\text{ km} - 1.5\text{ km} = \boxed{0.9\,\text{km}}$$

(*b*) Determine the angle θ subtended by the arc at the origin (campsite):

$$\theta_{\text{radians}} = \frac{\text{arc length}}{\text{radius}} = \frac{2.4\,\text{km}}{2.4\,\text{km}}$$

$$= 1\,\text{rad} = 57.3°$$

His direction from camp is 1 rad North of East.

(*c*) Express the total distance as the sum of the three parts of his walk:

$$d_{\text{tot}} = d_{\text{east}} + d_{\text{arc}} + d_{\text{toward camp}}$$

Substitute the given distances to find the total:

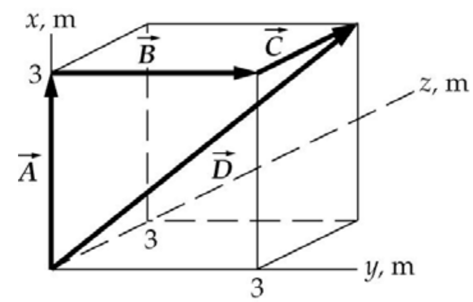
$$d_{\text{tot}} = 2.4\text{ km} + 2.4\text{ km} + 1.5\text{ km}$$
$$= 6.3\text{ km}$$

Express the ratio of the magnitude of his displacement to the total distance he walked and substitute to obtain a numerical value for this ratio:

$$\frac{\text{Magnitude of his displacement}}{\text{Total distance walked}} = \frac{0.9\,\text{km}}{6.3\,\text{km}} = \boxed{\frac{1}{7}}$$

43 •• [SSM] The faces of a cubical storage cabinet in your garage has 3.0-m-long edges that are parallel to the *xyz* coordinate planes. The cube has one corner at the origin. A cockroach, on the hunt for crumbs of food, begins at that corner and walks along three edges until it is at the far corner. (*a*) Write the roach's displacement using the set of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$ unit vectors, and (*b*) find the magnitude of its displacement.

Picture the Problem While there are several walking routes the cockroach could take to get from the origin to point C, its displacement will be the same for all of them. One possible route is shown in the figure.



(*a*) The roach's displacement $\vec{D}$ during its trip from the origin to point C is:

$$\vec{D} = \vec{A} + \vec{B} + \vec{C}$$
$$= \boxed{(3.0\,\text{m})\hat{\boldsymbol{i}} + (3.0\,\text{m})\hat{\boldsymbol{j}} + (3.0\,\text{m})\hat{\boldsymbol{k}}}$$

(*b*) The magnitude of the roach's displacement is given by:

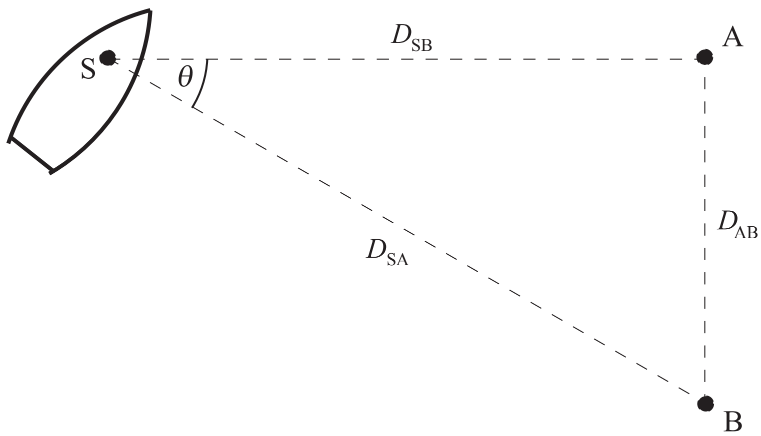
$$D = \sqrt{D_x^2 + D_y^2 + D_z^2}$$

Substitute numerical values for D_x, D_y, and D_z and evaluate D to obtain:

$$D = \sqrt{(3.0\,\text{m})^2 + (3.0\,\text{m})^2 + (3.0\,\text{m})^2}$$
$$= \boxed{5.2\,\text{m}}$$

44 • You are the navigator of a ship at sea. You receive radio signals from two transmitters A and B, which are 100 km apart, one due south of the other. The direction finder shows you that transmitter A is at a heading of 30º south of east from the ship, while transmitter B is due east. Calculate the distance between your ship and transmitter B.

Picture the Problem The diagram shows the locations of the transmitters relative to the ship and defines the distances separating the transmitters from each other and from the ship. We can find the distance between the ship and transmitter B using trigonometry.



Relate the distance between A and B to the distance from the ship to A and the angle θ:

$$\tan\theta = \frac{D_{AB}}{D_{SB}} \Rightarrow D_{SB} = \frac{D_{AB}}{\tan\theta}$$

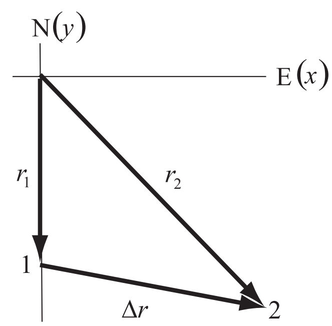
Substitute numerical values and evaluate D_{SB}:

$$D_{SB} = \frac{100\,\text{km}}{\tan 30°} = \boxed{1.7\times10^{5}\,\text{m}}$$

Velocity and Acceleration Vectors

45 • A stationary radar operator determines that a ship is 10 km due south of him. An hour later the same ship is 20 km due southeast. If the ship moved at constant speed and always in the same direction, what was its velocity during this time?

Picture the Problem For constant speed and direction, the instantaneous velocity is identical to the average velocity. Take the origin to be the location of the stationary radar and let the $+x$ direction be to the East and the $+y$ direction be to the North.



Express the average velocity:

$$\vec{\boldsymbol{v}}_{av} = \frac{\Delta\vec{\boldsymbol{r}}}{\Delta t} \qquad (1)$$

Determine the position vectors $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$:

$$\vec{\boldsymbol{r}}_1 = (-10\,\text{km})\hat{\boldsymbol{j}}$$

and

$$\vec{\boldsymbol{r}}_2 = (14.1\,\text{km})\hat{\boldsymbol{i}} + (-14.1\,\text{km})\hat{\boldsymbol{j}}$$

Find the displacement vector $\Delta\vec{r}$:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (14.1\,\text{km})\hat{i} + (-4.1\,\text{km})\hat{j}$$

Substitute for $\Delta\vec{r}$ and Δt in equation (1) to find the average velocity.

$$\vec{v}_{av} = \frac{(14.1\,\text{km})\hat{i} + (-4.1\,\text{km})\hat{j}}{1.0\,\text{h}} = \boxed{(14\,\text{km/h})\hat{i} + (-4.1\,\text{km/h})\hat{j}}$$

46 • A particle's position coordinates (x, y) are (2.0 m, 3.0 m) at $t = 0$; (6.0 m, 7.0 m) at $t = 2.0$ s; and (13 m, 14 m) at $t = 5.0$ s. (*a*) Find the magnitude of the average velocity from $t = 0$ to $t = 2.0$ s. (*b*) Find the magnitude of the average velocity from t = 0 to t = 5.0 s.

Picture the Problem The average velocity is the change in position divided by the elapsed time.

(*a*) The magnitude of the average velocity is given by:

$$|\vec{v}_{av}| = \frac{|\Delta\vec{r}|}{\Delta t}$$

Find the position vectors and the displacement vector:

$$\vec{r}_0 = (2.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j}$$

$$\vec{r}_2 = (6.0\text{ m})\hat{i} + (7.0\text{ m})\hat{j}$$

and

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (4.0\text{ m})\hat{i} + (4.0\text{ m})\hat{j}$$

Find the magnitude of the displacement vector for the interval between $t = 0$ and $t = 2.0$ s:

$$|\Delta\vec{r}_{02}| = \sqrt{(4.0\text{ m})^2 + (4.0\text{ m})^2} = 5.66\text{ m}$$

Substitute to determine $|\vec{v}_{av}|$:

$$|\vec{v}_{av}| = \frac{5.66\,\text{m}}{2.0\,\text{s}} = \boxed{2.8\,\text{m/s}}$$

θ is given by:

$$\theta = \tan^{-1}\left(\frac{\Delta r_{y,02}}{\Delta r_{x,02}}\right) = \tan^{-1}\left(\frac{4.0\,\text{m}}{4.0\,\text{m}}\right) = \boxed{45°}$$

measured from the positive x axis.

(*b*) Repeat (*a*), this time using the displacement between $t = 0$ and $t = 5.0$ s to obtain:

$$\vec{r}_5 = (13\,\text{m})\hat{\boldsymbol{i}} + (14\,\text{m})\hat{\boldsymbol{j}}$$

$$\Delta\vec{r}_{05} = \vec{r}_5 - \vec{r}_0 = (11\,\text{m})\hat{\boldsymbol{i}} + (11\,\text{m})\hat{\boldsymbol{j}}.$$

$$|\Delta\vec{r}_{05}| = \sqrt{(11\,\text{m})^2 + (11\,\text{m})^2} = 15.6\,\text{m}$$

$$|\vec{v}_{\text{av}}| = \frac{15.6\,\text{m}}{5.0\,\text{s}} = \boxed{3.1\,\text{m/s}}.$$

and

$$\theta = \tan^{-1}\left(\frac{11\,\text{m}}{11\,\text{m}}\right) = \boxed{45°}$$

measured from the $+x$ axis.

47 • **[SSM]** A particle moving at a velocity of 4.0 m/s in the $+x$ direction is given an acceleration of 3.0 m/s^2 in the $+y$ direction for 2.0 s. Find the final speed of the particle.

Picture the Problem The magnitude of the velocity vector at the end of the 2 s of acceleration will give us its speed at that instant. This is a constant-acceleration problem.

Find the final velocity vector of the particle:

$$\begin{aligned}\vec{v} &= v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} = v_{x0}\hat{\boldsymbol{i}} + a_y t\hat{\boldsymbol{j}}\\ &= (4.0\,\text{m/s})\hat{\boldsymbol{i}} + (3.0\,\text{m/s}^2)(2.0\,\text{s})\hat{\boldsymbol{j}}\\ &= (4.0\,\text{m/s})\hat{\boldsymbol{i}} + (6.0\,\text{m/s})\hat{\boldsymbol{j}}\end{aligned}$$

The magnitude of $\vec{v}$ is:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

Substitute for v_x and v_y and evaluate $|\vec{v}|$:

$$|\vec{v}| = \sqrt{(4.0\,\text{m/s})^2 + (6.0\,\text{m/s})^2} = \boxed{7.2\,\text{m/s}}$$

48 • Initially, a swift-moving hawk is moving due west with a speed of 30 m/s; 5.0 s later it is moving due north with a speed of 20 m/s. (*a*) What are the magnitude and direction of $\Delta\vec{v}_{\text{av}}$ during this 5.0-s interval? (*b*) What are the magnitude and direction of $\vec{a}_{\text{av}}$ during this 5.0-s interval?

Picture the Problem Choose a coordinate system in which north coincides with the $+y$ direction and east with the $+x$ direction. Expressing the hawk's velocity vectors is the first step in determining $\Delta\vec{v}$ and $\vec{a}_{\text{av}}$.

(*a*) The change in the hawk's velocity during this interval is:

$$\Delta\vec{v}_{\text{av}} = \vec{v}_{\text{N}} - \vec{v}_{\text{W}}$$

$\vec{v}_{\text{W}}$ and $\vec{v}_{\text{N}}$ are given by:

$$\vec{v}_{\text{W}} = -(30\text{ m/s})\hat{i} \text{ and } \vec{v}_{\text{N}} = (20\text{ m/s})\hat{j}$$

Substitute for $\vec{v}_{\text{W}}$ and $\vec{v}_{\text{N}}$ and evaluate $\Delta\vec{v}$:

$$\begin{aligned}\Delta\vec{v}_{\text{av}} &= (20\text{ m/s})\hat{j} - \left[-(30\text{ m/s})\hat{i}\right]\\ &= (30\text{ m/s})\hat{i} + (20\text{ m/s})\hat{j}\end{aligned}$$

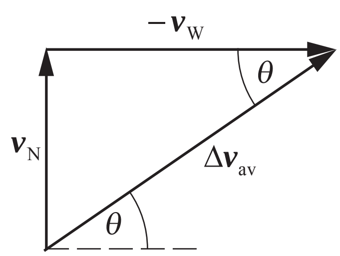
The magnitude of $\Delta\vec{v}_{\text{av}}$ is given by:

$$\left|\Delta\vec{v}_{\text{av}}\right| = \sqrt{\Delta v_x^2 + \Delta v_y^2}$$

Substitute numerical values and evaluate Δv:

$$\begin{aligned}\left|\Delta\vec{v}_{\text{av}}\right| &= \sqrt{(30\text{ m/s})^2 + (20\text{ m/s})^2}\\ &= \boxed{36\text{ m/s}}\end{aligned}$$

The direction of Δv is given by:



$$\theta = \tan^{-1}\left(\frac{\Delta v_y}{\Delta v_x}\right)$$

where θ is measured from the $+x$ axis.

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{20\text{ m/s}}{30\text{ m/s}}\right) = \boxed{34°}$$

(*b*) The hawk's average acceleration during this interval is:

$$\vec{a}_{\text{av}} = \frac{\Delta\vec{v}_{\text{av}}}{\Delta t}$$

Substitute for $\Delta\vec{v}$ and Δt to obtain:

$$\begin{aligned}\vec{a}_{\text{av}} &= \frac{(30\text{ m/s})\hat{i} + (20\text{ m/s})\hat{j}}{5.0\text{ s}}\\ &= \left(6.0\text{ m/s}^2\right)\hat{i} + \left(4.0\text{ m/s}^2\right)\hat{j}\end{aligned}$$

The magnitude of $\vec{a}_{\text{av}}$ is given by:

$$\left|\vec{a}_{\text{av}}\right| = \sqrt{a_x^2 + a_y^2}$$

Substitute numerical values and evaluate $\left|\vec{a}_{\text{av}}\right|$:

$$\begin{aligned}\left|\vec{a}_{\text{av}}\right| &= \sqrt{\left(6.0\text{ m/s}^2\right) + \left(4.0\text{ m/s}^2\right)}\\ &= \boxed{7.2\text{ m/s}^2}\end{aligned}$$

The direction of $\vec{a}_{\text{av}}$ is given by:

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{4.0\ \text{m/s}^2}{6.0\ \text{m/s}^2}\right) = \boxed{34°}$$

where θ is measured from the $+x$ axis.

Remarks: Because $\vec{a}_{\text{av}}$ is in the same direction as $\Delta\vec{v}_{\text{av}}$, the calculation of θ in Part (*b*) was not necessary.

49 • At $t = 0$, a particle located at the origin has a velocity of 40 m/s at $\theta = 45°$. At $t = 3.0$ s, the particle is at $x = 100$ m and $y = 80$ m and has a velocity of 30 m/s at $\theta = 50°$. Calculate (*a*) the average velocity and (*b*) the average acceleration of the particle during this 3.0-s interval.

Picture the Problem The initial and final positions and velocities of the particle are given. We can find the average velocity and average acceleration using their definitions by first calculating the given displacement and velocities using unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

(*a*) The average velocity of the particle is the ratio of its displacement to the elapsed time:

$$\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t}$$

The displacement of the particle during this interval of time is:

$$\Delta\vec{r} = (100\,\text{m})\hat{\boldsymbol{i}} + (80\,\text{m})\hat{\boldsymbol{j}}$$

Substitute to find the average velocity:

$$\vec{v}_{\text{av}} = \frac{(100\,\text{m})\hat{\boldsymbol{i}} + (80\,\text{m})\hat{\boldsymbol{j}}}{3.0\,\text{s}} = (33.3\,\text{m/s})\hat{\boldsymbol{i}} + (26.7\,\text{m/s})\hat{\boldsymbol{j}} = \boxed{(33\,\text{m/s})\hat{\boldsymbol{i}} + (27\,\text{m/s})\hat{\boldsymbol{j}}}$$

(*b*) The average acceleration is:

$$\vec{a}_{\text{av}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

$\vec{v}_1$ and $\vec{v}_2$ are given by:

$$\vec{v}_1 = [(40\,\text{m/s})\cos 45°]\hat{\boldsymbol{i}} + [(40\,\text{m/s})\sin 45°]\hat{\boldsymbol{j}} = (28.28\,\text{m/s})\hat{\boldsymbol{i}} + (28.28\,\text{m/s})\hat{\boldsymbol{j}}$$

and

$$\vec{v}_2 = [(30\,\text{m/s})\cos 50°]\hat{\boldsymbol{i}} + [(30\,\text{m/s})\sin 50°]\hat{\boldsymbol{j}} = (19.28\,\text{m/s})\hat{\boldsymbol{i}} + (22.98\,\text{m/s})\hat{\boldsymbol{j}}$$

Substitute for $\vec{v}_1$ and $\vec{v}_2$ and evaluate $\vec{a}_{\text{av}}$:

$$\vec{a}_{\text{av}} = \frac{(19.28\,\text{m/s})\hat{i} + (22.98\,\text{m/s})\hat{j} - \left[(28.28\,\text{m/s})\hat{i} + (28.28\,\text{m/s})\hat{j}\right]}{3.0\,\text{s}}$$

$$= \frac{(-9.00\,\text{m/s})\hat{i} + (-5.30\,\text{m/s})\hat{j}}{3.0\,\text{s}} = \boxed{(-3.0\,\text{m/s}^2)\hat{i} + (-1.8\,\text{m/s}^2)\hat{j}}$$

50 •• At time zero, a particle is at $x = 4.0$ m and $y = 3.0$ m and has velocity $\vec{v}$ = (2.0 m/s) $\hat{i}$ + (–9.0 m/s) $\hat{j}$. The acceleration of the particle is constant and is given by $\vec{a}$ = (4.0 m/s²) $\hat{i}$ + (3.0 m/s²) $\hat{j}$. (*a*) Find the velocity at $t = 2.0$ s. (*b*) Express the position vector at $t = 4.0$ s in terms of $\hat{i}$ and $\hat{j}$. In addition, give the magnitude and direction of the position vector at this time.

Picture the Problem The acceleration is constant so we can use the constant-acceleration equations in vector form to find the velocity at $t = 2.0$ s and the position vector at $t = 4.0$ s.

(*a*) The velocity of the particle, as a function of time, is given by:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

Substitute for $\vec{v}_0$ and $\vec{a}$ to find $\vec{v}(2.0\,\text{s})$:

$$\vec{v}(2.0\,\text{s}) = (2.0\,\text{m/s})\hat{i} + (-9.0\,\text{m/s})\hat{j} + \left[(4.0\,\text{m/s}^2)\hat{i} + (3.0\,\text{m/s}^2)\hat{j}\right](2.0\,\text{s})$$

$$= \boxed{(10\,\text{m/s})\hat{i} + (-3.0\,\text{m/s})\hat{j}}$$

(*b*) Express the position vector as a function of time:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

Substitute numerical values and evaluate $\vec{r}$:

$$\vec{r} = (4.0\,\text{m})\hat{i} + (3.0\,\text{m})\hat{j} + \left[(2.0\,\text{m/s})\hat{i} + (-9.0\,\text{m/s})\hat{j}\right](4.0\,\text{s})$$
$$+ \tfrac{1}{2}\left[(4.0\,\text{m/s}^2)\hat{i} + (3.0\,\text{m/s}^2)\hat{j}\right](4.0\,\text{s})^2$$
$$= (44.0\,\text{m})\hat{i} + (-9.0\,\text{m})\hat{j}$$
$$= \boxed{(44\,\text{m})\hat{i} + (-9.0\,\text{m})\hat{j}}$$

The magnitude of $\vec{r}$ is given by $r = \sqrt{r_x^2 + r_y^2}$. Substitute numerical values and evaluate r(4.0 s):

$$r(4.0\,\text{s}) = \sqrt{(44.0\,\text{m})^2 + (-9.0\,\text{m})^2}$$
$$= \boxed{45\,\text{m}}$$

The direction of $\vec{r}$ is given by $\theta = \tan^{-1}\left(\dfrac{r_y}{r_x}\right)$**.** Because $\vec{r}$ is in the 4th quadrant, its direction measured from the $+x$ axis is:

$$\theta = \tan^{-1}\left(\frac{-9.0\,\text{m}}{44.0\,\text{m}}\right) = -11.6° = \boxed{-12°}$$

51 •• [SSM] A particle has a position vector given by $\vec{r} = (30t)\hat{\boldsymbol{i}} + (40t - 5t^2)\hat{\boldsymbol{j}}$, where r is in meters and t is in seconds. Find the instantaneous-velocity and instantaneous-acceleration vectors as functions of time t.

Picture the Problem The velocity vector is the time-derivative of the position vector and the acceleration vector is the time-derivative of the velocity vector.

Differentiate $\vec{r}$ with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(30t)\hat{\boldsymbol{i}} + \left(40t - 5t^2\right)\hat{\boldsymbol{j}}\right]$$

$$= \boxed{30\hat{\boldsymbol{i}} + (40 - 10t)\hat{\boldsymbol{j}}}$$

where $\vec{v}$ has units of m/s if t is in seconds.

Differentiate $\vec{v}$ with respect to time:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[30\hat{\boldsymbol{i}} + (40 - 10t)\hat{\boldsymbol{j}}\right]$$

$$= \boxed{\left(-10\,\text{m/s}^2\right)\hat{\boldsymbol{j}}}$$

52 •• A particle has a constant acceleration of $\vec{a} = (6.0\ \text{m/s}^2)\hat{\boldsymbol{i}} + (4\ .0\ \text{m/s}^2)\hat{\boldsymbol{j}}$. At time $t = 0$, the velocity is zero and the position vector is $\vec{r}_0 = (10\ \text{m})\hat{\boldsymbol{i}}$. (*a*) Find the velocity and position vectors as functions of time t. (*b*) Find the equation of the particle's path in the xy plane and sketch the path.

Picture the Problem We can use constant-acceleration equations in vector form to find the velocity and position vectors as functions of time t. In (*b*), we can eliminate t from the equations giving the x and y components of the particle to find an expression for y as a function of x.

(*a*) Use $\vec{v} = \vec{v}_0 + \vec{a}t$ with $\vec{v}_0 = 0$ to find $\vec{v}$:

$$\vec{v} = \boxed{\left[\left(6.0\ \text{m/s}^2\right)\hat{\boldsymbol{i}} + \left(4.0\ \text{m/s}^2\right)\hat{\boldsymbol{j}}\right]t}$$

Use $\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ with $\vec{r}_0 = (10\,\text{m})\hat{\boldsymbol{i}}$ to find $\vec{r}$:

$$\vec{r} = \boxed{\left[(10\ \text{m}) + \left(3.0\ \text{m/s}^2\right)t^2\right]\hat{\boldsymbol{i}} + \left[\left(2.0\ \text{m/s}^2\right)t^2\right]\hat{\boldsymbol{j}}}$$

(*b*) Obtain the *x* and *y* components of the path from the vector equation in (*a*):

$$x = 10\ \text{m} + \left(3.0\ \text{m/s}^2\right)t^2$$

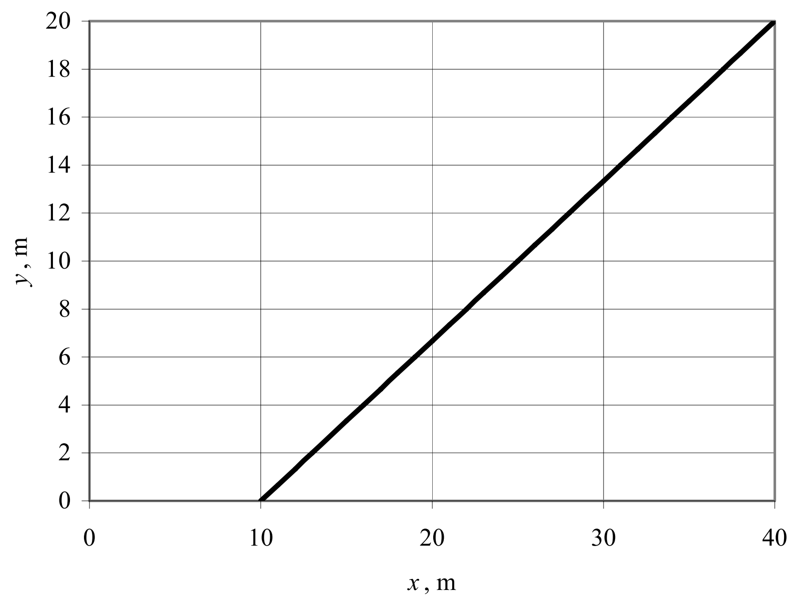
and

$$y = \left(2.0\ \text{m/s}^2\right)t^2$$

Eliminate *t* from these equations and solve for *y* to obtain:

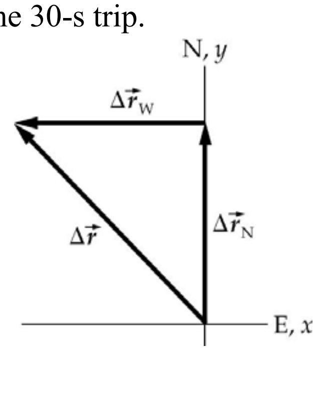
$$\boxed{y = \tfrac{2}{3}x - \tfrac{20}{3}\ \text{m}}$$

The graph of $y = \frac{2}{3}x - \frac{20}{3}\ \text{m}$ is shown below. Note that the path in the *xy* plane is a straight line.



53 •• Starting from rest at a dock, a motor boat on a lake heads north while gaining speed at a constant 3.0 m/s^2 for 20 s. The boat then heads west and continues for 10 s at the speed that it had at 20 s. (*a*) What is the average velocity of the boat during the 30-s trip? (*b*) What is the average acceleration of the boat during the 30-s trip? (*c*) What is the displacement of the boat during the 30-s trip?

Picture the Problem The displacements of the boat are shown in the following figure. Let the +*x* direction be to the east and the +*y* direction be to the north. We need to determine each of the displacements in order to calculate the average velocity of the boat during the 30-s trip.



(*a*) The average velocity of the boat is given by:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} \quad (1)$$

The total displacement of the boat is given by:

$$\Delta\vec{r} = \Delta\vec{r}_N + \Delta\vec{r}_W = \tfrac{1}{2}a_N(\Delta t_N)^2\,\hat{j} + v_W\Delta t_W(-\hat{i}) \quad (2)$$

To calculate the displacement we first have to find the speed after the first 20 s:

$$v_W = v_{N,f} = a_N\Delta t_N$$

Substitute numerical values and evaluate v_W:

$$v_W = (3.0\ \text{m/s}^2)(20\ \text{s}) = 60\ \text{m/s}$$

Substitute numerical values in equation (2) and evaluate $\Delta\vec{r}(30\ \text{s})$:

$$\Delta\vec{r}(30\ \text{s}) = \tfrac{1}{2}(3.0\ \text{m/s}^2)(20\ \text{s})^2\,\hat{j} - (60\ \text{m/s})(10\ \text{s})\,\hat{i} = (600\ \text{m})\hat{j} - (600\ \text{m})\hat{i}$$

Substitute numerical values in equation (1) to find the boat's average velocity:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(600\ \text{m})(-\hat{i}+\hat{j})}{30\ \text{s}} = \boxed{(20\ \text{m/s})(-\hat{i}+\hat{j})}$$

(*b*) The average acceleration of the boat is given by:

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

Substitute numerical values and evaluate $\vec{a}_{av}$:

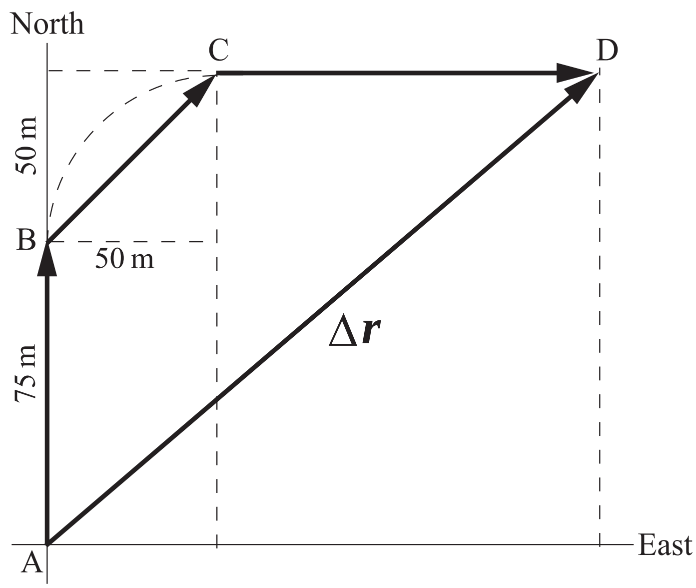
$$\vec{a}_{av} = \frac{(-60\ \text{m/s})\hat{i} - 0}{30\ \text{s}} = \boxed{(-2.0\ \text{m/s}^2)\hat{i}}$$

(*c*) The displacement of the boat from the dock at the end of the 30-s trip was one of the intermediate results we obtained in Part (*a*).

$$\Delta\vec{r} = (600\ \text{m})\hat{j} + (-600\ \text{m})\hat{i} = \boxed{(600\ \text{m})(-\hat{i}+\hat{j})}$$

54 •• Starting from rest at point A, you ride your motorcycle north to point B 75.0 m away, increasing speed at steady rate of 2.00 m/s^2. You then gradually turn toward the east along a circular path of radius 50.0 m at constant speed from B to point C until your direction of motion is due east at C. You then continue eastward, slowing at a steady rate of 1.00 m/s^2 until you come to rest at point D. (*a*) What is your average velocity and acceleration for the trip from A to D? (*b*) What is your displacement during your trip from A to C? (*c*) What distance did you travel for the entire trip from A to D?

Picture the Problem The following diagram summarizes the information given in the problem statement. Let the $+x$ direction be to the east and the $+y$ direction be to the north. To find your average velocity you'll need to find your displacement $\Delta\vec{r}$ and the total time required for you to make this trip. You can express $\Delta\vec{r}$ as the sum of your displacements $\Delta\vec{r}_{AB}$, $\Delta\vec{r}_{BC}$, and $\Delta\vec{r}_{CD}$. The total time for your trip is the sum of the times required for each of the segments. Because the acceleration is constant (but different) along each of the segments of your trip, you can use constant-acceleration equations to find each of these quantities.



(*a*) The average velocity for your trip is given by:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} \qquad (1)$$

The total displacement for your trip is the sum of the displacements along the three segments:

$$\Delta\vec{r} = \Delta\vec{r}_{AB} + \Delta\vec{r}_{BC} + \Delta\vec{r}_{CD} \qquad (2)$$

The total time for your trip is the sum of the times for the three segments:

$$\Delta t = \Delta t_{AB} + \Delta t_{BC} + \Delta t_{CD} \qquad (3)$$

The displacements of the three segments of the trip are:

$$\Delta\vec{r}_{AB} = 0\hat{\boldsymbol{i}} + (75.0\text{ m})\hat{\boldsymbol{j}}\,,$$

$$\Delta\vec{r}_{BC} = (50.0\text{ m})\hat{\boldsymbol{i}} + (50.0\text{ m})\hat{\boldsymbol{j}}\,,$$

and

$$\Delta\vec{r}_{CD} = \Delta r_{CD}\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$$

In order to find Δx_{CD}, you need to find the time for the C to D segment of the trip. Use a constant-acceleration equation to express Δx_{CD}:

$$\Delta x_{CD} = v_C \Delta t_{CD} + \tfrac{1}{2} a_{CD} (\Delta t_{CD})^2 \quad (4)$$

Because $v_C = v_B$:

$$v_C = v_B = v_A + a_{AB}\Delta t_{AB}$$

or, because $v_A = 0$,

$$v_C = a_{AB}\Delta t_{AB} \quad (5)$$

Use a constant-acceleration equation to relate Δt_{AB} to Δy_{AB}:

$$\Delta y_{AB} = v_A \Delta t_{AB} + \tfrac{1}{2} a_{AB} (\Delta t_{AB})^2$$

or, because $v_A = 0$,

$$\Delta y_{AB} = \tfrac{1}{2} a_{AB} (\Delta t_{AB})^2 \Rightarrow \Delta t_{AB} = \sqrt{\frac{2\Delta y_{AB}}{a_{AB}}}$$

Substitute numerical values and evaluate Δt_{AB}:

$$\Delta t_{AB} = \sqrt{\frac{2(75.0\text{ m})}{2.00\text{ m/s}^2}} = 8.660\text{ s}$$

Substitute for Δt_{AB} in equation (5) to obtain:

$$v_C = a_{AB}\sqrt{\frac{2\Delta y_{AB}}{a_{AB}}} = \sqrt{2a_{AB}\Delta y_{AB}}$$

Substitute numerical values and evaluate v_C:

$$v_C = \sqrt{2(2.00\text{ m/s}^2)(75.0\text{ m})} = 17.32\text{ m/s}$$

The time required for the circular segment BC is given by:

$$\Delta t_{BC} = \frac{\tfrac{1}{2}\pi r}{v_B}$$

Substitute numerical values and evaluate Δt_{BC}:

$$\Delta t_{BC} = \frac{\tfrac{1}{2}\pi (50.0\text{ m})}{17.32\text{ m/s}} = 4.535\text{ s}$$

The time to travel from C to D is given by:

$$\Delta t_{CD} = \frac{\Delta v_{CD}}{a_{CD}} = \frac{v_D - v_C}{a_{CD}}$$

or, because $v_D = 0$,

$$\Delta t_{CD} = \frac{-v_C}{a_{CD}}$$

Substitute numerical values and evaluate Δt_{CD}:

$$\Delta t_{CD} = \frac{-17.32\text{ m/s}}{-1.00\text{ m/s}^2} = 17.32\text{ s}$$

Now we can use equation (4) to evaluate Δt_{CD}:

$$\Delta x_{\mathrm{CD}} = (17.32\ \mathrm{m/s})(17.32\ \mathrm{s}) + \tfrac{1}{2}(-1.00\ \mathrm{m/s^2})(17.32\ \mathrm{s})^2 = 150\ \mathrm{m}$$

The three displacements that you need to add in order to get $\Delta\vec{r}$ are:

$\Delta\vec{r}_{\mathrm{AB}} = 0\hat{i} + (75.0\ \mathrm{m})\hat{j}$,
$\Delta\vec{r}_{\mathrm{BC}} = (50.0\ \mathrm{m})\hat{i} + (50.0\ \mathrm{m})\hat{j}$,
and
$\Delta\vec{r}_{\mathrm{CD}} = (150\ \mathrm{m})\hat{i} + 0\hat{j}$

Substitute in equation (2) to obtain:

$$\begin{aligned}\Delta\vec{r} &= 0\hat{i} + (75.0\ \mathrm{m})\hat{j} + (50.0\ \mathrm{m})\hat{i} + (50.0\ \mathrm{m})\hat{j} + (150\ \mathrm{m})\hat{i} + 0\hat{j}\\ &= (200\ \mathrm{m})\hat{i} + (125\ \mathrm{m})\hat{j}\end{aligned}$$

Substituting in equation (3) for Δt_{AB}, Δt_{BC}, and Δt_{CD} yields:

$$\Delta t = 8.660\ \mathrm{s} + 4.535\ \mathrm{s} + 17.32\ \mathrm{s} = 30.51\ \mathrm{s}$$

Use equation (1) to find $\vec{v}_{\mathrm{av}}$:

$$\begin{aligned}\vec{v}_{\mathrm{av}} &= \frac{(200\ \mathrm{m})\hat{i} + (125\ \mathrm{m})\hat{j}}{30.51\ \mathrm{s}}\\ &= \boxed{(6.56\ \mathrm{m/s})\hat{i} + (4.10\ \mathrm{m/s})\hat{j}}\end{aligned}$$

Because the final and initial velocities are zero, the average acceleration is zero.

(*b*) Express your displacement from A to C as the sum of the displacements from A to B and from B to C:

$$\Delta\vec{r}_{\mathrm{AC}} = \Delta\vec{r}_{\mathrm{AB}} + \Delta\vec{r}_{\mathrm{BC}}$$

From (*a*) you have:

$\Delta\vec{r}_{\mathrm{AB}} = 0\hat{i} + (75.0\ \mathrm{m})\hat{j}$
and
$\Delta\vec{r}_{\mathrm{BC}} = (50.0\ \mathrm{m})\hat{i} + (50.0\ \mathrm{m})\hat{j}$

Substitute and simplify to obtain:

$$\begin{aligned}\Delta\vec{r}_{\mathrm{AC}} &= 0\hat{i} + (75.0\ \mathrm{m})\hat{j} + (50.0\ \mathrm{m})\hat{i}\\ &\quad + (50.0\ \mathrm{m})\hat{j}\\ &= \boxed{(50.0\ \mathrm{m})\hat{i} + (125\ \mathrm{m})\hat{j}}\end{aligned}$$

(*c*) Express the total distance you traveled as the sum of the distances traveled along the three segments:

$d_{\mathrm{tot}} = d_{\mathrm{AB}} + d_{\mathrm{BC}} + d_{\mathrm{CD}}$
where $d_{\mathrm{BC}} = \tfrac{1}{2}\pi r$ and r is the radius of the circular arc.

Substituting for d_{BC} yields: $d_{tot} = d_{AB} + \frac{1}{2}\pi r + d_{CD}$

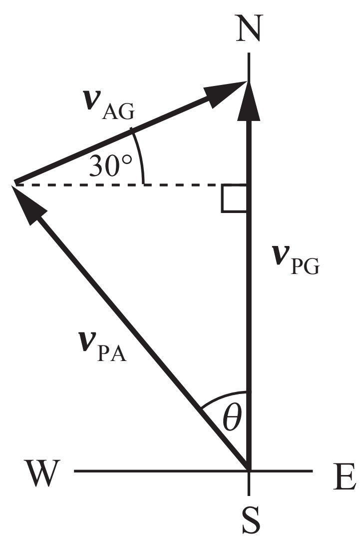
Substitute numerical values and evaluate d_{tot}:

$$d_{tot} = 75.0\text{ m} + \tfrac{1}{2}\pi(50.0\text{ m}) + 150\text{ m} = \boxed{304\text{ m}}$$

Relative Velocity

55 •• A plane flies at an airspeed of 250 km/h. A wind is blowing at 80 km/h toward the direction 60º east of north. (*a*) In what direction should the plane head in order to fly due north relative to the ground? (*b*) What is the speed of the plane relative to the ground?

Picture the Problem Choose a coordinate system in which north is the +*y* direction and east is the +*x* direction. Let θ be the angle between north and the direction of the plane's heading. The velocity of the plane relative to the ground, $\vec{v}_{PG}$, is the sum of the velocity of the plane relative to the air, $\vec{v}_{PA}$, and the velocity of the air relative to the ground, $\vec{v}_{AG}$. That is, $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$. The pilot must head in such a direction that the east-west component of $\vec{v}_{PG}$ is zero in order to make the plane fly due north.



(*a*) From the diagram one can see that: $v_{AG}\cos 30° = v_{PA}\sin\theta$

Solving for θ yields:

$$\theta = \sin^{-1}\left[\frac{v_{AG}\cos 30°}{v_{PA}}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{(80\text{ km/h})\cos 30°}{250\text{ km/h}}\right] = 16.1° \approx \boxed{16°\ \text{west of north}}$$

(*b*) Because the plane is headed due north, add the north components of $\vec{v}_{PA}$ and $\vec{v}_{AG}$ to determine the plane's ground speed:

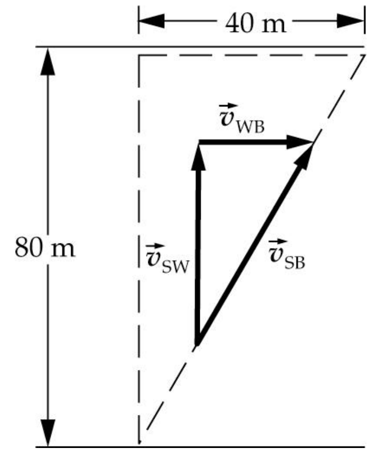
$$\left|\vec{v}_{PG}\right| = (250\text{ km/h})\cos 16.1° + (80\text{ km/h})\sin 30° = \boxed{280\text{ km/s}}$$

56 •• A swimmer *heads* directly across a river, swimming at 1.6 m/s relative to the water. She arrives at a point 40 m downstream from the point directly across the river, which is 80 m wide. (*a*) What is the speed of the river current? (*b*) What is the swimmer's speed relative to the shore? (*c*) In what direction should the swimmer head in order to arrive at the point directly opposite her starting point?

Picture the Problem Let $\vec{v}_{SB}$ represent the velocity of the swimmer relative to the shore; $\vec{v}_{SW}$ the velocity of the swimmer relative to the water; and $\vec{v}_{WB}$ the velocity of the water relative to the shore; i.e.,

$$\vec{v}_{SB} = \vec{v}_{SW} + \vec{v}_{WB}$$

The current of the river causes the swimmer to drift downstream.



(*a*) The triangles shown in the figure are similar right triangles. Set up a proportion between their sides and solve for the speed of the water relative to the bank:

$$\frac{v_{WB}}{v_{SW}} = \frac{40\,\text{m}}{80\,\text{m}}$$

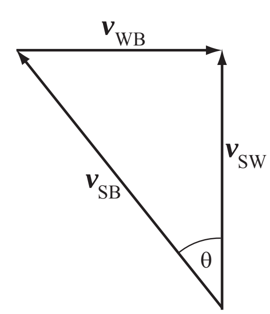
and

$$v_{WB} = \tfrac{1}{2}(1.6\,\text{m/s}) = \boxed{0.80\,\text{m/s}}$$

(*b*) Use the Pythagorean Theorem to solve for the swimmer's speed relative to the shore:

$$v_{SB} = \sqrt{v_{SW}^2 + v_{WB}^2} = \sqrt{(1.6\,\text{m/s})^2 + (0.80\,\text{m/s})^2} = \boxed{1.8\,\text{m/s}}$$

(*c*) The swimmer should head in a direction such that the upstream component of her velocity relative to the shore ($\vec{v}_{SB}$) is equal to the speed of the water relative to the shore ($\vec{v}_{WB}$):



Referring to the diagram, relate $\sin\theta$ to $\vec{v}_{WB}$ and $\vec{v}_{SB}$:

$$\sin\theta = \frac{v_{WB}}{v_{SB}} \Rightarrow \theta = \sin^{-1}\left(\frac{v_{WB}}{v_{SB}}\right)$$

Substitute numerical values and evaluate θ:

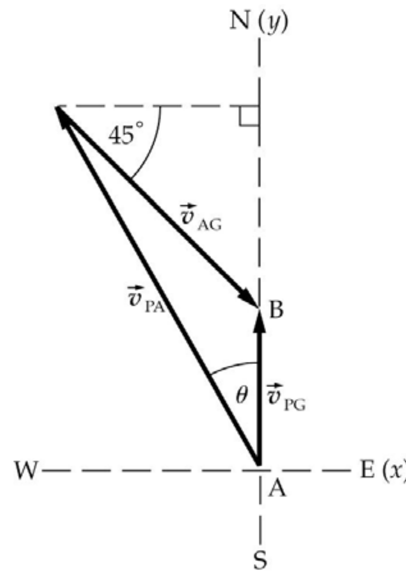
$$\theta = \sin^{-1}\left(\frac{0.80\,\text{m/s}}{1.8\,\text{m/s}}\right) = \boxed{26^\circ}$$

57 •• [SSM] A small plane departs from point A heading for an airport 520 km due north at point B. The airspeed of the plane is 240 km/h and there is a steady wind of 50 km/h blowing directly toward the southeast. Determine the proper heading for the plane and the time of flight.

Picture the Problem Let the velocity of the plane relative to the ground be represented by $\vec{v}_{\text{PG}}$; the velocity of the plane relative to the air by $\vec{v}_{\text{PA}}$, and the velocity of the air relative to the ground by $\vec{v}_{\text{AG}}$. Then

$$\vec{v}_{\text{PG}} = \vec{v}_{\text{PA}} + \vec{v}_{\text{AG}} \quad (1)$$

Choose a coordinate system with the origin at point A, the +*x* direction to the east, and the +*y* direction to the north. θ is the angle between north and the direction of the plane's heading. The pilot must head so that the east-west component of $\vec{v}_{\text{PG}}$ is zero in order to make the plane fly due north.



Use the diagram to express the condition relating the eastward component of $\vec{v}_{\text{AG}}$ and the westward component of $\vec{v}_{\text{PA}}$. This must be satisfied if the plane is to stay on its northerly course. [*Note*: this is equivalent to equating the *x*-components of equation (1).]

$$v_{\text{AG}} \cos 45° = v_{\text{PA}} \sin\theta$$

Now solve for θ to obtain:

$$\theta = \sin^{-1}\left[\frac{v_{\text{AG}} \cos 45°}{v_{\text{PA}}}\right]$$

Substitute numerical values and evaluate θ:

$$\boldsymbol{\theta} = \sin^{-1}\left[\frac{(50\text{ km/h})\cos 45°}{240\text{ km/h}}\right] = 8.47°$$

$$= \boxed{8.5°}$$

Add the north components of $\vec{v}_{\text{PA}}$ and $\vec{v}_{\text{AG}}$ to find the velocity of the plane relative to the ground:

$$v_{\text{PG}} + v_{\text{AG}} \sin 45° = v_{\text{PA}} \cos 8.47°$$

Solving for v_{PG} yields: $v_{PG} = v_{PA}\cos 8.47° - v_{AG}\sin 45°$

Substitute numerical values and evaluate v_{PG} to obtain:

$$v_{PG} = (240\ \text{km/h})\cos 8.47° - (50\ \text{km/h})\sin 45° = 202.0\ \text{km/h}$$

The time of flight is given by: $t_{flight} = \dfrac{\text{distance travelled}}{v_{PG}}$

Substitute numerical values and evaluate t_{flight}: $t_{flight} = \dfrac{520\ \text{km}}{202.0\ \text{km/h}} = \boxed{2.57\ \text{h}}$

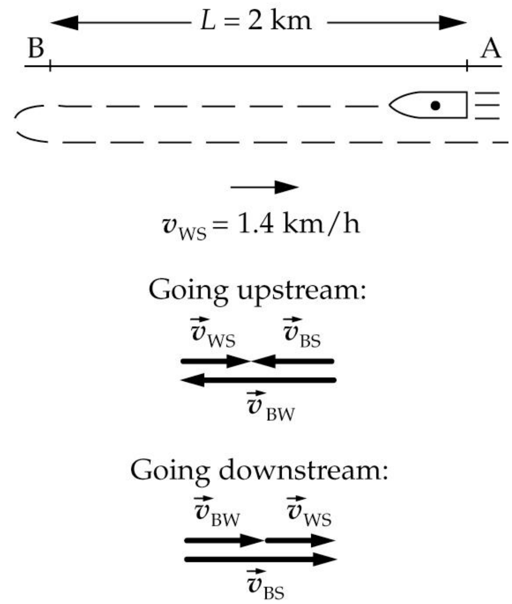
58 •• Two boat landings are 2.0 km apart on the same bank of a stream that flows at 1.4 km/h. A motorboat makes the round trip between the two landings in 50 min. What is the speed of the boat relative to the water?

Picture the Problem Let $\vec{v}_{BS}$ be the velocity of the boat relative to the shore; $\vec{v}_{BW}$ be the velocity of the boat relative to the water; and $\vec{v}_{WS}$ represent the velocity of the water relative to the shore. Independently of whether the boat is going upstream or downstream:

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}$$

Going upstream, the speed of the boat relative to the shore is reduced by the speed of the water relative to the shore.

Going downstream, the speed of the boat relative to the shore is increased by the same amount.



For the upstream leg of the trip: $v_{BS} = v_{BW} - v_{WS}$

For the downstream leg of the trip: $v_{BS} = v_{BW} + v_{WS}$

Express the total time for the trip in terms of the times for its upstream and downstream legs:

$$t_{total} = t_{upstream} + t_{downstream} = \frac{L}{v_{BW} - v_{WS}} + \frac{L}{v_{BW} + v_{WS}}$$

Multiply both sides of the equation by $(v_{\text{BW}} - v_{\text{WS}})(v_{\text{BW}} + v_{\text{WS}})$ (the product of the- denominators) and rearrange the terms to obtain:

$$v_{\text{BW}}^2 - \frac{2L}{t_{\text{total}}} v_{\text{BW}} - v_{\text{WS}}^2 = 0$$

Substituting numerical values gives:

$$v_{\text{BW}}^2 - \frac{2(2.0\,\text{km})}{\frac{5}{6}\,\text{h}} v_{\text{BW}} - (1.4\,\text{km/h})^2 = 0$$

or

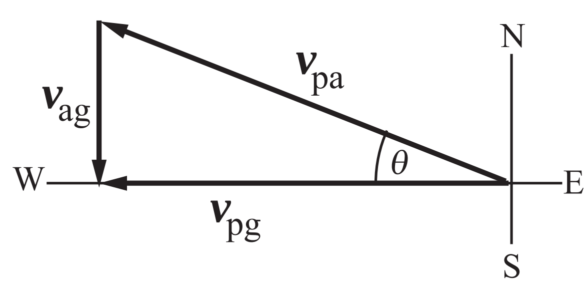
$$v_{\text{BW}}^2 - (4.80\,\text{km/h})v_{\text{BW}} - 1.96\,(\text{km})^2/\text{h}^2 = 0$$

Use the quadratic formula or your graphing calculator to solve the quadratic equation for v_{BW} (Note that only the positive root is physically meaningful.):

$$v_{\text{BW}} = \boxed{5.2\,\text{km/h}}$$

59 •• During a radio-controlled model-airplane competition, each plane must fly from the center of a 1.0-km-radius circle to any point on the circle and back to the center. The winner is the plane that has the shortest round-trip time. The contestants are free to fly their planes along any route as long as the plane begins at the center, travels to the circle, and then returns to the center. On the day of the race, a steady wind blows out of the north at 5.0 m/s. Your plane can maintain an air speed of 15 m/s. Should you fly your plane upwind on the first leg and downwind on the trip back, or across the wind flying east and then west? Optimize your chances by calculating the round-trip time for both routes using your knowledge of vectors and relative velocities. With this pre-race calculation, you can determine the best route and have a major advantage over the competition!

Picture the Problem Let $\vec{\boldsymbol{v}}_{\text{pg}}$ be the velocity of the plane relative to the ground; $\vec{\boldsymbol{v}}_{\text{ag}}$ be the velocity of the air relative to the ground; and $\vec{\boldsymbol{v}}_{\text{pa}}$ the velocity of the plane relative to the air. Then, $\vec{\boldsymbol{v}}_{\text{pg}} = \vec{\boldsymbol{v}}_{\text{pa}} + \vec{\boldsymbol{v}}_{\text{ag}}$. The wind will affect the flight times differently along these two paths.



The speed of the plane, relative to the ground, on its eastbound leg is equal to its speed on its westbound leg. Using the diagram, express the speed of the plane relative to the ground for both directions:

$$v_{pg} = \sqrt{v_{pa}^2 - v_{ag}^2}$$

Substitute numerical values and evaluate v_{pg}:

$$v_{pg} = \sqrt{(15\ \text{m/s})^2 - (5.0\ \text{m/s})^2} = 14.1\ \text{m/s}$$

Express the time for the east-west roundtrip in terms of the distances and velocities for the two legs:

$$t_{\text{roundtrip,EW}} = t_{\text{eastbound}} + t_{\text{westbound}} = \frac{\text{radius of the circle}}{v_{\text{pg,eastbound}}} + \frac{\text{radius of the circle}}{v_{\text{pg,westbound}}}$$

Substitute numerical values and evaluate $t_{\text{roundtrip,EW}}$:

$$t_{\text{roundtrip,EW}} = \frac{2\times 1.0\ \text{km}}{14.1\ \text{m/s}} = 142\ \text{s}$$

Use the distances and speeds for the two legs to express the time for the north-south roundtrip:

$$t_{\text{roundtrip,NS}} = t_{\text{northbound}} + t_{\text{southbound}} = \frac{\text{radius of the circle}}{v_{\text{pg,northbound}}} + \frac{\text{radius of the circle}}{v_{\text{pg,southbound}}}$$

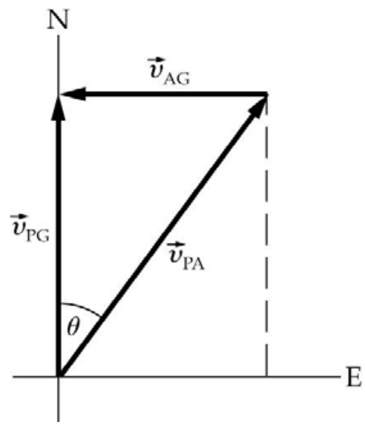
Substitute numerical values and evaluate $t_{\text{roundtrip,NS}}$:

$$t_{\text{roundtrip,NS}} = \frac{1.0\ \text{km}}{(15\ \text{m/s}) - (5.0\ \text{m/s})} + \frac{1.0\ \text{km}}{(15\ \text{m/s}) + (5.0\ \text{m/s})} = 150\ \text{s}$$

Because $t_{\text{roundtrip,EW}} < t_{\text{roundtrip,NS}}$, you should fly your plane across the wind.

60 • You are piloting a small plane that can maintain an air speed of 150 kt (knots, or nautical miles per hour) and you want to fly due north (azimuth = 000º) relative to the ground. (*a*) If a wind of 30 kt is blowing from the east (azimuth = 090º), calculate the heading (azimuth) you must ask your co-pilot to maintain. (*b*) At that heading, what will be your groundspeed?

Picture the Problem This is a relative velocity problem. The given quantities are the direction of the velocity of the plane relative to the ground and the velocity (magnitude and direction) of the air relative to the ground. Asked for is the direction of the velocity of the air relative to the ground. Using $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$, draw a vector addition diagram and solve for the unknown quantities.



(*a*) Referring to the diagram, relate the heading you must take to the wind speed and the speed of your plane relative to the air:

$$\theta = \sin^{-1}\left(\frac{v_{AG}}{v_{PA}}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left(\frac{30\text{ kts}}{150\text{ kts}}\right) = 11.5°$$

Because this is also the angle of the plane's heading clockwise from north, it is also its azimuth or the required true heading:

$$\text{Az} = (011.5°) = \boxed{(012°)}$$

(*b*) Referring to the diagram, relate your heading to your plane's speeds relative to the ground and relative to the air:

$$\cos\theta = \frac{v_{PG}}{v_{PA}}$$

Solve for the speed of your plane relative to the ground to obtain:

$$v_{PG} = v_{PA}\cos\theta$$

Substitute numerical values and evaluate v_{PG}:

$$v_{PG} = (150\text{ kt})\cos 11.5° = 147\text{ kt}$$
$$= \boxed{1.5\times10^{2}\text{ kt}}$$

61 •• **[SSM]** Car A is traveling east at 20 m/s toward an intersection. As car A crosses the intersection, car B starts from rest 40 m north of the intersection and moves south, steadily gaining speed at 2.0 m/s^2. Six seconds after A crosses the intersection find (*a*) the position of B relative to A, (*b*) the velocity of B relative to A, and (*c*) the acceleration of B relative to A. (*Hint: Let the unit vectors* $\hat{\boldsymbol{i}}$ *and* $\hat{\boldsymbol{j}}$ *be toward the east and north, respectively, and express your answers using* $\hat{\boldsymbol{i}}$ *and* $\hat{\boldsymbol{j}}$.)

Picture the Problem The position of B relative to A is the vector from A to B; that is,

$$\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$$

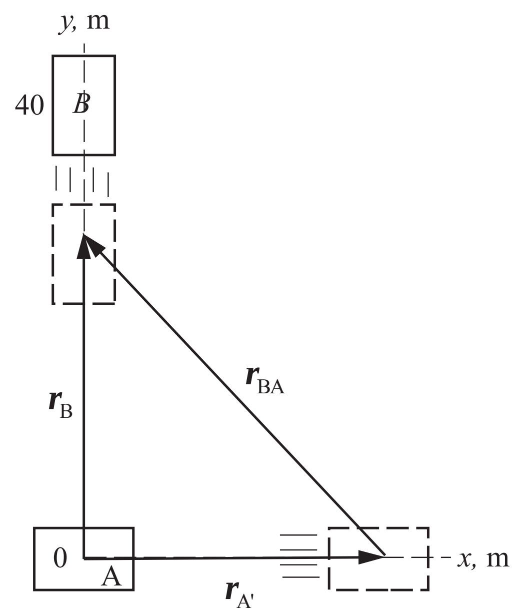
The velocity of B relative to A is

$$\vec{v}_{BA} = d\vec{r}_{BA}/dt$$

and the acceleration of B relative to A is

$$\vec{a}_{BA} = d\vec{v}_{BA}/dt$$

Choose a coordinate system with the origin at the intersection, the $+x$ direction to the east, and the $+y$ direction to the north.



(*a*) Find $\vec{r}_B$ and $\vec{r}_A$:

$$\vec{r}_B = \left[40\text{ m} - \tfrac{1}{2}\left(2.0\text{ m/s}^2\right)t^2\right]\hat{j}$$

and

$$\vec{r}_A = \left[(20\text{ m/s})t\right]\hat{i}$$

Use $\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$ to find $\vec{r}_{BA}$:

$$\vec{r}_{BA} = \left[(-20\text{ m/s})t\right]\hat{i} + \left[40\text{ m} - \tfrac{1}{2}\left(2.0\text{ m/s}^2\right)t^2\right]\hat{j}$$

Evaluate $\vec{r}_{BA}$ at $t = 6.0$ s:

$$\vec{r}_{BA}(6.0\text{ s}) = \left[(-20\text{ m/s})(6.0\text{ s})\right]\hat{i} + \left[40\text{ m} - \tfrac{1}{2}\left(2.0\text{ m/s}^2\right)(6.0\text{ s})^2\right]\hat{j}$$
$$= \boxed{(-1.2\times10^2\text{ m})\,\hat{i} + (4.0\text{ m})\,\hat{j}}$$

(*b*) Find $\vec{v}_{BA} = d\vec{r}_{BA}/dt$:

$$\vec{v}_{BA} = \frac{d\vec{r}_{BA}}{dt} = \frac{d}{dt}\left[\{(-20\text{ m/s})t\}\hat{i} + \left\{40\text{ m} - \tfrac{1}{2}\left(2.0\text{ m/s}^2\right)t^2\right\}\hat{j}\right]$$
$$= (-20\text{ m/s})\,\hat{i} + (-2.0\text{ m/s}^2)\,t\,\hat{j}$$

Evaluate $\vec{v}_{BA}$ at $t = 6.0$ s:

$$\vec{v}_{BA}(6.0\text{ s}) = (-20\text{ m/s})\,\hat{i} + (-2.0\text{ m/s}^2)(6.0\text{ s})\,\hat{j} = \boxed{(-20\text{ m/s})\hat{i} - (12\text{ m/s})\hat{j}}$$

(*c*) Find $\vec{a}_{\mathrm{BA}} = d\vec{v}_{\mathrm{BA}}/dt$:

$$\vec{a}_{\mathrm{BA}} = \frac{d}{dt}\left[(-20\ \mathrm{m/s})\,\hat{i} + (-2.0\ \mathrm{m/s^2})\,t\,\hat{j}\right]$$
$$= \boxed{\left(-2.0\,\mathrm{m/s^2}\right)\hat{j}}$$

Note that $\vec{a}_{\mathrm{BA}}$ is independent of time.

62 •• While walking between gates at an airport, you notice a child running along a moving walkway. Estimating that the child runs at a constant speed of 2.5 m/s relative to the surface of the walkway, you decide to try to determine the speed of the walkway itself. You watch the child run on the entire 21-m walkway in one direction, immediately turn around, and run back to his starting point. The entire trip taking a total elapsed time of 22 s. Given this information, what is the speed of the moving walkway relative to the airport terminal?

Picture the Problem We don't need to know which direction the child ran first, and which he ran second. Because we have the total length of the walkway, the elapsed time for the round trip journey, and the child's walking speed relative to the walkway, we are given sufficient information to determine the moving walkway's speed. The distance covered in the airport is 21 m, and this is covered in a total time of 22 s. When the child walks in the direction of the walkway, his velocity in the airport is the sum of his walking velocity, $\vec{v}_{\mathrm{child}}$, and the walkway velocity, $\vec{v}_{\mathrm{ww}}$. On the return trip, the velocity in the airport is the difference of these two velocities.

Express the total time for the child's run in terms of his running times with and against the moving walkway:

$$\Delta t_{\mathrm{tot}} = \Delta t_{\mathrm{with}} + \Delta t_{\mathrm{against}}$$

In terms of the length L of the walkway and the speeds, relative to the airport, of the child walking with and against the moving walkway:

$$\Delta t_{\mathrm{tot}} = \frac{L}{v_{\mathrm{with}}} + \frac{L}{v_{\mathrm{against}}} \qquad (1)$$

The speeds of the child relative to the airport, then, in each case, are:

$$v_{\mathrm{with}} = \left|\vec{v}_{\mathrm{child}}\right| + \left|\vec{v}_{\mathrm{ww}}\right| = v_{\mathrm{child}} + v_{\mathrm{ww}}$$

and

$$v_{\mathrm{against}} = \left|\vec{v}_{\mathrm{child}}\right| - \left|\vec{v}_{\mathrm{ww}}\right| = v_{\mathrm{child}} - v_{\mathrm{ww}}$$

Substitute for v_{with} and v_{against} in equation (1) to obtain:

$$\Delta t_{\mathrm{tot}} = \frac{L}{v_{\mathrm{child}} + v_{\mathrm{ww}}} + \frac{L}{v_{\mathrm{child}} + v_{\mathrm{ww}}}$$

Solving this equation for v_{ww} yields:

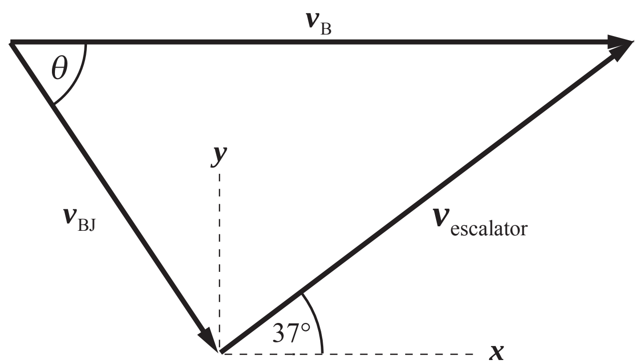
$$v_{\mathrm{ww}} = \sqrt{v_{\mathrm{child}}^2 - \frac{2v_{\mathrm{child}}L}{\Delta t_{\mathrm{tot}}}}$$

Substitute numerical values and evaluate v_{ww}:

$$v_{ww} = \sqrt{(2.5\text{ m/s})^2 - \frac{2(2.5\text{ m/s})(21\text{ m})}{22\text{ s}}} = \boxed{1.2\text{ m/s}}$$

63 •• [SSM] Ben and Jack are shopping in a department store. Ben leaves Jack at the bottom of the escalator and walks east at a speed of 2.4 m/s. Jack then stands on the escalator, which is inclined at an angle of 37° above the horizontal and travels eastward and upward at a speed of 2.0 m/s. (*a*) What is the velocity of Ben relative to Jack? (*b*) At what speed should Jack walk up the escalator so that he is always directly above Ben (until he reaches the top)?

Picture the Problem The velocity of Ben relative to Jack ($\vec{v}_{BJ}$) is the difference between the vectors $\vec{v}_B$ and $\vec{v}_{escalator}$. Choose the coordinate system shown and express these vectors using the unit vectors $\hat{i}$ and $\hat{j}$.



(*a*) The velocity of Ben relative to Jack is given by:

$$\vec{v}_{BJ} = \vec{v}_B - \vec{v}_{escalator}$$

The velocities of the floor walker and the escalator are:

$$\vec{v}_B = (2.4\text{ m/s})\,\hat{i}$$

and

$$\vec{v}_{escalator} = (2.0\text{ m/s})\cos 37°\,\hat{i} + (2.0\text{ m/s})\sin 37°\,\hat{j}$$

Substitute for $\vec{v}_B$ and $\vec{v}_{escalator}$ and simplify to obtain:

$$\begin{aligned}\vec{v}_{BJ} &= (2.4\text{ m/s})\,\hat{i} - \left[(2.0\text{ m/s})\cos 37°\,\hat{i} + (2.0\text{ m/s})\sin 37°\,\hat{j}\right] \\ &= (0.803\text{ m/s})\,\hat{i} - (1.20\text{ m/s})\,\hat{j} \\ &= \boxed{(0.80\text{ m/s})\,\hat{i} - (1.2\text{ m/s})\,\hat{j}}\end{aligned}$$

The magnitude and direction of $\vec{v}_{BJ}$ are given by:

$$\left|\vec{v}_{BJ}\right| = \sqrt{v_{BJ,x}^2 + v_{BJ,y}^2}$$

and

$$\theta = \tan^{-1}\left(\frac{v_{BJ,y}}{v_{BJ,x}}\right)$$

Substitute numerical values and evaluate $\left|\vec{v}_{BJ}\right|$ and θ:

$$\left|\vec{v}_{BJ}\right| = \sqrt{(0.803\text{ m/s})^2 + (-1.20\text{ m/s})^2} = \boxed{1.4\text{ m/s}}$$

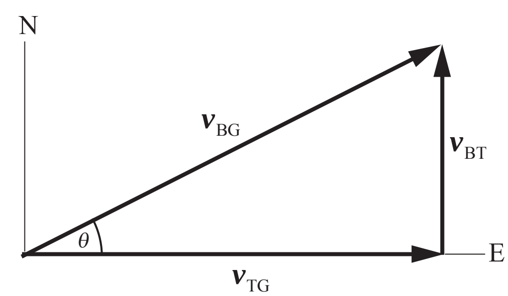
and

$$\theta = \tan^{-1}\left(\frac{-1.20\text{ m/s}}{0.803\text{ m/s}}\right) = -56.2°$$

$$= \boxed{56° \text{ below the horizontal}}$$

64 ••• A juggler traveling in a train on level track throws a ball straight up, relative to the train, with a speed of 4.90 m/s. The train has a velocity of 20.0 m/s due east. As observed by the juggler, (*a*) what is the ball's total time of flight, and (*b*) what is the displacement of the ball during its rise? According to a friend standing on the ground next to the tracks, (*c*) what is the ball's initial speed, (*d*) what is the angle of the launch, and (*e*) what is the displacement of the ball during its rise?

Picture the Problem The vector diagram shows the relationship between the velocity of the ball relative to the train $\vec{v}_{BT}$, the velocity of the train relative to the ground $\vec{v}_{TG}$, and the velocity of the ball relative to the ground $\vec{v}_{BG}$. Choose a coordinate system in which the +*x* direction is to the east and the +*y* direction is to the north.



(*a*) The ball's time-of-flight is twice the time it takes it to reach its maximum height:

$$t_{flight} = 2t_{max\ height} \qquad (1)$$

Use a constant-acceleration equation to relate the ball's speed to its initial speed and the time into its flight:

$$v_y = v_{0y} + a_y t$$

At its maximum height, $v_y = 0$:

$$0 = v_{0y} + a_y t_{\text{max height}}$$

or, because $a_y = -g$,

$$0 = v_{0y} - g t_{\text{max height}}$$

Solving for $t_{\text{max height}}$ yields:

$$t_{\text{max height}} = \frac{v_{0y}}{g}$$

Substitute for $t_{\text{max height}}$ in equation (1) to obtain:

$$t_{\text{flight}} = \frac{2v_{0y}}{g}$$

Substitute numerical values and evaluate t_{flight}:

$$t_{\text{flight}} = \frac{2(4.90\ \text{m/s})}{9.81\ \text{m/s}^2} = \boxed{1.00\ \text{s}}$$

(*b*) The displacement of the ball during its rise is given by:

$$\Delta\vec{\boldsymbol{y}}_{\text{BT}} = \Delta y_{\text{BT}}\hat{\boldsymbol{j}}$$

Use a constant-acceleration equation to relate the displacement of the ball during its ascent to its initial and final speeds:

$$v_x^2 = v_{0x}^2 - 2g\Delta y_{\text{BT}}$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 - 2g\Delta y_{\text{BT}} \Rightarrow \Delta y_{\text{BT}} = \frac{v_{0x}^2}{2g}$$

Substituting for Δy_{BT} gives:

$$\Delta\vec{\boldsymbol{y}}_{\text{BT}} = \frac{v_{0x}^2}{2g}\hat{\boldsymbol{j}}$$

Substitute numerical values and evaluate $\Delta\vec{\boldsymbol{y}}_{\text{BT}}$:

$$\Delta\vec{\boldsymbol{y}}_{\text{BT}} = \frac{(4.90\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)}\hat{\boldsymbol{j}} = \boxed{(1.22\ \text{m})\hat{\boldsymbol{j}}}$$

(*c*) $\vec{\boldsymbol{v}}_{\text{BG}}$ is the sum of $\vec{\boldsymbol{v}}_{\text{BT}}$ and $\vec{\boldsymbol{v}}_{\text{TG}}$:

$$\vec{\boldsymbol{v}}_{\text{BG}} = \vec{\boldsymbol{v}}_{\text{BT}} + \vec{\boldsymbol{v}}_{\text{TG}} \qquad (3)$$

Express $\vec{\boldsymbol{v}}_{\text{BT}}$ and $\vec{\boldsymbol{v}}_{\text{TG}}$ in terms of the unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$:

$$\vec{\boldsymbol{v}}_{\text{BT}} = 0\hat{\boldsymbol{i}} + (4.90\ \text{m/s})\hat{\boldsymbol{j}}$$

and

$$\vec{\boldsymbol{v}}_{\text{TG}} = (20.0\ \text{m/s})\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$$

Substitute for $\vec{\boldsymbol{v}}_{\text{BT}}$ and $\vec{\boldsymbol{v}}_{\text{TG}}$ in equation (1) to obtain:

$$\vec{\boldsymbol{v}}_{\text{BG}} = (20.0\ \text{m/s})\hat{\boldsymbol{i}} + (4.90\ \text{m/s})\hat{\boldsymbol{j}}$$

The magnitude of $\vec{\boldsymbol{v}}_{\text{BG}}$ is given by:

$$v_{\text{BG}} = \sqrt{(20.0\ \text{m/s})^2 + (4.90\ \text{m/s})^2} = \boxed{20.6\ \text{m/s}}$$

(*d*) Referring to the vector diagram, express the angle of the launch θ in terms of v_{BT} and v_{TG}:

$$\theta = \tan^{-1}\left(\frac{v_{BT}}{v_{TG}}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{4.90\text{ m/s}}{20.0\text{ m/s}}\right) = \boxed{13.8°}$$

(*e*) When the ball is at its peak it is 20.0 m east and 1.22 m above the friend observing from the ground. Hence its displacement is:

$$\vec{\boldsymbol{d}} = \boxed{(20.0\text{ m})\hat{\boldsymbol{i}} + (1.22\text{ m})\hat{\boldsymbol{j}}}$$

Circular Motion and Centripetal Acceleration

65 • What is the magnitude of the acceleration of the tip of the minute hand of the clock in Problem 38? Express it as a fraction of the magnitude of free-fall acceleration *g*.

Picture the Problem We can use the definition of centripetal acceleration to express a_c in terms of the speed of the tip of the minute hand. We can find the tangential speed of the tip of the minute hand by using the distance it travels each revolution and the time it takes to complete each revolution.

Express the magnitude of the acceleration of the tip of the minute hand of the clock as a function of the length of the hand and the speed of its tip:

$$a_c = \frac{v^2}{R} \qquad (1)$$

Use the distance the minute hand travels every hour to express its speed:

$$v = \frac{2\pi R}{T}$$

Substituting for *v* in equation (1) yields:

$$a_c = \frac{4\pi^2 R}{T^2}$$

Substitute numerical values and evaluate a_c:

$$a_c = \frac{4\pi^2(0.50\text{ m})}{(3600\text{ s})^2} = 1.52\times10^{-6}\text{ m/s}^2$$

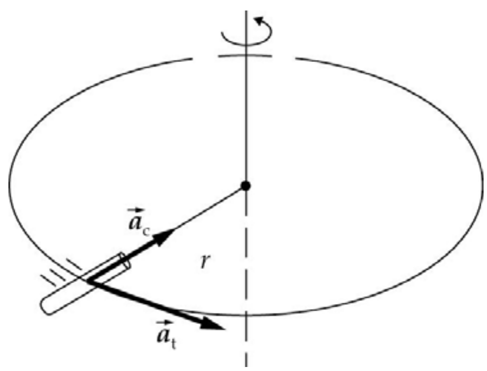
$$= \boxed{1.5\times10^{-6}\text{ m/s}^2}$$

Express the ratio of a_c to *g*:

$$\frac{a_c}{g} = \frac{1.52\times10^{-6}\text{ m/s}^2}{9.81\text{ m/s}^2} = \boxed{1.55\times10^{-7}}$$

66 • You are designing a centrifuge to spins at a rate of 15,000 rev/min. (*a*) Calculate the maximum centripetal acceleration that a test-tube sample held in the centrifuge arm 15 cm from the rotation axis must withstand. (*b*) It takes 1 min, 15 s for the centrifuge to spin up to its maximum rate of revolution from rest. Calculate the *magnitude* of the tangential acceleration of the centrifuge while it is spinning up, assuming that the tangential acceleration is constant.

Picture the Problem The following diagram shows the centripetal and tangential accelerations experienced by the test tube. The tangential acceleration will be zero when the centrifuge reaches its maximum speed. The centripetal acceleration increases as the tangential speed of the centrifuge increases. We can use the definition of centripetal acceleration to express a_c in terms of the speed of the test tube. We can find the tangential speed of the test tube by using the distance it travels each revolution and the time it takes to complete each revolution. The tangential acceleration can be found from the change in the tangential speed as the centrifuge is spinning up.



(*a*) Express the maximum centripetal acceleration of the centrifuge arm as a function of the length of its arm and the speed of the test tube:

$$a_{c,\max} = \frac{v^2}{R} \quad (1)$$

Use the distance the test tube travels every revolution to express its speed:

$$v = \frac{2\pi R}{T}$$

Substituting for v in equation (1) yields:

$$a_{c,\max} = \frac{4\pi^2 R}{T^2}$$

Substitute numerical values and evaluate a_c:

$$a_{c,\max} = \frac{4\pi^2(0.15\,\text{m})}{\left(\frac{1\,\text{min}}{15000\,\text{rev}} \times \frac{60\,\text{s}}{\text{min}}\right)^2} = \boxed{3.7\times10^5\ \text{m/s}^2}$$

(*b*) Express the tangential acceleration in terms of the difference between the final and initial tangential speeds:

$$a_t = \frac{v_f - v_i}{\Delta t} = \frac{\frac{2\pi R}{T} - 0}{\Delta t} = \frac{2\pi R}{T\Delta t}$$

Substitute numerical values and evaluate a_T:

$$a_t = \frac{2\pi(0.15\,\text{m})}{\left(\frac{1\,\text{min}}{15000\,\text{rev}} \times \frac{60\,\text{s}}{\text{min}}\right)(75\,\text{s})} = \boxed{3.1\,\text{m/s}^2}$$

67 ••• [SSM] Earth rotates on its axis once every 24 hours, so that objects on its surface that are stationary with respect to the surface execute uniform circular motion about the axis with a period of 24 hours. Consider only the effect of this rotation on the person on the surface. (Ignore Earth's orbital motion about the Sun.) (*a*) What is the speed and what is the magnitude of the acceleration of a person standing on the equator? (Express the magnitude of this acceleration as a percentage of *g*.) (*b*) What is the direction of the acceleration vector? (*c*) What is the speed and what is the magnitude of the centripetal acceleration of a person standing on the surface at 35°N latitude? (*d*) What is the angle between the direction of the acceleration of the person at 35°N latitude and the direction of the acceleration of the person at the equator if both persons are at the same longitude?

Picture the Problem The radius of Earth is 6370 km. Thus at the equator, a person undergoes circular motion with radius equal to Earth's radius, and a period of 24 h = 86400 s. At 35° N latitude, the person undergoes circular motion having radius $r\cos 35^\circ$ = 5220 km, and the same period.

The centripetal acceleration experienced by a person traveling with a speed v in a circular path of radius r is given by:

$$a = \frac{v^2}{r} \qquad (1)$$

The speed of the person is the distance the person travels in one revolution divided by the elapsed time (the period T):

$$v = \frac{2\pi r}{T}$$

Substitute for v in equation (1) to obtain:

$$a = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

(*a*) Substitute numerical values and evaluate v for the person at the equator:

$$v = \frac{2\pi(6370\,\text{km})}{86400\,\text{s}} = \boxed{463\,\text{m/s}}$$

Substitute numerical values and evaluate a for the person at the equator:

$$a = \frac{4\pi^2 (6370\ \text{km})}{(86400\ \text{s})^2} = 3.369 \times 10^{-2}\ \text{m/s}^2$$
$$= \boxed{3.37\ \text{cm/s}^2}$$

The ratio of a to g is:

$$\frac{a}{g} = \frac{3.369 \times 10^{-2}\ \text{m/s}^2}{9.81\ \text{m/s}^2} = \boxed{0.343\%}$$

(*b*) The acceleration vector points directly at the center of Earth.

(*c*) Substitute numerical values and evaluate v for the person at 35°N latitude:

$$v = \frac{2\pi (5220\ \text{km})}{86400\ \text{s}} = \boxed{380\ \text{m/s}}$$

Substitute numerical values and evaluate v for the person at 35°N latitude:

$$a = \frac{4\pi^2 (5220\ \text{km})}{(86400\ \text{s})^2} = \boxed{2.76\ \text{cm/s}^2}$$

(*d*) The plane of the person's path is parallel to the plane of the equator – the acceleration vector is in the plane – so that it is perpendicular to Earth's axis, pointing at the center of the person's revolution, rather than the center of Earth.

68 •• Determine the acceleration of the Moon toward Earth, using values for its mean distance and orbital period from the Terrestrial and Astronomical Data table in this book. Assume a circular orbit. Express the acceleration as a fraction of the magnitude of free-fall acceleration g.

Picture the Problem We can relate the acceleration of the Moon toward Earth to its orbital speed and distance from Earth. Its orbital speed can be expressed in terms of its distance from Earth and its orbital period. From the Terrestrial and Astronomical Data table, we find that the sidereal period of the Moon is 27.3 d and that its mean distance from Earth is 3.84×10^8 m.

Express the centripetal acceleration of the Moon:

$$a_c = \frac{v^2}{r} \qquad (1)$$

Express the orbital speed of the Moon:

$$v = \frac{2\pi r}{T}$$

Substituting for v in equation (1) yields:

$$a_c = \frac{4\pi^2 r}{T^2}$$

Substitute numerical values and evaluate a_c:

$$a_c = \frac{4\pi^2\left(3.84\times10^8\ \text{m}\right)}{\left(27.3\ \text{d}\times\frac{24\ \text{h}}{\text{d}}\times\frac{3600\ \text{s}}{\text{h}}\right)^2} = 2.72\times10^{-3}\ \text{m/s}^2 = \boxed{2.78\times10^{-4}\ g}$$

Remarks: Note that $\frac{\mathbf{a_c}}{\mathbf{g}} = \frac{\textbf{radius of Earth}}{\textbf{distance from Earth to Moon}}$ **(a_c is just the acceleration due to Earth's gravity evaluated at the moon's position). This is Newton's famous ″falling apple″ observation.**

69 •• (*a*) What are the period and speed of the motion of a person on a carousel if the person has an acceleration magnitude of 0.80 m/s^2 when she is standing 4.0 m from the axis? (*b*) What are her acceleration magnitude and speed if she then moves in to a distance of 2.0 m from the carousel center and the carousel keeps rotating with the same period?

Picture the Problem The person riding on this carousel experiences a centripetal acceleration due to the fact that her velocity is continuously changing. Use the expression for centripetal acceleration to relate her speed to her centripetal acceleration and the relationship between distance, speed, and time to find the period of her motion.

In general, the acceleration and period of any object moving in a circular path at constant speed are given by:

$$a_c = \frac{v^2}{r} \qquad (1)$$

and

$$T = \frac{2\pi r}{v} \qquad (2)$$

Solving equation (1) for v yields:

$$v = \sqrt{a_c r} \qquad (3)$$

Substituting for v in equation (2) yields:

$$T = \frac{2\pi r}{\sqrt{a_c r}} = 2\pi\sqrt{\frac{r}{a_c}} \qquad (4)$$

(*a*) Substitute numerical values in equation (3) and evaluate v for a person standing 4.0 m from the axis:

$$v = \sqrt{\left(0.80\ \text{m/s}^2\right)\left(4.0\ \text{m}\right)} = 1.79\ \text{m/s} = \boxed{1.8\ \text{m/s}}$$

Substitute numerical values in equation (4) and evaluate T for a person standing 4.0 m from the axis:

$$T = 2\pi\sqrt{\frac{4.0\ \text{m}}{0.80\ \text{m/s}^2}} = 14.05\ \text{s} = \boxed{14\ \text{s}}$$

(*b*) Solve equation (2) for v to obtain:

$$v = \frac{2\pi r}{T}$$

For a person standing 2.0 m from the axis:

$$v = \frac{2\pi(2.0\ \text{m})}{14.05\ \text{s}} = 0.894\ \text{m/s} = \boxed{0.89\ \text{m/s}}$$

From equation (1) we have, for her acceleration:

$$a_c = \frac{(0.894\ \text{m/s})^2}{2.0\ \text{m}} = \boxed{0.40\ \text{m/s}^2}$$

70 ••• Pulsars are neutron stars that emit X rays and other radiation in such a way that we on Earth receive pulses of radiation from the pulsars at regular intervals equal to the period that they rotate. Some of these pulsars rotate with periods as short as 1 ms! The Crab Pulsar, located inside the Crab Nebula in the constellation Orion, has a period currently of length 33.085 ms. It is estimated to have an equatorial radius of 15 km, which is an average radius for a neutron star. (*a*) What is the value of the centripetal acceleration of an object on the surface and at the equator of the pulsar? (*b*) Many pulsars are observed to have periods that lengthen slightly with time, a phenomenon called "spin down." The rate of slowing of the Crab Pulsar is 3.5×10^{-13} s per second, which implies that if this rate remains constant, the Crab Pulsar will stop spinning in 9.5×10^{10} s (about 3000 years). What is the tangential acceleration of an object on the equator of this neutron star?

Picture the Problem We need to know the speed of a point on the equator where the object sits. The circular path that the object at the equator undergoes has a radius equal to that of the star, and therefore is simply the equatorial circumference – and is traversed in a time interval of one period, 33.085 ms. In (*b*) we need only consider the change in speed of an object on the surface over the course of this time period.

(*a*) The centripetal acceleration of the object is given by:

$$a_c = \frac{v^2}{r} \qquad (1)$$

The speed of the object is equal to related to the radius of its path and the period of its motion:

$$v = \frac{2\pi r}{T}$$

Substituting for v in equation (1) yields:

$$a_c = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

Substitute numerical values and evaluate a_c:

$$a_c = \frac{4\pi^2(15\ \text{km})}{(33.085\ \text{ms})^2} = \boxed{5.4\times10^8\ \text{m/s}^2}$$

(*b*) The tangential acceleration of an object on the equator of the neutron star is given by:

$$a_{\mathrm{t}}=\frac{\Delta v_{\mathrm{t}}}{\Delta t_{\text{spin down}}}=\frac{v_{\text{t, final}}-v_{\text{t, initial}}}{\Delta t_{\text{spin down}}}$$

Because $\boldsymbol{v}_{\text{r,final}}=0$:

$$a_{\mathrm{t}}=\frac{-v_{\text{t, initial}}}{\Delta t_{\text{spin down}}}$$

We can obtain the initial speed of the object from equation (1):

$$v_{\text{t, initial}}=\frac{2\pi r}{T}$$

Substitute for $\boldsymbol{v}_{\text{r,final}}=0$ to obtain:

$$a_{\mathrm{t}}=\frac{-\dfrac{2\pi r}{T}}{\Delta t_{\text{spin down}}}=-\frac{2\pi r}{\Delta t_{\text{spin down}}T}$$

Substitute numerical values and evaluate $\boldsymbol{a}_{\mathrm{t}}$:

$$a_{\mathrm{t}}=-\frac{2\pi(15\text{ km})}{(9.5\times10^{10}\text{ s})(33.085\times10^{-3}\text{ s})}=\boxed{-3.0\times10^{-5}\text{ m/s}^2}$$

71 ••• Human blood contains plasma, platelets and blood cells. To separate the plasma from other components, centrifugation is used. Effective centrifugation requires subjecting blood to an acceleration of 2000*g* or more. In this situation, assume that blood is contained in test tubes that are 15 cm long and are full of blood. These tubes ride in the centrifuge tilted at an angle of 45.0° above the horizontal (See Figure 3-34.) (*a*) What is the distance of a sample of blood from the rotation axis of a centrifuge rotating at 3500 rpm, if it has an acceleration of 2000*g*? (*b*) If the blood at the center of the tubes revolves around the rotation axis at the radius calculated in Part (*a*), calculate the centripetal accelerations experienced by the blood at each end of the test tube. Express all accelerations as multiples of *g*.

Picture the Problem The equations that describe the centripetal acceleration and speed of objects moving in circular paths at constant speed can be used to find the distance of the sample of blood from the rotation axis as well as the accelerations experienced by the blood at each end of the test tube under the conditions described in the problem statement.

(*a*) The centripetal acceleration of the blood sample is given by:

$$a_{\mathrm{c}}=\frac{v^2}{r}$$

The speed of the blood's revolution is given by:

$$v=\frac{2\pi r}{T}=2\pi rf$$

where *f* is the frequency of revolution.

Substitute for v in the expression for a_c to obtain:

$$a_c = \frac{(2\pi rf)^2}{r} = 4\pi^2 f^2 r \qquad (1)$$

Solving for r yields:

$$r = \frac{a_c}{4\pi^2 f^2}$$

or, because $a_c = g$,

$$r = \frac{2000g}{4\pi^2 f^2}$$

Substitute numerical values and evaluate r:

$$r = \frac{2000(9.81\ \text{m/s}^2)}{4\pi^2\left(3500\ \frac{\text{rev}}{\text{min}} \times \frac{1\ \text{min}}{60\ \text{s}}\right)^2} = 0.146\ \text{m}$$

$$= \boxed{15\ \text{cm}}$$

(b) The center of the tube undergoes uniform circular motion with a radius of 15cm. Because the tubes are tilted at a 45 degree angle, the top surface of the blood is undergoing uniform circular motion with a radius of:

$$r_{\text{top}} = 14.6\ \text{cm} - (7.5\ \text{cm})\cos 45° = 9.30\ \text{cm}$$

The bottom surface is moving in a circle of radius:

$$r_{\text{bottom}} = 14.6\ \text{cm} + (7.5\ \text{cm})\cos 45° = 19.90\ \text{cm}$$

The range of accelerations can be found from equation (1):

$$a_c = 4\pi^2 f^2 r$$

The minimum acceleration corresponds to r_{top}:

$$a_{\min} = 4\pi^2 f^2 r_{\text{top}}$$

Substitute numerical values and evaluate $a_{\min}$:

$$a_{\min} = 4\pi^2\left(3500\ \frac{\text{rev}}{\text{min}} \times \frac{1\ \text{min}}{60\ \text{s}}\right)^2 (9.30\ \text{cm}) = 1.249\times10^4\ \text{m/s}^2 \approx \boxed{1300g}$$

The maximum acceleration corresponds to r_{bottom}:

$$a_{\max} = 4\pi^2 f^2 r_{\text{bottom}}$$

Substitute numerical values and evaluate a_{max}:

$$a_{max} = 4\pi^2\left(3500\,\frac{\text{rev}}{\text{min}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)^2(19.90\,\text{cm}) = 2.673\times10^4\ \text{m/s}^2 \approx \boxed{2700g}$$

The range of accelerations is 1300*g* to 2700*g*.

Projectile Motion and Projectile Range

72 • While trying out for the position of pitcher on your high school baseball team, you throw a fastball at 87 mi/h toward home plate, which is 18.4 m away. How far does the ball drop due to effects of gravity by the time it reaches home plate? (Ignore any effects due to air resistance.)

Picture the Problem Neglecting air resistance, the accelerations of the ball are constant and the horizontal and vertical motions of the ball are independent of each other. We can use the horizontal motion to determine the time-of-flight and then use this information to determine the distance the ball drops. Choose a coordinate system in which the origin is at the point of release of the ball, downward is the positive y direction, and the horizontal direction is the positive x direction.

Express the vertical displacement of the ball:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y} = 0$ and $a_y = g$,

$$\Delta y = \tfrac{1}{2}g(\Delta t)^2 \qquad (1)$$

Use $v_x = \Delta x/\Delta t$ to express the time of flight:

$$\Delta t = \frac{\Delta x}{v_x}$$

Substitute for Δt in equation (1) to obtain:

$$\Delta y = \tfrac{1}{2}g\left(\frac{\Delta x}{v_x}\right)^2 = \frac{g(\Delta x)^2}{2(v_x)^2}$$

Substitute numerical values and evaluate Δy:

$$\Delta y = \frac{\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(18.4\,\text{m})^2}{2\left(87\,\frac{\text{mi}}{\text{h}}\cdot\frac{0.4470\,\frac{\text{m}}{\text{s}}}{1\,\frac{\text{mi}}{\text{h}}}\right)^2} = \boxed{1.1\,\text{m}}$$

73 • A projectile is launched with speed v_0 at an angle of θ_0 above the horizontal. Find an expression for the maximum height it reaches above its starting point in terms of v_0, θ_0, and g. (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the maximum height achieved by a projectile depends on the vertical component of its initial velocity.

Use a constant-acceleration equation to relate the displacement of a projectile to its initial and final speeds and its acceleration:

$$v_y^2 = v_{0y}^2 + 2a_y\Delta y \Rightarrow \Delta y = \frac{v_y^2 - v_{0y}^2}{2a_y}$$

Because $v_y = 0$ and $a_y = -g$:

$$\Delta y = \frac{v_{0y}^2}{2g}$$

The vertical component of the projectile's initial velocity is:

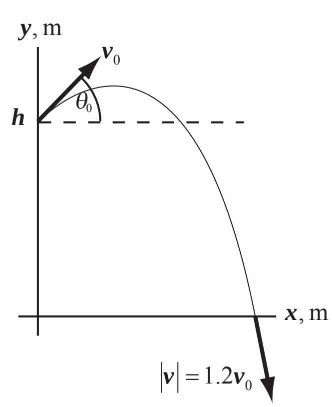
$$v_{0y} = v_0 \sin\theta_0$$

Setting $\Delta y = h$ and substituting for v_{0y} yields:

$$h = \boxed{\frac{(v_0 \sin\theta_0)^2}{2g}}$$

74 •• A cannonball is fired with initial speed v_0 at an angle 30º above the horizontal from a height of 40 m above the ground. The projectile strikes the ground with a speed of $1.2v_0$. Find v_0. (Ignore any effects due to air resistance.)

Picture the Problem Choose the coordinate system shown to the right. Because, in the absence of air resistance, the horizontal and vertical speeds are independent of each other, we can use constant-acceleration equations to relate the impact speed of the projectile to its components.



The horizontal and vertical velocity components are:

$$v_{0x} = v_x = v_0 \cos\theta_0$$

and

$$v_{0y} = v_0 \sin\theta_0$$

Using a constant-acceleration equation, relate the vertical component of the velocity to the vertical displacement of the projectile:

$$v_y^2 = v_{0y}^2 + 2a_y\Delta y$$

or, because $a_y = -g$ and $\Delta y = -h$,

$$v_y^2 = (v_0 \sin\theta_0)^2 + 2gh$$

Express the relationship between the magnitude of a velocity vector and its components, substitute for the components, and simplify to obtain:

$$\begin{aligned} v^2 &= v_x^2 + v_y^2 = (v_0 \cos\theta_0)^2 + v_y^2 \\ &= v_0^2(\sin^2\theta_0 + \cos^2\theta_0) + 2gh \\ &= v_0^2 + 2gh \end{aligned}$$

Substituting for v and solving for v_0 gives:

$$(1.2v_0)^2 = v_0^2 + 2gh \Rightarrow v_0 = \sqrt{\frac{2gh}{0.44}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{\frac{2(9.81\,\text{m/s}^2)(40\,\text{m})}{0.44}} = \boxed{42\,\text{m/s}}$$

Remarks: Note that v is independent of θ_0. This will be more obvious once conservation of energy has been studied.

75 •• In Figure 3.35, what is the minimum initial speed of the dart if it is to hit the monkey before the monkey hits the ground, which is 11.2 m below the initial position of the monkey, if x is 50 m and h = 10 m? (Ignore any effects due to air resistance.)

Picture the Problem Example 3-12 shows that the dart will hit the monkey unless the dart hits the ground before reaching the monkey's line of fall. What initial speed does the dart need in order to just reach the monkey's line of fall? First, we will calculate the fall time of the monkey, and then we will calculate the horizontal component of the dart's velocity.

Relate the horizontal speed of the dart to its launch angle and initial velocity v_0:

$$v_x = v_0 \cos\theta \Rightarrow v_0 = \frac{v_x}{\cos\theta}$$

Use the definition of v_x and the fact that, in the absence of air resistance, it is constant to obtain:

$$v_x = \frac{\Delta x}{\Delta t}$$

Substituting in the expression for v_0 yields:

$$v_0 = \frac{\Delta x}{(\cos\theta)\Delta t} \qquad (1)$$

Using a constant-acceleration equation, relate the monkey's fall distance to the fall time:

$$\Delta h = \tfrac{1}{2} g (\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta h}{g}}$$

Substituting for Δt in equation (1) yields:

$$v_0 = \frac{\Delta x}{\cos\theta}\sqrt{\frac{g}{2\Delta h}} \qquad (2)$$

Let θ be the angle the barrel of the dart gun makes with the horizontal. Then:

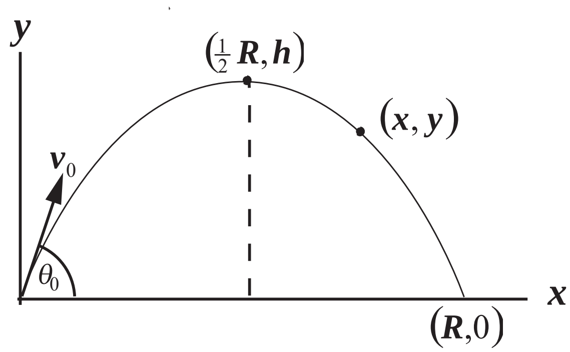
$$\theta = \tan^{-1}\left(\frac{10\,\text{m}}{50\,\text{m}}\right) = 11.3°$$

Substitute numerical values in equation (2) and evaluate v_0:

$$v_0 = \frac{50\text{ m}}{\cos 11.3°}\sqrt{\frac{9.81\,\text{m/s}^2}{2(11.2\,\text{m})}} = \boxed{34\text{ m/s}}$$

76 •• A projectile is launched from ground level with an initial speed of 53 m/s. Find the launch angle (the angle the initial velocity vector is above the horizontal) such that the maximum height of the projectile is equal to its horizontal range. (Ignore any effects due to air resistance.)

Picture the Problem Choose the coordinate system shown in the figure to the right. In the absence of air resistance, the projectile experiences constant acceleration in both the x and y directions. We can use the constant-acceleration equations to express the x and y coordinates of the projectile along its trajectory as functions of time. The elimination of the parameter t will yield an expression for y as a function of x that we can evaluate at (R, 0) and (R/2, h). Solving these equations simultaneously will yield an expression for θ.



Express the position coordinates of the projectile along its flight path in terms of the parameter t:

$$x = (v_0\cos\theta)t$$

and

$$y = (v_0\sin\theta)t - \tfrac{1}{2}gt^2$$

Eliminate t from these equations to obtain:

$$y = (\tan\theta)x - \frac{g}{2v_0^2\cos^2\theta}x^2 \qquad (1)$$

Evaluate equation (1) at $(R, 0)$ to obtain:

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$

Evaluate equation (1) at $(R/2, h)$ to obtain:

$$h = \frac{(v_0 \sin\theta)^2}{2g}$$

Equating R and h yields:

$$\frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{(v_0 \sin\theta)^2}{2g}$$

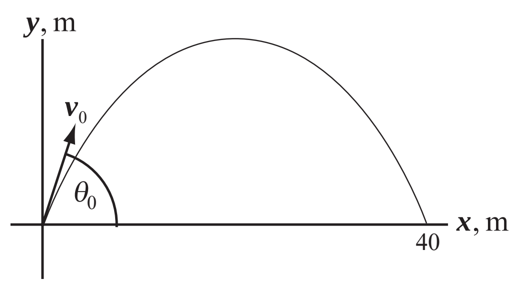
Solving for θ gives:

$$\theta = \tan^{-1}(4) = \boxed{76°}$$

Note that this result is independent of v_0.

77 •• [SSM] A ball launched from ground level lands 2.44 s later 40.0-m away from the launch point. Find the magnitude of the initial velocity vector and the angle it is above the horizontal. (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the motion of the ball is uniformly accelerated and its horizontal and vertical motions are independent of each other. Choose the coordinate system shown in the figure to the right and use constant-acceleration equations to relate the x and y components of the ball's initial velocity.



Express θ_0 in terms of v_{0x} and v_{0y}:

$$\theta_0 = \tan^{-1}\left(\frac{v_{0y}}{v_{0x}}\right) \quad (1)$$

Use the Pythagorean relationship between the velocity and its components to express v_0:

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} \quad (2)$$

Using a constant-acceleration equation, express the vertical speed of the projectile as a function of its initial upward speed and time into the flight:

$$v_y = v_{0y} + a_y t \Rightarrow v_{0y} = v_y - a_y t$$

Because $v_y = 0$ halfway through the flight (at maximum elevation) and $a_y = -g$:

$$v_{0y} = g t_{\text{max elevation}}$$

Substitute numerical values and evaluate v_{0y}:

$$v_{0y}=(9.81\text{ m/s}^2)(1.22\text{ s})=11.97\text{ m/s}$$

Because there is no acceleration in the horizontal direction, v_{0x} can be found from:

$$v_{0x}=\frac{\Delta x}{\Delta t}=\frac{40.0\text{ m}}{2.44\text{ s}}=16.39\text{ m/s}$$

Substitute for v_{0x} and v_{0y} in equation (2) and evaluate v_0:

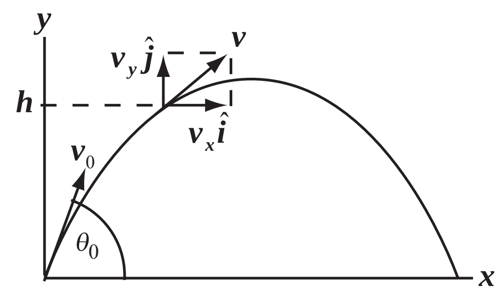
$$v_0=\sqrt{(16.39\text{ m/s})^2+(11.97\text{ m/s})^2}=\boxed{20.3\text{ m/s}}$$

Substitute for v_{0x} and v_{0y} in equation (1) and evaluate θ_0 :

$$\theta_0=\tan^{-1}\left(\frac{11.97\text{ m/s}}{16.39\text{ m/s}}\right)=\boxed{36.1°}$$

78 •• Consider a ball that is launched from ground level with initial speed v_0 at an angle θ_0 above the horizontal. If we consider its speed to be v at height h above the ground, show that for a given value of h, v is independent of θ_0. (Ignore any effects due to air resistance.)

Picture the Problem In the absence of friction, the acceleration of the ball is constant and we can use the constant-acceleration equations to describe its motion. The figure shows the launch conditions and an appropriate coordinate system. The speeds v, v_x, and v_y are related by the Pythagorean Theorem.



The squares of the vertical and horizontal components of the object's velocity are:

$$v_y^2=v_0^2\sin^2\theta_0-2gh$$

and

$$v_x^2=v_0^2\cos^2\theta_0$$

The relationship between these variables is:

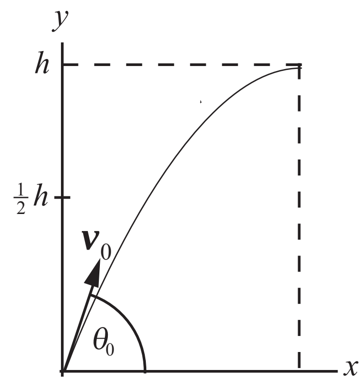
$$v^2=v_x^2+v_y^2$$

Substitute for v_x and v_y and simplify to obtain:

$$\begin{aligned}v^2&=v_0^2\cos^2\theta_0+v_0^2\sin^2\theta_0-2gh\\&=v_0^2(\cos^2\theta_0+\sin^2\theta_0)-2gh\\&=\boxed{v_0^2-2gh}\end{aligned}$$

79 ••• At $\frac{1}{2}$ of its maximum height, the speed of a projectile is $\frac{3}{4}$ of its initial speed. What was its launch angle? (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the projectile experiences constant acceleration during its flight and we can use constant-acceleration equations to relate the speeds at half the maximum height and at the maximum height to the launch angle θ_0 of the projectile.



The angle θ the initial velocity makes with the horizontal is related to the initial velocity components:

$$\theta_0 = \tan^{-1}\left(\frac{v_{0y}}{v_{0x}}\right) \quad (1)$$

Using a constant-acceleration equation, relate the vertical component of the speed to the projectile to its acceleration and displacement:

$$v_y^2 = v_{0y}^2 + 2a\Delta y \quad (2)$$

When the projectile is at the top of its trajectory $\Delta y = h$ and $v_y = 0$. Hence:

$$0 = v_{0y}^2 - 2gh \Rightarrow v_{0y}^2 = 2gh$$

When $\Delta y = \frac{1}{2}h$ equation (2) becomes:

$$v_y^2 = v_{0y}^2 - gh \quad (3)$$

Substituting for v_{0y}^2 and simplifying yields:

$$v_y^2 = 2gh - gh = gh$$

We are given that $v = \frac{3}{4}v_0$ when $\Delta y = \frac{1}{2}h$. Square both sides of $v = \frac{3}{4}v_0$ and express this using the components of the velocity.

$$v_{0x}^2 + v_y^2 = \left(\tfrac{3}{4}\right)^2\left(v_{0x}^2 + v_{0y}^2\right) \quad (4)$$

where we have used v_{0x} for v_x because the x component of the velocity remains constant.

Solving equation (2) for v_{0y}^2 gives:

Substitute for v_{0y}^2 in equation (3) to obtain:

$$v_y^2 = 2gh - gh = gh$$

Substituting for v_y^2 and v_{0y}^2 in equation (4) yields:

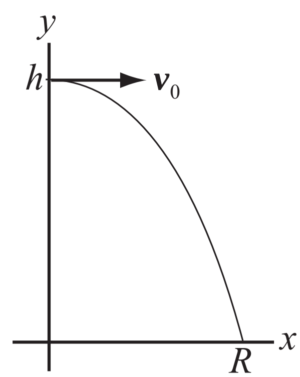
$$v_{0x}^2 + gh = \tfrac{9}{16}\left(v_{0x}^2 + 2gh\right) \Rightarrow v_{0x}^2 = \tfrac{2}{7}gh$$

Finally, substituting for v_{0y} and v_{0x} in equation (1) and simplifying gives:

$$\theta_0 = \tan^{-1}\left(\frac{\sqrt{2gh}}{\sqrt{\frac{2}{7}gh}}\right) = \tan^{-1}\left(\sqrt{7}\right) = \boxed{69.3°}$$

80 •• A cargo plane is flying horizontally at an altitude of 12 km with a speed of 900 km/h when a large crate falls out of the rear-loading ramp. (Ignore any effects due to air resistance.) (*a*) How long does it take the crate to hit the ground? (*b*) How far horizontally is the crate from the point where it fell off when it hits the ground? (*c*) How far is the crate from the aircraft when the crate hits the ground, assuming that the plane continues to fly with the same velocity?

Picture the Problem The horizontal speed of the crate, in the absence of air resistance, is constant and equal to the speed of the cargo plane. Choose a coordinate system in which the direction the plane is moving is the positive x direction and downward is the positive y direction and apply the constant-acceleration equations to describe the crate's displacements at any time during its flight.



(*a*) Using a constant-acceleration equation, relate the vertical displacement of the crate Δy to the time of fall Δt:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}g(\Delta t)^2$$

or, because $v_{0y} = 0$ and $\Delta y = h$,

$$h = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2h}{g}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{2(12\times10^3\ \text{m})}{9.81\,\text{m/s}^2}} = 49.46\,\text{s} = \boxed{49\,\text{s}}$$

(*b*) The horizontal distance traveled in time Δt is:

$$R = \Delta x = v_{0x}\Delta t$$

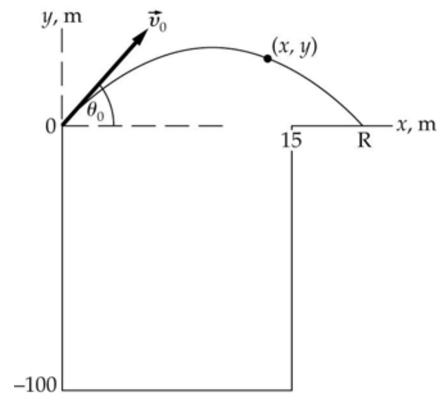
Substitute numerical values and evaluate R:

$$R = (900\,\text{km/h})\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)(49.46\,\text{s}) = \boxed{12\,\text{km}}$$

(*c*) Because the velocity of the plane is constant, it will be directly over the crate when it hits the ground; that is, the distance to the aircraft will be the elevation of the aircraft. Therefore $\Delta y = \boxed{12\,\text{km}}$.

81 •• [SSM] Wile E. Coyote (*Carnivorous hungribilous*) is chasing the Roadrunner (*Speedibus cantcatchmi*) yet again. While running down the road, they come to a deep gorge, 15.0 m straight across and 100 m deep. The Roadrunner launches himself across the gorge at a launch angle of 15º above the horizontal, and lands with 1.5 m to spare. (*a*) What was the Roadrunner's launch speed? (*b*) Wile E. Coyote launches himself across the gorge with the same initial speed, but at a different launch angle. To his horror, he is short the other lip by 0.50 m. What was his launch angle? (Assume that it was less than 15º.)

Picture the Problem In the absence of air resistance, the accelerations of both Wiley Coyote and the Roadrunner are constant and we can use constant-acceleration equations to express their coordinates at any time during their leaps across the gorge. By eliminating the parameter t between these equations, we can obtain an expression that relates their y coordinates to their x coordinates and that we can solve for their launch angles.



(*a*) Using constant-acceleration equations, express the x coordinate of the Roadrunner while it is in flight across the gorge:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $a_x = 0$ and $v_{0x} = v_0 \cos\theta_0$,

$$x = (v_0 \cos\theta_0)t \qquad (1)$$

Using constant-acceleration equations, express the y coordinate of the Roadrunner while it is in flight across the gorge:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $y_0 = 0$, $a_y = -g$ and $v_{0y} = v_0 \sin\theta_0$,

$$y = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \qquad (2)$$

Eliminate the variable t between equations (1) and (2) to obtain:

$$y = (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2 \qquad (3)$$

When $y = 0$, $x = R$ and equation (3) becomes:

$$0 = (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, solve for v_0:

$$v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{\frac{(16.5\,\text{m})(9.81\,\text{m/s}^2)}{\sin 30°}} = \boxed{18\,\text{m/s}}$$

(*b*) Letting *R* represent Wiley's range, solve equation (1) for his launch angle:

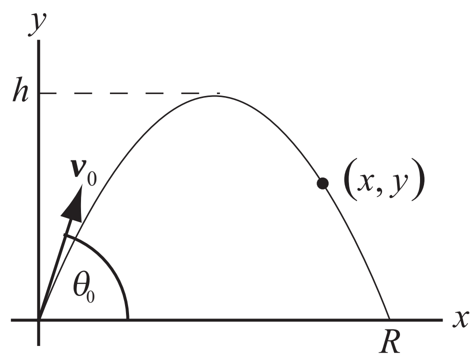
$$\theta_0 = \frac{1}{2}\sin^{-1}\left(\frac{Rg}{v_0^2}\right)$$

Substitute numerical values and evaluate θ_0:

$$\theta_0 = \frac{1}{2}\sin^{-1}\left[\frac{(14.5\,\text{m})(9.81\,\text{m/s}^2)}{(18.0\,\text{m/s})^2}\right] = \boxed{13°}$$

82 •• A cannon barrel is elevated 45º above the horizontal. It fires a ball with a speed of 300 m/s. (*a*) What height does the ball reach? (*b*) How long is the ball in the air? (*c*) What is the horizontal range of the cannon ball? (Ignore any effects due to air resistance.)

Picture the Problem Because, in the absence of air resistance, the vertical and horizontal accelerations of the cannonball are constant, we can use constant-acceleration equations to express the ball's position and velocity as functions of time and acceleration. The maximum height of the ball and its time-of-flight are related to the components of its launch velocity.



(*a*) Using a constant-acceleration equation, relate *h* to the initial and final speeds of the cannon ball:

$$v^2 = v_{0y}^2 + 2a_y\Delta y$$

or, because $v = 0$, $a_y = -g$, and $\Delta y = h$,

$$0 = v_{0y}^2 - 2gh \Rightarrow h = \frac{v_{0y}^2}{2g}$$

The vertical component of the firing speed is given by:

$$v_{0y} = v_0\sin\theta_0$$

Substituting for v_{0y} gives:

$$h = \frac{v_0^2\sin^2\theta_0}{2g}$$

Substitute numerical values and evaluate *h*:

$$h = \frac{(300\,\text{m/s})^2\sin^2 45°}{2(9.81\,\text{m/s}^2)} = \boxed{2.3\,\text{km}}$$

(*b*) The total flight time of the cannon ball is given by:

$$\Delta t = t_{\text{up}} + t_{\text{dn}} = 2t_{\text{up}} = 2\frac{v_{0y}}{g} = \frac{2v_0\sin\theta_0}{g}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(300\ \text{m/s})\sin 45°}{9.81\ \text{m/s}^2} = 43.2\ \text{s}$$
$$= \boxed{43\ \text{s}}$$

(*c*) Express the *x* coordinate of the cannon ball as a function of time:

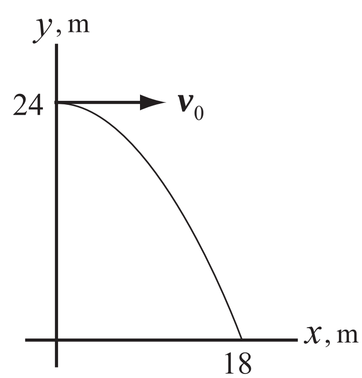
$$x = v_{0x}\Delta t = (v_0 \cos\theta_0)\Delta t$$

Evaluate *x* (= *R*) when Δt = 43.2 s:

$$x = [(300\ \text{m/s})\cos 45°](43.2\ \text{s})$$
$$= \boxed{9.2\ \text{km}}$$

83 •• [SSM] A stone thrown horizontally from the top of a 24-m tower hits the ground at a point 18 m from the base of the tower. (Ignore any effects due to air resistance.) (*a*) Find the speed with which the stone was thrown. (*b*) Find the speed of the stone just before it hits the ground.

Picture the Problem Choose a coordinate system in which the origin is at the base of the tower and the *x*- and *y*-axes are as shown in the figure to the right. In the absence of air resistance, the horizontal speed of the stone will remain constant during its fall and a constant-acceleration equation can be used to determine the time of fall. The final velocity of the stone will be the vector sum of its *x* and *y* components.



Because the stone is thrown horizontally:

$$v_x = v_{0x} = \frac{\Delta x}{\Delta t} \qquad (1)$$

(*a*) Using a constant-acceleration equation, express the vertical displacement of the stone as a function of the fall time:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because v_{0y} = 0 and $a = -g$,

$$\Delta y = -\tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{-\frac{2\Delta y}{g}}$$

Substituting for Δt in equation (1) and simplifying yields:

$$v_x = \Delta x\sqrt{\frac{g}{-2\Delta y}}$$

Substitute numerical values and evaluate v_x:

$$v_x = (18\ \text{m})\sqrt{\frac{9.81\ \text{m/s}^2}{-2(-24\ \text{m})}} = \boxed{8.1\ \text{m/s}}$$

(*b*) The speed with which the stone hits the ground is related to the *x* and *y* components of its speed:

$$v=\sqrt{v_x^2+v_y^2} \qquad (2)$$

The *y* component of the stone's velocity at time *t* is:

$$v_y=v_{0y}-gt$$

or, because $v_{0y}=0$,

$$v_y=-gt$$

Substitute for v_x and v_y in equation (2) and simplify to obtain:

$$v=\sqrt{\left(\Delta x\sqrt{\frac{g}{-2\Delta y}}\right)^2+(-gt)^2}$$

$$=\sqrt{\frac{g(\Delta x)^2}{-2\Delta y}+g^2t^2}$$

Substitute numerical values and evaluate *v*:

$$v=\sqrt{\frac{(9.81\,\text{m/s}^2)(18\,\text{m})^2}{-2(-24\,\text{m})}+(9.81\,\text{m/s}^2)^2(2.21\,\text{s})^2}=\boxed{23\,\text{m/s}}$$

84 •• A projectile is fired into the air from the top of a 200-m cliff above a valley (Figure 3-36). Its initial velocity is 60 m/s at 60º above the horizontal. Where does the projectile land? (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the acceleration of the projectile is constant and its horizontal and vertical motions are independent of each other. We can use constant-acceleration equations to express the horizontal and vertical displacements of the projectile in terms of its time-of-flight.

Using a constant-acceleration equation, express the horizontal displacement of the projectile as a function of time:

$$\Delta x=v_{0x}\Delta t+\tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x}=v_0\cos\theta_0$ and $a_x=0$,

$$\Delta x=(v_0\cos\theta_0)\Delta t$$

Using a constant-acceleration equation, express the vertical displacement of the projectile as a function of time:

$$\Delta y=v_{0y}\Delta t+\tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y}=v_0\sin\theta_0$ and $a_y=-g$,

$$\Delta y=(v_0\cos\theta_0)\Delta t-\tfrac{1}{2}g(\Delta t)^2$$

Substitute numerical values to obtain the quadratic equation:

$$-200\,\text{m}=(60\text{m/s})(\sin 60°)\Delta t-\tfrac{1}{2}(9.81\text{m/s}^2)(\Delta t)^2$$

Use the quadratic formula or your graphing calculator to obtain:

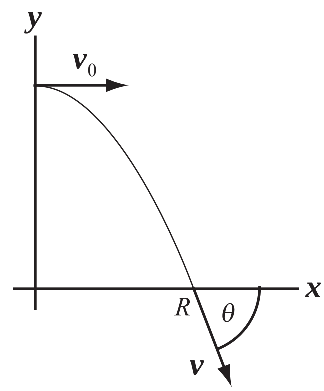
$\Delta t = 13.6\ \text{s}$

Substitute for Δt and evaluate the horizontal distance traveled by the projectile:

$$\Delta x = (60\ \text{m/s})(\cos 60°)(13.6\ \text{s}) = \boxed{0.41\,\text{km}}$$

85 •• The range of a cannonball fired horizontally from a cliff is equal to the height of the cliff. What is the direction of the velocity vector of the projectile as it strikes the ground? (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the acceleration of the cannonball is constant and its horizontal and vertical motions are independent of each other. Choose the origin of the coordinate system to be at the base of the cliff and the axes directed as shown and use constant-acceleration equations to describe both the horizontal and vertical displacements of the cannonball.



Express the direction of the velocity vector when the projectile strikes the ground:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad (1)$$

Express the vertical displacement using a constant-acceleration equation:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y} = 0$ and $a_y = -g$,

$$\Delta y = -\tfrac{1}{2}g(\Delta t)^2$$

Set $\Delta x = -\Delta y$ ($R = -h$) to obtain:

$$\Delta x = v_x\Delta t = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow v_x = \frac{\Delta x}{\Delta t} = \tfrac{1}{2}g\Delta t$$

Find the y component of the projectile as it hits the ground:

$$v_y = v_{0y} + a\Delta t = -g\Delta t = -2v_x$$

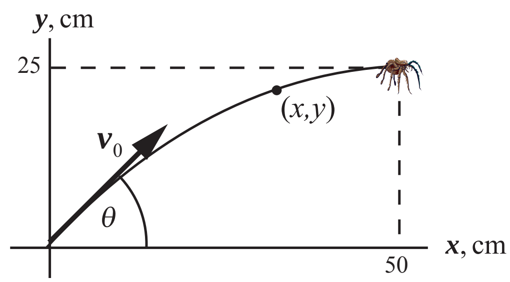
Substituting for v_y in equation (1) yields:

$$\theta = \tan^{-1}\left(\frac{-2v_x}{v_x}\right) = \tan^{-1}(-2) = \boxed{-63.4°}$$

86 •• An archerfish launches a droplet of water from the surface of a small lake at an angle of 60° above the horizontal. He is aiming at a juicy spider sitting on a leaf 50 cm to the east and on a branch 25 cm above the water surface. The

fish is trying to knock the spider into the water so that the fish may eat the spider. (*a*) What must the speed of the water droplet be for the fish to be successful? (*b*) When it hits the spider, is the droplet rising or falling?

Picture the Problem The diagram to the right shows the trajectory of the water droplet launched by the archerfish. We can use constant-acceleration equations to derive expressions for the required speed of the droplet and for the vertical velocity of the droplet as a function of its position.



(*a*) Use a constant-acceleration equation to express the *x*- and *y*-coordinates of the droplet:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$
and
$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

Because $x_0 = y_0 = 0$, $v_{0x} = v_0\cos\theta$, $v_{0y} = v_0\sin\theta$, and, in the absence of air resistance, $a_x = 0$:

$$x = (v_0 \cos\theta)t \qquad (1)$$
and
$$y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \qquad (2)$$

Solve equation (1) for *t* to obtain:

$$t = \frac{x}{v_0\cos\theta}$$

Substituting for *t* in equation (2) yields:

$$y = (v_0 \sin\theta)\left(\frac{x}{v_0\cos\theta}\right) - \tfrac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$$

Simplify this equation to obtain:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2$$

Solving for v_0 yields:

$$v_0 = \sqrt{\frac{g}{2(x\tan\theta - y)\cos^2\theta}}\,x$$

When the droplet hits the spider, its coordinates must be $x = 0.50$ m and $y = 0.25$ m. Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{\frac{9.81\,\text{m/s}^2}{2((0.50\,\text{m})\tan 60° - 0.25\,\text{m})\cos^2 60°}}(0.50\,\text{m}) = 2.822\,\text{m/s} = \boxed{2.8\,\text{m/s}}$$

(*b*) Use a constant-acceleration equation to express the y component of the speed of the droplet as a function of time:

$$v_y(t) = v_{0y} + a_y t$$

or, because $v_{0y} = v_0 \sin\theta$ and $a_y = -g$,

$$v_y(t) = v_0 \sin\theta - gt \qquad (3)$$

From equation (1), the time-to-the-target, $t_{\text{to target}}$, is given by:

$$t_{\text{to target}} = \frac{x_{\text{target}}}{v_x} = \frac{x_{\text{target}}}{v_{0x}} = \frac{x_{\text{target}}}{v_0 \cos\theta}$$

where x_{target} is the x-coordinate of the target.

Substitute in equation (3) to obtain:

$$v_y(t_{\text{to target}}) = v_0 \sin\theta - g t_{\text{to target}} = v_0 \sin\theta - \frac{g x_{\text{target}}}{v_0 \cos\theta}$$

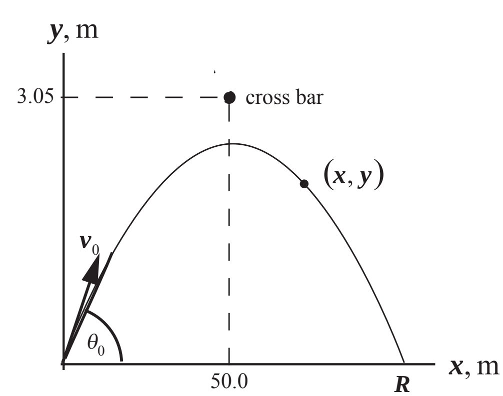
Substitute numerical values and evaluate $v_y(t_{\text{to target}})$:

$$v_y(t_{\text{to target}}) = (2.822\ \text{m/s})\sin 60° - \frac{(9.81\ \text{m/s}^2)(0.50\ \text{m})}{(2.822\ \text{m/s})\cos 60°} = -1.032\ \text{m/s}$$

Because the y-component of the velocity of the droplet is negative at the location of the spider, the droplet is falling and moving horizontally when it hits the spider.

87 •• [SSM] You are trying out for the position of place-kicker on a professional football team. With the ball teed up 50.0 m from the goalposts with a crossbar 3.05 m off the ground, you kick the ball at 25.0 m/s and 30° above the horizontal. (*a*) Is the field goal attempt good? (*b*) If so, by how much does it clear the bar? If not, by how much does it go under the bar? (*c*) How far behind the plane of the goalposts does the ball land?

Picture the Problem We can use constant-acceleration equations to express the x and y coordinates of the ball along its flight path. Eliminating t between the equations will leave us with an equation for y as a function of x that we can use to find the height of the ball when it has reached the cross bar. We can use this same equation to find the range of the ball and, hence, how far behind the plane of the goalposts the ball lands.



(*a*) Use a constant-acceleration equation to express the x coordinate of the ball as a function of time:

$$x(t) = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$,

$$x(t) = (v_0\cos\theta_0)t \Rightarrow t = \frac{x(t)}{v_0\cos\theta_0}$$

Use a constant-acceleration equation to express the y coordinate of the ball as a function of time:

$$y(t) = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $y_0 = 0$, $v_{0y} = v_0\sin\theta_0$, and $a_y = -g$,

$$y(t) = (v_0\sin\theta_0)t - \tfrac{1}{2}gt^2$$

Substituting for t yields:

$$y(x) = (v_0\sin\theta_0)\frac{x(t)}{v_0\cos\theta_0} - \tfrac{1}{2}g\left(\frac{x(t)}{v_0\cos\theta_0}\right)^2$$

Simplify to obtain:

$$y(x) = (\tan\theta_0)x(t) - \frac{g}{2v_0^2\cos^2\theta_0}(x(t))^2$$

Substitute numerical values and evaluate $y(50.0\text{ m})$:

$$y(50.0\text{ m}) = (\tan 30°)(50.0\text{ m}) - \frac{9.81\text{ m/s}^2}{2(25.0\text{ m/s})^2\cos^2 30°}(50.0\text{ m})^2 = 2.71\text{ m}$$

Because 2.71 m < 3.05 m, the ball goes under the crossbar and the kick is no good.

(*b*) The ball goes under the bar by:

$$d_{\text{under}} = 3.05\text{ m} - 2.71\text{ m} = \boxed{0.34\text{ m}}$$

(*c*) The distance the ball lands behind the goalposts is given by:

$$d_{\substack{\text{behind the}\\ \text{goal posts}}} = R - 50.0\text{ m} \qquad (3)$$

Evaluate the equation derived in (*a*) for $y = 0$ and $x(t) = R$:

$$0 = (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Solving for v_0 yields:

$$R = \frac{v_0^2\sin 2\theta_0}{g}$$

Substitute for R in equation (3) to obtain:

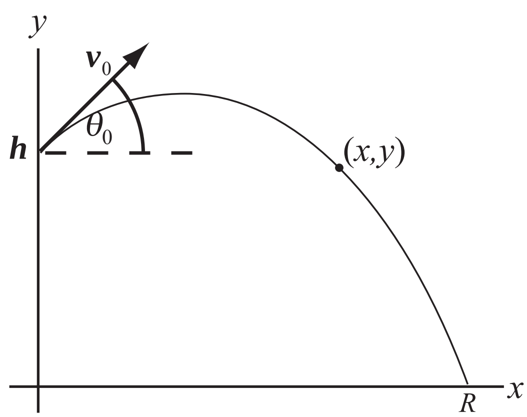
$$d_{\substack{\text{behind the}\\\text{goal posts}}} = \frac{v_0^2 \sin 2\theta_0}{g} - 50.0\text{ m}$$

Substitute numerical values and evaluate $d_{\substack{\text{behind the}\\\text{goal posts}}}$:

$$d_{\substack{\text{behind the}\\\text{goal posts}}} = \frac{(25.0\text{ m/s})^2 \sin 2(30°)}{9.81\text{ m/s}^2} - 50.0\text{ m} = \boxed{5.2\text{ m}}$$

88 •• The speed of an arrow fired from a compound bow is about 45.0 m/s. (*a*) A Tartar archer sits astride his horse and launches an arrow into the air, elevating the bow at an angle of 10º above the horizontal. If the arrow is 2.25 m above the ground at launch, what is the arrow's horizontal range? Assume that the ground is level, and ignore any effects due to air resistance. (*b*) Now assume that his horse is at full gallop and moving in the same direction as the direction the archer will fire the arrow. Also assume that the archer elevates the bow at the same elevation angle as in Part (*a*) and fires. If the horse's speed is 12.0 m/s, what is the arrow's horizontal range now?

Picture the Problem Choose a coordinate system in which the origin is at ground level. Let the positive x direction be to the right and the positive y direction be upward. We can apply constant-acceleration equations to obtain equations in time that relate the range to the initial horizontal speed and the height h to which the initial upward speed. Eliminating time from these equations will leave us with a quadratic equation in R, the solution to which will give us the range of the arrow. In (*b*), we'll find the launch speed and angle as viewed by an observer who is at rest on the ground and then use these results to find the arrow's range when the horse is moving at 12.0 m/s.



(*a*) Use constant-acceleration equations to express the horizontal and vertical coordinates of the arrow's motion:

$\Delta x = x - x_0 = v_{0x}t$

and

$y = h + v_{0y}t + \frac{1}{2}(-g)t^2$

where

$v_{0x} = v_0 \cos\theta_0$ and $v_{0y} = v_0 \sin\theta_0$

Solve the x-component equation for time:

$$t = \frac{\Delta x}{v_{0x}} = \frac{\Delta x}{v_0 \cos\theta_0}$$

Eliminating time from the y-component equation yields:

$$y = h + v_{0y}\frac{\Delta x}{v_{0x}} - \frac{1}{2}g\left(\frac{\Delta x}{v_{0x}}\right)^2$$

When $y = 0$, $\Delta x = R$ and:

$$0 = h + (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Solve for the range R to obtain:

$$R = \frac{v_0^2}{2g}\sin 2\theta_0\left(1 + \sqrt{1 + \frac{2gh}{v_0^2\sin^2\theta_0}}\right)$$

Substitute numerical values and evaluate R:

$$R = \frac{(45.0\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)}\sin 20°\left(1 + \sqrt{1 + \frac{2(9.81\,\text{m/s}^2)(2.25\,\text{m})}{(45.0\,\text{m/s})^2(\sin^2 10°)}}\right) = \boxed{82\,\text{m}}$$

(b) Express the speed of the arrow in the horizontal direction:

$$\begin{aligned} v_x &= v_{\text{arrow}} + v_{\text{archer}} \\ &= (45.0\,\text{m/s})\cos 10° + 12.0\,\text{m/s} \\ &= 56.32\,\text{m/s} \end{aligned}$$

Express the vertical speed of the arrow:

$$v_y = (45.0\,\text{m/s})\sin 10° = 7.814\,\text{m/s}$$

Express the angle of elevation from the perspective of someone on the ground:

$$\theta_0 = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Substitute numerical values and evaluate θ_0 :

$$\theta_0 = \tan^{-1}\left(\frac{7.814\,\text{m/s}}{56.32\,\text{m/s}}\right) = 7.899°$$

The arrow's speed relative to the ground is given by:

$$v_0 = \sqrt{v_x^2 + v_y^2}$$

Substitute numerical values and evaluate v_0:

$$\begin{aligned} v_0 &= \sqrt{(56.32\,\text{m/s})^2 + (7.814\,\text{m/s})^2} \\ &= 56.86\,\text{m/s} \end{aligned}$$

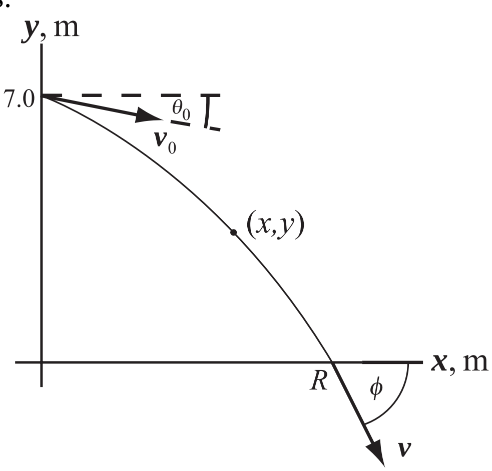
Substitute numerical values and evaluate R:

$$R=\frac{(56.86\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)}\sin(15.8°)\left(1+\sqrt{1+\frac{2(9.81\,\text{m/s}^2)(2.25\,\text{m})}{(56.86\,\text{m/s})^2(\sin^2 7.899°)}}\right)=\boxed{0.10\,\text{km}}$$

Remarks: An alternative solution for part (*b*) is to solve for the range in the reference frame of the archer and then add to it the distance the frame travels, relative to Earth, during the time of flight.

89 •• [SSM] The roof of a two-story house makes an angle of 30° with the horizontal. A ball rolling down the roof rolls off the edge at a speed of 5.0 m/s. The distance to the ground from that point is about two stories or 7.0 m. (*a*) How long is the ball in the air? (*b*) How far from the base of the house does it land? (*c*) What is its speed and direction just before landing?

Picture the Problem Choosing the coordinate system shown in the figure to the right, we can use a constant acceleration equation to express the *y* coordinate of the ball's position as a function of time. When the ball hits the ground, this position coordinate is zero and we can solve the resulting quadratic equation for the time-to-the-ground. Because, in the absence of air resistance, there is no acceleration in the horizontal direction we can find *R, v,* and ϕ using constant-acceleration equations.



(*a*) Use a constant-acceleration equation to relate the *y* coordinate of the ball to its time-in-flight:

$$y(t)=y_0+v_{0y}t+\tfrac{1}{2}a_y t^2$$

or, because $v_{0y}=v_0\sin\theta_0$ and $a_y=-g$,

$$y(t)=y_0+(v_0\sin\theta_0)t-\tfrac{1}{2}gt^2$$

$y(t_{\text{ground}})=0$ when the ball hits the ground and:

$$0=y_0+(v_0\sin\theta_0)t_{\text{ground}}-\tfrac{1}{2}gt_{\text{ground}}^2$$

Substitute numerical values to obtain:

$$0 = 7.0\text{ m} + (-5.0\text{ m/s})\sin 30^\circ\, t_{\text{ground}} - \tfrac{1}{2}(9.81\text{ m/s}^2)t_{\text{ground}}^2$$

Simplifying yields:

$$(4.91\text{ m/s}^2)t_{\text{ground}}^2 + (2.50\text{ m/s})t_{\text{ground}} - 7.0\text{ m} = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$t_{\text{ground}} = 0.966\text{ s or } -1.48\text{ s}$$

Because only the positive root has physical meaning:

$$t_{\text{ground}} = \boxed{0.97\text{ s}}$$

(*b*) The distance from the house *R* is related to the ball's horizontal speed and the time-to-ground:

$$R = v_{0x}t_{\text{ground}} = (v_0\cos\theta_0)t_{\text{ground}}$$

Substitute numerical values and evaluate *R*:

$$R = (5.0\text{ m/s})(\cos 30^\circ)(0.966\text{ s}) = \boxed{4.2\text{ m}}$$

(*c*) The speed and direction of the ball just before landing are given by:

$$v = \sqrt{v_x^2 + v_y^2} \qquad (1)$$

and

$$\phi = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad (2)$$

Express v_x and v_y in terms of v_0:

$$v_x = v_0\cos\theta_0$$

and

$$v_y = v_{0y} + a_y t = v_0\sin\theta_0 + gt$$

Substitute numerical values and evaluate v_x and v_y:

$$v_x = (5.0\text{ m/s})\cos 30^\circ = 4.330\text{ m/s}$$

and

$$v_y = (5.0\text{ m/s})\sin 30^\circ + (9.81\text{ m/s}^2)(0.966\text{ s}) = 11.98\text{ m/s}$$

Substituting numerical values in equations (1) and (2) yields:

$$v=\sqrt{(4.330\text{ m/s})^2+(11.98\text{ m/s})^2}=\boxed{13\text{ m/s}}$$

and

$$\phi=\tan^{-1}\left(\frac{11.98\text{ m/s}}{4.330\text{ m/s}}\right)=\boxed{70^\circ\text{ below the horizontal}}$$

90 •• Compute $dR/d\theta_0$ from $R=(v_0^2/g)\sin(2\theta_0)$ and show that setting $dR/d\theta_0=0$ gives $\theta_0=45°$ for the maximum range.

Picture the Problem An extreme value (i.e., a maximum or a minimum) of a function is determined by setting the appropriate derivative equal to zero. Whether the extremum is a maximum or a minimum can be determined by evaluating the second derivative at the point determined by the first derivative.

Evaluate $dR/d\theta_0$:

$$\frac{dR}{d\theta_0}=\frac{v_0^2}{g}\frac{d}{d\theta_0}\left[\sin(2\theta_0)\right]=\frac{2v_0^2}{g}\cos(2\theta_0)$$

Set $dR/d\theta_0=0$ for extrema:

$$\frac{2v_0^2}{g}\cos(2\theta_0)=0$$

Solve for θ_0 to obtain:

$$\theta_0=\tfrac{1}{2}\cos^{-1}(0)=45^\circ$$

Determine whether 45° corresponds to a maximum or a minimum value of R:

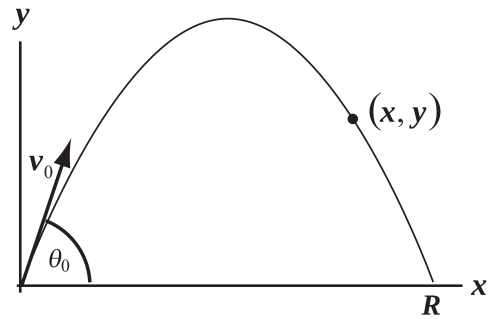
$$\left.\frac{d^2R}{d\theta_0^2}\right|_{\theta_0=45^\circ}=\left[-4\left(v_0^2/g\right)\sin 2\theta_0\right]_{\theta_0=45^\circ}<0$$

Therefore R is a maximum at $\theta_0=45°$.

91 •• In a science fiction short story written in the 1970s, Ben Bova described a conflict between two hypothetical colonies on the moon—one founded by the United States and the other by the USSR. In the story, colonists from each side started firing bullets at each other, only to find to their horror that their rifles had large enough muzzle velocities that the bullets went into orbit. (*a*) If the magnitude of free-fall acceleration on the moon is 1.67 m/s^2, what is the maximum range of a rifle bullet with a muzzle velocity of 900 m/s? (Assume the curvature of the surface of the moon is negligible.) (*b*) What would the muzzle velocity have to be to send the bullet into a circular orbit just above the surface of the moon?

Picture the Problem We can use constant-acceleration equations to express the x and y coordinates of a bullet in flight on the moon as a function of t. Eliminating this parameter will yield an expression for y as a function of x that we can use to

find the range of the bullet. The necessity that the centripetal acceleration of an object in orbit at the surface of a body equals the acceleration due to gravity at the surface will allow us to determine the required muzzle velocity for orbital motion.



(*a*) Using a constant-acceleration equation, express the *x* coordinate of a bullet in flight on the moon:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $a_x = 0$ and $v_{0x} = v_0\cos\theta_0$,

$$x = (v_0 \cos\theta_0)t$$

Using a constant-acceleration equation, express the *y* coordinate of a bullet in flight on the moon:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $y_0 = 0$, $a_y = -g_{\text{moon}}$ and $v_{0y} = v_0\sin\theta_0$,

$$y = (v_0 \sin\theta_0)t - \tfrac{1}{2}g_{\text{moon}}t^2$$

Using the *x*-component equation, eliminate time from the *y*-component equation to obtain:

$$y = (\tan\theta_0)x - \frac{g_{\text{moon}}}{2v_0^2\cos^2\theta_0}x^2$$

When $y = 0$ and $x = R$:

$$0 = (\tan\theta_0)R - \frac{g_{\text{moon}}}{2v_0^2\cos^2\theta_0}R^2$$

and

$$R = \frac{v_0^2}{g_{\text{moon}}}\sin 2\theta_0$$

Substitute numerical values and evaluate *R*:

$$R = \frac{(900\,\text{m/s})^2}{1.67\,\text{m/s}^2}\sin 90° = \boxed{485\,\text{km}}$$

This result is probably not very accurate because it is about 28% of the moon's radius (1740 km). This being the case, we can no longer assume that the ground is "flat" because of the curvature of the moon.

(*b*) Express the condition that the centripetal acceleration must satisfy for an object in orbit at the surface of the moon:

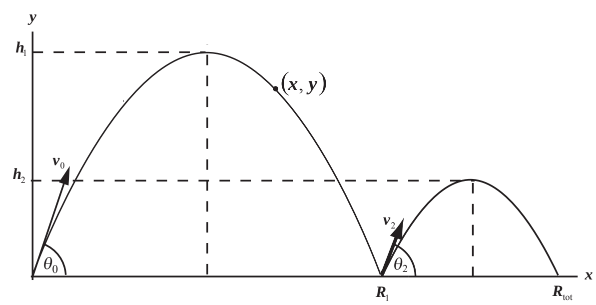
$$a_c = g_{\text{moon}} = \frac{v^2}{r} \Rightarrow v = \sqrt{g_{\text{moon}}r}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(1.67\,\text{m/s}^2)(1.74\times10^6\,\text{m})} = \boxed{1.70\,\text{km/s}}$$

92 ••• On a level surface, a ball is launched from ground level at an angle of 55° above the horizontal, with an initial speed of 22 m/s. It lands on a hard surface, and bounces, reaching a peak height of 75% of the height it reached on its first arc. (Ignore any effects due to air resistance.) (*a*) What is the maximum height reached in its first parabolic arc? (*b*) How far horizontally from the launch point did it strike the ground the first time? (*c*) How far horizontally from the launch point did the ball strike the ground the second time? Assume the horizontal component of the velocity remains constant during the collision of the ball with the ground. *Note: You cannot assume the angle with which the ball leaves the ground after the first collision is the same as the initial launch angle.*

Picture the Problem The following diagram shows a convenient coordinate system to use in solving this problem as well as the relevant distances and angles. We can use constant-acceleration equations to find h_1. In (*b*) we can use the same-elevation equation to find R_1. In (*c*) we know the speed with which the tennis ball hits the ground on its first bounce – this is simply the speed with which it was launched. As it collides with the ground, it accelerates such that it leaves the surface with a speed that is reduced and is directed generally upward again. We are told that it reaches a height of 75% of its first peak height, and to assume that there is no horizontal acceleration. Again, we can use constant-acceleration equations to find θ_2 and the components of $\vec{v}_2$ and hence the range of the ball after its bounce. We can then find the total range from the sum of the ranges before and after the bounce.



(*a*) Use constant-acceleration equations to express the x and y coordinates of the tennis ball along its first parabolic trajectory as functions of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad (1)$$

and

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \qquad (2)$$

Because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$, equation (1) becomes:

$$x = (v_0\cos\theta_0)t \qquad (3)$$

Because $y_0 = 0$, $v_{0y} = v_0 \sin\theta_0$, and $a_y = -g$, equation (2) becomes:

$$y = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \qquad (4)$$

Use a constant-acceleration equation to relate the vertical velocity of the ball to time:

$$v_y = v_{0y} + a_y t = v_0 \sin\theta_0 - gt$$

When the ball is at the top of its trajectory $v_y = 0$ and:

$$0 = v_0 \sin\theta_0 - gt_{\text{top of trajectory}}$$

Solving for $t_{\text{top of trajectory}}$ yields:

$$t_{\text{top of trajectory}} = \frac{v_0 \sin\theta_0}{g}$$

Substitute in equation (4) and simplify to obtain:

$$h_1 = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2 = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Substitute numerical values and evaluate h_1:

$$h_1 = \frac{(22\ \text{m/s})^2 \sin^2 55^\circ}{2(9.81\ \text{m/s}^2)} = 16.55\ \text{m}$$
$$= \boxed{17\ \text{m}}$$

(*b*) The same-elevation range equation is:

$$R = \frac{v_0^2}{g}\sin 2\theta_0$$

Substitute numerical values and evaluate R_1:

$$R_1 = \frac{(22\ \text{m/s})^2}{9.81\ \text{m/s}^2}\sin 2(55^\circ) = 46.36\ \text{m}$$
$$= \boxed{46\ \text{m}}$$

(*c*) The total range of the ball is the sum of the ranges R_1 and R_2:

$$R_{\text{tot}} = R_1 + R_2 \qquad (5)$$

Provided we can determine the initial velocity v_{20} and rebound angle θ_2 of the rebounding ball, we can use the same-elevation equation to find R_2:

$$R_2 = \frac{v_{20}^2}{g}\sin 2\theta_2 \qquad (6)$$

where

$$v_{20}^2 = v_{2x}^2 + v_{2y}^2 \qquad (7)$$

Substituting equation (6) in equation (7) yields:

$$R_2 = \frac{v_{2x}^2 + v_{2y}^2}{g}\sin 2\theta_2 \qquad (8)$$

Use a constant-acceleration equation to relate the rebound height h_2 to the y component of the rebound velocity:

$$v_2^2 = v_{2y}^2 + 2a_y h_2$$

or, because $v_2 = 0$ and $a_y = -g$,

$$0 = v_{2y}^2 - 2gh_2$$

Solving for v_{2y} yields:

$$v_{2y}^2 = 2gh_2$$

or, because $h_2 = 0.75h_1$,

$$v_{2y}^2 = 1.5gh_1 \qquad (9)$$

Because there is no horizontal acceleration, the ball continues with the same horizontal velocity it originally had:

$$v_{2x} = v_{1x} = v_0 \cos\theta_0$$

and

$$v_{2x}^2 = v_0^2 \cos^2\theta_0 \qquad (10)$$

The rebound angle θ_2 is given by:

$$\theta_2 = \tan^{-1}\left(\frac{v_{2y}}{v_{2x}}\right)$$

Substituting for v_{2y} and v_{2x} and simplifying yields:

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1.5gh_1}}{v_0 \cos\theta_0}\right)$$

Substitute numerical values and evaluate θ_2:

$$\boldsymbol{\theta}_2 = \tan^{-1}\left(\frac{\sqrt{1.5(9.81\text{ m/s}^2)(16.55\text{ m})}}{(22\text{ m/s})\cos 55°}\right) = 51.04°$$

Substitute (9) and (10) in equation (8) to obtain:

$$R_2 = \frac{v_0^2 \cos^2\theta_0 + 1.5gh_1}{g}\sin 2\theta_2$$

Substitute numerical values and evaluate R_2:

$$R_2 = \frac{(22\text{ m/s})^2 \cos^2 55° + 1.5\left(9.81\text{ m/s}^2\right)(16.55\text{ m})}{9.81\text{ m/s}^2}\sin 2\left(51.04°\right) = 40.15\text{ m}$$

Substitute numerical values for R_1 and R_2 in equation (5) and evaluate R_{tot}:

$$R_{\text{tot}} = 46.36\text{ m} + 40.15\text{ m} = \boxed{87\text{ m}}$$

93 ••• In the text, we calculated the range for a projectile that lands at the same elevation from which it is fired as $R = (v_0^2 / g)\sin 2\theta_0$. A golf ball hit from an elevated tee at 45.0 m/s and an angle of 35.0° lands on a green 20.0 m below the tee (Figure 3-37). (Ignore any effects due to air resistance.) (*a*) Calculate the range using the same elevation equation $R = (v_0^2 / g)\sin 2\theta_0$ even though the ball is hit from an elevated tee. (*b*) Show that the range for the more general problem

(Figure 3-37) is given by $R=\left(1+\sqrt{1-\dfrac{2gy}{v_0^2\sin^2\theta_0}}\right)\dfrac{v_0^2}{2g}\sin 2\theta_0$, where y is the height of the tee above the green. That is, $y=-h$. (*c*) Compute the range using this formula. What is the percentage error in neglecting the elevation of the green?

Picture the Problem The initial position is on the $+y$ axis and the point of impact is on the $+x$ axis, with the x axis horizontal and the y axis vertical. We can apply constant-acceleration equations to express the horizontal and vertical coordinates of the projectile as functions of time. Solving these equations simultaneously will leave us with a quadratic equation in R, the solution to which is the result we are required to establish.

(*a*) The same elevation equation is:

$$R_{\substack{\text{same}\\\text{elevation}}}=\frac{v_0^2}{g}\sin 2\theta_0$$

Substitute numerical values and evaluate $R_{\substack{\text{same}\\\text{elevation}}}$:

$$R_{\substack{\text{same}\\\text{elevation}}}=\frac{(45.0\text{ m/s})^2}{9.81\text{ m/s}^2}\sin 2(35.0°)=\boxed{194\text{ m}}$$

(*b*) Write the constant-acceleration equations for the horizontal and vertical components of the projectile's motion:

$$x=v_{0x}t$$

and

$$y=h+v_{0y}t+\tfrac{1}{2}(-g)t^2$$

where

$v_{0x}=v_0\cos\theta_0$ and $v_{0y}=v_0\sin\theta_0$

Solve the x-component equation for t to obtain:

$$t=\frac{x}{v_{0x}}=\frac{x}{v_0\cos\theta_0}$$

Using the x-component equation, eliminate time from the y-component equation to obtain:

$$y=h+(\tan\theta_0)x-\frac{g}{2v_0^2\cos^2\theta_0}x^2$$

When the projectile strikes the ground its coordinates are (R, 0) and our equation becomes:

$$0=h+(\tan\theta_0)R-\frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Using the plus sign in the quadratic formula to ensure a physically meaningful root (one that is positive), solve for the range to obtain:

$$R=\left(1+\sqrt{1+\frac{2gh}{v_0^2\sin^2\theta_0}}\right)\frac{v_0^2}{2g}\sin 2\theta_0$$

Because $y = -h$:

$$\boxed{R = \left(1 + \sqrt{1 - \frac{2gy}{v_0^2 \sin^2 \theta_0}}\right)\frac{v_0^2}{2g}\sin 2\theta_0}$$

(c) Substitute numerical values and evaluate R:

$$R = \left(1 + \sqrt{1 - \frac{2(9.81\,\text{m/s}^2)(-20.0\,\text{m})}{(45.0\,\text{m/s})^2 \sin^2 35.0°}}\right)\frac{(45.0\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)}\sin 2(35.0°) = \boxed{219\,\text{m}}$$

The percent error is given by:

$$\%\,\text{error} = \left|\frac{R - R_{\substack{\text{same}\\\text{elevation}}}}{R}\right|$$

Substitute numerical values and evaluate the percent error:

$$\%\,\text{error} = \left|\frac{219\,\text{m} - 194\,\text{m}}{219\,\text{m}}\right| = \boxed{11\%}$$

Remarks: Note that, for $y = -h$, the equation derived in Part (b) reduces to $R = (v_0^2/g)\sin 2\theta_0$, in agreement with the first equation in the problem statement.

94 ••• In the text, we calculated the range for a projectile that lands at the same elevation from which it is fired as $R = (v_0^2/g)\sin 2\theta_0$ if the effects of air resistance are negligible. (a) Show that for the same conditions the change in the range for a small change in free-fall acceleration g, and the same initial speed and angle, is given by $\Delta R/R = -\Delta g/g$. (b) What would be the length of a homerun at a high altitude where g is 0.50% less than at sea level if the homerun at sea level traveled 400 ft?

Picture the Problem We can show that $\Delta R/R = -\Delta g/g$ by differentiating R with respect to g and then using a differential approximation.

(a) Differentiate the range equation with respect to g:

$$\frac{dR}{dg} = \frac{d}{dg}\left(\frac{v_0^2}{g}\sin 2\theta_0\right) = -\frac{v_0^2}{g^2}\sin 2\theta_0$$
$$= -\frac{R}{g}$$

Approximate dR/dg by $\Delta R/\Delta g$:

$$\frac{\Delta R}{\Delta g} = -\frac{R}{g}$$

Separate the variables to obtain:

$$\frac{\Delta R}{R} = \boxed{-\frac{\Delta g}{g}}$$

That is, for small changes in gravity ($g \approx g \pm \Delta g$), the fractional change in R is linearly opposite to the fractional change in g.

(*b*) Solve the proportion derived in (*a*) for ΔR to obtain:

$$\Delta R = -\frac{\Delta g}{g} R$$

Substitute numerical values and evaluate ΔR:

$$\Delta R = -\frac{-0.0050g}{g}(400\text{ ft}) = 2\text{ ft}$$

The length of the homerun at this high-altitude location would be 402 ft.

Remarks: The result in (*a*) tells us that as gravity increases, the range will decrease, and vice versa. This is as it must be because *R* is inversely proportional to *g*.

95 ••• [SSM] In the text, we calculated the range for a projectile that lands at the same elevation from which it is fired as $R = \left(v_0^2/g\right)\sin 2\theta_0$ if the effects of air resistance are negligible. (*a*) Show that for the same conditions the change in the range for a small change in launch speed, and the same initial angle and free-fall acceleration, is given by $\Delta R/R = 2\Delta v_0/v_0$. (*b*) Suppose a projectile's range was 200 m. Use the formula in Part (*a*) to estimate its increase in range if the launch speed were increased by 20.0%. (*c*) Compare your answer in (*b*) to the increase in range by calculating the increase in range directly from $R = \left(v_0^2/g\right)\sin 2\theta_0$. If the results for Parts (*b*) and (*c*) are different, is the estimate too small or large, and why?

Picture the Problem We can show that $\Delta R/R = 2\Delta v_0/v_0$ by differentiating R with respect to v_0 and then using a differential approximation.

(*a*) Differentiate the range equation with respect to v_0:

$$\frac{dR}{dv_0} = \frac{d}{dv_0}\left(\frac{v_0^2}{g}\sin 2\theta_0\right) = \frac{2v_0}{g}\sin 2\theta_0$$
$$= 2\frac{R}{v_0}$$

Approximate dR/dv_0 by $\Delta R/\Delta v_0$:

$$\frac{\Delta R}{\Delta v_0} = 2\frac{R}{v_0}$$

Separate the variables to obtain:

$$\frac{\Delta R}{R}=\boxed{2\frac{\Delta v_0}{v_0}}$$

That is, for small changes in the launch velocity ($v_0 \approx v_0 \pm \Delta v_0$), the fractional change in R is twice the fractional change in v_0.

(*b*) Solve the proportion derived in (*a*) for ΔR to obtain:

$$\Delta R = 2R\frac{\Delta v_0}{v_0}$$

Substitute numerical values and evaluate ΔR:

$$\Delta R = 2(200\text{ m})\frac{0.20v_0}{v_0}=\boxed{80\text{ m}}$$

(*c*) The increase in range is given by:

$$\Delta R = R'-R \qquad (1)$$

The same-elevation equation is:

$$R=\frac{v_0^2}{g}\sin 2\theta_0$$

The longer range R' resulting from a 20.0% increase in the launch speed is given by:

$$R' = \frac{(v_0+0.20v_0)^2}{g}\sin 2\theta_0 = \frac{(1.20v_0)^2}{g}\sin 2\theta_0$$

Substituting for R and R' gives:

$$\Delta R = \frac{(1.20v_0)^2}{g}\sin 2\theta_0 - \frac{v_0^2}{g}\sin 2\theta_0 = \left[(1.20)^2-1\right]\frac{v_0^2}{g}\sin 2\theta_0 = 0.44R$$

Substitute the numerical value of R and evaluate ΔR:

$$\Delta R = 0.44(200\text{ m})=\boxed{88\text{ m}}$$

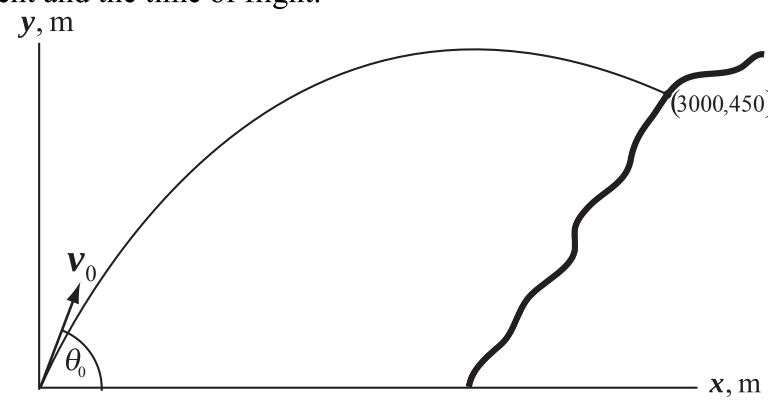
The approximate solution is smaller. The estimate ignores higher-order terms and they are important when the differences are not small.

Remarks: The result in (*a*) tells us that as launch velocity increases, the range will increase twice as fast, and vice versa.

96 ••• A projectile, fired with unknown initial velocity, lands 20.0 s later on the side of a hill, 3000 m away horizontally and 450 m vertically above its starting point. Neglect any effects due to air resistance. (*a*) What is the vertical component of its initial velocity? (*b*) What is the horizontal component of its initial velocity? (*c*) What was its maximum height above its launch point? (*d*) As it hit the hill,

what speed did it have and what angle did its velocity make with the vertical? (Ignore air resistance.)

Picture the Problem In the absence of air resistance, the horizontal and vertical displacements of the projectile are independent of each other and describable by constant-acceleration equations. Choose the origin at the firing location and with the coordinate axes as shown in the pictorial representation and use constant-acceleration equations to relate the vertical displacement to the vertical component of the initial velocity and the horizontal velocity to the horizontal displacement and the time of flight.



(*a*) Using a constant-acceleration equation, express the vertical displacement of the projectile as a function of its time of flight:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $a_y = -g$,

$$\Delta y = v_{0y}\Delta t - \tfrac{1}{2}g(\Delta t)^2 \quad (1)$$

Solve for v_{0y}:

$$v_{0y} = \frac{\Delta y + \frac{1}{2}g(\Delta t)^2}{\Delta t}$$

Substitute numerical values and evaluate v_{0y}:

$$v_{0y} = \frac{450\text{ m} + \frac{1}{2}(9.81\text{ m/s}^2)(20.0\text{ s})^2}{20.0\text{ s}}$$

$$= 120.6\text{ m/s} = \boxed{121\text{m/s}}$$

(*b*) The horizontal velocity remains constant, so:

$$v_{0x} = v_x = \frac{\Delta x}{\Delta t} = \frac{3000\text{ m}}{20.0\text{ s}} = \boxed{150\text{ m/s}}$$

(*c*) From equation (1) we have, when the projectile is at its maximum height *h*:

$$h = v_{0y}\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

or, because $v_{0y} = v_0\sin\theta$,

$$h = (v_0\sin\theta_0)\Delta t - \tfrac{1}{2}g(\Delta t)^2 \quad (2)$$

Use a constant-acceleration equation to relate the y component of the velocity of the projectile to its initial value and to the elapsed time:

$$v_y = v_{0y} + a_y\Delta t = v_0 \sin\theta_0 - g\Delta t \quad (3)$$

The projectile is at its maximum height when its upward velocity is zero. Under this condition we have:

$$0 = v_0 \sin\theta_0 - g\Delta t \Rightarrow \Delta t = \frac{v_0 \sin\theta_0}{g}$$

Substitute for Δt in equation (2) to obtain:

$$h = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

Simplifying yields:

$$h = \frac{v_0^2 \sin^2\theta_0}{2g} \qquad (4)$$

v_0^2 is given by:

$$v_0^2 = v_{0x}^2 + v_{0y}^2$$

Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{(150\ \text{m/s})^2 + (120.6\ \text{m/s})^2} = 192.5\ \text{m/s}$$

The horizontal displacement of the projectile is given by:

$$\Delta x = v_{0x}\Delta t = (v_0 \cos\theta_0)\Delta t$$

Solving for the launch angle θ_0 gives:

$$\theta_0 = \cos^{-1}\left(\frac{\Delta x}{v_0 \Delta t}\right)$$

Substitute numerical values and evaluate θ_0:

$$\boldsymbol{\theta}_0 = \cos^{-1}\left(\frac{3000\ \text{m}}{(192.5\ \text{m/s})(20.0\ \text{s})}\right) = 38.80°$$

Substitute numerical values in equation (3) and evaluate h:

$$h = \frac{(192.5\ \text{m/s})^2 \sin^2 38.80°}{2(9.81\ \text{m/s}^2)} = \boxed{742\ \text{m}}$$

(*d*) Use equation (3) to evaluate $v_y(20.0\ \text{s})$:

$$v_y(20.0\ \text{s}) = 120.6\ \text{m/s} - (9.81\ \text{m/s}^2)(20.0\ \text{s}) = -75.60\ \text{m/s}$$

The speed of the projectile at impact is given by:

$$v(20.0\ \text{s}) = \sqrt{(v_x(20.0\ \text{s}))^2 + (v_y(20.0\ \text{s}))^2}$$

Substitute numerical values and evaluate $v(20.0\text{ s})$:

$$v(20.0\,\text{s}) = \sqrt{(150\,\text{m/s})^2 + (-75.6\,\text{m/s})^2} = \boxed{168\,\text{m/s}}$$

The angle at impact is given by:

$$\theta_{\text{impact}} = \tan^{-1}\left(\frac{v_y(20.0\,\text{s})}{v_x(20.0\,\text{s})}\right)$$

Substitute numerical values and evaluate θ_{impact}:

$$\theta_{\text{impact}} = \tan^{-1}\left(\frac{-75.60\,\text{m/s}}{150\,\text{m/s}}\right) = -26.75°$$
$$= \boxed{63.3° \text{ with the vertical}}$$

97 ••• A projectile is launched over level ground at an initial elevation angle of θ. An observer standing at the launch site sees the projectile at the point of its highest elevation, and measures the angle ϕ shown in Figure 3-38. Show that $\tan\phi = \frac{1}{2}\tan\theta$. (Ignore any effects due to air resistance.)

Picture the Problem We can use trigonometry to relate the maximum height of the projectile to its range and the sighting angle at maximum elevation and the range equation to express the range as a function of the launch speed and angle. We can use a constant-acceleration equation to express the maximum height reached by the projectile in terms of its launch angle and speed. Combining these relationships will allow us to conclude that $\tan\phi = \frac{1}{2}\tan\theta$.

Referring to the figure, relate the maximum height of the projectile to its range and the sighting angle ϕ:

$$\tan\phi = \frac{h}{\frac{1}{2}R}$$

Express the range of the rocket and use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ to rewrite the expression as:

$$R = \frac{v_0^2}{g}\sin(2\theta) = 2\frac{v_0^2}{g}\sin\theta\cos\theta$$

Using a constant-acceleration equation, relate the maximum height h of a projectile to the vertical component of its launch speed:

$$v_y^2 = v_{0y}^2 - 2gh$$

or, because $v_y = 0$ and $v_{0y} = v_0\sin\theta$,

$$v_0^2\sin^2\theta = 2gh \Rightarrow h = \frac{v_0^2}{2g}\sin^2\theta$$

Substitute for R and h and simplify to obtain:

$$\tan\phi = \frac{2\left(\dfrac{v_0^2}{2g}\sin^2\theta\right)}{2\left(\dfrac{v_0^2}{g}\sin\theta\cos\theta\right)} = \boxed{\tfrac{1}{2}\tan\theta}$$

98 ••• A toy cannon is placed on a ramp that has a slope of angle ϕ. (*a*) If the cannonball is projected up the hill at an angle of θ_0 above the horizontal (Figure 3-39) and has a muzzle speed of v_0, show that the range R of the cannonball (as measured along the ramp) is given by $R = \dfrac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g\cos\phi}$. Neglect any effects due to air resistance.

Picture the Problem The equation of a particle's trajectory is derived in the text so we'll use it as our starting point in this derivation. We can relate the coordinates of the point of impact (x, y) to the angle ϕ and use this relationship to eliminate y from the equation for the cannonball's trajectory. We can then solve the resulting equation for x and relate the horizontal component of the point of impact to the cannonball's range.

The equation of the cannonball's trajectory is given in Equation 3-22:

$$y(x) = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2$$

Relate the x and y components of a point on the ground to the angle ϕ:

$$y(x) = (\tan\phi)x$$

Express the condition that the cannonball hits the ground:

$$(\tan\phi)x = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2$$

Solving for x yields:

$$x = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g}$$

Relate the range of the cannonball's flight R to the horizontal distance x:

$$x = R\cos\phi$$

Substitute to obtain:

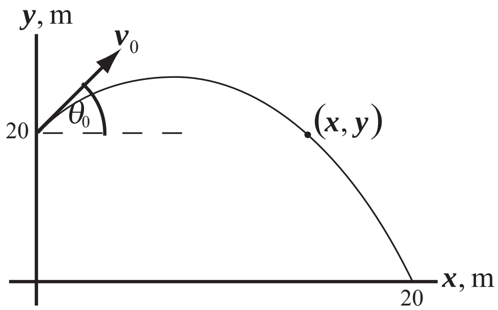
$$R\cos\phi = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g}$$

Solving for R yields:

$$R = \boxed{\frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g\cos\phi}}$$

99 ••• A rock is thrown from the top of a 20-m-high building at an angle of 53° above the horizontal. (*a*) If the horizontal range of the throw is equal to the height of the building, with what speed was the rock thrown? (*b*) How long is it in the air? (*c*) What is the velocity of the rock just before it strikes the ground? (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the acceleration of the rock is constant and the horizontal and vertical motions are independent of each other. Choose the coordinate system shown in the figure with the origin at the base of the building and the axes oriented as shown and apply constant-acceleration equations to relate the horizontal and vertical coordinates of the rock to the time into its flight.



(*a*) Using constant-acceleration equations, express the x and y coordinates of the rock as functions of the time into its flight:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$
and
$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

Because $x_0 = 0$, $a_x = 0$, and $a_y = -g$:

$$x = v_{0x}t = (v_0 \cos\theta_0)t \qquad (1)$$
and
$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2$$
$$= y_0 + (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2$$

Eliminating time between these equations yields:

$$y = y_0 + (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

When the rock hits the ground, $x = y_0$ and:

$$0 = y_0 + (\tan\theta_0)y_0 - \frac{g}{2v_0^2\cos^2\theta_0}y_0^2$$
or
$$0 = 1 + \tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}y_0$$

Solve for v_0 to obtain:

$$v_0 = \frac{1}{\cos\theta_0}\sqrt{\frac{gy_0}{2(1+\tan\theta_0)}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{1}{\cos 53^\circ}\sqrt{\frac{(9.81\ \text{m/s}^2)(20\ \text{m})}{2(1+\tan 53^\circ)}} = 10.79\ \text{m/s} \approx \boxed{11\ \text{m/s}}$$

(*b*) Solve equation (1) for the time-to-impact to obtain:

$$t = \frac{x}{v_0 \cos\theta_0}$$

Substitute numerical values and evaluate *t*:

$$t = \frac{20\,\text{m}}{(10.79\,\text{m/s})\cos 53°} = 3.08\,\text{s} = \boxed{3.1\,\text{s}}$$

(*c*) Using constant-acceleration equations, find v_y and v_x at impact:

$$v_x = v_{0x} = v_0 \cos\theta = (10.79\ \text{m/s})\cos 53° = 6.50\,\text{m/s}$$

and

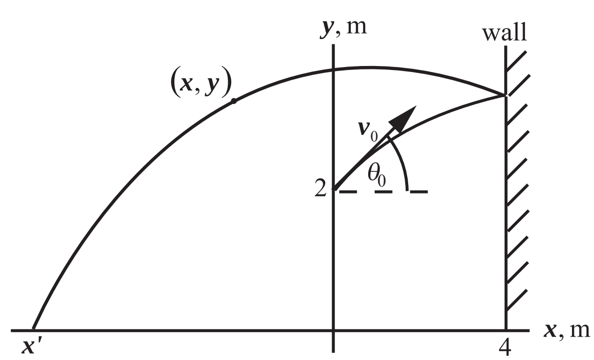
$$v_y = v_{0y} - gt = v_0 \sin\theta_0 - gt = (10.79\ \text{m/s})\sin 53° - (9.81\,\text{m/s}^2)\times(3.08\,\text{s}) = -21.6\,\text{m/s}$$

Expressing the velocity at impact in vector form gives:

$$\vec{\boldsymbol{v}} = \boxed{(6.5\,\text{m/s})\hat{\boldsymbol{i}} + (-22\ \text{m/s})\hat{\boldsymbol{j}}}$$

100 ••• A woman throws a ball at a vertical wall 4.0 m away (Figure 3-40). The ball is 2.0 m above ground when it leaves the woman's hand with an initial velocity of 14 m/s at 45º as shown. When the ball hits the wall, the horizontal component of its velocity is reversed; the vertical component remains unchanged. (*a*) Where does the ball hit the ground? (*b*) How long was the ball in the air before it hit the wall? (*c*) Where did the ball hit the wall? (*d*) How long was the ball in the air after it left the wall? Ignore any effects due to air resistance.

Picture the Problem Choose the coordinate system shown below with the positive *x* axis to the right and the positive *y* axis upward. The solid curve represents the trajectory of the ball as it bounces from the vertical wall. Because the ball experiences constant acceleration (except during its collision with the wall), we can use constant-acceleration equations to describe its motion.



(*a*) The ball hits the ground at a point x' whose coordinate is given by:

$$x' = \Delta x - 4.0\,\text{m} \qquad (1)$$

where Δx is the displacement of the ball after it hits the wall.

Using a constant-acceleration equation, express the vertical displacement of the ball as a function of Δt:

$$\Delta y = v_{0y}\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

or, because $v_{0y} = v_0\sin\theta_0$,

$$\Delta y = (v_0\sin\theta_0)\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

Because the y-component doesn't change on impact with the wall, the same equation for Δy applies for the entire motion of the ball. Substitute numerical values to obtain:

$$\Delta y = (14\,\text{m/s})(\sin 45°)\Delta t - \tfrac{1}{2}(9.81\,\text{m/s}^2)(\Delta t)^2$$

When the ball hits the ground, $\Delta y = -2.0$ m and $\Delta t = t_{\text{flight}}$:

$$-2.0\,\text{m} = (14\,\text{m/s})(\sin 45°)t_{\text{flight}} - \tfrac{1}{2}(9.81\,\text{m/s}^2)t_{\text{flight}}^2$$

Using the quadratic formula or your graphing calculator, solve for the time of flight:

$$t_{\text{flight}} = 2.203\,\text{s}$$

The horizontal distance traveled in this time is given by:

$$\Delta x = (v_{0x}\cos\theta_0)t_{\text{flight}}$$

Substitute numerical values and evaluate Δx:

$$\Delta \boldsymbol{x} = (14\,\text{m/s})(\cos 45°)(2.203\,\text{s}) = 21.8\,\text{m}$$

Use equation (1) to find where the ball hits the ground:

$$x' = 21.8\,\text{m} - 4.0\,\text{m} = \boxed{18\text{m}}$$

(b) The time-to-the-wall, Δt_{wall}, is related to the displacement of the ball when it hits the wall and to the x component of its velocity:

$$\Delta t_{\text{wall}} = \frac{\Delta x_{\text{wall}}}{v_{0x}} = \frac{\Delta x_{\text{wall}}}{v_0\cos\theta_0}$$

Substitute numerical values and evaluate Δt_{wall}:

$$\Delta t_{\text{wall}} = \frac{4.0\,\text{m}}{(14\,\text{m/s})\cos 45°} = 0.4041\,\text{s} = \boxed{0.40\,\text{s}}$$

(c) First we'll find the y component of the ball's velocity when it hits the wall. Use a constant-acceleration equation to express $v_y(\Delta t_{\text{wall}})$:

$$v_y(\Delta t_{\text{wall}}) = v_{0y} + a_y\Delta t_{\text{wall}}$$

or, because $v_{0y} = v_0\sin\theta_0$ and $a_y = -g$,

$$v_y(\Delta t_{\text{wall}}) = v_0\sin\theta_0 - g\Delta t_{\text{wall}}$$

Substitute numerical values and evaluate $v_y(0.4041\text{ s})$:

$$v_y(0.4041\,\text{s}) = (14\,\text{m/s})\sin 45° - (9.81\,\text{m/s}^2)(0.4041\,\text{s}) = 5.935\,\text{m/s}$$

Now we need to find the height of the ball when it hit the wall. Use a constant-acceleration equation to express $y(\Delta t_{\text{wall}})$:

$$y(\Delta t_{\text{wall}}) = y_0 + v_{0y}\Delta t_{\text{wall}} + \tfrac{1}{2}a_y(\Delta t_{\text{wall}})^2$$

Because $v_{0y} = v_0\sin\theta_0$ and $a_y = -g$:

$$y(\Delta t_{\text{wall}}) = y_0 + (v_0\sin\theta_0)\Delta t_{\text{wall}} - \tfrac{1}{2}g(\Delta t_{\text{wall}})^2$$

Substitute numerical values and evaluate $y(0.4041\text{ s})$:

$$\begin{aligned} y(0.4041\,\text{s}) &= 2.0\,\text{m} + (14\,\text{m/s})(\sin 45°)(0.4041\,\text{s}) - \tfrac{1}{2}(9.81\,\text{m/s}^2)(0.4041\,\text{s})^2 \\ &= 5.199\,\text{m} = \boxed{5.2\,\text{m}} \end{aligned}$$

(*d*) Letting Δt_{after} represent the time the ball is in the air after its collision with the wall, express its vertical position as a function of time:

$$y(\Delta t_{\text{after}}) = y_0 + v_y(0.4041\,\text{s})\Delta t_{\text{after}} - \tfrac{1}{2}g(\Delta t_{\text{after}})^2$$

Noting that $y(\Delta t_{\text{after}}) = 0$ when the ball hits the ground, substitute numerical values to obtain:

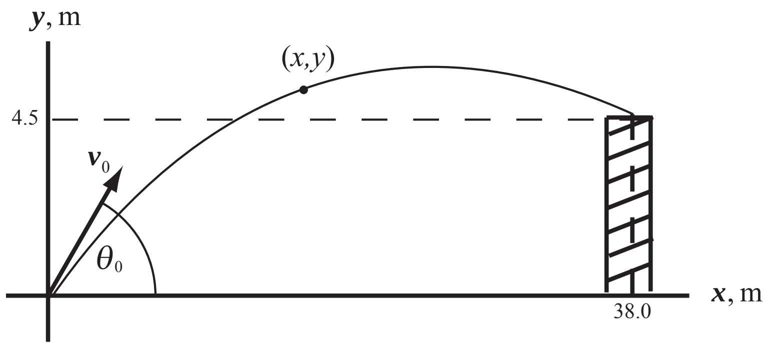
$$0 = 5.199\,\text{m} + (5.935\,\text{m/s})\Delta t_{\text{after}} - \tfrac{1}{2}(9.81\,\text{m/s}^2)(\Delta t_{\text{after}})^2$$

Solving this equation using the quadratic formula or your graphing calculator yields:

$$\Delta t_{\text{after}} = 1.798\,\text{s} = \boxed{1.8\,\text{s}}$$

101 ••• Catapults date from thousands of years ago, and were used historically to launch everything from stones to horses. During a battle in what is now Bavaria, inventive artillerymen from the united German clans launched giant *spaetzle* from their catapults toward a Roman fortification whose walls were 8.50 m high. The catapults launched *spaetzle* projectiles from a height of 4.00 m above the ground, and 38.0 m from the walls of the fortification, at an angle of 60.0 degrees above the horizontal. (Figure 3-41.) If the projectiles were to impact the top of the wall, splattering the Roman soldiers atop the wall with pulverized pasta, (*a*) what launch speed was necessary? (*b*) How long was the spaetzle in the air? (*c*) At what speed did the projectiles hit the wall? Ignore any effects due to air resistance.

Picture the Problem The simplest solution for this problem is to consider the equation relating y and x along the projectile's trajectory that is found in equation 3-17. This equation is valid for trajectories which begin at the origin. We'll let the origin of coordinates be the point of launch (as shown in the diagram). Thus the impact point on top of the walls is y = 4.50 m, rather than 8.50 m. Our launch angle θ_0 is 60.0°, and the position of the center of the wall is 38.0 m. We can solve Equation 3-17 for the launch speed.



(*a*) Equation 3-17 is:

$$y(x) = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2$$

Solving for v_0 yields:

$$v_0 = \frac{x}{\cos\theta_0}\sqrt{\frac{g}{2[x\tan\theta_0 - y(x)]}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{38.0\text{ m}}{\cos 60.0°}\sqrt{\frac{9.81\text{ m/s}^2}{2[(38.0\text{ m})\tan 60.0° - 4.50\text{ m}]}} = 21.50\text{ m/s} = \boxed{21.5\text{ m/s}}$$

(*b*) Relate the x component of the spaetzle's velocity to its time-of-flight:

$$t_{\text{flight}} = \frac{\Delta x}{v_{0x}} = \frac{\Delta x}{v_0\cos\theta_0}$$

Substitute numerical values and evaluate t_{flight}:

$$t_{\text{flight}} = \frac{38.0\text{ m}}{(21.50\text{ m/s})\cos 60.0°} = 3.535\text{ s}$$
$$= \boxed{3.53\text{ s}}$$

(*c*)The speed with which the projectiles hit the wall is related to its x and y components through the Pythagorean Theorem:

$$v = \sqrt{v_x^2 + v_y^2} \qquad (1)$$

Because the acceleration in the x direction is zero, the x component of the projectile's velocity on impact equals its initial x component:

$$v_x = v_{0x} = v_0 \cos\theta_0$$

Substitute numerical values and evaluate v_x:

$$v_x = (21.5\ \text{m/s})\cos 60.0° = 10.75\ \text{m/s}$$

Use a constant-acceleration equation to express the y component of the projectile's velocity at any time along its flight path:

$$v_y(t) = v_{0y} + a_y t$$

or, because $v_{0y} = v_0\sin\theta_0$ and $a_y = -g$,

$$v_y(t) = v_0 \sin\theta_0 - gt$$

Substitute numerical values and evaluate $v_y(3.535\ \text{s})$:

$$v_y(3.535\ \text{s}) = (21.5\ \text{m/s})\sin 60.0° - (9.81\ \text{m/s}^2)(3.535\ \text{s}) = -16.06\ \text{m/s}$$

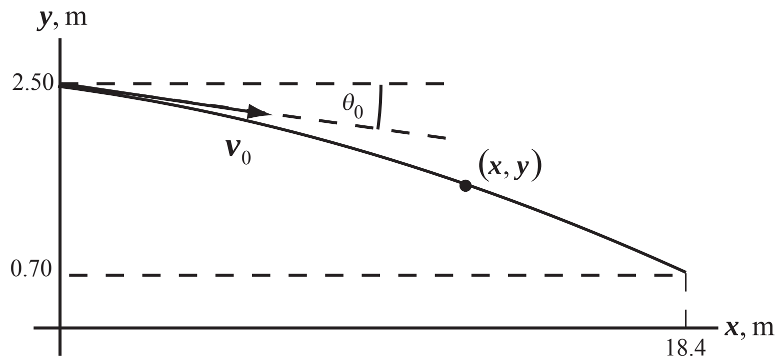
Substitute numerical values for v_x and v_y in equation (1) and evaluate v:

$$v = \sqrt{(10.75\ \text{m/s})^2 + (-16.06\ \text{m/s})^2} = \boxed{19.3\ \text{m/s}}$$

Hitting Targets and Related Problems

102 ••• **[SSM]** The distance from the pitcher's mound to home plate is 18.4 m. The mound is 0.20 m above the level of the field. A pitcher throws a fastball with an initial speed of 37.5 m/s. At the moment the ball leaves the pitcher's hand, it is 2.30 m above the mound. (*a*) What should the angle between $\vec{v}_0$ and the horizontal be so that the ball crosses the plate 0.70 m above ground? (Ignore any effects due to air resistance.) (*b*) With what speed does the ball cross the plate?

Picture the Problem The acceleration of the ball is constant (zero horizontally and $-g$ vertically) and the vertical and horizontal components are independent of each other. Choose the coordinate system shown in the figure and use constant-acceleration equations to express the x and y coordinates of the ball along its trajectory as functions of time. Eliminating t between these equations will lead to an equation that can be solved for θ_0. Differentiating the x and y coordinate equations with respect to time will yield expressions for the instantaneous speed of the ball along its trajectory that you can use to find its speed as it crosses the plate.



(*a*) Express the horizontal and vertical positions of the ball as functions of time:

$$x = (v_0 \cos\theta_0)t \quad (1)$$

and

$$y = y_0 + (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \quad (2)$$

Eliminating t between these equations gives:

$$y = y_0 - (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

When the ball crosses the plate:

$$0.70\text{ m} = 2.50\text{ m} - (\tan\theta_0)(18.4\text{ m}) - \frac{9.81\text{ m/s}^2}{2(37.5\text{ m/s})^2\cos^2\theta_0}(18.4\text{ m})^2$$

Simplifying yields:

$$\frac{1}{\cos^2\theta_0} + 15.58\tan\theta_0 = 1.524$$

Use your graphing calculator to find the magnitude of θ_0:

$$\theta_0 = 1.922° = \boxed{1.9°}$$

(*b*) The speed with which the ball crosses the plate is related to the x and y components of its speed through the Pythagorean Theorem:

$$v = \sqrt{v_x^2 + v_y^2} \quad (3)$$

The x- and y-components of the speed of the ball are given by:

$$v_x = v_0\cos\theta_0$$

and

$$v_y = v_0\sin\theta_0 - gt$$

Substituting for v_x and v_y and simplifying yields:

$$v = \sqrt{(v_0\cos\theta_0)^2 + (v_0\sin\theta_0 - gt)^2}$$
$$= v_0^2 - (2v_0\sin\theta_0 - gt)gt$$

From equation (1), the time required for the ball to reach home plate is:

$$t_{\text{to plate}} = \frac{x_{\text{to plate}}}{v_0 \cos\theta_0}$$

Substitute numerical values and evaluate $gt_{\text{to plate}}$:

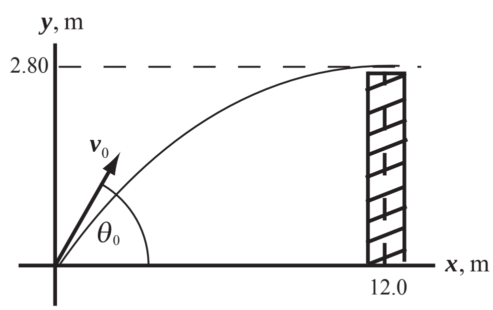
$$t_{\text{to plate}} = \frac{(9.81\ \text{m/s}^2)(18.4\ \text{m})}{(37.5\ \text{m/s})\cos 1.922°}$$
$$= 4.816\ \text{m/s}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(37.5\ \text{m/s})^2 - [2(37.5\ \text{m/s})\sin 1.922° - 4.186\ \text{m/s}](4.186\ \text{m/s})}$$
$$= \boxed{38\ \text{m/s}}$$

103 ••• You are watching your friend play hockey. In the course of the game, he strikes the puck is struck in such a way that, when it is at its highest point, it just clears the surrounding 2.80-m-high Plexiglas wall that is 12.0 m away. Find (*a*) the vertical component of its initial velocity, (*b*) the time it takes to reach the wall, and (*c*) the horizontal component of its initial velocity, its initial speed and angle. Ignore any effects due to air resistance.

Picture the Problem The acceleration of the puck is constant (zero horizontally and $-g$ vertically) and the vertical and horizontal components are independent of each other. Choose a coordinate system with the origin at the point of contact with the puck and the coordinate axes as shown in the figure and use constant-acceleration equations to relate the variables v_{0y}, the time t to reach the wall, v_{0x}, v_0, and θ_0.



(*a*) Using a constant-acceleration equation for the motion in the y direction, express v_{0y} as a function of the puck's displacement Δy:

$$v_y^2 = v_{0y}^2 + 2a_y\Delta y$$
or, because $v_y = 0$ and $a_y = -g$,
$$0 = v_{0y}^2 - 2g\Delta y$$

Solve for v_{0y} to obtain:

$$v_{0y} = \sqrt{2g\Delta y}$$

Substitute numerical values and evaluate v_{0y}:

$$v_{0y} = \sqrt{2(9.81\text{m/s}^2)(2.80\,\text{m})}$$
$$= 7.412\ \text{m/s} = \boxed{7.41\,\text{m/s}}$$

(*b*) The time to reach the top of the wall is related to the initial velocity in the *y* direction:

$$\Delta t = \frac{v_{0y}}{g}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{7.412\,\text{m/s}}{9.81\,\text{m/s}^2} = 0.7555\,\text{s} = \boxed{0.756\text{ s}}$$

(*c*) Use the definition of average velocity to express v_{0x}:

$$v_{0x} = v_x = \frac{\Delta x}{\Delta t}$$

Substitute numerical values and evaluate v_{0x}:

$$v_{0x} = \frac{12.0\,\text{m}}{0.7555\,\text{s}} = 15.88\,\text{m/s} = \boxed{15.9\,\text{m/s}}$$

v_0 is given by:

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{(15.88\,\text{m/s})^2 + (7.412\,\text{m/s})^2} = \boxed{17.5\,\text{m/s}}$$

θ_0 is related to v_{0y} and v_{0x}:

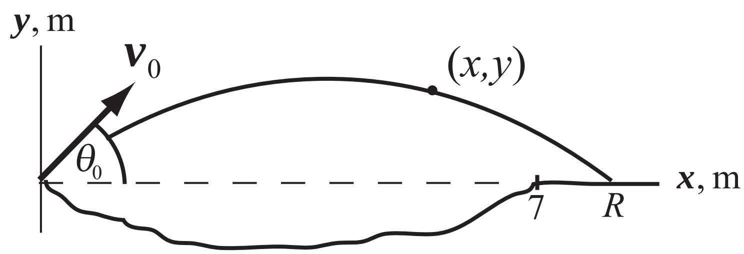
$$\theta_0 = \tan^{-1}\left(\frac{v_{0y}}{v_{0x}}\right)$$

Substitute numerical values and evaluate θ_0 :

$$\theta_0 = \tan^{-1}\left(\frac{7.412\,\text{m/s}}{15.88\,\text{m/s}}\right) = \boxed{25.0°}$$

104 ••• Carlos is on his trail bike, approaching a creek bed that is 7.0 m wide. A ramp with an incline of 10º has been built for daring people who try to jump the creek. Carlos is traveling at his bike's maximum speed, 40 km/h. (*a*) Should Carlos attempt the jump or brake hard? (*b*) What is the minimum speed a bike must have to make this jump? Assume equal elevations on either side of the creek. Ignore any effects of air resistance.

Picture the Problem In the absence of air resistance, the acceleration of Carlos and his bike is constant and we can use constant-acceleration equations to express his *x* and *y* coordinates as functions of time. Eliminating the parameter *t* between these equations will yield *y* as a function of *x* … an equation we can use to decide whether he can jump the creek bed as well as to find the minimum speed required to make the jump.



(a) Use a constant-acceleration equation to express Carlos' horizontal position as a function of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$,

$$x = (v_0\cos\theta_0)t$$

Use a constant-acceleration equation to express Carlos' vertical position as a function of time:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

Because $y_0 = 0$, $v_{0y} = v_0\sin\theta_0$, and $a_y = -g$:

$$y = (v_0\sin\theta_0)t - \tfrac{1}{2}gt^2$$

Eliminate t between the x- and y-equations to obtain:

$$y = (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

Substitute $y = 0$ and $x = R$ to obtain:

$$0 = (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Solving for R yields:

$$R = \frac{v_0^2}{g}\sin(2\theta_0)$$

Substitute numerical values and evaluate R:

$$R = \frac{\left(40\times10^3\ \frac{\text{m}}{\text{h}}\times\frac{1\ \text{h}}{3600\ \text{s}}\right)^2}{9.81\ \text{m/s}^2}\sin 2(10°)$$

$$= 4.3\ \text{m}$$

Because 4.3 m < 7 m, he should apply the brakes!

(b) Solve the equation we used in the previous step for $v_{0,\text{min}}$:

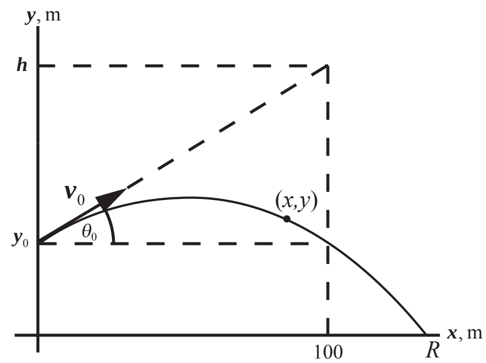
$$v_{0,\text{min}} = \sqrt{\frac{Rg}{\sin(2\theta_0)}}$$

Letting $R = 7$ m, evaluate $v_{0,\min}$:

$$v_{0,\min} = \sqrt{\frac{(7.0\text{ m})(9.81\text{m/s}^2)}{\sin 2(10°)}} = \boxed{14\text{ m/s} = 50\text{ km/h}}$$

105 ••• **[SSM]** If a bullet that leaves the muzzle of a gun at 250 m/s is to hit a target 100 m away at the level of the muzzle (1.7 m above the level ground), the gun must be aimed at a point above the target. (*a*) How far above the target is that point? (*b*) How far behind the target will the bullet strike the ground? Ignore any effects due to air resistance.

Picture the Problem In the absence of air resistance, the bullet experiences constant acceleration along its parabolic trajectory. Choose a coordinate system with the origin at the end of the barrel and the coordinate axes oriented as shown in the figure and use constant-acceleration equations to express the *x* and *y* coordinates of the bullet as functions of time along its flight path.



(*a*) Use a constant-acceleration equation to express the bullet's horizontal position as a function of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$,

$$x = (v_0\cos\theta_0)t$$

Use a constant-acceleration equation to express the bullet's vertical position as a function of time:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $v_{0y} = v_0\sin\theta_0$, and $a_y = -g$,

$$y = y_0 + (v_0\sin\theta_0)t - \tfrac{1}{2}gt^2$$

Eliminate *t* between the two equations to obtain:

$$y = y_0 + (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

Because $y = y_0$ when the bullet hits the target:

$$0 = (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

Solve for the angle above the horizontal that the rifle must be fired to hit the target:

$$\theta_0 = \tfrac{1}{2}\sin^{-1}\left(\frac{xg}{v_0^2}\right)$$

Substitute numerical values and evaluate θ_0:

$$\theta_0 = \tfrac{1}{2}\sin^{-1}\left[\frac{(100\,\text{m})(9.81\,\text{m/s}^2)}{(250\,\text{m/s})^2}\right] = 0.450°$$

Note: A second value for θ_0, 89.6° is physically unreasonable.

Referring to the diagram, relate h to θ_0:

$$\tan\theta_0 = \frac{h}{100\,\text{m}} \Rightarrow h = (100\,\text{m})\tan\theta_0$$

Substitute numerical values and evaluate h:

$$h = (100\,\text{m})\tan(0.450°) = \boxed{0.785\,\text{m}}$$

(*b*) The distance Δx behind the target where the bullet will strike the ground is given by:

$$\Delta x = R - 100\text{ m} \qquad (1)$$

where R is the range of the bullet.

When the bullet strikes the ground, $y = 0$ and $x = R$:

$$0 = y_0 + (\tan\theta_0)R - \frac{g}{2v_0^2\cos^2\theta_0}R^2$$

Substitute numerical values and simplify to obtain:

$$\frac{9.81\text{ m/s}^2}{2(250\text{ m/s})^2\cos^2 0.450°}R^2 - (\tan 0.450°)R - 1.7\text{ m} = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$R = 206\text{ m}$$

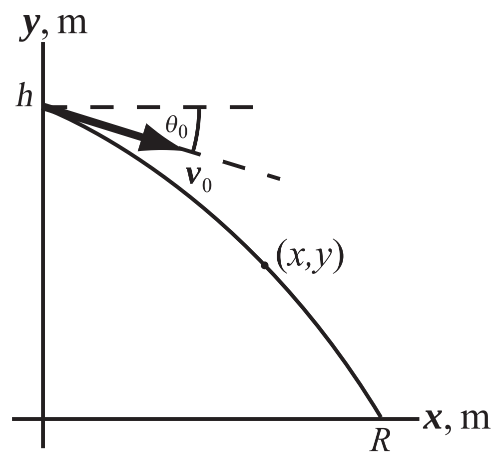
Substitute for R in equation (1) to obtain:

$$\Delta \boldsymbol{x} = 206\text{ m} - 100\text{ m} = \boxed{105\text{ m}}$$

General Problems

106 •• During a do-it-yourself roof repair project, you are on the roof of your house and accidentally drop your hammer. The hammer then slides down the roof at constant speed of 4.0 m/s. The roof makes an angle of 30º with the horizontal, and its lowest point is 10 m from the ground. (*a*) How long after leaving the roof does the hammer hit the ground? (*b*) What is the horizontal distance traveled by the hammer between the instant it leaves the roof and the instant it hits the ground? (Ignore any effects due to air resistance.)

Picture the Problem In the absence of air resistance, the hammer experiences constant acceleration as it falls. Choose a coordinate system with the origin and coordinate axes as shown in the figure and use constant-acceleration equations to describe the x and y coordinates of the hammer along its trajectory. We'll use the equation describing the vertical motion to find the time of flight of the hammer and the equation describing the horizontal motion to determine its range.



(a) Using a constant-acceleration equation, express the x coordinate of the hammer as a function of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$,

$$x = (v_0 \cos\theta_0)t$$

Using a constant-acceleration equation, express the y coordinate of the hammer as a function of time:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $y_0 = h$, $v_{0y} = v_0\sin\theta$, and $a_y = -g$,

$$y = h + (v_0 \sin\theta)t - \tfrac{1}{2}gt^2$$

Substituting numerical values yields:

$$y = 10\,\text{m} + (-4.0\,\text{m/s})(\sin 30°)t - \tfrac{1}{2}(9.81\,\text{m/s}^2)t^2$$

When the hammer hits the ground, $y = 0$ and our equation becomes:

$$0 = 10\,\text{m} - (4.0\,\text{m/s})\sin 30°\, t_{\text{fall}} - \tfrac{1}{2}(9.81\,\text{m/s}^2)t_{\text{fall}}^2$$

Use the quadratic formula or your graphing calculator to solve for the time of fall:

$$t_{\text{fall}} = 1.238\,\text{s} = \boxed{1.2\,\text{s}}$$

(b) Use the x-coordinate equation to express the horizontal distance traveled by the hammer as a function of its time-of-fall:

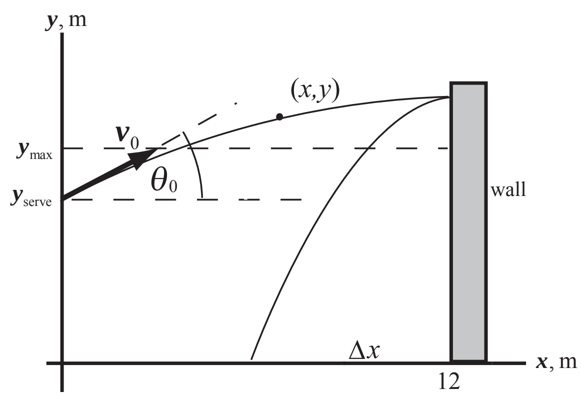
$$R = v_{0x}t_{\text{fall}} = (v_0\cos\theta_0)t_{\text{fall}}$$

Substitute numerical values and evaluate R:

$$R = (4.0\,\text{m/s})(\cos 30°)(1.238\,\text{s})$$
$$= \boxed{4.3\,\text{m}}$$

107 •• A squash ball typically rebounds from a surface with 25% of the speed with which it initially struck the surface. Suppose a squash ball is served in a shallow trajectory, from a height above the ground of 45 cm, at a launch angle of 6.0° degrees above the horizontal, and at a distance of 12 m from the front wall. (*a*) If it strikes the front wall exactly at the top of its parabolic trajectory, determine how high above the floor the ball strikes the wall. (*b*) How far horizontally from the wall does it strike the floor, after rebounding? Ignore any effects due to air resistance.

Picture the Problem The following diagram shows the trajectory of the ball, a coordinate system, and the distances relevant to the solution of the problem. This problem is another variation on the projectile motion problem in which we are given the launch angle and the horizontal distance to the point at which the ball reaches its peak. At that point, the ball's velocity is solely horizontal. In order to answer the question raised in (*b*), we'll need to find the (horizontal) speed with which the ball strikes the wall. We can then use this rebound speed to find the distance Δx by applying appropriate constant-acceleration equations.



(*a*) The height above the floor at which the ball strikes the wall is given by:

$$y_{max} = y_{serve} + h \qquad (1)$$

where h is the height above the service elevation to which the ball rises.

The distance h is related to the time Δt required for the ball to rise to its maximum height:

$$h = \tfrac{1}{2} g(\Delta t)^2 \qquad (2)$$

Δt is also related to v_{0y} and Δx_{serve}:

$$v_{0y} = g\Delta t \Rightarrow \Delta t = \frac{v_{0y}}{g} \qquad (3)$$

and

$$\Delta x_{serve} = v_{0x}\Delta t \Rightarrow \Delta t = \frac{\Delta x_{serve}}{v_{0x}} \qquad (4)$$

Substituting both (3) and (4) in equation (2) yields:

$$h = \tfrac{1}{2} g \left(\frac{v_{0y}}{g} \right) \left(\frac{\Delta x_{\text{serve}}}{v_{0x}} \right)$$

or, because $v_{0y} = v_0 \sin\theta_0$ and $v_{0x} = v_0 \cos\theta_0$,

$$h = \tfrac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g} \right) \left(\frac{\Delta x_{\text{serve}}}{v_0 \cos\theta_0} \right)$$

Simplifying this expression yields:

$$h = \frac{\Delta x_{\text{serve}}}{2} \tan\theta_0$$

Substitute for h in equation (1) to obtain:

$$y_{\max} = y_{\text{serve}} + \frac{\Delta x_{\text{serve}}}{2} \tan\theta_0$$

Substitute numerical values and evaluate $y_{\max}$:

$$y_{\max} = 0.45\ \text{m} + \frac{12\ \text{m}}{2} \tan 6.0° = 1.081\ \text{m}$$

$$= \boxed{1.1\ \text{m}}$$

(*b*) The distance Δx the ball lands away from the wall is given by

$$\Delta x = v_{x,\text{rebound}} \Delta t$$

where Δt is the time it takes for the ball to fall to the floor.

Because the ball rebounds with 25% of the speed with which it initially struck the surface:

$$v_{x,\text{rebound}} = 0.25 v_{0x}$$

and

$$\Delta x = 0.25 v_{0x} \Delta t \qquad (5)$$

Δt is related to distance the ball falls vertically before hitting the floor:

$$y_{\max} = \tfrac{1}{2} g (\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2 y_{\max}}{g}}$$

Substitute for Δt in equation (5) to obtain:

$$\Delta x = 0.25 v_{0x} \sqrt{\frac{2 y_{\max}}{g}} \qquad (6)$$

To find v_{0x}, equate equations (3) and (4):

$$\frac{v_{0y}}{g} = \frac{\Delta x_{\text{serve}}}{v_{0x}} \qquad (7)$$

From the diagram, we can see that:

$$v_{0y} = v_{0x} \tan\theta_0$$

Substituting for v_{0y} in equation (7) yields:

$$\frac{v_{0x} \tan\theta_0}{g} = \frac{\Delta x_{\text{serve}}}{v_{0x}}$$

Solving for v_{0x} yields:

$$v_{0x} = \sqrt{\frac{g\Delta x_{\text{serve}}}{\tan\theta_0}}$$

Substitute for v_{0x} in equation (6) to obtain:

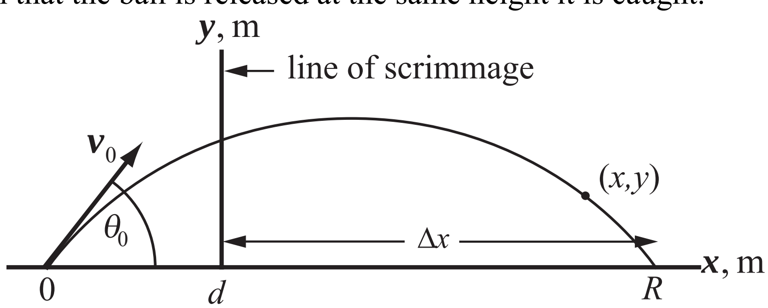
$$\Delta x = 0.25\sqrt{\frac{g\Delta x_{\text{serve}}}{\tan\theta_0}}\sqrt{\frac{2y_{\max}}{g}} = 0.25\sqrt{\frac{2\Delta x_{\text{serve}}y_{\max}}{\tan\theta_0}}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = 0.25\sqrt{\frac{2(12\text{ m})(1.081\text{ m})}{\tan 6.0°}} = \boxed{3.9\text{ m}}$$

108 •• A football quarterback throws a pass at an angle of 36.5° above the horizontal. He releases the pass 3.50 m behind the line of scrimmage. His receiver left the line of scrimmage 2.50 s earlier, goes straight downfield at a constant speed of 7.50 m/s. In order that the pass land gently in the receiver's hands without the receiver breaking stride, with what speed must the quarterback throw the pass? Assume that the ball is released at the same height it is caught and that the receiver is straight downfield from the quarterback at the time of the release. Ignore any effects due to air resistance.

Picture the Problem This problem deals with a moving target being hit by a projectile. The range required for the projectile is simply the distance that the receiver runs by the time the ball lands in his hands. We can use the equations developed in the text for both the range and time of flight of a projectile for any value of launch speed and angle, provided the launch and landing occur at the same height. The following diagram shows a convenient choice of a coordinate system as well as the trajectory of the ball. Note that the diagram includes the assumption that the ball is released at the same height it is caught.



The distance the receiver runs is given by:

$$\Delta x = v_{\text{rec}}\Delta t_{\text{tot}} \qquad (1)$$

The time-of-flight of the ball is given by:

$$\Delta t_{\text{ball}} = \Delta t_{\text{tot}} - \Delta t_{\text{delay}} \qquad (2)$$

where Δt_{delay} is the time the receiver ran before the quarterback released the ball.

The same-elevation range of the ball is:

$$R = \frac{v_0^2}{g}\sin 2\theta_0 \qquad (3)$$

If d is the distance behind the line of scrimmage that the ball is released:

$$R = \Delta x + d \qquad (4)$$

Combine equations (1), (3) and (4) to obtain:

$$v_{\text{rec}}\Delta t_{\text{tot}} + d = \frac{v_0^2}{g}\sin 2\theta_0$$

Solving for Δt_{tot} yields:

$$\Delta t_{\text{tot}} = \frac{v_0^2}{g v_{\text{rec}}}\sin 2\theta_0 - \frac{d}{v_{\text{rec}}}$$

The time-of-flight of the ball is given by:

$$\Delta t_{\text{ball}} = \frac{2v_0}{g}\sin\theta_0$$

Substitute for Δt_{tot} and Δt_{ball} in equation (2) to obtain:

$$\frac{2v_0}{g}\sin\theta_0 = \frac{v_0^2}{g v_{\text{rec}}}\sin 2\theta_0 - \Delta t_{\text{delay}} - \frac{d}{v_{\text{rec}}}$$

Rearrange and simplify to obtain:

$$\frac{\sin 2\theta_0}{g v_{\text{rec}}}v_0^2 - \frac{2\sin\theta_0}{g}v_0 - \Delta t_{\text{delay}} - \frac{d}{v_{\text{rec}}} = 0$$

Substituting numerical values yields:

$$\frac{\sin 2(36.5^\circ)}{(9.81\,\text{m/s}^2)(7.50\,\text{m/s})}v_0^2 - \frac{2\sin 36.5^\circ}{9.81\,\text{m/s}^2}v_0 - 2.50\,\text{s} - \frac{3.50\,\text{m}}{7.50\,\text{m/s}} = 0$$

or

$$\left(0.0130\,\frac{\text{s}^3}{\text{m}^2}\right)v_0^2 - \left(0.1213\,\frac{\text{s}^2}{\text{m}}\right)v_0 - 2.967\,\text{s} = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$v_0 = -11.1\,\text{m/s or } 20.5\,\text{m/s}$$

and, because the negative throwing speed is not physical,

$$v_0 = \boxed{20.5\,\text{m/s}}$$

109 •• Suppose a test pilot is able to safely withstand an acceleration of up to 5.0 times the acceleration due to gravity (that is, remain conscious and alert enough to fly). In the course of maneuvers, he is required to fly the plane in a horizontal circle at its top speed of 1900 mi/h, (*a*) What is the radius of the smallest circle he will be able to safely fly the plane in? (*b*) How long does it take him to go halfway around this minimum-radius circle?

Picture the Problem This is a simple application of centripetal acceleration.

(*a*) Express the radius of the pilot's circular path:

$$r = \frac{v^2}{a_c}$$

or, because $a_c = 5.0g$,

$$r = \frac{v^2}{5.0g}$$

Substitute numerical values and evaluate r:

$$r = \frac{\left(1900\,\frac{\text{mi}}{\text{h}} \times 1609\,\frac{\text{m}}{\text{mi}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{5.0\left(9.81\,\text{m/s}^2\right)}$$

$$= 14.70\,\text{km} = \boxed{15\,\text{km}}$$

(*b*) The time required to reverse his direction is equal to one-half the period of his motion:

$$t_{\text{rev}} = \tfrac{1}{2}T = \frac{2\pi r}{2v} = \frac{\pi r}{v}$$

Substitute numerical values and evaluate t_{rev}:

$$t_{\text{rev}} = \frac{\pi(14.70\,\text{km})}{1900\,\frac{\text{mi}}{\text{h}} \times 1609\,\frac{\text{m}}{\text{mi}} \times \frac{1\,\text{h}}{3600\,\text{s}}}$$

$$= \boxed{54\,\text{s}}$$

110 •• A particle moves in the *xy* plane with constant acceleration. At $t = 0$ the particle is at $\vec{r}_1 = (4.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j}$, with velocity $\vec{v}_1$. At $t = 2.0$ s, the particle has moved to $\vec{r}_2 = (10\text{ m})\hat{i} - (2.0\text{ m})\hat{j}$ and its velocity has changed to $\vec{v}_2 = (5.0\text{ m/s})\hat{i} - (6.0\text{ m/s})\hat{j}$. (*a*) Find $\vec{v}_1$. (*b*) What is the acceleration of the particle? (*c*) What is the velocity of the particle as a function of time? (*d*) What is the position vector of the particle as a function of time?

Picture the Problem Because the acceleration is constant; we can use the constant-acceleration equations in vector form and the definitions of average velocity and average (instantaneous) acceleration to solve this problem.

(*a*) The average velocity is given by:

$$\vec{\boldsymbol{v}}_{av} = \frac{\Delta\vec{\boldsymbol{r}}}{\Delta t} = \frac{\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1}{\Delta t}$$
$$= (3.0\ \text{m/s})\hat{\boldsymbol{i}} + (-2.5\ \text{m/s})\hat{\boldsymbol{j}}$$

The average velocity can also be expressed as:

$$\vec{\boldsymbol{v}}_{av} = \frac{\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2}{2} \Rightarrow \vec{v}_1 = 2\vec{v}_{av} - \vec{v}_2$$

Substitute numerical values to obtain:

$$\vec{\boldsymbol{v}}_1 = \boxed{(1.0\ \text{m/s})\hat{\boldsymbol{i}} + (1.0\ \text{m/s})\hat{\boldsymbol{j}}}$$

(*b*) The acceleration of the particle is given by:

$$\vec{\boldsymbol{a}} = \frac{\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1}{\Delta t}$$
$$= \boxed{(2.0\ \text{m/s}^2)\hat{\boldsymbol{i}} + (-3.5\ \text{m/s}^2)\hat{\boldsymbol{j}}}$$

(*c*) The velocity of the particle as a function of time is:

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{a}}t = \boxed{[(1.0\ \text{m/s}) + (2.0\ \text{m/s}^2)t]\hat{\boldsymbol{i}} + [(1.0\ \text{m/s}) + (-3.5\ \text{m/s}^2)t]\hat{\boldsymbol{j}}}$$

(*d*) Express the position vector as a function of time:

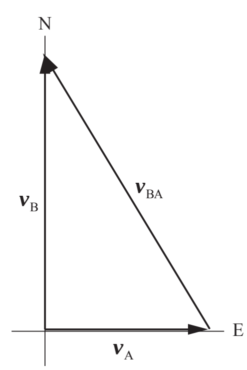
$$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_1 + \vec{\boldsymbol{v}}_1 t + \tfrac{1}{2}\vec{\boldsymbol{a}}t^2$$

Substitute numerical values and evaluate $\vec{\boldsymbol{r}}(t)$:

$$\vec{\boldsymbol{r}}(t) = \boxed{\begin{array}{l}[(4.0\ \text{m}) + (1.0\ \text{m/s})t + (1.0\ \text{m/s}^2)t^2]\hat{\boldsymbol{i}} \\ \qquad + [(3.0\ \text{m}) + (1.0\ \text{m/s})t + \left(-1.8\ \text{m/s}^2\right)t^2]\hat{\boldsymbol{j}}\end{array}}$$

111 •• [SSM] Plane A is headed due east, flying at an air speed of 400 mph. Directly below, at a distance of 4000 ft, plane B is headed due north, flying at an air speed of 700 mph. Find the velocity vector of plane B relative to A.

Picture the Problem Choose a coordinate system in which the +*x* axis is to the east and the +*y* axis is to the north and write the velocity vectors for the two airplanes using unit vector notation. The velocity vectors of the two planes and the velocity vector of plane B relative to plane A is shown in the vector diagram.



The velocity of plane B relative to plane A is given by:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{BA}$$

Solving for $\vec{v}_{\text{BA}}$ gives:

$$\vec{v}_{\text{BA}} = \vec{v}_{\text{B}} - \vec{v}_{\text{A}}$$

Using unit vector notation, write expressions for $\vec{v}_{\text{Bg}}$ and $\vec{v}_{\text{Ag}}$:

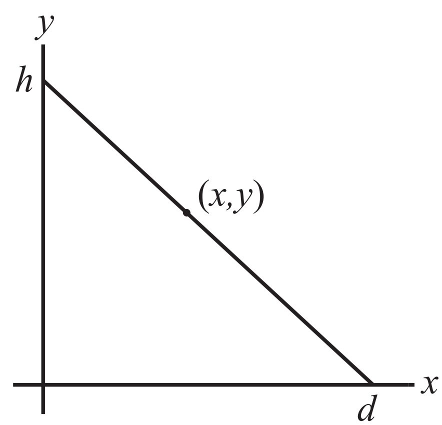
$$\vec{v}_{\text{B}} = (700\ \text{mi/h})\,\hat{\boldsymbol{j}}$$
and
$$\vec{v}_{\text{A}} = (400\ \text{mi/h})\,\hat{\boldsymbol{i}}$$

Substitute for $\vec{v}_{\text{Ba}}$ and $\vec{v}_{\text{Aa}}$ in equation (1) to obtain:

$$\vec{v}_{\text{BA}} = (700\ \text{mi/h})\,\hat{\boldsymbol{j}} - (400\ \text{mi/h})\,\hat{\boldsymbol{i}}$$
$$= \boxed{(-400\ \text{mi/h})\,\hat{\boldsymbol{i}} + (700\ \text{mi/h})\,\hat{\boldsymbol{j}}}$$

112 •• A diver steps off the cliffs at Acapulco, Mexico 30.0 m above the surface of the water. At that moment, he activates his rocket-powered backpack horizontally, which gives him a constant horizontal acceleration of 5.00 m/s^2 but does not affect his vertical motion. (*a*) How long does he take to reach the surface of the water? (*b*) How far out from the base of the cliff does he enter the water, assuming the cliff is vertical? (*c*) Show that his flight path is a straight line.

Picture the Problem Choose a coordinate system in which the origin is at the base of the cliff, the positive *x* axis is to the right, and the positive *y* axis is upward. Let the height of the cliff be *h* and the distance from the bottom of the cliff where the diver hits the water be *d.* Use constant-acceleration equations to obtain expressions for the *x* and *y* coordinates of the diver along his flight path, shown here as a straight line.



(*a*) Using a constant-acceleration equation, relate the vertical position of the diver to his initial position, initial speed, acceleration, and time of fall:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$
or, because $y_0 = h$, $v_{0y} = 0$, and $a_y = -g$,
$$y = h - \tfrac{1}{2}g t^2 \qquad (1)$$

When the diver reaches the surface of the water, $y = 0$ and:

$$0 = h - \tfrac{1}{2}g t^2 \Rightarrow t = \sqrt{\frac{2h}{g}} \qquad (2)$$

Substitute numerical values and evaluate *t*:

$$t = \sqrt{\frac{2(30.0\ \text{m})}{9.81\ \text{m/s}^2}} = \boxed{2.47\ \text{s}}$$

(*b*) Using a constant-acceleration equation, relate the horizontal position of the diver to his initial position, initial speed, acceleration, and time of fall:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$ and $v_{0x} = 0$,

$$x = \tfrac{1}{2}a_x t^2 \qquad (3)$$

When the diver hits the water, his x coordinate is d and the time is given by equation (2):

$$d = \tfrac{1}{2}a_x \frac{2h}{g} = \frac{a_x h}{g}$$

Substitute numerical values and evaluate d:

$$d = \frac{(5.00\ \text{m/s}^2)(30.0\ \text{m})}{9.81\ \text{m/s}^2} = \boxed{15.3\ \text{m}}$$

(*c*) Solving equation (3) for t^2 yields:

$$t^2 = \frac{2x}{a_x}$$

Substitute for t in equation (1) to obtain:

$$y = h - \tfrac{1}{2}g\frac{2x}{a_x} = h - \frac{g}{a_x}x$$

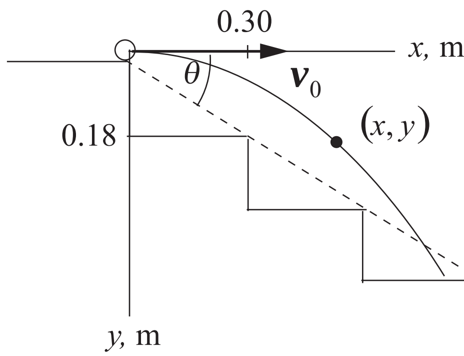
or

$$y = -\frac{g}{a_x}x + h \qquad (4)$$

Because equation (4) is of the form $y = mx + b$, the diver's trajectory into the water is a straight line.

113 •• [**SSM**] A small steel ball is projected horizontally off the top of a long flight of stairs. The initial speed of the ball is 3.0 m/s. Each step is 0.18 m high and 0.30 m wide. Which step does the ball strike first?

Picture the Problem In the absence of air resistance; the steel ball will experience constant acceleration. Choose a coordinate system with its origin at the initial position of the ball, the $+x$ direction to the right, and the $+y$ direction downward. In this coordinate system $y_0 = 0$ and $a = g$. Letting (x, y) be a point on the path of the ball, we can use constant-acceleration equations to express both x and y as functions of time and, using the geometry of the staircase, find an expression for the time of flight of the ball. Knowing its time of flight, we can find its range and identify the step it strikes first.



The angle of the steps, with respect to the horizontal, is:

$$\theta = \tan^{-1}\left(\frac{0.18\,\text{m}}{0.30\,\text{m}}\right) = 31.0°$$

Using a constant-acceleration equation, express the x coordinate of the steel ball in its flight:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_{0x} = 0$ and $a_x = 0$,

$$x = v_{0x}t = v_0 t \qquad (1)$$

Using a constant-acceleration equation, express the y coordinate of the steel ball in its flight:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $y_0 = 0$, $v_{0y} = 0$, and $a_y = g$,

$$y = \tfrac{1}{2}gt^2$$

The equation of the dashed line in the figure is:

$$\frac{y}{x} = \tan\theta = \frac{gt}{2v_0} \Rightarrow t = \frac{2v_0}{g}\tan\theta$$

Substitute for t in equation (1) to find the x coordinate of the landing position:

$$x = v_0 t = v_0\left(\frac{2v_0}{g}\tan\theta\right) = \frac{2v_0^2}{g}\tan\theta$$

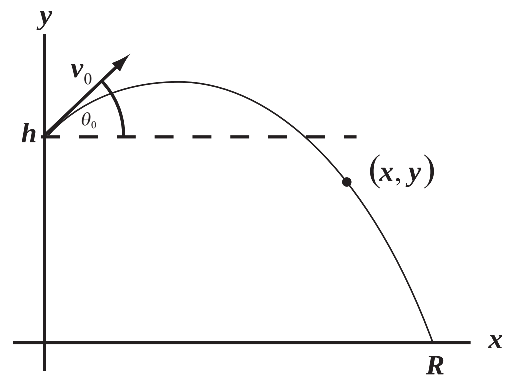
Substitute numerical values and evaluate x:

$$x = \frac{2(3.0\,\text{m/s})^2}{9.81\,\text{m/s}^2}\tan 31° = 1.1\,\text{m} \Rightarrow \text{The}$$

first step with $x > 1.1$ m is the 4th step.

114 •• Suppose you can throw a ball a maximum horizontal distance L when standing on level ground. How far can you throw it from the top of a building of height h if you throw it at (*a*) 0º, (*b*) 30º, (*c*) 45º? Ignore any effects due to air resistance.

Picture the Problem Ignoring the influence of air resistance, the acceleration of the ball is constant once it has left your hand and we can use constant-acceleration equations to express the x and y coordinates of the ball. Elimination of t will yield an equation from which we can determine v_0. We can then use the y equation to express the time of flight of the ball and the x equation to express its range in terms of x_0, v_0, θ and the time of flight.



Use a constant-acceleration equation to express the ball's horizontal position as a function of time:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

or, because $x_0 = 0$, $v_{0x} = v_0\cos\theta_0$, and $a_x = 0$,

$$x = (v_0\cos\theta_0)t \qquad (1)$$

Use a constant-acceleration equation to express the ball's vertical position as a function of time:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2$$

or, because $v_{0y} = v_0\sin\theta_0$, and $a_y = -g$,

$$y = y_0 + (v_0\sin\theta_0)t - \tfrac{1}{2}gt^2 \qquad (2)$$

Eliminate t between equations (1) and (2) to obtain:

$$y = x_0 + (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2$$

For the throw while standing on level ground we have:

$$0 = (\tan\theta_0)L - \frac{g}{2v_0^2\cos^2\theta_0}L^2$$

and

$$L = \frac{v_0^2}{g}\sin 2\theta_0 = \frac{v_0^2}{g}\sin 2(45°) = \frac{v_0^2}{g}$$

Solving for v_0 gives:

$$v_0 = \sqrt{gL}$$

At impact equation (2) becomes:

$$0 = x_0 + \left(\sqrt{gL}\sin\theta_0\right)t_{\text{flight}} - \tfrac{1}{2}gt_{\text{flight}}^2$$

Solve for the time of flight:

$$t_{\text{flight}} = \sqrt{\frac{L}{g}}\left(\sin\theta_0 + \sqrt{\sin^2\theta_0 + 2}\right)$$

Substitute for v_0 and t_{flight} in equation (1) to express the range of the ball when thrown from an elevation x_0 at an angle θ_0 with the horizontal:

$$R = \left(\sqrt{gL}\cos\theta_0\right)t_{flight} = \left(\sqrt{gL}\cos\theta_0\right)\sqrt{\frac{L}{g}}\left(\sin\theta_0 + \sqrt{\sin^2\theta_0 + 2}\right)$$
$$= L\cos\theta\left(\sin\theta_0 + \sqrt{\sin^2\theta_0 + 2}\right)$$

(*a*) Substitute $\theta = 0°$ and evaluate R: $R(0°) = \boxed{1.41L}$

(*b*) Substitute $\theta = 30°$ and evaluate R: $R(30°) = \boxed{1.73L}$

(*c*) Substitute $\theta = 45°$ and evaluate R: $R(45°) = \boxed{1.62L}$

115 ••• Darlene is a stunt motorcyclist in a traveling circus. For the climax of her show, she takes off from the ramp at angle θ_0, clears a ditch of width L, and lands on an elevated ramp (height h) on the other side (Figure 3-42). (*a*) For a given height h, find the minimum necessary takeoff speed v_{min} needed to make the jump successfully. (*b*) What is v_{min} for $\theta_0 = 30°$, $L = 8.0$ m and $h = 4.0$ m? (*c*) Show that no matter what her takeoff speed, the maximum height of the platform is $h < L\tan\theta_0$. Interpret this result physically. (Neglect any effects due to air resistance and treat the rider and the bike as if they were a single particle.)

Picture the Problem Choose a coordinate system with its origin at the point where the bike becomes airborne and with the positive x direction to the right and the positive y direction upward. With this choice of coordinate system we can relate the x and y coordinates of the rider and bike (which we're treating as a single particle) using Equation 3-17.

(*a*) The path of the motorcycle is given by Equation 3-17:

$$y(x) = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2$$

For the jump to be successful, $h < y(x)$. Solving for v_0 with $x = L$, we find that:

$$v_{min} > \boxed{\frac{L}{\cos\theta_0}\sqrt{\frac{g}{2(L\tan\theta_0 - h)}}}$$

(*b*) Substitute numerical values and evaluate v_{min}:

$$v_{min} > \frac{8.0\text{ m}}{\cos 30°}\sqrt{\frac{9.81\text{ m/s}^2}{2[(8.0\text{ m})\tan 30° - 4.0\text{ m}]}} = \boxed{26\text{ m/s}}$$

(*c*) In order for our expression for v_{min} to be real valued; that is, to predict values for v_{min} that are physically meaningful, $L\tan\theta_0 - h > 0$. Hence $h < L\tan\theta_0$.

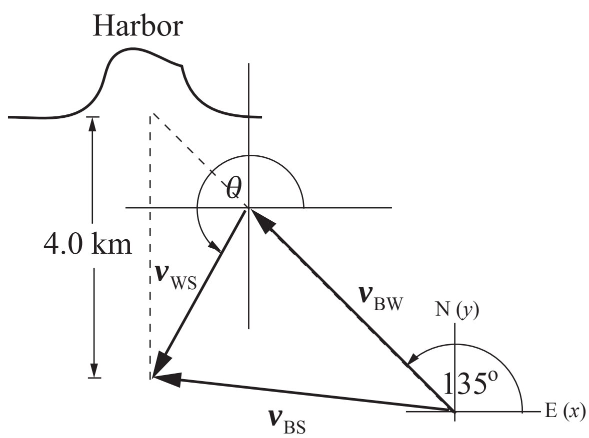
The interpretation is that the bike "falls away" from traveling on a straight-line path due to the free-fall acceleration downward. No matter what the initial speed of the bike, it must fall a little bit before reaching the other side of the ditch.

116 ••• A small boat is headed for a harbor 32 km directly northwest of its current position when it is suddenly engulfed in heavy fog. The captain maintains a compass bearing of northwest and a speed of 10 km/h relative to the water. The fog lifts 3.0 h later and the captain notes that he is now exactly 4.0 km south of the harbor. (*a*) What was the average velocity of the current during those 3.0 h? (*b*) In what direction should the boat have been heading to reach its destination along a straight course? (*c*) What would its travel time have been if it had followed a straight course?

Picture the Problem Let the origin be at the position of the boat when it was engulfed by the fog. Take the *x* and *y* directions to be east and north, respectively. Let $\vec{v}_{BW}$ be the velocity of the boat relative to the water, $\vec{v}_{BS}$ be the velocity of the boat relative to the shore, and $\vec{v}_{WS}$ be the velocity of the water with respect to the shore. Then

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}.$$

θ is the angle of $\vec{v}_{WS}$ with respect to the *x* (east) direction.



(*a*) The position of the boat at $t = 3.0$ h is:

$$\begin{aligned}\vec{r}_{\text{boat}} &= [(32\text{ km})(\cos 135°)]\hat{\boldsymbol{i}} + [(32\text{ km})(\sin 135°) - 4.0\text{ km}]\hat{\boldsymbol{j}} \\ &= (-22.63\text{ km})\hat{\boldsymbol{i}} + (18.63\text{ km})\hat{\boldsymbol{j}}\end{aligned}$$

The coordinates of the boat at $t = 3.0$ h are also given by:

$$r_x = [(10\text{ km/h})\cos 135° + v_{WS}\cos\theta](3.0\text{ h}) = -22.63\text{ km}$$

and

$$r_y = [(10\text{ km/h})\sin 135° + v_{WS}\sin\theta](3.0\text{ h}) = 18.63\text{ km}$$

Solving for $v_{WS}\cos\theta$ and $v_{WS}\sin\theta$ gives:

$$v_{WS}\cos\theta = -0.4723\text{ km/h} \qquad (1)$$

and

$$v_{WS}\sin\theta = -0.8611\text{ km/h} \qquad (2)$$

Divide the second of these equations by the first and solve for θ to obtain:

$$\theta = \tan^{-1}\left(\frac{-0.8611\text{ km/h}}{-0.4723\text{ km/h}}\right) = 61.3° \text{ or } 241.3°$$

One can solve either equation (1) or equation (2) for v_{WS}. Using equation (1) yields:

$$v_{WS} = \frac{-0.4723\text{ km/h}}{\cos\theta}$$

Because the boat has drifted south, use $\theta = 241.3°$ to obtain:

$$v_{WS} = \frac{-0.4723\text{ km/h}}{\cos(241.3°)} = \boxed{0.98\text{ km/h at } \theta = 241°}$$

(*b*) Letting ϕ be the angle between east and the proper heading for the boat, express the components of the velocity of the boat with respect to the shore:

$$v_{BS,x} = (10\text{ km/h})\cos\phi + (0.982\text{ km/h})\cos(241.3°) \qquad (3)$$

and

$$v_{BS,y} = (10\text{ km/h})\sin\phi + (0.982\text{ km/h})\sin(241.3°)$$

For the boat to travel northwest:

$$v_{BS,x} = -v_{BS,y}$$

Substitute the velocity components, square both sides of the equation, and simplify the expression to obtain the equations:

$$\sin\phi + \cos\phi = 0.133 \text{ and}$$

$$\sin^2\phi + \cos^2\phi + 2\sin\phi\cos\phi = 0.0177$$

or

$$1 + \sin(2\phi) = 0.0177$$

Solving for ϕ yields:

$$\phi = \tfrac{1}{2}\sin^{-1}(0.0177 - 1) = -39.6°$$

Because the current pushes south, the boat must head more northerly than 135°:

The correct heading is $\boxed{39.6° \text{ west of north}}$.

(*c*) Substitute for ϕ in equation (3) and evaluate $v_{BS,x}$ to obtain:

$$v_{BS,x} = -6.846\text{ km/h}$$

From the diagram:

$$v_{BS} = \frac{v_{BS,x}}{\cos 135°} = \frac{-6.846\text{ km/h}}{\cos 135°} = 9.681\text{ km/h}$$

To find the time to travel 32 km, divide the distance by the boat's actual speed:

$$t = \frac{32\ \text{km}}{9.681\ \text{km/h}} = \boxed{3.3\,\text{h} \approx 3\,\text{h}\ 18\,\text{min}}$$

117 •• [**SSM**] Galileo showed that, if any effects due to air resistance are neglected, the ranges for projectiles (on a level field) whose angles of projection exceed or fall short of 45º by the same amount are equal. Prove Galileo's result.

Picture the Problem In the absence of air resistance, the acceleration of the projectile is constant and the equation of a projectile for equal initial and final elevations, which was derived from the constant-acceleration equations, is applicable. We can use the equation giving the range of a projectile for equal initial and final elevations to evaluate the ranges of launches that exceed or fall short of 45° by the same amount.

Express the range of the projectile as a function of its initial speed and angle of launch:

$$R = \frac{v_0^2}{g}\sin 2\theta_0$$

Letting $\theta_0 = 45° \pm \theta$ yields:

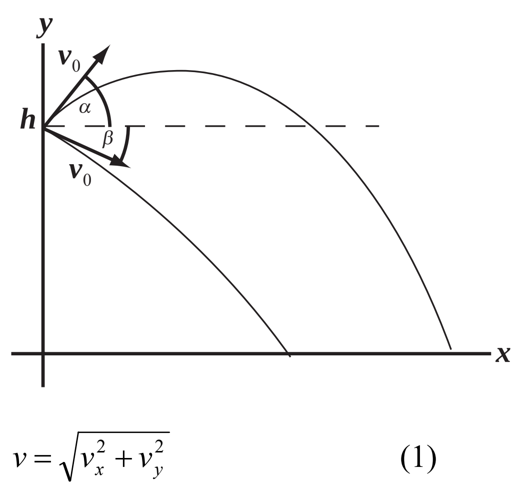
$$R = \frac{v_0^2}{g}\sin(90° \pm 2\theta) = \frac{v_0^2}{g}\cos(\pm 2\theta)$$

Because cos(–*θ*) = cos(+*θ*) (the cosine function is an *even* function):

$$\boxed{R(45° + \theta) = R(45° - \theta)}$$

118 •• Two balls are thrown with equal speeds from the top of a cliff of height *h*. One ball is thrown at an angle of α above the horizontal. The other ball is thrown at an angle of β below the horizontal. Show that each ball strikes the ground with the same speed, and find that speed in terms of *h* and the initial speed v_0. Neglect any effects due to air resistance.

Picture the Problem In the absence of air resistance, the acceleration of both balls is that due to gravity and the horizontal and vertical motions are independent of each other. Choose a coordinate system with the origin at the base of the cliff and the coordinate axes oriented as shown and use constant-acceleration equations to relate the *x* and *y* components of the ball's speed.



The speed of each ball on impact is:

$$v = \sqrt{v_x^2 + v_y^2} \qquad (1)$$

Independently of whether a ball is thrown upward at the angle α or downward at β, the vertical motion is described by:

$$v_y^2 = v_{0y}^2 + 2a_y\Delta y$$
$$= v_{0y}^2 - 2gh$$

The horizontal component of the speed of each ball is:

$$v_x = v_{0x}$$

Substitute for v_x and v_y in equation (1) and simplify to obtain:

$$v = \sqrt{v_{0x}^2 + v_{0y}^2 - 2gh} = \boxed{\sqrt{v_0^2 - 2gh}}$$

119 •• In his car, a driver tosses an egg vertically from chest height so that the peak of its path is just below the ceiling of the passenger compartment, which is 65 cm above his release point. He catches the egg at the same height at which he released it. If you are a roadside observer, and measure the horizontal distance between catch and release points to be 19 m, (*a*) how fast is the car moving? (*b*) In your reference frame, at what angle above the horizontal was the egg thrown?

Picture the Problem For the first part of the question, we simply need to determine the time interval for the ball's "flight." In order to determine that, we need only consider the time required for an egg to fall 65 cm under the influence of gravity and then double that time. The path taken by the egg, according to the one observing on the side of the road, is a parabola – in fact, its path is indistinguishable from that of a projectile. We can obtain the apparent angle of launch by noting that the horizontal component of the egg's motion is simply given by the speed of the car, while the vertical component is given by the speed of the toss necessary to reach a height of 0.65 m.

(*a*) The speed of the car is given by:

$$v_{\text{car}} = \frac{\Delta x}{\Delta t_{\text{flight}}} \qquad (1)$$

where Δx is the distance the car moves during the time Δt_{flight} the egg is in flight.

The fall time, Δt_{fall}, for the egg to fall (from rest) and the distance Δh it falls are related according to:

$$\Delta h = \tfrac{1}{2}g(\Delta t_{\text{fall}})^2 \Rightarrow \Delta t_{\text{fall}} = \sqrt{\frac{2\Delta h}{\text{g}}}$$

The time-of-flight is twice the egg's fall time:

$$\Delta t_{\text{flight}} = 2\Delta t_{\text{fall}}$$

Substitute for Δt_{fall} to obtain:

$$\Delta t_{\text{flight}} = 2\sqrt{\frac{2\Delta h}{\text{g}}}$$

Substituting for Δt_{flight} in equation (1) yields:

$$v_{\text{car}} = \frac{\Delta x}{2\sqrt{\frac{2\Delta h}{g}}} = \frac{\Delta x}{2}\sqrt{\frac{g}{2\Delta h}} \quad (2)$$

Substitute numerical values and evaluate v_{car}:

$$v_{\text{car}} = \frac{19\text{ m}}{2}\sqrt{\frac{9.81\text{ m/s}^2}{2(0.65\text{ m})}} = \boxed{26\text{ m/s}}$$

(*b*) The angle θ_0 at which the egg was thrown is given by:

$$\theta_0 = \tan^{-1}\left(\frac{v_{0y}}{v_{0x}}\right) \quad (3)$$

Use a constant-acceleration equation to relate the final upward speed of the egg to its initial speed and its maximum height Δh:

$$v_y^2 = v_{0y}^2 + 2a_y\Delta h$$

or, because $v_y = 0$ and $a_y = -g$,

$$0 = v_{0y}^2 - 2g\Delta h$$

Solve for v_{0y} to obtain:

$$v_{0y} = \sqrt{2g\Delta h}$$

v_{0x} is the speed of the car. Substituting in equation (2) yields:

$$v_{\text{car}} = v_{0x} = \frac{\Delta x}{2}\sqrt{\frac{g}{2\Delta h}}$$

Substitute for v_{0x} in equation (3) to obtain:

$$\theta_0 = \tan^{-1}\left(\frac{v_{0y}}{\frac{\Delta x}{2}\sqrt{\frac{g}{2\Delta h}}}\right) = \tan^{-1}\left(\frac{4\Delta h}{\Delta x}\right)$$

Substitute numerical values and evaluate θ_0:

$$\theta_0 = \tan^{-1}\left(\frac{4(0.65\text{ m})}{19\text{ m}}\right) = \boxed{7.8°}$$

120 ••• A straight line is drawn on the surface of a 16-cm-radius turntable from the center to the perimeter. A bug crawls along this line from the center outward as the turntable spins counterclockwise at a constant 45 rpm. Its walking speed relative to the turntable is a steady 3.5 cm/s. Let its initial heading be in the positive *x*-direction. As the bug reaches the edge of the turntable, and falls off, (still traveling at 3.5 cm/s radially, relative to the turntable, what are the *x* and *y* components of the velocity of the bug?

Picture the Problem In order to determine the total velocity vector of the bug we need to consider the two components that must be added together, namely, the bug's velocity relative to the turntable at its point of departure from the turntable, and the velocity of the point on the turntable from which the bug steps off relative to the ground.

Express the velocity of the bug relative to the ground $\vec{v}_{\text{Bg}}$:

$$\vec{v}_{\text{Bg}} = \vec{v}_{\text{BT}} + \vec{v}_{\text{Tg}} \qquad (1)$$

where $\vec{v}_{\text{BT}}$ is the velocity of the bug relative to the turntable and $\vec{v}_{\text{Tg}}$ is the velocity of the turntable relative to the ground.

The angle through which the turntable has rotated when the bug reaches its edge is given by:

$$\Delta\theta = \left(\frac{360°}{1\text{ rev}}\right)\Delta t = \left(\frac{360°}{1\text{ rev}}\right)\left(\frac{R}{v}\right)$$

where R is the radius of the turntable and v is the speed of the bug.

Substitute numerical values and evaluate $\Delta\theta$:

$$\Delta\theta = \left(\frac{360°}{1\text{ rev}} \times \frac{45\text{ rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{16\text{ cm}}{3.5\text{ cm/s}}\right) = 1234°$$

The number of degrees beyond an integer multiple of revolutions is:

$$\theta = \Delta\theta - 360°n = 1234° - 360°(3)$$
$$= 154°$$

Now we can express the velocity of the bug relative to the turntable:

$$\vec{v}_{\text{BT}} = (v_{\text{B}}\cos\theta)\hat{\boldsymbol{i}} + (v_{\text{B}}\sin\theta)\hat{\boldsymbol{j}}$$

Substitute numerical values and evaluate $\vec{v}_{\text{BT}}$:

$$\begin{aligned}\vec{v}_{\text{BT}} &= ((3.5\text{ cm/s})\cos 154°)\hat{\boldsymbol{i}} + ((3.5\text{ cm/s})\sin 154°)\hat{\boldsymbol{j}} \\ &= (-3.146\text{ cm/s})\hat{\boldsymbol{i}} + (1.534\text{ cm/s})\hat{\boldsymbol{j}}\end{aligned}$$

The point on the turntable from which the bug leaves the turntable is moving with a velocity vector pointed 90° ahead of $\vec{v}_{\text{BT}}$; that is, tangent to the turntable:

$$\vec{v}_{\text{Tg}} = v_{\text{turntable}}\cos(\theta + 90°)\hat{\boldsymbol{i}} + v_{\text{turntable}}\sin(\theta + 90°)\hat{\boldsymbol{j}}$$

The velocity of the turntable is given by:

$$v_{\text{turntable}} = \frac{2\pi R}{T}$$

where T is the period of the turntable's motion.

Substitute numerical values and evaluate $v_{\text{turntable}}$:

$$v_{\text{turntable}} = \frac{2\pi(16\text{ cm/s})}{\dfrac{60\text{ s}}{45\text{ rev}}} = 75.40\text{ cm/s}$$

Substitute in the expression for $\vec{v}_{\text{Tg}}$ and simplify to obtain:

$$\begin{aligned}\vec{v}_{\text{Tg}} &= (75.40\text{ cm/s})\cos(154° + 90°)\hat{i} + (75.40\text{ cm/s})\sin(154° + 90°)\hat{j} \\ &= (-33.05\text{ cm/s})\hat{i} + (-67.77\text{ cm/s})\hat{j}\end{aligned}$$

Finally, substitute for $\vec{v}_{\text{BT}}$ and $\vec{v}_{\text{Tg}}$ in equation (1) and simplify:

$$\begin{aligned}\vec{v}_{\text{Bg}} &= (-3.146\text{ cm/s})\hat{i} + (1.534\text{ cm/s})\hat{j} + (-33.05\text{ cm/s})\hat{i} + (-67.77\text{ cm/s})\hat{j} \\ &= (-36\text{ cm/s})\hat{i} + (-66\text{ cm/s})\hat{j}\end{aligned}$$

The x and y components of the velocity of the bug are:

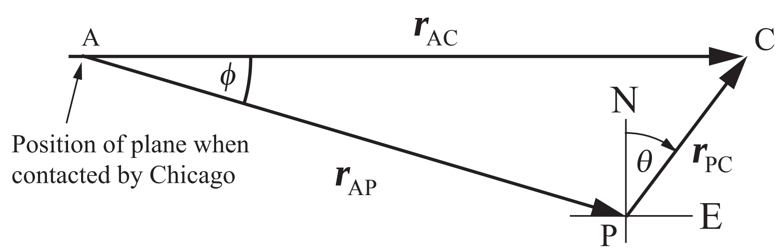
$$v_{x,\text{Bug}} = \boxed{-36\text{ cm/s}}$$

and

$$v_{y,\text{Bug}} = \boxed{-66\text{ cm/s}}$$

121 ••• On a windless day, a stunt pilot is flying his vintage World War I Sopwith Camel from Dubuque, Iowa, to Chicago, Illinois for an air show. Unfortunately, he is unaware that his plane's ancient magnetic compass has a serious problem in that what it records as "north" is in fact 16.5° east of true north. At one moment during his flight, the airport in Chicago notifies him that he is, in reality, 150 km due west of the airport. He then turns due east, according to his plane's compass, and flies for 45 minutes at 150 km/hr. At that point, he expected to see the airport and begin final descent. What is the plane's actual distance from Chicago and what should be his heading if he is to fly directly to Chicago?

Picture the Problem The error in the guidance computer aboard the airplane implies that when the pilot flies the plane at what he thinks is a heading of due east, he is in fact flying at a heading of 16.5° south of east. When he has flown at 150 km/hr for 15 minutes along that heading he is therefore somewhat west and south of Chicago. In the following diagram, A identifies the position of the plane when contacted by the airport in Chicago, C the location of Chicago, and P the location of the plane after it has flown for 45 minutes at 150 km/h. Choose a coordinate system in which the $+x$ direction is to the east and the $+y$ direction is to the north. Expressing the vector $\vec{r}_{\text{CP}}$ in unit vector notation will allow us to find the plane's actual distance from Chicago (the magnitude of $\vec{r}_{\text{CP}}$) and the heading of Chicago (the direction of $-\vec{r}_{\text{CP}}$.



The plane's true distance from Chicago is given by the magnitude of vector $\vec{r}_{PC}$:

$$r_{PC} = \sqrt{r_{PC,x}^2 + r_{PC,y}^2} \quad (1)$$

The actual heading of Chicago relative to the plane's current position is given by:

$$\theta = \tan^{-1}\left(\frac{r_{PC,x}}{r_{PC,y}}\right) \quad (2)$$

Express the relationship between the vectors $\vec{r}_{AC}$, $\vec{r}_{PC}$, and $\vec{r}_{AP}$:

$$\vec{r}_{AP} + \vec{r}_{PC} = \vec{r}_{AC}$$
or
$$\vec{r}_{PC} = \vec{r}_{AC} - \vec{r}_{AP} \quad (3)$$

The vector $\vec{r}_{AP}$ is given by:

$$\vec{r}_{AP} = r_{AP}\cos\phi\,\hat{\boldsymbol{i}} - r_{AP}\sin\phi\,\hat{\boldsymbol{j}}$$

Substitute numerical values and simplify to obtain:

$$\begin{aligned}\vec{r}_{AP} &= \left[\left(150\,\frac{\text{km}}{\text{h}}\times 0.75\,\text{h}\right)\cos 16.5°\right]\hat{\boldsymbol{i}} - \left[\left(150\,\frac{\text{km}}{\text{h}}\times 0.75\,\text{h}\right)\sin 16.5°\right]\hat{\boldsymbol{j}}\\ &= (107.9\,\text{km})\hat{\boldsymbol{i}} - (31.95\,\text{km})\hat{\boldsymbol{j}}\end{aligned}$$

The vector $\vec{r}_{AC}$ is given by:

$$\vec{r}_{AC} = (150\,\text{km})\hat{\boldsymbol{i}}$$

Substitute for $\vec{r}_{AP}$ and $\vec{r}_{AC}$ in equation (3) to obtain:

$$\begin{aligned}\vec{r}_{PC} &= (150\,\text{km})\hat{\boldsymbol{i}} - \left[(107.9\,\text{km})\hat{\boldsymbol{i}} - (31.95\,\text{km})\hat{\boldsymbol{j}}\right]\\ &= (42.10\,\text{km})\hat{\boldsymbol{i}} + (31.95\,\text{km})\hat{\boldsymbol{j}}\end{aligned}$$

Substituting numerical values in equation (1) and evaluate r_{PC}:

$$r_{PC} = \sqrt{(42.10\,\text{km})^2 + (31.95\,\text{km})^2} = \boxed{52.9\,\text{km}}$$

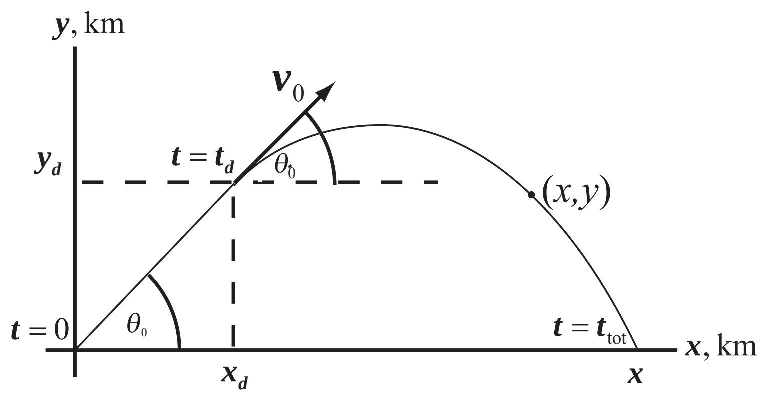
Substituting numerical values in equation (2) and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{42.10\text{ km}}{31.95\text{ km}}\right) = 52.8°$$

$$= \boxed{52.8° \text{ east of north}}$$

122 ••• A cargo plane in flight lost a package because somebody forgot to close the rear cargo doors. You are on the team of safety experts trying to analyze what happened. From the point of takeoff, while climbing to altitude, the airplane traveled in a straight line at a constant speed of 275 mi/h at an angle of 37° above the horizontal. During this ascent, the package slid off the back ramp. You found the package in a field a distance of 7.5 km from the takeoff point. To complete the investigation you need to know exactly how long after takeoff the package slid off the back ramp of the plane. (Consider the sliding speed to be negligible.) Calculate the time at which the package fell off the back ramp. Ignore any effects due to air resistance.

Picture the Problem We can solve this problem by first considering what the projectile is doing while it is still in the cargo plane and then considering its motion as a projectile. The following diagram shows a convenient choice of coordinate system and the trajectory of the package both in the plane and when it has become a projectile. We can use constant-acceleration equations to describe both parts of its motion.



Express the x- and y-coordinates of the drop point:

$$x_d = v_{x,d}t_d \quad (1)$$

and

$$y_d = v_{y,d}t_d \quad (2)$$

Express the x- and y-coordinates of the package along its parabolic trajectory (after it has fallen out of the plane):

$$x(t) = x_d + v_{x,d}(t - t_d)$$

and

$$y(t) = y_d + v_{y,d}(t - t_d) - \tfrac{1}{2}g(t - t_d)^2$$

Substitute for x_d and y_d from equations (1) and (2) and simplify to obtain:

$$x(t) = v_{x,d}t_d + v_{x,d}(t - t_d) = v_{x,d}t$$

and

$$y(t) = v_{y,d}t_d + v_{y,d}(t - t_d) - \tfrac{1}{2}g(t - t_d)^2$$
$$= v_{y,d}t - \tfrac{1}{2}g(t - t_d)^2$$

Express the x and y components of the plane's (and package's) velocity at the drop point:

$$v_{x,d} = v_{\text{plane}}\cos\theta_0$$

and

$$v_{y,d} = v_{\text{plane}}\sin\theta_0$$

$$v_{x,d} = \left(275\,\frac{\text{mi}}{\text{h}} \times 1609\,\frac{\text{m}}{\text{mi}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)\cos 37° = 98.16\ \text{m/s}$$

and

$$v_{y,d} = \left(275\,\frac{\text{mi}}{\text{h}} \times 1609\,\frac{\text{m}}{\text{mi}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)\sin 37° = 73.97\ \text{m/s}$$

At impact ($t = t_{\text{tot}}$):

$$x(t_{\text{tot}}) = x_\ell \text{ and } y(t_{\text{tot}}) = 0$$

The distance-to-impact is also given by:

$$x_\ell = v_{x,d}t_{\text{tot}} \Rightarrow t_{\text{tot}} = \frac{x_\ell}{v_{x,d}}$$

Substitute for t_{tot} in the $y(t)$ equation to obtain:

$$0 = v_{y,d}t_{\text{tot}} - \tfrac{1}{2}g(t_{\text{tot}} - t_d)^2$$

or, because $t_{\text{tot}} = x_\ell / v_{x,d}$,

$$0 = v_{y,d}\frac{x_\ell}{v_{x,d}} - \tfrac{1}{2}g\left(\frac{x_\ell}{v_{x,d}} - t_d\right)^2$$

Expand the binomial term and simplify the resulting expression to obtain the quadratic equation:

$$t_d^2 - \frac{2x_\ell}{v_{x,d}}t_d + \left[\frac{x_\ell^2}{v_{x,d}^2} - \frac{2x_\ell v_{y,d}}{g v_{x,d}}\right] = 0$$

Substitute numerical values to obtain:

$$t_d^2 - \frac{2(7.5\ \text{km})}{98.16\ \text{m/s}}t_d + \left[\frac{(7.5\ \text{km})^2}{(98.16\ \text{m/s})^2} - \frac{2(7.5\ \text{km})(73.97\ \text{m/s})}{(9.81\ \text{m/s}^2)(98.16\ \text{m/s})}\right] = 0$$

Simplification of this expression yields:

$$t_d^2 - (152.8\ \text{s})t_d + 4686\ \text{s}^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$t_d = 110\text{ s or }42\text{ s}$

Because 110 s gives too long a distance for the range of the package,

$t_d = \boxed{42\text{ s}}$

Chapter 4
Newton's Laws

Conceptual Problems

1 • While on a very smooth level transcontinental plane flight, your coffee cup sits motionless on your tray. Are there forces acting on the cup? If so, how do they differ from the forces that would be acting on the cup if it sat on your kitchen table at home?

Determine the Concept Yes, there are forces acting on it. They are the normal force of the table and the gravitational pull of Earth (weight). Because the cup is not accelerating relative to the ground, the forces are the same as those that would act on it if it was sitting on your table at home.

2 • You are passing another car on a highway and determine that, relative to you, the car you pass has an acceleration $\vec{a}$ to the west. However, the driver of the other car is maintaining a constant speed and direction relative to the road. Is the reference frame of your car an inertial one? If not, in which direction (east or west) is your car accelerating relative to the other car?

Determine the Concept No. You are in a non-inertial frame that is accelerating to the east, opposite the other car's apparent acceleration.

3 • [**SSM**] You are riding in a limousine that has opaque windows that do not allow you to see outside. The car is on a flat horizontal plane, so the car can accelerate by speeding up, slowing down, or turning. Equipped with just a small heavy object on the end of a string, how can you use it to determine if the limousine is changing either speed or direction? Can you determine the limousine's velocity?

Determine the Concept In the limo you hold one end of the string and suspend the object from the other end. If the string remains vertical, the reference frame of the limo is an inertial reference frame.

4 •• If only a single nonzero force acts on an object, does the object accelerate relative to all inertial reference frames? Is it possible for such an object to have zero velocity in some inertial reference frame and not in another? If so, give a specific example.

Determine the Concept An object accelerates when a *net* force acts on it. The fact that an object is accelerating tells us nothing about its velocity other than that it is always changing.

Yes, the object must have acceleration relative to all inertial frames of reference. According to Newton's first and second laws, an object must accelerate, relative to any inertial reference frame, in the direction of the net force. If there is ″only a single nonzero force,″ then this force is the net force.

Yes, the object's velocity may be momentarily zero in some inertial reference frame and not in another. During the period in which the force is acting, the object may be momentarily at rest, but its velocity cannot remain zero because it must continue to accelerate. Thus, its velocity is always changing.

5 •• A baseball is acted upon by a single known force. From this information alone, can you tell in which direction the baseball is moving relative to some reference frame? Explain.

Determine the Concept No. Predicting the direction of the subsequent motion correctly requires additional information (knowledge of the initial velocity as well as the acceleration). While the acceleration can be obtained from the net force through Newton's second law, the velocity can only be obtained by integrating the acceleration.

6 •• A truck moves directly away from you at constant velocity (as observed by you while standing in the middle of the road). It follows that (*a*) no forces act on the truck, (*b*) a constant force acts on the truck in the direction of its velocity, (*c*) the net force acting on the truck is zero, (*d*) the net force acting on the truck is its weight.

Determine the Concept An object in an inertial reference frame accelerates if there is a *net* force acting on it. Because the object is moving at constant velocity, the net force acting on it is zero. $\boxed{(c)}$ is correct.

7 •• Several space probes have been launched that are now far out in space *Pioneer 10*, for example, was launched in the 1970s and is still moving away from the Sun and its planets. Is the mass of *Pioneer 10* changing? Which of the known fundamental forces continue to act on it? Does it have a net force on it?

Determine the Concept No. The mass of the probe is constant. However, the solar system will attract the probe with a gravitational force. As the distance between *Pioneer 10* and the solar system becomes larger, the magnitude of the gravitational force becomes smaller. There is a net force on the probe because no other forces act on it.

8 •• Astronauts in apparent weightlessness during their stay on the International Space Station must carefully monitor their masses because significant loss of body mass is known to cause serious medical problems. Give an example of how you might design equipment to measure the mass of an astronaut on the orbiting space station.

Determine the Concept You could use a calibrated spring (a spring with a known stiffness constant) to pull on each astronaut and measure their resulting acceleration. Then you could use Newton's second law to calculate their mass.

9 •• [SSM] You are riding in an elevator. Describe two situations in which your apparent weight is greater than your true weight.

Determine the Concept Your apparent weight is the reading of a scale. If the acceleration of the elevator (and you) is directed upward, the normal force exerted by the scale on you is greater than your weight. You could be moving down but slowing or moving up and speeding up. In both cases your acceleration is upward.

10 •• Suppose you are in a train moving at constant velocity relative to the ground. You toss a ball to your friend several seats in front of you. Use Newton's second law to explain why you cannot use your observations of the tossed ball to determine the train's velocity relative to the ground.

Determine the Concept Because you are moving with constant velocity, your frame of reference is an inertial reference frame. In an inertial reference frame there are no fictitious forces. Thus moving or not moving, the ball will follow the same trajectory in your reference frame. The net force on the ball is the same, so its acceleration is the same.

11 •• Explain why, of the fundamental interactions, the gravitational interaction is the main concern in our everyday lives. One other on this list also plays an increasingly significant role in our rapidly advancing technology. Which one is that? Why are the others not obviously important?

Determine the Concept The most significant force in our everyday world is gravity. It literally keeps us on or near the ground. The other most common force is the electromagnetic force. It provides the ″glue″ to hold solids together and make them rigid. It is of great importance in electric circuits.

12 •• Give an example of an object that has three forces acting on it, and (*a*) accelerates, (*b*) moves at constant (non-zero) velocity, and (*c*) remains at rest.

(*a*) Any object for which the vector sum of the three forces doesn't add to zero. For example, a sled on a frictionless surface pulled horizontally. The normal force plus the weight plus the pulling force do not add to zero, so the sled accelerates.

(*b*) Pulling a fish vertically upward at constant velocity while it is still in the water. The forces acting on the fish are the pull, the gravitational force (weight of the fish), and water drag forces. These forces add up to zero.

(*c*) The three forces need to add vectorially to zero. An example is a picture hung by two wires.

13 •• [SSM] Suppose a block of mass m_1 rests on a block of mass m_2 and the combination rests on a table as shown in Figure 4-33. Tell the name of the force and its category (contact versus action-at-a-distance) for each of the following forces; (*a*) force exerted by m_1 on m_2, (*b*) force exerted by m_2 on m_1, (*c*) force exerted by m_2 on the table, (*d*) force exerted by the table on m_2, (*e*) force exerted by Earth on m_2. Which, if any, of these forces constitute a Newton's third law pair of forces?

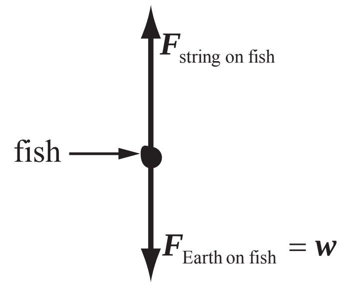
Determine the Concept

(*a*) The force exerted by m_1 on m_2.	Normal force, contact
(*b*) The force exerted by m_2 on m_1.	Normal force, contact
(*c*) The force exerted by m_2 on the table.	Normal force, contact
(*d*) The force exerted by the table on m_2.	Normal force, contact
(*e*) The force exerted by Earth on m_2.	Gravitational force, action-at-a-distance

The Newton's third law force pairs are the two normal forces between the two blocks and the two normal forces between the table and the bottom block. The gravitational force has a third law force pair that acts on Earth and so is not in the question set.

14 •• You yank a fish you have just caught on your line upward from rest into your boat. Draw a free-body diagram of the fish after it has left the water and as it gains speed as it rises. In addition, tell the type (tension, spring, gravity, normal, friction, etc.) and category (contact versus action-at-a-distance) for each force on your diagram. Which, if any, pairs of the forces on your diagram constitute a Newton's third law pair? Can you tell the relative magnitudes of the forces from the information given? Explain.

Determine the Concept A free-body diagram showing the forces acting on the fish is shown to the right. The forces do not constitute a Newton's 3rd law pair. A table summarizing the type and category of the forces is shown below.



Force	Type	Category
$\vec{F}_{\text{string on fish}}$	Tension	Contact
$\vec{F}_{\text{Earth on fish}}$	Gravity	Action-at-a-distance

Because the fish accelerates upward, the tension force must be greater in magnitude than the gravitational force acting on the fish.

15 • If you gently set a fancy plate on the table, it will not break. However if you drop it from a height, it might very well break. Discuss the forces that act on the plate (as it contacts the table) in both these situations. Use kinematics and Newton's second law to describe what is different about the second situation that causes the plate to break.

Determine the Concept When the plate is sitting on the table, the normal force F_{n} acting upward on it is exerted by the table and is the same size as the gravitational force F_g acting on the plate. Hence, the plate does not accelerate. However, to slow the plate down as it hits the table requires that $F_{\text{n}} > F_g$ (or $F_{\text{n}} >> F_g$ if the table is hard and the plate slows quickly). A large normal force exerted on delicate china can easily break it.

16 •• For each of the following forces, give what produces it, what object it acts on, its direction, and the reaction force. (*a*) The force you exert on your briefcase as you hold it while standing at the bus stop. (*b*) The normal force on the soles of your feet as you stand barefooted on a horizontal wood floor. (*c*) The gravitational force on you as you stand on a horizontal floor. (*d*) The horizontal force exerted on a baseball by a bat as the ball is hit straight up the middle towards center field for a single.

Determine the Concept

(*a*) The force you exert on your briefcase to hold it while standing at the bus stop:

You produce this force. It acts on the briefcase. It acts upward. The reaction force is the force the briefcase exerts on your hand.

(*b*) The normal force on the soles of your feet as you stand barefooted on a horizontal wood floor.	The floor produces this force. It acts on your feet. It acts upward. The reaction force is the force your feet exert on the floor.
(*c*) The gravitational force on you as you stand on a horizontal floor.	Earth produces this force. It acts on you. It acts downward. The reaction force is the gravitational force you exert on Earth.
(*d*) The horizontal force exerted on a baseball by a bat as the ball is hit straight up the middle towards center field for a single.	The bat produces this force. It acts on the ball. It acts horizontally. The reaction force is the force the ball exerts on the bat.

17 •• For each case, identify the force (including its direction) that causes the acceleration. (*a*) A sprinter at the very start of the race. (*b*) A hockey puck skidding freely but slowly coming to rest on the ice. (*c*) A long fly ball at the top of its arc. (*d*) A bungee jumper at the very bottom of her descent.

Determine the Concept

(*a*) A sprinter at the very start of the race:	The normal force of the block on the sprinter, in the forward direction.
(*b*) A hockey puck skidding freely but slowly coming to rest on the ice:	The frictional force by the ice on the puck, in the opposite direction to the velocity.
(*c*) A long fly ball at the top of its arc:	The gravitation force by Earth on the ball, in the downward direction.
(*d*) A bungee jumper at the very bottom of her descent:	The force exerted by the bungee cord on the jumper, in the upward direction.

18 • True or false:

(*a*) If two external forces that are both equal in magnitude and opposite in direction act on the same object, the two forces can never be a Newton's third law pair.
(*b*) The two forces of a Newton's third law pair are equal only if the objects involved are not accelerating.

(*a*) True. By definition, third law pairs cannot act on the same object.

(*b*) False. Action and reaction forces are equal independently of any motion of the involved objects.

19 •• An 80-kg man on ice skates is pushing his 40-kg son, also on skates, with a force of 100 N. Together, they move across the ice steadily gaining speed. (*a*) The force exerted by the boy on his father is (1) 200 N, (2) 100 N, (3) 50 N, or (4) 40 N. (*b*) How do the magnitudes of the two accelerations compare? (*c*) How do the directions of the two accelerations compare?

Determine the Concept

(*a*) (2) These forces are a Newton 3^{rd} law force pair, and so the force exerted by the boy on his father is 100 N.

(*b*) Because the father and son move together, their accelerations will be the same.

(*c*) The directions of their acceleration are the same.

20 •• A girl holds a stone in her hand and can move it up or down or keep it still. True or false: (*a*) The force exerted by her hand on the rock is always the same magnitude as the weight of the stone. (*b*) The force exerted by her hand on the rock is the reaction force to the pull of gravity on the stone. (*c*) The force exerted by her hand is always the same size the force her hand feels from the stone but in the opposite direction. (*d*) If the girl moves her hand down at a constant speed, then her upward force on the stone is less than the weight of the stone. (*e*) If the girl moves her hand downward but slows the stone to rest, the force of the stone on the girl's hand is the same magnitude as the pull of gravity on the stone.

(*a*) False. If the rock is accelerating, the force the girl exerts must be greater than the weight of the stone.

(*b*) False. The reaction force to the pull of gravity is the force the rock exerts on Earth.

(*c*) True. These forces constitute a Newton's third law pair.

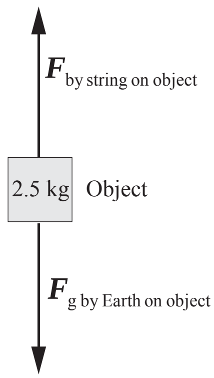
(*d*) False. If she moves the stone downward at a constant speed, the net force acting on the stone must be zero.

(*e*) False. If she is slowing the stone, it is experiencing acceleration and the net force acting on it can not be zero. The force of her hand on the stone, which has the same magnitude as the force of the stone on her hand, is greater than the force of gravity on the stone.

21 •• **[SSM]** A 2.5-kg object hangs at rest from a string attached to the ceiling. (*a*) Draw a free body diagram of the object, indicate the reaction force to each force drawn, and tell what object the reaction force acts on. (*b*) Draw a free body diagram of the string, indicate the reaction force to each force drawn, and tell what object each reaction force acts on. Do not neglect the mass of the string.

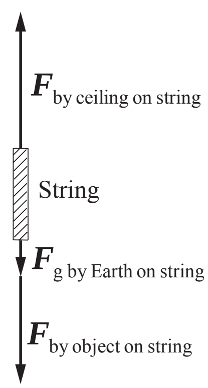
Determine the Concept The force diagrams will need to include forces exerted by the ceiling, on the string, on the object, and forces exerted by Earth.

(*a*)



Force	Third-Law Pair
$\vec{F}_{\text{by string on object}}$	$\vec{F}_{\text{by object on string}}$
$\vec{F}_{\text{g by Earth on object}}$	$\vec{F}_{\text{by object on Earth}}$

(*b*)



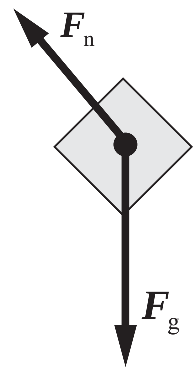
Force	Third-Law Pair
$\vec{F}_{\text{by ceiling on string}}$	$\vec{F}_{\text{by string on ceiling}}$
$\vec{F}_{\text{by Earth on string}}$	$\vec{F}_{\text{by string on Earth}}$
$\vec{F}_{\text{by object on string}}$	$\vec{F}_{\text{by string on object}}$

22 •• (*a*) Which of the free-body diagrams in Figure 4-34. represents a block sliding down a frictionless inclined surface? (*b*) For the correct figure, label the forces and tell which are contact forces and which are action-at-a-distance forces. (*c*) For each force in the correct figure, identify the reaction force, the object it acts on and its direction.

Determine the Concept Identify the objects in the block's environment that are exerting forces on the block and then decide in what directions those forces must be acting if the block is sliding *down* the inclined plane.

(*a*) Free-body diagram $\boxed{(c)}$ is correct.

(*b*) Because the incline is frictionless, the force $\vec{F}_n$ the incline exerts on the block must be normal to the surface and is a contact force. The second object capable of exerting a force on the block is Earth and its force; the gravitational force $\vec{F}_g$ acting on the block acts directly downward and is an action-at-a-distance force. The magnitude of the normal force is less than that of the weight because it supports only a portion of the weight.

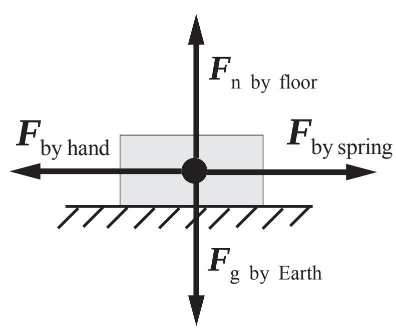


(*c*) The reaction to the normal force is the force the block exerts perpendicularly on the surface of the incline. The reaction to the gravitational force is the upward force the block exerts on Earth.

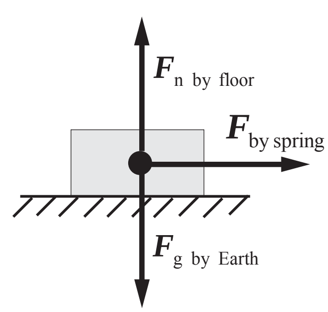
23 •• A wooden box on the floor is pressed against a compressed, horizontal spring that is attached to a wall. The horizontal floor beneath the box is frictionless. Draw the free body diagram of the box in the following cases. (*a*) The box is held at rest against the compressed spring. (*b*) The force holding the box against the spring no longer exists, but the box is still in contact with the spring. (*c*) When the box no longer has contact with the spring.

Determine the Concept In the following free-body diagrams we'll assume that the box is initially pushed to the left to compress the spring.

(*a*) Note that, in the free-body diagram to the right, that $\left|\vec{F}_{\text{n by floor}}\right| = \left|\vec{F}_{\text{g by Earth}}\right|$ and $\left|\vec{F}_{\text{by spring}}\right| = \left|\vec{F}_{\text{by hand}}\right|$.

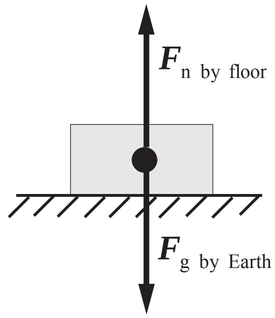


(*b*) Note that while $\vec{F}_{\text{n by floor}} = \left|\vec{F}_{\text{g by Earth}}\right|$, $\vec{F}_{\text{by spring}}$ is now the net force acting on the box. As the spring decompresses, $\vec{F}_{\text{by spring}}$ will become smaller.



(*c*) When the box separates from the spring, the force exerted by the spring on the box goes to zero. Note that it is still true that

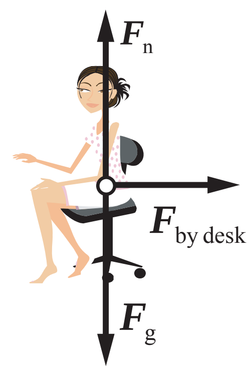
$\vec{F}_{\text{n by floor}} = \left|\vec{F}_{\text{g by Earth}}\right|$.



24 •• Imagine yourself seated on a wheeled desk chair at your desk. Consider any friction forces between the chair and the floor to be negligible. However, the friction forces between the desk and the floor are not negligible. When sitting at rest, you decide you need another cup of coffee. You push horizontally against the desk, and the chair rolls backward away from the desk. (*a*) Draw your free-body diagram of yourself during the push and clearly indicate which force was responsible for your acceleration. (*b*) What is the reaction force to the force that caused your acceleration? (*c*) Draw the free-body diagram of the desk and explain why it did not accelerate. Does this violate Newton's third law? Explain.

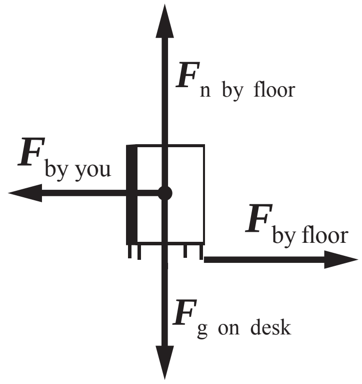
Determine the Concept In the following free-body diagrams we'll assume that the desk is to the left and that your motion is to the right.

(*a*) Newton's third law accounts for this as follows. When you push with your hands against the desk, the desk pushes back on your hands with a force of the same magnitude but opposite direction. This force accelerates you backward.



(*b*) The reaction force to the force that caused your acceleration is the force that you exerted on the desk.

(*c*) When you pushed on the desk, you did not apply sufficient force to overcome the force of friction between the desk and the floor. In terms of forces on the desk, you applied a force, and the floor applied a friction force that, when added as vectors, cancelled. The desk, therefore, did not accelerate and Newton's third law is not violated. The forces in the diagram do not constitute a Newton's third law pair of forces, even though they are equal in magnitude and opposite in direction.



25 ••• The same (net) horizontal force F is applied for a fixed time interval Δt to each of two objects, having masses m_1 and m_2, that sit on a flat, frictionless surface. (Let $m_1 > m_2$.) (*a*) Assuming the two objects are initially at rest, what is the ratio of their accelerations during the time interval in terms of F, m_1 and m_2? (*b*) What is the ratio of their speeds v_1 and v_2 at the end of the time interval? (*c*) How far apart are the two objects (and which is ahead) the end of the time interval?

Picture the Problem We can apply Newton's second law to find the ratios of the accelerations and speeds of the two objects and constant-acceleration equations to express the separation of the objects as a function of the elapsed time.

(*a*) Use Newton's second law to express the accelerations of the two objects:

$$a_1 = \frac{F}{m_1} \text{ and } a_2 = \frac{F}{m_2}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{a_1}{a_2} = \frac{\frac{F}{m_1}}{\frac{F}{m_2}} = \boxed{\frac{m_2}{m_1}}$$

(*b*) Because both objects started from rest, their speeds after time Δt has elapsed are:

$$v_1 = a_1\Delta t \text{ and } v_2 = a_2\Delta t$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{v_1}{v_2} = \frac{a_1\Delta t}{a_2\Delta t} = \frac{a_1}{a_2} = \boxed{\frac{m_2}{m_1}}$$

(*c*) The separation of the two objects at the end of the time interval is given by:

$$\Delta x = \Delta x_2 - \Delta x_1 \qquad (1)$$

Using a constant acceleration equation, express the distances traveled by the two objects when time Δt has elapsed:

$$\Delta x_1 = \tfrac{1}{2}a_1(\Delta t)^2$$
and
$$\Delta x_2 = \tfrac{1}{2}a_2(\Delta t)^2$$

Substitute for Δx_1 and Δx_2 in equation (1) and simplify to obtain:

$$\Delta x = \tfrac{1}{2}a_2(\Delta t)^2 - \tfrac{1}{2}a_1(\Delta t)^2 = \boxed{\tfrac{1}{2}F\left(\frac{1}{m_2} - \frac{1}{m_1}\right)(\Delta t)^2}$$

and, because $m_1 > m_2$, the object whose mass is m_2 is ahead.

Estimation and Approximation

26 •• Most cars have four springs attaching the body to the frame, one at each wheel position. Devise an experimental method of estimating the force constant of one of the springs using your known weight and the weights of several of your friends. Assume the four springs are identical. Use the method to estimate the force constant of your car's springs.

Picture the Problem Suppose you put in 800 lbs (or about 3600 N) of weight and the car sags several inches (or 6.00 cm). Then each spring supports about 900 N and we can use the definition of the force constant k to determine its value.

The force constant is the ratio of the compressing (or stretching) force to the compression (or stretch):

$$k = \frac{F_x}{\Delta x}$$

Substitute numerical values and evaluate k:

$$k = \frac{\tfrac{1}{4}(3600\ \text{N})}{6.00\ \text{cm}} \approx \boxed{150\ \text{N/cm}}$$

27 •• **[SSM]** Estimate the force exerted on the goalie's glove by the puck when he catches a hard slap shot for a save.

Picture the Problem Suppose the goalie's glove slows the puck from 60 m/s to zero as it recoils a distance of 10 cm. Further, assume that the puck's mass is 200 g. Because the force the puck exerts on the goalie's glove and the force the goalie's glove exerts on the puck are action-and-reaction forces, they are equal in magnitude. Hence, if we use a constant-acceleration equation to find the puck's acceleration and Newton's second law to find the force the glove exerts on the puck, we'll have the magnitude of the force exerted on the goalie's glove.

Apply Newton's second law to the puck as it is slowed by the goalie's glove to express the magnitude of the force the glove exerts on the puck:

$$F_{\text{glove on puck}} = \left| m_{\text{puck}} a_{\text{puck}} \right| \qquad (1)$$

Use a constant-acceleration equation to relate the initial and final speeds of the puck to its acceleration and stopping distance:

$$v^2 = v_0^2 + 2a_{\text{puck}} (\Delta x)_{\text{puck}}$$

Solving for a_{puck} yields:

$$a_{\text{puck}} = \frac{v^2 - v_0^2}{2(\Delta x)_{\text{puck}}}$$

Substitute for a_{puck} in equation (1) to obtain:

$$F_{\text{glove on puck}} = \left| \frac{m_{\text{puck}} (v^2 - v_0^2)}{2(\Delta x)_{\text{puck}}} \right|$$

Substitute numerical values and evaluate $F_{\text{glove on puck}}$:

$$F_{\text{glove on puck}} = \left| \frac{(0.200\ \text{kg})\left(0 - (60\ \text{m/s})^2\right)}{2(0.10\ \text{m})} \right| \approx \boxed{3.6\ \text{kN}}$$

Remarks: The force on the puck is about 1800 times its weight.

28 •• A baseball player slides into second base during a steal attempt. Assuming reasonable values for the length of the slide, the speed of the player at the beginning of the slide, and the speed of the player at the end of the slide, estimate the average force of friction acting on the player.

Picture the Problem Let's assume that the player's mass is 100 kg, that he gets going fairly quickly down the base path, and that his speed is 8.0 m/s when he begins his slide. Further, let's assume that he approaches the base at the end of the slide at 3.0 m/s. From these speeds, and the length of the slide, we can use Newton's second law and a constant-acceleration equation to find the force due to friction (which causes the slowing down).

Apply Newton's second law to the sliding runner:

$$\sum F_x = F_{\text{friction}} = ma \qquad (1)$$

Using a constant-acceleration equation, relate the runner's initial and final speeds to his acceleration and the length of his slide:

$$v_{\text{f}}^2 = v_{\text{i}}^2 + 2a\Delta x \Rightarrow a = \frac{v_{\text{f}}^2 - v_{\text{i}}^2}{2\Delta x}$$

Substituting for a in equation (1) yields:

$$F_{\text{friction}} = m\frac{v_{\text{f}}^2 - v_{\text{i}}^2}{2\Delta x}$$

Assuming the player slides 2.0 m, substitute numerical values and evaluate F_{friction} :

$$F_{\text{friction}} = (100\text{ kg})\frac{(3.0\text{ m/s})^2 - (8.0\text{ m/s})^2}{2(2.0\text{ m})}$$

$$\approx \boxed{-1.4\text{ kN}}$$

where the minus sign indicates that the force of friction opposes the runner's motion.

29 •• A race car skidding out of control manages to slow down to 90 km/h before crashing head-on into a brick wall. Fortunately, the driver is wearing a safety harness. Using reasonable values for the mass of the driver and the stopping distance, estimate the average force exerted on the driver by the safety harness, including its direction. Neglect any effects of frictional forces on the driver by the seat.

Picture the Problem Assume a crush distance of 1.0 m at 90 km/h (25 m/s) and a driver's mass of 55 kg. We can use a constant-acceleration equation (the definition of average acceleration) to find the acceleration of the driver and Newton's second law to find the force exerted on the driver by the seat belt.

Apply Newton's second law to the driver as she is brought to rest by her safety harness:

$$F_{\substack{\text{safety harness}\\ \text{on driver}}} = m_{\text{driver}} a_{\text{driver}} \qquad (1)$$

Use a constant-acceleration equation to relate the initial and final speeds of the driver to her acceleration and stopping distance:

$$v^2 = v_0^2 + 2a_{\text{driver}}(\Delta x)_{\text{driver}}$$

Solving for a_{driver} yields:

$$a_{\text{driver}} = \frac{v^2 - v_0^2}{2(\Delta x)_{\text{driver}}}$$

Substitute for a_{driver} in equation (1) to obtain:

$$F_{\substack{\text{safety harness}\\ \text{on driver}}} = m_{\text{driver}}\left(\frac{v^2 - v_0^2}{2(\Delta x)_{\text{driver}}}\right)$$

Substitute numerical values and evaluate $F_{\substack{\text{safety harness}\\ \text{on driver}}}$:

$$F_{\substack{\text{safety harness}\\ \text{on driver}}} = (55\text{ kg})\left(\frac{0 - (25\text{ m/s})^2}{2(1.0\text{ m})}\right)$$

$$\approx \boxed{-17\text{ kN}}$$

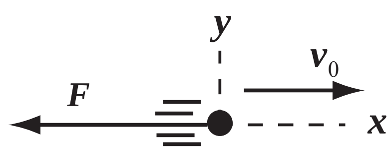
where the minus sign indicates that the force exerted by the safety harness is in the opposite direction from the driver's motion.

Remarks: The average force on the safety harness is about 32 times her weight.

Newton's First and Second Laws: Mass, Inertia, and Force

30 • A particle is traveling in a straight line at constant speed of 25.0 m/s. Suddenly a constant force of 15.0 N acts on it, bringing it to a stop in a distance of 62.5 m. (*a*) What is the direction of the force? (*b*) Determine the time it takes for the particle to come to a stop. (*c*) What is its mass?

Picture the Problem The acceleration of the particle, its stopping time, and its mass can be found using constant-acceleration equations and Newton's second law. A convenient coordinate system is shown in the following diagram.



(*a*) Because the constant force slows the particle, we can conclude that, as shown in the diagram, its direction is opposite the direction of the particle's motion.

(*b*) Use a constant-acceleration equation to relate the initial and final velocities of the particle to its acceleration and stopping time:

$$v_x = v_{0x} + a_x \Delta t$$

or, because $v_x = 0$,

$$0 = v_{0x} + a_x \Delta t \Rightarrow \Delta t = -\frac{v_{0x}}{a_x} \qquad (1)$$

Use a constant-acceleration equation to relate the initial and final velocities of the particle to its acceleration and stopping distance:

$$v_x^2 = v_{0x}^2 + 2a_x \Delta x$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 + 2a_x \Delta x \Rightarrow a_x = \frac{-v_{0x}^2}{2\Delta x}$$

Substituting for a_x in equation (1) and simplifying yields:

$$\Delta t = \frac{2\Delta x}{v_{0x}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(62.5\ \text{m})}{25.0\ \text{m/s}} = \boxed{5.00\ \text{s}}$$

(*c*) Apply Newton's second law to the particle to obtain:

$$\sum F_x = -F_{\text{net}} = ma_x$$

Solving for m yields:

$$m = \frac{-F_{\text{net}}}{a_x} \qquad (2)$$

Substitute for a_x in equation (2) to obtain:

$$m=\frac{2\Delta x F_{\text{net}}}{v_{0x}^2}$$

Substitute numerical values and evaluate m:

$$m=\frac{2(62.5\text{ m})(15.0\text{ N})}{(25.0\text{ m/s})^2}=\boxed{3.00\text{ kg}}$$

31 • An object has an acceleration of 3.0 m/s^2 when a single force of magnitude F_0 acts on it. (*a*) What is the magnitude of its acceleration when the magnitude of this force is doubled? (*b*) A second object has an acceleration of 9.0 m/s^2 under the influence of a single force of magnitude F_0. What is the ratio of the mass of the second object to that of the first object? (*c*) If the two objects are glued together to form a composite object, what acceleration magnitude will a single force of magnitude F_0 acting on the composite object produce?

Picture the Problem The acceleration of an object is related to its mass and the *net* force acting on it by $F_{\text{net}}=F_0=ma$.

(*a*) Use Newton's second law of motion to relate the acceleration of the object to the net force acting on it:

$$a=\frac{F_{\text{net}}}{m}$$

When $F_{\text{net}}=2F_0$:

$$a=\frac{2F_0}{m}=2a_0$$

Substitute numerical values and evaluate a:

$$a=2\left(3.0\text{ m/s}^2\right)=\boxed{6.0\text{ m/s}^2}$$

(*b*) Let the subscripts 1 and 2 distinguish the two objects. The ratio of the two masses is found from Newton's second law:

$$\frac{m_2}{m_1}=\frac{F_0/a_2}{F_0/a_1}=\frac{a_1}{a_2}=\frac{3.0\text{ m/s}^2}{9.0\text{ m/s}^2}=\boxed{\tfrac{1}{3}}$$

(*c*) The acceleration of the composite object is the net force divided by the total mass $m=m_1+m_2$ of the composite object:

$$a=\frac{F_{\text{net}}}{m}=\frac{F_0}{m_1+m_2}=\frac{F_0/m_1}{1+m_2/m_1}$$
$$=\frac{a_1}{1+\frac{1}{3}}=\tfrac{3}{4}a_1$$

Substitute for a_1 and evaluate a:

$$a=\tfrac{3}{4}\left(3.0\text{ m/s}^2\right)=\boxed{2.3\text{ m/s}^2}$$

32 • A tugboat tows a ship with a constant force of magnitude F_1. The increase in the ship's speed during a 10-s interval is 4.0 km/h. When a second tugboat applies an additional constant force of magnitude F_2 in the same direction, the speed increases by 16 km/h during a 10-s interval. How do the

magnitudes of F_1 and F_2 compare? (Neglect the effects of water resistance and air resistance.)

Picture the Problem The acceleration of an object is related to its mass and the *net* force acting on it by $F_{net} = ma$. Let m be the mass of the ship, a_1 be the acceleration of the ship when the net force acting on it is F_1, and a_2 be its acceleration when the net force is $F_1 + F_2$.

Using Newton's second law, express the net force acting on the ship when its acceleration is a_1:

$$F_1 = ma_1$$

Express the net force acting on the ship when its acceleration is a_2:

$$F_1 + F_2 = ma_2$$

Divide the second of these equations by the first and solve for the ratio F_2/F_1:

$$\frac{F_1 + F_2}{F_1} = \frac{ma_2}{ma_1} \Rightarrow \frac{F_2}{F_1} = \frac{a_2}{a_1} - 1$$

Substitute for the accelerations to determine the ratio of the accelerating forces and solve for F_2 to obtain:

$$\frac{F_2}{F_1} = \frac{\dfrac{16\,\text{km/h}}{10\,\text{s}}}{\dfrac{4.0\,\text{km/h}}{10\,\text{s}}} - 1 = 3 \Rightarrow F_2 = \boxed{3F_1}$$

33 • A single constant force of magnitude 12 N acts on a particle of mass m. The particle starts from rest and travels in a straight line a distance of 18 m in 6.0 s. Find m.

Picture the Problem The mass of the particle is related to its acceleration and the *net* force acting on it by Newton's second law of motion. Because the force is constant, we can use constant-acceleration formulas to calculate the acceleration. Choose a coordinate system in which the +x direction is the direction of motion of the particle.

The mass is related to the net force and the acceleration by Newton's second law:

$$m = \frac{\sum \vec{F}}{\vec{a}} = \frac{F_x}{a_x} \qquad (1)$$

Because the force is constant, the acceleration is constant. Use a constant-acceleration equation to relate the displacement of the particle to it's acceleration:

$$\Delta x = v_{0x}t + \tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_x(\Delta t)^2 \Rightarrow a_x = \frac{2\Delta x}{(\Delta t)^2}$$

Substitute for a_x in equation (1) to obtain:

$$m = \frac{F_x(\Delta t)^2}{2\Delta x}$$

Substitute numerical values and evaluate m:

$$m = \frac{(12\,\text{N})(6.0\,\text{s})^2}{2(18\,\text{m})} = \boxed{12\,\text{kg}}$$

34 • A net force of (6.0 N)$\hat{\boldsymbol{i}}$ – (3.0 N)$\hat{\boldsymbol{j}}$ acts on an object of mass 1.5 kg. Find the acceleration $\vec{\boldsymbol{a}}$.

Picture the Problem The acceleration of an object is related to its mass and the *net* force acting on it according to $\vec{\boldsymbol{a}} = \vec{\boldsymbol{F}}_{\text{net}}/m$.

Apply Newton's second law to the object to obtain:

$$\vec{\boldsymbol{a}} = \frac{\vec{\boldsymbol{F}}_{\text{net}}}{m}$$

Substitute numerical values and evaluate $\vec{\boldsymbol{a}}$:

$$\vec{\boldsymbol{a}} = \frac{(6.0\,\text{N})\hat{\boldsymbol{i}} - (3.0\,\text{N})\hat{\boldsymbol{j}}}{1.5\,\text{kg}}$$

$$= \boxed{(4.0\,\text{m/s}^2)\,\hat{\boldsymbol{i}} - (2.0\,\text{m/s}^2)\hat{\boldsymbol{j}}}$$

35 •• **[SSM]** A bullet of mass 1.80 × 10^{-3} kg moving at 500 m/s impacts a tree stump and penetrates 6.00 cm into the wood before coming to rest.
(*a*) Assuming that the acceleration of the bullet is constant, find the force (including direction) exerted by the wood on the bullet. (*b*) If the same force acted on the bullet and it had the same speed but half the mass, how far would it penetrate into the wood?

Picture the Problem Choose a coordinate system in which the +*x* direction is in the direction of the motion of the bullet and use Newton's second law and a constant- acceleration equation to express the relationship between F_{stopping} and the mass of the bullet and its displacement as it is brought to rest in the block of wood.

(*a*) Apply Newton's second law to the bullet to obtain:

$$\sum F_x = F_{\text{stopping}} = ma_x \qquad (1)$$

Use a constant-acceleration equation to relate the bullet's initial and final speeds, acceleration, and stopping distance:

$$v_{\text{f}x}^2 = v_{\text{i}x}^2 + 2a_x\Delta x$$

or, because $v_{\text{f}x} = 0$,

$$0 = v_{\text{i}x}^2 + 2a_x\Delta x \Rightarrow a_x = \frac{-v_{\text{i}x}^2}{2\Delta x}$$

Substitute for a_x in equation (1) to obtain:

$$F_{\text{stopping}} = -m\frac{v_{\text{i}x}^2}{2\Delta x} \qquad (2)$$

Substitute numerical values and evaluate F_{stopping}:

$$F_{\text{stopping}} = -\left(1.80\times10^{-3}\text{ kg}\right)\frac{(500\text{ m/s})^2}{2(6.00\text{ cm})} = \boxed{-3.8\text{ kN}}$$

where the minus sign indicates that F_{stopping} opposes the motion of the bullet.

(*b*) Solving equation (2) for Δx yields:

$$\Delta x = -m\frac{v_{ix}^2}{2F_{\text{stopping}}} \qquad (3)$$

For $m = m'$ and $\Delta x = \Delta x'$:

$$\Delta x' = -m'\frac{v_{ix}^2}{2F_{\text{stopping}}}$$

Evaluate this expression for $m' = \frac{1}{2}m$ to obtain:

$$\Delta x' = -m\frac{v_{ix}^2}{4F_{\text{stopping}}} \qquad (4)$$

Dividing equation (4) by equation (3) yields:

$$\frac{\Delta x'}{\Delta x} = \frac{-m\dfrac{v_{ix}^2}{4F_{\text{stopping}}}}{-m\dfrac{v_{ix}^2}{2F_{\text{stopping}}}} = \frac{1}{2}$$

or

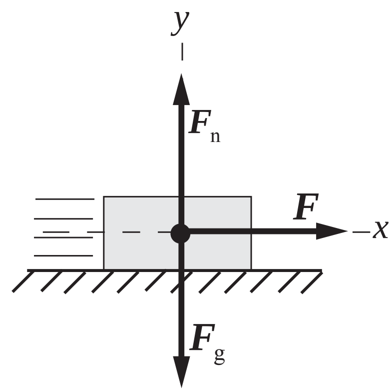
$$\Delta x' = \tfrac{1}{2}\Delta x$$

Substitute numerical values and evaluate $\Delta x'$:

$$\Delta x' = \tfrac{1}{2}(6.00\text{ cm}) = \boxed{3.00\text{ cm}}$$

36 •• A cart on a horizontal, linear track has a fan attached to it. The cart is positioned at one end of the track, and the fan is turned on. Starting from rest, the cart takes 4.55 s to travel a distance of 1.50 m. The mass of the cart plus fan is 355 g. Assume that the cart travels with constant acceleration. (*a*) What is the net force exerted on the cart-fan combination? (*b*) Mass is added to the cart until the total mass of the cart-fan combination is 722 g, and the experiment is repeated. How long does it take for the cart, starting from rest, to travel 1.50 m now? Ignore the effects due to friction.

Picture the Problem Choose the coordinate system shown in the diagram to the right. The force $\vec{F}$ acting on the cart-fan combination is the consequence of the fan blowing air to the left. We can use Newton's second law and a constant-acceleration equation to express the relationship between $\vec{F}$ and the mass of the cart-fan combination and the distance it travels in a given interval of time.



(*a*) Apply Newton's second law to the cart-fan combination to obtain:

$$\sum F_x = F = ma_x \qquad (1)$$

Using a constant-acceleration equation, relate the distance the cart-fan combination travels to its initial speed, acceleration, and the elapsed time:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_x(\Delta t)^2 \Rightarrow a_x = \frac{2\Delta x}{(\Delta t)^2}$$

Substitute for a_x in equation (1) to obtain:

$$F = m\frac{2\Delta x}{(\Delta t)^2} \qquad (2)$$

Substitute numerical values and evaluate F:

$$F = (0.355\text{ kg})\frac{2(1.50\text{ m})}{(4.55\text{ s})^2} = 0.05144\text{ N} = \boxed{0.0514\text{ N}}$$

(*b*) Solve equation (2) for Δt to obtain:

$$\Delta t = \sqrt{\frac{2m\Delta x}{F}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{2(0.722\text{ kg})(1.50\text{ m})}{0.05144\text{ N}}} = \boxed{6.49\text{ s}}$$

37 •• A horizontal force of magnitude F_0 causes an acceleration of 3.0 m/s^2 when it acts on an object of mass *m* sliding on a frictionless surface. Find the magnitude of the acceleration of the same object in the circumstances shown in Figure 4-35*a* and 4-35*b*.

Picture the Problem The acceleration of an object is related to its mass and the *net* force acting on it through Newton's second law. Choose a coordinate system in which the direction of $2F_0$ in (*b*) is the positive direction and the direction of the left-most F_0 in (*a*) is the positive direction. Find the resultant force in each case and then find the resultant acceleration.

(*a*) Apply Newton's second law to the object to obtain:

$$a = \frac{F_{net}}{m} \qquad (1)$$

The magnitude of the net force is given by:

$$F_{net} = \sqrt{F_x^2 + F_y^2} = \sqrt{2}F_0$$

Substitute for F_{net} in equation (1) and simplify to obtain:

$$a = \frac{\sqrt{2}F_0}{m} = \sqrt{2}a_0$$

Substitute the numerical value of a_0 and evaluate a:

$$a = \sqrt{2}\left(3.0\,\text{m/s}^2\right) = \boxed{4.2\,\text{m/s}^2}$$

(*b*)The magnitude of net force is given by:

$$F_{net} = \sqrt{F_x^2 + F_y^2} = \sqrt{\left(-F_0 \sin 45°\right)^2 + \left(2F_0 + F_0 \cos 45°\right)^2} = 2.80F_0$$

Substitute for F_{net} in equation (1) and simplify to obtain:

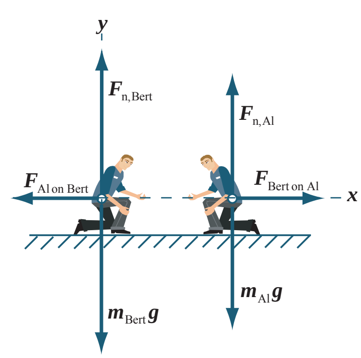
$$a = 2.80\frac{F_0}{m} = 2.80\frac{ma_0}{m} = 2.80a_0$$

Substitute the numerical value of a_0 and evaluate a:

$$a = 2.80\left(3.0\,\text{m/s}^2\right) = \boxed{8.4\,\text{m/s}^2}$$

38 •• Al and Bert stand in the middle of a large frozen lake (frictionless surface). Al pushes on Bert with a force of 20 N for 1.5 s. Bert's mass is 100 kg. Assume that both are at rest before Al pushes Bert. (*a*) What is the speed that Bert reaches as he is pushed away from Al? (*b*) What speed does Al reach if his mass is 80 kg?

Picture the Problem The speed of either Al or Bert can be obtained from their accelerations; in turn, they can be obtained from Newtons second law applied to each person. The free-body diagrams to the right show the forces acting on Al and Bert. The forces that Al and Bert exert on each other are action-and-reaction forces.



(*a*) Apply $\sum F_x = ma_x$ to Bert:

$$-F_{\text{Al on Bert}} = m_{\text{Bert}}a_{\text{Bert}} \Rightarrow a_{\text{Bert}} = \frac{-F_{\text{Al on Bert}}}{m_{\text{Bert}}}$$

Substitute numerical values and evaluate a_{Bert}:

$$a_{\text{Bert}} = \frac{-20\,\text{N}}{100\,\text{kg}} = -0.200\,\text{m/s}^2$$

Using a constant-acceleration equation, relate Bert's speed to his initial speed, speed after 1.5 s, and acceleration:

$$v_x = v_{0x} + a_{\text{Bert,x}}\Delta t$$

Substitute numerical values and evaluate Bert's speed at the end of 1.5 s:

$$v_x = 0 + \left(-0.200\ \text{m/s}^2\right)\left(1.5\ \text{s}\right) = \boxed{-0.30\ \text{m/s}}$$

(*b*) From Newton's 3rd law, an equal but oppositely directed force acts on Al while he pushes Bert. Because the ice is frictionless, Al speeds off in the opposite direction. Apply $\sum F_x = ma_x$ to Al:

$$\sum F_{x,\text{Al}} = F_{\text{Bert on Al},x} = m_{\text{Al}}a_{\text{Al},x}$$

Solving for Al's acceleration yields:

$$a_{\text{Al,x}} = \frac{F_{\text{Bert on Al},x}}{m_{\text{Al}}}$$

Substitute numerical values and evaluate $a_{\text{Al},x}$:

$$a_{\text{Al},x} = \frac{20\ \text{N}}{80\ \text{kg}} = 0.250\ \text{m/s}^2$$

Using a constant-acceleration equation, relate Al's speed to his initial speed, speed after 1.5 s, and acceleration:

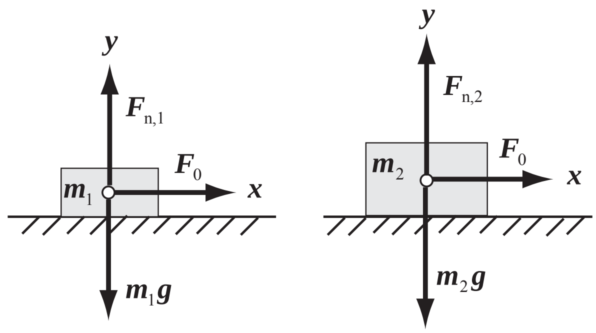
$$v_x = v_{0x} + a_{\text{Al},x}\Delta t$$

Substitute numerical values and evaluate Al's speed at the end of 1.5 s:

$$v_x(1.5\ \text{s}) = 0 + \left(0.250\ \text{m/s}^2\right)\left(1.5\ \text{s}\right) = \boxed{0.38\ \text{m/s}}$$

39 •• If you push a block whose mass is m_1 across a frictionless floor with a horizontal force of magnitude F_0, the block has an acceleration of 12 m/s^2. If you push on a different block whose mass is m_2 with a horizontal force of magnitude F_0, its acceleration is 3.0 m/s^2. (*a*) What acceleration will a horizontal force of magnitude F_0 give to a single block with mass $m_2 - m_1$? (*b*) What acceleration will a horizontal force of magnitude F_0 give to a single block with mass $m_2 + m_1$?

Picture the Problem The free-body diagrams show the forces acting on the two blocks. We can apply Newton's second law to the forces acting on the blocks and eliminate F to obtain a relationship between the masses. Additional applications of Newton's second law to the sum and difference of the masses will lead us to values for the accelerations of these combinations of mass.



(*a*) Apply $\sum F_x = ma_x$ to the two blocks:

$\sum F_{1,x} = F_0 = m_1 a_{1,x}$

and

$\sum F_{2,x} = F_0 = m_2 a_{2,x}$

Eliminate F_0 between the two equations and solve for m_2:

$$m_2 = \frac{a_{1,x}}{a_{2,x}} m_1$$

Substitute numerical values to obtain:

$$m_2 = \frac{12\,\text{m/s}^2}{3.0\,\text{m/s}^2} m_1 = 4.0\, m_1$$

Express the acceleration of an object whose mass is $m_2 - m_1$ when the net force acting on it is F_0:

$$a_x = \frac{F_0}{m_2 - m_1} = \frac{F_0}{4m_1 - m_1} = \frac{F_0}{3m_1} = \tfrac{1}{3} a_{1,x}$$

Substitute numerical values and evaluate a_x:

$$a_x = \tfrac{1}{3}\left(12\,\text{m/s}^2\right) = \boxed{4.0\,\text{m/s}^2}$$

(*b*) Express the acceleration of an object whose mass is $m_2 + m_1$ when the net force acting on it is F_0:

$$a_x = \frac{F_0}{m_2 + m_1} = \frac{F_0}{4m_1 + m_1} = \frac{F_0}{5m_1} = \tfrac{1}{5} a_{1,x}$$

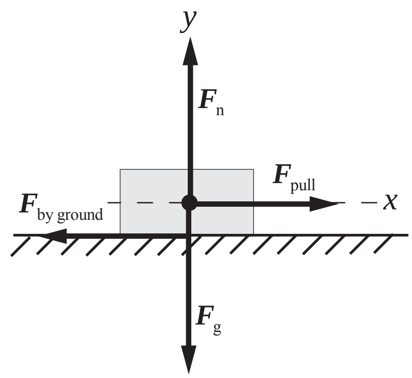
Substitute numerical values and evaluate a_x:

$$a_x = \tfrac{1}{5}\left(12\,\text{m/s}^2\right) = \boxed{2.4\,\text{m/s}^2}$$

40 •• To drag a 75.0-kg log along the ground at constant velocity, your tractor has to pull it with a horizontal force of 250 N. (*a*) Draw the free body diagram of the log. (*b*) Use Newton's laws to determine the force of friction on the log. (*c*) What is the normal force of the ground on the log? (*d*) What horizontal force must you exert if you want to give the log an acceleration of 2.00 m/s^2 assuming the force of friction does not change. Redraw the log's free body diagram for this situation.

Picture the Problem Because the velocity is constant, the net force acting on the log must be zero. Choose a coordinate system in which the positive *x* direction is the direction of motion of the log and apply Newton's second law to the log.

(*a*) The free-body diagram shows the forces acting on the log when it is being dragged in the +*x* direction at constant velocity.



(*b*) Apply $\sum F_x = ma_x$ to the log when it is moving at constant speed:

$$\sum F_x = F_{\text{pull}} - F_{\text{by ground}} = ma_x = 0$$
or
$$F_{\text{by ground}} = F_{\text{pull}}$$

Substitute for F_{pull} and evaluate the force of friction $F_{\text{by ground}}$:

$$F_{\text{by ground}} = F_{\text{pull}} = \boxed{250\text{ N}}$$

(*c*) Apply $\sum F_y = ma_y$ to the log to obtain:

$$\sum F_y = F_{\text{n}} - F_{\text{g}} = ma_y = 0$$
or
$$F_{\text{n}} = F_{\text{g}}$$

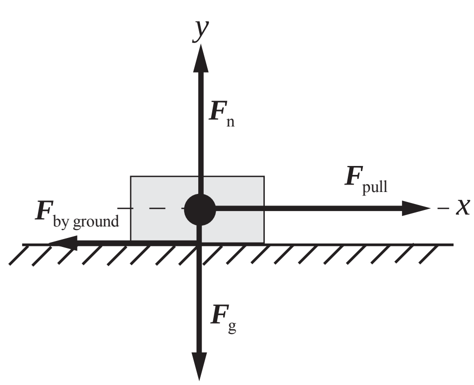
Because the gravitational force is given by $F_{\text{g}} = mg$:

$$F_{\text{n}} = mg$$

Substitute numerical values and evaluate F_{n}:

$$F_{\text{n}} = (75.0\text{ kg})(9.81\text{ m/s}^2) = \boxed{736\text{ N}}$$

(*d*) The free-body diagram shows the forces acting on the log when it is accelerating in the positive *x* direction.



Apply $\sum F_x = ma_x$ to the log when it is accelerating to the right:

$$\sum F_x = F_{\text{pull}} - F_{\text{by ground}} = ma_x$$

Solving for F_{pull} yields:

$$F_{\text{pull}} = ma_x + F_{\text{by ground}}$$

Substitute numerical values and evaluate F_{pull}:

$$F_{\text{pull}} = (75.0\ \text{kg})(2.00\ \text{m/s}^2) + 250\ \text{N} = \boxed{400\ \text{N}}$$

41 •• A 4.0-kg object is subjected to two constant forces, $\vec{F}_1 = (2.0\ \text{N})\,\hat{\boldsymbol{i}} + (-3.0\ \text{N})\,\hat{\boldsymbol{j}}$ and $\vec{F}_2 = (4.0\ \text{N})\,\hat{\boldsymbol{i}} - (11\ \text{N})\,\hat{\boldsymbol{j}}$. The object is at rest at the origin at time $t = 0$. (*a*) What is the object's acceleration? (*b*) What is its velocity at time $t = 3.0$ s? (c) Where is the object at time $t = 3.0$ s?

Picture the Problem The acceleration can be found from Newton's second law. Because both forces are constant, the net force and the acceleration are constant; hence, we can use the constant-acceleration equations to answer questions concerning the motion of the object at various times.

(*a*) Apply Newton's second law to the 4.0-kg object to obtain:

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

Substitute numerical values and simplify to evaluate $\vec{a}$:

$$\vec{a} = \frac{(2.0\ \text{N})\hat{\boldsymbol{i}} + (-3.0\ \text{N})\hat{\boldsymbol{j}} + (4.0\ \text{N})\hat{\boldsymbol{i}} + (-11\ \text{N})\hat{\boldsymbol{j}}}{4.0\ \text{kg}} = \frac{(6.0\ \text{N})\hat{\boldsymbol{i}} + (-14\ \text{N})\hat{\boldsymbol{j}}}{4.0\ \text{kg}}$$
$$= \boxed{(1.5\ \text{m/s}^2)\hat{\boldsymbol{i}} + (-3.5\ \text{m/s}^2)\hat{\boldsymbol{j}}}$$

(*b*) Using a constant-acceleration equation, express the velocity of the object as a function of time:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

Substitute numerical values and evaluate $\vec{v}(3.0\ \text{s})$:

$$\vec{v}(3.0\ \text{s}) = \left[(1.5\ \text{m/s}^2)\hat{\boldsymbol{i}} + (-3.5\ \text{m/s}^2)\hat{\boldsymbol{j}}\right](3.0\ \text{s}) = (4.5\ \text{m/s})\hat{\boldsymbol{i}} + (-10.5\ \text{m/s})\hat{\boldsymbol{j}}$$
$$= \boxed{(4.5\ \text{m/s})\hat{\boldsymbol{i}} + (-11\ \text{m/s})\hat{\boldsymbol{j}}}$$

(*c*) Express the position of the object in terms of its average velocity:

$$\vec{r} = \vec{v}_{\text{av}}t = \frac{\vec{v}_0 + \vec{v}}{2}t = \tfrac{1}{2}\vec{v}t$$

Substitute for $\vec{v}$ and evaluate this expression at $t = 3.0$ s:

$$\vec{r}(3.0\ \text{s}) = \tfrac{1}{2}\left[(4.5\ \text{m/s})\hat{\boldsymbol{i}} + (-10.5\ \text{m/s})\hat{\boldsymbol{j}}\right](3.0\ \text{s}) = (6.75\ \text{m})\hat{\boldsymbol{i}} + (-15.8\ \text{m})\hat{\boldsymbol{j}}$$
$$= \boxed{(6.8\ \text{m})\hat{\boldsymbol{i}} + (-16\ \text{m})\hat{\boldsymbol{j}}}$$

Mass and Weight

42 • On the moon, the acceleration due to gravity is only about 1/6 of that on Earth. An astronaut, whose weight on Earth is 600 N, travels to the lunar surface. His mass, as measured on the moon, will be (*a*) 600 kg, (*b*) 100 kg, (*c*) 61.2 kg, (*d*) 9.81 kg, (*e*) 360 kg.

Picture the Problem The mass of the astronaut is independent of gravitational fields and will be the same on the moon or, for that matter, out in deep space.

Express the mass of the astronaut in terms of his weight on Earth and the gravitational field at the surface of Earth:

$$m = \frac{w_{\text{earth}}}{g_{\text{earth}}} = \frac{600\,\text{N}}{9.81\,\text{N/kg}} = 61.2\,\text{kg}$$

and $\boxed{(c)}$ is correct.

43 • Find the weight of a 54-kg student in (*a*) newtons and (*b*) pounds.

Picture the Problem The weight of an object is related to its mass and the gravitational field through $F_g = mg$.

(*a*) The weight of the student is:

$$w = mg = (54\,\text{kg})(9.81\,\text{N/kg}) = 530\,\text{N} = \boxed{5.3\times10^{2}\ \text{N}}$$

(*b*) Convert newtons to pounds:

$$w = \frac{530\,\text{N}}{4.45\,\text{N/lb}} = 119\,\text{lb} \approx \boxed{1.2\times10^{2}\ \text{lb}}$$

44 • Find the mass of a 165-lb engineer in kilograms.

Picture the Problem The mass of an object is related to its weight and the gravitational field.

Convert the weight of the man into newtons:

$$165\,\text{lb} = (165\,\text{lb})(4.45\,\text{N/lb}) = 734\,\text{N}$$

Calculate the mass of the man from his weight and the gravitational field:

$$m = \frac{w}{g} = \frac{734\,\text{N}}{9.81\,\text{N/kg}} = \boxed{74.8\,\text{kg}}$$

45 •• **[SSM]** To train astronauts to work on the moon, where the acceleration due to gravity is only about 1/6 of that on Earth, NASA submerges them in a tank of water. If an astronaut, who is carrying a backpack, air conditioning unit, oxygen supply, and other equipment, has a total mass of 250 kg, determine the following quantities. (*a*) her weight, including her backpack, etc., on Earth, (*b*) her weight on the moon, (*c*) the required upward

buoyancy force of the water during her training for the moon's environment on Earth.

Picture the Problem We can use the relationship between weight (gravitational force) and mass, together with the given information about the acceleration due to gravity on the moon, to find the astronaut's weight on Earth and on the moon.

(*a*) Her weight on Earth is the product of her mass and the gravitational field at the surface of Earth:

$$w_{\text{Earth}} = mg$$

Substitute numerical values and evaluate *w*:

$$w_{\text{Earth}} = (250\text{ kg})(9.81\text{ m/s}^2) = 2.453\text{ kN}$$
$$= \boxed{2.45\text{ kN}}$$

(*b*) Her weight on the moon is the product of her mass and the gravitational field at the surface of the moon:

$$w_{\text{moon}} = mg_{\text{moon}} = \tfrac{1}{6}mg = \tfrac{1}{6}w_{\text{earth}}$$

Substitute for her weight on Earth and evaluate her weight on the moon:

$$w_{\text{moon}} = \tfrac{1}{6}(2.453\text{ kN}) = \boxed{409\text{ N}}$$

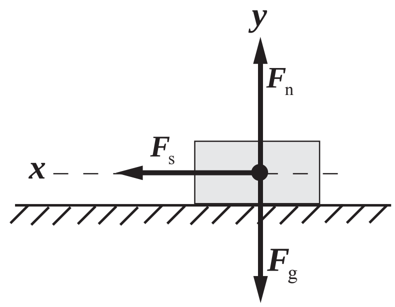
(*c*) The required upward buoyancy force of the water equals the difference between her weight on Earth and on the moon:

$$w_{\text{buoyancy}} = w_{\text{Earth}} - w_{\text{moon}}$$
$$= 2.45\text{ kN} - 0.41\text{ kN}$$
$$= \boxed{2.04\text{kN}}$$

46 •• It is the year 2075 and space travel is common. A physics professor brings his favorite teaching demonstration with him to the moon. The apparatus consists of a very smooth horizontal (frictionless) table and an object to slide on it. On Earth, when the professor attaches a spring (spring constant 50 N/m) to the object and pulls horizontally so the spring stretches 2.0 cm, the object accelerates at1.5 m/s^2. (*a*) Draw the free-body diagram of the object and use it and Newton's laws to determine the object's mass. (*b*) What would the object's acceleration be under identical conditions on the moon?

Picture the Problem The forces acting on the object are the normal force exerted by the table, the gravitational force exerted by Earth, and the force exerted by the stretched spring.

(*a*) The free-body diagram shown to the right assumes that the spring has been stretched to the right. Hence the force that the spring exerts on the object is to the left. Note that the +*x* direction has been chosen to be in the same direction as the force exerted by the spring.



Apply $\sum F_x = ma_x$ to the object to obtain:

$$\sum F_x = F_s = ma_x \Rightarrow m = \frac{F_s}{a_x} \quad (1)$$

The force exerted by the spring on the object is given by:

$$F_s = k\Delta x$$

where Δx is the amount by which the spring has been stretched or compressed and k is the force constant.

Substituting for F_s in equation (1) yields:

$$m = \frac{k\Delta x}{a_x}$$

Substitute numerical values and evaluate m:

$$m = \frac{(50\ \text{N/m})(2.0\ \text{cm})}{1.5\ \text{m/s}^2} = \boxed{0.67\ \text{kg}}$$

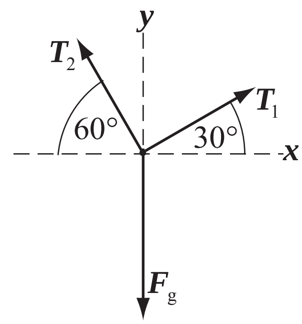
(*b*) Because the object's mass is the same on the moon as on Earth and the force exerted by the spring is the same, its acceleration on the moon would be the same as on Earth.

Free-Body Diagrams: Static Equilibrium

47 • A 35.0-kg traffic light is supported by two wires as in Figure 4-36. (*a*) Draw the light's free-body diagram and use it to answer the following question qualitatively: Is the tension in wire 2 greater than or less than the tension in wire 1? (*b*) Prove your answer by applying Newton's laws and solving for the two tensions.

Picture the Problem Because the traffic light is not accelerating, the *net* force acting on it must be zero; i.e., $\vec{T}_1 + \vec{T}_2 + \vec{F}_g = 0$.

(*a*) Construct a free-body diagram showing the forces acting on the support point:



Apply $\sum F_x = ma_x$ to the support point:

$$T_1 \cos 30° - T_2 \cos 60° = ma_x = 0$$

Solve for T_2 in terms of T_1:

$$T_2 = \frac{\cos 30°}{\cos 60°} T_1 = 1.732 T_1 \qquad (1)$$

Thus T_2 is greater than T_1.

(*b*) Apply $\sum F_y = ma_x$ to the support point:

$$T_1 \sin 30° + T_2 \sin 60° - F_g = ma_y = 0$$

Substitute for T_2 to obtain:

$$\tfrac{1}{2} T_1 + \tfrac{\sqrt{3}}{2}(1.732 T_1) - mg = 0$$

Solving for T_1 gives:

$$T_1 = \frac{mg}{2.000}$$

Substitute numerical values and evaluate T_1:

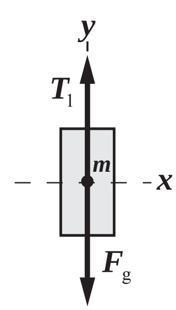
$$T_1 = \frac{(35.0\,\text{kg})(9.81\,\text{m/s}^2)}{2.000} = \boxed{172\,\text{N}}$$

From equation (1) we have:

$$T_2 = 1.732 T_1 = (1.732)(172\,\text{N}) = \boxed{298\,\text{N}}$$

48 • A 42.6-kg lamp is hanging from wires as shown in Figure 4-37. The ring has negligible mass. The tension T_1 in the vertical wire is (*a*) 209 N, (*b*) 418 N, (*c*) 570 N, (*d*) 360 N, (*e*) 730 N.

Picture the Problem From the figure, it is clear that T_1 supports the full weight of the lamp. Draw a free-body diagram showing the forces acting on the lamp and apply $\sum F_y = 0$.



Apply $\sum F_y = 0$ to the lamp to obtain:

$$\sum F_y = T_1 - F_g = 0$$

Solve for T_1 and substitute for F_g to obtain:

$$T_1 = F_g = mg$$

Substitute numerical values and evaluate T_1:

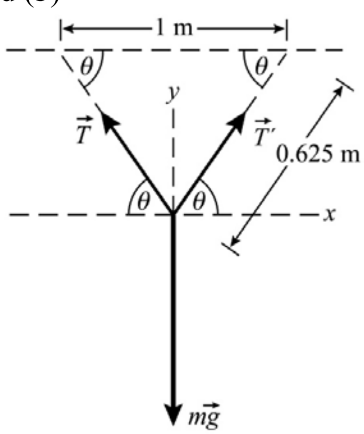
$$T_1 = (42.6\,\text{kg})(9.81\,\text{m/s}^2) = 418\,\text{N}$$

and $\boxed{(b)}$ is correct.

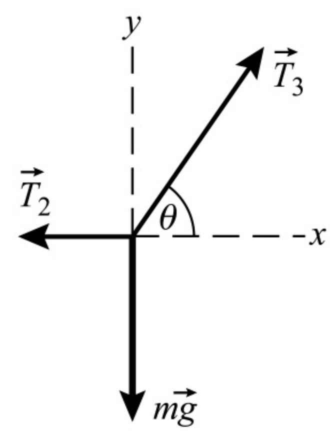
49 •• [SSM] In Figure 4-38*a*, a 0.500-kg block is suspended at the midpoint of a 1.25-m-long string. The ends of the string are attached to the ceiling at points separated by 1.00 m. (*a*) What angle does the string make with the ceiling? (*b*) What is the tension in the string? (*c*) The 0.500-kg block is removed and two 0.250-kg blocks are attached to the string such that the lengths of the three string segments are equal (Figure 4-38*b*). What is the tension in each segment of the string?

Picture the Problem The free-body diagrams for Parts (*a*), (*b*), and (*c*) are shown below. In both cases, the block is in equilibrium under the influence of the forces and we can use Newton's second law of motion and geometry and trigonometry to obtain relationships between θ and the tensions.

(*a*) and (*b*)



(*c*)



(*a*) Referring to the free-body diagram for Part (*a*), use trigonometry to determine θ:

$$\theta = \cos^{-1}\left(\frac{0.50\,\text{m}}{0.625\,\text{m}}\right) = 36.9° = \boxed{37°}$$

(*b*) Noting that $T = T'$, apply $\sum F_y = ma_y$ to the 0.500-kg block and solve for the tension T:

$$2T\sin\theta - mg = 0 \text{ because } a = 0$$

and

$$T = \frac{mg}{2\sin\theta}$$

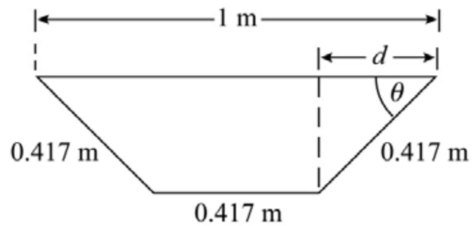
Substitute numerical values and evaluate T:

$$T = \frac{(0.500\,\text{kg})(9.81\,\text{m/s}^2)}{2\sin 36.9°} = \boxed{4.1\,\text{N}}$$

(*c*) The length of each segment is:

$$\frac{1.25\,\text{m}}{3} = 0.41667\,\text{m}$$

Find the distance *d*:



$$d = \frac{1.00\,\text{m} - 0.41667\text{m}}{2} = 0.29167\,\text{m}$$

Express θ in terms of *d* and solve for its value:

$$\theta = \cos^{-1}\left(\frac{d}{0.417\,\text{m}}\right) = \cos^{-1}\left(\frac{0.2917\,\text{m}}{0.4167\,\text{m}}\right) = 45.57°$$

Apply $\sum F_y = ma_y$ to the 0.250-kg block:

$$T_3 \sin\theta - mg = 0 \Rightarrow T_3 = \frac{mg}{\sin\theta}$$

Substitute numerical values and evaluate T_3:

$$T_3 = \frac{(0.250\,\text{kg})(9.81\,\text{m/s}^2)}{\sin 45.57°} = 3.434\text{ N} = \boxed{3.4\,\text{N}}$$

Apply $\sum F_x = ma_x$ to the 0.250-kg block and solve for the tension T_2:

$T_3 \cos\theta - T_2 = 0$ since $a = 0$.
and
$T_2 = T_3 \cos\theta$

Substitute numerical values and evaluate T_2:

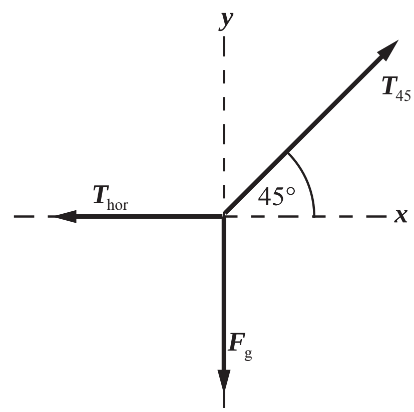
$$T_2 = (3.434\,\text{N})\cos 45.57° = \boxed{2.4\,\text{N}}$$

By symmetry:

$$T_1 = T_3 = \boxed{3.4\,\text{N}}$$

50 •• A ball weighing 100 N is shown suspended from a system of cords (Figure 4-39). What are the tensions in the horizontal and angled cords?

Picture the Problem The suspended body is in equilibrium under the influence of the forces $\vec{T}_{hor}$, $\vec{T}_{45}$, and $\vec{F}_g$. That is, $\vec{T}_{hor} + \vec{T}_{45} + \vec{F}_g = 0$. Draw the free-body diagram of the forces acting on the knot just above the 100-N body. Choose a coordinate system with the positive x direction to the right and the positive y direction upward. Apply the conditions for translational equilibrium to determine the tension in the horizontal cord.



Apply $\sum F_y = ma_y$ to the knot: $\sum F_y = T_{45}\sin 45° - F_g = ma_y = 0$

Solving for T_{45} yields: $T_{45} = \dfrac{F_g}{\sin 45°} = \dfrac{100\ \text{N}}{\sin 45°} = \boxed{141\ \text{N}}$

Apply $\sum F_x = ma_x$ to the knot: $\sum F_x = T_{45}\cos 45° - T_{hor} = ma_x = 0$

Solving for T_{hor} gives: $T_{hor} = T_{45}\cos 45° = (141\ \text{N})\cos 45° = \boxed{100\ \text{N}}$

51 •• **[SSM]** A 10-kg object on a frictionless table is subjected to two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, with magnitudes $F_1 = 20$ N and $F_2 = 30$ N, as shown in Figure 4-40. Find the third force $\vec{F}_3$ that must be applied so that the object is in static equilibrium.

Picture the Problem The acceleration of *any* object is directly proportional to the *net* force acting on it. Choose a coordinate system in which the positive x direction is the same as that of $\vec{F}_1$ and the positive y direction is to the right. Add the two forces to determine the net force and then use Newton's second law to find the acceleration of the object. If $\vec{F}_3$ brings the system into equilibrium, it must be true that $\vec{F}_3 + \vec{F}_1 + \vec{F}_2 = 0$.

Express $\vec{F}_3$ in terms of $\vec{F}_1$ and $\vec{F}_2$: $\vec{F}_3 = -\vec{F}_1 - \vec{F}_2$ (1)

Express $\vec{F}_1$ and $\vec{F}_2$ in unit vector notation:

$$\vec{F}_1 = (20\,\text{N})\hat{i}$$

and

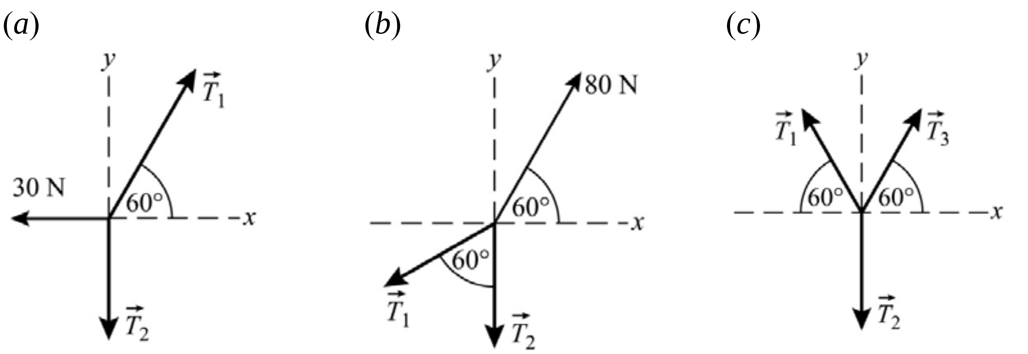
$$\vec{F}_2 = \{(-30\,\text{N})\sin 30°\}\hat{i} + \{(30\,\text{N})\cos 30°\}\hat{j} = (-15\,\text{N})\hat{i} + (26\,\text{N})\hat{j}$$

Substitute for $\vec{F}_1$ and $\vec{F}_2$ in equation (1) and simplify to obtain:

$$\vec{F}_3 = -(20\,\text{N})\hat{i} - [(-15\,\text{N})\hat{i} + (26\,\text{N})\hat{j}] = \boxed{(-5.0\,\text{N})\hat{i} + (-26\,\text{N})\hat{j}}$$

52 •• For the systems to be in equilibrium in Figure 4-41*a*, Figure 4-41*b*, and Figure 4-41*c* find the unknown tensions and masses.

Picture the Problem The free-body diagrams for the systems in equilibrium are shown below. Apply the conditions for translational equilibrium to find the unknown tensions.



(*a*) Apply $\sum F_x = 0$ and $\sum F_y = 0$ to the knot above the suspended mass to obtain:

$$\sum F_x = T_1 \cos 60° - 30\text{ N} = 0 \quad (1)$$

and

$$\sum F_y = T_1 \sin 60° - T_2 = 0 \quad (2)$$

Solving equation (1) for T_1 yields:

$$T_1 = \frac{30\text{ N}}{\cos 60°} = \boxed{60\text{ N}}$$

Solving equation (2) for T_2 yields:

$$T_2 = T_1 \sin 60° = (60\text{ N})\sin 60° = 51.96\text{ N} = \boxed{52\text{ N}}$$

Because T_2 is the weight of the object whose mass is m:

$$T_2 = F_g = mg \Rightarrow m = \frac{T_2}{g}$$

Substitute numerical values and evaluate m:

$$m = \frac{51.96\ \text{N}}{9.81\ \text{m/s}^2} = \boxed{5.3\ \text{kg}}$$

(*b*) Apply $\sum F_x = 0$ and $\sum F_y = 0$ to the knot above the suspended mass to obtain:

$$\sum F_x = (80\ \text{N})\cos 60° - T_1 \sin 60° = 0$$
and
$$\sum F_y = (80\ \text{N})\sin 60° - T_2 - T_1 \cos 60° = 0$$

Solving the first of these equations for T_1 yields:

$$T_1 = \frac{(80\ \text{N})\cos 60°}{\sin 60°} = 46.19\ \text{N}$$
$$= \boxed{46\ \text{N}}$$

Solving the second of these equations for T_2 yields:

$$T_2 = (80\ \text{N})\sin 60° - T_1 \cos 60°$$

Substitute numerical values and evaluate T_2:

$$T_2 = (80\ \text{N})\sin 60° - (46.19\ \text{N})\cos 60°$$
$$= 46.19\ \text{N} = \boxed{46\ \text{N}}$$

Because T_2 is the weight of the object whose mass is m:

$$T_2 = F_\text{g} = mg \Rightarrow m = \frac{T_2}{g}$$

Substitute numerical values and evaluate m:

$$m = \frac{46.19\ \text{N}}{9.81\ \text{m/s}^2} = \boxed{4.7\ \text{kg}}$$

(*c*) Apply $\sum F_x = 0$ and $\sum F_y = 0$ to the knot above the suspended mass to obtain:

$$\sum F_x = -T_1 \cos 60° + T_3 \cos 60° = 0$$
and
$$\sum F_y = T_1 \sin 60° + T_3 \sin 60° - F_\text{g} = 0$$

Solving the first of these equations for T_1 yields:

$$T_1 = T_3$$

Solving the second of these equations for T_1 yields:

$$T_1 = T_3 = \frac{F_\text{g}}{2\sin 60°} = \frac{mg}{2\sin 60°}$$

Substitute numerical values and evaluate T_1 and T_3:

$$T_1 = T_3 = \frac{(6.0\ \text{kg})(9.81\ \text{m/s}^2)}{2\sin 60°} = 33.98\ \text{N}$$
$$= \boxed{34\ \text{N}}$$

Because $T_2 = F_\text{g}$:

$$T_2 = (6.0\ \text{kg})(9.81\ \text{m/s}^2) = 58.9\ \text{N}$$
$$= \boxed{59\ \text{N}}$$

Because the effect of the pulley is to change the direction T_1 acts, T_1 is the weight of the object whose mass is m:

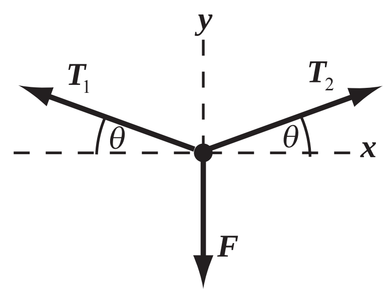
$$T_1 = mg \Rightarrow m = \frac{T_1}{g}$$

Substitute numerical values and evaluate m:

$$m = \frac{33.98\ \text{N}}{9.81\ \text{m/s}^2} = \boxed{3.5\ \text{kg}}$$

53 •• Your car is stuck in a mud hole. You are alone, but you have a long, strong rope. Having studied physics, you tie the rope tautly to a telephone pole and pull on it sideways, as shown in Figure 4-42. (*a*) Find the force exerted by the rope on the car when the angle θ is 3.00º and you are pulling with a force of 400 N but the car does not move. (*b*) How strong must the rope be if it takes a force of 600 N to move the car when θ is 4.00º?

Picture the Problem Construct the free-body diagram for that point in the rope at which you exert the force $\vec{F}$ and choose the coordinate system shown in the free-body diagram. We can apply Newton's second law to the rope to relate the tension to F.



(*a*) Noting that $T_1 = T_2 = T$ and that the car's acceleration is zero, apply $\sum F_y = ma_y$ to the car:

$$2T\sin\theta - F = ma_y = 0 \Rightarrow T = \frac{F}{2\sin\theta}$$

Substitute numerical values and evaluate T:

$$T = \frac{400\ \text{N}}{2\sin 3.00°} = \boxed{3.82\ \text{kN}}$$

(*b*) Proceed as in Part (*a*) to obtain:

$$T = \frac{600\ \text{N}}{2\sin 4.00°} = \boxed{4.30\ \text{kN}}$$

54 ••• Balloon arches are often seen at festivals or celebrations; they are made by attaching helium-filled balloons to a rope that is fixed to the ground at each end. The lift from the balloons raises the structure into the arch shape. Figure 4-43*a* shows the geometry of such a structure: N balloons are attached at equally spaced intervals along a massless rope of length L, which is attached to two supports at its ends. Each balloon provides a lift force F. The horizontal and vertical coordinates of the point on the rope where the ith balloon is attached are x_i and y_i, and T_i is the tension in the ith segment. (Note segment 0 is the segment between the point of attachment and the first balloon, and segment N is the segment between the last balloon and the other point of attachment). (*a*) Figure 4-43*b* shows a free-body diagram for the ith balloon. From this diagram, show that the horizontal component of the force T_i (call it T_H) is the same for all the

string segments. (*b*) By considering the vertical component of the forces, use Newton's laws to derive the following relationship between the tension in the *i*th and (i – 1)th segments: $T_{i-1}\sin\theta_{i-1} - T_i\sin\theta_i = F$ (*b*) Show that $\tan\theta_0 = -\tan\theta_{N+1} = NF/2T_H$. (*c*) From the diagram and the two expressions above, show that $\tan\theta_i = (N - 2i)F/2T_H$ and that $x_i = \dfrac{L}{N+1}\sum_{j=0}^{i-1}\cos\theta_j$,
$y_i = \dfrac{L}{N+1}\sum_{j=0}^{i-1}\sin\theta_j$.

Picture the Problem (*a*) Applying $\sum F_x = 0$ to the balloon shown in Figure 4-43 (*b*) will show that the horizontal component of the force T_i is the same for all string segments. In Part (*b*) we can apply $\sum F_y = 0$ to obtain the given expression for *F*. In (*c*) we can use a symmetry argument to find an expression for tan θ_0. Finally, in (*d*) we can use our results obtained in (*a*) and (*b*) to express x_i and y_i.

(*a*) Applying $\sum F_x = 0$ to the balloon shown in Figure 4-43 (*b*) gives:

$$T_i\cos\theta_i - T_{i-1}\cos\theta_{i-1} = 0$$

Because $T_H = T_i\cos\theta_i$:

$$T_H - T_{i-1}\cos\theta_{i-1} = 0 \Rightarrow T_H = T_{i-1}\cos\theta_{i-1}$$

independently of the string segment.

(*b*) Apply $\sum F_y = 0$ to the balloon shown in Figure 4-43 (*b*) gives:

$$F + T_i\sin\theta_i - T_{i-1}\sin\theta_{i-1} = 0$$

Solving for *F* yields:

$$F = \boxed{T_{i-1}\sin\theta_{i-1} - T_i\sin\theta_i}$$

(*c*) By symmetry, each support must balance half of the force acting on the entire arch. Therefore, the vertical component of the force on the support must be *NF*/2. The horizontal component of the tension must be T_H. Express tanθ_0 in terms of *NF*/2 and T_H:

$$\tan\theta_0 = \frac{NF/2}{T_H} = \frac{NF}{2T_H}$$

By symmetry, $\theta_{N+1} = -\theta_0$. Therefore, because the tangent function is odd:

$$\tan\theta_0 = \boxed{-\tan\theta_{N+1} = \frac{NF}{2T_H}}$$

(d) Using $T_H = T_i \cos\theta_i = T_{i-1}\cos\theta_{i-1}$, divide both sides of the result in (b) by T_H and simplify to obtain:

$$\frac{F}{T_H} = \frac{T_{i-1}\sin\theta_{i-1}}{T_{i-1}\cos\theta_{i-1}} - \frac{T_i \sin\theta_i}{T_i \cos\theta_i} = \tan\theta_{i-1} - \tan\theta_i$$

Using this result, express $\tan\theta_1$:

$$\tan\theta_1 = \tan\theta_0 - \frac{F}{T_H}$$

Substitute for $\tan\theta_0$ from (a):

$$\tan\theta_1 = \frac{NF}{2T_H} - \frac{F}{T_H} = (N-2)\frac{F}{2T_H}$$

Generalize this result to obtain:

$$\tan\theta_i = \boxed{(N-2i)\frac{F}{2T_H}}$$

Express the length of rope between two balloons:

$$\ell_{\substack{\text{between}\\\text{balloons}}} = \frac{L}{N+1}$$

Express the horizontal coordinate of the point on the rope where the ith balloon is attached, x_i, in terms of x_{i-1} and the length of rope between two balloons:

$$x_i = x_{i-1} + \frac{L}{N+1}\cos\theta_{i-1}$$

Sum over all the coordinates to obtain:

$$\boxed{x_i = \frac{L}{N+1}\sum_{j=0}^{i-1}\cos\theta_j}$$

Proceed similarly for the vertical coordinates to obtain:

$$\boxed{y_i = \frac{L}{N+1}\sum_{j=0}^{i-1}\sin\theta_j}$$

55 ••• (a) Consider a numerical solution to Problem 54. Write a **spreadsheet** program to make a graph of the shape of a balloon arch. Use the following parameters: $N = 10$ balloons each providing a lift force $F = 1.0$ N and each attached to a rope length $L = 11$ m, with a horizontal component of tension $T_H = 10$ N. How far apart are the two points of attachment? How high is the arch at its highest point? (b) Note that we haven't specified the spacing between the supports—it is determined by the other parameters. Vary T_H while keeping the other parameters the same until you create an arch that has a spacing of 8.0 m between the supports. What is T_H then? As you increase T_H, the arch should get flatter and more spread out. Does your spreadsheet model show this?

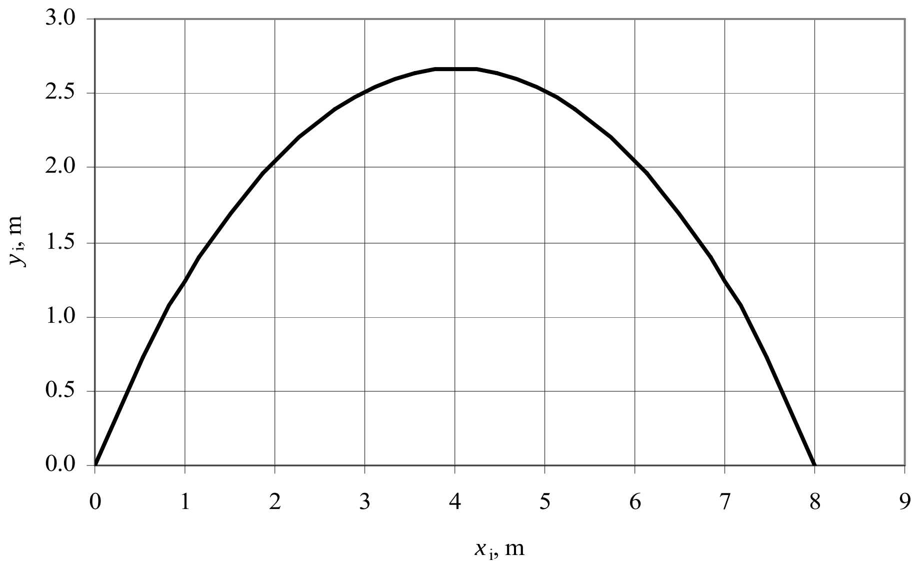
Picture the Problem

(*a*) A spreadsheet program is shown below. The formulas used to calculate the quantities in the columns are as follows:

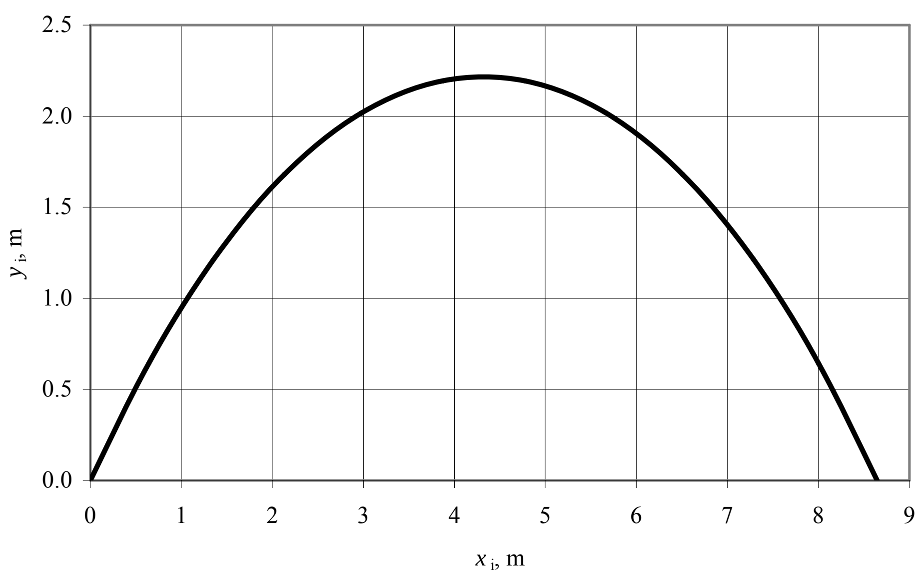
Cell	Content/Formula	Algebraic Form
C7	($B$2−2*B7)/(2*$B$4)	$(N-2\mathrm{i})\dfrac{F}{2T_{\mathrm{H}}}$
D7	SIN(ATAN(C7))	$\sin\left(\tan^{-1}\theta_{\mathrm{i}}\right)$
E7	COS(ATAN(C7))	$\cos\left(\tan^{-1}\theta_{\mathrm{i}}\right)$
F7	0.000	0
G7	0.000	0
F8	F7+$B$1/($B$2+1)*E7	$x_{\mathrm{i}-1}+\dfrac{L}{N+1}\cos\theta_{\mathrm{i}-1}$
G8	G7+$B$1/($B$2+1)*D7	$y_{\mathrm{i}-1}+\dfrac{L}{N+1}\cos\theta_{\mathrm{i}-1}$

	A	B	C	D	E	F	G
1	$L=$	11	m				
2	$N=$	10					
3	$F=$	1	N				
4	$T_{\mathrm{H}}=$	3.72	N				
5							
6		i	$\tan\theta_{\mathrm{i}}$	$\sin\theta_{\mathrm{i}}$	$\cos\theta_{\mathrm{i}}$	x_{i}	y_{i}
7		0	1.344	0.802	0.597	0.000	0.000
8		1	1.075	0.732	0.681	0.543	0.729
9		2	0.806	0.628	0.778	1.162	1.395
10		3	0.538	0.474	0.881	1.869	1.966
14		5	0.000	0.000	1.000	3.548	2.632
15		6	−0.269	−0.260	0.966	4.457	2.632
19		10	−1.344	−0.802	0.597	7.462	0.729
20		11				8.005	0.000

(b) A horizontal component of tension 3.72 N gives a spacing of 8 m. At this spacing, the arch is 2.63 m high, tall enough for someone to walk through. A graph of y_i as a function of x_i for the conditions specified follows:



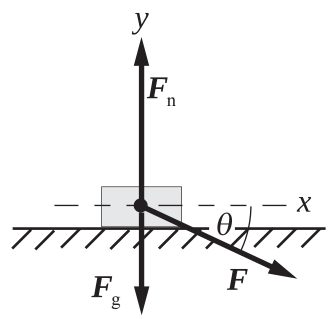
The spreadsheet graph shown below shows that changing the horizontal component of tension to 5 N broadens the arch and decreases its height as predicted by our mathematical model.



Free-Body Diagrams: Inclined Planes and the Normal Force

56 • A large box whose mass is 20.0 kg rests on a frictionless floor. A mover pushes on the box with a force of 250 N at an angle 35.0º below the horizontal. Draw the box's free body diagram and use it to determine the acceleration of the box.

Picture the Problem The free-body diagram shows the forces acting on the box as the man pushes it across the frictionless floor. We can apply Newton's second law to the box to find its acceleration.



Apply $\sum F_x = ma_x$ to the box:

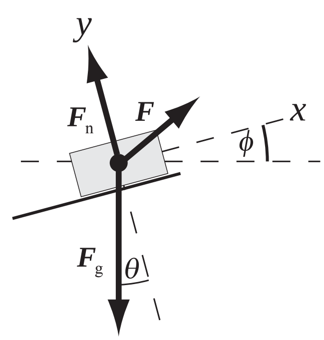
$$F\cos\theta = ma_x \Rightarrow a_x = \frac{F\cos\theta}{m}$$

Substitute numerical values and evaluate a_x:

$$a_x = \frac{(250\,\text{N})\cos 35.0°}{20.0\,\text{kg}} = \boxed{10.2\,\text{m/s}^2}$$

57 • A 20-kg box rests on a frictionless ramp with a 15.0º slope. The mover pulls on a rope attached to the box to pull it up the incline (Figure 4-44). If the rope makes an angle of 40.0º with the horizontal, what is the smallest force *F* the mover will have to exert to move the box up the ramp?

Picture the Problem The free-body diagram shows the forces acting on the box as the man pushes it up the frictionless incline. We can apply Newton's second law to the box to determine the smallest force that will move it up the incline at constant speed.



Letting $F_{min} = F$, apply $\sum F_x = ma_x$ to the box as it moves up the incline with constant speed:

$$F_{min}\cos(\phi - \theta) - F_g\sin\theta = 0$$

Solve for F_{min} to obtain:

$$F_{min} = \frac{F_g\sin\theta}{\cos(\phi - \theta)}$$

Because $F_g = mg$:

$$F_{min} = \frac{mg\sin\theta}{\cos(\phi - \theta)}$$

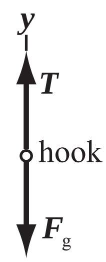
Substitute numerical values and evaluate F_{min}:

$$F_{min} = \frac{(20.0\,\text{kg})(9.81\,\text{m/s}^2)\sin 15°}{\cos(40.0° - 15.0°)} = \boxed{56.0\,\text{N}}$$

58 • In Figure 4-45, the objects are attached to spring scales calibrated in newtons. Give the readings of the balance(s) in each case, assuming that both the scales and the strings are massless.

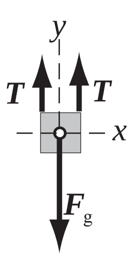
Picture the Problem The balance(s) indicate the tension in the string(s). Draw free-body diagrams for each of these systems and apply the condition(s) for equilibrium.

(*a*)



(*b*)

(*c*)



(*d*)

(*a*) Apply $\sum F_y = 0$ to the hook to obtain:

$\sum F_y = T - F_g = 0$
or, because $F_g = mg$,
$T = mg$

Substitute numerical values and evaluate T:

$$T = (10\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{98\,\text{N}}$$

(*b*) Apply $\sum F_x = 0$ to the balance to obtain:

$\sum F_x = T - T' = 0$
or, because $T' = mg$,
$T = T' = mg$

Substitute numerical values and evaluate T and T':

$$T = T' = (10\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{98\,\text{N}}$$

(c) Apply $\sum F_y = 0$ to the suspended object to obtain:

$$\sum F_y = 2T - F_g = 0$$

or, because $F_g = mg$,

$$2T - mg = 0 \Rightarrow T = \tfrac{1}{2}mg$$

Substitute numerical values and evaluate T:

$$T = \tfrac{1}{2}(10\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{49\,\text{N}}$$

(d) Apply $\sum F_x = 0$ to the balance to obtain:

$$\sum F_x = T - T' = 0$$

or, because $T' = mg$,

$$T = T' = mg$$

Substitute numerical values and evaluate T and T':

$$T = T' = (10\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{98\,\text{N}}$$

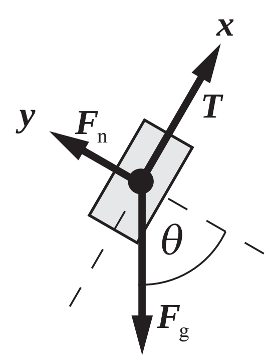
Remarks: Note that (*a*), (*b*), and (*d*) give the same answers … a rather surprising result until one has learned to draw FBDs and apply the conditions for translational equilibrium.

59 •• A box is held in position on a frictionless incline by a cable (Figure 4-46). (a) If $\theta = 60^\circ$ and $m = 50$ kg, find the tension in the cable and the normal force exerted by the incline. (b) Find the tension as a function of θ and m, and check your result for plausibility in the special cases when $\theta = 0^\circ$ and $\theta = 90^\circ$.

Picture the Problem Because the box is held in place (is in equilibrium) by the forces acting on it, we know that

$$\vec{T} + \vec{F}_n + \vec{F}_g = 0$$

Choose a coordinate system in which the $+x$ direction is in the direction of $\vec{T}$ and the $+y$ direction is in the direction of $\vec{F}_n$. Apply Newton's second law to the block to obtain expressions for $\vec{T}$ and $\vec{F}_n$.



(a) Apply $\sum F_x = 0$ to the box:

$$T - F_g \sin\theta = 0$$

or, because $F_g = mg$,

$$T - mg\sin\theta = 0 \Rightarrow T = mg\sin\theta \quad (1)$$

Substitute numerical values and evaluate T:

$$T = (50\,\text{kg})(9.81\,\text{m/s}^2)\sin 60° = \boxed{0.42\,\text{kN}}$$

Apply $\sum F_y = 0$ to the box:

$$F_n - F_g \cos\theta = 0$$

or, because $F_g = mg$,

$$F_n - mg\cos\theta = 0 \Rightarrow F_n = mg\cos\theta$$

Substitute numerical values and evaluate F_n:

$$F_n = (50\,\text{kg})(9.81\,\text{m/s}^2)\cos 60° = \boxed{0.25\,\text{kN}}$$

(*b*) The tension as a function of θ and m is given by equation (1):

$$T = \boxed{mg\sin\theta}$$

For $\theta = 90°$:

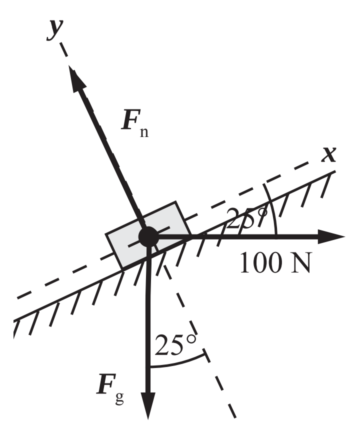
$T_{90°} = mg\sin 90° = \boxed{mg}$ … a result we know to be correct.

For $\theta = 0°$:

$T_{0°} = mg\sin 0° = \boxed{0}$ … a result we know to be correct.

60 •• A horizontal force of 100 N pushes a 12-kg block up a frictionless incline that makes an angle of 25º with the horizontal. (*a*) What is the normal force that the incline exerts on the block? (*b*) What is the acceleration of the block?

Picture the Problem Draw a free-body diagram for the box. Choose a coordinate system in which the positive x-axis is parallel to the inclined plane and the positive y-axis is in the direction of the normal force the incline exerts on the block. Apply Newton's second law of motion to find the normal force F_n that the incline exerts on the block and the acceleration a of the block.



(*a*) Apply $\sum F_y = ma_y$ to the block:

$$F_n - F_g\cos 25° - (100\,\text{N})\sin 25° = 0$$

or, because $F_g = mg$,

$$F_n - mg\cos 25° - (100\,\text{N})\sin 25° = 0$$

Solving for F_n yields:

$$F_n = mg\cos 25° + (100\,\text{N})\sin 25°$$

Substitute numerical values and evaluate F_n:

$$F_n = (12\,\text{kg})(9.81\,\text{m/s}^2)\cos 25° + (100\,\text{N})\sin 25° = \boxed{1.5\times10^2\ \text{N}}$$

(*b*) Apply $\sum F_x = ma_x$ to the block:

$$(100\,\text{N})\cos 25° - F_g \sin 25° = ma$$

or, because $F_g = mg$,

$$(100\,\text{N})\cos 25° - mg\sin 25° = ma$$

Solve for a to obtain:

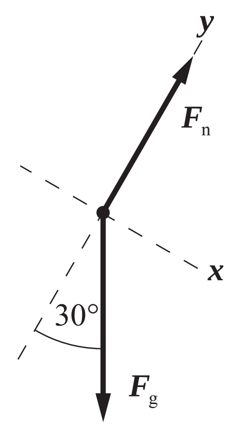
$$a = \frac{(100\,\text{N})\cos 25°}{m} - g\sin 25°$$

Substitute numerical values and evaluate a:

$$a = \frac{(100\,\text{N})\cos 25°}{12\,\text{kg}} - (9.81\,\text{m/s}^2)\sin 25° = \boxed{3.4\,\text{m/s}^2}$$

61 •• [SSM] A 65-kg student weighs himself by standing on a scale mounted on a skateboard that is rolling down an incline, as shown in Figure 4-47. Assume there is no friction so that the force exerted by the incline on the skateboard is normal to the incline. What is the reading on the scale if $\theta = 30°$?

Picture the Problem The scale reading (the boy's apparent weight) is the force the scale exerts on the boy. Draw a free-body diagram for the boy, choosing a coordinate system in which the positive x-axis is parallel to and down the inclined plane and the positive y-axis is in the direction of the normal force the incline exerts on the boy. Apply Newton's second law of motion in the y direction.



Apply $\sum F_y = ma_y$ to the boy to find F_n. Remember that there is no acceleration in the y direction:

$$F_n - F_g\cos 30° = 0$$

or, because $F_g = mg$,

$$F_n - mg\cos 30° = 0$$

Solving for F_n yields:

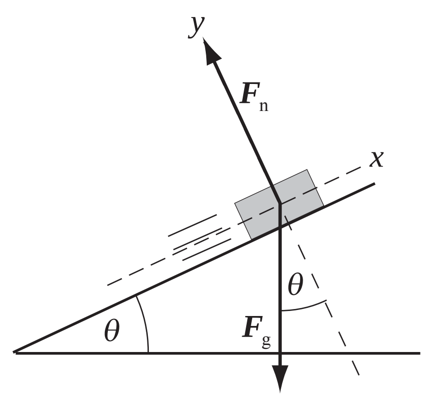
$$F_n = mg\cos 30°$$

Substitute numerical values and evaluate F_n:

$$F_n = (65\,\text{kg})(9.81\,\text{m/s}^2)\cos 30° = \boxed{0.55\,\text{kN}}$$

62 •• A block of mass m slides across a frictionless floor and then up a frictionless ramp (Figure 4-48). The angle of the ramp is θ and the speed of the block before it starts up the ramp is v_0. The block will slide up to some maximum height h above the floor before stopping. Show that h is independent of θ by deriving an expression for h in terms of v_0 and g.

Picture the Problem The free-body diagram for the block sliding up the incline is shown to the right. Applying Newton's second law to the forces acting in the x direction will lead us to an expression for a_x. Using this expression in a constant-acceleration equation will allow us to express h as a function of v_0 and g.



The height h is related to the distance Δx traveled up the incline:

$$h = \Delta x \sin\theta \qquad (1)$$

Using a constant-acceleration equation, relate the final speed of the block to its initial speed, acceleration, and distance traveled:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 + 2a_x\Delta x \Rightarrow \Delta x = \frac{-v_{0x}^2}{2a_x}$$

Substituting for Δx in equation (1) yields:

$$h = \frac{-v_{0x}^2}{2a_x}\sin\theta \qquad (2)$$

Apply $\sum F_x = ma_x$ to the block and solve for its acceleration:

$$-F_g \sin\theta = ma_x$$

Because $F_g = mg$:

$$-mg\sin\theta = ma_x \Rightarrow a_x = -g\sin\theta$$

Substitute for a_x in equation (2) and simplify to obtain:

$$h = \left(\frac{v_{0x}^2}{2g\sin\theta}\right)\sin\theta = \boxed{\frac{v_{0x}^2}{2g}}$$

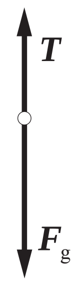
which is independent of the ramp's angle θ.

Free-Body Diagrams: Elevators

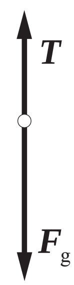
63 • **[SSM]** (*a*) Draw the free body diagram (with accurate relative force magnitudes) for an object that is hung by a rope from the ceiling of an elevator that is ascending but slowing. (*b*) Repeat Part (*a*) but for the situation in which the

elevator is descending and speeding up. (*c*) Can you tell the difference between the two diagrams? Explain why the diagrams do not tell anything about the object's velocity.

Picture the Problem
(*a*) The free body diagram for an object that is hung by a rope from the ceiling of an ascending elevator that is slowing down is shown to the right. Note that because $F_g > T$, the net force acting on the object is downward; as it must be if the object is slowing down as it is moving upward.



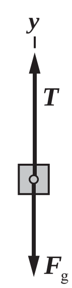
(*b*) The free body diagram for an object that is hung by a rope from the ceiling of an elevator that is descending and speeding up is shown to the right. Note that because $F_g > T$, the net force acting on the object is downward; as it must be if the object is speeding up as it descends.



(*c*) No, there is no difference. In both cases the acceleration is downward. You can only tell the direction of the acceleration, not the direction of the velocity.

64 • A 10.0-kg block is suspended from the ceiling of an elevator by a cord rated to withstand a tension of 150 N. Shortly after the elevator starts to ascend, the cord breaks. What was the minimum acceleration of the elevator when the cord broke?

Picture the Problem The free-body diagram shows the forces acting on the 10.0-kg block as the elevator accelerates upward. Apply Newton's second law to the block to find the minimum acceleration of the elevator required to break the cord.



Apply $\sum F_y = ma_y$ to the block:

$$T - F_g = ma_y$$

or, because $F_g = mg$,

$$T - mg = ma_y$$

Solve for a_y to determine the minimum breaking acceleration:

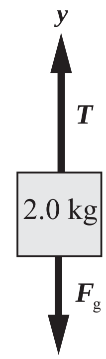
$$a_y = \frac{T - mg}{m} = \frac{T}{m} - g$$

Substitute numerical values and evaluate a_y:

$$a_y = \frac{150\,\text{N}}{10.0\,\text{kg}} - 9.81\,\text{m/s}^2 = \boxed{5.19\,\text{m/s}^2}$$

65 •• A 2.0-kg block hangs from a spring scale calibrated in newtons that is attached to the ceiling of an elevator (Figure 4-49). What does the scale read when (*a*) the elevator is ascending with a constant speed of 30 m/s, (*b*) the elevator is descending with a constant speed of 30 m/s, (*c*) the elevator is ascending at 20 m/s and gaining speed at a rate of 3.0 m/s^2? (*d*) Suppose that from $t = 0$ to $t = 5.0$ s, the elevator ascends at a constant speed of 10 m/s. Its speed is then steadily reduced to zero during the next 4.0 s, so that it is at rest at $t = 9.0$ s. Describe the reading of the scale during the interval $0 < t < 9.0$ s.

Picture the Problem The free-body diagram shows the forces acting on the 2-kg block as the elevator ascends at a constant velocity. Because the acceleration of the elevator is zero, the block is in equilibrium under the influence of $\vec{T}$ and $m\vec{g}$. Apply Newton's second law of motion to the block to determine the scale reading.



(*a*) Apply $\sum F_y = ma_y$ to the block to obtain:

$$T - F_g = ma_y$$

or, because $F_g = mg$,

$$T - mg = ma_y \qquad (1)$$

For motion with constant velocity, $a_y = 0$ and:

$$T - mg = 0 \Rightarrow T = mg$$

Substitute numerical values and evaluate T:

$$T = (2.0\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{20\,\text{N}}$$

(*b*) As in Part (*a*), for constant velocity, $a_y = 0$. Hence:

$$T - mg = 0$$

and

$$T = (2.0\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{20\,\text{N}}$$

(*c*) Solve equation (1) for T and simplify to obtain:

$$T = mg + ma_y = m(g + a_y) \qquad (2)$$

Because the elevator is ascending and its speed is increasing, we have $a_y = 3.0$ m/s^2. Substitute numerical values and evaluate T:

$$T = (2.0\,\text{kg})(9.81\,\text{m/s}^2 + 3.0\,\text{m/s}^2) = \boxed{26\,\text{N}}$$

(*d*) During the interval $0 < t < 5.0$ s, $a_y = 0$. Hence:

$$T_{0\to 5.0\,\text{s}} = \boxed{20\ \text{N}}$$

Using its definition, calculate a for $5.0\ \text{s} < t < 9.0$ s:

$$a = \frac{\Delta v}{\Delta t} = \frac{0 - 10\,\text{m/s}}{4.0\,\text{s}} = -2.5\,\text{m/s}^2$$

Substitute in equation (2) and evaluate T:

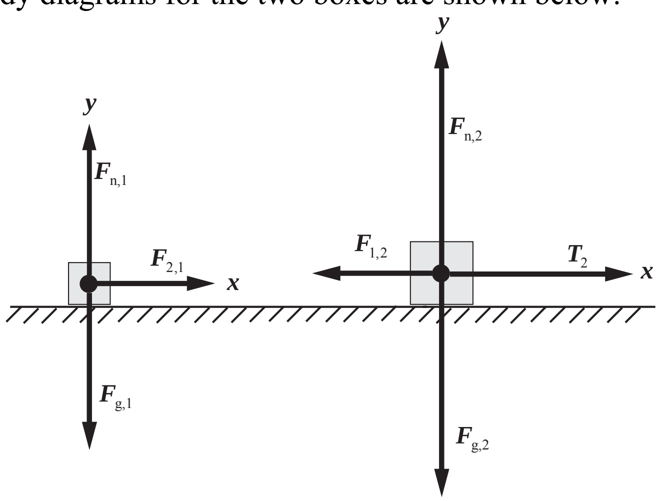
$$T_{5\text{s}\to 9\text{s}} = (2.0\,\text{kg})(9.81\,\text{m/s}^2 - 2.5\,\text{m/s}^2) = \boxed{15\,\text{N}}$$

Free-Body Diagrams: Several Objects and Newton's Third Law

66 •• Two boxes of mass m_1 and m_2 connected by a massless string are being pulled along a horizontal frictionless surface, as shown in Figure 4-50. (*a*) Draw the free body diagram of both boxes separately and show that $T_1/T_2 = m_1/(m_1 + m_2)$. (*b*) Is this result plausible? Explain. Does your answer make sense both in the limit that $m_2/m_1 >> 1$ and in the limit that $m_2/m_1 << 1$? Explain.

Picture the Problem Draw a free-body diagram for each box and apply Newton's second law. Solve the resulting simultaneous equations for the ratio of T_1 to T_2.

(*a*) The free-body diagrams for the two boxes are shown below:



Apply $\sum F_x = ma_x$ to the box on the left:

$F_{2,1} = m_1 a_{1x}$
or, because $F_{2,1} = T_1$,
$T_1 = m_1 a_{1x}$

Apply $\sum F_x = ma_x$ to the box on the right:

$T_2 - F_{1,2} = m_2 a_{2x}$
or, because $F_{1,2} = T_1$,
$T_2 - T_1 = m_2 a_{2x}$

Because the boxes have the same acceleration, we can divide the second equation by the first to obtain:

$$\frac{T_1}{T_2 - T_1} = \frac{m_1}{m_2} \Rightarrow \frac{T_1}{T_2} = \boxed{\frac{m_1}{m_1 + m_2}}$$

(*b*) Divide the numerator and denominator of the expression inside the parentheses by m_1 to obtain:

$$\frac{T_1}{T_2} = \frac{1}{1 + \frac{m_2}{m_1}}$$

For $m_2/m_1 << 1$:

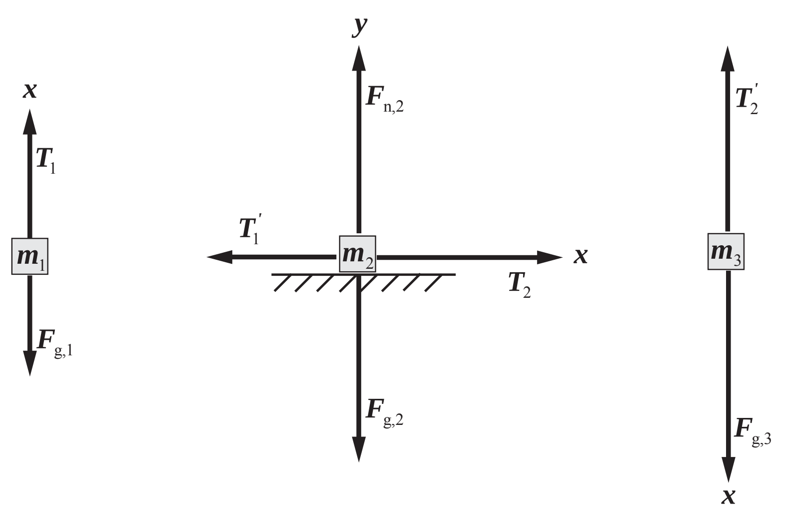
$$\frac{T_1}{T_2} \to \boxed{0}$$, as expected.

For $m_2/m_1 << 1$:

$$\frac{T_1}{T_2} \to \boxed{T_2}$$, as expected.

67 •• A box of mass $m_2 = 3.5$ kg rests on a frictionless horizontal shelf and is attached by strings to boxes of masses $m_1 = 1.5$ kg and $m_3 = 2.5$ kg as shown in Figure 4-51. Both pulleys are frictionless and massless. The system is released from rest. After it is released, find (*a*) the acceleration of each of the boxes and (*b*) the tension in each string.

Picture the Problem Call the common acceleration of the boxes *a*. Assume that box 1 moves upward, box 2 to the right, and box 3 downward and take this direction to be the positive *x* direction. Draw free-body diagrams for each of the boxes, apply Newton's second law of motion, and solve the resulting equations simultaneously. Note that $T_1 = T_1^{'}$ and $T_2 = T_2^{'}$.



(*a*) Apply $\sum F_x = ma_x$ to the box whose mass is m_1:

$$T_1 - F_{g,1} = m_1 a_x$$

or, because $F_{g,1} = m_1 g$,

$$T_1 - m_1 g = m_1 a_x \qquad (1)$$

Apply $\sum F_x = ma_x$ to the box whose mass is m_2:

$$T_2 - T_1' = m_2 a_x$$

or, because $T_1 = T_1'$,

$$T_2 - T_1 = m_2 a \qquad (2)$$

Noting that $T_2 = T_2'$, apply $\sum F_x = ma_x$ to the box whose mass is m_3:

$$F_{g,3} - T_2 = m_3 a_x$$

or, because $F_{g,3} = m_3 g$,

$$m_3 g - T_2 = m_3 a_x \qquad (3)$$

Add equations (1), (2), and (3) to obtain:

$$m_3 g - m_1 g = m_1 a_x + m_2 a_x + m_3 a_x$$

Solving for a_x gives:

$$a_x = \frac{(m_3 - m_1)g}{m_1 + m_2 + m_3}$$

Substitute numerical values and evaluate a_x:

$$a_x = \frac{(2.5\,\text{kg} - 1.5\,\text{kg})(9.81\,\text{m/s}^2)}{1.5\,\text{kg} + 3.5\,\text{kg} + 2.5\,\text{kg}} = 1.31\,\text{m/s}^2 = \boxed{1.3\,\text{m/s}^2}$$

(*b*) From equation (1) we have:

$$T_1 = m_1 a_x + m_1 g = m_1(a_x + g)$$

Substitute numerical values and evaluate T_1:

$$T_1 = (1.5\,\text{kg})(1.31\,\text{m/s}^2 + 9.81\,\text{m/s}^2) = \boxed{17\,\text{N}}$$

From equation (3) we have:

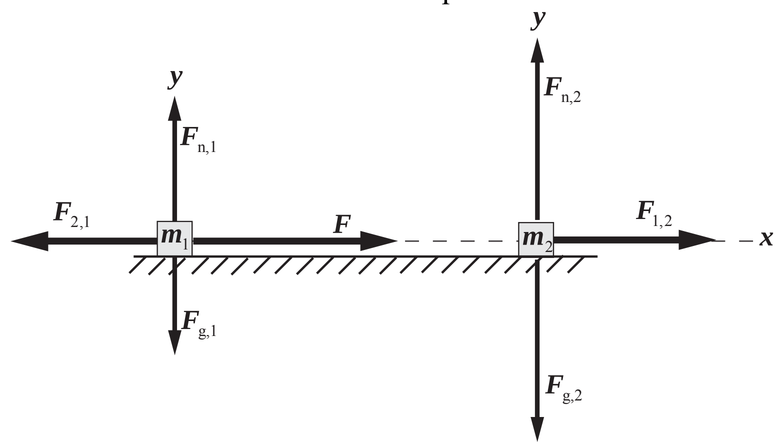
$$T_2 = m_3 g - m_3 a_x = m_3(g - a_x)$$

Substitute numerical values and evaluate T_2:

$$T_2 = (2.5\,\text{kg})(9.81\,\text{m/s}^2 - 1.31\,\text{m/s}^2) = \boxed{21\,\text{N}}$$

68 •• Two blocks are in contact on a frictionless horizontal surface. The blocks are accelerated by a single horizontal force $\vec{F}$ applied to one of them (Figure 4-52). Find the acceleration and the contact force of block 1 on block 2 (*a*) in terms of F, m_1, and m_2, and (*b*) for the specific values F = 3.2 N, m_1 = 2.0 kg, and m_2 = 6.0 kg.

Picture the Problem Choose a coordinate system in which the $+x$ direction is to the right and the $+y$ direction is upward. Let $\vec{F}_{2,1}$ be the contact force exerted by the block whose mass is m_2 on the block whose mass is m_1 and $\vec{F}_{1,2}$ be the force exerted by the block whose mass is m_1 on the block whose mass is m_2. These forces are equal and opposite so $\vec{F}_{2,1} = -\vec{F}_{1,2}$. The free- body diagrams for the blocks are shown below. Apply Newton's second law to each block separately and use the fact that their accelerations are equal.



(a) Apply $\sum F_x = ma_x$ to the block whose mass is m_1:

$F - F_{2,1} = m_1 a_{1x}$

or, because the blocks have a common acceleration a_x,

$$F - F_{2,1} = m_1 a_x \qquad (1)$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$$F_{1,2} = m_2 a_x \qquad (2)$$

Adding equations (1) and (2) yields:

$$F = m_1 a_x + m_2 a_x = (m_1 + m_2)a_x$$

Solve for a_x to obtain:

$$a_x = \boxed{\frac{F}{m_1 + m_2}}$$

Substitute for a in equation (1) to obtain:

$$F_{1,2} = m_2\left(\frac{F}{m_1 + m_2}\right) = \boxed{\frac{Fm_2}{m_1 + m_2}}$$

(b) Substitute numerical values in the equations derived in Part (a) and evaluate a_x and $F_{1,2}$:

$$a_x = \frac{3.2\,\text{N}}{2.0\,\text{kg} + 6.0\,\text{kg}} = \boxed{0.40\,\text{m/s}^2}$$

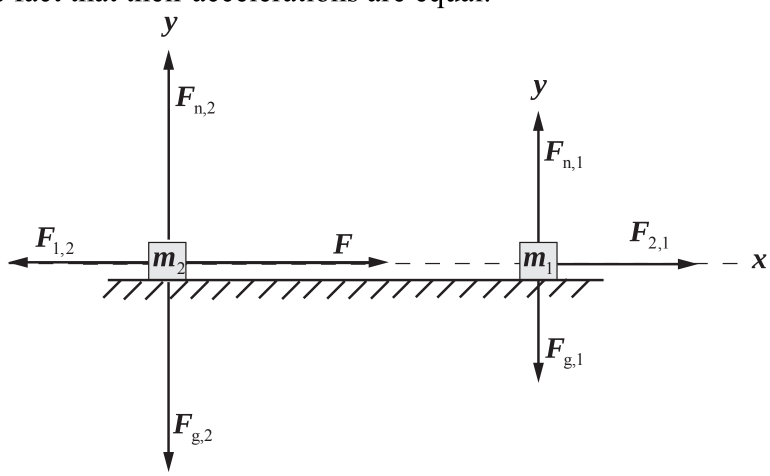
and

$$F_{1,2} = \frac{(3.2\,\text{N})(6.0\,\text{kg})}{2.0\,\text{kg} + 6.0\,\text{kg}} = \boxed{2.4\,\text{N}}$$

Remarks: Note that our results for the acceleration are the same as if the force $\boldsymbol{F}$ had acted on a single object whose mass is equal to the sum of the masses of the two blocks. In fact, because the two blocks have the same acceleration, we can consider them to be a single system with mass $m_1 + m_2$.

69 •• Repeat Problem 68, but with the two blocks interchanged. Are your answers the same as in Problem 68? Explain.

Picture the Problem Choose a coordinate system in which the +*x* direction is to the right and the +*y* direction is upward. Let $\vec{F}_{2,1}$ be the contact force exerted by the block whose mass is m_2 on the block whose mass is m_1 and $\vec{F}_{1,2}$ be the force exerted by the block whose mass is m_1 on the block whose mass is m_2. These forces are equal and opposite so $\vec{F}_{2,1} = -\vec{F}_{1,2}$. The free- body diagrams for the blocks are shown. We can apply Newton's second law to each block separately and use the fact that their accelerations are equal.



(*a*) Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$F - F_{1,2} = m_2 a_{2x}$

or, because the blocks have a common acceleration a_x,

$$F - F_{1,2} = m_2 a_x \quad (1)$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_1:

$$F_{2,1} = m_1 a_x \quad (2)$$

Add these equations to eliminate $F_{2,1}$ and $F_{1,2}$:

$F - F_{1,2} + F_{2,1} = m_2 a_x + m_1 a_x$

or, because $F_{1,2} = F_{2,1}$,

$F = (m_2 + m_1)a_x$

Solve for a_x to obtain:

$$a_x = \boxed{\frac{F}{m_1 + m_2}}$$

Substituting for a into equation (2) yields:

$$F_{2,1} = m_1\left(\frac{F}{m_1 + m_2}\right) = \boxed{\frac{Fm_1}{m_1 + m_2}}$$

(b) Substitute numerical values in the equations derived in Part (a) and evaluate a_x and $F_{2,1}$:

$$a = \frac{3.2\,\text{N}}{2.0\,\text{kg} + 6.0\,\text{kg}} = \boxed{0.40\,\text{m/s}^2}$$

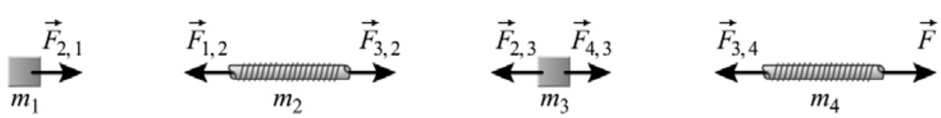
and

$$F_{2,1} = \frac{(3.2\,\text{N})(2.0\,\text{kg})}{2.0\,\text{kg} + 6.0\,\text{kg}} = \boxed{0.80\,\text{N}}$$

Remarks: Note that our results for the acceleration are the same as if the force *F* had acted on a single object whose mass is equal to the sum of the masses of the two blocks. In fact, because the two blocks have the same acceleration, we can consider them to be a single system with mass $m_1 + m_2$.

70 •• Two 100-kg boxes are dragged along a horizontal frictionless surface at a constant acceleration of 1.00 m/s^2, as shown in Figure 4-53. Each rope has a mass of 1.00 kg. Find the force F and the tension in the ropes at points A, B, and C.

Picture the Problem The free-body diagrams for the boxes and the ropes are below. Because the vertical forces have no bearing on the problem they have not been included. Let the numeral 1 denote the 100-kg box to the left, the numeral 2 the rope connecting the boxes, the numeral 3 the box to the right and the numeral 4 the rope to which the force $\vec{F}$ is applied. $\vec{F}_{3,4}$ is the tension force exerted by the box whose mass is m_3 on the rope whose mass is m_4, $\vec{F}_{4,3}$ is the tension force exerted by the rope whose mass is m_4 on the box whose mass is m_3, $\vec{F}_{2,3}$ is the tension force exerted by the rope whose mass is m_2 on the box whose mass is m_3, $\vec{F}_{3,2}$ is the tension force exerted by the box whose mass is m_3 on the rope whose mass is m_2, $\vec{F}_{1,2}$ is the tension force exerted by the box whose mass is m_1 on the rope whose mass is m_2, and $\vec{F}_{2,1}$ is the tension force exerted by the rope whose mass is m_2 on the box whose mass is m_1. The equal and opposite pairs of forces are $\vec{F}_{2,1} = -\vec{F}_{1,2}$, $\vec{F}_{3,2} = -\vec{F}_{2,3}$, and $\vec{F}_{4,3} = -\vec{F}_{3,4}$. We can apply Newton's second law to each box and rope separately and use the fact that their accelerations are equal.



Apply $\sum \vec{F} = m\vec{a}$ to the box whose mass is m_1:

$$F_{2,1} = m_1 a_1 = m_1 a \qquad (1)$$

Apply $\sum\vec{F} = m\vec{a}$ to the rope whose mass is m_2:

$$F_{3,2} - F_{1,2} = m_2 a_2 = m_2 a \qquad (2)$$

Apply $\sum\vec{F} = m\vec{a}$ to the box whose mass is m_3:

$$F_{4,3} - F_{2,3} = m_3 a_3 = m_3 a \qquad (3)$$

Apply $\sum\vec{F} = m\vec{a}$ to the rope whose mass is m_4:

$$F - F_{3,4} = m_4 a_4 = m_4 a$$

Add these equations to eliminate $F_{2,1}$, $F_{1,2}$, $F_{3,2}$, $F_{2,3}$, $F_{4,3}$, and $F_{3,4}$ and solve for F:

$$F = (m_1 + m_2 + m_3 + m_4)a$$

Substitute numerical values and evaluate F:

$$F = (202\,\text{kg})(1.00\,\text{m/s}^2) = \boxed{202\,\text{N}}$$

Use equation (1) to find the tension at point A:

$$F_{2,1} = (100\,\text{kg})(1.00\,\text{m/s}^2) = \boxed{100\,\text{N}}$$

Use equation (2) to express the tension at point B:

$$F_{3,2} = F_{1,2} + m_2 a$$

Substitute numerical values and evaluate $F_{3,2}$:

$$F_{3,2} = 100\,\text{N} + (1.00\,\text{kg})(1.00\,\text{m/s}^2) = \boxed{101\,\text{N}}$$

Use equation (3) to express the tension at point C:

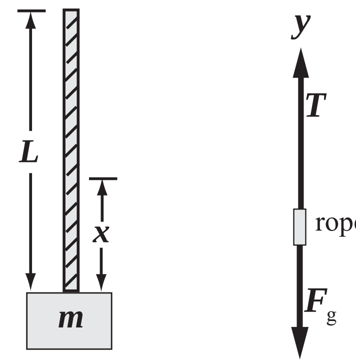
$$F_{4,3} = F_{2,3} + m_3 a$$

Substitute numerical values and evaluate $F_{4,3}$:

$$F_{4,3} = 101\,\text{N} + (100\,\text{kg})(1.00\,\text{m/s}^2) = \boxed{201\,\text{N}}$$

71 •• [SSM] A block of mass m is being lifted vertically by a rope of mass M and length L. The rope is being pulled upward by a force applied at its top end, and the rope and block are accelerating upward with an acceleration of magnitude a. The distribution of mass in the rope is uniform. Show that the tension in the rope at a distance x (where $x < L$) above the block is $(a + g)[m + (x/L)M]$.

Picture the Problem Because the distribution of mass in the rope is uniform, we can express the mass m' of a length x of the rope in terms of the total mass of the rope M and its length L. We can then express the total mass that the rope must support at a distance x above the block and use Newton's second law to find the tension as a function of x.



Set up a proportion expressing the mass m' of a length x of the rope as a function of M and L and solve for m':

$$\frac{m'}{x} = \frac{M}{L} \Rightarrow m' = \frac{M}{L}x$$

Express the total mass that the rope must support at a distance x above the block:

$$m + m' = m + \frac{M}{L}x$$

Apply $\sum F_y = ma_y$ to the block and a length x of the rope:

$$T - F_g = T - (m + m')g = (m + m')a_y$$

Substituting for $m + m'$ yields:

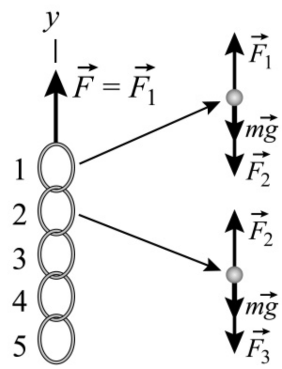
$$T - \left(m + \frac{M}{L}x\right)g = \left(m + \frac{M}{L}x\right)a_y$$

Solve for T and simplify to obtain:

$$T = \boxed{(a_y + g)\left(m + \frac{M}{L}x\right)}$$

72 •• A chain consists of 5 links, each having a mass of 0.10 kg. The chain is being pulled upward by a force applied by your hand to its top link, giving the chain an upward acceleration of 2.5 m/s^2. Find (*a*) the force magnitude F exerted on the top link by your hand; (*b*) the net force on each link; and (*c*) the magnitude of the force each link exerts on the link below it.

Picture the Problem Choose a coordinate system with the +y direction upward and denote the top link with the numeral 1, the second with the numeral 2, etc. The free-body diagrams show the forces acting on links 1 and 2. We can apply Newton's second law to each link to obtain a system of simultaneous equations that we can solve for the force each link exerts on the link below it. Note that the net force on each link is the product of its mass and acceleration.



(*a*) Apply $\sum F_y = ma_y$ to the top link and solve for *F*:

$$F - 5mg = 5ma$$

and

$$F = 5m(g + a)$$

Substitute numerical values and evaluate *F*:

$$F = 5(0.10\,\text{kg})(9.81\,\text{m/s}^2 + 2.5\,\text{m/s}^2) = \boxed{6.2\,\text{N}}$$

(*b*) Apply $\sum F_y = ma_y$ to a single link:

$$F_{1\,\text{link}} = m_{1\,\text{link}}a = (0.10\,\text{kg})(2.5\,\text{m/s}^2) = \boxed{0.25\,\text{N}}$$

(*c*) Apply $\sum F_y = ma_y$ to the 1st through 5th links to obtain:

$$F - F_2 - mg = ma\,, \quad (1)$$
$$F_2 - F_3 - mg = ma\,, \quad (2)$$
$$F_3 - F_4 - mg = ma\,, \quad (3)$$
$$F_4 - F_5 - mg = ma\,, \text{ and} \quad (4)$$
$$F_5 - mg = ma \quad (5)$$

Add equations (2) through (5) to obtain:

$$F_2 - 4mg = 4ma$$

Solve for F_2 to obtain:

$$F_2 = 4mg + 4ma = 4m(a + g)$$

Substitute numerical values and evaluate F_2:

$$F_2 = 4(0.10\,\text{kg})(9.81\,\text{m/s}^2 + 2.5\,\text{m/s}^2) = \boxed{4.9\,\text{N}}$$

Substitute for F_2 to find F_3, and then substitute for F_3 to find F_4:

$$F_3 = \boxed{3.7\,\text{N}} \text{ and } F_4 = \boxed{2.5\,\text{N}}$$

Solve equation (5) for F_5:

$$F_5 = m(g + a)$$

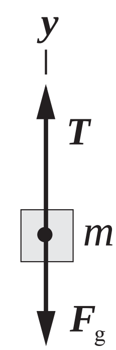
Substitute numerical values and evaluate F_5:

$$F_5 = (0.10\,\text{kg})(9.81\,\text{m/s}^2 + 2.5\,\text{m/s}^2) = \boxed{1.2\,\text{N}}$$

73 •• [SSM] A 40.0-kg object is supported by a vertical rope. The rope, and thus the object, is then accelerated from rest upward so that it attains a speed of 3.50 m’s in 0.700 s. (*a*) Draw the object’s free body diagram with the relative lengths of the vectors showing the relative magnitudes of the forces. (*b*) Use the free-body diagram and Newton’s laws to determine the tension in the rope.

Picture the Problem A *net* force is required to accelerate the object. In this problem the net force is the difference between $\vec{T}$ and $\vec{F}_g\,(= m\vec{g})$.

(*a*) The free-body diagram of the object is shown to the right. A coordinate system has been chosen in which the upward direction is positive. The magnitude of $\vec{T}$ is approximately 1.5 times the length of $\vec{F}_g$.



(*b*) Apply $\sum F = ma_y$ to the object to obtain:

$$T - mg = ma_y$$

Solving for T yields:

$$T = ma + mg = m(a_y + g)$$

Using its definition, substitute for a_y to obtain:

$$T = m\left(\frac{\Delta v_y}{\Delta t} + g\right)$$

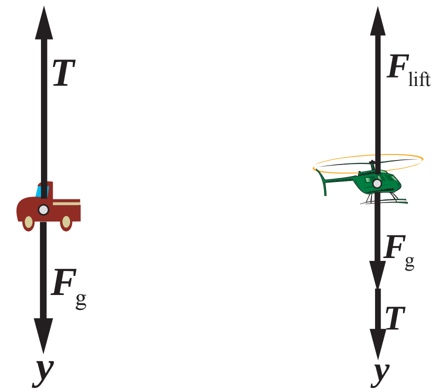
Substitute numerical values and evaluate T:

$$T = (40.0\,\text{kg})\left(\frac{3.50\,\text{m/s}}{0.700\,\text{s}} + 9.81\,\text{m/s}^2\right) = \boxed{592\,\text{N}}$$

74 •• A 15 000-kg helicopter is lowering a 4000-kg truck by a cable of fixed length. The truck, helicopter, and cable are descending at 15.0 m/s and must be slowed to 5.00 m/s in the next 50.0 m of descent to prevent damaging the truck. Assume a constant rate of slowing. (*a*) Draw the free-body diagram of the truck. (*b*) Determine the tension in the cable. (*c*) Determine the lift force of the helicopter blades.

Picture the Problem A net force in the upward direction is required to slow the truck's descent. This net force is the difference between $\vec{T}$ and $\vec{F}_g$. We can use Newton's second law and a constant-acceleration equation to find the tension in the cable that supports the truck.

(*a*) Free-body diagrams showing the forces acting on the truck and on the helicopter are shown to the right. A coordinate system in which the downward direction is positive has been chosen. Note that, because it is hovering, the helicopter is in equilibrium under the influence of the forces $\vec{F}_{\text{lift}}$, $\vec{F}_g$, and $\vec{T}$.



(*b*) Apply $\sum F_y = ma_y$ to the truck to obtain:

$$F_g - T = m_t a_y$$

or, because $F_g = m_t g$,

$$m_t g - T = m_t a_y$$

Solve for the tension in the lower portion of the cable:

$$T = m_t g - m_t a_y = m_t(g - a_y) \quad (1)$$

Use a constant-acceleration equation to relate the truck's initial and final speeds to its displacement and acceleration:

$$v_y^2 = v_{0y}^2 + 2a_y \Delta y \Rightarrow a_y = \frac{v_y^2 - v_{0y}^2}{2\Delta y}$$

Substitute numerical values and evaluate a_y:

$$a_y = \frac{(5.00\text{ m/s})^2 - (15.0\text{ m/s})^2}{2(50.0\text{ m})} = -2.000\text{ m/s}^2$$

Substitute numerical values in equation (1) and evaluate T:

$$T = (4000\text{ kg})(9.81\text{ m/s}^2 + 2.000\text{ m/s}^2) = 47.24\text{ kN} = \boxed{47.2\text{ kN}}$$

(*c*) Apply $\sum F_y = ma_y$ to the helicopter to obtain:

$$m_h g + T - F_{\text{lift}} = m_h a_y$$

Solving for F_{lift} yields:

$$F_{\text{lift}} = m_h g + T - m_h a_y = m_h(g - a_y) + T$$

Substitute for T from Part (a):

$$F_{\text{lift}} = m_{\text{h}}(g - a_y) + m_{\text{t}}(g - a_y) = (m_{\text{t}} + m_{\text{h}})(g - a_y)$$

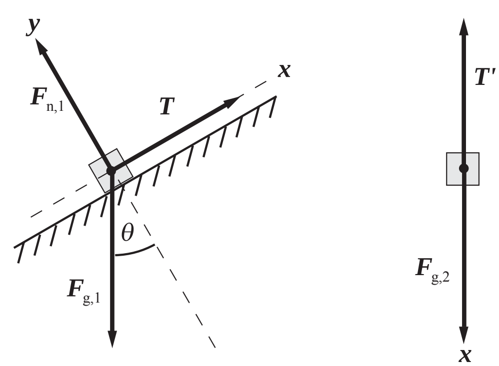
Substitute numerical values and evaluate F_{lift}:

$$F_{\text{lift}} = (4000\text{ kg} + 15000\text{ kg})(9.81\text{ m/s}^2 + 2.00\text{ m/s}^2) = \boxed{224\text{ kN}}$$

75 •• Two objects are connected by a massless string, as shown in Figure 4-54. The incline and the massless pulley are frictionless. Find the acceleration of the objects and the tension in the string for (a) in terms of θ, m_1 and m_2, and (b) $\theta = 30°$ and $m_1 = m_2 = 5.0$ kg.

Picture the Problem Because the string does not stretch or become slack, the two objects must have the same speed and therefore the magnitude of the acceleration is the same for each object. Choose a coordinate system in which up the incline is the $+x$ direction for the object of mass m_1 and downward is the $+x$ direction for the object of mass m_2. This idealized pulley acts like a piece of polished pipe; i.e., its only function is to change the direction the tension in the massless string acts. Draw a free-body diagram for each of the two objects, apply Newton's second law to both objects, and solve the resulting equations simultaneously.

(a) The free-body diagrams for the two objects are shown to the right:



Apply $\sum F_x = ma_x$ to the object whose mass is m_1:

$$T - F_{g,1}\sin\theta = m_1 a_{1x}$$

or, because $F_{g,1} = m_1 g$ and the two objects have a common acceleration a_x,

$$T - m_1 g\sin\theta = m_1 a_x \qquad (1)$$

Apply $\sum F_x = ma_x$ to the object whose mass is m_2 to obtain:

$$m_2 g - T = m_2 a_x \qquad (2)$$

Adding equations (1) and (2) yields:

$$m_2 g - m_1 g\sin\theta = m_1 a_x + m_2 a_x$$

Solve for a_x to obtain:

$$a_x = \boxed{\frac{g(m_2 - m_1\sin\theta)}{m_1 + m_2}} \qquad (3)$$

Substitute for a_x in either of the equations containing the tension and solve for T to obtain:

$$T = \boxed{\frac{gm_1m_2(1+\sin\theta)}{m_1+m_2}} \qquad (4)$$

(*b*) Substitute numerical values in equation (3) and evaluate a_x:

$$a_x = \frac{(9.81\text{ m/s}^2)(5.0\text{ kg} - (5.0\text{ kg})\sin 30°)}{5.0\text{ kg} + 5.0\text{ kg}} = \boxed{2.5\text{ m/s}^2}$$

Substitute numerical values in equation (4) and evaluate T:

$$T = \frac{(9.81\text{ m/s}^2)(5.0\text{ kg})^2(1+\sin 30°)}{5.0\text{ kg} + 5.0\text{ kg}} = \boxed{37\text{ N}}$$

76 •• During a stage production of *Peter Pan*, the 50-kg actress playing Peter has to fly in vertically (descend). To be in time with the music, she must be lowered, starting from rest, a distance of 3.2 m in 2.2 s at constant acceleration. Backstage, a smooth surface sloped at 50º supports a counterweight of mass m, as shown in Figure 4-55. Show the calculations that the stage manager must perform to find (*a*) the mass of the counterweight that must be used and (*b*) the tension in the wire.

Picture the Problem The magnitudes of the accelerations of Peter and the counterweight are the same. Choose a coordinate system in which up the incline is the +x direction for the counterweight and downward is the +x direction for Peter. The pulley changes the direction the tension in the rope acts. Let Peter's mass be m_P. Ignoring the mass of the rope, draw free-body diagrams for the counterweight and Peter, apply Newton's second law to each of them, and solve the resulting equations simultaneously.

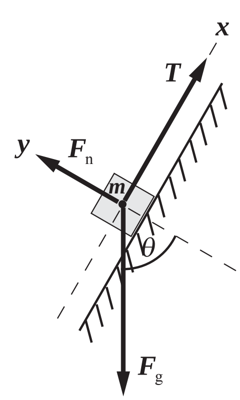
(*a*) Using a constant-acceleration equation, relate Peter's displacement to her acceleration and descent time:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_x(\Delta t)^2 \Rightarrow a_x = \frac{2\Delta x}{(\Delta t)^2}$$

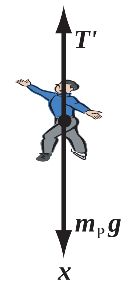
The free-body diagram for the counterweight is shown to the right:



Apply $\sum F_x = ma_x$ to the counterweight:

$$T - mg\sin 50° = ma_x$$

The free-body diagram for Peter is shown to the right:



Noting that $T' = T$, apply $\sum F_x = ma_x$ to Peter:

$$m_P g - T = m_P a_x$$

Adding the two equations and solving for m yields:

$$m = \frac{m_P(g - a_x)}{a_x + g\sin 50°}$$

Substituting for a_x yields:

$$m = \frac{m_P\left(g - \dfrac{2\Delta x}{(\Delta t)^2}\right)}{\dfrac{2\Delta x}{(\Delta t)^2} + g\sin 50°}$$

Substitute numerical values and evaluate m:

$$m = \frac{(50\text{ kg})\left(9.81\text{ m/s}^2 - \dfrac{2(3.2\text{ m})}{(2.2\text{ s})^2}\right)}{\dfrac{2(3.2\text{ m})}{(2.2\text{ s})^2} + 9.81\text{ m/s}^2\sin 50°}$$
$$= 48.0\text{ kg} = \boxed{48\text{ kg}}$$

(*b*) Substitute for m in the force equation for the counterweight and solve for T to obtain:

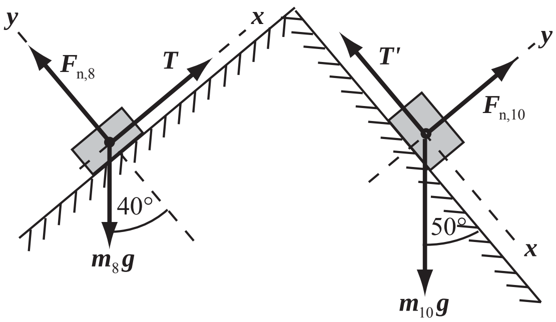
$$T = m(a_x + g\sin 50°)$$

Substitute numerical values and evaluate T:

$$T = (48.0\text{kg})\left[1.32\text{ m/s}^2 + (9.81\text{ m/s}^2)\sin 50°\right] = \boxed{0.42\text{ kN}}$$

77 •• An 8.0-kg block and a 10-kg block, connected by a rope that passes over a frictionless peg, slide on frictionless inclines (Figure 4-56). (*a*) Find the acceleration of the blocks and the tension in the rope. (*b*) The two blocks are replaced by two others of masses m_1 and m_2 such that there is no acceleration. Find whatever information you can about the masses of these two new blocks.

Picture the Problem The magnitude of the accelerations of the two blocks are the same. Choose a coordinate system in which up the incline is the $+x$ direction for the 8.0-kg object and downward is the $+x$ direction for the 10-kg object. The peg changes the direction the tension in the rope acts. Draw free-body diagrams for each object, apply Newton's second law of motion to both of them, and solve the resulting equations simultaneously.



(*a*) Apply $\sum F_x = ma_x$ to the 8.0-kg block:

$$T - m_8 g\sin 40° = m_8 a_x \qquad (1)$$

Noting that $T = T'$, apply $\sum F_x = ma_x$ to the 10-kg block:

$$m_{10} g\sin 50° - T = m_{10} a_x \qquad (2)$$

Add equations (1) and (2) to obtain:

$$m_{10} g\sin 50° - m_8 g\sin 40° = (m_{10} + m_8)a_x$$

Solving for a_x yields:

$$a_x = \frac{g(m_{10}\sin 50° - m_8\sin 40°)}{m_8 + m_{10}}$$

Substitute numerical values and evaluate a_x:

$$a_x = \frac{(9.81\text{ m/s}^2)((10\text{ kg})\sin 50° - (8.0\text{ kg})\sin 40°)}{8.0\text{ kg} + 10\text{ kg}} = 1.37\text{ m/s}^2 = \boxed{1.4\text{ m/s}^2}$$

Solving the first of the two force equations for T yields:

$$T = m_8(g\sin 40° + a_x)$$

Substitute numerical values and evaluate T:

$$T = (8.0\text{ kg})\left[(9.81\text{ m/s}^2)\sin 40° + 1.37\text{ m/s}^2\right] = \boxed{61\text{ N}}$$

(*b*) Because the system is in equilibrium, $a_x = 0$, and equations (1) and (2) become:

$$T - m_1 g\sin 40° = 0 \qquad (3)$$

and

$$m_2 g\sin 50° - T = 0 \qquad (4)$$

Adding equations (3) and (4) yields:

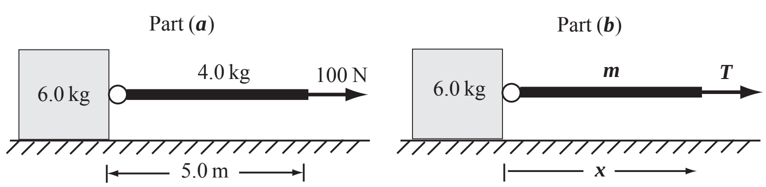
$$m_2 g\sin 50° - m_1 g\sin 40° = 0$$

Solve for and evaluate the ratio m_1/m_2 to obtain:

$$\frac{m_1}{m_2} = \frac{\sin 50°}{\sin 40°} = \boxed{1.19}$$

78 •• A heavy rope of length 5.0 m and mass 4.0 kg lies on a frictionless horizontal table. One end is attached to a 6.0-kg block. The other end of the rope is pulled by a constant horizontal 100-N force. (*a*) What is the acceleration of the system? (*b*) Give the tension in the rope as a function of position along the rope.

Picture the Problem The pictorial representations shown below summarize the information given in this problem. While the mass of the rope is distributed over its length, the rope and the 6.0-kg block have a common acceleration. Choose a coordinate system in which the direction of the 100-N force is the +*x* direction. Because the surface is horizontal and frictionless, the only force that influences our solution is the 100-N force.



(*a*) Apply $\sum F_x = ma_x$ to the system shown for Part (*a*):

$$100\text{ N} = (m_1 + m_2)a_x$$

Solve for a_x to obtain:

$$a_x = \frac{100\text{ N}}{m_1 + m_2}$$

Substitute numerical values and evaluate a_x:

$$a_x = \frac{100\text{ N}}{4.0\text{ kg} + 6.0\text{ kg}} = \boxed{10\text{ m/s}^2}$$

(*b*) Let *m* represent the mass of a length *x* of the rope. Assuming that the mass of the rope is uniformly distributed along its length:

$$\frac{m}{x} = \frac{m_2}{L_{\text{rope}}} = \frac{4.0\text{kg}}{5.0\text{m}} \Rightarrow m = \left(\frac{4.0\text{kg}}{5.0\text{m}}\right)x$$

Apply $\sum F_x = ma_x$ to the block in Part (*b*) to obtain:

$$T = (m_1 + m)a_x$$

Substituting for *m* yields:

$$T = \left(m_1 + \left(\frac{4.0\text{kg}}{5.0\text{m}}\right)x\right)a_x$$

Substitute numerical values and evaluate T:

$$T = \left[6.0\,\text{kg} + \left(\frac{4.0\,\text{kg}}{5.0\,\text{m}}\right)x\right]\left(10\,\text{m/s}^2\right)$$
$$= \boxed{60\,\text{N} + (8.0\,\text{N/m})x}$$

79 •• **[SSM]** A 60-kg housepainter stands on a 15-kg aluminum platform. The platform is attached to a rope that passes through an overhead pulley, which allows the painter to raise herself and the platform (Figure 4-57). (*a*) To accelerate herself and the platform at a rate of 0.80 m/s^2, with what force F must she pull down on the rope? (*b*) When her speed reaches 1.0 m/s, she pulls in such a way that she and the platform go up at a constant speed. What force is she exerting on the rope now? (Ignore the mass of the rope.)

Picture the Problem Choose a coordinate system in which the upward direction is the positive y direction. Note that $\vec{F}$ is the force exerted by the painter on the rope and that $\vec{T}$ is the resulting tension in the rope. Hence the net upward force on the painter-plus-platform is $2\vec{T}$.

(*a*) Letting $m_{\text{tot}} = m_{\text{frame}} + m_{\text{painter}}$, apply $\sum F_y = ma_y$ to the frame-plus-painter:

$$2T - m_{\text{tot}}g = m_{\text{tot}}a_y$$

Solving for T gives:

$$T = \frac{m_{\text{tot}}\left(a_y + g\right)}{2}$$

Substitute numerical values and evaluate T:

$$T = \frac{(75\,\text{kg})\left(0.80\,\text{m/s}^2 + 9.81\,\text{m/s}^2\right)}{2}$$
$$= 398\,\text{N}$$

Because $F = T$:

$$F = 398\text{ N} = \boxed{0.40\,\text{kN}}$$

(*b*) Apply $\sum F_y = 0$ to obtain:

$$2T - m_{\text{tot}}g = 0 \Rightarrow T = \tfrac{1}{2}m_{\text{tot}}g$$

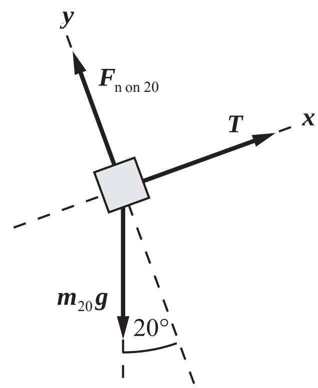
Substitute numerical values and evaluate T:

$$T = \tfrac{1}{2}(75\,\text{kg})\left(9.81\,\text{m/s}^2\right) = \boxed{0.37\,\text{kN}}$$

80 ••• Figure 4-58 shows a 20-kg block sliding on a 10-kg block. All surfaces are frictionless and the pulley is massless and frictionless. Find the acceleration of each block and the tension in the string that connects the blocks.

Picture the Problem Choose a coordinate system in which up the incline is the $+x$ direction and draw free-body diagrams for each block. Noting that $\vec{a}_{20} = -\vec{a}_{10}$, apply Newton's second law to each block and solve the resulting equations simultaneously.

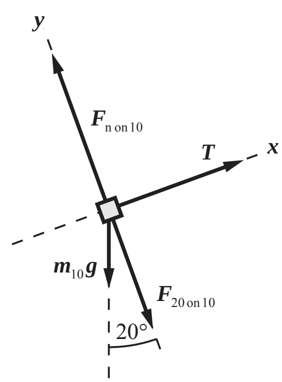
Draw a free-body diagram for the 20-kg block:



Apply $\sum F_x = ma_x$ to the block to obtain:

$$T - m_{20}g\sin 20° = m_{20}a_{20,x} \qquad (1)$$

Draw a free-body diagram for the 10-kg block. Because all the surfaces, including the surfaces between the blocks, are frictionless, the force the 20-kg block exerts on the 10-kg block must be normal to their surfaces as shown to the right.



Apply $\sum F_x = ma_x$ to the block to obtain:

$$T - m_{10}g\sin 20° = m_{10}a_{10,x} \qquad (2)$$

Because the blocks are connected by a taut string:

$$a_{20,x} = -a_{10,x}$$

Substituting for $a_{20,x}$ in equation (1) yields:

$$T - m_{20}g\sin 20° = -m_{20}a_{10,x} \qquad (3)$$

Eliminate T between equations (2) and (3) to obtain:

$$a_{10,x} = \left(\frac{m_{20} - m_{10}}{m_{20} + m_{10}}\right)g\sin 20°$$

Substitute numerical values and evaluate $a_{10,x}$:

$$a_{10,x} = \left(\frac{20\text{ kg} - 10\text{ kg}}{20\text{ kg} + 10\text{ kg}}\right)\left(9.81\text{ m/s}^2\right)\sin 20° = \boxed{1.1\text{ m/s}^2}$$

Because $a_{20,x} = -a_{10,x}$:

$$a_{20,x} = \boxed{-1.1\text{ m/s}^2}$$

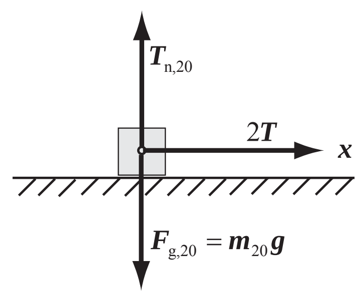
Substitute for either of the accelerations in the force equations and solve for T:

$$T = \boxed{45\text{ N}}$$

81 ••• A 20-kg block with a pulley attached slides along a frictionless ledge. It is connected by a massless string to a 5.0-kg block via the arrangement shown in Figure 4-59. Find (*a*) the acceleration of each block and (*b*) the tension in the connecting string.

Picture the Problem Choose a coordinate system in which the +*x* direction is to the right and draw free-body diagrams for each block. Because of the pulley, the string exerts a force of 2*T*. Apply Newton's second law of motion to both blocks and solve the resulting equations simultaneously.

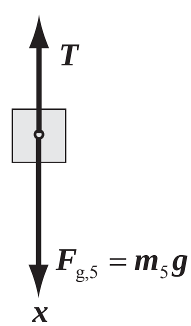
(*a*) Draw a free-body diagram for the 20-kg block:



Apply $\sum F_x = ma_x$ to the block to obtain:

$$2T = m_{20}a_{20,x}$$

Draw a free-body diagram for the 5.0-kg block:



Apply $\sum F_x = ma_x$ to the block to obtain:

$$m_5 g - T = m_5 a_{5,x}$$

Using a constant-acceleration equation, relate the displacement of the 5.0-kg block to its acceleration and the time during which it is accelerated:

$$\Delta x_5 = \tfrac{1}{2} a_{5,x} (\Delta t)^2$$

Using a constant-acceleration equation, relate the displacement of the 20-kg block to its acceleration and acceleration time:

$$\Delta x_{20} = \tfrac{1}{2} a_{20,x} (\Delta t)^2$$

Divide the first of these equations by the second to obtain:

$$\frac{\Delta x_5}{\Delta x_{20}} = \frac{\tfrac{1}{2} a_{5,x} (\Delta t)^2}{\tfrac{1}{2} a_{20,x} (\Delta t)^2} = \frac{a_{5,x}}{a_{20,x}}$$

Use the result of Part (*a*) to obtain:

$$a_{5,x} = 2a_{20,x}$$

Let $a_{20,x} = a$. Then $a_{5,x} = 2a$ and the force equations become:

$$2T = m_{20} a$$

and

$$m_5 g - T = m_5 (2a)$$

Eliminate T between the two equations to obtain:

$$a = a_{20,x} = \frac{m_5 g}{2m_5 + \tfrac{1}{2} m_{20}}$$

Substitute numerical values and evaluate $a_{20,x}$ and $a_{5,x}$:

$$a_{20,x} = \frac{(5.0\,\text{kg})(9.81\,\text{m/s}^2)}{2(5.0\,\text{kg}) + \tfrac{1}{2}(20\,\text{kg})} = 2.45\,\text{m/s}^2$$

$$= \boxed{2.5\,\text{m/s}^2}$$

and

$$a_5 = 2(2.45\,\text{m/s}^2) = \boxed{4.9\,\text{m/s}^2}$$

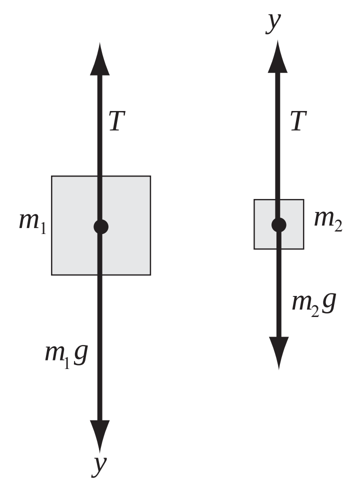
Substitute for either of the accelerations in either of the force equations and solve for T:

$$T = \boxed{25\,\text{N}}$$

82 •• The apparatus in Figure 4-60 is called an *Atwood's machine* and is used to measure the free-fall acceleration g by measuring the acceleration of the two blocks connected by a string over a pulley. Assume a massless, frictionless pulley and a massless string. (*a*) Draw a free-body diagram of each block. (*b*) Use the free-body diagrams and Newton's laws to show that the magnitude of the acceleration of either block and the tension in the string are $a = (m_1 - m_2)g/(m_1 + m_2)$ and $T = 2m_1 m_2 g/(m_1 + m_2)$. (*c*) Do these expressions give plausible results if $m_1 = m_2$, in the limit that $m_1 >> m_2$, and in the limit that $m_1 << m_2$? Explain.

Picture the Problem Assume that $m_1 > m_2$. Choose a coordinate system in which the $+y$ direction is downward for the block whose mass is m_1 and upward for the block whose mass is m_2 and draw free-body diagrams for each block. Apply Newton's second law to both blocks and solve the resulting equations simultaneously.

(*a*) The free-body diagrams for the two blocks are shown to the right:



(*b*) Apply $\sum F_y = ma_y$ to the block whose mass is m_2:

$$T - m_2 g = m_2 a_{2y} \quad (1)$$

Apply $\sum F_y = ma_y$ to the block whose mass is m_1:

$$m_1 g - T = m_1 a_{1y} \quad (2)$$

Because the blocks are connected by a taut string, they have the same acceleration. Let a represent their common acceleration. Then equations (1) and (2) become:

$$T - m_2 g = m_2 a$$
and
$$m_1 g - T = m_1 a$$

Adding these equations eliminates T and yields:

$$m_1 g - m_2 g = m_1 a + m_2 a$$

Solve for a to obtain:

$$a = \boxed{\frac{m_1 - m_2}{m_1 + m_2} g} \quad (3)$$

Substituting for a in either of the force equations and solving for T yields:

$$T = \boxed{\frac{2m_1 m_2 g}{m_1 + m_2}} \quad (4)$$

(*c*) For $m_1 = m_2$, equations (3) and (4) become:

$a = \dfrac{0}{m_1 + m_2} g = 0$ and $T = mg$ … as expected.

Divide the numerators and denominators if equations (3) and (4) by m_1 to obtain:

$$a = \left(\frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\right) g \text{ and } T = \left(\frac{2m_2}{1 + \frac{m_2}{m_1}}\right) g$$

For $m_1 >> m_2$ these equations become:

$a = g$ and $T = 2m_2 g$ … as expected.

Divide the numerators and denominators if equations (3) and (4) by m_2 to obtain:

$$a = \left(\frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1}\right) g \text{ and } T = \frac{2m_1 g}{\frac{m_1}{m_2} + 1}$$

For $m_1 << m_2$ these equations become:

$a = -g$ and $T = 2m_1 g$ … as expected.

83 •• If one of the masses of the Atwood's machine in Figure 4-60 is 1.2 kg, what should be the other mass so that the displacement of either mass during the first second following release is 0.30 m? Assume a massless, frictionless pulley and a massless string.

Picture the Problem The acceleration can be found from the given displacement during the first second. The ratio of the two masses can then be found from the acceleration using the first of the two equations derived in Problem 82 relating the acceleration of the Atwood's machine to its masses.

From Problem 82 we have:

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

Solving for m_1 yields:

$$m_1 = m_2 \left(\frac{g + a}{g - a}\right) \quad (1)$$

Using a constant-acceleration equation, relate the displacement of the masses to their acceleration and solve for the acceleration:

$$\Delta y = v_{0y} t + \tfrac{1}{2} a_y (\Delta t)^2$$

or, because $v_{0y} = 0$,

$$\Delta y = \tfrac{1}{2} a_y (\Delta t)^2 \Rightarrow a_y = \frac{2\Delta y}{(\Delta t)^2}$$

Substitute numerical values and evaluate a:

$$a_y = \frac{2(0.30\,\text{m})}{(1.0\,\text{s})^2} = 0.600\,\text{m/s}^2$$

Substitute numerical values in equation (1) to obtain:

$$m_1 = m_2\left(\frac{9.81\text{ m/s}^2 + 0.600\text{ m/s}^2}{9.81\text{ m/s}^2 - 0.600\text{ m/s}^2}\right) = 1.13m_2$$

Find the second value for m_2 for m_1 = 1.2 kg:

$$m_2 = \boxed{1.4\text{ kg or }1.1\text{ kg}}$$

84 ••• The acceleration of gravity g can be determined by measuring the time t it takes for a mass m_2 in an Atwood's machine described in Problem 82 to fall a distance L, starting from rest. (*a*) Using the results of Problem 82 (Note the acceleration is constant.), find an expression for g in terms of m_1, m_2, L, and t. (*b*) Show that a small error in the time measurement dt, will lead to an error in g by an amount dg given by $dg/g = -2dt/t$. (*c*) Assume that the only significant uncertainty in the experimental measurements is the time of fall. If L = 3.00 m and m_1 is 1.00 kg, find the value of m_2 such that g can be measured with an accuracy of ±5% with a time measurement that is accurate to ±0.1 s.

Picture the Problem Use a constant-acceleration equation to relate the displacement of the descending (or rising) mass as a function of its acceleration and then use one of the results from Problem 82 to relate a to g. Differentiation of our expression for g will allow us to relate uncertainty in the time measurement to uncertainty in the measured value for g … and to the values of m_2 that would yield an experimental value for g that is good to within ±5%.

(*a*) From Problem 82 we have:

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

Solving for g yields:

$$g = a\left(\frac{m_1 + m_2}{m_1 - m_2}\right) \qquad (1)$$

Using a constant-acceleration equation, express the displacement, L, as a function of t:

$$\Delta y = L = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y} = 0$ and $\Delta t = t$,

$$L = \tfrac{1}{2}a_y t^2 \Rightarrow a_y = \frac{2L}{t^2} \qquad (2)$$

Substitute for a_y in equation (1) to obtain:

$$g = \boxed{\frac{2L}{t^2}\left(\frac{m_1 + m_2}{m_1 - m_2}\right)}$$

(*b*) Evaluate *dg*/*dt* to obtain:

$$\frac{dg}{dt} = -4Lt^{-3}\left(\frac{m_1 + m_2}{m_1 - m_2}\right) = \frac{-2}{t}\left[\frac{2L}{t^2}\right]\left(\frac{m_1 + m_2}{m_1 - m_2}\right) = \frac{-2g}{t}$$

Divide both sides of this expression by *g* and multiply both sides by *dt* to separate the variables:

$$\frac{dg}{g} = \boxed{-2\frac{dt}{t}}$$

(*c*) Because $\frac{dg}{g} = \pm 0.05$:

$$t = \left|\frac{-2dt}{dg/g}\right| = \left|\frac{-2(\pm 0.1\,\text{s})}{\pm 0.05}\right| = 4\,\text{s}$$

Substitute numerical values in equation (2) and evaluate *a*:

$$a_y = \frac{2(3.00\,\text{m})}{(4\,\text{s})^2} = 0.375\,\text{m/s}^2$$

Solve equation (1) for m_2 to obtain:

$$m_2 = \frac{g - a}{g + a}m_1$$

Evaluate m_2 with m_1 = 1.00 kg:

$$m_2 = \frac{9.81\,\text{m/s}^2 - 0.375\,\text{m/s}^2}{9.81\,\text{m/s}^2 + 0.375\,\text{m/s}^2}(1.00\,\text{kg}) \approx 0.9\,\text{kg}$$

Solve equation (1) for m_1 to obtain:

$$m_1 = m_2\left(\frac{g + a_y}{g - a_y}\right)$$

Substitute numerical values to obtain:

$$m_1 = (0.926\,\text{kg})\left(\frac{9.81\,\text{m/s}^2 + 0.375\,\text{m/s}^2}{9.81\,\text{m/s}^2 - 0.375\,\text{m/s}^2}\right) \approx 1\,\text{kg}$$

Because the masses are interchangeable:

$$m_2 = \boxed{0.9\,\text{kg or }1\,\text{kg}}$$

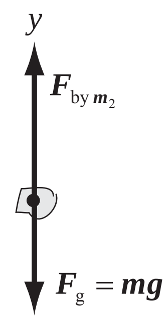
General Problems

85 •• A pebble of mass *m* rests on the block of mass m_2 of the ideal Atwood's machine in Figure 4-60. Find the force exerted by the pebble on the block of mass m_2.

Picture the Problem We can apply Newton's second law to the pebble and use the expression for the acceleration derived in Problem 82 to find the force $\vec{F}_{\text{by}\,m_2}$

exerted by the block of mass m_2 on the pebble. The forces $\vec{\boldsymbol{F}}_{\text{on}\,\boldsymbol{m}_2}$ exerted by the pebble on the block of mass m_2 and $\vec{\boldsymbol{F}}_{\text{by}\,\boldsymbol{m}_2}$ exerted by the block on the pebble constitute a third law pair and are equal in magnitude.

(*a*) The upward direction has been chosen as the $+y$ direction in the free-body diagram for the pebble shown to the right. The forces acting on the pebble are the force exerted by the object whose mass is m_2 ($\vec{\boldsymbol{F}}_{\text{by}\ m_2}$) and the gravitational force ($\vec{\boldsymbol{F}}_{\text{g}} = m\vec{\boldsymbol{g}}$) exerted by Earth.



(*b*) Apply $\sum F_y = ma_y$ to the pebble to obtain:

$$F_{\text{by}\,m_2} - mg = ma \Rightarrow F_{\text{by}\,m_2} = m(a+g)$$

Because $\vec{\boldsymbol{F}}_{\text{by}\,\boldsymbol{m}_2}$ and $\vec{\boldsymbol{F}}_{\text{on}\ m_2}$ constitute a third law pair:

$$F_{\text{on}\,m_2} = m(a+g) \qquad (1)$$

From Problem 82, the acceleration of the blocks is given by:

$$a = \frac{m_1 - m_2}{m_1 + m_2}g$$

With a pebble of mass m resting on the block of mass m_2, the expression for the acceleration becomes:

$$a = \frac{m_1 - (m+m_2)}{m_1 + (m+m_2)}g$$

Substituting for a in equation (1) and simplifying yields:

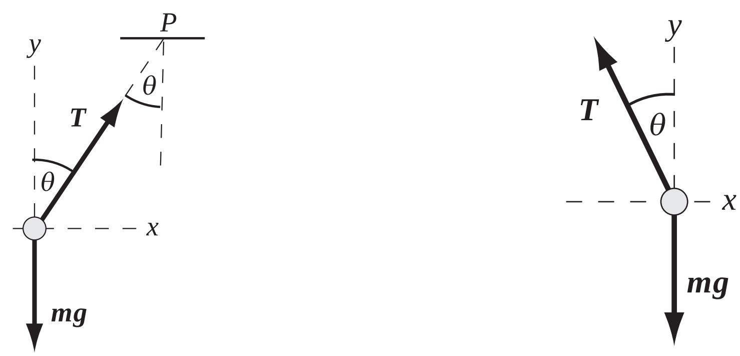
$$F_{\text{on}\,m_2} = m\left[\frac{m_1 - (m+m_2)}{m_1 + (m+m_2)}g + g\right] = mg\left[\frac{m_1 - m - m_2}{m_1 + m + m_2} + 1\right]$$

$$= \boxed{\left(\frac{2mm_1}{m + m_1 + m_2}\right)g}$$

86 •• A simple accelerometer can be made by suspending a small massive object from a string attached to a fixed point on an accelerating object. Suppose such an accelerometer is attached to point P on the ceiling of an automobile traveling in a straight line on a flat surface at constant acceleration. Due to the acceleration, the string will make an angle θ with the vertical. (*a*) Show that the magnitude of the acceleration a is related to the angle θ by $a = g \tan \theta$. (*b*) Suppose the automobile brakes steadily to rest from 50 km/h over a distance of 60 m. What angle will the string make with the vertical? Will the suspended

object be positioned below and ahead or below and behind point P during the braking?

Picture the Problem The free-body diagram shown below and to the left shows the forces acting on an object suspended from the ceiling of a car that is gaining speed to the right. Choose the coordinate system shown and use Newton's second law and constant- acceleration equations to describe the influence of the forces acting on the suspended object on its motion. The forces acting on the object are the gravitational force $m\vec{g}$ and the tension in the string $\vec{T}$. The free-body diagram shown to the right shows the forces acting on an object suspended from the ceiling of a car that is braking while moving to the right.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the object:

$$\sum F_x = T\sin\theta = ma_x$$

and

$$\sum T_y = T\cos\theta - mg = 0$$

Take the ratio of these two equations to eliminate T and m:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{ma_x}{mg}$$

or

$$\tan\theta = \frac{a_x}{g} \Rightarrow a_x = \boxed{g\tan\theta}$$

(*b*) Solve the equation derived in (*a*) for θ:

$$\theta = \tan^{-1}\left(\frac{a_x}{g}\right) \qquad (2)$$

Using a constant-acceleration equation, express the velocity of the car in terms of its acceleration and solve for the acceleration:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 + 2a_x\Delta x \Rightarrow a_x = \frac{-v_{0x}^2}{2\Delta x}$$

Substitute for a_x in equation (2) and simplify to obtain:

$$\theta = \tan^{-1}\left(\frac{\dfrac{-v_{0x}^2}{2\Delta x}}{g}\right) = \tan^{-1}\left(\frac{-v_{0x}^2}{2g\Delta x}\right)$$

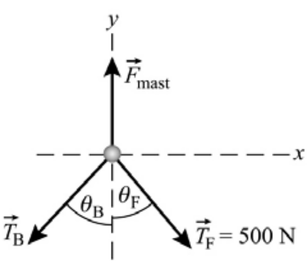
Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{-\left(50\,\frac{\text{km}}{\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{2\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(60\,\text{m})}\right)$$
$$= \boxed{9.3°}$$

The suspended object will be positioned below and ahead (by 9.3°) of point P during the braking.

87 •• **[SSM]** The mast of a sailboat is supported at bow and stern by stainless steel wires, the forestay and backstay, anchored 10 m apart (Figure 4-61). The 12.0-m-long mast weighs 800 N and stands vertically on the deck of the boat. The mast is positioned 3.60 m behind where the forestay is attached. The tension in the forestay is 500 N. Find the tension in the backstay and the force that the mast exerts on the deck.

Picture the Problem The free-body diagram shows the forces acting at the top of the mast. Choose the coordinate system shown and use Newton's second and third laws of motion to analyze the forces acting on the deck of the sailboat.



Apply $\sum F_x = ma_x$ to the top of the mast:

$$T_\text{F}\sin\theta_\text{F} - T_\text{B}\sin\theta_\text{B} = 0$$

Find the angles that the forestay and backstay make with the vertical:

$$\theta_\text{F} = \tan^{-1}\left(\frac{3.60\,\text{m}}{12.0\,\text{m}}\right) = 16.7°$$

and

$$\theta_\text{B} = \tan^{-1}\left(\frac{6.40\,\text{m}}{12.0\,\text{m}}\right) = 28.1°$$

Solving the x-direction equation for T_B yields:

$$T_\text{B} = T_\text{F}\frac{\sin\theta_\text{F}}{\sin\theta_\text{B}}$$

Substitute numerical values and evaluate T_B:

$$T_B = (500\,\text{N})\frac{\sin 16.7°}{\sin 28.1°} = \boxed{305\,\text{N}}$$

Apply $\sum F_y = 0$ to the mast:

$$\sum F_y = F_{\text{mast}} - T_F \cos\theta_F - T_B \cos\theta_B = 0$$

Solve for F_{mast} to obtain:

$$F_{\text{mast}} = T_F \cos\theta_F + T_B \cos\theta_B$$

Substitute numerical values and evaluate F_{mast}:

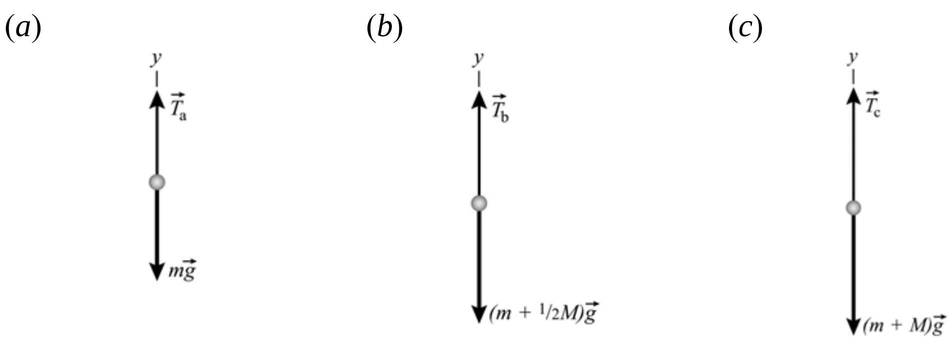
$$F_{\text{mast}} = (500\,\text{N})\cos 16.7° + (305\,\text{N})\cos 28.1° = 748\,\text{N}$$

The force that the mast exerts on the deck is the sum of its weight and the downward forces exerted on it by the forestay and backstay:

$$F_{\substack{\text{mast on}\\\text{the deck}}} = 748\,\text{N} + 800\,\text{N} = \boxed{1.55\,\text{kN}}$$

88 •• A 50-kg block is suspended from a uniform chain that is hanging from the ceiling. The mass of the chain itself is 20 kg, and the length of the chain is 1.5 m. Determine the tension in the chain (*a*) at the point where the chain is attached to the block, (*b*) midway up the chain, and (*c*) at the point where the chain is attached to the ceiling.

Picture the Problem Let m be the mass of the block and M be the mass of the chain. The free-body diagrams shown below display the forces acting at the locations identified in the problem. We can apply Newton's second law with $a_y = 0$ to each of the segments of the chain to determine the tensions.



(*a*) Apply $\sum F_y = ma_y$ to the block and solve for T_a:

$$T_a - mg = ma_y$$

or, because $a_y = 0$,

$$T_a = mg$$

Substitute numerical values and evaluate T_a:

$$T_a = (50\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{0.49\,\text{kN}}$$

(*b*) Apply $\sum F_y = ma_y$ to the block and half the chain and solve for T_b:

$$T_b - \left(m + \frac{M}{2}\right)g = ma_y$$

or, because $a_y = 0$,

$$T_b = \left(m + \frac{M}{2}\right)g$$

Substitute numerical values and evaluate T_b:

$$T_b = (50\,\text{kg} + 10\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{0.59\,\text{kN}}$$

(*c*) Apply $\sum F_y = ma_y$ to the block and chain and solve for T_c:

$$T_c - (m + M)g = ma_y$$

or, because $a_y = 0$,

$$T_c = (m + M)g$$

Substitute numerical values and evaluate T_c:

$$T_c = (50\,\text{kg} + 20\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{0.69\,\text{kN}}$$

89 •• The speed of the head of a redheaded woodpecker reaches 5.5 m/s before impact with the tree. If the mass of the head is 0.060 kg and the average force on the head during impact is 0.060 N, find (*a*) the acceleration of the head (assuming constant acceleration), (*b*) the depth of penetration into the tree, and (*c*) the time it takes for the head to come to a stop.

Picture the Problem (*a*) Choose a coordinate system in which the +*x* direction is the direction in which the woodpecker's head is moving and apply Newton's second law to the woodpecker's head to find its acceleration. (*b*) Because we've assumed constant acceleration, we can use a constant-acceleration equation to find the depth of penetration in the tree. In Part (*c*), we can the constant-acceleration equation $v_x = v_{0x} + a_x\Delta t$ to find the time it takes for the head to come to a stop.

(*a*) Apply $\sum F_x = ma_x$ to the woodpecker's head to obtain:

$$F_{\text{on the head}} = ma_x \Rightarrow a_x = \frac{F_{\text{on the head}}}{m}$$

Substitute numerical values and evaluate a_x:

$$a_x = \frac{-6.0\,\text{N}}{0.060\,\text{kg}} = \boxed{-0.10\,\text{km/s}^2}$$

(*b*) Use a constant-acceleration equation to relate the depth of penetration to the initial speed of the head and its acceleration:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x \Rightarrow \Delta x = \frac{v_x^2 - v_{0x}^2}{2a_x}$$

Because $v_x = 0$:

$$\Delta x = \frac{-v_{0x}^2}{2a_x}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{-(5.5\ \text{m/s})^2}{2(-0.10\ \text{km/s}^2)} = \boxed{15\ \text{cm}}$$

(*c*) Apply a constant-acceleration equation to relate the stopping time of the head to its initial and final speeds and to its acceleration:

$$v_x = v_{0x} + a_x\Delta t \Rightarrow \Delta t = \frac{v_x - v_{0x}}{a_x}$$

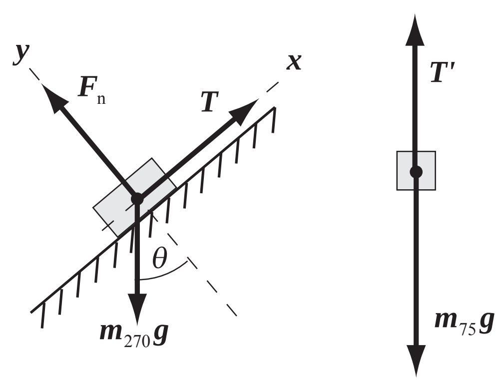
Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{0 - 5.5\ \text{m/s}}{-0.10\ \text{km/s}^2} = \boxed{55\ \text{ms}}$$

90 •• A frictionless surface is inclined at an angle of 30.0º to the horizontal. A 270-g block on the ramp is attached to a 75.0-g block using a pulley, as shown in Figure 4-62. (*a*) Draw two free-body diagrams, one for the 270-g block and the other for the 75.0-g block. (*b*) Find the tension in the string and the acceleration of the 270-g block. (*c*) The 270-g block is released from rest. How long does it take for it to slide a distance of 1.00 m along the surface? Will it slide up the incline, or down the incline?

Picture the Problem The application of Newton's second law to the block and the hanging weight will lead to simultaneous equations in their common acceleration *a* and the tension *T* in the cord that connects them. Once we know the acceleration
of this system, we can use a constant-acceleration equation to predict how long it takes the block to travel 1.00 m from rest. Note that the magnitudes of $\vec{T}$ and $\vec{T}'$ are equal.

(*a*) The free-body diagrams are shown to the right. m_{270} represents the mass of the 270-g block and m_{75} the mass of the 75.0-g block.



(*b*) Apply $\sum F_x = ma_x$ to the block and the suspended mass:

$$T - m_{270}g\sin\theta = m_{270}a_{1x}$$
and
$$m_{75}g - T = m_{75}a_{2x}$$

Letting a represent the common acceleration of the two objects, eliminate T between the two equations and solve a:

$$a = \frac{m_{75} - m_{270}\sin\theta}{m_{270} + m_{75}} g$$

Substitute numerical values and evaluate a:

$$a = \frac{0.0750\,\text{kg} - (0.270\,\text{kg})\sin 30°}{0.0750\,\text{kg} + 0.270\,\text{kg}}\left(9.81\,\text{m/s}^2\right) = -1.706\,\text{m/s}^2 = \boxed{-1.71\,\text{m/s}^2}$$

where the minus sign indicates that the acceleration is down the incline.

Substitute for a in either of the force equations to obtain:

$$T = \boxed{0.864\,\text{N}}$$

(c) Using a constant-acceleration equation, relate the displacement of the block down the incline to its initial speed and acceleration:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_x(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta x}{a_x}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{2(-1.00\,\text{m})}{-1.706\,\text{m/s}^2}} = \boxed{1.08\,\text{s}}$$

Because the block is released from rest and its acceleration is negative, it will slide down the incline.

91 •• A box of mass m_1 is pulled along a frictionless horizontal surface by a horizontal force F that is applied to the end of a rope of mass m_2 (see Figure 4-63). Neglect any sag of the rope. (a) Find the acceleration of the rope and block, assuming them to be one object. (b) What is the net force acting on the rope? (c) Find the tension in the rope at the point where it is attached to the block.

Picture the Problem Note that, while the mass of the rope is distributed over its length, the rope and the block have a common acceleration. Because the surface is horizontal and smooth, the only force that influences our solution is $\vec{F}$. The figure misrepresents the situation in that each segment of the rope experiences a gravitational force; the combined effect of which is that the rope must sag.

(a) Apply $\vec{a} = \vec{F}_{\text{net}}/m_{\text{tot}}$ to the rope-block system to obtain:

$$a = \boxed{\frac{F}{m_1 + m_2}}$$

(*b*) Apply $\sum \vec{F} = m\vec{a}$ to the rope, substitute the acceleration of the system obtained in (*a*), and simplify to obtain:

$$F_{net} = m_2 a = m_2\left(\frac{F}{m_1 + m_2}\right)$$

$$= \boxed{\frac{m_2}{m_1 + m_2}F}$$

(*c*) Apply $\sum \vec{F} = m\vec{a}$ to the block, substitute the acceleration of the system obtained in (*a*), and simplify to obtain:

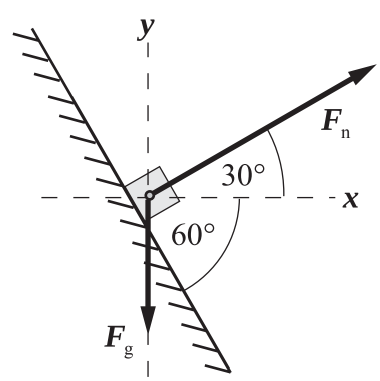
$$T = m_1 a = m_1\left(\frac{F}{m_1 + m_2}\right)$$

$$= \boxed{\frac{m_1}{m_1 + m_2}F}$$

92 •• A 2.0-kg block rests on a frictionless wedge that has a 60º incline and an acceleration $\vec{a}$ to the right such that the mass remains stationary relative to the wedge (Figure 4-64). (*a*) Draw the free body diagram of the block and use it to determine the magnitude of the acceleration. (*b*) What would happen if the wedge were given an acceleration larger than this value? Smaller than this value?

Picture the Problem We can apply Newton's second law to the 2-0-kg block to determine the magnitude of its acceleration.

(*a*) The free-body diagram of the 2.0-kg block is shown to the right. Because the surface of the wedge is frictionless, the force it exerts on the block must be normal to its surface. The forces acting on the block are the normal force $\vec{F}_n$ exerted by the frictionless surface and the gravitational force $\vec{F}_g$. The direction of the acceleration of the wedge has been chosen as the +*x* direction.



(*a*) Apply $\sum F_y = ma_y$ to the block to obtain:

$$F_n \sin 30° - F_g = ma_y$$

or, because $a_y = 0$ and $F_g = mg$,

$$F_n \sin 30° = mg \qquad (1)$$

Apply $\sum F_x = ma_x$ to the block:

$$F_n \cos 30° = ma_x \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{a_x}{g} = \cot 30° \Rightarrow a_x = \frac{g}{\tan 30°}$$

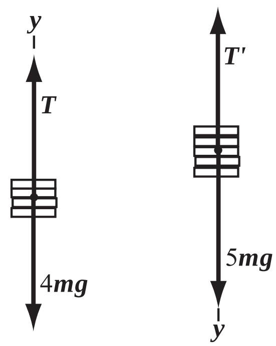
Substitute numerical values and evaluate a_x:

$$a_x = \frac{9.81\,\text{m/s}^2}{\tan 30°} = \boxed{17\,\text{m/s}^2}$$

(*b*) An acceleration of the wedge greater than $g/\tan 30°$ would require that the normal force exerted on the body by the wedge be greater than that given in Part (*a*); that is, $F_n > mg/\sin 30°$. Under this condition, there would be a net force in the *y* direction and the block would accelerate up the wedge. With an acceleration less than $g/\tan 30°$, the block would accelerate down the wedge.

93 ••• [SSM] The masses attached to each side of an ideal Atwood's machine consist of a stack of five washers, each of mass *m*, as shown in Figure 4-65. The tension in the string is T_0. When one of the washers is removed from the left side, the remaining washers accelerate and the tension decreases by 0.300 N. (*a*) Find *m*. (*b*) Find the new tension and the acceleration of each mass when a second washer is removed from the left side.

Picture the Problem Because the system is initially in equilibrium, it follows that $T_0 = 5mg$. When one washer is removed from the left side the remaining washers on the left side will accelerate upward (and those on the right side downward) in response to the net force that results. The free-body diagrams show the forces under this unbalanced condition. Applying Newton's second law to each collection of washers will allow us to determine both the acceleration of the system and the mass of a single washer.



(*a*) Apply $\sum F_y = ma_y$ to the rising washers:

$$T - 4mg = (4m)a_y \qquad (1)$$

Noting that $T = T'$, apply $\sum F_y = ma_y$ to the descending masses:

$$5mg - T = (5m)a_y \qquad (2)$$

Eliminate *T* between these equations to obtain:

$$a_y = \tfrac{1}{9}g$$

Use this acceleration in equation (1) or equation (2) to obtain:

$$T = \frac{40}{9}mg$$

Expressing the difference ΔT between T_0 and *T* yields:

$$\Delta T = 5mg - \frac{40}{9}mg \Rightarrow m = \frac{\frac{9}{5}\Delta T}{g}$$

Substitute numerical values and evaluate m:

$$m = \frac{\frac{9}{5}(0.300\,\text{N})}{9.81\,\text{m/s}^2} = \boxed{55.0\,\text{g}}$$

(b) Proceed as in (a) to obtain:

$$T - 3mg = 3ma_y \text{ and } 5mg - T = 5ma_y$$

Add these equations to eliminate T and solve for a_y to obtain:

$$a_y = \tfrac{1}{4}g$$

Substitute numerical values and evaluate a_y:

$$a_y = \tfrac{1}{4}(9.81\,\text{m/s}^2) = \boxed{2.45\,\text{m/s}^2}$$

Eliminate a_y in either of the motion equations and solve for T to obtain:

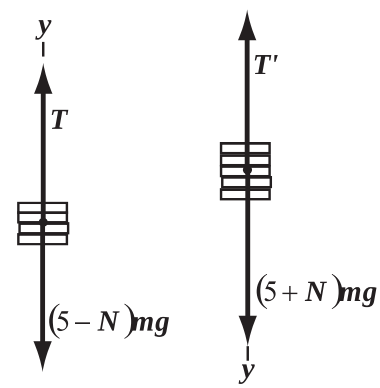
$$T = \tfrac{15}{4}mg$$

Substitute numerical values and evaluate T:

$$T = \tfrac{15}{4}(0.05505\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{2.03\,\text{N}}$$

94 •• Consider the ideal Atwood's machine in Figure 4-65. When N washers are transferred from the left side to the right side, the right side drops 47.1 cm in 0.40 s. Find N.

Picture the Problem The free-body diagram represents the Atwood's machine with N washers moved from the left side to the right side. Application of Newton's second law to each collection of washers will result in two equations that can be solved simultaneously to relate N, a, and g. The acceleration can then be found from the given data.



Apply $\sum F_y = ma_y$ to the rising washers:

$$T - (5 - N)mg = (5 - N)ma_y$$

Noting that $T = T'$, apply $\sum F_y = ma_y$ to the descending washers:

$$(5 + N)mg - T = (5 + N)ma_y$$

Add these equations to eliminate T:

$$(5 + N)mg - (5 - N)mg = (5 - N)ma_y + (5 + N)ma_y$$

Solving for N yields:

$$N = \frac{5a_y}{g}$$

Using a constant-acceleration equation, relate the distance the washers fell to their time of fall:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y} = 0$,

$$\Delta y = \tfrac{1}{2}a_y(\Delta t)^2 \Rightarrow a_y = \frac{2\Delta y}{(\Delta t)^2}$$

Substitute numerical values and evaluate a_y:

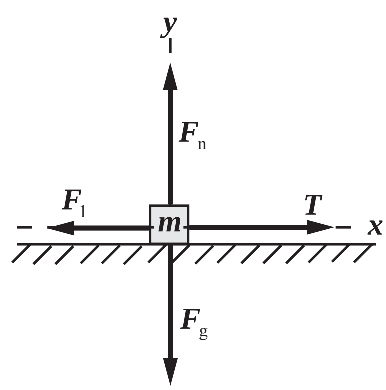
$$a_y = \frac{2(0.471\,\text{m})}{(0.40\,\text{s})^2} = 5.89\,\text{m/s}^2$$

Substitute in the expression for N:

$$N = 5\left(\frac{5.89\,\text{m/s}^2}{9.81\,\text{m/s}^2}\right) = \boxed{3}$$

95 •• Blocks of mass m and $2m$ are on a horizontal frictionless surface (Figure 4-66). The blocks are connected by a string. In addition, forces F_1 and F_2 are applied to the blocks as shown. (*a*) If the forces shown are constant, find the tension in the connecting string. (*b*) If the forces vary with time as $F_1 = Ct$ and $F_2 = 2Ct$, where $C = 5.00$ N/s and t is time, find the time t_0 at which the tension in the string equals 10.0 N.

Picture the Problem Draw the free-body diagram for the block of mass m and apply Newton's second law to obtain the acceleration of the system and then the tension in the rope connecting the two blocks.



(*a*) Apply $\sum F_x = ma_x$ to the block of mass m:

$$T - F_1 = ma_x \qquad (1)$$

Apply $\sum F_x = ma_x$ to both blocks:

$$F_2 - F_1 = (m + 2m)a = 3ma_x$$

Solving for a gives:

$$a_x = \frac{F_2 - F_1}{3m}$$

Substitute for a_x in the equation (1) to obtain:

$$T - F_1 = m\left(\frac{F_2 - F_1}{3m}\right)$$

Solving for T yields: $T = \boxed{\tfrac{1}{3}(F_2 + 2F_1)}$

(b) Substitute for F_1 and F_2 in the equation derived in Part (a):

$$T = \tfrac{1}{3}(2Ct + 2(Ct)) = \tfrac{4}{3}Ct \Rightarrow t = \frac{3T}{4C}$$

Substitute numerical values and evaluate $t = t_0$:

$$t_0 = \frac{3(10.0\ \text{N})}{4(5.00\ \text{N/s})} = \boxed{1.50\ \text{s}}$$

97 •• Elvis Presley, has supposedly been sighted numerous times after he passed away on August 16, 1977. The following is a chart of what Elvis' weight would be if he were sighted on the surface of other objects in our solar system. Use the chart to determine: (a) Elvis' mass on Earth, (b) Elvis' mass on Pluto, and (c) the acceleration due to gravity on Mars. (d) Compare the free-fall acceleration on Pluto to the free-fall acceleration on the moon.

planet	Elvis's weight (N)
Mercury	431
Venus	1031
Earth	1133
Mars	431
Jupiter	2880
Saturn	1222
Pluto	58
Moon	191

Picture the Problem Elvis' mass is the ratio of his weight on a given planet to the acceleration of gravity on that planet. In (b), (c), and (d) we can use the relationship between the gravitational force (weight) acting on an object, its mass (independent of location), and the local value of the free-fall acceleration.

(a) Elvis' mass on Earth is given by:

$$m = \frac{w_{\text{Earth}}}{g_{\text{Earth}}}$$

Substitute numerical values and evaluate m:

$$m = \frac{1131\ \text{N}}{9.81\ \text{m/s}^2} = 115.3\ \text{kg} = \boxed{115\ \text{kg}}$$

(b) Because his mass is independent of his location, Elvis' mass on Pluto is 115 kg.

(*c*) The free-fall acceleration on Mars is the ratio of the weight of an object on Mars to the mass of the object:

$$g_{\text{Mars}} = \frac{w_{\text{on Mars}}}{m}$$

Substitute numerical values for Elvis and evaluate g_{Mars}:

$$g_{\text{Mars}} = \frac{431\ \text{N}}{115.3\ \text{kg}} = \boxed{3.74\ \text{m/s}^2}$$

(*d*) The free-fall acceleration on Pluto is given by:

$$g_{\text{Pluto}} = \frac{w_{\text{on Pluto}}}{m} \qquad (1)$$

The free-fall acceleration on the moon is given by:

$$g_{\text{moon}} = \frac{w_{\text{on moon}}}{m} \qquad (2)$$

Divide equation (1) by equation (2) and simplify to obtain:

$$\frac{g_{\text{Pluto}}}{g_{\text{moon}}} = \frac{\dfrac{w_{\text{on Pluto}}}{m}}{\dfrac{w_{\text{on moon}}}{m}} = \frac{w_{\text{on Pluto}}}{w_{\text{on moon}}}$$

Substituting numerical values yields:

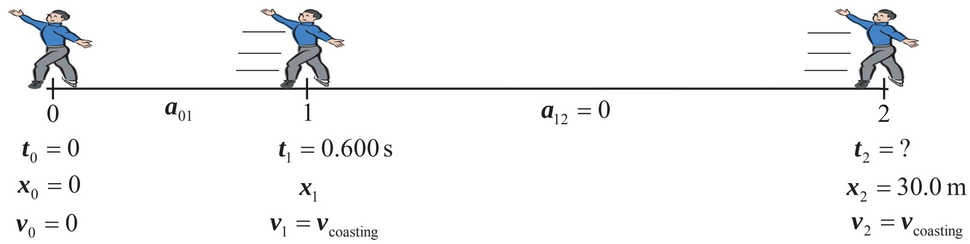
$$\frac{g_{\text{Pluto}}}{g_{\text{moon}}} = \frac{58\ \text{N}}{191\ \text{N}} = 0.30$$

or

$$g_{\text{Pluto}} = \boxed{0.30 g_{\text{moon}}}$$

97 ••• As a prank, your friends have kidnapped you in your sleep, and transported you out onto the ice covering a local pond. When you wake up you are 30.0 m from the nearest shore. The ice is so slippery (i.e. frictionless) that you can not seem to get yourself moving. You realize that you can use Newton's third law to your advantage, and choose to throw the heaviest thing you have, one boot, in order to get yourself moving. Take your weight to be 595 N. (*a*) What direction should you throw your boot so that you will most quickly reach the shore? (*b*) If you throw your 1.20-kg boot with an average force of 420 N, and the throw takes 0.600 s (the time interval over which you apply the force), what is the magnitude of the average force that the boot exerts on you? (Assume constant acceleration.) (*c*) How long does it take you to reach shore, including the short time in which you were throwing the boot?

Picture the Problem The diagram shown below summarizes the information about your trip to the shore and will be helpful in solving Part (*c*) of the problem.



(*a*) You should throw your boot in the direction away from the closest shore.

(*b*) The magnitude of the average force you exert on the boot equals the magnitude of the average force the boot exerts on you:

$$F_{\text{av, on you}} = \boxed{420\text{ N}}$$

(*c*) The time required for you to reach the shore is the sum of your travel time while accelerating and your travel time while coasting:

$$\Delta t_{\text{total}} = \Delta t_{01} + \Delta t_{12}$$

or, because $\Delta t_{01} = 0.600$ s,

$$\Delta t_{\text{total}} = 0.600\,\text{s} + \Delta t_{12} \qquad (1)$$

Use a constant-acceleration equation to relate your displacement Δx_{01} to your acceleration time Δt_{01}:

$$\Delta x_{01} = \tfrac{1}{2} a_{01} (\Delta t_{01})^2 \qquad (2)$$

Apply Newton's second law to express your acceleration during this time interval:

$$a_{01} = \frac{F_{\text{net}}}{m} = \frac{F_{\text{av}}}{w/g} = \frac{F_{\text{av}}g}{w}$$

Substitute numerical values and evaluate a_{01}:

$$a_{01} = \frac{(420\text{ N})(9.81\text{ m/s}^2)}{595\text{ N}} = 6.925\text{ m/s}^2$$

Substitute numerical values in equation (2) and evaluate Δx_{01}:

$$\Delta x_{01} = \tfrac{1}{2}(6.925\text{ m/s}^2)(0.600\text{ s})^2 = 1.246\text{ m}$$

Your coasting time is the ratio of your displacement while coasting to your speed while coasting:

$$\Delta t_{12} = \frac{\Delta x_{12}}{v_1}$$

or, because $\Delta \boldsymbol{x}_{12} = 30.0\text{ m} - \Delta \boldsymbol{x}_{01}$,

$$\Delta t_{12} = \frac{30.0\text{ m} - \Delta x_{01}}{v_1} \qquad (3)$$

Use a constant-acceleration equation to relate your terminal speed (your speed after the interval of acceleration) to your acceleration and displacement during this interval:

$$v_1 = v_0 + a_{01}\Delta t_{01}$$

or, because $v_0 = 0$,

$$v_1 = a_{01}\Delta t_{01}$$

Substitute for v_1 in equation (3) to obtain:

$$\Delta t_{12} = \frac{30.0\text{ m} - \Delta x_{01}}{a_{01}\Delta t_{01}}$$

Substituting for $\Delta t_{\text{coasting}}$ in equation (1) yields:

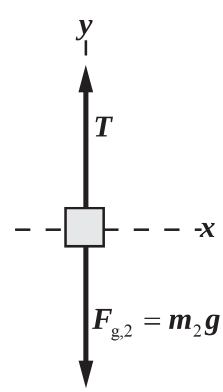
$$\Delta t_{\text{total}} = 0.600\text{ s} + \frac{30.0\text{ m} - \Delta x_{01}}{a_{01}\Delta t_{01}}$$

Substitute numerical values and evaluate Δt_{total}:

$$\Delta t_{\text{total}} = 0.600\text{ s} + \frac{30.0\text{ m} - 1.246\text{ m}}{(6.925\text{ m/s}^2)(0.600\text{ s})} = \boxed{7.52\text{ s}}$$

98 ••• The pulley of an ideal Atwood's machine is given an upward acceleration *a*, as shown in Figure 4-67. Find the acceleration of each mass and the tension in the string that connects them.

Picture the Problem Because a constant-upward acceleration has the same effect as an increase in the acceleration due to gravity, we can use the result of Problem 82 (for the tension) with *a* replaced by *a* + *g*. The application of Newton's second law to the object whose mass is m_2 will connect the acceleration of this body to tension from Problem 82.



In Problem 82 it is given that, when the support pulley is not accelerating, the tension in the rope and the acceleration of the masses are related according to:

$$\boxed{T = \frac{2m_1m_2}{m_1+m_2}g}$$

Replace *a* with *a* + *g*:

$$T = \frac{2m_1m_2}{m_1+m_2}(a+g)$$

Apply $\sum F_y = ma_y$ to the object whose mass is m_2:

$$T - m_2 g = m_2 a_2 \Rightarrow a_2 = \frac{T - m_2 g}{m_2}$$

Substitute for T and simplify to obtain:

$$a_2 = \boxed{\frac{(m_1 - m_2)g + 2m_1 a}{m_1 + m_2}}$$

The expression for a_1 is the same as for a_2 with all subscripts interchanged (note that a positive value for a_1 represents acceleration upward):

$$a_1 = \boxed{\frac{(m_2 - m_1)g + 2m_2 a}{m_1 + m_2}}$$

99 ••• You are working for an automotive magazine and putting a certain new automobile (mass 650 kg) through its paces. While accelerating from rest, its onboard computer records its speed as a function of time as follows:

v (m/s):	0	10	20	30	40	50
t (s):	0	1.8	2.8	3.6	4.9	6.5

(*a*) Using a **spreadsheet**, find the average acceleration over each 1.8-s time interval, and graph the velocity versus time and acceleration versus time for this car. (*b*) Where on the graph of velocity versus time is the net force on the car highest and lowest? Explain your reasoning. (*c*) What is the average net force on the car over the whole trip? (*d*) From the graph of velocity versus time, estimate the total distance covered by the car.

Picture the Problem

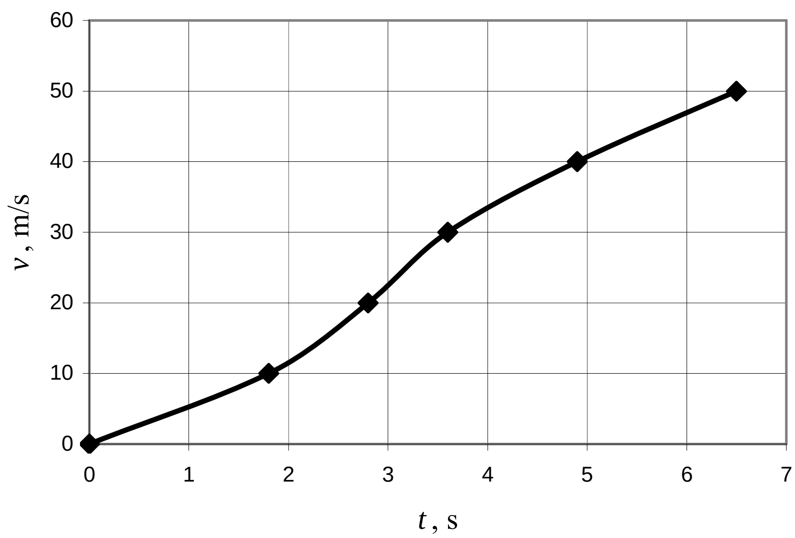
(*a*) A spreadsheet program is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
C1	650	m
D5	(B5−B4)/2	$\frac{1}{2}\Delta t$
E5	(C5−C4)/D5	$\frac{\Delta v}{\Delta t}$
F5	$D$1*E5	ma
F10	(F5+F6+F7+F8+F9)/5	F_{ave}

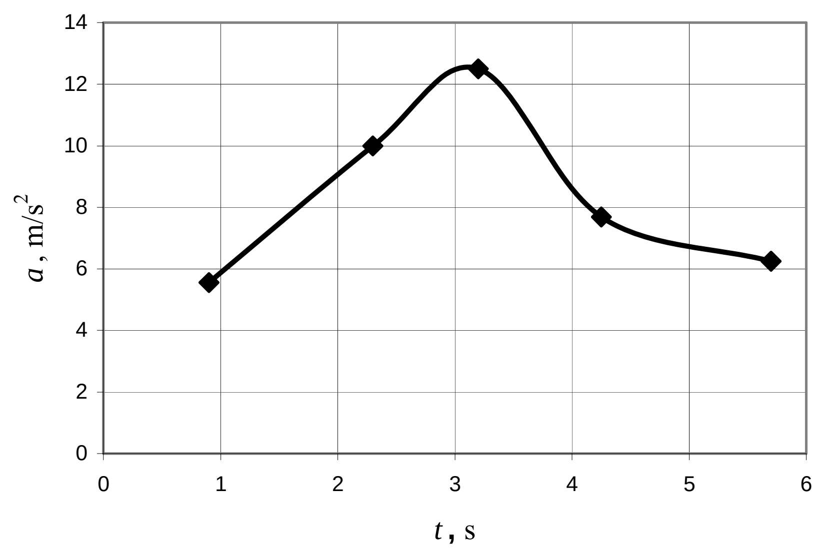
	A	B	C	D	E	F
1			m=	650	kg	
2						
3		t (s)	v (m/s)	t_{midpt} (s)	a (m/s^2)	$F = ma$ (N)
4		0.0	0			
5		1.8	10	0.90	5.56	3611
6		2.8	20	2.30	10.00	6500
7		3.6	30	3.20	12.50	8125
8		4.9	40	4.25	7.69	5000

9		6.5	50	5.70	6.25	4063
10						5460
11						

A graph of velocity as a function of time follows:



A graph of acceleration as a function of time is shown below:



(b) Because the net force is lowest where the acceleration is lowest, we can see from the graph of velocity versus time that its slope (the acceleration) is smallest in the interval from 0 to 1.8 s. Because the net force is highest where the acceleration is highest, we can see from the graph of velocity versus time that its slope (the acceleration) is greatest in the interval from 2.8 s to 3.6 s.

(*c*) From the table we see that:

$$F_{\text{ave}} \approx \boxed{5500\ \text{N}}$$

(*d*) The distance covered by the car is the area under the graph of velocity versus time. Because the graph of speed versus time is approximately linear, we can estimate the total distance covered by the car by finding the area of the triangular region under it.

$$A_{\text{triangle}} = \tfrac{1}{2}bh = \tfrac{1}{2}(6.5\ \text{s})(50\ \text{m/s})$$
$$\approx \boxed{160\ \text{m}}$$

Chapter 5
Additional Applications of Newton's Laws

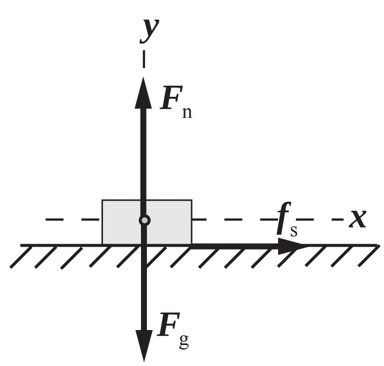
Conceptual Problems

1 • [SSM] Various objects lie on the bed of a truck that is moving along a straight horizontal road. If the truck gradually speeds up, what force acts on the objects to cause them to speed up too? Explain why some of the objects might stay stationary on the floor while others might slip backward on the floor.

Determine the Concept Static and kinetic frictional forces are responsible for the accelerations. If the coefficient of static friction between the truck bed and the object is sufficiently large, then the object will not slip on the truck bed. The larger the acceleration of the truck, the larger the coefficient of static friction that is needed to prevent slipping.

2 • Blocks made of the same material but differing in size lie on the bed of a truck that is moving along a straight horizontal road. All of the blocks will slide if the truck's acceleration is sufficiently great. How does the minimum acceleration at which a small block slips compare with the minimum acceleration at which a much heavier block slips?

Determine the Concept The forces acting on an object are the normal force exerted by the floor of the truck, the gravitational force exerted by Earth, and the friction force; also exerted by the floor of the truck. Of these forces, the only one that acts in the direction of the acceleration (chosen to be to the right) is the static friction force. Apply Newton's second law to the object to determine how the critical acceleration depends on its weight.



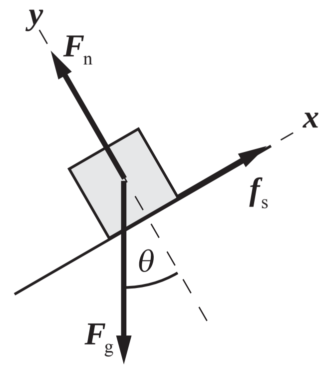
Taking the $+x$ direction to be to the right, apply $\Sigma F_x = ma_x$ to the object:

$$f_s = \mu_s F_g = \mu_s mg = ma_x \Rightarrow a_x = \mu_s g$$

Because a_x is independent of m and F_g, the critical accelerations are the same.

3 • A block of mass m rests on a plane that is inclined at an angle θ with the horizontal. It follows that the coefficient of static friction between the block and plane is (*a*) $\mu_s \geq g$, (*b*) $\mu_s = \tan\theta$, (*c*) $\mu_s \leq \tan\theta$, (*d*) $\mu_s \geq \tan\theta$.

Determine the Concept The forces acting on the block are the normal force $\vec{F}_n$ exerted by the incline, the weight of the block $\vec{F}_g$ exerted by Earth, and the static friction force $\vec{f}_s$ also exerted by the incline. We can use the definition of μ_s and the conditions for equilibrium to determine the relationship between μ_s and θ.



Apply $\sum F_x = ma_x$ to the block:

$$f_s - F_g \sin\theta = 0$$

or, because $F_g = mg$,

$$f_s - mg\sin\theta = 0 \qquad (1)$$

Apply $\sum F_y = ma_y$ in the y direction:

$$F_n - mg\cos\theta = 0 \qquad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\tan\theta = \frac{f_s}{F_n}$$

Substitute for f_s ($\le \mu_s F_n$) and simplify to obtain:

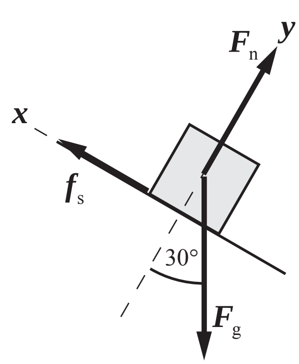
$$\tan\theta \le \frac{\mu_s F_n}{F_n} = \mu_s$$ and $\boxed{(d)}$ is correct.

4 • A block of mass m is at rest on a plane that is inclined at an angle of 30º with the horizontal, as shown in Figure 5-56. Which of the following statements about the magnitude of the static frictional force f_s is necessarily true? (a) $f_s > mg$. (b) $f_s > mg\cos 30º$. (c) $f_s = mg\cos 30º$. (d) $f_s = mg\sin 30°$. (e) None of these statements is true.

Determine the Concept The block is in equilibrium under the influence of $\vec{F}_n$, $\vec{F}_g$, and $\vec{f}_s$; that is,

$$\vec{F}_n + \vec{F}_g + \vec{f}_s = \mathbf{0}$$

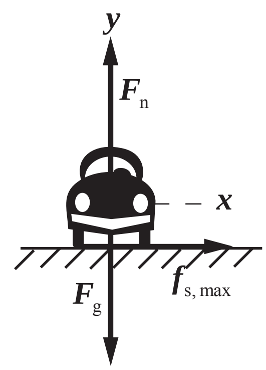
We can apply Newton's second law in the x direction to determine the relationship between f_s and $F_g = mg$.



Apply $\sum F_x = 0$ to the block to obtain: $f_s - mg\sin\theta = 0 \Rightarrow f_s = mg\sin\theta$ and $\boxed{(d)}$ is correct.

5 •• On an icy winter day, the coefficient of friction between the tires of a car and a roadway is reduced to one-quarter of its value on a dry day. As a result, the maximum speed $v_{\text{max dry}}$ at which the car can safely negotiate a curve of radius R is reduced. The new value for this speed is (*a*) $v_{\text{max dry}}$, (*b*) $0.71v_{\text{max dry}}$, (*c*) $0.50v_{\text{max dry}}$, (*d*) $0.25v_{\text{max dry}}$, (*e*) reduced by an unknown amount depending on the car's mass.

Picture the Problem The forces acting on the car as it rounds a curve of radius R at maximum speed are shown on the free-body diagram to the right. The centripetal force is the static friction force exerted by the roadway on the tires. We can apply Newton's second law to the car to derive an expression for its maximum speed and then compare the speeds under the two friction conditions described.



Apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = f_{\text{s,max}} = m\frac{v_{\text{max}}^2}{R}$$

and

$$\sum F_y = F_n - mg = 0$$

From the y equation we have: $F_n = mg$

Express $f_{\text{s,max}}$ in terms of F_n in the x equation and solve for v_{max} to obtain:

$$v_{\text{max}} = \sqrt{\mu_s gR} \qquad (1)$$

When $\mu_s = \mu'_s$:

$$v'_{\text{max}} = \sqrt{\mu'_s gR} \qquad (2)$$

Dividing equation (2) by equation (1) yields:

$$\frac{v'_{\text{max}}}{v_{\text{max}}} = \frac{\sqrt{\mu'_s gR}}{\sqrt{\mu_s gR}} = \sqrt{\frac{\mu'_s}{\mu_s}}$$

Solve for v'_{max} to obtain:

$$v'_{\text{max}} = \sqrt{\frac{\mu'_s}{\mu_s}}v_{\text{max}}$$

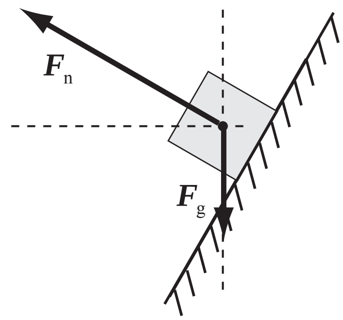
Evaluate $v'_{\max}$ for $\mu'_{s} = \frac{1}{4}\mu_{s}$:

$$v'_{\max} = \sqrt{\tfrac{1}{4}}v_{\max} = 0.5v_{\max} = 50\%v_{\max}$$

and $\boxed{(c)}$ is correct.

6 •• If it is started properly on the frictionless inside surface of a cone (Figure 5-57), a block is capable of maintaining uniform circular motion. Draw the free-body diagram of the block and identify clearly which force (or forces or force components) is responsible for the centripetal acceleration of the block.

Determine the Concept The forces acting on the block are the normal force $\vec{F}_{n}$ exerted by the surface of the cone and the gravitational force $\vec{F}_{g}$ exerted by Earth. The horizontal component of $\vec{F}_{n}$ is responsible for the centripetal force on the block.



7 •• Here is an interesting experiment that you can perform at home: take a wooden block and rest it on the floor or some other flat surface. Attach a rubber band to the block and pull gently and steadily on the rubber band in the horizontal direction. Keep your hand moving at constant speed. At some point, the block will start moving, but it will not move smoothly. Instead, it will start moving, stop again, start moving again, stop again, and so on. Explain why the block moves this way. (The start-stop motion is sometimes called ″stick-slip″ motion.)

Determine the Concept As the spring is extended, the force exerted by the spring on the block increases. Once that force exceeds the maximum value of the force of static friction, the block will slip. As it does, it will shorten the length of the spring, decreasing the force that the spring exerts. The force of kinetic friction then slows the block to a stop, which starts the cycle over again.

8 • Viewed from an inertial reference frame, an object is seen to be moving in a circle. Which, if any, of the following statements must be true. (*a*) A nonzero net force acts on the object. (*b*) The object cannot have a radially outward force acting on it. (*c*) At least one of the forces acting on the object must point directly toward the center of the circle.

(*a*) True. The velocity of an object moving in a circle is continually changing independently of whether the object's speed is changing. The change in the velocity vector and the acceleration vector and the net force acting on the object all point toward the center of circle. This center-pointing force is called a centripetal force.

(*b*) False. The only condition that must be satisfied in order that the object move along a circular path is that the *net* force acting on it be radially inward.

(*c*) False. The only condition that must be satisfied in order that the object move along a circular path is that the *net* force acting on it be radially inward.

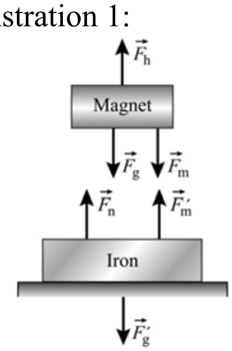
9 •• A particle is traveling in a vertical circle at constant speed. One can conclude that the magnitude of its _____ is(are) constant. (*a*) velocity, (*b*) acceleration, (*c*) net force, (*d*) apparent weight.

Determine the Concept A particle traveling in a vertical circle experiences a downward gravitational force plus an additional force that constrains it to move along a circular path. Because the speed of the particle is constant, the magnitude of its velocity is constant. Because the magnitude of its velocity is constant, its acceleration must be constant. Because the magnitude of its acceleration is constant, the magnitude of the net force acting on it must be constant. Therefore, $\boxed{(a)}$, $\boxed{(b)}$, and $\boxed{(c)}$ are correct.

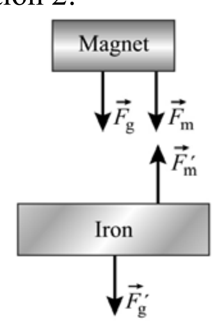
10 •• You place a lightweight piece of iron on a table and hold a small kitchen magnet above the iron at a distance of 1.00 cm. You find that the magnet cannot lift the iron, even though there is obviously a force between the iron and the magnet. Next, again holding the magnet 1.00 cm above the iron, you drop them from arm's length, releasing them from rest simultaneously. As they fall, the magnet and the piece of iron bang into each other before hitting the floor. (*a*) Draw free-body diagrams illustrating all of the forces on the magnet and the iron for each demonstration. (*b*) Explain why the magnet and iron move closer together while they are falling, even though the magnet cannot lift the piece of iron when it is sitting on the table.

Determine the Concept We can analyze these demonstrations by drawing force diagrams for each situation. In both diagrams, h denotes ″hand″, g denotes ″gravitational″, m denotes ″magnetic″, and n denotes ″normal.″

(*a*) Demonstration 1:



Demonstration 2:

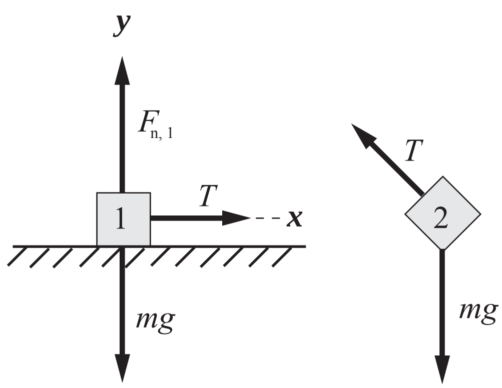


(*b*) Because the magnet doesn't lift the iron in the first demonstration, the force exerted on the iron must be less than its (the iron's) weight. This is still true when the two are falling, but the motion of the iron is not restrained by the table, and the

motion of the magnet is not restrained by the hand. Looking at the second diagram, the net force pulling the magnet down is greater than its weight, implying that its acceleration is greater than g. The opposite is true for the iron: the magnetic force acts upwards, slowing it down, so its acceleration will be less than g. Because of this, the magnet will catch up to the iron piece as they fall.

11 ••• [SSM] The following question is an excellent "braintwister" invented by Boris Korsunsky. Two identical blocks are attached by a massless string running over a pulley as shown in Figure 5-58. The rope initially runs over the pulley at the rope's midpoint, and the surface that block 1 rests on is frictionless. Blocks 1 and 2 are initially at rest when block 2 is released with the string taut and horizontal. Will block 1 hit the pulley before or after block 2 hits the wall? (Assume that the initial distance from block 1 to the pulley is the same as the initial distance from block 2 to the wall.) There is a very simple solution.

Picture the Problem The following free-body diagrams show the forces acting on the two objects some time after block 2 is dropped. Note that the tension in the string is the same on both sides of the pulley. The only force pulling block 2 to the left is the horizontal component of the tension. Because this force is smaller than the T and blocks 1 and 2 have the same mass, the acceleration of block 1 to the right will always be greater than the acceleration of block 2 to the left.



Because the initial distance from block 1 to the pulley is the same as the initial distance of block 2 to the wall, block 1 will hit the pulley before block 2 hits the wall.

12 •• In class, most professors do the following experiment while discussing the conditions under which air drag can be neglected while analyzing free-fall. First, a flat piece of paper and a small lead weight are dropped next to each other, and clearly the paper's acceleration is less than that of the lead weight. Then, the paper is crumpled into a small wad and the experiment repeated. Over the distance of a meter or two, it is clear the acceleration of the paper is now very close to that of the lead weight. To your dismay, the professor calls on you to explain why the paper's acceleration changed so dramatically. Repeat your explanation here!

Determine the Concept Air drag depends on the frontal area presented. Reducing it by crumpling the paper makes the force of air drag a lot less so that gravity is the most important force. The paper will thus accelerate at approximately *g* (until speeds are high enough for drag forces to come back into play in spite of the reduced area).

13 •• [SSM] Jim decides to attempt to set a record for terminal speed in skydiving. Using the knowledge he has gained from a physics course, he makes the following plans. He will be dropped from as high an altitude as possible (equipping himself with oxygen), on a warm day and go into a ″knife″ position in which his body is pointed vertically down and his hands are pointed ahead. He will outfit himself with a special sleek helmet and rounded protective clothing. Explain how each of these factors helps Jim attain the record.

Determine the Concept Air drag is proportional to the density of air and to the cross-sectional area of the object. On a warm day the air is less dense. The air is also less dense at high altitudes. Pointing his hands results in less area being presented to air drag forces and, hence, reduces them. Rounded and sleek clothing has the same effect as pointing his hands.

14 •• You are sitting in the passenger seat in a car driving around a circular, horizontal, flat racetrack at a high speed. As you sit there, you ″feel″ a ″force″ pushing you toward the outside of the track. What is the true direction of the force acting on you, and where does it come from? (Assume that you do not slide across the seat.) Explain the *sensation* of an "outward force" on you in terms of the Newtonian perspective.

Determine the Concept In your frame of reference (the accelerating reference frame of the car), the direction of the force must point toward the center of the circular path along which you are traveling; that is, in the direction of the centripetal force that keeps you moving in a circle. The friction between you and the seat you are sitting on supplies this force. The reason you seem to be "pushed" to the outside of the curve is that your body's inertia "wants", in accordance with Newton's first law (the law of inertia), to keep it moving in a straight line–that is, tangent to the curve.

15 • [SSM] The mass of the moon is only about 1% of that of Earth. Therefore, the force that keeps the moon in its orbit around Earth (*a*) is much smaller than the gravitational force exerted on the moon by Earth, (*b*) is much greater than the gravitational force exerted on the moon by Earth, (*c*) is the gravitational force exerted on the moon by Earth, (*d*) cannot be answered yet, because we have not yet studied Newton's law of gravity.

Determine the Concept The centripetal force that keeps the moon in its orbit around Earth is provided by the gravitational force Earth exerts on the moon. As described by Newton's 3rd law, this force is equal in magnitude to the force the

moon exerts on Earth. $\boxed{(c)}$ is correct.

16 • A block is sliding on a frictionless surface along a loop-the-loop, as in Figure 5-59. The block is moving fast enough so that it never loses contact with the track. Match the points along the track to the appropriate free-body diagrams in the figure.

Determine the Concept The only forces acting on the block are its weight and the force the surface exerts on it. Because the loop-the-loop surface is frictionless, the force it exerts on the block must be perpendicular to its surface.

At point A the weight is downward and the normal force is to the right. The normal force is the centripetal force. Free-body diagram 3 matches these forces.

At point B the weight is downward, the normal force is upward, and the normal force is greater than the weight so that their difference is the centripetal force. Free-body diagram 4 matches these forces.

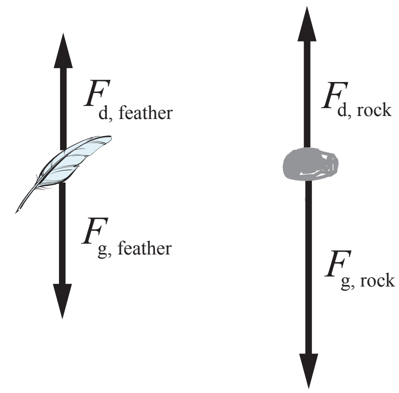
At point C the weight is downward and the normal force is to the left. The normal force is the centripetal force. Free-body diagram 5 matches these forces.

At point D both the weight and the normal forces are downward. Their sum is the centripetal force. Free-body diagram 2 matches these forces.

17 •• [SSM] (*a*) A rock and a feather held at the same height above the ground are simultaneously dropped. During the first few milliseconds following release, the drag force on the rock is smaller than the drag force on the feather, but later on during the fall the *opposite* is true. Explain. (*b*) In light of this result, explain how the rock's acceleration can be so obviously larger than that of the feather. *Hint*: *Draw a free-body diagram of each object*.

Determine the Concept The drag force

acting on the objects is given by $F_\text{d} = \frac{1}{2}CA\rho v^2$, where A is the projected surface area, v is the object's speed, ρ is the density of air, and C is a dimensionless coefficient. We'll assume that, over the height of the fall, the density of air ρ is constant. The free-body diagrams for a feather and a rock several milliseconds into their fall are shown to the right. The forces acting on both objects are the downward gravitational force and an upward drag force.



(a) The drag force is proportional to the area presented and some power of the speed. The drag force on the feather is larger because the feather presents a larger area than does the rock. As the rock gains speed, the drag force on it increases. The drag force on the rock eventually exceeds the drag force on the feather because the drag force on the feather cannot exceed the gravitational force on the feather.

A short time after being released the feather reaches terminal speed. During the rest of its fall the drag force on it is equal to the gravitational force acting on it. However, the gravitational force on the rock is much greater than the drag force on the feather, so the rock continues to gain speed long after the feather reaches terminal speed. As the rock continues to gain speed, the drag force on it continues to increase. As a result, the drag force on the rock eventually exceeds the drag force on the feather.

(b) Terminal speed is much higher for the rock than for the feather. The acceleration of the rock will remain high until its speed approaches its terminal speed.

18 •• Two pucks of masses m_1 and m_2 are lying on a frictionless table and are connected by a massless spring of force constant k. A horizontal force F_1 directed away from m_2 is then exerted on m_1. What is the magnitude of the resulting acceleration of the center of mass of the two-puck system? (a) F_1/m_1. (b) $F_1/(m_1 + m_2)$. (c) $(F_1 + kx)/(m_1 + m_2)$, where x is the amount the spring is stretched. (d) $(m_1 + m_2)F_1/m_1m_2$.

Determine the Concept The acceleration of the center of mass of a system of particles is described by $\vec{\boldsymbol{F}}_\text{net,ext} = \sum_\text{i} \vec{\boldsymbol{F}}_\text{i,ext} = M\vec{\boldsymbol{a}}_\text{cm}$, where M is the total mass of the system.

Express the acceleration of the center of mass of the two pucks:

$$a_{cm} = \frac{F_{net,ext}}{M} = \frac{F_1}{m_1 + m_2}$$

because the spring force is an internal force. $\boxed{(b)}$ is correct.

19 •• The two pucks in Problem 18 lie unconnected on a frictionless table. A horizontal force F_1 directed away from m_2 is then exerted on m_1. How does the magnitude of the resulting acceleration of the center of mass of the two-puck system compare to the magnitude of the acceleration of m_1? Explain your reasoning.

Determine the Concept The magnitude of the acceleration of the puck whose mass is m_1 is related to the net force F_1 acting on it through Newton's second law.

Because the pucks are no longer connected, the magnitude of the acceleration of the center of mass is:

$$a_{\text{CM, disconnected}} = \frac{F_1}{m_1}$$

From Problem 18:

$$a_{\text{CM, connected}} = \frac{F_1}{m_1 + m_2}$$

Because $\frac{F_1}{m_1} > \frac{F_1}{m_1 + m_2}$ the magnitude of the acceleration of m_1 is greater.

20 •• If only external forces can cause the center of mass of a system of particles to accelerate, how can a car on level ground ever accelerate? We normally think of the car's engine as supplying the force needed to accelerate the car, but is this true? Where does the external force that accelerates the car come from?

Determine the Concept There is only one force that can cause the car to move forward−the friction of the road! The car's engine causes the tires to rotate, but if the road were frictionless (as is closely approximated by icy conditions) the wheels would simply spin without the car moving anywhere. Because of friction, the car's tire pushes backwards against the road−from Newton's third law, the frictional force acting on the tire must then push it forward. This may seem odd, as we tend to think of friction as being a retarding force only, but it is true.

21 •• When we push on the brake pedal to slow down a car, a brake pad is pressed against the rotor so that the friction of the pad slows the wheel's rotation. However, the friction of the pad against the rotor *cannot be* the force that slows the car down, because it is an internal force (both the rotor and the wheel are parts of the car, so any forces between them are purely internal to the system). What is

the external force that slows down the car? Give a detailed explanation of how this force operates.

Determine the Concept The friction of the tire against the road causes the car to slow down. This is rather subtle, as the tire is in contact with the ground without slipping at all times, and so as you push on the brakes harder, the force of static friction of the road against the tires must increase.

22 •• Give an example of each of the following. (*a*) A three-dimensional object that has no matter at its center of mass. (*b*) A solid object whose center of mass is outside of it. (*c*) A solid sphere whose center of mass does not lie at its geometrical center. (*d*) A long wooden stick whose center of mass does not lie at its middle.

(*a*) A solid spherical shell, or donut, or tire.

(*b*) A solid hemispherical shell.

(*c*) Any sphere with one side a different density than the other, or a density variation that isn't radially symmetric.

(*d*) Any stick with a non-uniform and non-symmetric density variation. A baseball bat is a good example of such a stick.

23 •• **[SSM]** When you are standing upright, your center of mass is located within the volume of your body. However, as you bend over (say to pick up a package), its location changes. Approximately where is it when you are bent over at right angles and what change in your body caused the center of mass location to change? Explain.

Determine the Concept Relative to the ground, your center of mass moves downward. This is because some of your mass (hips) moved backward, some of your mass (your head and shoulders) moved forward, and the top half of your body moved downward.

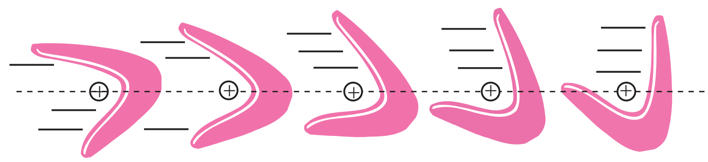
24 •• Early on their three-day (one-way) trip to the moon, the Apollo team (late 1960s to early 1970s) would explosively separate the lunar ship from the third-stage booster (that provided the final ″boost″) while still fairly close to Earth. During the explosion, how did the velocity of each of the two pieces of the system change? How did the velocity of the center of mass of the system change?

Determine the Concept The spacecraft speed increased toward the moon. The speed of the third-stage booster decreased, but the booster continued to move away from Earth and toward the moon. Right after the explosion, the center of

mass velocity was the same as before. After a while, however, the backward pull of gravity of Earth will cause it to decrease because the speeds of both the lunar ship and the booster decrease.

25 •• You throw a boomerang and for a while it "flies" horizontally in a straight line at a constant speed, while spinning rapidly. Draw a series of pictures, as viewed vertically down from overhead, of the boomerang in different rotational positions as it moves parallel to the surface of Earth. On each picture, indicate the location of the boomerang's center of mass and connect the dots to trace the trajectory of its center of mass. What is the center of mass's acceleration during this part of the flight?

Determine the Concept The diagram shows a spinning boomerang with its center of mass at the location of the circle. As viewed from above, the center of mass moves in a straight line as the boomerang spins about it. The acceleration of the center of mass is zero.

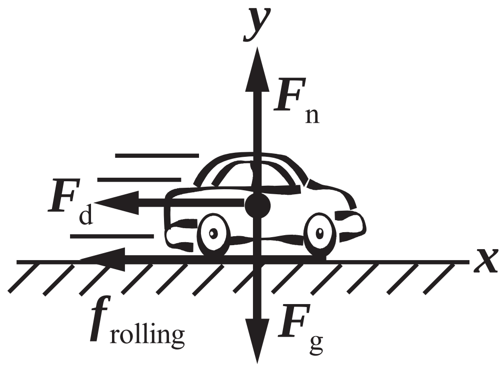


Estimation and Approximation

26 •• To determine the aerodynamic drag on a car, automotive engineers often use the "coast-down" method. The car is driven on a long, flat road at some convenient speed (60 mi/h is typical), shifted into neutral, and allowed to coast to a stop. The time that it takes for the speed to drop by successive 5-mi/h intervals is measured and used to compute the net force slowing the car down. (*a*) One day, a group measured that a Toyota Tercel with a mass of 1020 kg coasted down from 60.0 mi/h to 55.0 mi/h in 3.92 s. Estimate the average net force slowing the car down in this speed range. (*b*) If the coefficient of rolling friction for this car is known to be 0.020, what is the force of rolling friction that is acting to slow it down? Assuming that the only two forces acting on the car are rolling friction and aerodynamic drag, what is the average drag force acting on the car? (*c*) The drag force has the form $\frac{1}{2}C\rho Av^2$, where A is the cross-sectional area of the car facing into the air, v is the car's speed, ρ is the density of air, and C is a dimensionless constant of order 1. If the cross-sectional area of the car is 1.91 m^2, determine C from the data given. (The density of air is 1.21 kg/m^3; use 57.5 mi/h for the speed of the car in this computation.)

Picture the Problem The forces acting on the Tercel as it slows from 60 to 55 mi/h are a rolling-friction force exerted by the roadway, an air-drag force exerted by the air, the normal force exerted by the roadway, and the gravitational force exerted by Earth. The car is moving in the positive x direction. We can use Newton's second law to calculate the average force from the rate at which the

car's speed decreases and the rolling force from its definition. The drag force can be inferred from the average- and rolling-friction forces and the drag coefficient from the defining equation for the drag force.



(*a*) Apply $\sum F_x = ma_x$ to the car to relate the average force acting on it to its average acceleration:

$$F_{av} = ma_{av} = m\frac{\Delta v}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = (1020\,\text{kg})\frac{5\dfrac{\text{mi}}{\text{h}}\times 1.609\dfrac{\text{km}}{\text{mi}}\times\dfrac{1\,\text{h}}{3600\,\text{s}}\times\dfrac{1000\,\text{m}}{\text{km}}}{3.92\,\text{s}} = 581\,\text{N} = \boxed{0.58\,\text{kN}}$$

(*b*) The rolling-friction force is the product of the coefficient of rolling friction and the normal force:

$$f_{rolling} = \mu_{rolling}F_n = \mu_{rolling}mg$$

Substitute numerical values and evaluate $f_{rolling}$:

$$f_{rolling} = (0.020)(1020\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{0.20\,\text{kN}}$$

Assuming that only two forces are acting on the car opposite to the direction of its motion gives:

$$F_{av} = F_d + f_{rolling} \Rightarrow F_d = F_{av} - f_{rolling}$$

Substitute numerical values and evaluate F_d:

$$F_d = 581\,\text{N} - 200\,\text{N} = \boxed{0.38\,\text{kN}}$$

(*c*) Using the definition of the drag force and its calculated value from (*b*) and the average speed of the car during this 5 mph interval, solve for C:

$$F_d = \tfrac{1}{2}C\rho Av^2 \Rightarrow C = \frac{2F_d}{\rho Av^2}$$

Substitute numerical values and evaluate C:

$$C = \frac{2(381\,\text{N})}{(1.21\,\text{kg/m}^3)(1.91\,\text{m}^2)\left(57.5\frac{\text{mi}}{\text{h}}\times\frac{1.609\,\text{km}}{\text{mi}}\times\frac{1\,\text{h}}{3600\,\text{s}}\times\frac{10^3\,\text{m}}{\text{km}}\right)^2} = \boxed{0.50}$$

27 •• [SSM] Using dimensional analysis, determine the units and dimensions of the constant b in the retarding force bv^n if (*a*) $n = 1$ and (*b*) $n = 2$. (*c*) Newton showed that the air resistance of a falling object with a circular cross section should be approximately $\frac{1}{2}\rho\pi r^2 v^2$, where $\rho = 1.20$ kg/m^3, the density of air. Show that this is consistent with your dimensional analysis for Part (*b*). (*d*) Find the terminal speed for a 56.0-kg skydiver; approximate his cross-sectional area as a disk of radius 0.30 m. The density of air near the surface of Earth is 1.20 kg/m^3. (*e*) The density of the atmosphere decreases with height above the surface of Earth; at a height of 8.0 km, the density is only 0.514 kg/m^3. What is the terminal velocity at this height?

Picture the Problem We can use the dimensions of force and speed to determine the dimensions of the constant b and the dimensions of ρ, r, and v to show that, for $n = 2$, Newton's expression is consistent dimensionally with our result from part (*b*). In Parts (*d*) and (*e*), we can apply Newton's second law under terminal speed conditions to find the terminal speed of the sky diver near the surface of Earth and at a height of 8 km. Assume that $g = 9.81$ m/s^2 remains constant. (Note: At 8 km, $g = 9.78$ m/s^2. However, it will not affect the result in Part (*e*).)

(*a*) Solve the drag force equation for b with $n = 1$:

$$b = \frac{F_\text{d}}{v}$$

Substitute the dimensions of F_d and v and simplify to obtain:

$$[b] = \frac{\frac{\text{ML}}{\text{T}^2}}{\frac{\text{L}}{\text{T}}} = \boxed{\frac{\text{M}}{\text{T}}}$$

and the units of b are $\boxed{\text{kg/s}}$

(*b*) Solve the drag force equation for b with $n = 2$:

$$b = \frac{F_\text{d}}{v^2}$$

Substitute the dimensions of F_d and v and simplify to obtain:

$$[b] = \frac{\frac{\text{ML}}{\text{T}^2}}{\left(\frac{\text{L}}{\text{T}}\right)^2} = \boxed{\frac{\text{M}}{\text{L}}}$$

and the units of b are $\boxed{\text{kg/m}}$

(*c*) Express the dimensions of Newton's expression:

$$[F_d] = \left[\tfrac{1}{2}\rho\pi r^2 v^2\right] = \left(\frac{\text{M}}{\text{L}^3}\right)(\text{L})^2\left(\frac{\text{L}}{\text{T}}\right)^2 = \boxed{\frac{\text{ML}}{\text{T}^2}}$$

From Part (*b*) we have:

$$[F_d] = \left[bv^2\right] = \left(\frac{\text{M}}{\text{L}}\right)\left(\frac{\text{L}}{\text{T}}\right)^2 = \boxed{\frac{\text{ML}}{\text{T}^2}}$$

(*d*) Letting the downward direction be the $+y$ direction, apply $\sum F_y = ma_y$ to the sky diver:

$$mg - \tfrac{1}{2}\rho\pi r^2 v_t^2 = 0 \Rightarrow v_t = \sqrt{\frac{2mg}{\rho\pi r^2}}$$

Substitute numerical values and evaluate v_t:

$$v_t = \sqrt{\frac{2(56\,\text{kg})(9.81\,\text{m/s}^2)}{\pi(1.2\,\text{kg/m}^3)(0.30\,\text{m})^2}} = \boxed{57\,\text{m/s}}$$

(*e*) Evaluate v_t at a height of 8 km:

$$v_t = \sqrt{\frac{2(56\,\text{kg})(9.81\,\text{m/s}^2)}{\pi(0.514\,\text{kg/m}^3)(0.30\,\text{m})^2}} = \boxed{87\,\text{m/s}}$$

28 •• Estimate the terminal velocity of an average sized raindrop and a golf-ball- sized hailstone. (*Hint*: See Problems 26 and 27.)

Picture the Problem From Newton's second law, the equation describing the motion of falling raindrops and large hailstones is $mg - F_d = ma$ where $F_d = \frac{1}{2}\rho\pi r^2 v^2 = bv^2$ is the drag force. Under terminal speed conditions ($a = 0$), the drag force is equal to the weight of the falling object. Take the radius of a raindrop to be 0.50 mm and the radius of a golf-ball sized hailstone to be 2.0 cm.

Express the relationship between v_t and the weight of a falling object under terminal speed:

$$bv_t^2 = mg \Rightarrow v_t = \sqrt{\frac{mg}{b}} \qquad (1)$$

Using $b = \frac{1}{2}\pi\rho r^2$, evaluate b_r:

$$b_r = \tfrac{1}{2}\pi(1.2\,\text{kg/m}^3)(0.50\times10^{-3}\,\text{m})^2 = 4.71\times10^{-7}\,\text{kg/m}$$

Evaluating b_h yields:

$$b_h = \tfrac{1}{2}\pi(1.2\,\text{kg/m}^3)(2.0\times10^{-2}\,\text{m})^2 = 7.54\times10^{-4}\,\text{kg/m}$$

Express the mass of a sphere in terms of its volume and density:

$$m = \rho V = \frac{4\pi r^3 \rho}{3}$$

Using $\rho_r = 1.0 \times 10^3$ kg/m^3, evaluate m_r:

$$m_r = \frac{4\pi\left(0.50\times10^{-3}\ \text{m}\right)^3\left(1.0\times10^3\ \text{kg/m}^3\right)}{3} = 5.24\times10^{-7}\ \text{kg}$$

Using $\rho_h = 920$ kg/m^3, evaluate m_h:

$$m_h = \frac{4\pi\left(2.0\times10^{-2}\ \text{m}\right)^3\left(920\,\text{kg/m}^3\right)}{3} = 3.08\times10^{-2}\ \text{kg}$$

Substitute numerical values in equation (1) and evaluate $v_{t,r}$:

$$v_{t,r} = \sqrt{\frac{\left(5.24\times10^{-7}\ \text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{4.71\times10^{-7}\ \text{kg/m}}} = \boxed{3.3\,\text{m/s}}$$

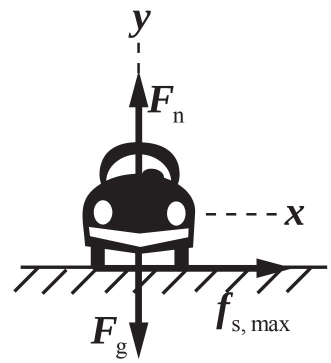
Substitute numerical values in equation (1) and evaluate $v_{t,h}$:

$$v_{t,h} = \sqrt{\frac{\left(3.08\times10^{-2}\ \text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{7.54\times10^{-4}\ \text{kg/m}}} = \boxed{20\,\text{m/s}}$$

29 •• Estimate the minimum coefficient of static friction needed between a car's tires and the pavement in order to complete a left turn at a city street intersection at the posted straight-ahead speed limit of 25 mph and on narrow inner-city streets. Comment on the wisdom of attempting such a turn at that speed.

Picture the Problem In order to perform this estimate, we need to determine a rough radius of curvature for the car's turn in a normal city intersection. Assuming the car goes from right-hand lane to right-hand lane, and assuming fairly normal dimensions of 40 feet for the width of the street, the center of the car's path travels along a circle of, say, 30 feet in radius. The net centripetal force is provided by the force of static friction and the acceleration of the car is equal to this net force divided by the mass of the car. Finally, we solve for the coefficient of static friction.

A diagram showing the forces acting on the car as it rounds the curve is shown to the right.



Apply $\sum F_x = ma_x$ to the car's tires:

$$f_{s,\,max} = ma_x = m\frac{v^2}{r}$$

or, because $f_{s,\,max} = \mu_s F_n$

$$\mu_s F_n = m\frac{v^2}{r} \Rightarrow \mu_s = \frac{mv^2}{rF_n} \quad (1)$$

Apply $\sum F_y = ma_y$ to the car's tires:

$$F_n - F_g = 0$$

or, because $F_g = mg$,

$$F_n - mg = 0 \Rightarrow F_n = mg$$

Substituting for F_n in equation (1) yields:

$$\mu_s = \frac{mv^2}{rmg} = \frac{v^2}{rg}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{\left(25\,\frac{\text{mi}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}} \times \frac{1609\,\text{m}}{\text{mi}}\right)^2}{\left(30\,\text{ft} \times 0.3048\,\frac{\text{m}}{\text{ft}}\right)\left(9.81\,\text{m/s}^2\right)}$$

$$= \boxed{1.4}$$

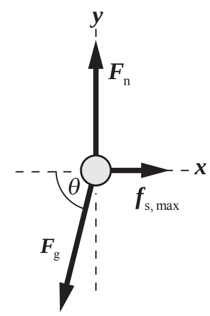
This is probably not such a good idea. Tires on asphalt or concrete have a maximum coefficient of static friction of about 1.

30 •• Estimate the widest stance you can take when standing on a dry, icy surface. That is, how wide can you safely place your feet and not slip into an undesired "split?" Let the coefficient of static friction of rubber on ice be roughly 0.25.

Picture the Problem We need to estimate the forces active at the place of each foot. Assuming a symmetrical stance, with the defining angle being the angle between each leg and the ground, θ, we can then draw a force diagram and apply Newton's second law to your foot. The free-body diagram shows the normal

force, exerted by the icy surface, the maximum static friction force, also exerted by the icy surface, and the force due to gravity exerted on your foot.

A free-body diagram showing the forces acting on one foot is shown to the right.



Apply $\sum \vec{F} = m\vec{a}$ to one of your feet:

$$\sum F_x = f_{s,max} - F_g \cos\theta = 0 \qquad (1)$$

and

$$\sum F_y = F_n - F_g \sin\theta = 0 \qquad (2)$$

Substituting $f_{s,\,max} = \mu_s F_n$ in equation (1) gives:

$$\mu_s F_n - F_g \cos\theta = 0$$

or

$$\mu_s F_n = F_g \cos\theta \qquad (3)$$

Solving equation (2) for F_n yields:

$$F_n = F_g \sin\theta \qquad (4)$$

Divide equation (4) by equation (3) to obtain:

$$\tan\theta = \frac{1}{\mu_s} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{\mu_s}\right)$$

Substitute the numerical value of μ_s and evaluate θ:

$$\theta = \tan^{-1}\left(\frac{1}{0.25}\right) = 76°$$

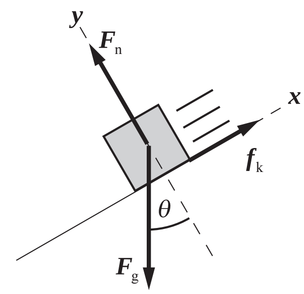
This angle corresponds to an angle between your legs of about 28°.

Friction

31 • [SSM] A block of mass m slides at constant speed down a plane inclined at an angle of θ with the horizontal. It follows that (*a*) $\mu_k = mg \sin\theta$, (*b*) $\mu_k = \tan\theta$, (*c*) $\mu_k = 1 - \cos\theta$, (*d*) $\mu_k = \cos\theta - \sin\theta$.

Picture the Problem The block is in equilibrium under the influence of $\vec{F}_n$, $m\vec{g}$, and $\vec{f}_k$; that is, $\vec{F}_n + m\vec{g} + \vec{f}_k = 0$. We can apply Newton's second law to determine the relationship between f_k, θ, and mg.

A pictorial representation showing the forces acting on the sliding block is shown to the right.



Using its definition, express the coefficient of kinetic friction:

$$\mu_k = \frac{f_k}{F_n} \quad (1)$$

Apply $\sum F_x = ma_x$ to the block:

$$f_k - mg\sin\theta = ma_x$$

or, because $a_x = 0$,

$$f_k = mg\sin\theta$$

Apply $\sum F_y = ma_y$ to the block:

$$F_n - mg\cos\theta = ma_y$$

or, because $a_y = 0$,

$$F_n = mg\cos\theta$$

Substitute for f_k and F_n in equation (1) and simplify to obtain:

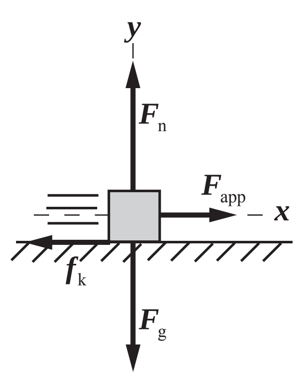
$$\mu_k = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta$$

and $\boxed{(b)}$ is correct.

32 • A block of wood is pulled at constant velocity by a horizontal string across a horizontal surface with a force of 20 N. The coefficient of kinetic friction between the surfaces is 0.3. The force of friction is (*a*) impossible to determine without knowing the mass of the block, (*b*) impossible to determine without knowing the speed of the block, (*c*) 0.30 N, (*d*) 6.0 N, or (*e*) 20 N.

Picture the Problem The block is in equilibrium under the influence of $\vec{F}_n$, $\vec{F}_g$, $\vec{F}_{app}$, and $\vec{f}_k$; that is

$$\vec{F}_n + \vec{F}_g + \vec{F}_{app} + \vec{f}_k = 0$$



We can apply Newton's second law to determine f_k.

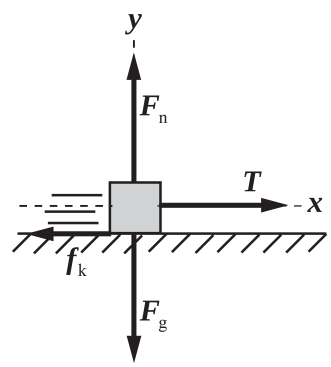
Apply $\sum F_x = ma_x$ to the block: $F_{\text{app}} - f_{\text{k}} = ma_x$

or, because $a_x = 0$,

$f_{\text{k}} = F_{\text{app}} = 20\ \text{N}$ and $\boxed{(e)}$ is correct.

33 • **[SSM]** A block weighing 20-N rests on a horizontal surface. The coefficients of static and kinetic friction between the surface and the block are $\mu_s = 0.80$ and $\mu_k = 0.60$. A horizontal string is then attached to the block and a constant tension T is maintained in the string. What is the subsequent force of friction acting on the block if (*a*) $T = 15$ N or (*b*) $T = 20$ N?

Picture the Problem Whether the friction force is that due to static friction or kinetic friction depends on whether the applied tension is greater than the maximum static friction force. We can apply the definition of the maximum static friction to decide whether $f_{\text{s,max}}$ or T is greater.



Noting that $F_{\text{n}} = F_{\text{g}}$, calculate the maximum static friction force:

$$f_{\text{s,max}} = \mu_{\text{s}} F_{\text{n}} = \mu_{\text{s}} F_{\text{g}} = (0.80)(20\ \text{N}) = 16\ \text{N}$$

(*a*) Because $f_{\text{s,max}} > T$:

$$f = f_{\text{s}} = T = \boxed{15\ \text{N}}$$

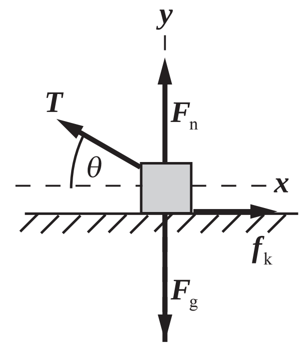
(*b*) Because $T > f_{\text{s,max}}$:

$$f = f_{\text{k}} = \mu_{\text{k}} F_{\text{n}} = \mu_{\text{k}} F_{\text{g}} = (0.60)(20\ \text{N}) = \boxed{12\ \text{N}}$$

34 • A block of mass m is pulled at a constant velocity across a horizontal surface by a string as shown in Figure 5-60. The magnitude of the frictional force is (*a*) $\mu_k mg$, (*b*) $T\cos\theta$, (*c*) $\mu_k(T - mg)$, (*d*) $\mu_k T\sin\theta$, or (*e*) $\mu_k(mg - T\sin\theta)$.

Picture the Problem The block is in equilibrium under the influence of the forces $\vec{T}$, $\vec{f}_{\text{k}}$, $\vec{F}_n$, and $\vec{F}_g$; that is $\vec{T} + \vec{f}_{\text{k}} + \vec{F}_g + \vec{F}_n = 0$. We can apply Newton's second law to determine the relationship between T and f_{k}.

A free-body diagram showing the forces acting on the block is shown to the right.

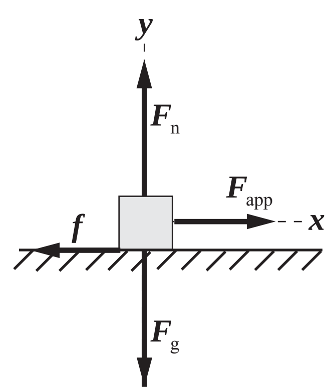


Apply $\sum F_x = ma_x$ to the block: $-T\cos\theta + f_k = ma_x$

Because $a_x = 0$: $f_k = T\cos\theta$ and $\boxed{(b)}$ is correct.

35 • **[SSM]** A 100-kg crate rests on a thick-pile carpet. A weary worker then pushes on the crate with a horizontal force of 500 N. The coefficients of static and kinetic friction between the crate and the carpet are 0.600 and 0.400, respectively. Find the subsequent frictional force exerted by the carpet on the crate.

Picture the Problem Whether the friction force is that due to static friction or kinetic friction depends on whether the applied tension is greater than the maximum static friction force. If it is, then the box moves and the friction force is the force of kinetic friction. If it is less, the box does not move.



The maximum static friction force is given by:

$$f_{s,max} = \mu_s F_n$$

or, because $F_n = F_g = mg$,

$$f_{s,max} = \mu_s mg$$

Substitute numerical values and evaluate $f_{s,max}$:

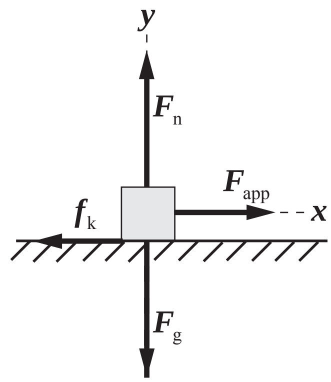
$$f_{s,max} = (0.600)(100\,\text{kg})(9.81\,\text{m/s}^2) = 589\,\text{N}$$

Because $f_{s,\,max} > F_{app}$, the box does not move and :

$$F_{app} = f_s = \boxed{500\,\text{N}}$$

36 • A box weighing 600 N is pushed along a horizontal floor at constant velocity with a force of 250 N parallel to the floor. What is the coefficient of kinetic friction between the box and the floor?

Picture the Problem Because the box is moving with constant velocity, its acceleration is zero and it is in equilibrium under the influence of $\vec{F}_{\text{app}}$, $\vec{F}_{\text{n}}$, $\vec{F}_{\text{g}}$, and $\vec{f}_{\text{k}}$; that is, $\vec{F}_{\text{app}} + \vec{F}_{\text{n}} + \vec{F}_{\text{g}} + \vec{f}_{\text{k}} = 0$. We can apply Newton's second law to determine the relationship between f_{k} and mg.



The definition of μ_{k} is:

$$\mu_{\text{k}} = \frac{f_{\text{k}}}{F_{\text{n}}} \qquad (1)$$

Apply $\sum F_y = ma_y$ to the box:

$$F_{\text{n}} - F_{\text{g}} = ma_y$$

or, because $a_y = 0$,

$$F_{\text{n}} - F_{\text{g}} = 0 \Rightarrow F_{\text{n}} = F_{\text{g}} = mg = 600\text{ N}$$

Apply $\sum F_x = ma_x$ to the box:

$$F_{\text{app}} - f_{\text{k}} = ma_x$$

or, because $a_x = 0$,

$$F_{\text{app}} = f_{\text{k}} = 250\text{ N}$$

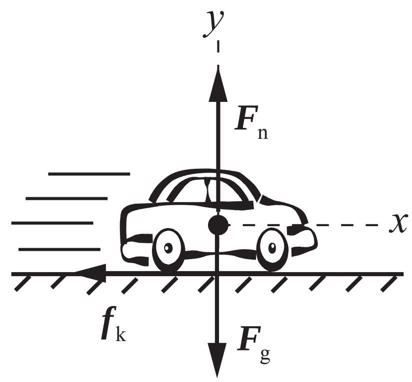
Substitute numerical values in equation (1) and evaluate μ_{k}:

$$\mu_{\text{k}} = \frac{250\text{ N}}{600\text{ N}} = \boxed{0.417}$$

37 • **[SSM]** The coefficient of static friction between the tires of a car and a horizontal road is 0.60. Neglecting air resistance and rolling friction, (*a*) what is the magnitude of the maximum acceleration of the car when it is braked? (*b*) What is the shortest distance in which the car can stop if it is initially traveling at 30 m/s?

Picture the Problem Assume that the car is traveling to the right and let the positive x direction also be to the right. We can use Newton's second law of motion and the definition of μ_{s} to determine the maximum acceleration of the car. Once we know the car's maximum acceleration, we can use a constant-acceleration equation to determine the least stopping distance.

(*a*) A diagram showing the forces acting on the car is shown to the right.



Apply $\sum F_x = ma_x$ to the car: $\quad -f_{s,max} = -\mu_s F_n = ma_x \qquad (1)$

Apply $\sum F_y = ma_y$ to the car and solve for F_n:

$$F_n - F_g = ma_y$$

$$F_n - w = ma_y = 0$$

or, because $a_y = 0$ and $F_g = mg$,

$$F_n = mg \qquad (2)$$

Substitute for Fn in equation (1) to obtain:

$$-f_{s,max} = -\mu_s mg = ma_x$$

Solving for $a_{x,max}$ yields:

$$a_{x,max} = \mu_s g$$

Substitute numerical values and evaluate $a_{x,max}$:

$$a_{x,max} = (0.60)(9.81\ \text{m/s}^2) = 5.89\ \text{m/s}^2 = \boxed{5.9\ \text{m/s}^2}$$

(b) Using a constant-acceleration equation, relate the stopping distance of the car to its initial speed and its acceleration:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 + 2a_x\Delta x \Rightarrow \Delta x = \frac{-v_{0x}^2}{2a_x}$$

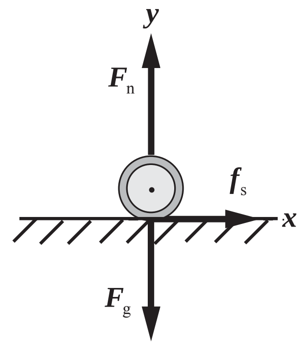
Using ax = −5.89 m/s2 because the acceleration of the car is to the left, substitute numerical values and evaluate Δx:

$$\Delta x = \frac{-(30\ \text{m/s})^2}{2(-5.89\ \text{m/s}^2)} = \boxed{76\ \text{m}}$$

38 • The force that accelerates a car along a flat road is the frictional force exerted by the road on the car's tires. (*a*) Explain why the acceleration can be greater when the wheels do not slip. (*b*) If a car is to accelerate from 0 to 90 km/h in 12 s, what is the minimum coefficient of friction needed between the road and tires? Assume that the drive wheels support exactly half the weight of the car.

Picture the Problem We can use the definition of acceleration and apply Newton's second law to the horizontal and vertical components of the forces to determine the minimum coefficient of friction between the road and the tires.

(*a*) The free-body diagram shows the forces acting on the tires on the drive wheels, the tires we're assuming support half the weight of the car.



Because $\mu_s > \mu_k$, f will be greater if the wheels do not slip.

(*b*) Apply $\sum F_x = ma_x$ to the car: $f_s = \mu_s F_n = ma_x$ (1)

Apply $\sum F_y = ma_y$ to the car and solve for F_n: $F_n - F_g = ma_y$
or, because $a_y = 0$ and $F_g = \frac{1}{2}mg$,
$F_n = \frac{1}{2}mg$

Substituting for F_n in equation (1) yields: $\frac{1}{2}\mu_s mg = ma_x \Rightarrow \mu_s = \frac{2a_x}{g}$

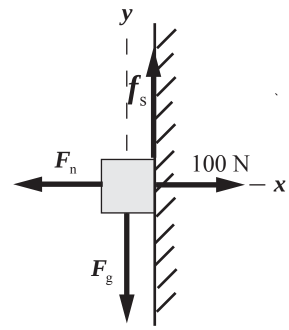
Substitute for a_x to obtain: $\mu_s = \frac{2\Delta v}{g\Delta t}$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{2\left(90\,\frac{\text{km}}{\text{h}} \times \frac{10^3\ \text{m}}{\text{km}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)}{\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(12\,\text{s})} = \boxed{0.42}$$

39 •• A 5.00-kg block is held at rest against a vertical wall by a horizontal force of 100 N. (*a*) What is the frictional force exerted by the wall on the block? (*b*) What is the minimum horizontal force needed to prevent the block from falling if the static coefficient of friction between the wall and the block is 0.400?

Picture the Problem The block is in equilibrium under the influence of the forces shown on the force diagram. We can use Newton's second law and the definition of μ_s to solve for f_s and F_n.



(*a*) Apply $\sum F_y = ma_y$ to the block:

$f_s - mg = ma_y$

or, because $a_y = 0$,

$f_s - mg = 0 \Rightarrow f_s = mg$

Substitute numerical values and evaluate f_s :

$$f_s = (5.00\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{49.1\,\text{N}}$$

(*b*) Use the definition of μ_s to express F_n:

$$F_n = \frac{f_{s,\max}}{\mu_s}$$

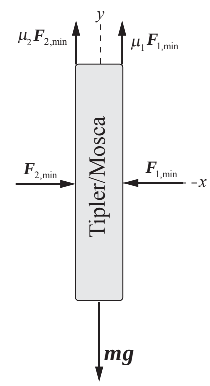
Substitute numerical values and evaluate F_n:

$$F_n = \frac{49.1\,\text{N}}{0.400} = \boxed{123\,\text{N}}$$

40 •• A tired and overloaded student is attempting to hold a large physics textbook wedged under his arm, as shown in Figure 5-61. The textbook has a mass of 3.2 kg, while the coefficient of static friction of the textbook against the student's underarm is 0.320 and the coefficient of static friction of the book against the student's shirt is 0.160. (*a*) What is the minimum horizontal force that the student must apply to the textbook to prevent it from falling? (*b*) If the student can only exert a force of 61 N, what is the acceleration of the textbook as it slides from under his arm? The coefficient of kinetic friction of arm against textbook is 0.200, while that of shirt against textbook is 0.090.

Picture the Problem We can apply Newton's second law to relate the minimum force required to hold the book in place to its mass and to the coefficients of static friction. In Part (*b*), we can proceed similarly to relate the acceleration of the book to the coefficients of kinetic friction.

(*a*) The force diagram shows the forces acting on the book. The normal force is the net force the student exerts in squeezing the book. Let the horizontal direction be the +*x* direction and upward the +*y* direction. Note that the normal force is the same on either side of the book because it is not accelerating in the horizontal direction. The book could be accelerating downward.



Apply $\sum \vec{F} = m\vec{a}$ to the book:

$$\sum F_x = F_{2,\min} - F_{1,\min} = 0$$

and

$$\sum F_y = \mu_{s,1} F_{1,\min} + \mu_{s,2} F_{2,\min} - mg = 0$$

Noting that $F_{1,\min} = F_{2,\min}$, solve the *y* equation for $F_{\min}$:

$$F_{\min} = \frac{mg}{\mu_{s,1} + \mu_{s,2}}$$

Substitute numerical values and evaluate $F_{\min}$:

$$F_{\min} = \frac{(3.2\,\text{kg})(9.81\,\text{m/s}^2)}{0.320 + 0.160} = \boxed{65\,\text{N}}$$

(*b*) Apply $\sum F_y = ma_y$ with the book accelerating downward, to obtain:

$$\sum F_y = \mu_{k,1} F + \mu_{k,2} F - mg = ma$$

Solving for a_y yields:

$$a_y = \frac{\mu_{k,1} + \mu_{k,2}}{m} F - g$$

Substitute numerical values and evaluate a_y:

$$a_y = \left| \left(\frac{0.200 + 0.090}{3.2\,\text{kg}} \right) (61\,\text{N}) - 9.81\,\text{m/s}^2 \right|$$

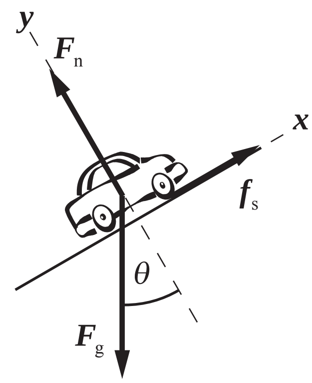
$$= \boxed{4.3\,\text{m/s}^2\text{, downward}}$$

41 •• You are racing in a rally on a snowy day when the temperature is near the freezing point. The coefficient of static friction between a car's tires and an icy road is 0.080. Your crew boss is concerned about some of the hills on the course and wants you to think about switching to studded tires. To address the issue, he wants to compare the actual hill angles on the course to see which of them your car can negotiate. (*a*) What is the angle of the steepest incline that a vehicle with four-wheel drive can climb at constant speed? (*b*) Given that the hills are icy, what is the steepest possible hill angle for the same four-wheel drive car to descend at constant speed?

Picture the Problem We can use the definition of the coefficient of static friction

and Newton's second law to relate the angle of the incline to the forces acting on the car.

(*a*) The free-body diagram shows the forces acting on the car when it is either moving up the hill or down the hill without acceleration. The friction force that the ground exerts on the tires is the force f_s shown acting up the incline.



Apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = f_s - F_g \sin\theta = 0 \qquad (1)$$

and

$$\sum F_y = F_n - F_g \cos\theta = 0 \qquad (2)$$

Because $F_g = mg$, equations (1) and (2) become:

$$f_s - mg\sin\theta = 0 \qquad (3)$$

and

$$F_n - mg\cos\theta = 0 \qquad (4)$$

Solving equation (3) for f_s and equation (4) for F_n yields:

$$f_s = mg\sin\theta$$

and

$$F_n = mg\cos\theta$$

Use the definition of μ_s to relate f_s and F_n:

$$\mu_s = \frac{f_s}{F_n} = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta$$

Solving for θ yields:

$$\theta = \tan^{-1}(\mu_s) = \tan^{-1}(0.080) = \boxed{4.6°}$$

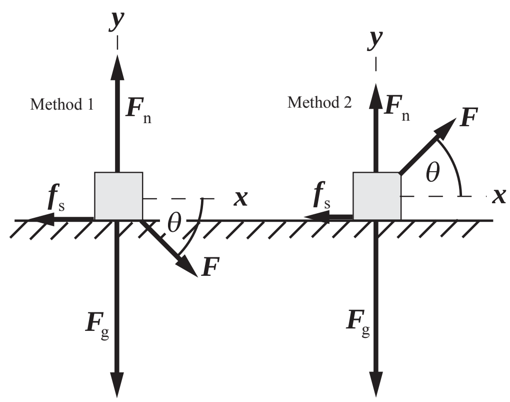
(*b*) Proceed exactly as in (*a*) to obtain:

$$\theta = \tan^{-1}(0.080) = \boxed{4.6°}$$

42 •• A 50-kg box that is resting on a level floor must be moved. The coefficient of static friction between the box and the floor is 0.60. One way to move the box is to push down on the box at an angle θ below the horizontal. Another method is to pull up on the box at an angle θ above the horizontal. (*a*) Explain why one method requires less force than the other. (*b*) Calculate the minimum force needed to move the box by each method if θ = 30º and compare the answer with the results when θ = 0°.

Picture the Problem The free-body

diagrams for the two methods are shown to the right. Method 1 results in the box being pushed into the floor, increasing the normal force and the static friction force. Method 2 partially lifts the box,, reducing the normal force and the static friction force. We can apply Newton's second law to obtain expressions that relate the maximum static friction force to the applied force $\vec{F}$.



(*a*) Method 2 is preferable because it reduces F_n and, therefore, f_s.

(*b*) Apply $\sum F_x = ma_x$ to the box:

$$F\cos\theta - f_x = F\cos\theta - \mu_s F_n = 0$$

Method 1: Apply $\sum F_y = ma_y$ to the block and solve for F_n:

$$F_n - mg - F\sin\theta = 0$$
and
$$F_n = mg + F\sin\theta$$

Relate $f_{s,max}$ to F_n:

$$f_{s,max} = \mu_s F_n = \mu_s(mg + F\sin\theta) \quad (1)$$

Method 2: Apply $\sum F_y = ma_y$ to the forces in the y direction and solve for F_n:

$$F_n - mg + F\sin\theta = 0$$
and
$$F_n = mg - F\sin\theta$$

Relate $f_{s,max}$ to F_n:

$$f_{s,max} = \mu_s F_n = \mu_s(mg - F\sin\theta) \quad (2)$$

Express the condition that must be satisfied to move the box by either method:

$$f_{s,max} = F\cos\theta \quad (3)$$

Method 1: Substitute (1) in (3) and solve for F:

$$F_1 = \frac{\mu_s mg}{\cos\theta - \mu_s \sin\theta} \quad (4)$$

Method 2: Substitute (2) in (3) and solve for F:

$$F_2 = \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta} \quad (5)$$

Substitute numerical values and evaluate equations (4) and (5) with $\theta = 30°$:

$$F_1(30°) = \frac{(0.60)(50\text{ kg})(9.81\text{ m/s}^2)}{\cos 30° - (0.60)\sin 30°} = \boxed{0.52\text{ kN}}$$

and

$$F_2(30°) = \frac{(0.60)(50\text{ kg})(9.81\text{ m/s}^2)}{\cos 30° + (0.60)\sin 30°} = \boxed{0.25\text{ kN}}$$

Evaluate equations (4) and (5) with $\theta = 0°$:

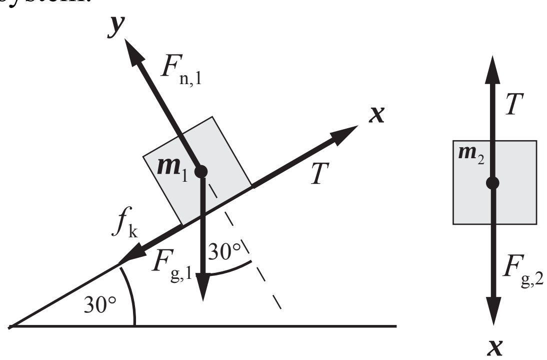
$$F_1(0°) = \frac{(0.60)(50\text{ kg})(9.81\text{ m/s}^2)}{\cos 0° - (0.60)\sin 0°} = \boxed{0.29\text{ kN}}$$

and

$$F_2(0°) = \frac{(0.60)(50\text{ kg})(9.81\text{ m/s}^2)}{\cos 0° + (0.60)\sin 0°} = \boxed{0.29\text{ kN}}$$

43 •• [SSM] A block of mass $m_1 = 250$ g is at rest on a plane that makes an angle of $\theta = 30°$ with the horizontal. The coefficient of kinetic friction between the block and the plane is 0.100. The block is attached to a second block of mass $m_2 = 200$ g that hangs freely by a string that passes over a frictionless, massless pulley (Figure 5-62). When the second block has fallen 30.0 cm, what will be its speed?

Picture the Problem Choose a coordinate system in which the $+x$ direction is up the incline for the block whose mass is m_1 and downward for the block whose mass is m_2. We can find the speed of the system when it has moved a given distance by using a constant-acceleration equation. We'll assume that the string is massless and that it does not stretch. Under the influence of the forces shown in the free-body diagrams, the blocks will have a common acceleration a. The application of Newton's second law to each block, followed by the elimination of the tension T and the use of the definition of f_k, will allow us to determine the acceleration of the system.



Using a constant-acceleration equation, relate the speed of the system to its acceleration and displacement:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x$$

and, because $v_{0x} = 0$,

$$v_x^2 = 2a_x\Delta x \Rightarrow v_x = \sqrt{2a_x\Delta x} \qquad (1)$$

Apply $\sum\vec{F} = m\vec{a}$ to the block whose mass is m_1:

$$\sum F_x = T - f_k - F_{g,1}\sin 30° = m_1 a_x \qquad (2)$$

and

$$\sum F_y = F_{n,1} - F_{g,1}\cos 30° = 0 \qquad (3)$$

Because $F_{g,1} = m_1 g$, equations (2) and (3) can be written as:

$$T - f_k - m_1 g\sin 30° = m_1 a_x \qquad (4)$$

and

$$F_{n,1} = m_1 g\cos 30° \qquad (5)$$

Using $f_k = \mu_k F_{n,1}$, substitute equation (5) in equation (4) to obtain:

$$T - \mu_k m_1 g\cos 30° - m_1 g\sin 30° = m_1 a_x \qquad (6)$$

Applying $\sum F_x = ma_x$ to the block whose mass is m_2 gives:

$$F_{g,2} - T = m_2 a_x$$

or, because $F_{g,2} = m_2 g$,

$$m_2 g - T = m_2 a_x \qquad (7)$$

Add equations (6) and (7) to eliminate T and then solve for a_x to obtain:

$$a_x = \frac{(m_2 - \mu_k m_1\cos 30° - m_1\sin 30°)g}{m_1 + m_2}$$

Substituting for a_x in equation (1) and simplifying yields:

$$v_x = \sqrt{\frac{2[m_2 - m_1(\mu_k\cos 30° + \sin 30°)]g\Delta x}{m_1 + m_2}}$$

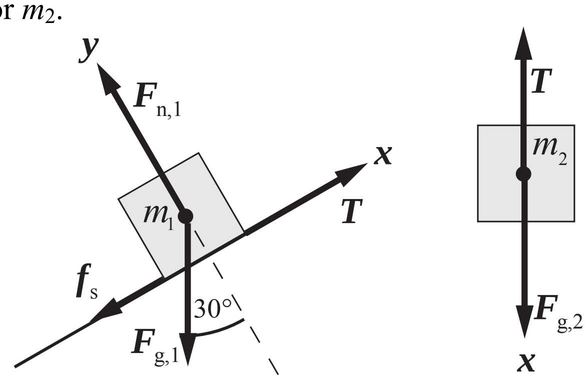
Substitute numerical values and evaluate v_x:

$$v_x = \sqrt{\frac{2[0.200\text{ kg} - (0.250\text{ kg})((0.100)\cos 30° + \sin 30°)](9.81\text{ m/s}^2)(0.300\text{ m})}{0.250\text{ kg} + 0.200\text{ kg}}}$$

$$= \boxed{84\text{ cm/s}}$$

44 •• In Figure 5-62 $m_1 = 4.0$ kg and the coefficient of static friction between the block and the incline is 0.40. (*a*) Find the range of possible values for m_2 for which the system will be in static equilibrium. (*b*) Find the frictional force on the 4.0-kg block if $m_2 = 1.0$ kg?

Picture the Problem Choose a coordinate system in which the +*x* direction is up the incline for the block whose mass is m_1 and downward for the block whose mass is m_2. We'll assume that the string is massless and that it does not stretch. Under the influence of the forces shown in the free-body diagrams, the blocks are in static equilibrium. While f_s can be either up or down the incline, the free-body diagram shows the situation in which motion is impending up the incline. The application of Newton's second law to each block, followed by the elimination of the tension T and the use of the definition of f_s, will allow us to determine the range of values for m_2.



(*a*) Noting that $F_{g,1} = m_1 g$, apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_1:

$$\sum F_x = T \pm f_{s,max} - m_1 g \sin 30° = 0 \quad (1)$$

and

$$\sum F_y = F_{n,1} - m_1 g \cos 30° = 0 \quad (2)$$

Using $f_{s,\,max} = \mu_s F_n$, substitute equation (2) in equation (1) to obtain:

$$T \pm \mu_s m_1 g \cos 30° - m_1 g \sin 30° = 0 \quad (3)$$

Noting that $F_{g,2} = m_2 g$, apply $\sum F_x = m a_x$ to the block whose mass is m_2:

$$m_2 g - T = 0 \quad (4)$$

Add equations (3) and (4) to eliminate T and solve for m_2:

$$m_2 = m_1(\pm\mu_s\cos 30° + \sin 30°) \quad (5)$$

Substitute numerical values to obtain:

$$m_2 = (4.0\,\text{kg})[\pm(0.40)\cos 30° + \sin 30°]$$

Denoting the value of m_2 with a plus sign as $m_{2,+}$ and the value of m_2 with the minus sign as $m_{2,-}$ determine the range of values of m_2 for which the system is in static equilibrium:

$m_{2,+} = 3.4\,\text{kg}$ and $m_{2,-} = 0.61\,\text{kg}$

and

$$\boxed{0.61\,\text{kg} \le m_2 \le 3.4\,\text{kg}}$$

(*b*) With m_2 = 1 kg, the impending motion is down the incline and the static friction force is up the incline. Apply $\sum F_x = ma_x$ to the block whose mass is m_1:

$$T + f_s - m_1 g\sin 30° = 0 \quad (6)$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$$m_2 g - T = 0 \quad (7)$$

Add equations (6) and (7) and solve for f_s to obtain:

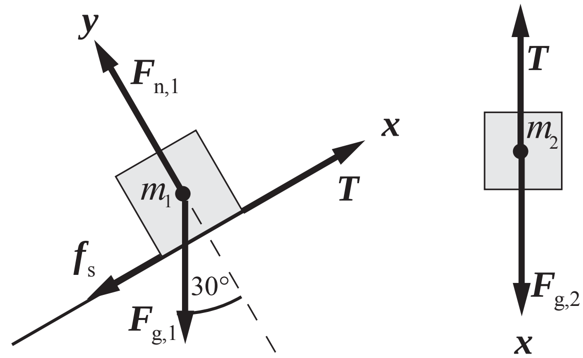
$$f_s = (m_1\sin 30° - m_2)g$$

Substitute numerical values and evaluate f_s:

$$f_s = [(4.0\,\text{kg})\sin 30° - 1.0\,\text{kg}](9.81\,\text{m/s}^2) = \boxed{9.8\,\text{N}}$$

45 •• In Figure 5-62, m_1 = 4.0 kg, m_2 = 5.0 kg, and the coefficient of kinetic friction between the inclined plane and the 4.0-kg block is μ_k = 0.24. Find the magnitude of the acceleration of the masses and the tension in the cord.

Picture the Problem Choose a coordinate system in which the +x direction is up the incline for the block whose mass is m_1 and downward for the block whose mass is m_2. We'll assume that the string is massless and that it does not stretch. Under the influence of the forces shown in the free-body diagrams, the blocks will have a common acceleration a. The application of Newton's second law to each block, followed by the elimination of the tension T and the use of the definition of f_k, will allow us to determine the acceleration of the system. Finally, we can substitute for the tension in either of the motion equations to determine the acceleration of the masses.



Noting that $F_{g,1} = m_1 g$, apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_1:

$$\sum F_x = T - f_k - m_1 g \sin 30° = m_1 a_x \quad (1)$$

and

$$\sum F_y = F_{n,1} - m_1 g \cos 30° = 0 \quad (2)$$

Using $f_k = \mu_k F_n$, substitute equation (2) in equation (1) to obtain:

$$T - \mu_k m_1 g \cos 30° - m_1 g \sin 30° = m_1 a_x \quad (3)$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$$m_2 g - T = m_2 a_x \quad (4)$$

Add equations (3) and (4) to eliminate T and solve for a_x to obtain:

$$a_x = \frac{(m_2 - \mu_k m_1 \cos 30° - m_1 \sin 30°)g}{m_1 + m_2}$$

Substituting numerical values and evaluating a_x yields:

$$a_x = \frac{[5.0\text{ kg} - (0.24)(4.0\text{ kg})\cos 30° - (4.0\text{ kg})\sin 30°](9.81\text{ m/s}^2)}{4.0\text{ kg} + 5.0\text{ kg}} = \boxed{2.4\text{ m/s}^2}$$

Solving equation (3) for T yields:

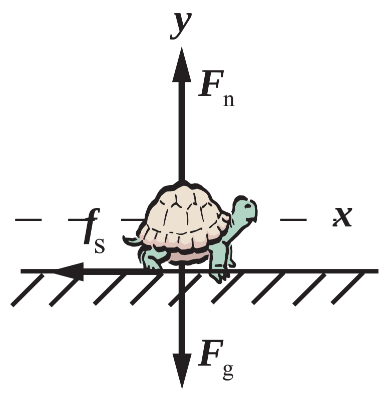
$$T = m_1 a_x + [\mu_k \cos 30° + \sin 30°]m_1 g$$

Substitute numerical values and evaluate T:

$$T = (4.0\text{ kg})(2.36\text{ m/s}^2) + [(0.24)\cos 30° + \sin 30°](4.0\text{ kg})(9.81\text{ m/s}^2) = \boxed{37\text{ N}}$$

46 •• A 12-kg turtle rests on the bed of a zookeeper's truck, which is traveling down a country road at 55 mi/h. The zookeeper spots a deer in the road, and slows to a stop in 12 s. Assuming constant acceleration, what is the minimum coefficient of static friction between the turtle and the truck bed surface that is needed to prevent the turtle from sliding?

Picture the Problem We can determine the acceleration necessary for the truck and turtle by considering the displacement of both during the given time interval. The static friction force must provide the necessary acceleration for the turtle. The turtle, if it is not to slip, must have this acceleration which is produced by the static friction force acting on it



The required coefficient of static friction is given by:

$$\mu_s = \frac{f_s}{F_n} \qquad (1)$$

Letting m represent the mass of the turtle, apply $\sum \vec{F} = m\vec{a}$ to the turtle:

$$\sum F_x = -f_s = ma_x \qquad (2)$$

and

$$\sum F_y = F_n - F_g = ma_y \qquad (3)$$

Solving equation (2) for f_s yields:

$$f_s = -ma_x$$

Because $a_y = 0$ and $F_g = mg$, equation (3) becomes:

$$F_n = F_g = mg$$

Substituting for f_s and F_n in equation (1) and simplifying yields:

$$\mu_s = \frac{-ma_x}{mg} = \frac{-a_x}{g} \qquad (4)$$

The acceleration of the truck and turtle is given by:

$$a_x = \frac{\Delta v}{\Delta t} = \frac{v_{f,x} - v_{i,x}}{\Delta t}$$

or, because $v_{f,x} = 0$,

$$a_x = \frac{-v_{i,x}}{\Delta t}$$

Substitute for a_x in equation (4) to obtain:

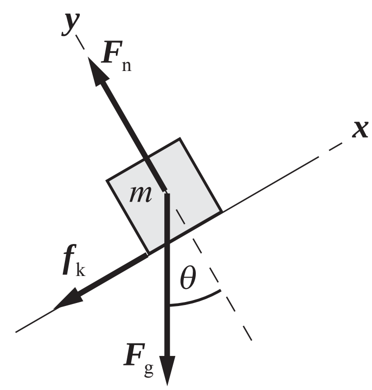
$$\mu_s = \frac{v_{i,x}}{g\Delta t}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{55\,\frac{\text{mi}}{\text{h}} \times \frac{1609\text{ m}}{\text{mi}} \times \frac{1\text{ h}}{3600\text{ s}}}{\left(9.81\text{ m/s}^2\right)\left(12\text{ s}\right)} = \boxed{0.21}$$

47 •• [SSM] A 150-g block is projected up a ramp with an initial speed of 7.0 m/s. The coefficient of kinetic friction between the ramp and the block is 0.23. (*a*) If the ramp is inclined 25° with the horizontal, how far along the surface of the ramp does the block slide before coming to a stop? (*b*) The block then slides back down the ramp. What is the coefficient of static friction between the block and the ramp if the block is not to slide back down the ramp?

Picture the Problem The force diagram shows the forces acting on the block as it slides up the ramp. Note that the block is accelerated by $\vec{f}_k$ and the x component of $\vec{F}_g$. We can use a constant-acceleration equation to express the displacement of the block up the ramp as a function of its acceleration and Newton's second law to find the acceleration of the block as it slides up the ramp.



(*a*) Use a constant-acceleration equation to relate the distance the block slides up the incline to its initial speed and acceleration:

$$v_x^2 = v_{0x}^2 + 2a_x\Delta x$$

or, because $v_x = 0$,

$$0 = v_{0x}^2 + 2a_x\Delta x \Rightarrow \Delta x = \frac{-v_{0x}^2}{2a_x} \quad (1)$$

Apply $\sum\vec{F} = m\vec{a}$ to the block:

$$\sum F_x = -f_k - F_g\sin\theta = ma_x \quad (2)$$

and

$$\sum F_y = F_n - F_g\cos\theta = 0 \quad (3)$$

Substituting $f_k = \mu_k F_n$ and $F_g = mg$ in equations (2) and (3) yields:

$$-\mu_k F_n - mg\sin\theta = ma_x \quad (4)$$

and

$$F_n - mg\cos\theta = 0 \quad (5)$$

Eliminate F_n between equations (4) and (5) to obtain:

$$-\mu_k mg\cos\theta - mg\sin\theta = ma_x$$

Solving for a_x yields:

$$a_x = -(\mu_k\cos\theta + \sin\theta)g$$

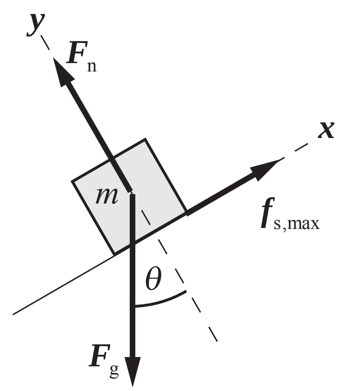
Substitute for a in equation (1) to obtain:

$$\Delta x = \frac{v_{0x}^2}{2(\mu_k\cos\theta + \sin\theta)g}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{(7.0\ \text{m/s})^2}{2\left[(0.23)\cos 25° + \sin 25°\right](9.81\ \text{m/s}^2)} = 3.957\ \text{m} = \boxed{4.0\ \text{m}}$$

(*b*) At the point at which the block is instantaneously at rest, static friction becomes operative and, if the static friction coefficient is too high, the block will not resume motion, but will remain at the high point. We can determine the maximum possible value of μs for which the block still slides back down the incline by considering the equality of the static friction force and the component of the weight of the block down the ramp.



Apply $\sum\vec{F} = m\vec{a}$ to the block when it is in equilibrium at the point at which it is momentarily at rest:

$$\sum F_x = f_{\text{s,max}} - F_g \sin\theta = 0 \quad (5)$$

and

$$\sum F_y = F_n - F_g \cos\theta = 0 \quad (6)$$

Solving equation (6) for F_n yields:

$$F_n = F_g \cos\theta$$

Because $f_{\text{s,max}} = \mu_s F_n$, equation (5) becomes:

$$\mu_s F_g \cos\theta - F_g \sin\theta = 0$$

or

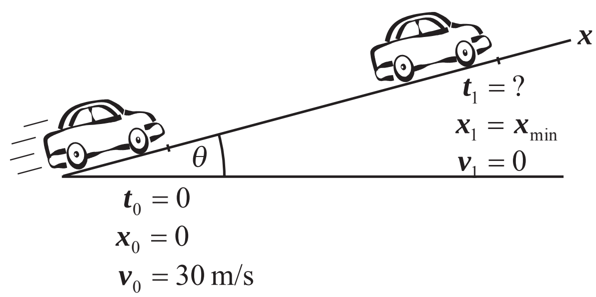
$$\mu_s \cos\theta - \sin\theta = 0 \Rightarrow \mu_s = \tan\theta$$

Substitute the numerical value of θ and evaluate μ_s:

$$\mu_s = \tan 25° = \boxed{0.47}$$

48 •• An automobile is going up a 15º grade at a speed of 30 m/s. The coefficient of static friction between the tires and the road is 0.70. (*a*) What minimum distance does it take to stop the car? (*b*) What minimum distance would it take to stop if the car were going down the grade?

Picture the Problem We can find the stopping distances by applying Newton's second law to the automobile and then using a constant-acceleration equation. The friction force the road exerts on the tires and the component of the car's weight along the incline combine to provide the net force that stops the car. The pictorial representation summarizes what we know about the motion of the car. We can use Newton's second law to determine the acceleration of the car and a constant-acceleration equation to obtain its stopping distance.



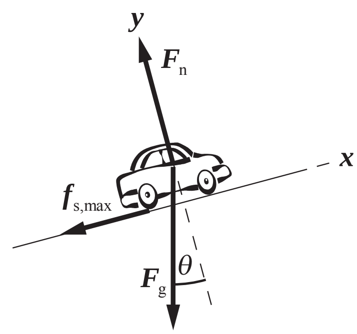
(*a*) Using a constant-acceleration equation, relate the final speed of the car to its initial speed, acceleration, and displacement; solve for its displacement:

$$v_{1x}^2 = v_{0x}^2 + 2a_{max,x}x_{min}$$

or, because $v_{1x} = 0$,

$$x_{min} = \frac{-v_{0x}^2}{2a_{max,x}} \quad (1)$$

Draw the free-body diagram for the car going up the incline:



Noting that $F_g = mg$, apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = -f_{s,max} - mg\sin\theta = ma_x \quad (2)$$

and

$$\sum F_y = F_n - mg\cos\theta = 0 \quad (3)$$

Substitute $f_{s,\,max} = \mu_s F_n$ and F_n from equation (3) in equation (2) and solve for $a_{max,x}$:

$$a_{max,x} = -g(\mu_s\cos\theta + \sin\theta)$$

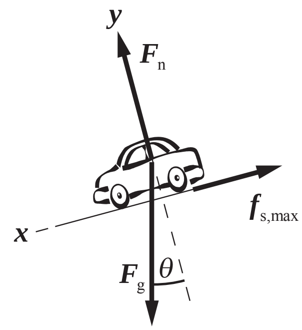
Substituting for $a_{max,x}$ in equation (1) yields:

$$x_{min} = \frac{v_{0x}^2}{2g(\mu_s\cos\theta + \sin\theta)}$$

Substitute numerical values and evaluate x_{min}:

$$x_{min} = \frac{(30\text{ m/s})^2}{2(9.81\text{ m/s}^2)((0.70)\cos 15° + \sin 15°)} = \boxed{49\text{ m}}$$

(*b*) When the car is going down the incline, the static friction force is up the incline as shown in the free-body diagram to the right. Note the change in coordinate system from Part (*a*).



Apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = mg\sin\theta - f_{\text{s,max}} = ma_x$$

and

$$\sum F_y = F_\text{n} - mg\cos\theta = 0$$

Proceed as in (*a*) to obtain:

$$a_{\max,x} = g(\sin\theta - \mu_\text{s}\cos\theta)$$

Substituting for $a_{\max,x}$ in equation (1) yields:

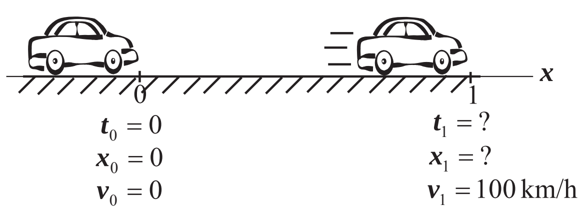
$$x_{\min} = \frac{-v_{0x}^2}{2g(\sin\theta - \mu_\text{s}\cos\theta)}$$

Substitute numerical values and evaluate $x_{\min}$:

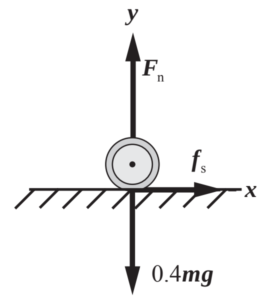
$$x_{\min} = \frac{-(30\text{ m/s})^2}{2(9.81\text{ m/s}^2)(\sin 15° - (0.70)\cos 15°)} = \boxed{0.11\text{ km}}$$

49 •• A rear-wheel-drive car supports 40 percent of its weight on its two drive wheels and has a coefficient of static friction of 0.70 with a horizontal straight road. (*a*) Find the vehicle's maximum acceleration. (*b*) What is the shortest possible time in which this car can achieve a speed of 100 km/h? (Assume the engine has unlimited power.)

Picture the Problem The friction force the road exerts on the tires provides the net force that accelerates the car. The pictorial representation summarizes what we know about the motion of the car. We can use Newton's second law to determine the acceleration of the car and a constant-acceleration equation to calculate how long it takes it to reach 100 km/h.



(*a*) Because 40% of the car's weight is on its two drive wheels and the accelerating friction forces act just on these wheels, the free-body diagram shows just the forces acting on the drive wheels.



Apply $\sum \vec{F} = m\vec{a}$ to the drive wheels:

$$\sum F_x = f_{s,max} = ma_{max,x} \quad (1)$$

and

$$\sum F_y = F_n - 0.4mg = 0 \quad (2)$$

Use the definition of $f_{s,\,max}$ in equation (1) and eliminate F_n between the two equations to obtain:

$$a_{max,x} = 0.4\mu_s g$$

Substitute numerical values and evaluate $a_{max,\,x}$:

$$a_{max,x} = 0.4(0.70)(9.81\,\text{m/s}^2) = 2.747\ \text{m/s}^2 = \boxed{2.7\,\text{m/s}^2}$$

(*b*) Using a constant-acceleration equation, relate the initial and final velocities of the car to its acceleration and the elapsed time; solve for the time:

$$v_{1,x} = v_{0,x} + a_{max,x}\Delta t$$

or, because $v_{0,x} = 0$ and $\Delta t = t_1$,

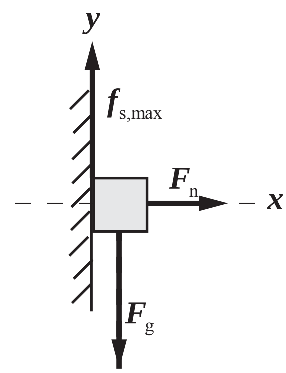
$$t_1 = \frac{v_{1,x}}{a_{max,x}}$$

Substitute numerical values and evaluate t_1:

$$t_1 = \frac{100\dfrac{\text{km}}{\text{h}} \times \dfrac{1\,\text{h}}{3600\,\text{s}} \times \dfrac{1000\,\text{m}}{\text{km}}}{2.747\,\text{m/s}^2} = \boxed{10\,\text{s}}$$

50 •• You and your best pal make a friendly bet that you can place a 2.0-kg box against the side of a cart, as in Figure 5-63, and that the box will not fall to the ground, even though you guarantee to use no hooks, ropes, fasteners, magnets, glue, or adhesives of any kind. When your friend accepts the bet, you begin pushing the cart in the direction shown in the figure. The coefficient of static friction between the box and the cart is 0.60. (*a*) Find the minimum acceleration for which you will win the bet. (*b*) What is the magnitude of the frictional force in this case? (*c*) Find the force of friction on the box if the acceleration is twice the minimum needed for the box not to fall. (*d*) Show that, for a box of any mass, the box will not fall if the magnitude of the forward acceleration is $a \geq g/\mu_s$, where μ_s is the coefficient of static friction.

Picture the Problem To hold the box in place, the acceleration of the cart and box must be great enough so that the static friction force acting on the box will equal the weight of the box. We can use Newton's second law to determine the minimum acceleration required.



(*a*) Noting that $F_g = mg$, apply $\sum \vec{F} = m\vec{a}$ to the box:

$$\sum F_x = F_n = ma_{\min,x} \quad (1)$$

and

$$\sum F_y = f_{s,max} - mg = 0 \quad (2)$$

Substituting $\mu_s F_n$ for $f_{s,\,max}$ in equation (2) yields:

$$\mu_s F_n - mg = 0$$

Substitute for F_n from equation (1) to obtain:

$$\mu_s (ma_{\min,x}) - mg = 0 \Rightarrow a_{\min,x} = \frac{g}{\mu_s}$$

Substitute numerical values and evaluate $a_{\min,x}$:

$$a_{\min,x} = \frac{9.81\,\text{m/s}^2}{0.60} = \boxed{16\,\text{m/s}^2}$$

(*b*) From equation (2) we have:

$$f_{s,max} = mg$$

Substitute numerical values and evaluate $f_{s,\,max}$:

$$f_{s,max} = (2.0\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{20\,\text{N}}$$

(*c*) If a is twice that required to hold the box in place, f_s will still have its maximum value given by:

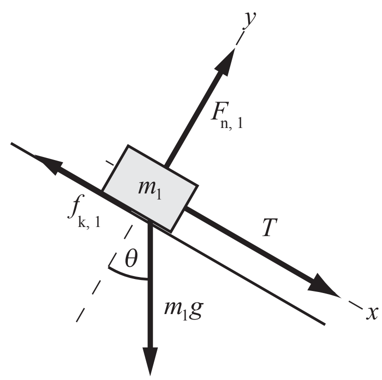
$$f_{s,max} = \boxed{20\,\text{N}}$$

(*d*) Because $a_{\min,x} = g/\mu_s$, the box will not fall if $a \geq g/\mu_s$.

51 •• Two blocks attached by a string (Figure 5-64) slide down a 10º incline. Block 1 has mass $m_1 = 0.80$ kg and block 2 has mass $m_2 = 0.25$ kg. In addition, the kinetic coefficients of friction between the blocks and the incline are 0.30 for block 1 and 0.20 for block 2. Find (*a*) the magnitude of the acceleration of the blocks, and (*b*) the tension in the string.

Picture the Problem Assume that the string is massless and does not stretch. Then the blocks have a common acceleration and the tension in the string acts on both blocks in accordance with Newton's third law of motion. Let down the incline be the +x direction. Draw the free-body diagrams for each block and apply Newton's second law of motion and the definition of the kinetic friction force to each block to obtain simultaneous equations in a_x and T.

Draw the free-body diagram for the block whose mass is m_1:



Apply $\sum\vec{F} = m\vec{a}$ to the upper block:

$$\sum F_x = -f_{k,1} + T + m_1 g\sin\theta = m_1 a \quad (1)$$

and

$$\sum F_y = F_{n,1} - m_1 g\cos\theta = 0 \quad (2)$$

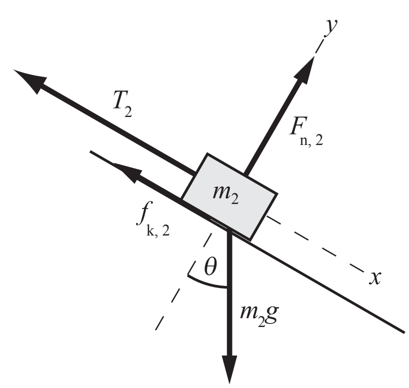
The relationship between $f_{k,1}$ and $F_{n,1}$ is:

$$f_{k,1} = \mu_{k,1}F_{n,1} \quad (3)$$

Eliminate $f_{k,1}$ and $F_{n,1}$ between (1), (2), and (3) to obtain:

$$-\mu_{k,1}m_1 g\cos\theta + T + m_1 g\sin\theta = m_1 a \quad (4)$$

Draw the free-body diagram for the block whose mass is m_2:



Apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = -f_{k,2} - T + m_2 g \sin\theta = m_2 a \quad (5)$$

and

$$\sum F_y = F_{n,2} - m_2 g \cos\theta = 0 \quad (6)$$

The relationship between $f_{k,2}$ and $F_{n,2}$ is:

$$f_{k,2} = \mu_{k,2} F_{n,2} \quad (7)$$

Eliminate $f_{k,2}$ and $F_{n,2}$ between (5), (6), and (7) to obtain:

$$-\mu_{k,2} m_2 g \cos\theta - T + m_2 g \sin\theta = m_1 a \quad (8)$$

Add equations (4) and (8) to eliminate T, and solve for $|a|$:

$$|a| = \left|\left[\sin\theta - \frac{\mu_{k,1} m_1 + \mu_{k,2} m_2}{m_1 + m_2}\cos\theta\right] g\right|$$

Substitute numerical values and evaluate $|a|$:

$$|a| = \left|\left[\sin 10^\circ - \frac{(0.20)(0.25\,\text{kg}) + (0.30)(0.80\,\text{kg})}{0.25\,\text{kg} + 0.80\,\text{kg}}\cos 10^\circ\right](9.81\,\text{m/s}^2)\right| = \boxed{0.96\,\text{m/s}^2}$$

(*b*) Eliminate *a* between equations (4) and (8) and solve for $T = T_1 = T_2$ to obtain:

$$T = \frac{m_1 m_2 (\mu_{k,2} - \mu_{k,1}) g \cos\theta}{m_1 + m_2}$$

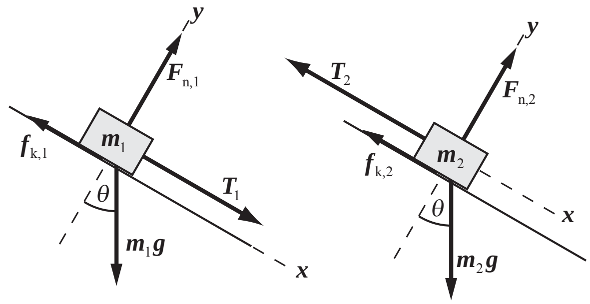
Substitute numerical values and evaluate T:

$$T = \frac{(0.25\,\text{kg})(0.80\,\text{kg})(0.30 - 0.20)(9.81\,\text{m/s}^2)\cos 10^\circ}{0.25\,\text{kg} + 0.80\,\text{kg}} = \boxed{0.18\,\text{N}}$$

52 •• Two blocks of masses m_1 and m_2 are sliding down an incline as shown in Figure 5-64. They are connected by a massless *rod*. The coefficients of kinetic friction between the block and the surface are μ_1 for block 1 and μ_2 for block 2. (*a*) Determine the acceleration of the two blocks. (*b*) Determine the force that the rod exerts on each of the two blocks. Show that the forces are both 0 when $\mu_1 = \mu_2$ and give a simple, nonmathematical argument why this is true.

Picture the Problem The free-body diagrams show the forces acting on the two blocks as they slide down the incline. Down the incline has been chosen as the +*x* direction. *T* is the force transmitted by the rod; it can be either tensile ($T > 0$) or

compressive ($T < 0$). By applying Newton's second law to these blocks, we can obtain equations in T and a_x from which we can eliminate either by solving them simultaneously. Once we have expressed T, the role of the rod will become apparent.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to block 1:

$$\sum F_x = T_1 + m_1 g \sin\theta - f_{k,1} = m_1 a_x$$

and

$$\sum F_y = F_{n,1} - m_1 g \cos\theta = 0$$

Apply $\sum \vec{F} = m\vec{a}$ to block 2:

$$\sum F_x = m_2 g \sin\theta - T_2 - f_{k,2} = m_2 a_x$$

and

$$\sum F_y = F_{n,2} - m_2 g \cos\theta = 0$$

Letting $T_1 = T_2 = T$, use the definition of the kinetic friction force to eliminate $f_{k,1}$ and $F_{n,1}$ between the equations for block 1 and $f_{k,2}$ and $F_{n,1}$ between the equations for block 2 to obtain:

$$m_1 a_x = m_1 g \sin\theta + T - \mu_1 m_1 g \cos\theta \quad (1)$$

and

$$m_2 a_x = m_2 g \sin\theta - T - \mu_2 m_2 g \cos\theta \quad (2)$$

Add equations (1) and (2) to eliminate T and solve for a_x:

$$a_x = \boxed{g\left(\sin\theta - \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}\cos\theta\right)}$$

(*b*) Rewrite equations (1) and (2) by dividing both sides of (1) by m_1 and both sides of (2) by m_2 to obtain.

$$a_x = g\sin\theta + \frac{T}{m_1} - \mu_1 g\cos\theta \quad (3)$$

and

$$a_x = g\sin\theta - \frac{T}{m_2} - \mu_2 g\cos\theta \quad (4)$$

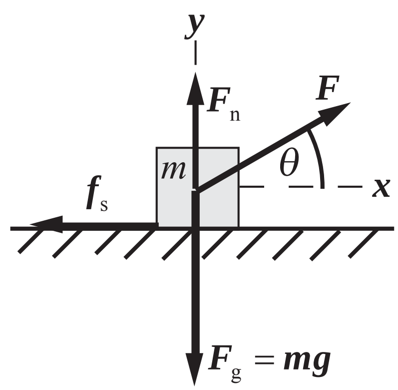
Subtracting (4) from (3) and rearranging yields:

$$T = \boxed{\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\mu_1 - \mu_2) g \cos\theta}$$

If $\mu_1 = \mu_2$, $T = 0$ and the blocks move down the incline with the same acceleration of $g(\sin\theta - \mu\cos\theta)$. Inserting a stick between them can't change this; therefore, the stick must exert no force on either block.

53 •• **[SSM]** A block of mass m rests on a horizontal table (Figure 5-65). The block is pulled by a massless rope with a force $\vec{F}$ at an angle θ. The coefficient of static friction is 0.60. The minimum value of the force needed to move the block depends on the angle θ. (*a*) Discuss qualitatively how you would expect this force to depend on θ. (*b*) Compute the force for the angles $\theta = 0°$, 10°, 20°, 30°, 40°, 50°, and 60°, and make a plot of F versus θ for $mg = 400$ N. From your plot, at what angle is it most efficient to apply the force to move the block?

Picture the Problem The vertical component of $\vec{F}$ reduces the normal force; hence, the static friction force between the surface and the block. The horizontal component is responsible for any tendency to move and equals the static friction force until it exceeds its maximum value. We can apply Newton's second law to the box, under equilibrium conditions, to relate F to θ.



(*a*) The static-friction force opposes the motion of the object, and the maximum value of the static-friction force is proportional to the normal force F_N. The normal force is equal to the weight minus the vertical component F_V of the force F. Keeping the magnitude F constant while increasing θ from zero results in an increase in F_V and a decrease in F_n; thus decreasing the maximum static-friction force f_{max}. The object will begin to move if the horizontal component F_H of the force F exceeds f_{max}. An increase in θ results in a decrease in F_H. As θ increases from 0, the decrease in F_N is larger than the decrease in F_H, so the object is more and more likely to slip. However, as θ approaches 90°, F_H approaches zero and no movement will be initiated. If F is large enough and if θ increases from 0, then at some value of θ the block will start to move.

(*b*) Apply $\sum\vec{F} = m\vec{a}$ to the block:

$$\sum F_x = F\cos\theta - f_s = 0 \qquad (1)$$

and

$$\sum F_y = F_n + F\sin\theta - mg = 0 \qquad (2)$$

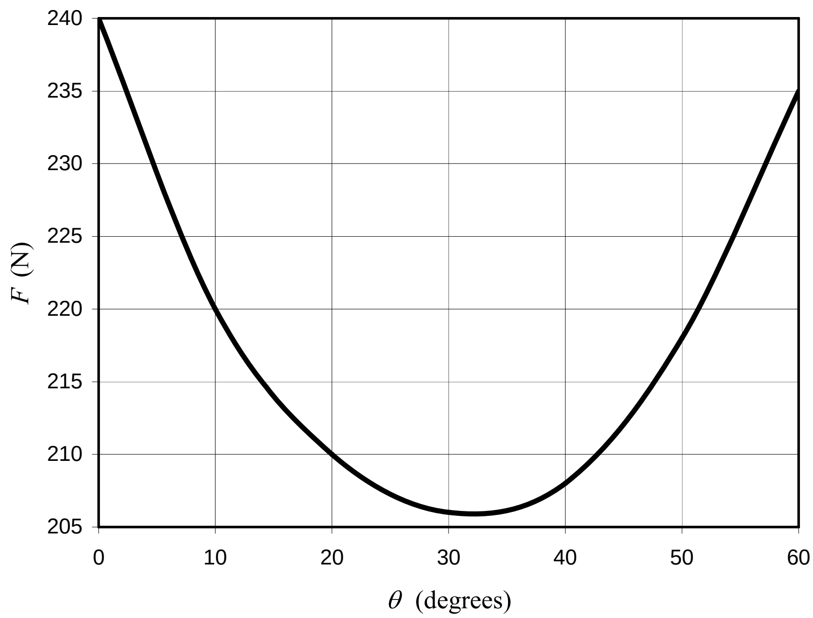
Assuming that $f_s = f_{s,max}$, eliminate f_s and F_n between equations (1) and (2) and solve for F:

$$F = \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta}$$

Use this function with mg = 400 N to generate the following table:

θ	(deg)	0	10	20	30	40	50	60
F	(N)	240	220	210	206	208	218	235

The following graph of $F(\theta)$ was plotted using a spreadsheet program.

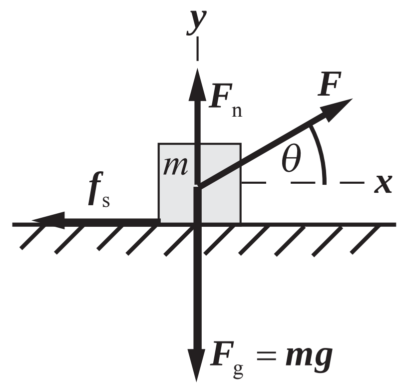


From the graph, we can see that the minimum value for F occurs when $\theta \approx 32°$.

Remarks: An alternative to manually plotting F as a function of θ or using a spreadsheet program is to use a graphing calculator to enter and graph the function.

54 •• Consider the block in Problem 53. Show that, in general, the following results hold for a block of mass m resting on a horizontal surface whose coefficient of static friction is μ_s. (*a*) If you want to apply the *minimum* possible force to move the block, you should apply it with the force pulling upward at an angle $\theta_{min} = \tan^{-1}\mu_s$, and (*b*) the minimum force necessary to start the block moving is $F_{min} = \left(\mu_s/\sqrt{1-\mu_s^2}\right)mg$. (*c*) Once the block starts moving, if you want to apply the least possible force to *keep* it moving, should you keep the angle at which you are pulling the same? increase it? decrease it?

Picture the Problem The free-body diagram shows the forces acting on the block. We can apply Newton's second law, under equilibrium conditions, to relate F to θ and then set its derivative with respect to θ equal to zero to find the value of θ that minimizes F.



(a) Apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = F\cos\theta - f_s = 0 \quad (1)$$

and

$$\sum F_y = F_n + F\sin\theta - mg = 0 \quad (2)$$

Assuming that $f_s = f_{s,max}$, eliminate f_s and F_n between equations (1) and (2) and solve for F:

$$F = \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta} \quad (3)$$

To find θ_{min}, differentiate F with respect to θ and set the derivative equal to zero for extrema of the function:

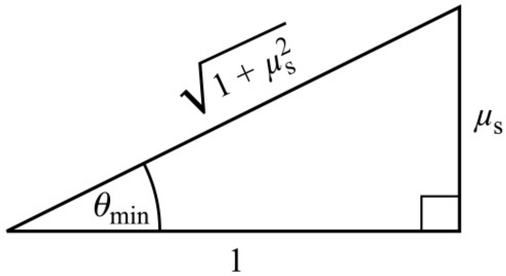
$$\frac{dF}{d\theta} = \frac{(\cos\theta + \mu_s \sin\theta)\frac{d}{d\theta}(\mu_s mg)}{(\cos\theta + \mu_s \sin\theta)^2} - \frac{\mu_s mg \frac{d}{d\theta}(\cos\theta + \mu_s \sin\theta)}{(\cos\theta + \mu_s \sin\theta)^2}$$

$$= \frac{\mu_s mg(-\sin\theta + \mu_s \cos\theta)}{(\cos\theta + \mu_s \sin\theta)^2} = 0 \text{ for extrema}$$

Solving for θ_{min} yields:

$$\theta_{min} = \boxed{\tan^{-1}\mu_s}$$

(b) Use the reference triangle shown below to substitute for $\cos\theta$ and $\sin\theta$ in equation (3):



$$F_{min} = \frac{\mu_s mg}{\frac{1}{\sqrt{1+\mu_s^2}} + \mu_s \frac{\mu_s}{\sqrt{1+\mu_s^2}}}$$

$$= \frac{\mu_s mg}{\frac{1+\mu_s^2}{\sqrt{1+\mu_s^2}}}$$

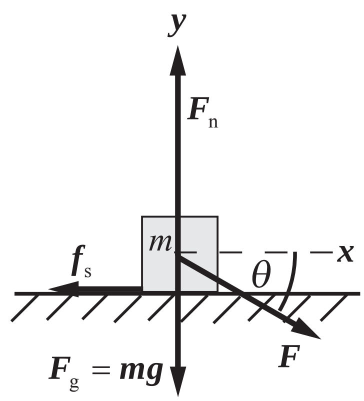
$$= \boxed{\frac{\mu_s}{\sqrt{1+\mu_s^2}} mg}$$

(*c*) The coefficient of kinetic friction is less than the coefficient of static friction. An analysis identical to the one above shows that the minimum force one should apply to keep the block moving should be applied at an angle given by $\theta_{\text{min}} = \tan^{-1}\mu_{\text{k}}$. Therefore, once the block is moving, the coefficient of friction will decrease, so the angle can be decreased.

55 •• Answer the questions in Problem 54, but for a force $\vec{F}$ that pushes down on the block at an angle θ *below* the horizontal.

Picture the Problem The vertical component of $\vec{F}$ increases the normal force and the static friction force between the surface and the block. The horizontal component is responsible for any tendency to move and equals the static friction force until it exceeds its maximum value. We can apply Newton's second law to the box, under equilibrium conditions, to relate F to θ.

(*a*) As θ increases from zero, F increases the normal force exerted by the surface and the static friction force. As the horizontal component of F decreases with increasing θ, one would expect F to continue to increase.



(*b*) Apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = F\cos\theta - f_{\text{s}} = 0 \qquad (1)$$

and

$$\sum F_y = F_{\text{n}} - F\sin\theta - mg = 0 \qquad (2)$$

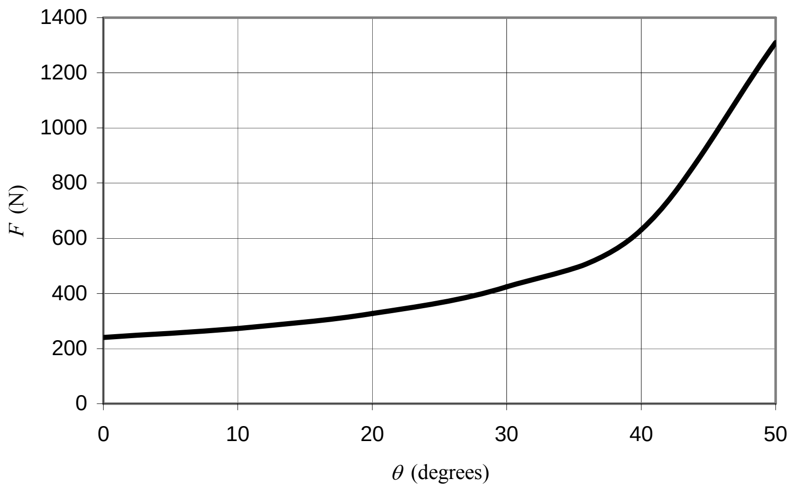
Assuming that $f_{\text{s}} = f_{\text{s,max}}$, eliminate f_{s} and F_{n} between equations (1) and (2) and solve for F:

$$F = \frac{\mu_{\text{s}} mg}{\cos\theta - \mu_{\text{s}}\sin\theta} \qquad (3)$$

Use this function with mg = 400 N to generate the table shown below.

θ	(deg)	0	10	20	30	40	50	60
F	(N)	240	273	327	424	631	1310	very large

The following graph of F as a function of θ, plotted using a spreadsheet program, confirms our prediction that F continues to increase with θ.



(*a*) From the graph we see that: $\theta_{min} = \boxed{0°}$

(*b*) Evaluate equation (3) for $\theta = 0°$ to obtain:

$$F = \frac{\mu_s mg}{\cos 0° - \mu_s \sin 0°} = \boxed{\mu_s mg}$$

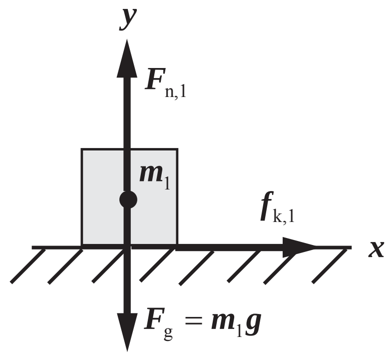
(*c*) You should keep the angle at 0°.

Remarks: An alternative to the use of a spreadsheet program is to use a graphing calculator to graph the function.

56 •• A 100-kg mass is pulled along a frictionless surface by a horizontal force $\vec{F}$ such that its acceleration is $a_1 = 6.00\ \text{m/s}^2$ (Figure 5-66). A 20.0-kg mass slides along the top of the 100-kg mass and has an acceleration of $a_2 = 4.00\ \text{m/s}^2$. (It thus slides backward relative to the 100-kg mass.) (*a*) What is the frictional force exerted by the 100-kg mass on the 20.0-kg mass? (*b*) What is the net force acting on the 100-kg mass? What is the force F? (*c*) After the 20.0-kg mass falls off the 100-kg mass, what is the acceleration of the 100-kg mass? (Assume that the force F does not change.)

Picture the Problem The forces acting on each of these masses are shown in the free-body diagrams below. m_1 represents the mass of the 20.0-kg mass and m_2 that of the 100-kg mass. As described by Newton's 3rd law, the normal reaction force $F_{n,1}$ and the friction force $f_{k,1}$ ($= f_{k,2}$) act on both masses but in opposite directions. Newton's second law and the definition of kinetic friction forces can be used to determine the various forces and the acceleration called for in this problem.

(*a*) Draw a free-body diagram showing the forces acting on the block whose mass is 20 kg:



Apply $\sum \vec{F} = m\vec{a}$ to this mass:

$$\sum F_x = f_{k,1} = m_1 a_{1x} \quad (1)$$

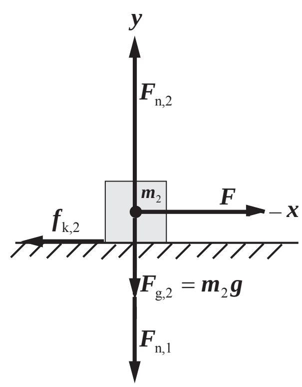
and

$$\sum F_y = F_{n,1} - m_1 g = 0 \quad (2)$$

Substitute numerical values in equation (1) and evaluate $f_{k,1}$:

$$f_{k,1} = (20.0\text{ kg})(4.00\text{ m/s}^2) = \boxed{80.0\text{ N}}$$

(*b*) Draw a free-body diagram showing the forces acting on the block whose mass is 100 kg:



Apply $\sum F_x = ma_x$ to the 100-kg object and evaluate F_{net}:

$$F_{\text{net}} = m_2 a_{2x} = (100\text{ kg})(6.00\text{ m/s}^2)$$
$$= \boxed{600\text{ N}}$$

Express F in terms of F_{net} and $f_{k,2}$:

$$F = F_{\text{net}} + f_{k,2}$$

Substitute numerical values and evaluate F:

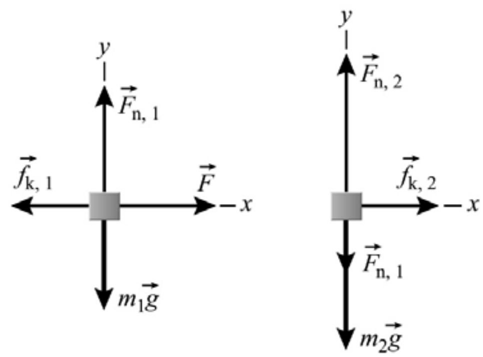
$$F = 600\text{ N} + 80.0\text{ N} = \boxed{680\text{ N}}$$

(*c*) When the 20.0-kg object falls off, the 680-N force acts just on the 100-kg object and its acceleration is given by Newton's second law:

$$a = \frac{F_{\text{net}}}{m} = \frac{680\text{ N}}{100\text{ kg}} = \boxed{6.80\text{ m/s}^2}$$

57 •• A 60.0-kg block slides along the top of a 100-kg block. The 60.0-kg block has an acceleration of 3.0 m/s^2 when a horizontal force of 320 N is applied, as in Figure 5-67. There is no friction between the 100-kg block and a horizontal frictionless surface, but there is friction between the two blocks. (*a*) Find the coefficient of kinetic friction between the blocks. (*b*) Find the acceleration of the 100-kg block during the time that the 60.0-kg block remains in contact.

Picture the Problem The forces acting on each of these blocks are shown in the free-body diagrams to the right. m_1 represents the mass of the 60-kg block and m_2 that of the 100-kg block. As described by Newton's 3rd law, the normal reaction force $F_{n,1}$ and the friction force $f_{k,1}$ (= $f_{k,2}$) act on both objects but in opposite directions. Newton's second law and the definition of kinetic friction forces can be used to determine the coefficient of kinetic friction and acceleration of the 100-kg block.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the 60-kg block:

$$\sum F_x = F - f_{k,1} = m_1 a_{1,x} \quad (1)$$

and

$$\sum F_y = F_{n,1} - m_1 g = 0 \quad (2)$$

Apply $\sum F_x = ma_x$ to the 100-kg block:

$$f_{k,2} = m_2 a_{2,x} \quad (3)$$

Using equation (2), express the relationship between the kinetic friction forces $\vec{f}_{k,1}$ and $\vec{f}_{k,2}$:

$$f_{k,1} = f_{k,2} = f_k = \mu_k F_{n,1} = \mu_k m_1 g \quad (4)$$

Substitute for $f_{k,1}$ in equation (1) and solve for μ_k:

$$\mu_k = \frac{F - m_1 a_1}{m_1 g}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{320\,\text{N} - (60\,\text{kg})(3.0\,\text{m/s}^2)}{(60\,\text{kg})(9.81\,\text{m/s}^2)} = 0.238 = \boxed{0.24}$$

(*b*) Substitute for $f_{k,2}$ in equation (3) and solve for a_2 to obtain:

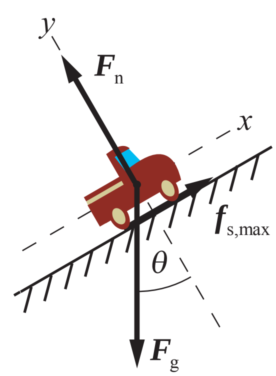
$$a_{2,x} = \frac{\mu_k m_1 g}{m_2}$$

Substitute numerical values and evaluate $a_{2,x}$:

$$a_{2,x} = \frac{(0.238)(60\,\text{kg})(9.81\,\text{m/s}^2)}{100\,\text{kg}} = \boxed{1.4\,\text{m/s}^2}$$

58 •• The coefficient of static friction between a rubber tire and the road surface is 0.85. What is the maximum acceleration of a 1000-kg four-wheel-drive truck if the road makes an angle of 12º with the horizontal and the truck is (*a*) climbing and (*b*) descending?

Picture the Problem Choose a coordinate system in which the $+x$ direction is up the incline and apply Newton's second law of motion. The free-body diagram shows the truck climbing the incline with maximum acceleration.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the truck when it is climbing the incline:

$$\sum F_x = f_{s,max} - mg\sin\theta = ma_x \quad (1)$$

and

$$\sum F_y = F_n - mg\cos\theta = 0 \quad (2)$$

Solve equation (2) for F_n and use the definition $f_{s,max}$ to obtain:

$$f_{s,max} = \mu_s mg\cos\theta \quad (3)$$

Substitute $f_{s,max}$ in equation (1) and solve for a_x:

$$a_x = g(\mu_s\cos\theta - \sin\theta)$$

Substitute numerical values and evaluate a_x:

$$a_x = (9.81\,\text{m/s}^2)[(0.85)\cos 12° - \sin 12°] = \boxed{6.1\,\text{m/s}^2}$$

(*b*) When the truck is descending the incline with maximum acceleration, the static friction force points down the incline; that is, its direction is reversed Apply $\sum F_x = ma_x$ to the truck under these conditions to obtain:

$$-f_{s,max} - mg\sin\theta = ma_x \qquad (4)$$

Substituting for $f_{s,max}$ in equation (4) and solving for a_x gives:

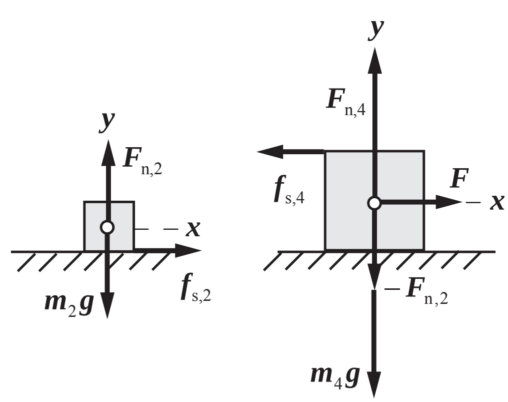
$$a_x = -g(\mu_s \cos\theta + \sin\theta)$$

Substitute numerical values and evaluate a_x:

$$a_x = (-9.81\,\text{m/s}^2)[(0.85)\cos 12° + \sin 12°] = \boxed{-10\,\text{m/s}^2}$$

59 •• A 2.0-kg block sits on a 4.0-kg block that is on a frictionless table (Figure 5-68). The coefficients of friction between the blocks are $\mu_s = 0.30$ and $\mu_k = 0.20$. (*a*) What is the maximum horizontal force *F* that can be applied to the 4.0-kg block if the 2.0-kg block is not to slip? (*b*) If *F* has half this value, find the acceleration of each block and the force of friction acting on each block. (*c*) If *F* is twice the value found in (*a*), find the acceleration of each block..

Picture the Problem The forces acting on each of the blocks are shown in the free-body diagrams to the right. m_2 represents the mass of the 2.0-kg block and m_4 that of the 4.0-kg block. As described by Newton's 3rd law, the normal reaction force $F_{n,2}$ and the friction force $f_{s,2}$ (= $f_{s,4}$) act on both objects but in opposite directions. Newton's second law and the definition of the maximum static friction force can be used to determine the maximum force acting on the 4.0-kg block for which the 2.0-kg block does not slide.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the 2.0-kg block:

$$\sum F_x = f_{s,2,max} = m_2 a_{2,max} \qquad (1)$$

and

$$\sum F_y = F_{n,2} - m_2 g = 0 \qquad (2)$$

Apply $\sum \vec{F} = m\vec{a}$ to the 4.0-kg block:

$$\sum F_x = F - f_{\text{s,2,max}} = m_4 a_{4,\text{max}} \quad (3)$$

and

$$\sum F_y = F_{\text{n},4} - F_{\text{n},2} - m_4 g = 0 \quad (4)$$

Using equation (2), express the relationship between the static friction forces $\vec{f}_{\text{s,2,max}}$ and $\vec{f}_{\text{s,4,max}}$:

$$f_{\text{s,2,max}} = f_{s,4,\text{max}} = \mu_\text{s} m_2 g \quad (5)$$

Substitute for $\vec{f}_{\text{s,2,max}}$ in equation (1) and solve for $a_{2,\text{max}}$:

$$a_{2,\text{max}} = \mu_\text{s} g$$

Solve equation (3) for $F = F_\text{max}$ and substitute for $a_{2,\text{max}}$ and $a_{4,\text{max}}$ to obtain:

$$F_\text{max} = m_4 \mu_\text{s} g + \mu_\text{s} m_2 g = (m_4 + m_2)\mu_\text{s} g$$

Substitute numerical values and evaluate F_max:

$$F_\text{max} = (4.0\,\text{kg} + 2.0\,\text{kg})(0.30)(9.81\,\text{m/s}^2) = 17.66\,\text{N} = \boxed{18\,\text{N}}$$

(*b*) Use Newton's second law to express the acceleration of the blocks moving as a unit:

$$a_x = \frac{F}{m_1 + m_2} = \frac{\frac{1}{2}F_\text{max}}{m_1 + m_2}$$

Substitute numerical values and evaluate *a*:

$$a_x = \frac{\frac{1}{2}(17.66\,\text{N})}{2.0\,\text{kg} + 4.0\,\text{kg}} = 1.472\,\text{m/s}^2 = \boxed{1.5\,\text{m/s}^2}$$

Because the friction forces are an action-reaction pair, the friction force acting on each block is given by:

$$f_\text{s} = m_1 a_x$$

Substitute numerical values and evaluate f_s:

$$f_\text{s} = (2.0\,\text{kg})(1.472\,\text{m/s}^2) = \boxed{2.9\,\text{N}}$$

(c) If $F = 2F_{max}$, then the 2.0-kg block slips on the 4.0-kg block and the friction force (now kinetic) is given by:

$$f = f_k = \mu_k m_2 g$$

Use $\sum F_x = ma_x$ to relate the acceleration of the 2.0-kg block to the net force acting on it and solve for $a_{2,x}$:

$$f_k = \mu_k m_1 g = m_2 a_{2,x}$$

and

$$a_{2,x} = \mu_k g$$

Substitute numerical values and evaluate $a_{2,x}$:

$$a_{2,x} = (0.20)(9.81\ \text{m/s}^2) = 1.96\ \text{m/s}^2 = \boxed{2.0\ \text{m/s}^2}$$

Use $\sum F_x = ma_x$ to relate the acceleration of the 4.0-kg block to the net force acting on it:

$$F - \mu_k m_2 g = m_4 a_{4,x}$$

Solving for $a_{4,x}$ yields:

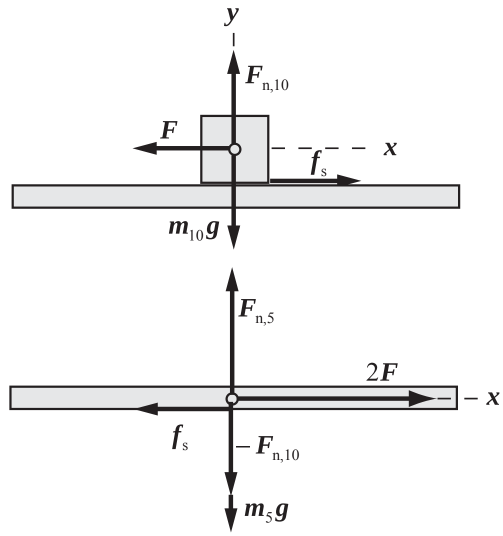
$$a_{4,x} = \frac{F - \mu_k m_2 g}{m_4}$$

Substitute numerical values and evaluate $a_{4,x}$:

$$a_{4,x} = \frac{2(17.66\ \text{N}) - (0.20)(2.0\ \text{kg})(9.81\ \text{m/s}^2)}{4.0\ \text{kg}} = \boxed{7.8\ \text{m/s}^2}$$

60 •• A 10.0-kg block rests on a 5.0-kg bracket, as shown in Figure 5-69. The 5.0-kg bracket sits on a frictionless surface. The coefficients of friction between the 10.0-kg block and the bracket on which it rests are $\mu_s = 0.40$ and $\mu_k = 0.30$. (a) What is the maximum force F that can be applied if the 10.0-kg block is not to slide on the bracket? (b) What is the corresponding acceleration of the 5.0-kg bracket?

Picture the Problem The top diagram shows the forces action on the 10-kg block and the bottom diagram shows the forces acting on the 5.0-kg bracket. If the block is not to slide on the bracket, the maximum value for $\vec{F}$ must equal the maximum value of f_s. This value for $\vec{F}$ will produce the maximum acceleration of block and bracket system. We can apply Newton's second law and the definition of $f_{s,max}$ to first calculate the maximum acceleration and then the maximum value of F.



Apply $\sum\vec{F} = m\vec{a}$ to the 10-kg block when it is experiencing its maximum acceleration:

$$\sum F_x = f_{s,max} - F = m_{10}a_{10,max} \quad (1)$$

and

$$\sum F_y = F_{n,10} - m_{10}g = 0 \quad (2)$$

Express the static friction force acting on the 10-kg block:

$$f_{s,max} = \mu_s F_{n,10} \quad (3)$$

Eliminate $f_{s,max}$ and $F_{n,10}$ from equations (1), (2) and (3) to obtain:

$$\mu_s m_{10}g - F = m_{10}a_{10,max} \quad (4)$$

Apply $\sum F_x = ma_x$ to the bracket to obtain:

$$2F - \mu_s m_{10}g = m_5 a_{5,max} \quad (5)$$

Because $a_{5,max} = a_{10,max}$, denote this acceleration by a_{max}. Eliminate F from equations (4) and (5) and solve for a_{max}:

$$a_{max} = \frac{\mu_s m_{10}g}{m_5 + 2m_{10}}$$

(b) Substitute numerical values and evaluate a_{max}:

$$a_{max} = \frac{(0.40)(10\,\text{kg})(9.81\,\text{m/s}^2)}{5.0\,\text{kg} + 2(10\,\text{kg})} = 1.57\,\text{m/s}^2 = \boxed{1.6\,\text{m/s}^2}$$

Solve equation (4) for $F = F_{max}$:

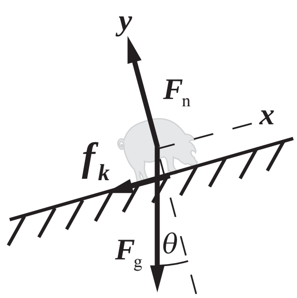
$$F = \mu_s m_{10}g - m_{10}a_{max} = m_{10}(\mu_s g - a_{max})$$

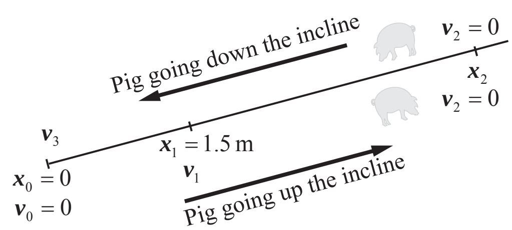
(a) Substitute numerical values and evaluate F:

$$F = (10\,\text{kg})\left[(0.40)(9.81\,\text{m/s}^2) - 1.57\,\text{m/s}^2\right] = \boxed{24\,\text{N}}$$

61 ••• You and your friends push a 75.0-kg greased pig up an aluminum slide at the county fair, starting from the low end of the slide. The coefficient of kinetic friction between the pig and the slide is 0.070. (a) All of you pushing together (parallel to the incline) manage to accelerate the pig from rest at the constant rate of 5.0 m/s^2 over a distance of 1.5 m, at which point you release the pig. The pig continues up the slide, reaching a maximum *vertical height* above its release point of 45 cm. What is the angle of inclination of the slide? (b) At the maximum height the pig turns around and begins to slip down once slide, how fast is it moving when it arrives at the low end of the slide?

Picture the Problem The free-body diagram shows the forces acting on the pig sometime after you and your friends have stopped pushing on it but before it has momentarily stopped moving up the slide. We can use constant-acceleration equations and Newton's second law to find the angle of inclination of the slide and the pig's speed when it returns to bottom of the slide. The pictorial representation assigns a coordinate system and names to the variables that we need to consider in solving this problem.





(a) Apply $\sum \vec{F} = m\vec{a}$ to the pig:

$$\sum F_x = -f_k - F_g \sin\theta = ma_x \qquad (1)$$

and

$$\sum F_y = F_n - F_g \cos\theta = 0 \qquad (2)$$

Substitute $f_k = \mu_k F_n$ and $F_g = mg$ in equation (1) to obtain:

$$-\mu_k F_n - mg\sin\theta = ma_x \qquad (3)$$

Solving equation (2) for F_n yields:

$$F_n = F_g \cos\theta$$

Substitute for F_n in equation (3) to obtain:

$$-\mu_k mg\cos\theta - mg\sin\theta = ma_x$$

Solving for a_x yields:

$$a_x = -g(\mu_k \cos\theta + \sin\theta) \qquad (4)$$

Use a constant-acceleration equation to relate the distance d the pig slides up the ramp to its speed after you and your friends have stopped pushing, its final speed, and its acceleration:

$$v_2^2 = v_1^2 + 2a_x d$$

or, because $v_2 = 0$,

$$0 = v_1^2 + 2a_x d$$

Substituting for a_x from equation (4) yields:

$$0 = v_1^2 - 2g(\mu_k \cos\theta + \sin\theta)d \qquad (5)$$

Use a constant-acceleration equation to relate the pig's speed v_1 after being accelerated to the distance Δx over which it was accelerated:

$$v_1^2 = v_0^2 + 2a_x \Delta x$$

or, because $v_0 = 0$,

$$v_1^2 = 2a_x \Delta x$$

Substitute for v_1^2 in equation (5) to obtain:

$$0 = 2a_x \Delta x - 2g(\mu_k \cos\theta + \sin\theta)d \qquad (6)$$

Use trigonometry to relate the vertical distance h risen by the pig during its slide to the distance d it moves up the slide:

$$h = d\sin\theta \Rightarrow d = \frac{h}{\sin\theta}$$

Substituting for d in equation (6) gives:

$$0 = 2a_x \Delta x - 2g(\mu_k \cos\theta + \sin\theta)\frac{h}{\sin\theta}$$

Solve for θ to obtain:

$$\theta = \tan^{-1}\left[\frac{\mu_k h}{\dfrac{a_x \Delta x}{g} - h}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{(0.070)(0.45\ \text{m})}{\dfrac{(5.0\ \text{m/s}^2)(1.5\ \text{m})}{9.81\ \text{m/s}^2} - 0.45\ \text{m}}\right]$$

$$= 5.719° = \boxed{5.7°}$$

(b) Use a constant-acceleration equation to express v_3 as a function of the pig's acceleration down the incline a_{down}, its initial speed v_2, the distance Δd it slides down the incline:

$$v_3^2 = v_2^2 + 2a_{\text{down}} \Delta d$$

or, because $v_2 = 0$,

$$v_3^2 = 2a_{\text{down}} \Delta d \Rightarrow v_3 = \sqrt{2a_{\text{down}} \Delta d} \qquad (7)$$

When the pig is sliding down the incline, the kinetic friction force shown in the free-body diagram is reversed (it points up the incline … opposing the pig's motion) and the pig's acceleration is:

$$a_{\text{down}} = g(\sin\theta - \mu_{\text{k}}\cos\theta)$$

Substitute for a_{down} in equation (7) to obtain:

$$v_3 = \sqrt{2g(\sin\theta - \mu_{\text{k}}\cos\theta)\Delta d} \quad (8)$$

Δd is the sum of the distances the pig was pushed and then slid to a momentary stop:

$$\Delta d = 1.5\text{ m} + \frac{d}{\sin\theta}$$

Substituting for Δd in equation (8) yields:

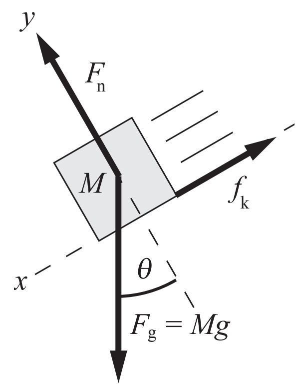
$$v_3 = \sqrt{2g(\sin\theta - \mu_{\text{k}}\cos\theta)\left(1.5\text{ m} + \frac{d}{\sin\theta}\right)}$$

Substitute numerical values and evaluate v_3:

$$v_3 = \sqrt{2(9.81\text{ m/s}^2)[\sin 5.719° - (0.070)\cos 5.719°]\left(1.5\text{ m} + \frac{0.45\text{ m}}{\sin 5.719°}\right)} = \boxed{1.9\text{ m/s}}$$

62 ••• A 100-kg block on an inclined plane is attached to another block of mass *m* via a string, as in Figure 5-70. The coefficients of static and kinetic friction for the block and the incline are $\mu_{\text{s}} = 0.40$ and $\mu_{\text{k}} = 0.20$ and the plane is inclined 18º with horizontal. (*a*) Determine the range of values for *m*, the mass of the hanging block, for which the 100-kg block will not move unless disturbed, but if nudged, will slide down the incline. (*b*) Determine a range of values for *m* for which the 100-kg block will not move unless nudged, but if nudged will slide up the incline.

Picture the Problem The free-body diagram shows the block sliding down the incline under the influence of a friction force, its weight, and the normal force exerted on it by the inclined surface. We can find the range of values for *m* for the two situations described in the problem statement by applying Newton's second law of motion to, first, the conditions under which the block will not slide if pushed, and secondly, if pushed, the block will slide up the incline.



(*a*) Assume that the block is sliding down the incline with a constant velocity and with no hanging weight ($m = 0$) and apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = -f_k + Mg\sin\theta = 0$$
and
$$\sum F_y = F_n - Mg\cos\theta = 0$$

Using $f_k = \mu_k F_n$, eliminate F_n between the two equations and solve for the net force acting on the block:

$$F_{net} = -\mu_k Mg\cos\theta + Mg\sin\theta$$

If the block is moving, this net force must be nonnegative and:

$$(-\mu_k\cos\theta + \sin\theta)Mg \geq 0$$

This condition requires that:

$$\mu_k \leq \tan\theta = \tan 18° = 0.325$$

Because $\mu_k = 0.40 > \tan 18°$, the block will not move unless pushed.

Because $\mu_k = 0.2$, this condition is satisfied and:

$$m_{min} = 0$$

To find the maximum value, note that the maximum possible value for the tension in the rope is mg. For the block to move down the incline, the component of the block's weight parallel to the incline minus the frictional force must be greater than or equal to the tension in the rope:

$$Mg\sin\theta - \mu_k Mg\cos\theta \geq mg$$

Solving for m_{max} yields:

$$m_{max} \leq M(\sin\theta - \mu_k\cos\theta)$$

Substitute numerical values and evaluate m_{max}:

$$m_{max} \leq (100\,\text{kg})[\sin 18° - (0.20)\cos 18°]$$
$$= 11.9\,\text{kg}$$

The range of values for m is: $\boxed{0 \le m \le 12\,\text{kg}}$

(b) If the block is being dragged up the incline, the frictional force will point down the incline, and:

$$Mg\sin\theta + \mu_k Mg\cos\theta < m_{\min} g$$

Solve for $m_{\min}$:

$$m_{\min} > M(\sin\theta + \mu_k \cos\theta)$$

Substitute numerical values and evaluate $m_{\min}$:

$$m_{\min} > (100\,\text{kg})[\sin 18° + (0.20)\cos 18°] = 49.9\,\text{kg}$$

If the block is not to move unless pushed:

$$Mg\sin\theta + \mu_s Mg\cos\theta > m_{\max} g$$

Solve for $m_{\max}$:

$$m_{\max} < M(\sin\theta + \mu_s \cos\theta)$$

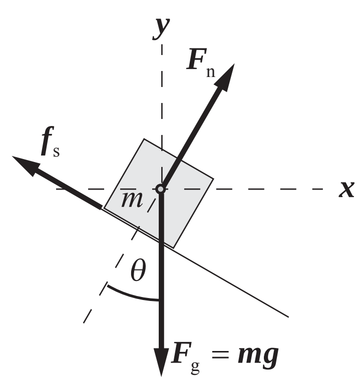
Substitute numerical values and evaluate $m_{\max}$:

$$m_{\max} < (100\,\text{kg})[\sin 18° + (0.40)\cos 18°] = 68.9\,\text{kg}$$

The range of values for m is: $\boxed{50\,\text{kg} \le m \le 69\,\text{kg}}$

63 ••• A block of mass 0.50 kg rests on the inclined surface of a wedge of mass 2.0 kg, as in Figure 5-71. The wedge is acted on by a horizontal applied force $\vec{F}$ and slides on a frictionless surface. (a) If the coefficient of static friction between the wedge and the block is $\mu_s = 0.80$ and the wedge is inclined 35º with the horizontal, find the maximum and minimum values of the applied force for which the block does not slip. (b) Repeat part (a) with $\mu_s = 0.40$.

Picture the Problem The free-body diagram shows the forces acting on the 0.50-kg block when the acceleration is a minimum. Note the choice of coordinate system that is consistent with the direction of $\vec{F}$. Apply Newton's second law to the block and to the block-plus-incline and solve the resulting equations for $F_{\min}$ and $F_{\max}$.



(*a*) The maximum and minimum values of the applied force for which the block does not slip are related to the maximum and minimum accelerations of the block:

$$F_{max} = m_{tot} a_{x,max} \quad (1)$$

and

$$F_{min} = m_{tot} a_{x,min} \quad (2)$$

where m_{tot} is the combined mass of the block and incline.

Apply $\sum \vec{F} = m\vec{a}$ to the 0.50-kg block:

$$\sum F_x = F_n \sin\theta - f_s \cos\theta = ma_x \quad (3)$$

and

$$\sum F_y = F_n \cos\theta + f_s \sin\theta - mg = 0 \quad (4)$$

Under minimum acceleration, $f_s = f_{s,\,max}$. Express the relationship between $f_{s,\,max}$ and F_n:

$$f_{s,max} = \mu_s F_n$$

Substitute $f_{s,\,max}$ for f_s in equation (4) and solve for F_n:

$$F_n = \frac{mg}{\cos\theta + \mu_s \sin\theta}$$

Substitute for F_n and $f_{s,\,max}$ in equation (3) and solve for $a_x = a_{x,min}$:

$$a_{x,min} = g\frac{\sin\theta - \mu_s \cos\theta}{\cos\theta + \mu_s \sin\theta}$$

Substitute for $a_{x,\,min}$ in equation (2) to obtain:

$$F_{min} = m_{tot} g \left(\frac{\sin\theta - \mu_s \cos\theta}{\cos\theta + \mu_s \sin\theta}\right)$$

Substitute numerical values and evaluate F_{min}:

$$F_{min} = \left|(2.5\text{ kg})(9.81\text{ m/s}^2)\left[\frac{\sin 35° - (0.80)\cos 35°}{\cos 35° + (0.80)\sin 35°}\right]\right| = \boxed{1.6\text{ N}}$$

To find the maximum acceleration, reverse the direction of $\vec{f}_s$ and apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = F_n \sin\theta + f_s \cos\theta = ma_x$$

and

$$\sum F_y = F_n \cos\theta - f_s \sin\theta - mg = 0$$

Proceed as above to obtain:

$$a_{x,max} = g\left(\frac{\sin\theta + \mu_s \cos\theta}{\cos\theta - \mu_s \sin\theta}\right)$$

Substitute for $a_{x,\,max}$ in equation (1) to obtain:

$$F_{max} = m_{tot} g \left(\frac{\sin\theta + \mu_s \cos\theta}{\cos\theta - \mu_s \sin\theta}\right)$$

Substitute numerical values and evaluate $F_{\max}$:

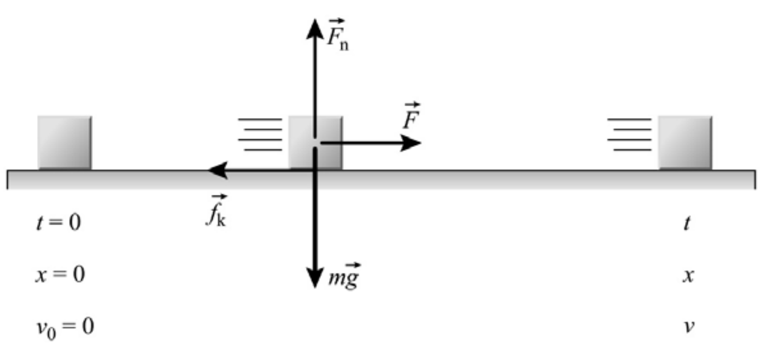
$$F_{\max} = (2.5\ \text{kg})(9.81\ \text{m/s}^2)\left[\frac{\sin 35° + (0.80)\cos 35°}{\cos 35° - (0.80)\sin 35°}\right] = \boxed{84\ \text{N}}$$

(*b*) Repeat Part (*a*) with $\mu_s = 0.40$ to obtain:

$$F_{\min} = \boxed{5.8\ \text{N}} \text{ and } F_{\max} = \boxed{37\ \text{N}}$$

64 ••• In your physics lab, you and your lab partners push a block of wood with a mass of 10.0 kg (starting from rest), with a constant horizontal force of 70 N across a wooden floor. In the previous week's laboratory meeting, your group determined that the coefficient of kinetic friction was not exactly constant, but instead was found to vary with the object's speed according to $\mu_k = 0.11/(1 + 2.3 \times 10^{-4}\, v^2)^2$. Write a **spreadsheet** program using Euler's method to calculate and graph both the speed and the position of the block as a function of time from 0 to 10 s. Compare this result to the result you would get if you assumed the coefficient of kinetic friction had a constant value of 0.11.

Picture the Problem The kinetic friction force f_k is the product of the coefficient of sliding friction μ_k and the normal force F_n the surface exerts on the sliding object. By applying Newton's second law in the vertical direction, we can see that, on a horizontal surface, the normal force is the weight of the sliding object. We can apply Newton's second law in the horizontal (x) direction to relate the block's acceleration to the net force acting on it. In the spreadsheet program, we'll find the acceleration of the block from this net force (which is speed dependent), calculate the increase in the block's speed from its acceleration and the elapsed time and add this increase to its speed at end of the previous time interval, determine how far it has moved in this time interval, and add this distance to its previous position to find its current position. We'll also calculate the position of the block x_2, under the assumption that $\mu_k = 0.11$, using a constant-acceleration equation.

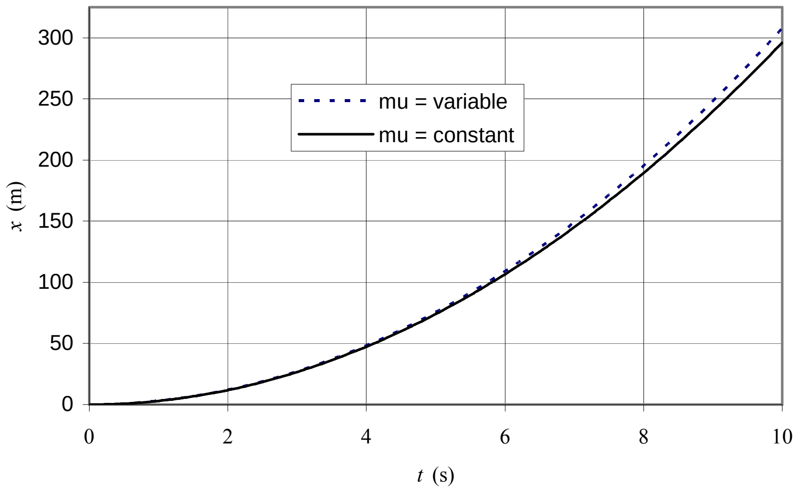


The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

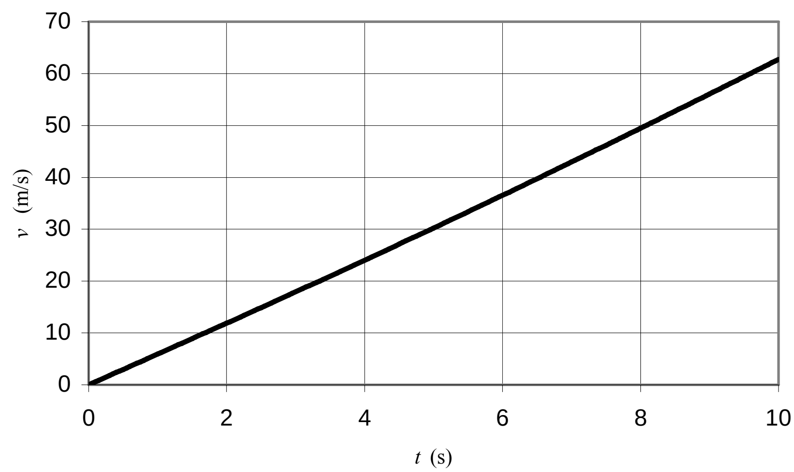
Cell	Formula/Content	Algebraic Form
C9	C8+$B$6	$t+\Delta t$
D9	D8+F9*$B$6	$v+a\Delta t$
E9	$B$5−($B$3)*($B$2)*$B$5/ (1+$B$4*D9^2)^2	$F-\dfrac{\mu_k mg}{\left(1+2.34\times10^{-4}v^2\right)^2}$
F9	E10/$B$5	F_{net}/m
G9	G9+D10*$B$6	$x+v\Delta t$
K9	0.5*5.922*I10^2	$\frac{1}{2}at^2$
L9	J10-K10	$x-x_2$

	A	B	C	D	E	F	G	H	I	J
1	g=	9.81	m/s^2							
2	Coeff1=	0.11								
3	Coeff2=	2.30E−04								
4	m=	10	kg							
5	F_{app}=	70	N							
6	Δt=	0.05	s							
7										
8							t	x	x_2	$x-x_2$
9	t	v	F_{net}	a,	x			$\mu_{variable}$	$\mu_{constant}$	
10	0.00	0.00			0.00		0.00	0.00	0.00	0.00
11	0.05	0.30	59.22	5.92	0.01		0.05	0.01	0.01	0.01
12	0.10	0.59	59.22	5.92	0.04		0.10	0.04	0.03	0.01
13	0.15	0.89	59.22	5.92	0.09		0.15	0.09	0.07	0.02
14	0.20	1.18	59.22	5.92	0.15		0.20	0.15	0.12	0.03
15	0.25	1.48	59.23	5.92	0.22		0.25	0.22	0.19	0.04
205	9.75	61.06	66.84	6.68	292.37		9.75	292.37	281.48	10.89
206	9.80	61.40	66.88	6.69	295.44		9.80	295.44	284.37	11.07
207	9.85	61.73	66.91	6.69	298.53		9.85	298.53	287.28	11.25
208	9.90	62.07	66.94	6.69	301.63		9.90	301.63	290.21	11.42
209	9.95	62.40	66.97	6.70	304.75		9.95	304.75	293.15	11.61
210	10.00	62.74	67.00	6.70	307.89		10.00	307.89	296.10	11.79

The position of the block as a function of time, for a constant coefficient of friction ($\mu_k = 0.11$) is shown as a solid line on the following graph and for a variable coefficient of friction, is shown as a dotted line. Because the coefficient of friction decreases with increasing particle speed, the particle travels slightly farther when the coefficient of friction is variable.

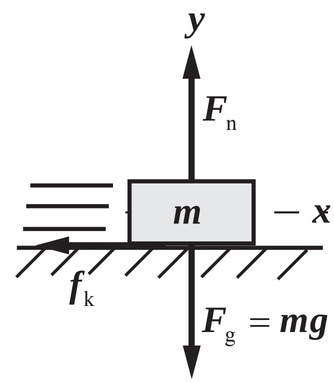


The speed of the block as a function of time, with variable coefficient of kinetic friction, is shown in the following graph.



65 ••• In order to determine the coefficient of kinetic friction of a block of wood on a horizontal table surface, you are given the following assignment: Take the block of wood and give it an initial velocity across the surface of the table. Using a stopwatch, measure the time Δt it takes for the block to come to a stop and the total displacement Δx the block slides following the push. (*a*) Using Newton's laws and a free-body diagram of the block, show that the expression for the coefficient of kinetic friction is given by $\mu_k = 2\Delta x/[(\Delta t)^2 g]$. (*b*) If the block slides a distance of 1.37 m in 0.97 s, calculate μ_k. (*c*) What was the initial speed of the block?

Picture the Problem The free-body diagram shows the forces acting on the block as it moves to the right. The kinetic friction force will slow the block and, eventually, bring it to rest. We can relate the coefficient of kinetic friction to the stopping time and distance by applying Newton's second law and then using constant-acceleration equations.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the block of wood:

$$\sum F_x = -f_k = ma_x$$

and

$$\sum F_y = F_n - mg = 0$$

Using the definition of f_k, eliminate F_n between the two equations to obtain:

$$a_x = -\mu_k g \qquad (1)$$

Use a constant-acceleration equation to relate the acceleration of the block to its displacement and its stopping time:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2 \qquad (2)$$

Relate the initial speed of the block, v_0, to its displacement and stopping distance:

$$\Delta x = v_{av}\Delta t = \left(\frac{v_{0x} + v}{2}\right)\Delta t$$

or, because $v = 0$,

$$\Delta x = \tfrac{1}{2}v_{0x}\Delta t \qquad (3)$$

Use this result to eliminate v_{0x} in equation (2):

$$\Delta x = -\tfrac{1}{2}a_x(\Delta t)^2 \qquad (4)$$

Substitute equation (1) in equation (4) and solve for μ_k:

$$\mu_k = \boxed{\frac{2\Delta x}{g(\Delta t)^2}}$$

(*b*) Substitute for $\Delta x = 1.37$ m and $\Delta t = 0.97$ s to obtain:

$$\mu_k = \frac{2(1.37\,\text{m})}{(9.81\,\text{m/s}^2)(0.97\,\text{s})^2} = \boxed{0.30}$$

(*c*) Use equation (3) to find v_0:

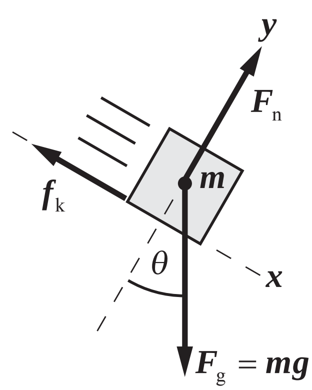
$$v_0 = \frac{2\Delta x}{\Delta t} = \frac{2(1.37\,\text{m})}{0.97\,\text{s}} = \boxed{2.8\,\text{m/s}}$$

66 ••• (*a*) A block is sliding down an inclined plane. The coefficient of kinetic friction between the block and the plane is μ_k. Show that a graph of $a_x/\cos\theta$ versus $\tan\theta$ (where a_x is the acceleration down the incline and θ is the angle the plane is inclined with the horizontal) would be a straight line with slope g and intercept $-\mu_k g$. (*b*) The following data show the acceleration of a block sliding down an inclined plane as a function of the angle θ that the plane is inclined with the horizontal:

θ (degrees)	Acceleration (m/s^2)
25.0	1.69
27.0	2.10
29.0	2.41
31.0	2.89
33.0	3.18
35.0	3.49
37.0	3.79
39.0	4.15
41.0	4.33
43.0	4.72
45.0	5.11

Using a **spreadsheet** program, graph these data and fit a straight line (a Trendline in Excel parlance) to them to determine μ_k and g. What is the percentage difference between the obtained value of g and the commonly specified value of 9.81 m/s^2?

Picture the Problem The free-body diagram shows the forces acting on the block as it slides down an incline. We can apply Newton's second law to these forces to obtain the acceleration of the block and then manipulate this expression algebraically to show that a graph of $a/\cos\theta$ versus $\tan\theta$ will be linear with a slope equal to the acceleration due to gravity and an intercept whose absolute value is the coefficient of kinetic friction.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the block as it slides down the incline:

$$\sum F_x = mg\sin\theta - f_k = ma_x$$

and

$$\sum F_y = F_n - mg\cos\theta = 0$$

Substitute $\mu_k F_n$ for f_k and eliminate F_n between the two equations to obtain:

$$a_x = g(\sin\theta - \mu_k \cos\theta)$$

Divide both sides of this equation by $\cos\theta$ to obtain:

$$\frac{a_x}{\cos\theta} = \boxed{g\tan\theta - g\mu_k}$$

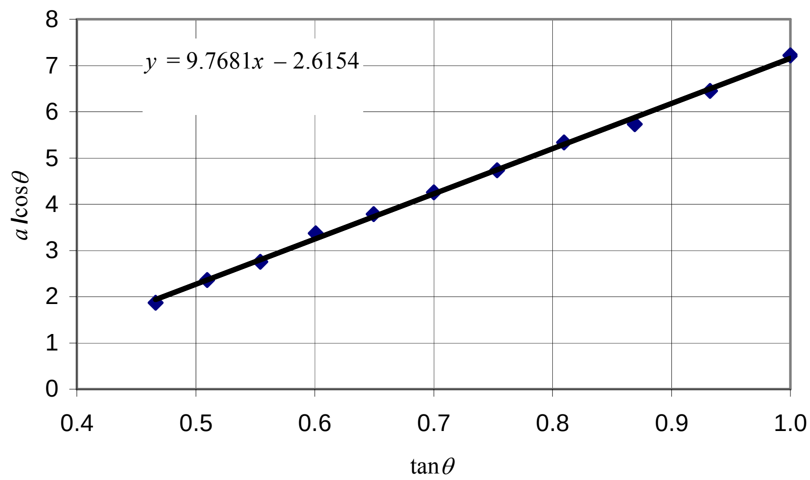
Note that this equation is of the form $y = mx + b$. Thus, if we graph $a/\cos\theta$ versus $\tan\theta$, we should get a straight line with slope g and y-intercept $-g\mu_k$.

(*b*) A spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
C7	θ	
D7	a	
E7	TAN(C7*PI()/180)	$\tan\left(\theta\times\frac{\pi}{180}\right)$
F7	D7/COS(C7*PI()/180)	$\frac{a}{\cos\left(\theta\times\frac{\pi}{180}\right)}$

	C	D	E	F
6	θ	a	$\tan\theta$	$a/\cos\theta$
7	25.0	1.69	0.466	1.866
8	27.0	2.10	0.510	2.362
9	29.0	2.41	0.554	2.751
10	31.0	2.89	0.601	3.370
11	33.0	3.18	0.649	3.786
12	35.0	3.49	0.700	4.259
13	37.0	3.78	0.754	4.735
14	39.0	4.15	0.810	5.338
15	41.0	4.33	0.869	5.732
16	43.0	4.72	0.933	6.451
17	45.0	5.11	1.000	7.220

A graph of $a/\cos\theta$ versus $\tan\theta$ follows. From the curve fit (Excel's Trendline was used), $g = \boxed{9.77\ \text{m/s}^2}$ and $\mu_k = \dfrac{2.62\ \text{m/s}^2}{9.77\ \text{m/s}^2} = \boxed{0.268}$.



The percentage error in g from the commonly accepted value of $9.81\ \text{m/s}^2$ is:

$$100\left(\frac{9.81\ \text{m/s}^2 - 9.77\ \text{m/s}^2}{9.81\ \text{m/s}^2}\right) = \boxed{0.408\%}$$

Drag Forces

67 • **[SSM]** A Ping-Pong ball has a mass of 2.3 g and a terminal speed of 9.0 m/s. The drag force is of the form bv^2. What is the value of b?

Picture the Problem The ping-pong ball experiences a downward gravitational force exerted by Earth and an upward drag force exerted by the air. We can apply Newton's second law to the Ping-Pong ball to obtain its equation of motion. Applying terminal speed conditions will yield an expression for b that we can evaluate using the given numerical values. Let the downward direction be the $+y$ direction.

Apply $\sum F_y = ma_y$ to the Ping-Pong ball:

$$mg - bv^2 = ma_y$$

When the Ping-Pong ball reaches its terminal speed $v = v_t$ and $a_y = 0$:

$$mg - bv_t^2 = 0 \Rightarrow b = \frac{mg}{v_t^2}$$

Substitute numerical values and evaluate b:

$$b = \frac{\left(2.3\times10^{-3}\ \text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{\left(9.0\,\text{m/s}\right)^2} = \boxed{2.8\times10^{-4}\ \text{kg/m}}$$

68 • A small pollution particle settles toward Earth in still air. The terminal speed is 0.30 mm/s, the mass of the particle is 1.0 × 10^{-10} g and the drag force is of the form bv. What is the value of b?

Picture the Problem The pollution particle experiences a downward gravitational force exerted by Earth and an upward drag force exerted by the air. We can apply Newton's second law to the particle to obtain its equation of motion. Applying terminal speed conditions will yield an expression for b that we can evaluate using the given numerical values. Let the downward direction by the $+y$ direction.

Apply $\sum F_y = ma_y$ to the particle:

$$mg - bv = ma_y$$

When the particle reaches its terminal speed $v = v_t$ and $a_y = 0$:

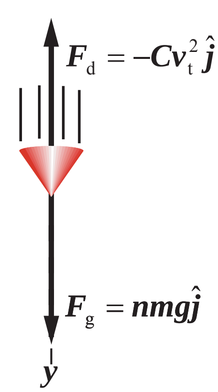
$$mg - bv_t = 0 \Rightarrow b = \frac{mg}{v_t}$$

Substitute numerical values and evaluate b:

$$b = \frac{\left(1.0\times10^{-13}\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{3.0\times10^{-4}\ \text{m/s}} = \boxed{3.3\times10^{-9}\ \text{kg/s}}$$

69 •• **[SSM]** A common classroom demonstration involves dropping basket-shaped coffee filters and measuring the time required for them to fall a given distance. A professor drops a single basket-shaped coffee filter from a height h above the floor, and records the time for the fall as Δt. How far will a stacked set of n identical filters fall during the same time interval Δt? Consider the filters to be so light that they instantaneously reach their terminal velocities. Assume a drag force that varies as the square of the speed and assume the filters are released oriented right-side up.

Picture the Problem The force diagram shows n coffee filters experiencing an upward drag force exerted by the air that varies with the square of their terminal velocity and a downward gravitational force exerted by Earth. We can use the definition of average velocity and Newton's second law to establish the dependence of the distance of fall on n.



Express the distance fallen by 1 coffee filter, falling at its terminal speed, in time Δt:

$$d_{1\text{ filter}} = v_{\text{t, 1 filter}}\Delta t \qquad (1)$$

Express the distance fallen by n coffee filters, falling at their terminal speed, in time Δt:

$$d_{n\text{ filters}} = v_{\text{t, }n\text{ filters}}\Delta t \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{d_{n\text{ filters}}}{d_{1\text{ filter}}} = \frac{v_{\text{t, }n\text{ filters}}\Delta t}{v_{\text{t, 1 filter}}\Delta t} = \frac{v_{\text{t, }n\text{ filters}}}{v_{\text{t, 1 filter}}}$$

Solving for $d_{n\text{ filters}}$ yields:

$$d_{n\text{ filters}} = \left(\frac{v_{\text{t, }n\text{ filters}}}{v_{\text{t, 1 filter}}}\right)d_{1\text{ filter}} \qquad (3)$$

Apply $\sum F_y = ma_y$ to the coffee filters:

$$nmg - Cv_\text{t}^2 = ma_y$$

or, because $a_y = 0$,

$$nmg - Cv_\text{t}^2 = 0$$

Solving for v_t yields:

$$v_{\text{t, }n\text{ filters}} = \sqrt{\frac{nmg}{C}} = \sqrt{\frac{mg}{C}}\sqrt{n}$$

or, because $v_{\text{t, 1 filter}} = \sqrt{\frac{mg}{C}}$,

$$v_{\text{t, }n\text{ filters}} = v_{\text{t, 1 filter}}\sqrt{n}$$

Substitute for $v_{\text{t, }n\text{ filters}}$ in equation (3) to obtain:

$$d_{n\text{ filters}} = \left(\frac{v_{\text{t, 1 filter}}\sqrt{n}}{v_{\text{t, 1 filter}}}\right)d_{1\text{ filter}} = \boxed{\sqrt{n}\,d_{1\text{ filter}}}$$

This result tells us that n filters will fall farther, in the same amount of time, than 1 filter by a factor of $\sqrt{n}$.

70 •• A skydiver of mass 60.0 kg can slow herself to a constant speed of 90 km/h by orienting her body horizontally, looking straight down with arms and legs extended. In this position she presents the maximum cross-sectional area and thus maximizes the air-drag force on her. (*a*) What is the magnitude of the drag force on the skydiver? (*b*) If the drag force is given by bv^2, what is the value of b? (*c*) At some instant she quickly flips into a ″knife″ position, orienting her body vertically with her arms straight down. Suppose this reduces the value of b to 55 percent of the value in Parts (*a*) and (*b*). What is her acceleration at the instant she achieves the ″knife″ position?

Picture the Problem The skydiver experiences a downward gravitational force exerted by Earth and an upward drag force exerted by the air. Let the upward direction be the $+y$ direction and apply Newton's second law to the sky diver.

(*a*) Apply $\sum F_y = ma_y$ to the sky diver:

$$F_d - mg = ma_y \tag{1}$$

or, because $a_y = 0$,

$$F_d = mg \tag{2}$$

Substitute numerical values and evaluate F_d:

$$F_d = (60.0\,\text{kg})(9.81\,\text{m/s}^2) = 588.6\,\text{N} = \boxed{589\,\text{N}}$$

(*b*) Substitute $F_d = bv_t^2$ in equation (2) to obtain:

$$bv_t^2 = mg \Rightarrow b = \frac{mg}{v_t^2} = \frac{F_d}{v_t^2}$$

Substitute numerical values and evaluate b:

$$b = \frac{588.6\,\text{N}}{\left(90\,\frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2} = 0.9418\,\text{kg/m} = \boxed{0.94\,\text{kg/m}}$$

(*c*) Solving equation (1) for a_y yields:

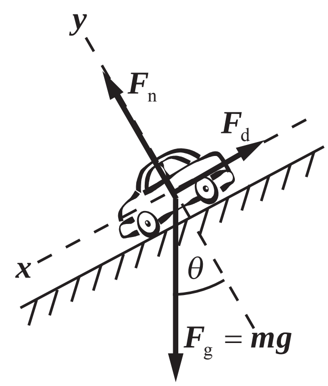
$$a_y = \left|\frac{F_d}{m} - g\right|$$

Because b is reduced to 55 percent of its value in Part (*b*), so is F_d. Substitute numerical values and evaluate a_y:

$$a_y = \left|\frac{(0.55)(588.6\,\text{N})}{60.0\,\text{kg}} - 9.81\,\text{m/s}^2\right| = \boxed{4.41\,\text{m/s}^2\text{, downward}}$$

71 •• Your team of test engineers is to release the parking brake so an 800-kg car will roll down a very long 6.0% grade in preparation for a crash test at the bottom of the incline. (On a 6.0 % grade the change in altitude is 6.0% of the horizontal distance traveled.) The total resistive force (air drag plus rolling friction) for this car has been previously established to be F_d = 100 N + (1.2 N·s^2/m^2)v^2, where m and v are the mass and speed of the car. What is the terminal speed for the car rolling down this grade?

Picture the Problem The free-body diagram shows the forces acting on the car as it descends the grade with its terminal speed and a convenient coordinate system. The application of Newton's second law with $a = 0$ and F_d equal to the given function will allow us to solve for the terminal speed of the car.



Apply $\sum F_x = ma_x$ to the car:

$$mg\sin\theta - F_d = ma_x$$

or, because $v = v_t$ and $a_x = 0$,

$$mg\sin\theta - F_d = 0$$

Substitute for F_d to obtain:

$$mg\sin\theta - 100\,\text{N} - \left(1.2\,\text{N}\cdot\text{s}^2/\text{m}^2\right)v_t^2 = 0$$

Solving for v_t yields:

$$v_t = \sqrt{\frac{mg\sin\theta - 100\,\text{N}}{1.2\,\text{N}\cdot\text{s}^2/\text{m}^2}}$$

Substitute numerical values and evaluate v_t:

$$v_t = \sqrt{\frac{(800\,\text{kg})\left(9.81\,\text{m/s}^2\right)\sin 6.0° - 100\,\text{N}}{1.2\,\text{N}\cdot\text{s}^2/\text{m}^2}} = \boxed{25\,\text{m/s}}$$

72 ••• Small slowly moving spherical particles experience a drag force given by Stokes' law: $F_d = 6\pi\eta rv$, where r is the radius of the particle, v is its speed, and η is the coefficient of viscosity of the fluid medium. (*a*) Estimate the terminal speed of a spherical pollution particle of radius 1.00×10^{-5} m and density of 2000 kg/m^3. (*b*) Assuming that the air is still and that η is 1.80×10^{-5} N·s/m^2, estimate the time it takes for such a particle to fall from a height of 100 m.

Picture the Problem Let the downward direction be the +y direction and apply Newton's second law to the particle to obtain an equation from which we can find the particle's terminal speed.

(*a*) Apply $\sum F_y = ma_y$ to a pollution particle:

$$mg - 6\pi\eta rv = ma_y$$

or, because $a_y = 0$,

$$mg - 6\pi\eta rv_t = 0 \Rightarrow v_t = \frac{mg}{6\pi\eta r}$$

Express the mass of a sphere in terms of its volume:

$$m = \rho V = \rho\left(\frac{4\pi r^3}{3}\right)$$

Substitute for m to obtain:

$$v_t = \frac{2r^2\rho g}{9\eta}$$

Substitute numerical values and evaluate v_t:

$$v_t = \frac{2(1.00\times10^{-5}\ \text{m})^2(2000\,\text{kg/m}^3)(9.81\,\text{m/s}^2)}{9(1.80\times10^{-5}\ \text{N}\cdot\text{s/m}^2)} = 2.422\ \text{cm/s} = \boxed{2.42\,\text{cm/s}}$$

(*b*) Use distance equals average speed times the fall time to find the time to fall 100 m at 2.42 cm/s:

$$t = \frac{10^4\ \text{cm}}{2.422\,\text{cm/s}} = 4.128\times10^3\ \text{s} = \boxed{1.15\,\text{h}}$$

73 ••• [SSM] You are on an environmental chemistry internship, and are in charge of a sample of air that contains pollution particles of the size and density given in Problem 72. You capture the sample in an 8.0-cm-long test tube. You then place the test tube in a centrifuge with the midpoint of the test tube 12 cm from the center of the centrifuge. You set the centrifuge to spin at 800 revolutions per minute. (*a*) Estimate the time you have to wait so that nearly all of the pollution particles settle to the end of the test tube. (*b*) Compare this to the time required for a pollution particle to fall 8.0 cm under the action of gravity and subject to the drag force given in Problem 72.

Picture the Problem The motion of the centrifuge will cause the pollution particles to migrate to the end of the test tube. We can apply Newton's second law and Stokes' law to derive an expression for the terminal speed of the sedimentation particles. We can then use this terminal speed to calculate the sedimentation time. We'll use the 12 cm distance from the center of the centrifuge as the average radius of the pollution particles as they settle in the test tube. Let *R* represent the radius of a particle and *r* the radius of the particle's circular path in the centrifuge.

(*a*) Express the sedimentation time in terms of the sedimentation speed v_t:

$$\Delta t_{\text{sediment}} = \frac{\Delta x}{v_t} \qquad (1)$$

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to a pollution particle:

$$6\pi\eta R v_t = ma_c$$

Express the mass of the particle in terms of its radius R and density ρ:

$$m = \rho V = \tfrac{4}{3}\pi R^3 \rho$$

Express the acceleration of the pollution particles due to the motion of the centrifuge in terms of their orbital radius r and period T:

$$a_c = \frac{v^2}{r} = \frac{\left(\dfrac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

Substitute for m and a_c and simplify to obtain:

$$6\pi\eta R v_t = \tfrac{4}{3}\pi R^3 \rho\left(\frac{4\pi^2 r}{T^2}\right) = \frac{16\pi^3 \rho r R^3}{3T^2}$$

Solving for v_t yields:

$$v_t = \frac{8\pi^2 \rho r R^2}{9\eta T^2}$$

Substitute for v_t in equation (1) and simplify to obtain:

$$\Delta t_{\text{sediment}} = \frac{\Delta x}{\dfrac{8\pi^2 \rho r R^2}{9\eta T^2}} = \frac{9\eta T^2 \Delta x}{8\pi^2 \rho r R^2}$$

Substitute numerical values and evaluate $\Delta t_{\text{sediment}}$:

$$\Delta t_{\text{sediment}} = \frac{9\left(1.8\times10^{-5}\,\dfrac{\text{N}\cdot\text{s}}{\text{m}^2}\right)\left(\dfrac{1}{800\,\dfrac{\text{rev}}{\text{min}}\cdot\dfrac{1\,\text{min}}{60\,\text{s}}}\right)^2 (8.0\,\text{cm})}{8\pi^2\left(2000\,\dfrac{\text{kg}}{\text{m}^3}\right)(0.12\,\text{m})\left(10^{-5}\,\text{m}\right)^2} = 38.47\text{ ms} \approx \boxed{38\text{ ms}}$$

(*b*) In Problem 72 it was shown that the rate of fall of the particles in air is 2.42 cm/s. Find the time required to fall 8.0 cm in air under the influence of gravity:

$$\Delta t_{\text{air}} = \frac{\Delta x}{v} = \frac{8.0\,\text{cm}}{2.42\,\text{cm/s}} = 3.31\,\text{s}$$

Find the ratio of the two times:

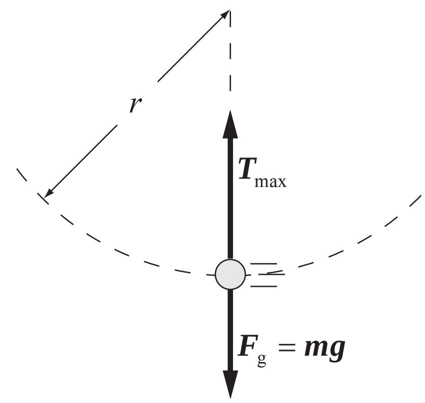
$$\frac{\Delta t_{\text{air}}}{\Delta t_{\text{sediment}}} = \frac{3.31\,\text{s}}{38.47\,\text{ms}} \approx 86$$

With the drag force in Problem 72 it takes about 86 times longer than it does using the centrifuge.

Motion Along a Curved Path

74 • A rigid rod with a 0.050-kg ball at one end rotates about the other end so the ball travels at constant speed in a vertical circle with a radius of 0.20 m. What is the maximum speed of the ball so that the force of the rod on the ball does not exceed 10 N?

Picture the Problem The force of the rod on the ball is a maximum when the ball is at the bottom of the vertical circle. We can use Newton's second law to relate the force of the rod on the ball to the mass *m* of the ball, the radius *r* of its path, and the constant speed of the ball along its circular path. The diagram to the right shows the forces acting on the ball when it is at the bottom of the vertical circle.



Apply $\sum F_{radial} = ma_{radial}$ to the ball:

$$T_{max} - mg = m\frac{v_{max}^2}{r}$$

Solving for v_{max} yields:

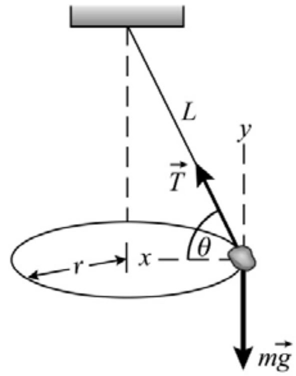
$$v_{max} = \sqrt{\left(\frac{T_{max}}{m} - g\right)r}$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = \sqrt{\left(\frac{10\ \text{N}}{0.050\ \text{kg}} - 9.81\ \text{m/s}^2\right)(0.20\ \text{m})} = \boxed{6.2\ \text{m/s}}$$

75 • **[SSM]** A 95-g stone is whirled in a horizontal circle on the end of an 85-cm-long string. The stone takes 1.2 s to make one complete revolution. Determine the angle that the string makes with the horizontal.

Picture the Problem The only forces acting on the stone are the tension in the string and the gravitational force. The centripetal force required to maintain the circular motion is a component of the tension. We'll solve the problem for the general case in which the angle with the horizontal is θ by applying Newton's second law of motion to the forces acting on the stone.



Apply $\sum \vec{F} = m\vec{a}$ to the stone:

$$\sum F_x = T\cos\theta = ma_c = m\frac{v^2}{r} \quad (1)$$

and

$$\sum F_y = T\sin\theta - mg = 0 \quad (2)$$

Use the right triangle in the diagram to relate r, L, and θ:

$$r = L\cos\theta \quad (3)$$

Eliminate T and r between equations (1), (2) and (3) and solve for v_2:

$$v^2 = gL\cot\theta\cos\theta \quad (4)$$

Express the speed of the stone in terms of its period:

$$v = \frac{2\pi r}{t_{1\,\text{rev}}} \quad (5)$$

Eliminate v between equations (4) and (5) and solve for θ:

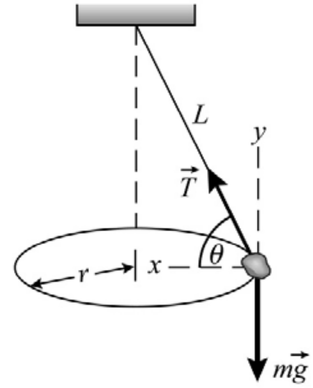
$$\theta = \sin^{-1}\left(\frac{gt_{1\,\text{rev}}^2}{4\pi^2 L}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{(9.81\,\text{m/s}^2)(1.2\,\text{s})^2}{4\pi^2(0.85\,\text{m})}\right] = \boxed{25°}$$

76 •• A 0.20-kg stone is whirled in a horizontal circle on the end of an 0.80-m-long string. The string makes an angle of 20º with the horizontal. Determine the speed of the stone.

Picture the Problem The only forces acting on the stone as it moves in a horizontal circle are the tension in the string and the gravitational force. The centripetal force required to maintain the circular motion is a component of the tension. We'll solve the problem for the general case in which the angle with the horizontal is θ by applying Newton's second law of motion to the forces acting on the stone.



Apply $\sum \vec{F} = m\vec{a}$ to the stone:

$$\sum F_x = T\cos\theta = ma_c = m\frac{v^2}{r} \quad (1)$$

and

$$\sum F_y = T\sin\theta - mg = 0 \quad (2)$$

Use the right triangle in the diagram to relate r, L, and θ:

$$r = L\cos\theta \quad (3)$$

Eliminate T and r between equations (1), (2), and (3) and solve for v:

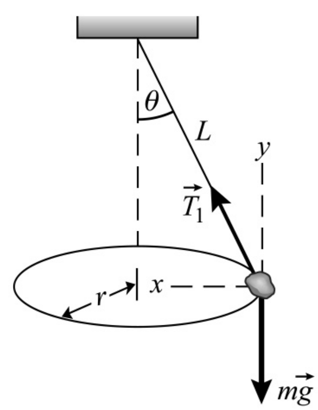
$$v = \sqrt{gL\cot\theta\cos\theta}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(9.81\,\text{m/s}^2)(0.80\,\text{m})\cot 20°\cos 20°} = \boxed{4.5\,\text{m/s}}$$

77 •• A 0.75-kg stone attached to a string is whirled in a horizontal circle of radius 35 cm as in the tetherball Example 5-11. The string makes an angle of 30º with the vertical. (*a*) Find the speed of the stone. (*b*) Find the tension in the string.

Picture the Problem The only forces acting on the stone are the tension in the string and the gravitational force. The centripetal force required to maintain the circular motion is a component of the tension. We'll solve the problem for the general case in which the angle with the vertical is θ by applying Newton's second law of motion to the forces acting on the stone.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the stone:

$$\sum F_x = T\sin\theta = ma_c = m\frac{v^2}{r} \quad (1)$$

and

$$\sum F_y = T\cos\theta - mg = 0 \quad (2)$$

Eliminate T between equations (1) and (2) and solve for v:

$$v = \sqrt{rg\tan\theta}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(0.35\,\text{m})(9.81\,\text{m/s}^2)\tan 30°} = \boxed{1.4\,\text{m/s}}$$

(*b*) Solving equation (2) for T yields:

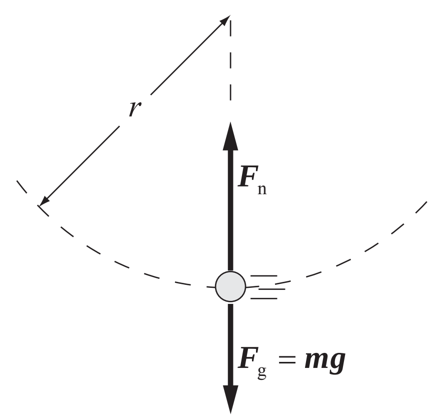
$$T = \frac{mg}{\cos\theta}$$

Substitute numerical values and evaluate T:

$$T = \frac{(0.75\,\text{kg})(9.81\,\text{m/s}^2)}{\cos 30°} = \boxed{8.5\,\text{N}}$$

78 •• A pilot with a mass of 50 kg comes out of a vertical dive in a circular arc such that at the bottom of the arc her upward acceleration is 3.5*g*. (*a*) How does the magnitude of the force exerted by the airplane seat on the pilot at the bottom of the arc compare to her weight? (*b*) Use Newton's laws of motion to explain why the pilot might be subject to a blackout. This means that an above normal volume of blood ″pools″ in her lower limbs. How would an inertial reference frame observer describe the cause of the blood pooling?

Picture the Problem The diagram shows the forces acting on the pilot when her plane is at the lowest point of its dive. $\vec{F}_n$ is the force the airplane seat exerts on her. We'll apply Newton's second law for circular motion to determine F_n and the radius of the circular path followed by the airplane.



(*a*) Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the pilot:

$$F_n - mg = ma_c \Rightarrow F_n = m(g + a_c)$$

Because $a_c = 3.5g$:

$$F_n = m(g + 3.5g) = 4.5mg$$

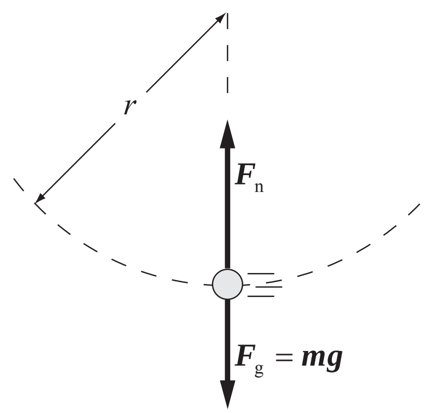
The ratio of F_n to her weight is:

$$\frac{F_n}{mg} = \frac{4.5mg}{mg} = \boxed{4.5}$$

(*b*) An observer in an inertial reference frame would see the pilot's blood continue to flow in a straight line tangent to the circle at the lowest point of the arc. The pilot accelerates upward away from this lowest point and therefore it appears, from the reference frame of the plane, as though the blood accelerates downward.

79 •• A 80.0-kg airplane pilot pulls out of a dive by following, at a constant speed of 180 km/h, the arc of a circle whose radius is 300 m. (*a*) At the bottom of the circle, what are the direction and magnitude of his acceleration? (*b*) What is the net force acting on him at the bottom of the circle? (*c*) What is the force exerted on the pilot by the airplane seat?

Picture the Problem The diagram shows the forces acting on the pilot when his plane is at the lowest point of its dive. $\vec{F}_n$ is the force the airplane seat exerts on the pilot. We'll use the definitions of centripetal acceleration and centripetal force and apply Newton's second law to calculate these quantities and the normal force acting on the pilot.



(*a*) The pilot's acceleration is centripetal and given by:

$$a_c = \frac{v^2}{r}\text{, upward}$$

Substitute numerical values and evaluate a_c:

$$a_c = \frac{\left(180\,\frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{300\,\text{m}} = 8.333\,\text{m/s}^2$$

$$= \boxed{8.33\,\text{m/s}^2\text{, upward}}$$

(*b*) The net force acting on her at the bottom of the circle is the force responsible for her centripetal acceleration:

$$F_{net} = ma_c = (80.0\,\text{kg})(8.333\,\text{m/s}^2)$$

$$= \boxed{667\,\text{N, upward}}$$

(*c*) Apply $\sum F_y = ma_y$ to the pilot:

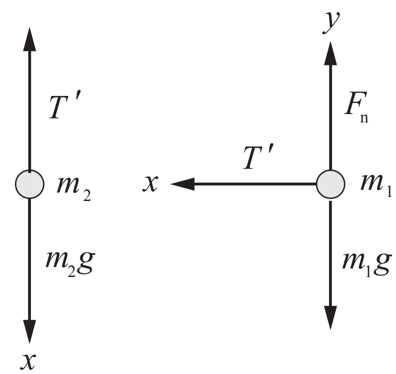
$$F_n - mg = ma_c \Rightarrow F_n = m(g + a_c)$$

Substitute numerical values and evaluate F_n:

$$F_n = (80.0\text{ kg})(9.81\text{ m/s}^2 + 8.33\text{ m/s}^2) = \boxed{1.45\text{ kN, upward}}$$

80 •• A small object of mass m_1 moves in a circular path of radius r on a frictionless horizontal tabletop (Figure 5-72). It is attached to a string that passes through a small frictionless hole in the center of the table. A second object with a mass of m_2 is attached to the other end of the string. Derive an expression for r in terms of m_1, m_2, and the time T for one revolution.

Picture the Problem Assume that the string is massless and that it does not stretch. The free-body diagrams for the two objects are shown to the right. The hole in the table changes the direction the tension in the string (which provides the centripetal force required to keep the object moving in a circular path) acts. The application of Newton's second law and the definition of centripetal force will lead us to an expression for r as a function of m_1, m_2, and the time T for one revolution.



Apply $\sum F_x = ma_x$ to both objects and use the definition of centripetal acceleration to obtain:

$$m_2 g - T' = 0$$

and

$$T' = m_1 a_c = m_1 \frac{v^2}{r}$$

Eliminating T between these equations gives:

$$m_2 g - m_1 \frac{v^2}{r} = 0 \qquad (1)$$

Express the speed v of the object in terms of the distance it travels each revolution and the time T for one revolution:

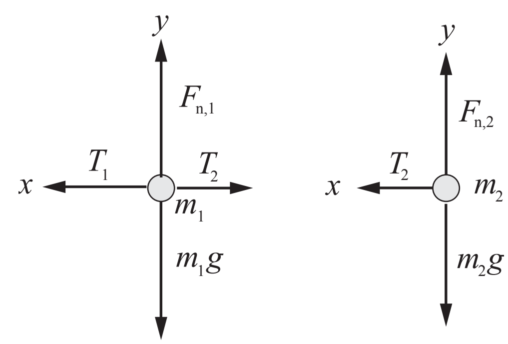
$$v = \frac{2\pi r}{T}$$

Substitute for v in equation (1) to obtain:

$$m_2 g - m_1 \frac{4\pi^2 r}{T^2} = 0 \Rightarrow r = \boxed{\frac{m_2 g T^2}{4\pi^2 m_1}}$$

81 •• **[SSM]** A block of mass m_1 is attached to a cord of length L_1, which is fixed at one end. The block moves in a horizontal circle on a frictionless tabletop. A second block of mass m_2 is attached to the first by a cord of length L_2 and also moves in a circle on the same frictionless tabletop, as shown in Figure 5-73. If the period of the motion is T, find the tension in each cord in terms of the given symbols.

Picture the Problem The free-body diagrams show the forces acting on each block. We can use Newton's second law to relate these forces to each other and to the masses and accelerations of the blocks.



Apply $\sum F_x = ma_x$ to the block whose mass is m_1:

$$T_1 - T_2 = m_1 \frac{v_1^2}{L_1}$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$$T_2 = m_2 \frac{v_2^2}{L_1 + L_2}$$

Relate the speeds of each block to their common period T and their distance from the center of the circle:

$$v_1 = \frac{2\pi L_1}{T} \text{ and } v_2 = \frac{2\pi(L_1 + L_2)}{T}$$

In the second force equation, substitute for v_2, and simplify to obtain:

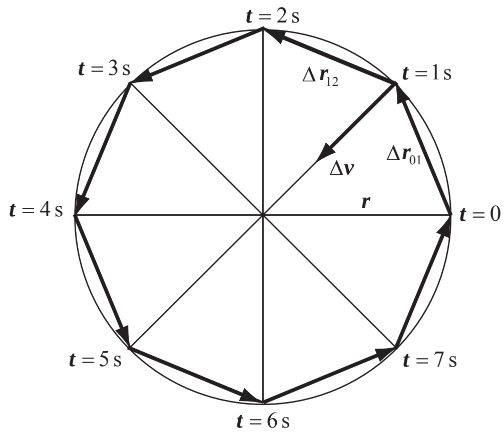
$$T_2 = \boxed{[m_2(L_1 + L_2)]\left(\frac{2\pi}{T}\right)^2}$$

Substitute for T_2 and v_1 in the first force equation to obtain:

$$T_1 = \boxed{[m_2(L_1 + L_2) + m_1L_1]\left(\frac{2\pi}{T}\right)^2}$$

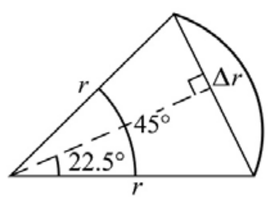
82 •• A particle moves with constant speed in a circle of radius 4.0 cm. It takes 8.0 s to complete each revolution. (*a*) Draw the path of the particle to scale, and indicate the particle's position at 1.0-s intervals. (*b*) Sketch the displacement vectors for each interval. These vectors also indicate the directions for the average-velocity vectors for each interval. (*c*) Graphically find the magnitude of the change in the average velocity $|\Delta\vec{v}|$ for two consecutive 1-s intervals. Compare $|\Delta\vec{v}|/\Delta t$, measured in this way, with the magnitude of the instantaneous acceleration computed from $a_c = v^2/r$.

Picture the Problem (*a*) and (*b*) The path of the particle and its position at 1-s intervals are shown in the following diagram. The displacement vectors are also shown. The velocity vectors for the average velocities in the first and second intervals are along $\vec{r}_{01}$ and $\vec{r}_{12}$, respectively. $\Delta\vec{v}$ points toward the center of the circle.



(*c*) Use the diagram below to show that:

$$\Delta r = 2r\sin 22.5°$$

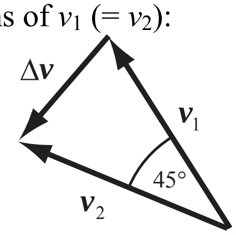


Express the magnitude of the average velocity of the particle along the chords:

$$|\vec{v}_{\text{av}}| = \frac{\Delta r}{\Delta t} = \frac{2r\sin 22.5°}{\Delta t}$$

Using the diagram below, express Δv in terms of v_1 (= v_2):

$$\Delta v = 2v_1 \sin 22.5°$$



Express Δv using v_{av} as v_1:

$$\Delta v = 2\left(\frac{2r\sin 22.5°}{\Delta t}\right)\sin 22.5°$$

$$= \frac{4r\sin^2 22.5°}{\Delta t}$$

Express $a=\dfrac{\Delta v}{\Delta t}$:

$$a=\frac{\dfrac{4r\sin^2 22.5°}{\Delta t}}{\Delta t}=\frac{4r\sin^2 22.5°}{(\Delta t)^2}$$

Substitute numerical values and evaluate a:

$$a=\frac{4(4.0\text{ cm})\sin^2 22.5°}{(1.0\text{ s})^2}=2.34\text{ cm/s}^2$$
$$=\boxed{2.3\text{ cm/s}^2}$$

The radial acceleration of the particle is given by:

$$a_c=\frac{v^2}{r}$$

Express the speed v (= $v_1 = v_2$ …) of the particle along its circular path:

$$v=\frac{2\pi r}{T}$$

Substituting for v in the expression for a_c yields:

$$a_c=\frac{\left(\dfrac{2\pi r}{T}\right)^2}{r}=\frac{4\pi^2 r}{T^2}$$

Substitute numerical values and evaluate a_c:

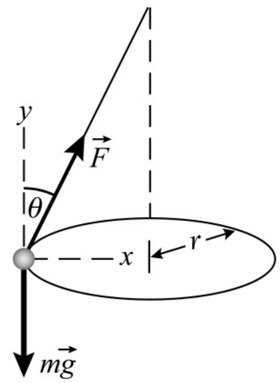
$$a_c=\frac{4\pi^2(4.0\text{ cm})}{(8.0\text{ s})^2}=2.47\text{ cm/s}^2$$
$$=\boxed{2.5\text{ cm/s}^2}$$

Compare a_c and a by taking their ratio:

$$\frac{a_c}{a}=\frac{2.47\text{ cm/s}^2}{2.34\text{ cm/s}^2}=1.06\Rightarrow a_c=\boxed{1.1a}$$

83 •• You are swinging your younger sister in a circle of radius 0.75 m, as shown in Figure 5-74. If her mass is 25 kg and you arrange it so she makes one revolution every 1.5 s, (*a*) what are the magnitude and direction of the force that must be exerted by you on her? (Model her as a point particle.) (*b*) What are the magnitude and direction of the force she exerts on you?

Picture the Problem The following diagram shows the free-body diagram for the child superimposed on a pictorial representation of her motion. The force you exert on your sister is $\vec{F}$ and the angle it makes with respect to the direction we've chosen as the positive y direction is θ. We can infer her speed from the given information concerning the radius of her path and the period of her motion. Applying Newton's second law will allow us to find both the direction and magnitude of $\vec{F}$.



(*a*) Apply $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to the child:

$$\sum F_x = F\sin\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F\cos\theta - mg = 0$$

Eliminate *F* between these equations and solve for θ to obtain:

$$\theta = \tan^{-1}\left[\frac{v^2}{rg}\right]$$

Express *v* in terms of the radius and period of the child's motion:

$$v = \frac{2\pi r}{T}$$

Substitute for *v* in the expression for θ to obtain:

$$\theta = \tan^{-1}\left[\frac{4\pi^2 r}{gT^2}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{4\pi^2(0.75\,\text{m})}{(9.81\,\text{m/s}^2)(1.5\,\text{s})^2}\right] = 53.3°$$

$$= \boxed{53°}\ \text{above horizontal}$$

Solve the *y* equation for *F*:

$$F = \frac{mg}{\cos\theta}$$

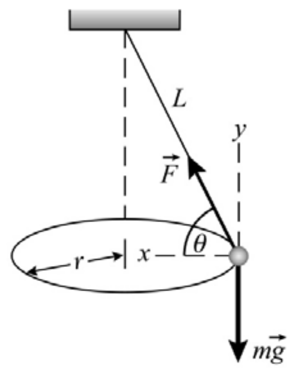
Substitute numerical values and evaluate *F*:

$$F = \frac{(25\,\text{kg})(9.81\,\text{m/s}^2)}{\cos 53.3°} = \boxed{0.41\,\text{kN}}$$

(*b*) The force your sister exerts on you is the reaction force to the force you exert on her. Thus its magnitude is the same as the force you exert on her (0.41 kN) and its direction is 53° below horizontal.

84 •• The string of a conical pendulum is 50.0 cm long and the mass of the bob is 0.25 kg. (*a*) Find the angle between the string and the horizontal when the tension in the string is six times the weight of the bob. (*b*) Under those conditions, what is the period of the pendulum?

Picture the Problem The following diagram shows the free-body diagram for the bob of the conical pendulum superimposed on a pictorial representation of its motion. The tension in the string is $\vec{F}$ and the angle it makes with respect to the direction we've chosen as the +*x* direction is θ. We can find θ from the *y* equation and the information provided about the tension. Then, by using the definition of the speed of the bob in its orbit and applying Newton's second law as it describes circular motion, we can find the period *T* of the motion.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the pendulum bob:

$$\sum F_x = F\cos\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F\sin\theta - mg = 0$$

Using the given information that $F = 6mg$, solve the *y* equation for θ and simplify to obtain:

$$\theta = \sin^{-1}\left(\frac{mg}{F}\right) = \sin^{-1}\left(\frac{mg}{6mg}\right) = \boxed{9.6°}$$

(*b*) With $F = 6mg$, solve the *x* equation for *v*:

$$v = \sqrt{6rg\cos\theta}$$

Relate the period *T* of the motion to the speed of the bob and the radius of the circle in which it moves:

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{6rg\cos\theta}}$$

From the diagram, one can see that:

$$r = L\cos\theta$$

Substitute for r in the expression for the period and simplify to obtain:

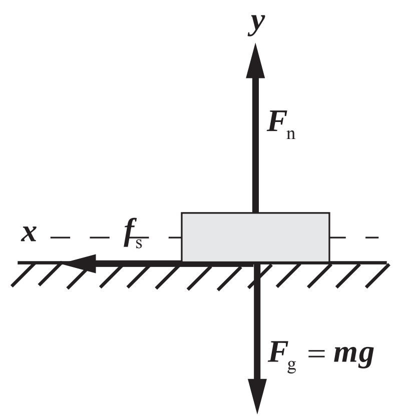
$$T = 2\pi\sqrt{\frac{L}{6g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{0.50\text{ m}}{6\left(9.81\text{ m/s}^2\right)}} = \boxed{0.58\text{ s}}$$

85 •• A 100-g coin sits on a horizontally rotating turntable. The turntable makes exactly 1.00 revolution each second. The coin is located 10 cm from the axis of rotation of the turntable. (*a*) What is the frictional force acting on the coin? (*b*) If the coin slides off the turntable when it is located more than 16.0 cm from the axis of rotation, what is the coefficient of static friction between the coin and the turntable?

Picture the Problem The static friction force f_s is responsible for keeping the coin from sliding on the turntable. Using Newton's second law of motion, the definition of the period of the coin's motion, and the definition of the maximum static friction force, we can find the magnitude of the friction force and the value of the coefficient of static friction for the two surfaces.



(*a*) Apply $\sum\vec{F} = m\vec{a}$ to the coin:

$$\sum F_x = f_s = m\frac{v^2}{r} \qquad (1)$$

and

$$\sum F_y = F_n - mg = 0$$

If T is the period of the coin's motion, its speed is given by:

$$v = \frac{2\pi r}{T}$$

Substitute for v in equation (1) and simplify to obtain:

$$f_s = \frac{4\pi^2 mr}{T^2}$$

Substitute numerical values and evaluate f_s:

$$f_s = \frac{4\pi^2\left((0.100\text{ kg})(0.10\text{ m})\right)}{(1.00\text{ s})^2} = \boxed{0.40\text{ N}}$$

(*b*) Determine F_n from the y equation: $F_n = mg$

If the coin is on the verge of sliding at r = 16 cm, $f_s = f_{s,max}$. Solve for μ_s in terms of $f_{s,max}$ and F_n:

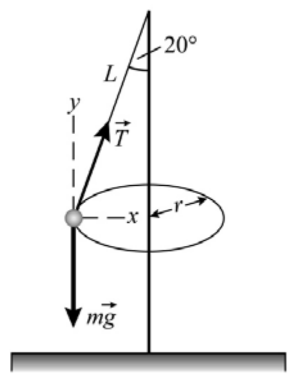
$$\mu_s = \frac{f_{s,max}}{F_n} = \frac{\frac{4\pi^2 mr}{T^2}}{mg} = \frac{4\pi^2 r}{gT^2}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{4\pi^2(0.160\,\text{m})}{(9.81\,\text{m/s}^2)(1.00\,\text{s})^2} = \boxed{0.644}$$

86 •• A 0.25-kg tether ball is attached to a vertical pole by a 1.2-m cord. Assume the radius of the ball is negligible. If the ball moves in a horizontal circle with the cord making an angle of 20º with the vertical, (*a*) what is the tension in the cord? (*b*) What is the speed of the ball?

Picture the Problem The forces acting on the tetherball are shown superimposed on a pictorial representation of the motion. The horizontal component of $\vec{T}$ is the centripetal force. Applying Newton's second law of motion and solving the resulting equations will yield both the tension in the cord and the speed of the ball.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the tetherball:

$$\sum F_x = T\sin 20° = m\frac{v^2}{r}$$

and

$$\sum F_y = T\cos 20° - mg = 0$$

Solve the y equation for T:

$$T = \frac{mg}{\cos 20°}$$

Substitute numerical values and evaluate T:

$$T = \frac{(0.25\,\text{kg})(9.81\,\text{m/s}^2)}{\cos 20°} = \boxed{2.6\,\text{N}}$$

(*b*) Eliminate T between the force equations and solve for v to obtain:

$$v = \sqrt{rg\tan 20°}$$

Note from the diagram that:

$$r = L\sin 20°$$

Substitute for r in the expression for v to obtain:

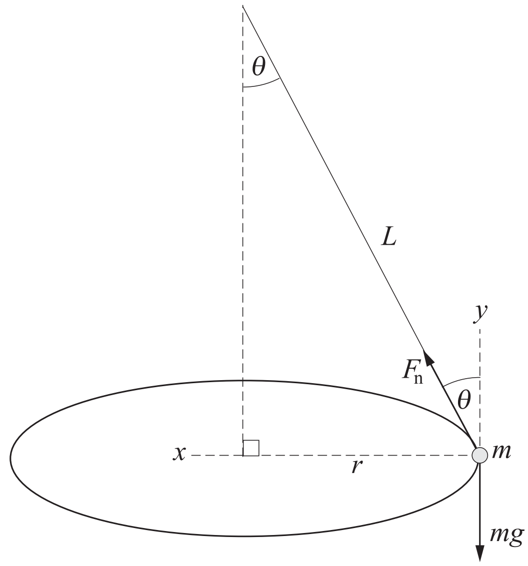
$$v = \sqrt{gL\sin 20°\tan 20°}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{(9.81\,\text{m/s}^2)(1.2\,\text{m})\sin 20^\circ \tan 20^\circ}=\boxed{1.2\,\text{m/s}}$$

87 ••• A small bead with a mass of 100 g (Figure 5-75) slides without friction along a semicircular wire with a radius of 10 cm that rotates about a vertical axis at a rate of 2.0 revolutions per second. Find the value of θ for which the bead will remain stationary relative to the rotating wire.

Picture the Problem The semicircular wire of radius 10 cm limits the motion of the bead in the same manner as would a 10-cm string attached to the bead and fixed at the center of the semicircle. The horizontal component of the normal force the wire exerts on the bead is the centripetal force. The application of Newton's second law, the definition of the speed of the bead in its orbit, and the relationship of the frequency of a circular motion to its period will yield the angle at which the bead will remain stationary relative to the rotating wire.



Apply $\sum \vec{F} = m\vec{a}$ to the bead:

$$\sum F_x = F_n \sin\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - mg = 0$$

Eliminate F_n from the force equations to obtain:

$$\tan\theta = \frac{v^2}{rg}$$

The frequency of the motion is the reciprocal of its period T. Express the speed of the bead as a function of the radius of its path and its period:

$$v=\frac{2\pi r}{T}$$

Using the diagram, relate r to L and θ:

$$r = L\sin\theta$$

Substitute for r and v in the expression for $\tan\theta$ and solve for θ:

$$\theta = \cos^{-1}\left[\frac{gT^2}{4\pi^2 L}\right]$$

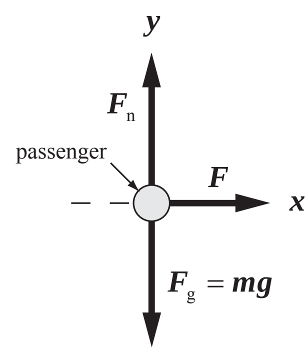
Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left[\frac{\left(9.81\,\text{m/s}^2\right)\left(0.50\,\text{s}\right)^2}{4\pi^2\left(0.10\,\text{m}\right)}\right] = \boxed{52°}$$

Centripetal Force

88 • A car speeds along the curved exit ramp of a freeway. The radius of the curve is 80.0 m. A 70.0-kg passenger holds the armrest of the car door with a 220-N force in order to keep from sliding across the front seat of the car. (Assume the exit ramp is not banked and ignore friction with the car seat.) What is the car's speed?

Picture the Problem The force $\vec{F}$ the passenger exerts on the armrest of the car door is the radial force required to maintain the passenger's speed around the curve and is related to that speed through Newton's second law of motion.



Apply $\sum F_x = ma_x$ to the forces acting on the passenger:

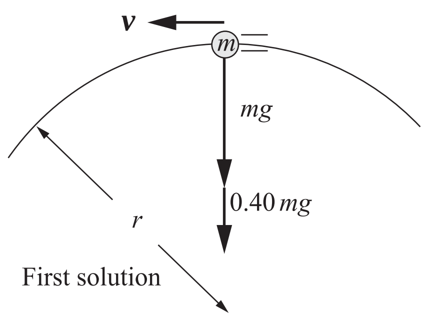
$$F = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{rF}{m}}$$

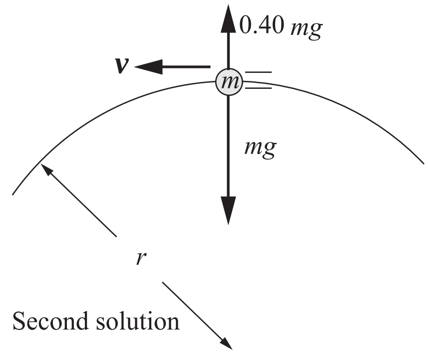
Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{\left(80.0\,\text{m}\right)\left(220\,\text{N}\right)}{70.0\,\text{kg}}} = \boxed{15.9\,\text{m/s}}$$

89 • **[SSM]** The radius of curvature of the track at the top of a loop-the-loop on a roller-coaster ride is 12.0 m. At the top of the loop, the force that the seat exerts on a passenger of mass m is $0.40mg$. How fast is the roller-coaster car moving as it moves through the highest point of the loop.

Picture the Problem There are two solutions to this problem. Pictorial diagrams that include a free-body diagram for each solution are shown below. The speed of the roller coaster is embedded in the expression for its radial acceleration and the radial acceleration is determined by the net radial force force exerted on the passenger by the seat of the roller coaster. We can use Newton's second law to relate the net force on the passenger to the speed of the roller coaster.





First solution

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the passenger:

$$mg + 0.40mg = m\frac{v^2}{r} \Rightarrow v = \sqrt{1.40gr}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(1.40)(9.81\,\text{m/s}^2)(12.0\,\text{m})} = \boxed{12.8\,\text{m/s}}$$

Second solution

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the passenger:

$$mg - 0.40mg = m\frac{v^2}{r} \Rightarrow v = \sqrt{0.60gr}$$

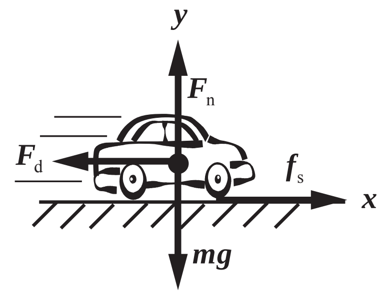
Substitute numerical values and evaluate v:

$$v = \sqrt{(0.60)(9.81\,\text{m/s}^2)(12.0\,\text{m})} = \boxed{8.4\,\text{m/s}}$$

90 •• On a runway of a decommissioned airport, a 2000-kg car travels at a constant speed of 100 km/h. At 100-km/h the air drag on the car is 500 N. Assume that rolling friction is negligible. (*a*) What is the force of static friction exerted on the car by the runway surface, and what is the minimum coefficient of static friction necessary for the car to sustain this speed? (*b*) The car continues to travel at 100 km/h, but now along a path with radius of curvature *r*. For what value of *r* will the angle between the static frictional force vector and the velocity vector equal 45.0°, and for what value of *r* will it equal 88.0°? (*c*) What is the minimum coefficient of static friction necessary for the car to hold this last radius of curvature without skidding?

Picture the Problem (*a*) We can apply Newton's second law to the car to find the force of static friction exerted on the car by the surface and use the definition of the coefficient of static friction to find the minimum coefficient of static friction necessary for the car to sustain its speed. In Part (*b*), we can again apply Newton's second law, this time in both tangential and radial form, to find the values of *r* for the given angles.

(*a*) The forces acting on the car as it moves at a constant speed of 100 km/h are shown in the pictorial representation to the right.



Apply $\sum\vec{F} = m\vec{a}$ to the car to obtain:

$$\sum F_x = f_s - F_d = 0 \qquad (1)$$

and

$$\sum F_y = F_n - mg = 0 \qquad (2)$$

Solving equation (1) for f_s yields:

$$f_s = F_d = \boxed{500\text{ N}}$$

Use the definition of the coefficient of static friction to obtain:

$$\mu_{s,\,min} = \frac{f_s}{F_n}$$

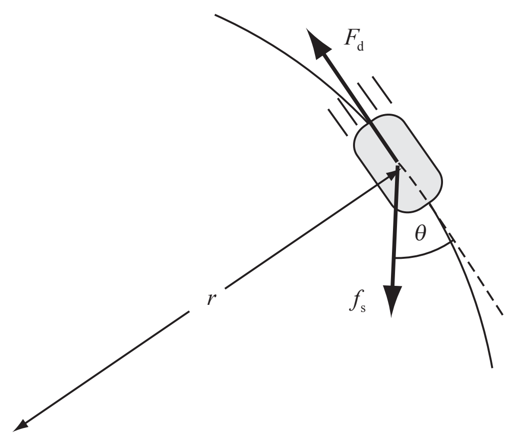
From equation (2), $F_n = mg$; hence:

$$\mu_{s,\,min} = \frac{f_s}{mg}$$

Substitute numerical values and evaluate $\mu_{s,\,min}$:

$$\mu_{s,\,min} = \frac{500\text{ N}}{(2000\text{ kg})(9.81\text{ m/s}^2)} = \boxed{0.0255}$$

(*b*) The horizontal forces acting on the car (shown as a top view) as it travels clockwise along a path of radius of curvature *r* are shown in the diagram to the right.



Apply $\sum F_{radial} = ma_c$ to the car to obtain:

$$f_s \sin\theta = m\frac{v^2}{r} \qquad (1)$$

Appling $\sum F_{\text{tangential}} = ma_{\text{t}}$ to the car yields:

$$f_{\text{s}}\cos\theta - F_{\text{d}} = 0$$

or

$$f_{\text{s}}\cos\theta = F_{\text{d}} \qquad (2)$$

Divide equation (1) by equation (2):

$$\tan\theta = \frac{mv^2}{rF_{\text{d}}} \Rightarrow r = \frac{mv^2}{F_{\text{d}}\tan\theta}$$

Substitute numerical values and evaluate r for $\theta = 45.0°$:

$$r = \frac{(2000\ \text{kg})\left(100\frac{\text{km}}{\text{h}}\cdot\frac{10^3\ \text{m}}{\text{km}}\cdot\frac{1\ \text{h}}{3600\ \text{s}}\right)^2}{(500\ \text{N})\tan 45.0°} = \boxed{3.09\ \text{km}}$$

Substitute numerical values and valuate r for $\theta = 88.0°$:

$$r = \frac{(2000\ \text{kg})\left(100\frac{\text{km}}{\text{h}}\cdot\frac{10^3\ \text{m}}{\text{km}}\cdot\frac{1\ \text{h}}{3600\ \text{s}}\right)^2}{(500\ \text{N})\tan 88.0°} = 107.8\ \text{m} = \boxed{108\ \text{m}}$$

(*c*) The minimum coefficient of static friction necessary for the car to hold this last radius of curvature without skidding is given by:

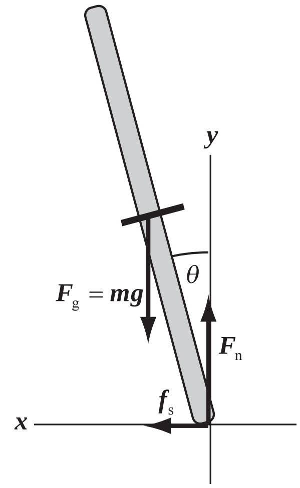
$$\mu_{\text{s, min}} = \frac{f_{\text{s, min}}}{mg} = \frac{m\frac{v^2}{r}}{mg} = \frac{v^2}{rg}$$

Substitute numerical values and evaluate $\mu_{\text{s, min}}$:

$$\mu_{\text{s, min}} = \frac{\left(100\frac{\text{km}}{\text{h}}\cdot\frac{10^3\ \text{m}}{\text{km}}\cdot\frac{1\ \text{h}}{3600\ \text{s}}\right)^2}{(107.8\ \text{m})(9.81\ \text{m/s}^2)} = \boxed{0.730}$$

91 •• Suppose you ride a bicycle in a 20-m-radius circle on a horizontal surface. The resultant force exerted by the surface on the bicycle (normal force plus frictional force) makes an angle of 15º with the vertical. (*a*) What is your speed? (*b*) If the frictional force on the bicycle is half its maximum possible value, what is the coefficient of static friction?

Picture the Problem The forces acting on the bicycle are shown in the force diagram. The static friction force is the centripetal force exerted by the surface on the bicycle that allows it to move in a circular path. $\vec{F}_n + \vec{f}_s$ makes an angle θ with the vertical direction. The application of Newton's second law will allow us to relate this angle to the speed of the bicycle and the coefficient of static friction.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the bicycle:

$$\sum F_x = f_s = \frac{mv^2}{r}$$

and

$$\sum F_y = F_n - mg = 0$$

Relate F_n and f_s to θ:

$$\tan\theta = \frac{f_s}{F_n} = \frac{\frac{mv^2}{r}}{mg} = \frac{v^2}{rg}$$

Solving for v yields:

$$v = \sqrt{rg\tan\theta}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(20\,\text{m})(9.81\,\text{m/s}^2)\tan 15°}$$
$$= \boxed{7.3\,\text{m/s}}$$

(*b*) Relate f_s to μ_s and F_n:

$$f_s = \tfrac{1}{2} f_{s,\max} = \tfrac{1}{2}\mu_s mg$$

Solve for μ_s and substitute for f_s to obtain:

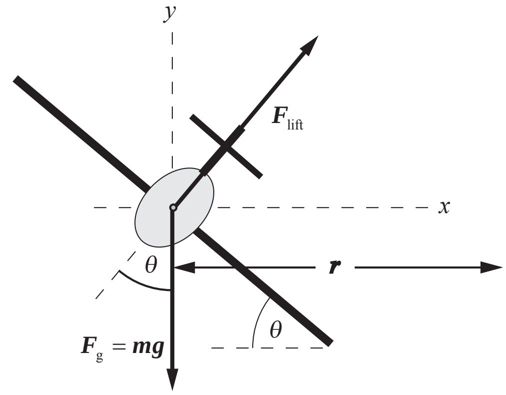
$$\mu_s = \frac{2f_s}{mg} = \frac{2v^2}{rg}$$

Substitute numerical values and evaluate μ_s

$$\mu_s = \frac{2(7.25\,\text{m/s})^2}{(20\,\text{m})(9.81\,\text{m/s}^2)} = \boxed{0.54}$$

92 •• An airplane is flying in a horizontal circle at a speed of 480 km/h. The plane is banked for this turn, its wings tilted at an angle of 40º from the horizontal (Figure 5-76). Assume that a lift force acting perpendicular to the wings acts on the aircraft as it moves through the air. What is the radius of the circle in which the plane is flying?

Picture the Problem The diagram shows the forces acting on the plane as it flies in a horizontal circle of radius r. We can apply Newton's second law to the plane and eliminate the lift force in order to obtain an expression for r as a function of v and θ.



Apply $\sum \vec{F} = m\vec{a}$ to the plane:

$$\sum F_x = F_{\text{lift}} \sin\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F_{\text{lift}} \cos\theta - mg = 0$$

Eliminate F_{lift} between these equations to obtain:

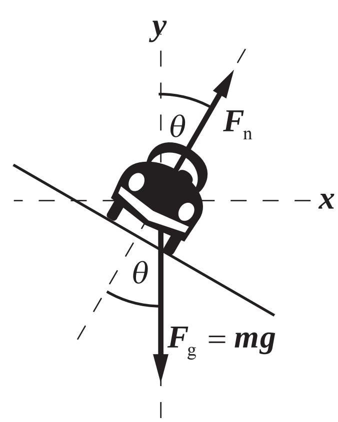
$$\tan\theta = \frac{v^2}{rg} \Rightarrow r = \frac{v^2}{g\tan\theta}$$

Substitute numerical values and evaluate r:

$$r = \frac{\left(480\frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{\left(9.81\,\text{m/s}^2\right)\tan 40°} = \boxed{2.2\,\text{km}}$$

93 •• An automobile club plans to race a 750-kg car at the local racetrack. The car needs to be able to travel around several 160-m-radius curves at 90 km/h. What should the banking angle of the curves be so that the force of the pavement on the tires of the car is in the normal direction? (*Hint: What does this requirement tell you about the frictional force?*)

Picture the Problem Under the conditions described in the problem statement, the only forces acting on the car are the normal force exerted by the road and the gravitational force exerted by Earth. The horizontal component of the normal force is the centripetal force. The application of Newton's second law will allow us to express θ in terms of v, r, and g.



Apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = F_n \sin\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - mg = 0$$

Eliminate F_n from the force equations to obtain:

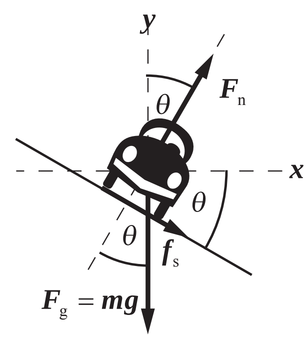
$$\tan\theta = \frac{v^2}{rg} \Rightarrow \theta = \tan^{-1}\left[\frac{v^2}{rg}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left\{\frac{\left(90\,\frac{\text{km}}{\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{(160\,\text{m})(9.81\,\text{m/s}^2)}\right\} = \boxed{22°}$$

94 •• A curve of radius 150 m is banked at an angle of 10º. An 800-kg car negotiates the curve at 85 km/h without skidding. Neglect the effects of air drag and rolling friction. Find (*a*) the normal force exerted by the pavement on the tires, (*b*) the frictional force exerted by the pavement on the tires, (*c*) the minimum coefficient of static friction between the pavement and the tires.

Picture the Problem Both the normal force and the static friction force contribute to the centripetal force in the situation described in this problem. We can apply Newton's second law to relate f_s and F_n and then solve these equations simultaneously to determine each of these quantities.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the car:

$$\sum F_x = F_n \sin\theta + f_s \cos\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - f_s \sin\theta - mg = 0$$

Multiply the x equation by $\sin\theta$ and the y equation by $\cos\theta$ to obtain:

$$f_s \sin\theta\cos\theta + F_n \sin^2\theta = m\frac{v^2}{r}\sin\theta$$

and

$$F_n \cos^2\theta - f_s \sin\theta\cos\theta - mg\cos\theta = 0$$

Add these equations to eliminate f_s:

$$F_n - mg\cos\theta = m\frac{v^2}{r}\sin\theta$$

Solve for F_n:

$$F_n = mg\cos\theta + m\frac{v^2}{r}\sin\theta$$
$$= m\left(g\cos\theta + \frac{v^2}{r}\sin\theta\right)$$

Substitute numerical values and evaluate F_n:

$$F_n = (800\,\text{kg})\left[(9.81\,\text{m/s}^2)\cos 10° + \frac{\left(85\,\frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{150\,\text{m}}\sin 10°\right] = 8.245\,\text{kN}$$
$$= \boxed{8.3\,\text{kN}}$$

(*b*) Solve the *y*-equation for f_s:

$$f_s = \frac{F_n\cos\theta - mg}{\sin\theta}$$

Substitute numerical values and evaluate f_s:

$$f_s = \frac{(8.245\,\text{kN})\cos 10° - (800\,\text{kg})(9.81\,\text{m/s}^2)}{\sin 10°} = 1.565\,\text{kN} = \boxed{1.6\,\text{kN}}$$

(*c*) Express $\mu_{s,\min}$ in terms of f_s and F_n:

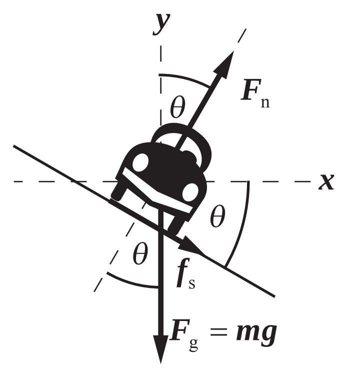
$$\mu_{s,\min} = \frac{f_s}{F_n}$$

Substitute numerical values and evaluate $\mu_{s,\min}$:

$$\mu_{s,\min} = \frac{1.565\,\text{kN}}{8.245\,\text{kN}} = \boxed{0.19}$$

95 •• On another occasion, the car in Problem 94 negotiates the curve at 38 km/h. Neglect the effects of air drag and rolling friction. Find (*a*) the normal force exerted on the tires by the pavement, and (*b*) the frictional force exerted on the tires by the pavement.

Picture the Problem Both the normal force and the static friction force contribute to the centripetal force in the situation described in this problem. We can apply Newton's second law to relate f_s and F_n and then solve these equations simultaneously to determine each of these quantities.



(a) Apply $\sum\vec{F} = m\vec{a}$ to the car:

$$\sum F_x = F_n \sin\theta + f_s \cos\theta = m\frac{v^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - f_s \sin\theta - mg = 0$$

Multiply the x equation by $\sin\theta$ and the y equation by $\cos\theta$:

$$f_s \sin\theta\cos\theta + F_n \sin^2\theta = m\frac{v^2}{r}\sin\theta$$

$$F_n \cos^2\theta - f_s \sin\theta\cos\theta - mg\cos\theta = 0$$

Add these equations to eliminate f_s:

$$F_n - mg\cos\theta = m\frac{v^2}{r}\sin\theta$$

Solving for F_n yields:

$$F_n = mg\cos\theta + m\frac{v^2}{r}\sin\theta = m\left(g\cos\theta + \frac{v^2}{r}\sin\theta\right)$$

Substitute numerical values and evaluate F_n:

$$F_n = (800\,\text{kg})\left[(9.81\,\text{m/s}^2)\cos 10^\circ + \frac{\left(38\,\frac{\text{km}}{\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{150\,\text{m}}\sin 10^\circ\right] = 7.83\,\text{kN}$$

$$= \boxed{7.8\,\text{kN}}$$

(b) Solve the y equation for f_s:

$$f_s = \frac{F_n\cos\theta - mg}{\sin\theta} = F_n\cot\theta - \frac{mg}{\sin\theta}$$

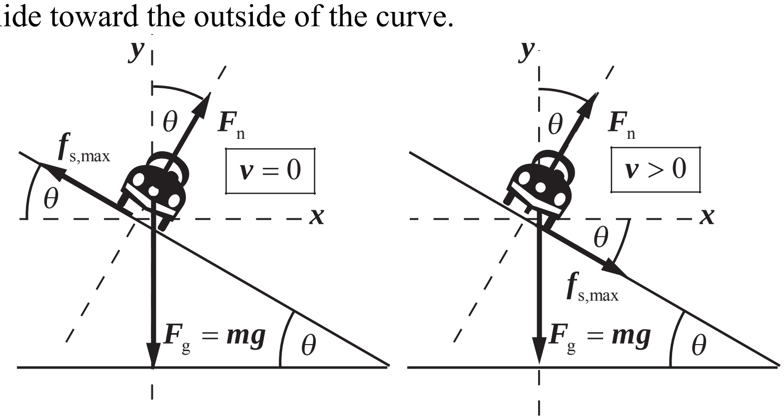
Substitute numerical values and evaluate f_s:

$$f_s = (7.83\ \text{kN})\cot 10° - \frac{(800\ \text{kg})(9.81\ \text{m/s}^2)}{\sin 10°} = \boxed{-0.78\ \text{kN}}$$

The negative sign tells us that f_s points upward along the inclined plane rather than as shown in the force diagram.

96 ••• As a civil engineer intern during one of your summers in college, you are asked to design a curved section of roadway that meets the following conditions: When ice is on the road, and the coefficient of static friction between the road and rubber is 0.080, a car at rest must not slide into the ditch and a car traveling less than 60 km/h must not skid to the outside of the curve. Neglect the effects of air drag and rolling friction. What is the minimum radius of curvature of the curve and at what angle should the road be banked?

Picture the Problem The free-body diagram to the left is for the car at rest. The static friction force up the incline balances the downward component of the car's weight and prevents it from sliding. In the free-body diagram to the right, the static friction force points in the opposite direction as the tendency of the moving car is to slide toward the outside of the curve.



Apply $\sum \vec{F} = m\vec{a}$ to the car that is at rest:

$$\sum F_y = F_n \cos\theta + f_s \sin\theta - mg = 0 \quad (1)$$

and

$$\sum F_x = F_n \sin\theta - f_s \cos\theta = 0 \quad (2)$$

Substitute $f_s = f_{s,max} = \mu_s F_n$ in equation (2) and solve for the maximum allowable value of θ:

$$\theta = \tan^{-1}(\mu_s)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}(0.080) = 4.57° = \boxed{4.6°}$$

Apply $\sum \vec{F} = m\vec{a}$ to the car that is moving with speed v:

$$\sum F_y = F_n \cos\theta - f_s \sin\theta - mg = 0 \quad (3)$$

$$\sum F_x = F_n \sin\theta + f_s \cos\theta = m\frac{v^2}{r} \quad (4)$$

Substitute $f_s = \mu_s F_n$ in equations (3) and (4) and simplify to obtain:

$$F_n\left(\cos\theta - \mu_s \sin\theta\right) = mg \quad (5)$$

and

$$F_n\left(\mu_s \cos\theta + \sin\theta\right) = m\frac{v^2}{r} \quad (6)$$

Substitute numerical values in equations (5) and (6) to obtain:

$$0.9904F_n = mg$$

and

$$0.1595F_n = m\frac{v^2}{r}$$

Eliminate F_n between these equations and solve for r:

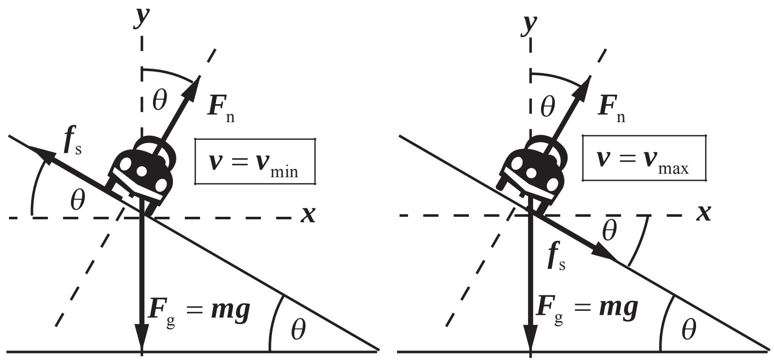
$$r = \frac{v^2}{0.1610g}$$

Substitute numerical values and evaluate r:

$$r = \frac{\left(60\,\frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{0.1610\left(9.81\,\text{m/s}^2\right)} = \boxed{0.18\,\text{km}}$$

97 ••• A curve of radius 30 m is banked so that a 950-kg car traveling at 40.0 km/h can round it even if the road is so icy that the coefficient of static friction is approximately zero. You are commissioned to tell the local police the range of speeds at which a car can travel around this curve without skidding. Neglect the effects of air drag and rolling friction. If the coefficient of static friction between the road and the tires is 0.300, what is the range of speeds you tell them?

Picture the Problem The free-body diagram to the left is for the car rounding the curve at the minimum (not sliding down the incline) speed. The combination of the horizontal components of the frictional force and the normal force provides the net force toward the center of the curve. In the free-body diagram to the right, the static friction force points in the opposite direction as the tendency of the car moving with the maximum safe speed is to slide toward the outside of the curve. Application of Newton's second law and the simultaneous solution of the force equations will yield v_{min} and v_{max}.



Apply $\sum \vec{F} = m\vec{a}$ to a car traveling around the curve when the coefficient of static friction is zero:

$$\sum F_x = F_n \sin\theta = m\frac{v_{min}^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - mg = 0$$

Divide the first of these equations by the second to obtain:

$$\tan\theta = \frac{v^2}{rg} \Rightarrow \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

Substitute numerical values and evaluate the banking angle:

$$\theta = \tan^{-1}\left[\frac{\left(40\frac{\text{km}}{\text{h}}\times\frac{10^3\,\text{m}}{\text{km}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{(30\,\text{m})(9.81\,\text{m/s}^2)}\right] = 22.76°$$

Apply $\sum \vec{F} = m\vec{a}$ to the car traveling around the curve at minimum speed:

$$\sum F_x = F_n \sin\theta - f_s \cos\theta = m\frac{v_{min}^2}{r}$$

and

$$\sum F_y = F_n \cos\theta + f_s \sin\theta - mg = 0$$

Substitute $f_s = f_{s,max} = \mu_s F_n$ in the force equations and simplify to obtain:

$$F_n(\sin\theta - \mu_s \cos\theta) = m\frac{v_{min}^2}{r}$$

and

$$F_n(\cos\theta + \mu_s \sin\theta) = mg$$

Evaluate these equations for θ = 22.876 and μ_s = 0.300:

$$0.1102F_n = m\frac{v_{min}^2}{r} \text{ and } 1.038F_n = mg$$

Eliminate F_n between these two equations and solve for v_{min}:

$$v_{min} = \sqrt{0.106rg}$$

Substitute numerical values and evaluate v_{min}:

$$v_{min} = \sqrt{0.106(30\,\text{m})(9.81\,\text{m/s}^2)} = 5.6\,\text{m/s} \approx 20\,\text{km/h}$$

Apply $\sum \vec{F} = m\vec{a}$ to the car traveling around the curve at maximum speed:

$$\sum F_x = F_n \sin\theta + f_s \cos\theta = m\frac{v_{max}^2}{r}$$

and

$$\sum F_y = F_n \cos\theta - f_s \sin\theta - mg = 0$$

Substitute $f_s = f_{s,\,max} = \mu_s F_n$ in the force equations and simplify to obtain:

$$F_n(\mu_s \cos\theta + \sin\theta) = m\frac{v_{max}^2}{r}$$

and

$$F_n(\cos\theta - \mu_s \sin\theta) = mg$$

Evaluate these equations for $\theta = 22.76°$ and $\mu_s = 0.300$:

$$0.6635F_n = m\frac{v_{max}^2}{r}$$

and

$$0.8061F_n = 0.8061mg$$

Eliminate F_n between these two equations and solve for v_{max}:

$$v_{max} = \sqrt{0.8231rg}$$

Substitute numerical values and evaluate v_{max}:

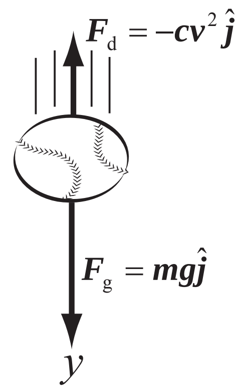
$$v_{max} = \sqrt{(0.8231)(30\,\text{m})(9.81\,\text{m/s}^2)} = 16\,\text{m/s} \approx 56\,\text{km/h}$$

You should tell them that the safe range of speeds is $\boxed{20\,\text{km/m} \le v \le 56\,\text{km/h}}$.

Euler's Method

98 •• You are riding in a hot air balloon when you throw a baseball straight down with an initial speed of 35.0 km/h. The baseball falls with a terminal speed of 150 km/h. Assuming air drag is proportional to the speed squared, use Euler's method (**spreadsheet**) to estimate the speed of the ball after 10.0 s. What is the uncertainty in this estimate? You drop a second baseball, this one released from rest. How long does it take for it to reach 99 percent of its terminal speed? How far does it fall during this time?

Picture the Problem The free-body diagram shows the forces acting on the baseball sometime after it has been thrown downward but before it has reached its terminal speed. In order to use Euler's method, we'll need to determine how the acceleration of the ball varies with its speed. We can do this by applying Newton's second law to the ball and using its terminal speed to express the constant in the acceleration equation in terms of the ball's terminal speed. We can then use $v_{n+1} = v_n + a_n\Delta t$ to find the speed of the ball at any given time.



Apply Newton's second law to the ball to obtain:

$$mg - bv^2 = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{b}{m}v^2 \quad (1)$$

When the ball reaches its terminal speed, its acceleration is zero:

$$0 = g - \frac{b}{m}v_{\rm t}^2 \Rightarrow \frac{b}{m} = \frac{g}{v_{\rm t}^2}$$

Substitute in equation (1) to obtain:

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_{\rm t}^2}\right)$$

Express the position of the ball to obtain:

$$x_{n+1} = x_n + \frac{v_{n+1} + v_n}{2}\Delta t$$

Letting a_n be the acceleration of the ball at time t_n, express its speed when $t = t_{n+1}$:

$$v_{n+1} = v_n + a_n\Delta t$$

where $a_n = g\left(1 - \dfrac{v_n^2}{v_{\rm t}^2}\right)$ and Δt is an arbitrarily small interval of time.

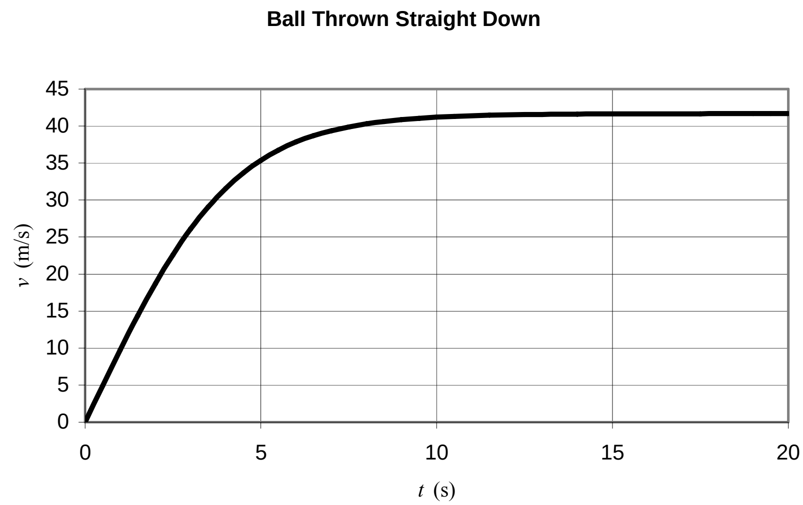
A spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A10	B9+$B$1	$t + \Delta t$
B10	B9+0.5*(C9+C10)*$B$1	$x_{n+1} = x_n + \frac{v_{n+1} + v_n}{2}\Delta t$
C10	C9+D9*$B$1	$v_{n+1} = v_n + a_n\Delta t$
D10	$B$4*(1−C10^2/$B$5^2)	$a_n = g\left(1 - \frac{v_n^2}{v_{\rm t}^2}\right)$

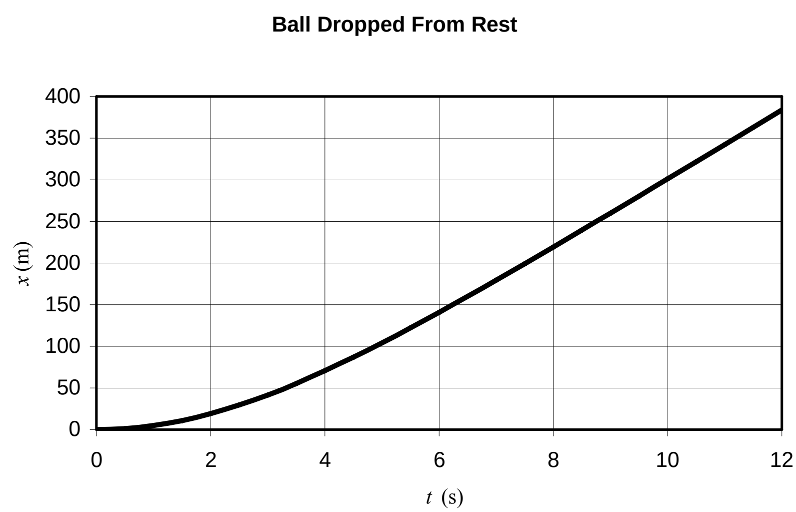
	A	B	C	D
1	Δt=	0.5	s	
2	x_0=	0	m	
3	v_0=	9.722	m/s	
4	a_0=	9.81	m/s^2	
5	v_t=	41.67	m/s	
6				
7	t	x	v	a
8	(s)	(m)	(m/s)	(m/s^2)
9	0.0	0	9.7	9.28
10	0.5	6	14.4	8.64
11	1.0	14	18.7	7.84
12	1.5	25	22.6	6.92
28	9.5	317	41.3	0.17
29	10.0	337	41.4	0.13
30	10.5	358	41.5	0.10
38	14.5	524	41.6	0.01
39	15.0	545	41.7	0.01
40	15.5	566	41.7	0.01
41	16.0	587	41.7	0.01
42	16.5	608	41.7	0.00

From the table we can see that the speed of the ball after 10 s is approximately $\boxed{41.4\,\text{m/s.}}$ We can estimate the uncertainty in this result by halving Δt and recalculating the speed of the ball at t = 10 s. Doing so yields v(10 s) ≈ 41.3 m/s, a difference of about $\boxed{0.02\%.}$

The following graph shows the velocity of the ball thrown straight down as a function of time.

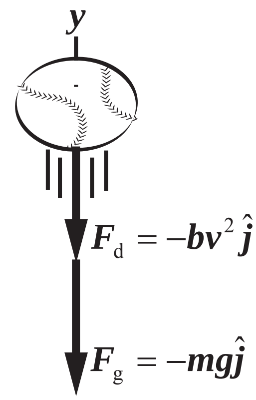


Reset Δt to 0.5 s and set $v_0 = 0$. Ninety-nine percent of 41.67 m/s is approximately 41.3 m/s. Note that the ball will reach this speed in about $\boxed{10.5\,\text{s}}$ and that the distance it travels in this time is about $\boxed{322\,\text{m.}}$ The following graph shows the distance traveled by the ball dropped from rest as a function of time.



99 •• [SSM] You throw a baseball straight up with an initial speed of 150 km/h. The ball's terminal speed when falling is also 150 km/h. (*a*) Use Euler's method (**spreadsheet**) to estimate its height 3.50 s after release. (*b*) What is the maximum height it reaches? (*c*) How long after release does it reach its maximum height? (*d*) How much later does it return to the ground? (*e*) Is the time the ball spends on the way up less than, the same as, or greater than the time it spends on the way down?

Picture the Problem The free-body diagram shows the forces acting on the baseball after it has left your hand. In order to use Euler's method, we'll need to determine how the acceleration of the ball varies with its speed. We can do this by applying Newton's second law to the baseball. We can then use $v_{n+1} = v_n + a_n\Delta t$ and $x_{n+1} = x_n + v_n\Delta t$ to find the speed and position of the ball.



Apply $\sum F_y = ma_y$ to the baseball:

$$-bv|v| - mg = m\frac{dv}{dt}$$

where $|v| = v$ for the upward part of the flight of the ball and $|v| = -v$ for the downward part of the flight.

Solve for dv/dt to obtain:

$$\frac{dv}{dt} = -g - \frac{b}{m}v|v|$$

Under terminal speed conditions ($|v| = -v_\mathrm{t}$):

$$0 = -g + \frac{b}{m}v_\mathrm{t}^2 \text{ and } \frac{b}{m} = \frac{g}{v_\mathrm{t}^2}$$

Substituting for b/m yields:

$$\frac{dv}{dt} = -g - \frac{g}{v_\mathrm{t}^2}v|v| = -g\left(1 + \frac{v|v|}{v_\mathrm{t}^2}\right)$$

Letting a_n be the acceleration of the ball at time t_n, express its position and speed when $t = t_n + 1$:

$$y_{n+1} = y_n + \tfrac{1}{2}\left(v_n + v_{n-1}\right)\Delta t$$

and

$$v_{n+1} = v_n + a_n\Delta t$$

where $a_n = -g\left(1 + \dfrac{v_n|v_n|}{v_\mathrm{t}^2}\right)$ and Δt is an arbitrarily small interval of time.

A spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
D11	D10+$B$6	$t + \Delta t$
E10	41.7	v_0
E11	E10−$B$4*	$v_{n+1} = v_n + a_n\Delta t$

	(1+E10*ABS(E10)/($B$5^2))*$B$6	
F10	0	y_0
F11	F10+0.5*(E10+E11)*$B$6	$y_{n+1} = y_n + \frac{1}{2}(v_n + v_{n-1})\Delta t$
G10	0	y_0
G11	$E$10*D11–0.5*$B$4*D11^2	$v_0 t - \frac{1}{2} g t^2$

	A	B	C	D	E	F	G
4	$g=$	9.81	m/s^2				
5	$v_t=$	41.7	m/s				
6	$\Delta t=$	0.1	s				
7							
8							
9				t	v	y	$y_{\text{no drag}}$
10				0.00	41.70	0.00	0.00
11				0.10	39.74	4.07	4.12
12				0.20	37.87	7.95	8.14
40				3.00	3.01	60.13	81.00
41				3.10	2.03	60.39	82.18
42				3.20	1.05	60.54	83.26
43				3.30	0.07	60.60	84.25
44				3.40	–0.91	60.55	85.14
45				3.50	–1.89	60.41	85.93
46				3.60	–2.87	60.17	86.62
78				6.80	–28.34	6.26	56.98
79				6.90	–28.86	3.41	54.44
80				7.00	–29.37	0.49	51.80
81				7.10	–29.87	–2.47	49.06

(*a*) When $t = 3.50$ s, the height of the ball is about $\boxed{60.4\,\text{m}}$.

(*b*) The maximum height reached by the ball is $\boxed{60.6\,\text{m}}$.

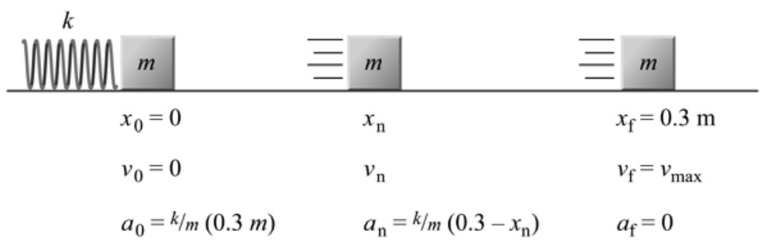
(*c*) The time the ball takes to reach its maximum height is about $\boxed{3.0\,\text{s}}$.

(*d*) The ball hits the ground at about $t = \boxed{7.0\,\text{s}}$

(*e*) Because the time the ball takes to reach its maximum height is less than half its time of flight, the time the ball spends on the way up less than the time it spends on the way down

100 ••• A 0.80-kg block on a horizontal frictionless surface is held against a massless spring, compressing it 30 cm. The force constant of the spring is 50 N/m. The block is released and the spring pushes it 30 cm. Use Euler's method (**spreadsheet**) with $\Delta t = 0.005$ s to estimate the time it takes for the spring to push the block the 30 cm. How fast is the block moving at this time? What is the uncertainty in this speed?

Picture the Problem The pictorial representation shows the block in its initial position against the compressed spring, later as the spring accelerates it to the right, and finally when it has reached its maximum speed at $x_f = 0.30$ m. In order to use Euler's method, we'll need to determine how the acceleration of the block varies with its position. We can do this by applying Newton's second law to the block. We can then use $v_{n+1} = v_n + a_n\Delta t$ and $x_{n+1} = x_n + v_n\Delta t$ to find the speed and position of the block.



Apply $\sum F_x = ma_x$ to the block:

$$k(0.30\,\text{m} - x_n) = ma_n$$

Solve for a_n:

$$a_n = \frac{k}{m}(0.30\,\text{m} - x_n)$$

Express the position and speed of the block when $t = t_n + 1$:

$$x_{n+1} = x_n + v_n\Delta t$$

and

$$v_{n+1} = v_n + a_n\Delta t$$

where $a_n = \dfrac{k}{m}(0.30\,\text{m} - x_n)$ and Δt is an arbitrarily small interval of time.

A spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A10	A9+$B$1	$t + \Delta t$
B10	B9+C10*$B$1	$x_n + v_n\Delta t$
C10	C9+D9*$B$1	$v_n + a_n\Delta t$
D10	($B$4/$B$5)*(0.30–B10)	$\frac{k}{m}(0.30 - x_n)$

	A	B	C	D
1	Δt=	0.005	s	
2	x_0=	0	m	
3	v_0=	0	m/s	
4	k =	50	N/m	
5	m =	0.80	kg	
6				
7	t	x	v	a
8	(s)	(m)	(m/s)	(m/s^2)
9	0.000	0.00	0.00	18.75
10	0.005	0.00	0.09	18.72
11	0.010	0.00	0.19	18.69
12	0.015	0.00	0.28	18.63
45	0.180	0.25	2.41	2.85
46	0.185	0.27	2.42	2.10
47	0.190	0.28	2.43	1.34
48	0.195	0.29	2.44	0.58
49	0.200	0.30	2.44	−0.19

From the table we can see that it took about $\boxed{0.200\,\text{s}}$ for the spring to push the block 30 cm and that it was traveling about $\boxed{2.44\,\text{m/s}}$ at that time. We can estimate the uncertainty in this result by halving Δt and recalculating the speed of the ball at t = 10 s. Doing so yields v(0.20 s) ≈ 2.41 m/s, a difference of about $\boxed{1.2\%.}$

Finding the Center of Mass

101 • Three point masses of 2.0 kg each are located on the x axis. One is at the origin, another at x = 0.20 m, and another at x = 0.50 m. Find the center of mass of the system.

Picture the Problem We can use its definition to find the center of mass of this system.

The x coordinate of the center of mass is given by:

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{(2.0\,\text{kg})(0) + (2.0\,\text{kg})(0.20\,\text{m}) + (2.0\,\text{kg})(0.50\,\text{m})}{2.0\,\text{kg} + 2.0\,\text{kg} + 2.0\,\text{kg}} = 0.23\,\text{m}$$

Because the point masses all lie along the x axis:

$y_{cm} = 0$ and the center of mass of this system of particles is at $\boxed{(0.23\,\text{m}, 0)}$.

102 • On a weekend archeological dig, you discover an old club-ax that consists of a symmetrical 8.0-kg stone attached to the end of a uniform 2.5-kg stick. You measure the dimensions of the club-ax as shown in Figure 5-77. How far is the center of mass of the club-ax from the handle end of the club-ax?

Picture the Problem Let the left end of the handle be the origin of our coordinate system. We can disassemble the club-ax, find the center of mass of each piece, and then use these coordinates and the masses of the handle and stone to find the center of mass of the club-ax.

Express the center of mass of the handle plus stone system:

$$x_{cm} = \frac{m_{stick}x_{cm,stick} + m_{stone}x_{cm,stone}}{m_{stick} + m_{stone}}$$

Assume that the stone is drilled and the stick passes through it. Use symmetry considerations to locate the center of mass of the stick:

$$x_{cm,stick} = 49\,\text{cm}$$

Use symmetry considerations to locate the center of mass of the stone:

$$x_{cm,stone} = 89\,\text{cm}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{cm} = \frac{(2.5\,\text{kg})(49\,\text{cm}) + (8.0\,\text{kg})(89\,\text{cm})}{2.5\,\text{kg} + 8.0\,\text{kg}} = \boxed{79\,\text{cm}}$$

103 • Three balls A, B, and C, with masses of 3.0 kg, 1.0 kg, and 1.0 kg, respectively, are connected by massless rods, as shown in Figure 5-78. What are the coordinates of the center of mass of this system?

Picture the Problem We can treat each of balls as though they are point objects and apply the definition of the center of mass to find (x_{cm}, y_{cm}).

The x coordinate of the center of mass is given by:

$$x_{cm} = \frac{m_A x_A + m_B x_B + m_C x_C}{m_A + m_B + m_C}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{(3.0\,\text{kg})(2.0\,\text{m}) + (1.0\,\text{kg})(1.0\,\text{m}) + (1.0\,\text{kg})(3.0\,\text{m})}{3.0\,\text{kg} + 1.0\,\text{kg} + 1.0\,\text{kg}} = 2.0\,\text{m}$$

The y coordinate of the center of mass is given by:

$$y_{\text{cm}} = \frac{m_{\text{A}} y_{\text{A}} + m_{\text{B}} y_{\text{B}} + m_{\text{C}} y_{\text{C}}}{m_{\text{A}} + m_{\text{B}} + m_{\text{C}}}$$

Substitute numerical values and evaluate y_{cm}:

$$y_{\text{cm}} = \frac{(3.0\,\text{kg})(2.0\,\text{m}) + (1.0\,\text{kg})(1.0\,\text{m}) + (1.0\,\text{kg})(0)}{3.0\,\text{kg} + 1.0\,\text{kg} + 1.0\,\text{kg}} = 1.4\,\text{m}$$

The center of mass of this system of balls is at $\boxed{(2.0\,\text{m}, 1.4\,\text{m})}$.

Alternate Solution Using Vectors

Picture the Problem We can use the vector expression for the center of mass to find (x_{cm}, y_{cm}).

The vector expression for the center of mass is:

$$M\vec{\boldsymbol{r}}_{\text{cm}} = \sum_i m_i \vec{\boldsymbol{r}}_i \text{ or } \vec{\boldsymbol{r}}_{\text{cm}} = \frac{\sum_i m_i \vec{\boldsymbol{r}}_i}{M} \qquad (1)$$

where

$$\vec{\boldsymbol{r}}_{\text{cm}} = x_{\text{cm}}\hat{\boldsymbol{i}} + y_{\text{cm}}\hat{\boldsymbol{j}} + z_{\text{cm}}\hat{\boldsymbol{k}}$$

The position vectors for the objects located at A, B, and C are:

$$\vec{\boldsymbol{r}}_{\text{A}} = \left(2.0\hat{\boldsymbol{i}} + 2.0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\right)\text{m},$$

$$\vec{\boldsymbol{r}}_{\text{B}} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\,,$$

and

$$\vec{\boldsymbol{r}}_{\text{C}} = \left(3.0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\right)\text{m}$$

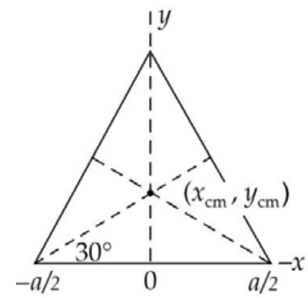
Substitute numerical values in (1) and simplify to obtain:

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{1}{(3.0\,\text{kg} + 1.0\,\text{kg} + 1.0\,\text{kg})}\Big[(3.0\,\text{kg})\left(2.0\hat{\boldsymbol{i}} + 2.0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\right)\text{m} + (1.0\,\text{kg})\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\right)\text{m} \\ &\qquad + (1.0\,\text{kg})\left(3.0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\right)\text{m}\Big] \\ &= (2.0\,\text{m})\hat{\boldsymbol{i}} + (1.4\,\text{m})\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}\end{aligned}$$

The center of mass of this system of balls is at $\boxed{(2.0\,\text{m}, 1.4\,\text{m}, 0)}$.

104 • By symmetry, locate the center of mass of an equilateral triangle with edges of length a. The triangle has one vertex on the y axis and the others at $(-a/2, 0)$ and $(+a/2, 0)$.

Picture the Problem The figure shows an equilateral triangle with its y-axis vertex above the x axis. The bisectors of the vertex angles are also shown. We can find x coordinate of the center-of-mass by inspection and the y coordinate using trigonometry.



From symmetry considerations: $x_{cm} = 0$

Express the trigonometric relationship between $a/2$, 30°, and y_{cm}: $\tan 30° = \dfrac{y_{cm}}{\frac{1}{2}a}$

Solve for y_{cm} and simplify to obtain: $y_{cm} = \frac{1}{2}a\tan 30° = 0.29a$

The center of mass of an equilateral triangle oriented as shown above is at $\boxed{(0, 0.29a)}$.

105 •• **[SSM]** Find the center of mass of the uniform sheet of plywood in Figure 5-79. Consider this as a system of effectively two sheets, letting one have a ″negative mass″ to account for the cutout. Thus, one is a square sheet of 3.0-m edge length and mass m_1 and the second is a rectangular sheet measuring 1.0 m × 2.0 m with a mass of $-m_2$. Let the coordinate origin be at the lower left corner of the sheet.

Picture the Problem Let the subscript 1 refer to the 3.0-m by 3.0-m sheet of plywood before the 2.0-m by 1.0-m piece has been cut from it. Let the subscript 2 refer to 2.0-m by 1.0-m piece that has been removed and let σ be the area density of the sheet. We can find the center-of-mass of these two regions; treating the missing region as though it had negative mass, and then finding the center-of-mass of the U-shaped region by applying its definition.

Express the coordinates of the center of mass of the sheet of plywood:

$$x_{\text{cm}} = \frac{m_1 x_{\text{cm},1} - m_2 x_{\text{cm},2}}{m_1 - m_2}$$

and

$$y_{\text{cm}} = \frac{m_1 y_{\text{cm},1} - m_2 y_{\text{cm},2}}{m_1 - m_2}$$

Use symmetry to find $x_{\text{cm},1}$, $y_{\text{cm},1}$, $x_{\text{cm},2}$, and $y_{\text{cm},2}$:

$$x_{\text{cm},1} = 1.5\,\text{m},\ y_{\text{cm},1} = 1.5\,\text{m}$$

and

$$x_{\text{cm},2} = 1.5\,\text{m},\ y_{\text{cm},2} = 2.0\,\text{m}$$

Determine m_1 and m_2:

$$m_1 = \sigma A_1 = 9\sigma\,\text{kg}$$

and

$$m_2 = \sigma A_2 = 2\sigma\,kg$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{(9\sigma\,\text{kg})(1.5\,\text{m}) - (2\sigma\,\text{kg})(1.5\,\text{kg})}{9\sigma\,\text{kg} - 2\sigma\,\text{kg}}$$
$$= 1.5\,\text{m}$$

Substitute numerical values and evaluate y_{cm}:

$$\boldsymbol{y}_{\text{cm}} = \frac{(9\boldsymbol{\sigma}\,\text{kg})(1.5\,\text{m}) - (2\boldsymbol{\sigma}\,\text{kg})(2.0\,\text{m})}{9\boldsymbol{\sigma}\,\text{kg} - 2\boldsymbol{\sigma}\,\text{kg}}$$
$$= 1.4\,\text{m}$$

The center of mass of the U-shaped sheet of plywood is at $\boxed{(1.5\,\text{m}, 1.4\,\text{m})}$.

106 •• A can in the shape of a symmetrical cylinder with mass *M* and height *H* is filled with water. The initial mass of the water is *M*, the same mass as the can. A small hole is punched in the bottom of the can, and the water drains out. (*a*) If the height of the water in the can is *x*, what is the height of the center of mass of the can plus the water remaining in the can? (*b*) What is the minimum height of the center of mass as the water drains out?

Picture the Problem We can use its definition to find the center of mass of the can plus water. By setting the derivative of this function equal to zero, we can find the value of *x* that corresponds to the minimum height of the center of mass of the water as it drains out and then use this extreme value to express the minimum height of the center of mass.

(*a*) Using its definition, express the location of the center of mass of the can + water:

$$x_{\text{cm}} = \frac{M\left(\frac{H}{2}\right) + m\left(\frac{x}{2}\right)}{M + m} \qquad (1)$$

Let the cross-sectional area of the can be A and use the definition of density to relate the mass m of water remaining in the can at any given time to its depth x:

$$\rho = \frac{M}{AH} = \frac{m}{Ax} \Rightarrow m = \frac{x}{H}M$$

Substitute for m in equation (1) and simplify to obtain:

$$x_{\text{cm}} = \frac{M\left(\frac{H}{2}\right) + \left(\frac{x}{H}M\right)\left(\frac{x}{2}\right)}{M + \frac{x}{H}M} = \boxed{\frac{H}{2}\left(\frac{1+\left(\frac{x}{H}\right)^2}{1+\frac{x}{H}}\right)}$$

(b) Differentiate x_{cm} with respect to x and set the derivative equal to zero for extrema:

$$\frac{dx_{\text{cm}}}{dx} = \frac{H}{2}\frac{d}{dx}\left(\frac{1+\left(\frac{x}{H}\right)^2}{1+\frac{x}{H}}\right) = \frac{H}{2}\left\{\frac{\left(1+\frac{x}{H}\right)2\left(\frac{x}{H}\right)\left(\frac{1}{H}\right)}{\left(1+\frac{x}{H}\right)^2} - \frac{\left[1+\left(\frac{x}{H}\right)^2\right]\left(\frac{1}{H}\right)}{\left(1+\frac{x}{H}\right)^2}\right\} = 0$$

Simplify this expression to obtain a quadratic equation in x/H:

$$\left(\frac{x}{H}\right)^2 + 2\left(\frac{x}{H}\right) - 1 = 0$$

Solving for x/H yields:

$$x = H\left(\sqrt{2}-1\right) \approx 0.414H$$

where we've kept the positive solution because a negative value for x/H would make no sense.

Use your graphing calculator to convince yourself that the graph of x_{cm} as a function of x is concave upward at $x \approx 0.414H$ and that, therefore, the minimum value of x_{cm} occurs at $x \approx 0.414H$.

Evaluate x_{cm} at $x = H\left(\sqrt{2}-1\right)$ to obtain:

$$x_{\text{cm}}\Big|_{x=H\left(\sqrt{2}-1\right)} = \frac{H}{2}\left(\frac{1+\left(\frac{H\left(\sqrt{2}-1\right)}{H}\right)^2}{1+\frac{H\left(\sqrt{2}-1\right)}{H}}\right)$$

$$= \boxed{H\left(\sqrt{2}-1\right)}$$

107 •• **[SSM]** Two identical thin uniform rods each of length L are glued together so that the angle at the joint is 90°. Determine the location of the center of mass (in terms of L) of this configuration relative to the origin taken to be at the joint. (*Hint: You do not need the mass of the rods (why?), but you should start by assuming a mass m and see that it cancels out.*)

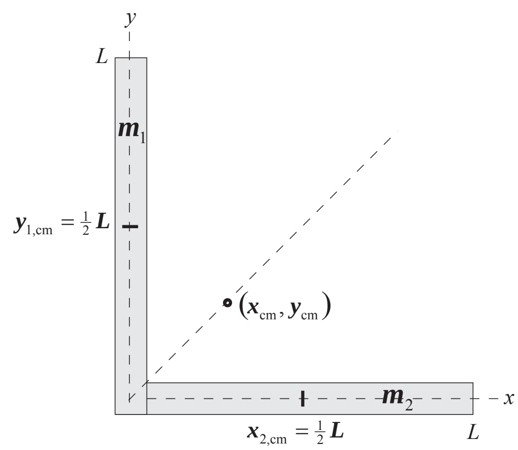
Picture the Problem A pictorial representation of the system is shown to the right. The x and y coordinates of the two rods are

$$(x_{1,\text{cm}}, y_{1,\text{cm}}) = (0, \tfrac{1}{2}L)$$

and

$$(x_{2,\text{cm}}, y_{2,\text{cm}}) = (\tfrac{1}{2}L, 0).$$

We can use the definition of the center of mass to find the coordinates $(x_{\text{cm}}, y_{\text{cm}})$.



The x coordinate of the center of mass is given by:

$$x_{\text{cm}} = \frac{x_{1,\text{cm}} m_1 + x_{2,\text{cm}} m_2}{m_1 + m_2}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{(0)m_1 + (\tfrac{1}{2}L)m_2}{m_1 + m_2}$$

or, because $m_1 = m_2 = m$,

$$x_{\text{cm}} = \frac{(0)m + (\tfrac{1}{2}L)m}{m + m} = \tfrac{1}{4}L$$

The y coordinate of the center of mass is given by:

$$y_{\text{cm}} = \frac{y_{1,\text{cm}} m_1 + y_{2,\text{cm}} m_2}{m_1 + m_2}$$

Substitute numerical values and evaluate y_{cm}:

$$y_{\text{cm}} = \frac{(\tfrac{1}{2}L)m_1 + (0)m_2}{m_1 + m_2}$$

or, because $m_1 = m_2 = m$,

$$y_{\text{cm}} = \frac{(\tfrac{1}{2}L)m + (0)m}{m + m} = \tfrac{1}{4}L$$

The center of mass of this system is located at $\boxed{(\tfrac{1}{4}L, \tfrac{1}{4}L)}$.

Remarks: Note that the center of mass is located at a distance $d = \tfrac{1}{2}L\cos 45°$ from the vertex on the axis of symmetry bisecting the two arms.

108 ••• Repeat the analysis of Problem 107 with a general angle θ at the joint instead of 90°. Does your answer agree with the specific 90°- angle answer in Problem 107 if you set θ equal to 90°? Does your answer give plausible results for angles of zero and 180°?

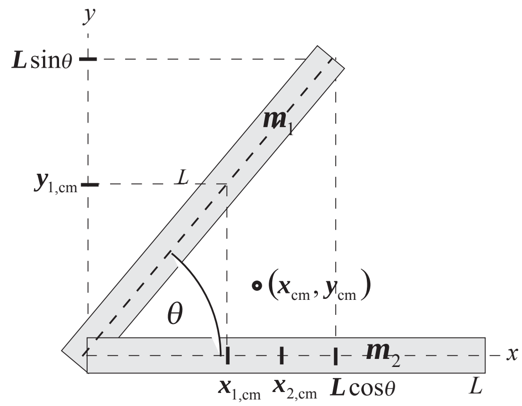
Picture the Problem The pictorial representation of the system is shown to the right. The x and y coordinates of the two rods are

$$(x_{1,\text{cm}}, y_{1,\text{cm}}) = \left(\tfrac{1}{2}L\cos\theta, \tfrac{1}{2}L\sin\theta\right)$$

and

$$(x_{2,\text{cm}}, y_{2,\text{cm}}) = \left(\tfrac{1}{2}L, 0\right).$$

We can use the definition of the center of mass to find the coordinates $(x_{\text{cm}}, y_{\text{cm}})$.



The x coordinate of the center of mass is given by:

$$x_{\text{cm}} = \frac{x_{1,\text{cm}}m_1 + x_{2,\text{cm}}m_2}{m_1 + m_2}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{\left(\tfrac{1}{2}L\cos\theta\right)m_1 + \left(\tfrac{1}{2}L\right)m_2}{m_1 + m_2}$$

or, because $m_1 = m_2 = m$,

$$x_{\text{cm}} = \frac{\left(\tfrac{1}{2}L\cos\theta\right)m + \left(\tfrac{1}{2}L\right)m}{m + m} = \tfrac{1}{4}L(1 + \cos\theta)$$

The y coordinate of the center of mass is given by:

$$y_{\text{cm}} = \frac{y_{1,\text{cm}}m_1 + y_{2,\text{cm}}m_2}{m_1 + m_2}$$

Substitute numerical values and evaluate y_{cm}:

$$y_{\text{cm}} = \frac{\left(\tfrac{1}{2}L\sin\theta\right)m_1 + (0)m_2}{m_1 + m_2}$$

or, because $m_1 = m_2 = m$,

$$y_{\text{cm}} = \frac{\left(\tfrac{1}{2}L\sin\theta\right)m + (0)m}{m + m} = \tfrac{1}{4}L\sin\theta$$

The center of mass of this system of rods is located at $\boxed{\left(\tfrac{1}{4}L(1 + \cos\theta), \tfrac{1}{4}L\sin\theta\right)}$.

For $\theta = 0°$:

$$\left(\tfrac{1}{4}L(1 + \cos\theta), \tfrac{1}{4}L\sin\theta\right) = \left(\tfrac{1}{4}L(1 + \cos 0°), \tfrac{1}{4}L\sin 0°\right) = \boxed{\left(\tfrac{1}{2}L, 0\right)} \text{ as expected.}$$

For $\theta = 90°$:

$$\left(\tfrac{1}{4}L(1+\cos\theta), \tfrac{1}{4}L\sin\theta\right) = \left(\tfrac{1}{4}L(1+\cos 90°), \tfrac{1}{4}L\sin 90°\right)$$
$$= \boxed{\left(\tfrac{1}{4}L, \tfrac{1}{4}L\right)} \text{ in agreement with Problem 107.}$$

For $\theta = 180°$:

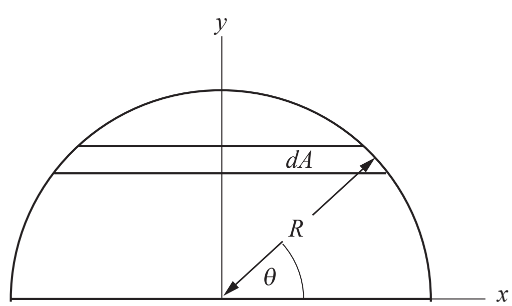
$$\left(\tfrac{1}{4}L(1+\cos\theta), \tfrac{1}{4}L\sin\theta\right) = \left(\tfrac{1}{4}L(1+\cos 180°), \tfrac{1}{4}L\sin 180°\right) = \boxed{(0,0)} \text{ as expected.}$$

Remarks: Note that the center of mass is located at a distance $d = \frac{1}{4}\sqrt{2}L\sqrt{1+\cos\theta}$ from the vertex on an axis that makes an angle $\phi = \tan^{-1}\left[\dfrac{\sin\theta}{1+\cos\theta}\right]$ with the *x* axis.

*Finding the Center of Mass by Integration

109 •• Show that the center of mass of a uniform semicircular disk of radius R is at a point $4R/(3\pi)$ from the center of the circle.

Picture the Problem A semicircular disk and a surface element of area dA is shown in the diagram. Because the disk is a continuous object, we'll use $M\vec{r}_{\text{cm}} = \int \vec{r}\,dm$ and symmetry to find its center of mass.



Express the coordinates of the center of mass of the semicircular disk:

$x_{\text{cm}} = 0$ by symmetry.

$$y_{\text{cm}} = \frac{\int y\sigma\, dA}{M} \qquad (1)$$

Express y as a function of r and θ:

$$y = r\sin\theta$$

Express dA in terms of r and θ:

$$dA = 2r\cos\theta\, dy$$
$$= (2R\cos\theta)(R\cos\theta)d\theta$$
$$= 2R^2\cos^2\theta d\theta$$

Express M as a function of r and θ:

$$M = \sigma A_{\text{half disk}} = \tfrac{1}{2}\sigma\pi R^2 \Rightarrow \sigma = \frac{2M}{\pi R^2}$$

Substitute for y, σ and dA in equation (1) to obtain:

$$y_{\text{cm}} = \frac{2}{\pi R^2}\int_0^{\frac{\pi}{2}} (R\sin\theta)\left(2R^2\cos^2\theta\right)d\theta$$

Simplifying yields:

$$y_{\text{cm}} = \frac{4R}{\pi}\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta$$

Change variables by letting $u = \cos\theta$. Then:

$$du = -\sin\theta\, d\theta$$

and

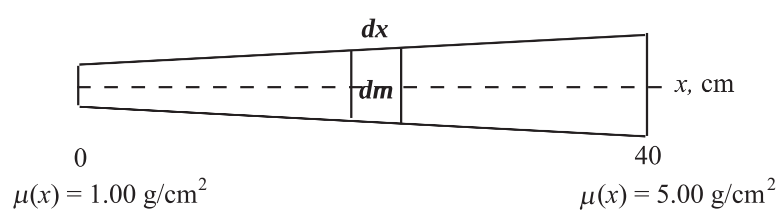
$$y_{\text{cm}} = -\frac{4R}{\pi}\int_{\ell_1}^{\ell_2} u^2\, du = -\frac{4R}{\pi}\left[\frac{u^3}{3}\right]_{\ell_1}^{\ell_2}$$

Substituting for u gives:

$$y_{\text{cm}} = -\frac{4R}{\pi}\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} = -\frac{4R}{\pi}\left[-\frac{1}{3}\right] = \boxed{\frac{4R}{3\pi}}$$

110 •• Find the location of the center of mass of a nonuniform rod 0.40 m in length if its density varies linearly from 1.00 g/cm at one end to 5.00 g/cm at the other end. Specify the center-of-mass location relative to the less-massive end of the rod.

Picture the Problem The pictorial representation summarizes the information we're given about the non-uniform rod. We can use the definition of the center of mass for a continuous object to find the center of mass of the non-uniform rod.



The x coordinate of the center of mass of the non-uniform rod is given by:

$$x_{cm} = \frac{\int x dm}{\int dm}$$

or, because $dm = \mu(x)dx$,

$$x_{cm} = \frac{\int x\mu(x)dx}{\int \mu(x)dx} \qquad (1)$$

By symmetry:

$$y_{cm} = 0$$

Use the given information regarding the linear variation in the density of the non-uniform rod to express $\mu(x)$:

$$\mu(x) = 1.00\ \text{g/cm} + (0.10\ \text{g/cm}^2)x$$

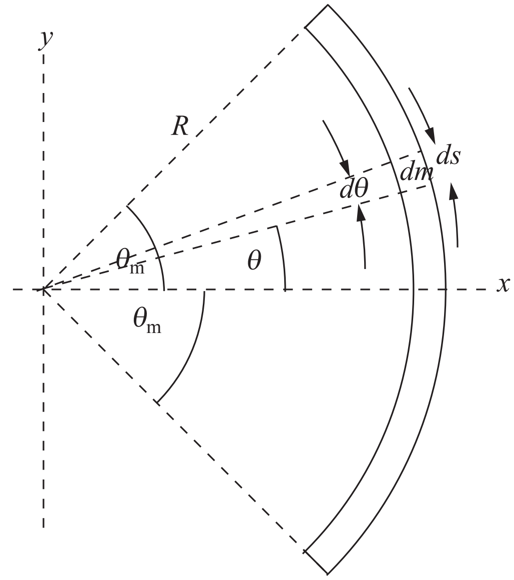
Substituting for $\mu(x)$ in equation (1) and evaluating the integrals yields:

$$x_{cm} = \frac{\int_0^{40\ \text{cm}} x\left[1.00\ \text{g/cm} + (0.10\ \text{g/cm}^2)x\right]xdx}{\int_0^{40\ \text{cm}} \left[1.00\ \text{g/cm} + (0.10\ \text{g/cm}^2)x\right]dx} = 24\ \text{cm}$$

The coordinates of the center of mass of the non-uniform rod are $\boxed{(24\ \text{cm}, 0)}$.

111 ••• You have a thin uniform wire bent into part of a circle that is described by a radius R and angle θ_m (see Figure 5-80). Show that the location of its center of mass is on the x axis and located a distance $x_{cm} = (R \sin \theta_m)/\theta_m$, where θ_m is expressed in radians. Double check by showing that this answer gives the physically expected limit for $\theta_m = 180°$. Verify that your answer gives you the result in the text (in the subsection Finding the Center of Mass by Integration) for the special case of $\theta_m = 90°$.

Picture the Problem We can use the definition of the center of mass for a continuous object to find the center of mass of the non-uniform rod.



The x coordinate of the center of mass of the thin uniform wire is given by:

$$x_{cm} = \frac{\int x dm}{\int dm}$$

or, because $dm = \lambda ds = \lambda R d\theta$,

$$x_{cm} = \frac{\int_{-\theta_m}^{\theta_m} x\lambda R d\theta}{\int_{-\theta_m}^{\theta_m} \lambda R d\theta} = \frac{\int_{-\theta_m}^{\theta_m} x d\theta}{\int_{-\theta_m}^{\theta_m} d\theta} \qquad (1)$$

By symmetry:

$$y_{cm} = 0$$

Because $x = R\cos\theta$, equation (1) becomes:

$$x_{cm} = \frac{\int_{-\theta_m}^{\theta_m} R\cos\theta \, d\theta}{\int_{-\theta_m}^{\theta_m} d\theta} = \frac{R\int_{-\theta_m}^{\theta_m} \cos\theta \, d\theta}{\int_{-\theta_m}^{\theta_m} d\theta}$$

Evaluate these integrals to obtain:

$$x_{cm} = \frac{R\sin\theta_m}{\theta_m}$$

The coordinates of the center of mass of the thin uniform wire are

$$\boxed{\left(\frac{R\sin\theta_m}{\theta_m}, 0\right)}.$$

For $\boldsymbol{\theta}_m = 180° = \boldsymbol{\pi}$ radians:

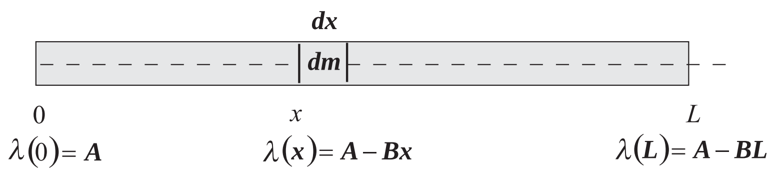
$$x_{cm} = \frac{R\sin\pi}{\pi} = \boxed{0} \text{ as expected.}$$

For $\theta_m = 90° = \frac{\pi}{2}$ radians:

$$x_{cm} = \frac{R\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \boxed{\frac{2R}{\pi}} \text{ as expected.}$$

112 ••• A long, thin wire of length L has a linear mass density given by $A - Bx$, where A and B are positive constants and x is the distance from the more massive end. (*a*) A condition for this problem to be realistic is that $A > BL$. Explain why. (*b*) Determine x_{cm} in terms of L, A, and B. Does your answer makes sense if $B = 0$? Explain.

Picture the Problem The pictorial representation summarizes the information we're given about the long thin wire. We can use the definition of the center of mass for a continuous object to find the center of mass of the non-uniform rod.



(*a*) At the end, the density has to be positive, so $A - BL > 0$ or $A > BL$.

(*b*) The x coordinate of the center of mass of the non-uniform rod is given by:

$$x_{cm} = \frac{\int x\,dm}{\int dm} \qquad (1)$$

By symmetry:

$$y_{cm} = 0$$

The density of the long thin wire decreases with distance according to:

$$dm = (A - Bx)dx$$

Substituting for dm in equation (1) yields:

$$x_{cm} = \frac{\int_0^L x(A - Bx)dx}{\int_0^L (A - Bx)dx}$$

Evaluate these integrals to obtain:

$$x_{cm} = \frac{L}{2}\left(\frac{1 - \frac{2BL}{3A}}{1 - \frac{BL}{2A}}\right)$$

The coordinates of the center of mass of the long thin wire are

$$\boxed{\left(\frac{L}{2}\left(\frac{1-\frac{2BL}{3A}}{1-\frac{BL}{2A}}\right),0\right)}.$$

Because $A > BL$, both the numerator and denominator are positive. Because the denominator is always larger than the numerator, it follows that $x_{cm} < \frac{1}{2}L$. This makes physical sense because the mass of the rod decreases with distance and so most of it is to the left of the midpoint of the rod. Note also that if B = 0, our result predicts a uniform density (of A) and the center of mass is at the midpoint of the rod (as it should be).

Motion of the Center of Mass

113 • **[SSM]** Two 3.0-kg particles have velocities $\vec{v}_1 = (2.0\text{ m/s})\vec{i} + (3.0\text{ m/s})\vec{j}$ and $\vec{v}_1 = (4.0\text{ m/s})\vec{i} - (6.0\text{ m/s})\vec{j}$. Find the velocity of the center of mass of the system.

Picture the Problem We can use Equation 5-24 ($M\vec{v}_{cm} = \sum_i m_i\vec{v}_i$) to find the velocity of the center of mass of this system of two particles.

From Equation 5-24:

$$\vec{v}_{cm} = \frac{\sum_i m_i\vec{v}_i}{M} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

Substitute numerical values and evaluate $\vec{v}_{cm}$:

$$\vec{v}_{cm} = \frac{(3.0\text{ kg})\left[(2.0\text{ m/s})\hat{i} + (3.0\text{ m/s})\hat{j}\right] + (3.0\text{ kg})\left[(4.0\text{ m/s})\hat{i} - (6.0\text{ m/s})\hat{j}\right]}{3.0\text{ kg} + 3.0\text{ kg}}$$

$$= \boxed{(3.0\text{ m/s})\hat{i} - (1.5\text{ m/s})\hat{j}}$$

114 • A 1500-kg car is moving westward with a speed of 20.0 m/s, and a 3000-kg truck is traveling east with a speed of 16.0 m/s. Find the velocity of the center of mass of the car-truck system.

Picture the Problem Choose a coordinate system in which east is the +x direction and use Equation 5-24 ($M\vec{v}_{cm} = \sum_i m_i\vec{v}_i$) to find the velocity of the center of mass of this system of the two cars.

From Equation 5-24:

$$\vec{v}_{cm}=\frac{\sum_i m_i\vec{v}_i}{M}=\frac{m_t\vec{v}_t+m_c\vec{v}_c}{m_t+m_c}$$

Express the velocity of the truck: $\vec{v}_t=(16.0\,\text{m/s})\hat{i}$

Express the velocity of the car: $\vec{v}_c=(-20.0\,\text{m/s})\hat{i}$

Substitute numerical values and evaluate $\vec{v}_{cm}$:

$$\vec{v}_{cm}=\frac{(3000\,\text{kg})(16.0\,\text{m/s})\,\hat{i}+(1500\,\text{kg})(-20.0\,\text{m/s})\,\hat{i}}{3000\,\text{kg}+1500\,\text{kg}}=\boxed{(4.00\,\text{m/s})\,\hat{i}}$$

115 • A force $\vec{F}$ = 12 N $\hat{i}$ is applied to the 3.0-kg ball in Figure 5-78 in Problem 103. (No forces act on the other two balls.) What is the acceleration of the center of mass of the three-ball system?

Picture the Problem The acceleration of the center of mass of the ball is related to the net external force through Newton's second law: $\vec{F}_{net,ext}=M\vec{a}_{cm}$.

Use Newton's second law to express the acceleration of the ball: $\vec{a}_{cm}=\frac{\vec{F}_{net,ext}}{M}$

Substitute numerical values and evaluate $\vec{a}_{cm}$:

$$\vec{a}_{cm}=\frac{(12\,\text{N})\hat{i}}{3.0\,\text{kg}+1.0\,\text{kg}+1.0\,\text{kg}}=\boxed{(2.4\,\text{m/s}^2)\hat{i}}$$

116 •• A block of mass *m* is attached to a string and suspended inside an otherwise empty box of mass *M*. The box rests on a scale that measures the system's weight. (*a*) If the string breaks, does the reading on the scale change? Explain your reasoning. (*b*) Assume that the string breaks and the mass *m* falls with constant acceleration *g*. Find the magnitude and direction of the acceleration of the center of mass of the box–block system. (*c*) Using the result from (*b*), determine the reading on the scale while *m* is in free-fall.

Picture the Problem Choose a coordinate system in which upward is the +*y* direction. We can use Newton's second law, $\vec{F}_{net,ext}=M\vec{a}_{cm}$, to find the acceleration of the center of mass of this two-body system.

(*a*) Yes; initially the scale reads $(M + m)g$. While the block is in free fall, the reading is Mg.

(*b*) Using Newton's second law, express the acceleration of the center of mass of the system:

$$\vec{a}_{cm} = \frac{\vec{F}_{net,ext}}{m_{tot}}$$

The box has zero acceleration. Hence:

$$\vec{a}_{cm} = -\left(\frac{mg + M(0)}{M+m}\right)\hat{j} = \boxed{-\left(\frac{mg}{M+m}\right)\hat{j}}$$

(*c*) Let F_n be the normal force exerted by the scale (the scale reading) on the box. Use Newton's second law to express the net force acting on the system while the block is falling:

$$F_{net,ext} = F_n - mg - Mg = (M+m)a_{cm}$$

Solving for F_n yields:

$$F_n = (M+m)a_{cm} + mg + Mg$$

Substitute for a_{cm} and simplify to obtain:

$$F_n = (M+m)\left(-\frac{mg}{M+m}\right) + (M+m)g$$

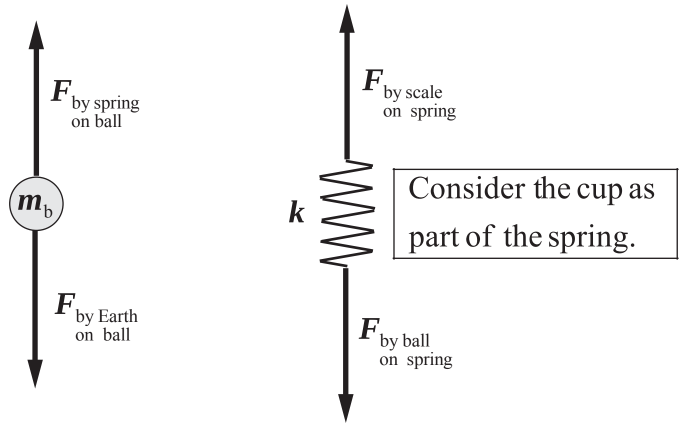
$$= \boxed{Mg}$$

as expected, given our answer to (*a*).

117 •• **[SSM]** The bottom end of a massless, vertical spring of force constant k rests on a scale and the top end is attached to a massless cup, as in Figure 5-81. Place a ball of mass m_b gently into the cup and ease it down into an equilibrium position where it sits at rest in the cup. (*a*) Draw the separate free-body diagrams for the ball and the spring. (*b)* Show that in this situation, the spring compression d is given by $d = m_b g/k$. (*c*) What is the scale reading under these conditions?

Picture the Problem (*b*) We can apply Newton's second law to the ball to find an expression for the spring's compression when the ball is at rest in the cup. (*c*) The scale reading is the force exerted by the spring on the scale and can be found from the application of Newton's second law to the cup (considered as part of the spring).

(*a*) The free-body diagrams for the ball and spring follow. Note that, because the ball has been eased down into the cup, both its speed and acceleration are zero.



(*b*) Letting the upward direction be the positive *y* direction, apply $\sum F_y = ma_y$ to the ball when it is at rest in the cup and the spring has been compressed a distance *d*:

$$F_{\substack{\text{by spring}\\\text{on ball}}} - F_{\substack{\text{by Earth}\\\text{on ball}}} = m_b a_y$$

or, because $a_y = 0$,

$$F_{\substack{\text{by spring}\\\text{on ball}}} - F_{\substack{\text{by Earth}\\\text{on ball}}} = 0 \qquad (1)$$

Because $F_{\substack{\text{by spring}\\\text{on ball}}} = kd$ and $F_{\substack{\text{by Earth}\\\text{on ball}}} = m_b g$:

$$kd - m_b g = 0 \Rightarrow d = \boxed{\frac{m_b g}{k}}$$

(*c*) Apply $\sum F_y = ma_y$ to the spring:

$$F_{\substack{\text{by scale}\\\text{on spring}}} - F_{\substack{\text{by ball}\\\text{on spring}}} = m_{\text{spring}} a_y \qquad (2)$$

or, because $a_y = 0$,

$$F_{\substack{\text{by scale}\\\text{on spring}}} - F_{\substack{\text{by ball}\\\text{on spring}}} = 0$$

Because $F_{\substack{\text{by spring}\\\text{on ball}}} = F_{\substack{\text{by ball}\\\text{on spring}}}$, adding equations (1) and (2) yields:

$$F_{\substack{\text{by scale}\\\text{on spring}}} - F_{\substack{\text{by Earth}\\\text{on ball}}} = 0$$

Solving for $F_{\substack{\text{by scale}\\\text{on spring}}}$ yields:

$$F_{\substack{\text{by scale}\\\text{on spring}}} = F_{\substack{\text{by Earth}\\\text{on ball}}} = \boxed{m_b g}$$

118 ••• In the Atwood's machine in Figure 5-82 the string passes over a fixed cylinder of mass m_c. The cylinder does not rotate. Instead, the string slides on its frictionless surface. (*a*) Find the acceleration of the center of mass of the two-block-cylinder-string system. (*b*) Use Newton's second law for systems to find the force *F* exerted by the support. (*c*) Find the tension *T* in the string connecting the blocks and show that $F = m_c g + 2T$.

Picture the Problem Assume that the object whose mass is m_1 is moving downward and take that direction to be the positive direction. We'll use Newton's second law for a system of particles to relate the acceleration of the center of mass to the acceleration of the individual particles.

(*a*) Relate the acceleration of the center of mass to m_1, m_2, m_c and their accelerations:

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_c\vec{a}_c$$

Because m_1 and m_2 have a common acceleration of magnitude a and $a_c = 0$:

$$a_{cm} = a\left(\frac{m_1 - m_2}{m_1 + m_2 + m_c}\right)$$

From Problem 4-84 we have::

$$a = g\frac{m_1 - m_2}{m_1 + m_2}$$

Substitute to obtain:

$$a_{cm} = \left(\frac{m_1 - m_2}{m_1 + m_2}g\right)\left(\frac{m_1 - m_2}{m_1 + m_2 + m_c}\right) = \boxed{\frac{(m_1 - m_2)^2}{(m_1 + m_2)(m_1 + m_2 + m_c)}g}$$

(*b*) Use Newton's second law for a system of particles to obtain:

$$-F + Mg = Ma_{cm}$$

where $M = m_1 + m_2 + m_c$ and F is upwards.

Solve for F, substitute for a_{cm} from Part (*a*), and simplify to obtain:

$$F = Mg - Ma_{cm} = Mg - \frac{(m_1 - m_2)^2}{m_1 + m_2}g = \boxed{\left[\frac{4m_1m_2}{m_1 + m_2} + m_c\right]g}$$

(*c*) From Problem 4-84 we have:

$$T = \frac{2m_1m_2}{m_1 + m_2}g$$

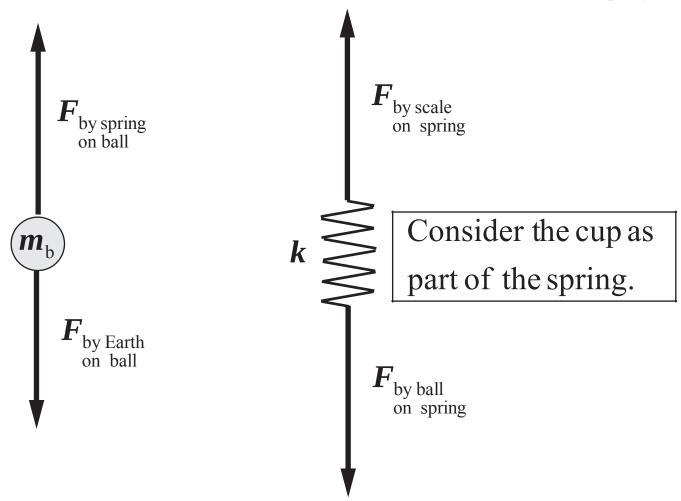
Substitute in our result from Part (*b*) and simplify to obtain:

$$F = \left[2\frac{2m_1m_2}{m_1 + m_2} + m_c\right]g = \left[2\frac{T}{g} + m_c\right]g = \boxed{2T + m_cg}$$

119 ••• Starting with the equilibrium situation in Problem 117, the whole system (scale spring, cup, and ball) is now subjected to an upward acceleration of magnitude a (for example, in an elevator). Repeat the free-body diagrams and calculations in Problem 117.

Picture the Problem Because the whole system is accelerating upward, the net upward force acting on the system must be upward. Because the spring is massless, the two forces acting on it remain equal and are oppositely directed. (*b*) We can apply Newton's second law to the ball to find an expression for the spring's compression under the given conditions. (*c*) The scale reading, as in Problem 117, is the force the scale exerts on the spring and can be found from the application of Newton's second law to the spring.

(*a*) The free-body diagrams for the ball and spring follow. Note that, because the system is accelerating upward, $F_{\substack{\text{by spring}\\\text{on ball}}} > F_{\substack{\text{by Earth}\\\text{on ball}}}$ whereas $F_{\substack{\text{by scale}\\\text{on spring}}} = F_{\substack{\text{by ball}\\\text{on spring}}}$.



(*b*) Letting the upward direction be the positive y direction, apply $\sum F_y = ma_y$ to the ball when the spring is compressed a distance d':

$$F_{\substack{\text{by spring}\\\text{on ball}}} - F_{\substack{\text{by Earth}\\\text{on ball}}} = m_b a_y \qquad (1)$$

Because $F_{\substack{\text{by spring}\\\text{on ball}}} = kd'$, $a_y = a$, and $F_{\substack{\text{by Earth}\\\text{on ball}}} = m_b g$:

$$kd' - m_b g = m_b a \Rightarrow d' = \boxed{\frac{m_b(g+a)}{k}}$$

(*c*) Apply $\sum F_y = ma_y$ to the spring:

$$F_{\substack{\text{by scale}\\\text{on spring}}} - F_{\substack{\text{by ball}\\\text{on spring}}} = 0 \qquad (2)$$

Because $F_{\substack{\text{by spring}\\\text{on ball}}} = F_{\substack{\text{by ball}\\\text{on spring}}}$ and $a_y = a$, adding equations (1) and (2) yields:

$$F_{\substack{\text{by scale}\\\text{on spring}}} - F_{\substack{\text{by Earth}\\\text{on ball}}} = m_b a$$

Solving for $F_{\substack{\text{by scale}\\\text{on spring}}}$ yields:

$$F_{\substack{\text{by scale}\\\text{on spring}}} = m_b a + F_{\substack{\text{by Earth}\\\text{on ball}}}$$

Substitute for $F_{\substack{\text{by Earth}\\\text{on ball}}}$ and simplify to obtain:

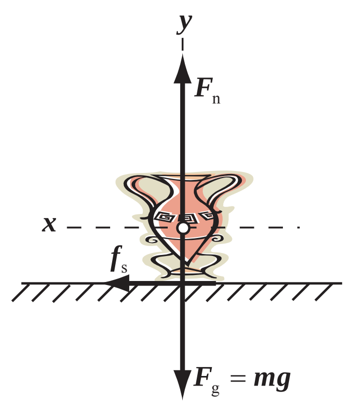
$$F_{\substack{\text{by scale}\\\text{on spring}}} = \boxed{m_b(a+g)}$$

Remarks: Note that the two forces acting on the spring and the upward force acting on the ball, while still equal, are larger (because the system is accelerating upward) than they were in Problem 117.

General Problems

120 • In designing your new house in California, you are prepared for it to withstand a maximum horizontal acceleration of 0.50g. What is the minimum coefficient of static friction between the floor and your prized Tuscan vase so that the vase does not slip on the floor under these conditions?

Picture the Problem The forces acting on the vase are the gravitational force $\vec{F}_g = m\vec{g}$ exerted by Earth, the normal force $\vec{F}_n$ exerted by the floor, and the static friction force $\vec{f}_s$, also exerted by the floor. We can apply Newton's second law to find the minimum coefficient of static friction that will prevent the vase from slipping.



Apply $\sum\vec{F} = m\vec{a}$ to the vase to obtain:

$$\sum F_x = f_s = ma_x \qquad (1)$$

and

$$\sum F_y = F_n - mg = 0$$

Relate the static friction force $\vec{f}_s$ to the minimum coefficient of static friction $\mu_{s,min}$ that will prevent the vase from slipping:

$$f_s = \mu_{s,min} F_n$$

or, because $F_n = mg$,

$$f_s = \mu_{s,min} mg$$

Substituting for f_s in equation (1) yields:

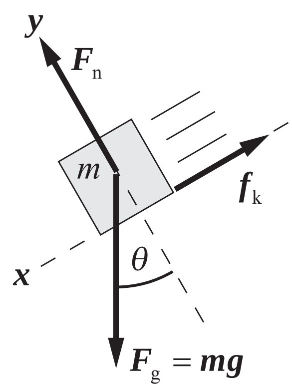
$$\mu_{s,\min} mg = ma_x \Rightarrow \mu_{s,\min} = \frac{a_x}{g}$$

Because $a_x = 0.50g$:

$$\mu_{s,\min} = \frac{0.50g}{g} = \boxed{0.50}$$

121 • A 4.5-kg block slides down an inclined plane that makes an angle of 28° with the horizontal. Starting from rest, the block slides a distance of 2.4 m in 5.2 s. Find the coefficient of kinetic friction between the block and plane.

Picture the Problem The forces that act on the block as it slides down the incline are shown on the free-body diagram to the right. The acceleration of the block can be determined from the distance-and-time information given in the problem. The application of Newton's second law to the block will lead to an expression for the coefficient of kinetic friction as a function of the block's acceleration and the angle of the incline.



Apply $\sum \vec{F} = m\vec{a}$ to the block:

$$\sum F_x = mg\sin\theta - f_k = ma_x \quad (1)$$

and

$$\sum F_y = F_n - mg\cos\theta = 0 \quad (2)$$

Set $f_k = \mu_k F_n$ in equation (1) to obtain:

$$mg\sin\theta - \mu_k F_n = ma_x \quad (3)$$

Solve equation (2) for F_n and substitute in equation (3) to obtain:

$$mg\sin\theta - \mu_k mg\cos\theta = ma_x$$

Solving for μ_k yields:

$$\mu_k = \frac{g\sin\theta - a_x}{g\cos\theta} \quad (4)$$

Using a constant-acceleration equation, relate the distance the block slides to its sliding time:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_x(\Delta t)^2 \Rightarrow a_x = \frac{2\Delta x}{(\Delta t)^2}$$

Substitute for a_x in equation (4) to obtain:

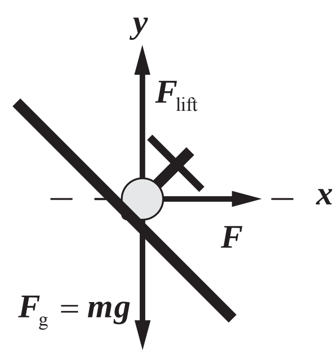
$$\mu_k = \frac{g\sin\theta - \dfrac{2\Delta x}{(\Delta t)^2}}{g\cos\theta}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{(9.81\ \text{m/s}^2)\sin 28° - \dfrac{2(2.4\ \text{m})}{(5.2\ \text{s})^2}}{(9.81\ \text{m/s}^2)\cos 28°} = \boxed{0.51}$$

122 •• You plan to fly a model airplane of mass 0.400 kg that is attached to a horizontal string. The plane will travel in a horizontal circle of radius 5.70 m. (Assume the weight of the plane is balanced by the upward "lift" force of the air on the wings of the plane.) The plane will make 1.20 revolutions every 4.00 s. (*a*) Find the speed at which you must fly the plane. (*b*) Find the force exerted on your hand as you hold the string (assume the string is massless).

Picture the Problem The force exerted on your hand as you hold the string is the reaction force to the tension $\vec{F}$ in the string and, hence, has the same magnitude. The speed of the plane can be calculated from the data concerning the radius of its path and the time it takes to make one revolution. The application of Newton's second law will give us the tension F in the string.



(*a*) Express the speed of the airplane in terms of the circumference of the circle in which it is flying and its period:

$$v = \frac{2\pi r}{T}$$

Substitute numerical values and evaluate v:

$$v = \frac{2\pi(5.70\ \text{m})}{\dfrac{4.00\ \text{s}}{1.20\ \text{rev}}} = \boxed{10.7\ \text{m/s}}$$

(*b*) Apply $\sum F_x = ma_x$ to the model airplane:

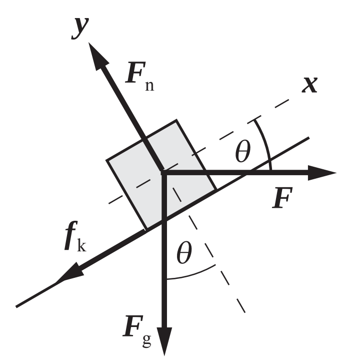
$$F = m\frac{v^2}{r}$$

Substitute numerical values and evaluate F:

$$F=\frac{(0.400\,\text{kg})(10.74\,\text{m/s})^2}{5.70\,\text{m}}=\boxed{8.10\,\text{N}}$$

123 •• Your moving company is to load a crate of books on a truck with the help of some planks that slope upward at 30.0º. The mass of the crate is 100 kg, and the coefficient of sliding friction between it and the planks is 0.500. You and your employees push *horizontally* with a combined net force $\vec{F}$. Once the crate has started to move, how large must F be in order to keep the crate moving at constant speed?

Picture the Problem The free-body diagram shows the forces acting on the crate of books. The kinetic friction force opposes the motion of the crate up the incline. Because the crate is moving at constant speed in a straight line, its acceleration is zero. We can determine F by applying Newton's second law to the crate, substituting for f_k, eliminating the normal force, and solving for the required force.



Apply $\sum\vec{F}=m\vec{a}$ to the crate, with both a_x and a_y equal to zero, to the crate:

$$\sum F_x=F\cos\theta-f_k-mg\sin\theta=0$$

and

$$\sum F_y=F_n-F\sin\theta-mg\cos\theta=0$$

Because $f_k=\mu_kF_n$, the x-equation becomes:

$$F\cos\theta-\mu_kF_n-mg\sin\theta=0 \quad (1)$$

Solving the y-equation for F_n yields:

$$F_n=F\sin\theta+mg\cos\theta$$

Substitute for F_n in equation (1) and solve for F to obtain:

$$F=\frac{mg(\sin\theta+\mu_k\cos\theta)}{\cos\theta-\mu_k\sin\theta}$$

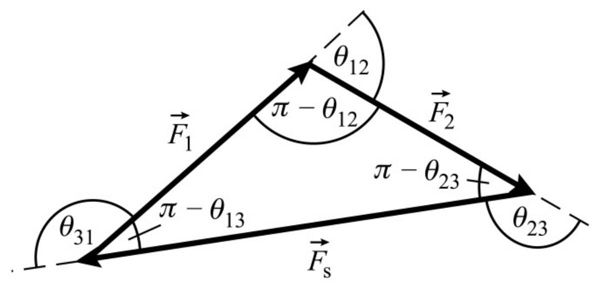
Substitute numerical values and evaluate F:

$$F=\frac{(100\,\text{kg})(9.81\,\text{m/s}^2)(\sin30°+(0.500)\cos30.0°)}{\cos30.0°-(0.500)\sin30°}=\boxed{1.49\,\text{kN}}$$

124 •• Three forces act on an object in static equilibrium (Figure 5-83). (*a*) If F_1, F_2, and F_3 represent the magnitudes of the forces acting on the object, show that $F_1/\sin\theta_{23} = F_2/\sin\theta_{31} = F_3/\sin\theta_{12}$. (*b*) Show that $F_1^2 = F_2^2 + F_3^2 + 2F_2F_3\cos\theta_{23}$.

Picture the Problem The fact that the object is in static equilibrium under the influence of the three forces means that $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$. Drawing the corresponding force triangle will allow us to relate the forces to the angles between them through the law of sines and the law of cosines.

(*a*) Using the fact that the object is in static equilibrium, redraw the force diagram connecting the forces head-to-tail:



Appling the law of sines to the triangle yields:

$$\frac{F_1}{\sin(\pi - \theta_{23})} = \frac{F_2}{\sin(\pi - \theta_{13})} = \frac{F_3}{\sin(\pi - \theta_{12})}$$

Use the trigonometric identity $\sin(\pi - \alpha) = \sin\alpha$ to obtain:

$$\boxed{\frac{F_1}{\sin\theta_{23}} = \frac{F_2}{\sin\theta_{13}} = \frac{F_3}{\sin\theta_{12}}}$$

(*b*) Appling the law of cosines to the triangle yields:

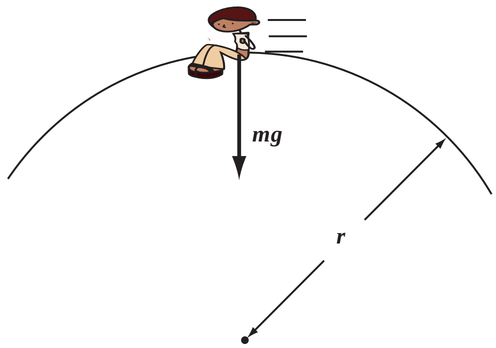
$$F_1^2 = F_2^2 + F_3^2 - 2F_2F_3\cos(\pi - \theta_{23})$$

Use the trigonometric identity $\cos(\pi - \alpha) = -\cos\alpha$ to obtain:

$$F_1^2 = \boxed{F_2^2 + F_3^2 + 2F_2F_3\cos\theta_{23}}$$

125 •• In a carnival ride, you sit on a seat in a compartment that rotates with constant speed in a vertical circle of radius 5.0 m. The ride is designed so your head always points toward the center of the circle. (*a*) If the ride completes one full circle in 2.0 s, find the direction and magnitude of your acceleration. (*b*) Find the slowest rate of rotation (in other words, the longest time T_m to complete one full circle) if the seat belt is to exert no force on you at the top of the ride.

Picture the Problem We can calculate your acceleration from your speed that, in turn, is a function of the period of the motion. To determine the longest period of the motion, we focus our attention on the situation at the very top of the ride when the seat belt is exerting no force on you. We can use Newton's second law to relate the period of the motion to your acceleration and speed.



(*a*) Because the motion is at constant speed, your acceleration is entirely radial and is given by:

$$a_c = \frac{v^2}{r}$$

Express your speed as a function of the radius of the circle and the period of the motion:

$$v = \frac{2\pi r}{T}$$

Substitute for v in the expression for a_c and simplify to obtain:

$$a_c = \frac{4\pi^2 r}{T^2}$$

Substitute numerical values and evaluate a_c:

$$a_c = \frac{4\pi^2(5.0\,\text{m})}{(2.0\,\text{s})^2} = \boxed{49\,\text{m/s}^2}$$

(*b*) Apply $\sum \vec{F} = m\vec{a}$ to yourself when you are at the top of the circular path and the seat belt is exerting no force on you:

$$\sum F_r = mg = m\frac{v^2}{r} \Rightarrow v = \sqrt{gr}$$

Express the period of your motion as a function of the radius of the circle:

$$T_m = \frac{2\pi r}{v}$$

Substituting for v and simplifying yields:

$$T_m = \frac{2\pi r}{\sqrt{gr}} = 2\pi\sqrt{\frac{r}{g}}$$

The slowest rate of rotation is the reciprocal of T_m:

$$T_m^{-1} = \frac{1}{2\pi}\sqrt{\frac{g}{r}}$$

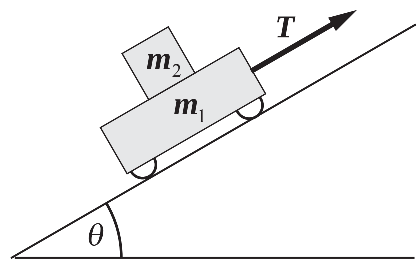
Substitute numerical values and evaluate $\boldsymbol{T}_{\mathrm{m}}^{-1}$:

$$T_{\mathrm{m}}^{-1} = \frac{1}{2\pi}\sqrt{\frac{9.81\,\mathrm{m/s^2}}{5.0\,\mathrm{m}}}$$
$$= 0.2229\,\mathrm{s}^{-1} \times \frac{60\,\mathrm{s}}{\mathrm{min}}$$
$$\approx \boxed{13\,\mathrm{rev/min}}$$

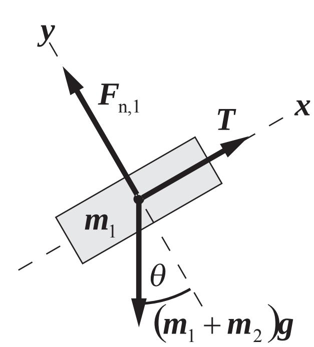
Remarks: You are ″weightless″ under the conditions described in Part (*b*).

126 •• A flat-topped toy cart moves on frictionless wheels, pulled by a rope under tension T. The mass of the cart is m_1. A load of mass m_2 rests on top of the cart with the coefficient of static friction μ_s between the cart and the load. The cart is pulled up a ramp that is inclined at angle θ above the horizontal. The rope is parallel to the ramp. What is the maximum tension T that can be applied without causing the load to slip?

Picture the Problem The pictorial representation to the right shows the cart and its load on the inclined plane. The load will not slip provided its maximum acceleration is not exceeded. We can find that maximum acceleration by applying Newton's second law to the load. We can then apply Newton's second law to the cart-plus-load system to determine the tension in the rope when the system is experiencing its maximum acceleration.



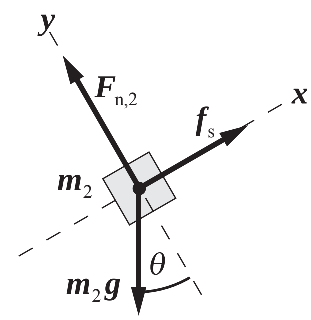
Draw the free-body diagram for the cart and its load:



Apply $\sum F_x = ma_x$ to the cart plus its load:

$$T - (m_1 + m_2)g\sin\theta = (m_1 + m_2)a_{x,\max} \qquad (1)$$

Draw the free-body diagram for the load of mass m_2 on top of the cart:



Apply $\sum\vec{F} = m\vec{a}$ to the load on top of the cart:

$$\sum F_x = f_{s,max} - m_2 g\sin\theta = m_2 a_{x,max}$$

and

$$\sum F_y = F_{n,2} - m_2 g\cos\theta = 0$$

Because $f_{s,max} = \mu_s F_{n,2}$, the x equation becomes:

$$\mu_s F_{n,2} - m_2 g\sin\theta = m_2 a_{x,max} \qquad (2)$$

Solving the y equation for $F_{n,2}$ yields:

$$F_{n,2} = m_2 g\cos\theta$$

Substitute for $F_{n,2}$ in equation (2) to obtain:

$$\mu_s m_2 g\cos\theta - m_2 g\sin\theta = m_2 a_{x,max}$$

Solving for $a_{x,max}$ and simplifying yields:

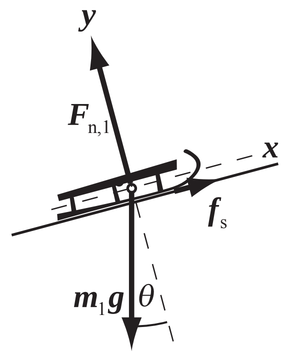
$$a_{x,max} = g(\mu_s\cos\theta - \sin\theta) \qquad (3)$$

Substitute for $a_{x,max}$ in equation (1) and solve for T to obtain:

$$T = \boxed{(m_1 + m_2)g\mu_s\cos\theta}$$

127 ••• A sled weighing 200 N that is held in place by static friction, rests on a 15º incline (Figure 5-84). The coefficient of static friction between the sled and the incline is 0.50. (*a*) What is the magnitude of the normal force on the sled? (*b*) What is the magnitude of the static frictional force on the sled? (*c*) The sled is now pulled up the incline at constant speed by a child walking up the incline ahead of the sled. The child weighs 500 N and pulls on the rope with a constant force of 100 N. The rope makes an angle of 30º with the incline and has negligible mass. What is the magnitude of the kinetic frictional force on the sled? (*d*) What is the coefficient of kinetic friction between the sled and the incline? (*e*) What is the magnitude of the force exerted on the child by the incline?

Picture the Problem The free-body diagram for the sled while it is held stationary by the static friction force is shown to the right. We can solve this problem by repeatedly applying Newton's second law under the conditions specified in each part of the problem.



(*a*) Apply $\sum F_y = ma_y$ to the sled: $F_{n,1} - m_1 g\cos\theta = 0$

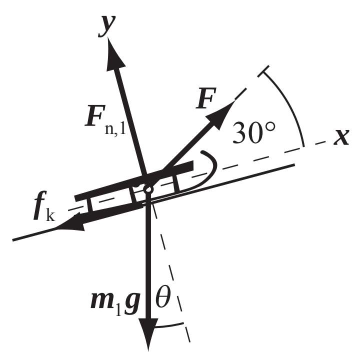
Solve for $F_{n,1}$: $F_{n,1} = m_1 g\cos\theta$

Substitute numerical values and evaluate $F_{n,1}$: $F_{n,1} = (200\ \text{N})\cos 15° = \boxed{0.19\ \text{kN}}$

(*b*) Apply $\sum F_x = ma_x$ to the sled: $f_s - m_1 g\sin\theta = 0 \Rightarrow f_s = m_1 g\sin\theta$

Substitute numerical values and evaluate f_s: $f_s = (200\ \text{N})\sin 15° = \boxed{52\ \text{N}}$

(*c*) Draw the free-body diagram for the sled when it is moving up the incline at constant speed:



Apply $\sum F_x = 0$ to the sled to obtain: $F\cos 30° - m_1 g\sin\theta - f_k = 0$

Solving for f_k yields: $f_k = F\cos 30° - m_1 g\sin\theta$

Substitute numerical values and evaluate f_k:

$$f_k = (100\ \text{N})\cos 30° - (200\ \text{N})\sin 15° = 34.84\ \text{N} = \boxed{35\ \text{N}}$$

(*d*) The coefficient of kinetic friction between the sled and the incline is given by:

$$\mu_k = \frac{f_k}{F_{n,1}} \qquad (1)$$

Apply $\sum F_y = 0$ to the sled to obtain:

$$F_{n,1} + F\sin 30° - m_1 g\cos\theta = 0$$

Solve for $F_{n,1}$:

$$F_{n,1} = m_1 g\cos\theta - F\sin 30°$$

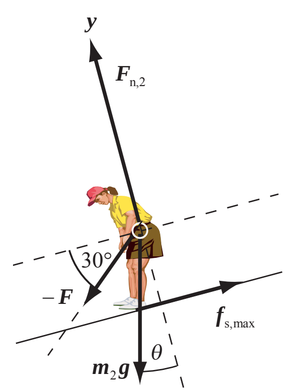
Substituting for $F_{n,1}$ in equation (1) yields:

$$\mu_k = \frac{f_k}{m_1 g\cos\theta - F\sin 30°}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{34.84\text{ N}}{(200\text{ N})\cos 15° - (100\text{ N})\sin 30°} = \boxed{0.24}$$

(*e*) Draw the free body diagram for the child:



Express the net force F_c exerted on the child by the incline:

$$F_c = \sqrt{F_{n2}^2 + f_{s,max}^2} \qquad (1)$$

Noting that the child is stationary, apply $\sum \vec{F} = m\vec{a}$ to the child:

$$\sum F_x = f_{s,max} - F\cos 30° - m_2 g\sin 15° = 0$$

and

$$\sum F_y = F_{n2} - m_2 g\cos 15° - F\sin 30° = 0$$

Solve the x equation for $f_{s,max}$ and the y equation for $F_{n,2}$:

$$f_{s,max} = F\cos 30° + m_2 g\sin 15°$$

and

$$F_{n,2} = m_2 g\cos 15° + F\sin 30°$$

Substitute numerical values and evaluate $f_{s,max}$ and $F_{n,2}$:

$$f_{s,max} = (100\,N)\cos 30° + (500\,N)\sin 15° = 216.0\,N$$

and

$$F_{n,2} = (500\,N)\cos 15° + (100\,N)\sin 30° = 533.0\,N$$

Substitute numerical values in equation (1) and evaluate F:

$$F_c = \sqrt{(216.0\,N)^2 + (533.0\,N)^2} = \boxed{0.58\,kN}$$

128 •• In 1976, Gerard O'Neill proposed that large space stations be built for human habitation in orbit around Earth and the moon. Because prolonged free-fall has adverse medical effects, he proposed making the stations in the form of long cylinders and spinning them around the cylinder axis to provide the inhabitants with the sensation of gravity. One such O'Neill colony is to be built 5.0 miles long, with a diameter of 0.60 mi. A worker on the inside of the colony would experience a sense of ″gravity,″ because he would be in an accelerated frame of reference due to the rotation. (*a*) Show that the ″acceleration of gravity″ experienced by the worker in the O'Neill colony is equal to his centripetal acceleration. (*Hint:* Consider someone ″looking in″ from outside the colony.) (*b*) If we assume that the space station is composed of several decks that are at varying distances (radii) from the axis of rotation, show that the ″acceleration of gravity″ becomes weaker the closer the worker gets to the axis. (*c*) How many revolutions per minute would this space station have to make to give an ″acceleration of gravity″ of 9.8 m/s^2 at the outermost edge of the station?

Picture the Problem Let v represent the speed of rotation of the station, and r the distance from the center of the station. Because the O'Neill colony is, presumably, in deep space, the only acceleration one would experience in it would be that due to its rotation.

(*a*) The acceleration of anyone who is standing inside the station is $a = v^2/r$. This acceleration is directed toward the axis of rotation. If someone inside the station drops an apple, the apple will not have any forces acting on it once released, but will move along a straight line at constant speed. However, from the point of view of our observer inside the station, if he views himself as unmoving, the apple is perceived to have an acceleration of v^2/r directed away from the axis of rotation (a "centrifugal" acceleration).

(*b*) Each deck must rotate about the central axis with the same period T. Relate the speed of a person on a particular deck to his/her distance r from the center:

$$v = \frac{2\pi r}{T}$$

Express the "acceleration of gravity" perceived by someone a distance r from the center:

$$a_c = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$$, a result that tells us that the "acceleration of gravity" decreases as r decreases.

(c) Relate the desired acceleration to the radius of the space station and its period:

$$a = \frac{4\pi^2 r}{T^2} \Rightarrow T = \sqrt{\frac{4\pi^2 r}{a}}$$

Substitute numerical values and evaluate T:

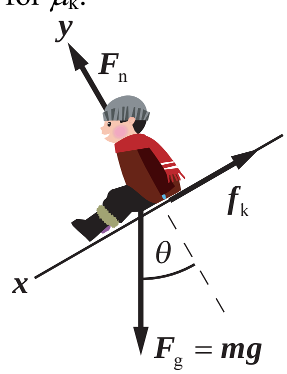
$$T = \sqrt{\frac{4\pi^2\left(0.30\,\text{mi} \times \frac{1.609\,\text{km}}{\text{mi}}\right)}{9.8\,\text{m/s}^2}} = 44\,\text{s} \approx 0.74\,\text{min}$$

Take the reciprocal of this time to find the number of revolutions per minute Babylon 5 has to make in order to provide this "Earth-like" acceleration:

$$T^{-1} = \boxed{1.4\,\text{rev/min}}$$

129 •• A child of mass m slides down a slide inclined at 30º in time t_1. The coefficient of kinetic friction between her and the slide is μ_k. She finds that if she sits on a small sled (also of mass m) with frictionless runners, she slides down the same slide in time $\frac{1}{2}t_1$. Find μ_k.

Picture the Problem The following free-body diagram shows the forces acting on the child as she slides down the incline. We'll first use Newton's second law to derive an expression for μ_k in terms of her acceleration and then use Newton's second law to find her acceleration when riding the frictionless cart. Using a constant-acceleration equation, we'll relate these two accelerations to her descent times and solve for her acceleration when sliding. Finally, we can use this acceleration in the expression for μ_k.



Apply $\sum \vec{F} = m\vec{a}$ to the child as she slides down the incline:

$$\sum F_x = mg\sin\theta - f_k = ma_{1,x}$$
and
$$\sum F_y = F_n - mg\cos\theta = 0$$

Because $f_k = \mu_k F_n$, the x-equation can be written:

$$mg\sin\theta - \mu_k F_n = ma_{1,x} \qquad (1)$$

Solving the y-equation for F_n yields:

$$F_n = mg\cos\theta$$

Substitute for F_n in equation (1) to obtain:

$$mg\sin\theta - \mu_k mg\cos\theta = ma_{1,x}$$

Solving for μ_k yields:

$$\mu_k = \tan 30° - \frac{a_{1,x}}{g\cos 30°} \qquad (2)$$

Apply $\sum F_x = ma_x$ to the child as she rides the frictionless cart down the incline and solve for her acceleration $a_{2,x}$:

$$mg\sin 30° = ma_{2,x}$$
and
$$a_{2,x} = g\sin 30°$$

Letting s represent the distance she slides down the incline, use a constant-acceleration equation to relate her sliding times to her accelerations and distance traveled down the slide :

$$s = v_{0x}t_1 + \tfrac{1}{2}a_{1,x}t_1^2 \text{ where } v_{0x} = 0$$
and
$$s = v_{0x}t_2 + \tfrac{1}{2}a_{2,x}t_2^2 \text{ where } v_{0x} = 0$$

Equate these expressions, substitute $t_2 = \frac{1}{2}t_1$ and solve for $a_{1,x}$:

$$a_{1,x} = \tfrac{1}{4}a_{2,x} = \tfrac{1}{4}g\sin 30°$$

Substitute for $a_{1,x}$ in equation (2) to obtain:

$$\mu_k = \tan 30° - \frac{\frac{1}{4}g\sin 30°}{g\cos 30°} = \tfrac{3}{4}\tan 30°$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \tfrac{3}{4}\tan 30° = \boxed{0.43}$$

130 ••• The position of a particle of mass $m = 0.80$ kg as a function of time is given by $\vec{r} = x\hat{i} + y\hat{j} = (R\sin\omega t)\hat{i} + (R\cos\omega t)\hat{j}$, where $R = 4.0$ m and $\omega = 2\pi\,\text{s}^{-1}$. (*a*) Show that the path of this particle is a circle of radius R, with its center at the origin of the xy plane. (*b*) Compute the velocity vector. Show that $v_x/v_y = -y/x$.

(*c*) Compute the acceleration vector and show that it is directed toward the origin and has the magnitude v^2/R. (*d*) Find the magnitude and direction of the net force acting on the particle.

Picture the Problem The path of the particle is a circle if *r* is constant. Once we have shown that it is, we can calculate its value from its components and determine the particle's velocity and acceleration by differentiation. The direction of the net force acting on the particle can be determined from the direction of its acceleration.

(*a*) Express the magnitude of $\vec{r}$ in terms of its components:

$$r = \sqrt{r_x^2 + r_y^2}$$

Evaluate *r* with $r_x = R\sin\omega t$ and $r_y = R\cos\omega t$:

$$r = \sqrt{[R\sin\omega t]^2 + [R\cos\omega t]^2}$$
$$= \sqrt{R^2\left(\sin^2\omega t + \cos^2\omega t\right)} = R$$

This result shows that the path of the particle is a circle of radius *R* centered at the origin.

(*b*) Differentiate $\vec{r}$ with respect to time to obtain $\vec{v}$:

$$\vec{v} = \frac{d\vec{r}}{dt} = [R\omega\cos\omega t]\,\hat{i} + [-R\omega\sin\omega t]\,\hat{j}$$
$$= \boxed{[(8.0\pi\cos 2\pi t)\text{m/s}]\,\hat{i} - [(8.0\pi\sin 2\pi t)\text{m/s}]\,\hat{j}}$$

Express the ratio $\dfrac{v_x}{v_y}$:

$$\frac{v_x}{v_y} = \frac{8.0\pi\cos\omega t}{-8.0\pi\sin\omega t} = -\cot\omega t \quad (1)$$

Express the ratio $-\dfrac{y}{x}$:

$$-\frac{y}{x} = -\frac{R\cos\omega t}{R\sin\omega t} = -\cot\omega t \quad (2)$$

From equations (1) and (2) we have:

$$\boxed{\frac{v_x}{v_y} = -\frac{y}{x}}$$

(*c*) Differentiate $\vec{v}$ with respect to time to obtain $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \boxed{\left[\left(-16\pi^2\text{m/s}^2\right)\sin\omega t\right]\hat{i} + \left[\left(-16\pi^2\text{m/s}^2\right)\cos\omega t\right]\hat{j}}$$

Factor $-4.0\pi^2/\text{s}^2$ from $\vec{a}$ to obtain:

$$\vec{\boldsymbol{a}} = \left(-4.0\boldsymbol{\pi}^2\ \text{s}^{-2}\right)\left[\left(4.0\sin\boldsymbol{\omega t}\right)\hat{\boldsymbol{i}} + \left(4.0\cos\boldsymbol{\omega t}\right)\hat{\boldsymbol{j}}\right] = \boxed{\left(-4.0\boldsymbol{\pi}^2\ \text{s}^{-2}\right)\vec{\boldsymbol{r}}}$$

Because $\vec{\boldsymbol{a}}$ is in the opposite direction from $\vec{\boldsymbol{r}}$, it is directed toward the center of the circle in which the particle is traveling.

Find the ratio $\dfrac{v^2}{R}$:

$$a = \frac{v^2}{R} = \frac{\left(8.0\pi\ \text{m/s}\right)^2}{4.0\,\text{m}} = \boxed{16\pi^2\ \text{m/s}^2}$$

(*d*) Apply $\sum\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to the particle:

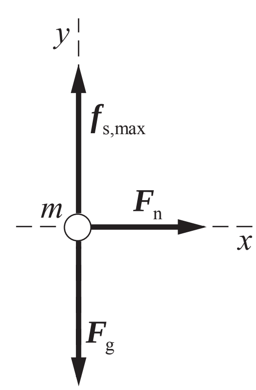
$$F_{\text{net}} = ma = \left(0.80\,\text{kg}\right)\left(16\pi^2\ \text{m/s}^2\right) = \boxed{13\pi^2\ \text{N}}$$

Because the direction of $\vec{\boldsymbol{F}}_{\text{net}}$ is the same as that of $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{F}}_{\text{net}}$ is toward the center of the circle.

131 ••• You are on an amusement park ride with your back against the wall of a spinning vertical cylinder. The floor falls away and you are held up by static friction. Assume your mass is 75 kg. (*a*) Draw a free-body diagram of yourself. (*b*) Use this diagram with Newton's laws to determine the force of friction on you. (*c*) The radius of the cylinder is 4.0 m and the coefficient of static friction between you and the wall is 0.55. What is the minimum number of revolutions per minute necessary to prevent you from sliding down the wall? Does this answer hold only for you? Will other, more massive, patrons fall downward? Explain.

Picture the Problem The application of Newton's second law and the definition of the maximum static friction force will be used to determine the period T of the motion. The reciprocal of the period will give us the minimum number of revolutions required per unit time to hold you in place.

(*a*) The free-body diagram showing the forces acting on you when you are being held in place by the maximum static friction force is shown to the right.



(*b*) Apply $\sum \vec{F} = m\vec{a}$ to yourself while you are held in place by friction:

$$\sum F_x = F_n = m\frac{v^2}{r} \qquad (1)$$

and

$$\sum F_y = f_{s,max} - mg = 0 \qquad (2)$$

Solve equation (2) for $f_{s,max}$:

$$f_{s,max} = mg$$

Substitute numerical values and evaluate $f_{s,max}$:

$$f_{s,max} = (75\text{ kg})(9.81\text{ m/s}^2) = \boxed{0.74\text{ kN}}$$

(*c*) The number of revolutions per minute *N* is the reciprocal of the period in minutes:

$$N = \frac{1}{T} \qquad (3)$$

Because $f_{s,max} = \mu_s F_n$, equation (1) can be written:

$$\frac{f_{s,max}}{\mu_s} = \frac{mg}{\mu_s} = m\frac{v^2}{r} \qquad (4)$$

Your speed is related to the period of your motion:

$$v = \frac{2\pi r}{T}$$

Substitute for *v* in equation (4) to obtain:

$$\frac{mg}{\mu_s} = m\frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 mr}{T^2}$$

Solving for *T* yields:

$$T = 2\pi\sqrt{\frac{\mu_s r}{g}}$$

Substitute for *T* in equation (3) to obtain:

$$N = \frac{1}{2\pi\sqrt{\frac{\mu_s r}{g}}} = \frac{1}{2\pi}\sqrt{\frac{g}{\mu_s r}}$$

Substitute numerical values and evaluate *N*:

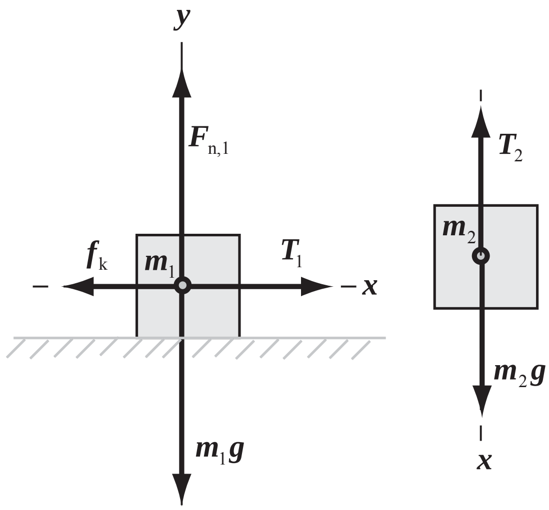
$$N = \frac{1}{2\pi}\sqrt{\frac{9.81\text{ m/s}^2}{(0.55)(4.0\text{ m})}} = 0.336\text{ rev/s}$$

$$= \boxed{20\text{ rev/min}}$$

Because your mass does not appear in the expression for *N,* this result holds for all patrons, regardless of their mass.

132 ••• A block of mass m_1 is on a horizontal table. The block is attached to a 2.5-kg block (m_2) by a light string that passes over a pulley at the edge of the table. The block of mass m_2 dangles 1.5 m above the ground (Figure 5-85). The string that connects them passes over a frictionless, massless pulley. This system is released from rest at $t = 0$ and the 2.5-kg block strikes the ground at $t = 0.82$ s. The system is now placed in its initial configuration and a 1.2-kg block is placed on top of the block of mass m_1. Released from rest, the 2.5-kg block now strikes the ground 1.3 s later. Determine the mass m_1 and the coefficient of kinetic friction between the block whose mass is m_1 and the table.

Picture the Problem The free-body diagrams below show the forces acting on the blocks whose masses are m_1 and m_2. The application of Newton's second law and the use of a constant-acceleration equation will allow us to find a relationship between the coefficient of kinetic friction and m_1. The repetition of this procedure with the additional block on top of the block whose mass is m_1 will lead us to a second equation that, when solved simultaneously with the former equation, leads to a quadratic equation in m_1. Finally, its solution will allow us to substitute in an expression for μ_k and determine its value.



Using a constant-acceleration equation, relate the displacement of the system in its first configuration as a function of its acceleration and fall time:

$$\Delta x = v_{0x}\Delta t + \tfrac{1}{2}a_{1x}(\Delta t)^2$$

or, because $v_{0x} = 0$,

$$\Delta x = \tfrac{1}{2}a_{x,1}(\Delta t)^2 \Rightarrow a_{x,1} = \frac{2\Delta x}{(\Delta t)^2}$$

Substitute numerical values and evaluate $a_{x,1}$:

$$a_{x,1} = \frac{2(1.5\,\text{m})}{(0.82\,\text{s})^2} = 4.4616\,\text{m/s}^2$$

Apply $\sum F_x = ma_x$ to the block whose mass is m_2:

$$m_2 g - T_1 = m_2 a_{x,1} \Rightarrow T_1 = m_2(g - a_{1x})$$

Substitute numerical values and evaluate T_1:

$$T_1 = (2.5\,\text{kg})(9.81\,\text{m/s}^2 - 4.4616\,\text{m/s}^2) = 13.371\,\text{N}$$

Apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_1:

$$\sum F_x = T_1 - f_\text{k} = m_1 a_{x,1}$$
and
$$\sum F_y = F_{\text{n},1} - m_1 g = 0$$

Using $f_\text{k} = \mu_\text{k} F_\text{n}$, eliminate F_n between the two equations to obtain:

$$T_1 - \mu_\text{k} m_1 g = m_1 a_{x,1} \quad (1)$$

Find the acceleration $a_{x,2}$ for the second run:

$$a_{x,2} = \frac{2\Delta x}{(\Delta t)^2} = \frac{2(1.5\,\text{m})}{(1.3\,\text{s})^2} = 1.775\,\text{m/s}^2$$

T_2 is given by:

$$T_2 = m_2(g - a_{x,2}) = (2.5\,\text{kg})(9.81\,\text{m/s}^2 - 1.775\,\text{m/s}^2) = 20.1\,\text{N}$$

Apply $\sum F_x = ma_x$ to the system with the 1.2-kg block in place:

$$T_2 - \mu_\text{k}(m_1 + 1.2\,\text{kg})g = (m_1 + 1.2\,\text{kg})a_{x,2} \quad (2)$$

Solve equation (1) for μ_k:

$$\mu_\text{k} = \frac{T_1 - m_1 a_{x,1}}{m_1 g} \quad (3)$$

Substitute for μ_k in equation (2) and simplify to obtain the quadratic equation in m_1:

$$2.686 m_1^2 + 9.940 m_1 - 16.05 = 0$$

Use your graphing calculator or the quadratic formula to obtain:

$$m_1 = 1.215\,\text{kg} = \boxed{1.2\,\text{kg}}$$

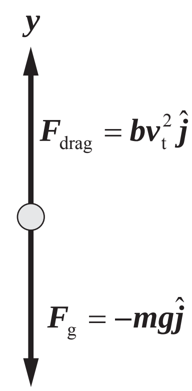
Substitute numerical values in equation (3) and evaluate μ_k:

$$\mu_\text{k} = \frac{13.375\,\text{N} - (1.215\,\text{kg})(4.4616\,\text{m/s}^2)}{(1.215\,\text{kg})(9.81\,\text{m/s}^2)} = \boxed{0.67}$$

133 ••• Sally claims flying squirrels do not really fly; they jump and use folds of skin that connect their forelegs and their back legs like a parachute to allow them to glide from tree to tree. Liz decides to test Sally's hypothesis by calculating the terminal speed of a falling outstretched flying squirrel. If the constant b in the drag force is proportional to the area of the object facing the air

flow, use the results of Example 5-8 and some assumptions about the size of the squirrel to estimate its terminal (downward) speed. Is Sally's claim supported by Liz's calculation?

Picture the Problem A free-body diagram showing the forces acting on the squirrel under terminal-speed conditions is shown to the right. We'll assume that b is proportional to the squirrel's frontal area and that this area is about 0.1 that of a human being. Further, we'll assume that the mass of a squirrel is about 1.0 kg. Applying Newton's second law will lead us to an expression for the squirrel's terminal speed.



Apply $\sum F_y = ma_y$ to the squirrel:

$$F_d - F_g = ma_y$$

or, because $a_y = 0$,

$$F_d - F_g = 0$$

Substituting for F_d and F_g yields:

$$bv_t^2 - mg = 0 \Rightarrow v_t = \sqrt{\frac{mg}{b}} \qquad (1)$$

Assuming that $A_{\text{squirrel}} = (0.1)A_{\text{human}}$:

$$b = (0.1)b_{\text{human}}$$

From Example 5-8:

$$b_{\text{human}} = 0.251\,\text{kg/m}$$

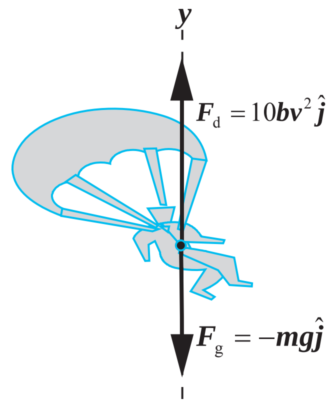
Substitute numerical values in equation (1) and evaluate v_t:

$$v_t = \sqrt{\frac{(1.0\,\text{kg})(9.81\,\text{m/s}^2)}{(0.1)(0.251\,\text{kg/m})}} \approx \boxed{20\,\text{m/s}}$$

Because the squirrel's terminal speed is approximately 80 km/h and this value is less than half that of the skydiver in Example 5-8, Sally's claim seems to be supported by Liz's calculation.

134 ••• After a parachutist jumps from an airplane (but before he pulls the rip cord to open his parachute), a downward speed of up to 180 km/h can be reached. When the parachute is finally opened, the drag force is increased by about a factor of 10, and this can create a large jolt on the jumper. Suppose this jumper falls at 180 km/h before opening his chute. (*a*) Determine the parachutist's acceleration when the chute is just opened, assuming his mass is 60 kg. (*b*) If rapid accelerations greater than 5.0g can harm the structure of the human body, is this a safe practice?

Picture the Problem The free-body diagram shows the drag force $\vec{F}_d$ exerted by the air and the gravitational force $\vec{F}_g$ exerted by Earth acting on on the parachutist just after his chute has opened. We can apply Newton's second law to the parachutist to obtain an expression for his acceleration as a function of his speed and then evaluate this expression for $v = v_t$.



(*a*) Apply $\sum F_y = ma_y$ to the parachutist immediately after the chute opens:

$$10bv^2 - mg = ma_{\substack{\text{chute}\\\text{open}}}$$

Solving for $a_{\substack{\text{chute}\\\text{open}}}$ yields:

$$a_{\substack{\text{chute}\\\text{open}}} = 10\frac{b}{m}v^2 - g \qquad (1)$$

Before the chute opened:

$$bv^2 - mg = ma_y$$

Under terminal speed conditions, $a_y = 0$ and:

$$bv_t^2 - mg = 0 \Rightarrow \frac{b}{m} = \frac{g}{v_t^2}$$

Substitute for b/m in equation (1) to obtain:

$$a_{\substack{\text{chute}\\\text{open}}} = 10\left(g\frac{1}{v_t^2}\right)v^2 - g$$

$$= \left[10\left(\frac{1}{v_t^2}\right)v^2 - 1\right]g$$

Evaluating $a_{\substack{\text{chute}\\\text{open}}}$ for $v = v_t$ yields:

$$a_{\substack{\text{chute}\\\text{open}}} = \left[10\left(\frac{1}{v_t^2}\right)v_t^2 - 1\right]g = \boxed{9g}$$

(*b*) Because this acceleration exceeds the safe acceleration by 4*g*, this is not a safe practice.

135 • Find the location of the center of mass of the Earth–moon system relative to the center of Earth. Is it inside or outside the surface of Earth?

Picture the Problem We can use the definition of the location of the center of mass of a system of particles to find the location of the center of mass of the Earth-moon system. The following pictorial representation is, of course, not shown to scale.

The center of mass of the Earth-moon system is given by:

$$x_{cm} = \frac{m_E x_{cm,E} + m_m x_{cm,m}}{m_E + m_m}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{cm} = \frac{(5.98\times10^{24}\text{ kg})(0) + (7.36\times10^{22}\text{ kg})(3.84\times10^{8}\text{ m})}{5.98\times10^{24}\text{ kg} + 7.36\times10^{22}\text{ kg}} = \boxed{4.67\times10^{6}\text{ m}}$$

Because the radius of Earth is 6.37×10^6 m, the center of mass of the Earth-moon system is inside Earth.

136 •• A circular plate of radius R has a circular hole of radius $R/2$ cut out of it (Figure 5-86). Find the center of mass of the plate after the hole has been cut. *Hint: The plate can be modeled as two disks superimposed, with the hole modeled as a disk of negative mass.*

Picture the Problem By symmetry, $x_{cm} = 0$. Let σ be the mass per unit area of the disk. The mass of the modified disk is the difference between the mass of the whole disk and the mass that has been removed.

Start with the definition of y_{cm}:

$$y_{cm} = \frac{\sum_i m_i y_i}{M - m_{hole}} = \frac{m_{disk} y_{disk} - m_{hole} y_{hole}}{M - m_{hole}}$$

Express the mass of the complete disk:

$$M = \sigma A = \sigma\pi r^2$$

Express the mass of the material removed:

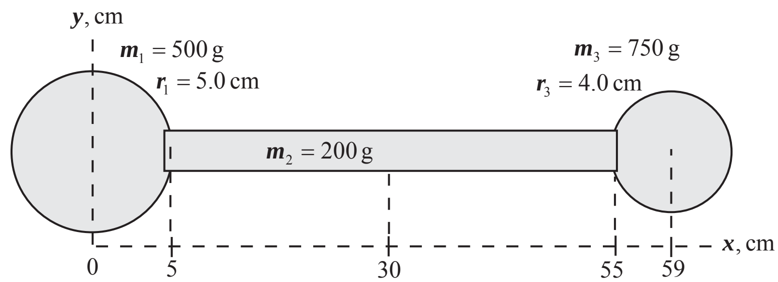
$$m_{hole} = \sigma\pi\left(\frac{r}{2}\right)^2 = \tfrac{1}{4}\sigma\pi r^2 = \tfrac{1}{4}M$$

Substitute and simplify to obtain:

$$y_{cm} = \frac{M(0) - \left(\frac{1}{4}M\right)\left(-\frac{1}{2}r\right)}{M - \frac{1}{4}M} = \boxed{\tfrac{1}{6}r}$$

137 •• An unbalanced baton consists of a 50-cm-long uniform rod of mass 200 g. At one end there is a 10-cm-diameter uniform solid sphere of mass 500 g, and at the other end there is a 8.0-cm-diameter uniform solid sphere of mass 750 g. (The center-to-center distance between the spheres is 59 cm.) (*a*) Where, relative to the center of the light sphere, is the center of mass of this baton? (*b*) If this baton is tossed straight up (but spinning) so that its initial center of mass speed is 10.0 m/s, what is the velocity of the center of mass 1.5 s later? (*c*) What is the net external force on the baton while in the air? (*d*) What is the baton's acceleration 1.5 s following its release?

Picture the Problem The pictorial representation summarizes the information concerning the unbalanced baton and shows a convenient choice for a coordinate system.



(*a*) The *x* coordinate of the center of mass of the unbalanced baton is given by:

$$x_{cm} = \frac{m_1 x_{cm,1} + m_2 x_{cm,2} + m_3 x_{cm,3}}{m_1 + m_2 + m_3}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{cm} = \frac{(500\text{ g})(0) + (200\text{ g})(30\text{ cm}) + (750\text{ g})(59\text{ cm})}{500\text{ g} + 200\text{ g} + 750\text{ g}} = \boxed{35\text{ cm}}$$

(*b*) Because the center of mass of the baton acts like a point particle, use a constant-acceleration equation to express is velocity as a function of time:

$$v(t) = v_{0,y} + a_y t$$

or, because $a_y = -g$,

$$v(t) = v_{0,y} - gt$$

Substitute numerical values and evaluate $v(1.5\text{ s})$:

$$\begin{aligned} v(1.5\text{ s}) &= 10.0\text{ m/s} - (9.81\text{ m/s}^2)(1.5\text{ s}) \\ &= -4.7\text{ m/s} \\ &= \boxed{4.7\text{ m/s downward}} \end{aligned}$$

(*c*) The net external force acting on the baton is the gravitational force (weight) exerted on it by Earth:

$$F_{net} = F_g = (m_1 + m_2 + m_3)g$$

Substitute numerical values and evaluate F_{net}:

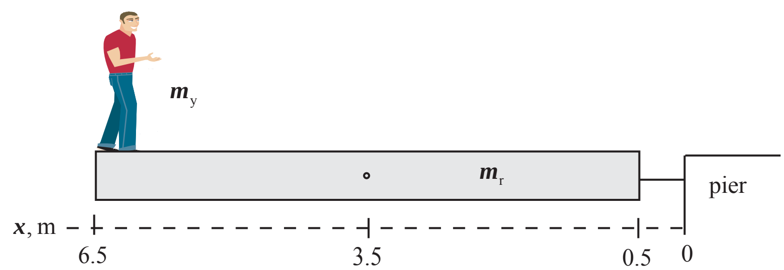
$$F_{net} = (500\text{ g} + 200\text{ g} + 750\text{ g})(9.81\text{ m/s}^2) = \boxed{14.2\text{ kN}}$$

(*d*) Again, because the center of mass acts like a point particle, its acceleration is that of a point particle in flight near the surface of Earth:

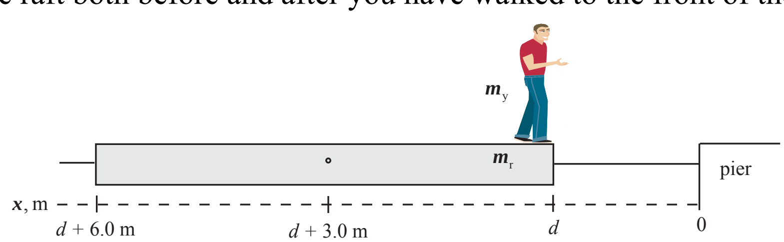
$$a_{cm} = g = \boxed{-9.81\text{ m/s}^2}$$

138 •• You are standing at the very rear of a 6.0-m-long, 120-kg raft that is at rest in a lake with its prow only 0.50 m from the end of the pier (Figure 5-87). Your mass is 60 kg. Neglect frictional forces between the raft and the water. (*a*) How far from the end of the pier is the center of mass of the you-raft system? (*b*) You walk to the front of the raft and then stop. How far from the end of the pier is the center of mass now? (*c*) When you are at the front of the raft, how far are you from the end of the pier?

Picture the Problem Let the origin be at the edge of the pier and the +*x* direction be to the right as shown in the pictorial representation immediately below.



In the following pictorial representation, *d* is the distance of the end of the raft from the pier after you have walked to its front. The raft moves to the left as you move to the right; with the center of mass of the you-raft system remaining fixed (because $F_{ext,net} = 0$). The diagram shows the initial ($x_{y,i}$) and final ($x_{y,f}$) positions of yourself as well as the initial ($x_{r\_cm,i}$) and final ($x_{r\_cm,f}$) positions of the center of mass of the raft both before and after you have walked to the front of the raft.



(*a*) x_{cm} before you walk to the front of the raft is given by:

$$x_{cm} = \frac{m_y x_{y,i} + m_r x_{r\_cm,i}}{m_y + m_r}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{cm}=\frac{(60\ \text{kg})(6.5\ \text{m})+(120\ \text{kg})(3.5\ \text{m})}{60\ \text{kg}+120\ \text{kg}}=\boxed{4.5\ \text{m}}$$

(*b*) Because $F_{ext,net}=0$, the center of mass remains fixed:

$$x'_{cm}=x_{cm}=\boxed{4.5\ \text{m}}$$

(*c*) Call the distance of the raft from the pier after you have walked to the front of the raft d and express the new location x'_{cm} of the center of mass of the system:

$$x'_{cm}=\frac{m_y x_{y,f}+m_r x_{r\_cm,f}}{m_y+m_r}=\frac{m_y d+m_r(d+3.0\ \text{m})}{m_y+m_r}=4.5\ \text{m}$$

Solving for d yields:

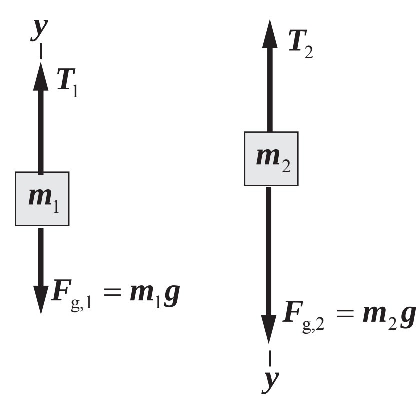
$$d=\frac{(4.5\ \text{m})m_y+(1.5\ \text{m})m_r}{m_y+m_r}$$

Substitute numerical values and evaluate x_{cm}:

$$d=\frac{(4.5\ \text{m})(60\ \text{kg})+(1.5\ \text{m})(120\ \text{kg})}{60\ \text{kg}+120\ \text{kg}}=\boxed{2.5\ \text{m}}$$

139 •• An Atwood's machine that has a frictionless massless pulley and massless strings has a 2.00-kg object hanging from one side and 4.00-kg object hanging from the other side. (*a*) What is the speed of each object 1.50 s after they are released from rest? (*b*) At that time, what is the velocity of the center of mass of the two objects? (*c*) At that time, what is the acceleration of the center of mass of the two objects?

Picture the Problem The forces acting on the 2.00-kg (m_1) and 4.00-kg (m_2) objects are shown in the free-body diagrams to the right. Note that the two objects have the same acceleration and common speeds. We can use constant-acceleration equations and Newton's second law in the analysis of the motion of these objects.



(*a*) Use a constant-acceleration equation to relate the speed of the objects to their common acceleration:

$$v(t) = v_{0y} + a_y t$$

or, because $v_{0y} = 0$,

$$v(t) = a_y t \qquad (1)$$

Apply $\sum F_y = ma_y$ to the object whose mass is m_1:

$$T_1 - m_1 g = m_1 a_y$$

Apply $\sum F_y = ma_y$ to the object whose mass is m_2:

$$m_2 g - T_2 = m_2 a_y$$

Because $T_1 = T_2$, adding these equations yields:

$$m_2 g - m_1 g = m_2 a_y + m_1 a_y$$

Solve for a_y to obtain:

$$a_y = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g \qquad (2)$$

Substituting for a_y in equation (1) yields:

$$v(t) = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g\,t$$

Substitute numerical values and evaluate $v(1.50\ \text{s})$:

$$v(1.50\ \text{s}) = \left(\frac{4.00\ \text{kg} - 2.00\ \text{kg}}{4.00\ \text{kg} + 2.00\ \text{kg}}\right)(9.81\ \text{m/s}^2)(1.50\ \text{s}) = 4.905\ \text{m/s} = \boxed{4.91\ \text{m/s}}$$

(*b*) The velocity of the center of mass is given by:

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{m_1 \vec{\boldsymbol{v}}_1 + m_2 \vec{\boldsymbol{v}}_2}{m_1 + m_2}$$

Substitute numerical values and evaluate $v_{\text{cm}}(1.50\ \text{s})$:

$$v_{\text{cm}}(1.50\ \text{s}) = \frac{(2.00\ \text{kg})(4.905\ \text{m/s})\hat{j} + (4.00\ \text{kg})(-4.905\ \text{m/s})\hat{j}}{2.00\ \text{kg} + 4.00\ \text{kg}} = (-1.64\ \text{m/s})\hat{j}$$

The velocity of the center of mass is $\boxed{\text{1.64 m/s downward.}}$

(*c*) The acceleration of the center of mass is given by:

$$\vec{\boldsymbol{a}}_{\text{cm}} = \frac{m_1 \vec{\boldsymbol{a}}_1 + m_2 \vec{\boldsymbol{a}}_2}{m_1 + m_2}$$

Substituting for a_y from equation (2) and simplifying yields:

$$\vec{\boldsymbol{a}}_{\text{cm}} = \frac{m_1\left(\dfrac{m_2 - m_1}{m_2 + m_1}\right)g\,\hat{\boldsymbol{j}} + m_2\left(-\dfrac{m_2 - m_1}{m_2 + m_1}\right)g\,\hat{\boldsymbol{j}}}{m_1 + m_2} = -\left(\frac{m_2 - m_1}{m_2 + m_1}\right)^2 g\,\hat{\boldsymbol{j}}$$

Substitute numerical values and evaluate a_{cm}:

$$\vec{\boldsymbol{a}}_{\text{cm}} = -\left(\frac{4.00\text{ kg} - 2.00\text{ kg}}{4.00\text{ kg} + 2.00\text{ kg}}\right)^2\left(9.81\text{ m/s}^2\right)\hat{\boldsymbol{j}} = \left(-1.09\text{ m/s}^2\right)\hat{\boldsymbol{j}}$$

The acceleration of the center of mass is $\boxed{1.09\text{ m/s}^2\text{ downward.}}$

Chapter 6
Work and Kinetic Energy

Conceptual Problems

1 • True or false: (*a*) If the net or total work done on a particle was not zero, then its speed must have changed. (*b*) If the net or total work done on a particle was not zero, then its velocity must have changed. (*c*) If the net or total work done on a particle was not zero, then its direction of motion could not have changed. (*d*) No work is done by the forces acting on a particle if it remains at rest. (*e*) A force that is always perpendicular to the velocity of a particle never does work on the particle.

Determine the Concept A force does work on an object when its point of application moves through some distance and there is a component of the force along the line of motion.

(*a*) True. The total work done on a particle is equal to the change in its kinetic energy (the work-kinetic energy theorem).

(*b*) True. The total work done on the particle changes the kinetic energy of the particle, which means that its velocity must change.

(*c*) False. If the total work done on the particle is negative, it could be decreasing the particle's speed and, eventually, reversing its direction of motion.

(*d*) True. The object could be at rest in one reference frame and moving in another. If we consider only the frame in which the object is at rest, then, because it must undergo a displacement in order for work to be done on it, we would conclude that the statement is true.

(*e*) True. A force that is always perpendicular to the velocity of a particle changes the direction the particle is moving but does no work on the particle.

2 • You push a heavy box in a straight line along the top of a rough horizontal table. The box starts at rest and ends at rest. Describe the work done on it (including sign) by each force acting on it and the net work done on it.

Determine the Concept The normal and gravitational forces do zero work because they act at right angles to the box's direction of motion. You do positive work on the box and friction does negative work. Overall the net work is zero because its kinetic energy doesn't change (zero to zero).

3 • You are riding on a Ferris wheel that is rotating at constant speed. True or false: During any fraction of a revolution: (*a*) None of the forces acting on you does work on you. (*b*) The total work done by all forces acting on you is zero. (*c*) There is zero net force on you. (*d*) You are accelerating.

(*a*) False. Both the force exerted by the surface on which you are sitting and the gravitational force acting on you do work on you.

(*b*) False. Because you are rotating at constant speed, your kinetic energy does not change. However, since your height is continuously changing, your potential energy also changes continuously.

(*c*) False. Because you are continuously changing direction, and therefore accelerating, there is a net force acting on you that does work on you.

(*d*) True. Because the direction you are moving is continually changing, your velocity is continually changing and, hence, you are experiencing acceleration.

4 • By what factor does the kinetic energy of a particle change if its speed is doubled but its mass is cut in half?

Determine the Concept The kinetic energy of a particle is proportional to the square of its speed. Because $K = \frac{1}{2}mv^2$, replacing v by $2v$ and m by $\frac{1}{2}m$ yields

$$K' = \tfrac{1}{2}\left(\tfrac{1}{2}m\right)(2v)^2 = 2\left(\tfrac{1}{2}mv^2\right) = 2K\,.$$

Thus doubling the speed of a particle and halving its mass doubles its kinetic energy.

5 • Give an example of a particle that has constant kinetic energy but is accelerating. Can a non-accelerating particle have a changing kinetic energy? If so, give an example.

Determine the Concept A particle moving along a circular path at constant speed has constant kinetic energy but is accelerating (because its velocity is continually changing). No, because if the particle is not accelerating, the net force acting on it must be zero and, consequently, no work is done on it and its kinetic energy must be constant.

6 • A particle initially has kinetic energy *K*. Later it is found to be moving in the opposite direction with three times its initial speed. What is the kinetic energy now? (*a*) *K*, (*b*) 3*K*, (*c*) 23*K*, (*d*) 9*K*, (*e*) –9*K*

Determine the Concept The kinetic energy of a particle is proportional to the square of its speed and is always positive. Because $K = \frac{1}{2}mv^2$, replacing v by $3v$

yields $K' = \frac{1}{2}m(3v)^2 = 9\left(\frac{1}{2}mv^2\right) = 9K$. Hence tripling the speed of a particle increases its kinetic energy by a factor of 9 and $\boxed{(d)}$ is correct.

7 • **[SSM]** How does the work required to stretch a spring 2.0 cm from its unstressed length compare with the work required to stretch it 1.0 cm from its unstressed length?

Determine the Concept The work required to stretch or compress a spring a distance x is given by $W = \frac{1}{2}kx^2$ where k is the spring's stiffness constant. Because $W \propto x^2$, doubling the distance the spring is stretched will require four times as much work.

8 • A spring is first stretched 2.0 cm from its unstressed length. It is then stretched an additional 2.0 cm. How does the work required for the second stretch compare to the work required for the first stretch (give a ratio of second to first)?

Picture the Problem The work required to stretch or compress a spring a distance x is given by $W = \frac{1}{2}kx^2$ where k is the spring's stiffness constant.

Letting x_1 represent the length of the first stretch, express the work (the energy stored in the spring after the stretch) done on the spring during the first stretch:

$$W_1 = \tfrac{1}{2}kx_1^2$$

The work done in stretching the spring from x_1 to x_2 is given by:

$$\begin{aligned} W_2 &= \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}k(2x_1)^2 - \tfrac{1}{2}kx_1^2 \\ &= 4\left(\tfrac{1}{2}kx_1^2\right) - \tfrac{1}{2}kx_1^2 = 3\left(\tfrac{1}{2}kx_1^2\right) \end{aligned}$$

Express the ratio of W_2 to W_1 to obtain:

$$\frac{W_2}{W_1} = \frac{3\left(\frac{1}{2}kx_1^2\right)}{\frac{1}{2}kx_1^2} = \boxed{3}$$

9 • The dimension of power is (*a*) M·L^2·T^2, (*b*) M·L^2/T, (*c*) M·L^2/T^2, (*d*) M·L^2/T^3.

Determine the Concept We can use the definition of power as the scalar product of force and velocity to express the dimension of power.

Power is defined to be the product of force and velocity:

$$P = \vec{F} \cdot \vec{v}$$

Express the dimension of force:

$$\frac{\mathrm{M \cdot L}}{\mathrm{T}^2}$$

Express the dimension of velocity: $\dfrac{\text{L}}{\text{T}}$

Express the dimension of power in terms of those of force and velocity: $\dfrac{\text{M}\cdot\text{L}}{\text{T}^2}\cdot\dfrac{\text{L}}{\text{T}}=\dfrac{\text{M}\cdot\text{L}^2}{\text{T}^3}\Rightarrow\boxed{(d)}$ is correct.

10 • Show that the SI units of the force constant of a spring can be written as kg/s^2 .

Picture the Problem We can use the relationship $F = kx$ to establish the SI units of *k*.

Solve $F = kx$ for *k* to obtain: $k=\dfrac{F}{x}$

Substitute the units of *F*/*x* and simplify to obtain: $\dfrac{\text{N}}{\text{m}}=\dfrac{\dfrac{\text{kg}\cdot\text{m}}{\text{s}^2}}{\text{m}}=\boxed{\text{kg}/\text{s}^2}$

11 • True or false: (*a*) The gravitational force cannot do work on an object, because it is not a contact force. (*b*) Static friction can never do work on an object. (*c*) As a negatively charged electron in an atom is pulled from a positively charged nucleus, the force between the electron and the nucleus does work that has a positive value. (*d*) If a particle is moving along a circular path, the total work being done on it is necessarily zero.

Comment: DAVID: The answer to part (d) is now FALSE.

(*a*) False. The definition of work is not limited to displacements caused by contact forces. Consider the work done by the gravitational force on an object in freefall.

(*b*) False. The force that Earth exerts on your foot as you walk (or run) is a static friction force. Another example is the force that Earth exerts on the tire mounted on a drive wheel of an automobile.

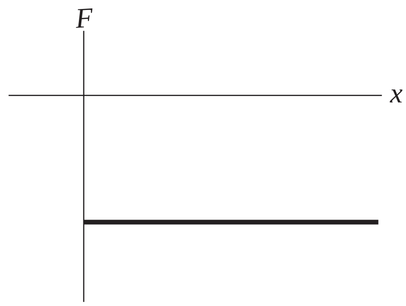
(*c*) False. The direction of the electric force on the electron during its removal from the atom is opposite that of the displacement of the electron.

(*d*) False. If the particle is accelerating, there must be a net force acting on it and, hence, work is done on it.

12 •• A hockey puck has an initial velocity in the +*x* direction on a horizontal sheet of ice. Qualitatively sketch the force-versus-position graph for the (constant) horizontal force that would need to act on the puck to bring it to rest. Assume that the puck is located at *x* = 0 when the force begins to act. Show

that the sign of the area under the curve agrees with the sign of the change in the puck's kinetic energy and interpret this in terms of the work–kinetic–energy theorem.

Determine the Concept The graph of the force F acting on the puck as a function of its position x is shown to the right. Note that, because the force is negative, the area bounded by it and the x axis is negative and, hence, the net work done by the force is negative. In accordance with the work-kinetic energy theorem, the change in kinetic energy is negative and the puck loses all of its initial kinetic energy.



13 •• [SSM] True or false: (*a*) The scalar product cannot have units. (*b*) If the scalar product of two nonzero vectors is zero, then they are parallel. (*c*) If the scalar product of two nonzero vectors is equal to the product of their magnitudes, then the two vectors are parallel. (*d*) As an object slides up an incline, the sign of the scalar product of the force of gravity on it and its displacement is negative.

(*a*) False. Work is the scalar product of force and displacement.

(*b*) False. Because $\vec{A}\cdot\vec{B} = AB\cos\theta,$ where θ is the angle between $\vec{A}$ and $\vec{B},$ if the scalar product of the vectors is zero, then θ must be 90° (or some odd multiple of 90°) and the vectors are perpendicular.

(*c*) True. Because $\vec{A}\cdot\vec{B} = AB\cos\theta,$ where θ is the angle between $\vec{A}$ and $\vec{B},$ if the scalar product of the vectors is equal to the product of their magnitudes, then θ must be 0° and the vectors are parallel.

(*d*) True. Because the angle between the gravitational force and the displacement of the object is greater than 90°, its cosine is negative and, hence, the scalar product is negative.

14 •• (*a*) Must the scalar product of two perpendicular unit vectors always be zero? If not, give an example. (*b*) An object has a velocity $\vec{v}$ at some instant. Interpret $\sqrt{\vec{v}\cdot\vec{v}}$ physically. (*c*) A ball rolls off a horizontal table. What is the scalar product between its velocity and its acceleration the instant after it leaves the table? Explain. (*d*) In Part (*c*), what is the sign of the scalar product of its velocity and acceleration the instant before it impacts the floor?

Determine the Concept
(a) Yes. Any unit vectors that are not perpendicular to each other (as are $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$) will have a scalar product different from zero.

(b) $\sqrt{\vec{v}\cdot\vec{v}}=\sqrt{v^2\cos 0°}=v$ is the object's speed.

(c) Zero, because at the instant the ball leaves the horizontal table, $\vec{v}$ and $\vec{a}(=\vec{g})$ are perpendicular.

(d) Positive, because the final velocity of the ball has a downward component in the direction of the acceleration.

15 •• You lift a package vertically upward a distance L in time Δt. You then lift a second package that has twice the mass of the first package vertically upward the same distance while providing the same power as required for the first package. How much time does lifting the second package take (answer in terms of Δt)?

Picture the Problem Power is the rate at which work is done.

Express the power you exert in lifting the package one meter in Δt seconds:

$$P_1=\frac{W_1}{\Delta t_1}=\frac{W_1}{\Delta t}$$

Express the power you develop in lifting a package of twice the mass one meter in t seconds:

$$P_2=\frac{W_2}{\Delta t_2}=\frac{2W_1}{\Delta t_2}$$

Because you exert the same power in lifting both packages:

$$\frac{W_1}{\Delta t}=\frac{2W_1}{\Delta t_2}\Rightarrow \Delta t_2=\boxed{2\Delta t}$$

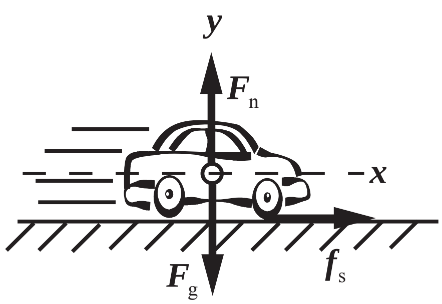
16 •• There are lasers that output more than 1.0 GW of power. A typical large modern electric generation plant typically produces 1.0 GW of electrical power. Does this mean the laser outputs a huge amount of energy? Explain. *Hint: These high-power lasers are pulsed on and off, so they are not outputting power for very long time intervals.*

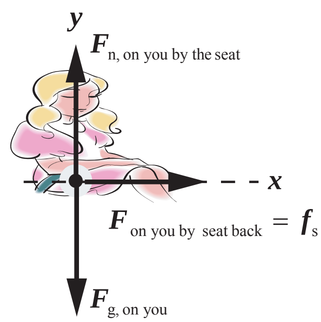
Determine the Concept No, the laser power may only last for a short time interval.

17 •• **[SSM]** You are driving a car that accelerates from rest on a level road without spinning its wheels. Use the center-of-mass work–translational-kinetic-energy relation and free-body diagrams to clearly explain which force (or

forces) is (are) directly responsible for the gain in translational kinetic energy of both you and the car. *Hint: The relation refers to external forces only, so the car's engine is not the answer. Pick your ″system″ correctly for each case.*

Determine the Concept The car shown in the free-body diagram is accelerating in the $+x$ direction, as are you (shown to the right). The only external force (neglecting air resistance) that does center-of-mass work on the car is the static friction force $\vec{f}_s$ exerted by the road on the tires. The positive center-of-mass work this friction force does is translated into a gain of kinetic energy.





Estimation and Approximation

18 •• [SSM] (*a*) Estimate the work done on you by gravity as you take an elevator from the ground floor to the top of the Empire State Building, a building 102 stories high. (*b*) Estimate the amount of work the normal force of the floor did on you. *Hint: The answer is not zero.* (*c*) Estimate the average power of the force of gravity.

Picture the Problem According to the work-kinetic energy theorem, the work done on you by gravity and the normal force exerted by the floor equals your change in kinetic energy.

(*a*) The definition of work is:

$$W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} \qquad (1)$$

Assuming a coordinate system in which the upward direction is the $+y$ direction, $\vec{F}$ and $d\vec{s}$ are given by:

$$\vec{F} = -mg\,\hat{j}$$

and

$$d\vec{s} = dy\,\hat{j}$$

Substitute for $\vec{F}$ and $d\vec{s}$ in equation (1) to obtain:

$$W_{\text{by gravity}} = \int_0^h -mg\,\hat{j} \cdot dy\,\hat{j} = -mg\int_0^h dy$$

where h is the height of the Empire State building.

Evaluating this integral yields:

$$W_{\text{by gravity}} = -mgh$$

Assume that your mass is 70 kg and that the height of the Empire State building is 300 m (102 floors). Then:

$$W_{\text{by gravity}} = -(70\text{ kg})(9.81\text{ m/s}^2)(300\text{ m}) = -2.060\times10^5\text{ J} \approx \boxed{-2.1\times10^5\text{ J}}$$

(*b*) Apply the work-kinetic energy theorem to obtain:

$$W_{\text{by gravity}} + W_{\text{by floor}} = \Delta K$$

or, because your change in kinetic energy is zero,

$$W_{\text{by gravity}} + W_{\text{by floor}} = 0$$

Solve for $W_{\text{by floor}}$ to obtain:

$$W_{\text{by floor}} = -W_{\text{by gravity}}$$

Substituting for $W_{\text{by gravity}}$ yields:

$$W_{\text{by floor}} \approx -(-2.060\times10^5\text{ J}) = \boxed{2.1\times10^5\text{ J}}$$

(*c*) If it takes 1 min to ride the elevator to the top floor, then the average power of the force of gravity is:

$$P_{\text{av}} = \frac{\Delta W}{\Delta t} = \frac{2.060\times10^5\text{ J}}{60\text{ s}} = \boxed{3.4\text{ kW}}$$

Remarks: Some books say the normal force cannot do work. Having solved this problem, what do you think about that statement?

19 •• The nearest stars, apart from the Sun, are light-years away from Earth. If we are to investigate these stars, our space ships will have to travel at an appreciable fraction of the speed of light. (*a*) You are in charge of estimating the energy required to accelerate a 10,000-kg capsule from rest to 10 percent of the speed of light in one year. What is the minimum amount of energy that is required? Note that at velocities approaching the speed of light, the kinetic energy formula $\frac{1}{2}mv^2$ is not correct. However, it gives a value that is within 1% of the correct value for speeds up to 10% of the speed of light. (*b*) Compare your estimate to the amount of energy that the United States uses in a year (about 5×10^{20} J). (*c*) Estimate the minimum average power required of the propulsion system.

Comment: DAVID: Wording changes will likely impact the solution.

Picture the Problem We can find the kinetic energy *K* of the spacecraft from its definition and compare its energy to the annual consumption in the United States by examining the ratio *K*/*E*. Finally, we can use the definition of power to find the average power required of the engines.

(*a*) The work required by the propulsion system equals the kinetic energy of the spacecraft:

$$W = \Delta K = K_f = \tfrac{1}{2}m(0.10c)^2$$

Substitute numerical values and evaluate *W*:

$$W = \tfrac{1}{2}(10^4\text{ kg})[(0.10)(2.998\times10^8\text{ m/s})]^2 = \boxed{4.5\times10^{18}\text{ J}}$$

(*b*) Express this amount of energy as a percentage of the annual consumption in the United States:

$$\frac{K}{E} \approx \frac{4.5\times10^{18}\text{ J}}{5\times10^{20}\text{ J}} \approx \boxed{1\%}$$

(*c*) The average power is the rate at which work is done (or energy is delivered):

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{\Delta K}{\Delta t}$$

Substitute numerical values and evaluate $P_{av,\,min}$:

$$P_{av,\,min} = \frac{4.5\times10^{18}\text{ J}}{1\text{ y}\times\dfrac{3.156\times10^7\text{ s}}{\text{y}}} = \boxed{1.4\times10^{11}\text{ W}}$$

20 •• The mass of the Space Shuttle orbiter is about 8×10^4 kg and the period of its orbit is 90 min. Estimate the kinetic energy of the orbiter and the work done on it by gravity between launch and orbit. (Although the force of gravity decreases with altitude, this affect is small in low-Earth orbit. Use this fact to make the necessary approximation; you do not need to do an integral.) The orbits are about 250 miles above the surface of Earth.

Picture the Problem We can find the orbital speed of the Shuttle from the radius of its orbit and its period and its kinetic energy from $K = \tfrac{1}{2}mv^2$. We'll ignore the variation in the acceleration due to gravity to estimate the change in the potential energy of the orbiter between its value at the surface of Earth and its orbital value.

Apply the work-kinetic energy theorem to the orbiter to obtain:

$$W_{by\ gravity} = \Delta K = K_{orbital} - K_{launch}$$

or, because $K_{launch} = 0$,

$$W_{by\ gravity} = K_{orbital} \quad (1)$$

The kinetic energy of the orbiter in orbit is given by:

$$K_{orbital} = \tfrac{1}{2}mv^2 \quad (2)$$

Relate the orbital speed of the orbiter to its radius *r* and period *T*:

$$v = \frac{2\pi r}{T}$$

Substitute for v in equation (2) and simplify to obtain:

$$K_{\text{orbital}} = \tfrac{1}{2}m\left(\frac{2\pi r}{T}\right)^2 = \frac{2\pi^2 mr^2}{T^2}$$

Substitute numerical values and evaluate K_{orbital}:

$$K_{\text{orbital}} = \frac{2\pi^2(8\times10^4\text{ kg})[(250\text{ mi}+3960\text{ mi})(1.609\text{ km/mi})]^2}{[(90\text{ min})(60\text{ s/min})]^2} \approx \boxed{2\text{ TJ}}$$

Substitute for K_{orbital} in equation (1) to obtain:

$$W_{\text{by gravity}} \approx \boxed{2\text{ TJ}}$$

21 • Ten inches of snow have fallen during the night, and you must shovel out your 50-ft-long driveway (Figure 6-29). Estimate how much work you do on the snow by completing this task. Make a plausible guess of any value(s) needed (the width of the driveway, for example), and state the basis for each guess.

Picture the Problem Let's assume that the width of the driveway is 18 ft. We'll also assume that you lift each shovel full of snow to a height of 1 m, carry it to the edge of the driveway, and drop it. We'll ignore the fact that you must slightly accelerate each shovel full as you pick it up and as you carry it to the edge of the driveway. While the density of snow depends on the extent to which it has been compacted, one liter of freshly fallen snow is approximately equivalent to 100 mL of water.

Express the work you do in lifting the snow a distance h:

$$W = \Delta U = mgh = \rho_{\text{snow}}V_{\text{snow}}gh$$

where ρ is the density of the snow.

Using its definition, express the densities of water and snow:

$$\rho_{\text{snow}} = \frac{m_{\text{snow}}}{V_{\text{snow}}} \text{ and } \rho_{\text{water}} = \frac{m_{\text{water}}}{V_{\text{water}}}$$

Divide the first of these equations by the second to obtain:

$$\frac{\rho_{\text{snow}}}{\rho_{\text{water}}} = \frac{V_{\text{water}}}{V_{\text{snow}}} \Rightarrow \rho_{\text{snow}} = \rho_{\text{water}}\frac{V_{\text{water}}}{V_{\text{snow}}}$$

Substitute numerical values and evaluate ρ_{snow}:

$$\rho_{\text{snow}} = (10^3\text{ kg/m}^3)\frac{100\text{ mL}}{\text{L}} = 100\text{ kg/m}^3$$

Calculate the volume of snow covering the driveway:

$$\begin{aligned} V_{\text{snow}} &= (50\text{ ft})(18\text{ ft})\left(\frac{10}{12}\text{ ft}\right) \\ &= 750\text{ ft}^3 \times \frac{28.32\text{ L}}{\text{ft}^3} \times \frac{10^{-3}\text{ m}^3}{\text{L}} \\ &= 21.2\text{ m}^3 \end{aligned}$$

Substitute numerical values in the expression for W to obtain an estimate (a lower bound) for the work you would do on the snow in removing it:

$$W = (100\,\text{kg/m}^3)(21.2\,\text{m}^3)(9.81\,\text{m/s}^2)(1\,\text{m}) = \boxed{21\,\text{kJ}}$$

Remarks: Note that the work done on the snow is zero when it is carried to the end of the driveway because F_{applied} is perpendicular to the displacement.

Work, Kinetic Energy and Applications

22 • A 15-g piece of space junk has a speed of 1.2 km/s. (*a*) What is its kinetic energy? (*b*) What is its kinetic energy if its speed is halved? (*c*) What is its kinetic energy if its speed is doubled?

Comment: DAVID: Please purge "bullet" from your solution.

Picture the Problem We can use $\frac{1}{2}mv^2$ to find the kinetic energy of the piece of space junk.

(*a*) The kinetic energy of the piece of space junk is given by:

$$K = \tfrac{1}{2}mv^2$$

Substitute numerical values and evaluate K:

$$K = \tfrac{1}{2}(0.015\,\text{kg})(1.2\,\text{km/s})^2 = \boxed{11\,\text{kJ}}$$

(*b*) Because $K \propto v^2$:

$$K' = \tfrac{1}{4}K = \boxed{2.7\,\text{kJ}}$$

(*c*) Because $K \propto v^2$:

$$K' = 4K = \boxed{43\,\text{kJ}}$$

23 • Find the kinetic energy of (*a*) a 0.145-kg baseball moving with a speed of 45.0 m/s, and (*b*) a 60.0-kg jogger running at a steady pace of 9.00 min/mi.

Picture the Problem We can use $\frac{1}{2}mv^2$ to find the kinetic energy of the baseball and the jogger.

(*a*) Use the definition of K to find the kinetic energy of the baseball:

$$K_{\text{baseball}} = \tfrac{1}{2}(0.145\,\text{kg})(45.0\,\text{m/s})^2 = \boxed{147\,\text{J}}$$

(*b*) Convert the jogger's pace of 9.00 min/mi into a speed:

$$v = \left(\frac{1\,\text{mi}}{9.00\,\text{min}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right)\left(\frac{1609\,\text{m}}{1\,\text{mi}}\right) = 2.98\,\text{m/s}$$

Use the definition of K to find the jogger's kinetic energy:

$$K_{\text{jogger}} = \tfrac{1}{2}(60.0\,\text{kg})(2.98\,\text{m/s})^2 = \boxed{266\,\text{J}}$$

24 • A 6.0-kg box is raised a distance of 3.0 m from rest by a vertical applied force of 80 N. Find (*a*) the work done on the box by the applied force, (*b*) the work done on the box by gravity, and (*c*) the final kinetic energy of the box.

Picture the Problem The work done by the force acting on the box is the scalar product of the force producing the displacement and the displacement of the box. Because the weight of an object is the gravitational force acting on it and this force acts downward, the work done by gravity is negative. We can use the work-kinetic energy theorem to find the final kinetic energy of the box.

(*a*) Use the definition of work to obtain:

$$W_{\text{on the box}} = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{y}} \qquad (1)$$

$\vec{\boldsymbol{F}}$ and $\Delta\vec{\boldsymbol{y}}$ are given by:

$$\vec{\boldsymbol{F}} = (80\,\text{N})\hat{\boldsymbol{j}}$$

and

$$\Delta\vec{\boldsymbol{y}} = (3.0\,\text{m})\hat{\boldsymbol{j}}$$

Substitute for $\vec{\boldsymbol{F}}$ and $\Delta\vec{\boldsymbol{y}}$ in equation (1) and evaluate $W_{\text{on the box}}$:

$$W_{\text{on the box}} = (80\,\text{N})\hat{\boldsymbol{j}} \cdot (3.0\,\text{m})\hat{\boldsymbol{j}} = \boxed{0.24\,\text{kJ}}$$

(*b*) Apply the definition of work again to obtain:

$$W_{\text{by gravity}} = \vec{\boldsymbol{F}}_{\text{g}} \cdot \Delta\vec{\boldsymbol{y}} \qquad (2)$$

$\vec{\boldsymbol{F}}$ and $\Delta\vec{\boldsymbol{y}}$ are given by:

$$\vec{\boldsymbol{F}}_{\text{g}} = -mg\hat{\boldsymbol{j}} = -(6.0\,\text{kg})(9.81\,\text{m/s}^2)\hat{\boldsymbol{j}} = (-58.9\,\text{N})\hat{\boldsymbol{j}}$$

and

$$\Delta\vec{\boldsymbol{y}} = (3.0\,\text{m})\hat{\boldsymbol{j}}$$

Substituting for $\vec{\boldsymbol{F}}$ and $\Delta\vec{\boldsymbol{y}}$ and simplifying yields:

$$W_{\text{by gravity}} = (-58.9\,\text{N})\hat{\boldsymbol{j}} \cdot (3.0\,\text{m})\hat{\boldsymbol{j}} = \boxed{-0.18\,\text{kJ}}$$

(*c*) According to the work-kinetic energy theorem:

$$\Delta K = W_{\text{on the box}} + W_{\text{by gravity}}$$

or, because $K_{\text{i}} = 0$,

$$K_{\text{f}} = W_{\text{on the box}} + W_{\text{by gravity}}$$

Substitute numerical values and evaluate K_f:

$$K_f = 0.24\ \text{kJ} - 0.18\ \text{kJ} = \boxed{0.06\ \text{kJ}}$$

25 • A constant 80-N force acts on a 5.0 kg box. The box initially is moving at 20 m/s in the direction of the force, and 3.0 s later the box is moving at 68 m/s. Determine both the work done by this force and the average power delivered by the force during the 3.0-s interval.

Comment: DAVID: The new wording give 2 sig figs to the time interval.

Comment: AU: OK to add for clarity?

Picture the Problem The constant force of 80 N is the net force acting on the box and the work it does is equal to the *change* in the kinetic energy of the box. The power of the force is the rate at which it does work on the box.

Using the work-kinetic energy theorem, relate the work done by the constant force to the *change* in the kinetic energy of the box:

$$W = K_f - K_i = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right)$$

Substitute numerical values and evaluate W:

$$W = \tfrac{1}{2}(5.0\,\text{kg})\left[(68\,\text{m/s})^2 - (20\,\text{m/s})^2\right] = 10.6\,\text{kJ} = \boxed{11\,\text{kJ}}$$

The average power delivered by the force is given by:

$$P = \frac{\Delta W}{\Delta t}$$

Substitute numerical values and evaluate P:

$$P = \frac{10.6\ \text{kJ}}{3.0\ \text{s}} = \boxed{3.5\ \text{kW}}$$

26 •• You run a race with a friend. At first you each have the same kinetic energy, but she is running faster than you are. When you increase your speed by 25 percent, you are running at the same speed she is. If your mass is 85 kg, what is her mass?

Picture the Problem We can use the definition of kinetic energy to find the mass of your friend.

Using the definition of kinetic energy and letting ″1″ denote your mass and speed and ″2″ your girlfriend’s, express the equality of your kinetic energies:

$$\tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m_2v_2^2 \Rightarrow m_2 = m_1\left(\frac{v_1}{v_2}\right)^2$$

Express the condition on your speed that enables you to run at the same speed as your girlfriend:

$$v_2 = 1.25v_1$$

Substitute for v_2 and simplify to obtain:

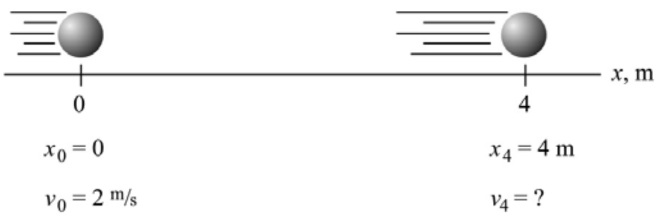
$$m_2 = m_1\left(\frac{v_1}{1.25v_1}\right)^2 = m_1\left(\frac{1}{1.25}\right)^2$$

Substitute the numerical value of m_1 and evaluate m_2:

$$m_2 = (85\,\text{kg})\left(\frac{1}{1.25}\right)^2 = \boxed{54\,\text{kg}}$$

27 •• [SSM] A 3.0-kg particle moving along the x axis has a velocity of +2.0 m/s as it passes through the origin. It is subjected to a single force, F_x, that varies with position, as shown in Figure 6-30. (*a*) What is the kinetic energy of the particle as it passes through the origin? (*b*) How much work is done by the force as the particle moves from x = 0.0 m to x = 4.0 m? (*c*) What is the speed of the particle when it is at x = 4.0 m?

Picture the Problem The pictorial representation shows the particle as it moves along the positive x axis. The particle's kinetic energy increases because work is done on it. We can calculate the work done on it from the graph of F_x vs. x and relate its kinetic energy when it is at x = 4.0 m to its kinetic energy when it was at the origin and the work done on it by using the work-kinetic energy theorem.



(*a*) Calculate the kinetic energy of the particle when it is at x = 0:

$$K_0 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(3.0\,\text{kg})(2.0\,\text{m/s})^2 = \boxed{6.0\,\text{J}}$$

(*b*) Because the force and displacement are parallel, the work done is the area under the curve. Use the formula for the area of a triangle to calculate the area under the $F(x)$ graph:

$$W_{0\to 4} = \tfrac{1}{2}(\text{base})(\text{altitude}) = \tfrac{1}{2}(4.0\,\text{m})(6.0\,\text{N}) = \boxed{12\,\text{J}}$$

(*c*) Express the kinetic energy of the particle at $x = 4.0$ m in terms of its speed and mass:

$$K_{4\,\text{s}} = \tfrac{1}{2}mv_{4\,\text{s}}^2 \Rightarrow v_{4\,\text{s}} = \sqrt{\frac{2K_{4\,\text{s}}}{m}} \quad (1)$$

Using the work-kinetic energy theorem, relate the work done on the particle to its *change* in kinetic energy:

$$W_{0\to4.0\,\text{m}} = K_{4\,\text{s}} - K_0$$

Solve for the particle's kinetic energy at $x = 4.0$ m:

$$K_{4\,\text{s}} = K_0 + W_{0\to4}$$

Substitute numerical values and evaluate $K_{4\,\text{s}}$:

$$K_{4\,\text{s}} = 6.0\,\text{J} + 12\,\text{J} = 18\,\text{J}$$

Substitute numerical values in equation (1) and evaluate v_4:

$$v_4 = \sqrt{\frac{2(18\,\text{J})}{3.0\,\text{kg}}} = \boxed{3.5\,\text{m/s}}$$

28 •• A 3.0-kg object moving along the *x* axis has a velocity of 2.4 m/s as it passes through the origin. It is acted on by a single force, F_x, that varies with *x*, as shown in Figure 6-31. (*a*) Find the work done by the force from $x = 0.0$ m to $x = 2.0$ m. (*b*) What is the kinetic energy of the object at $x = 2.0$ m? (*c*) What is the speed of the object at $x = 2.0$ m? (*d*) What is the work done on the object from $x = 0.0$ to $x = 4.0$ m? (*e*) What is the speed of the object at $x = 4.0$ m?

Picture the Problem The work done on an object can be determined by finding the area bounded by its graph of F_x as a function of *x* and the *x* axis. We can find the kinetic energy and the speed of the particle at any point by using the work-kinetic energy theorem.

(*a*) Express *W*, the area under the curve, in terms of the area of one square, A_{square}, and the number of squares *n*:

$$W_{0\to2\,\text{m}} = nA_{\text{square}} \quad (1)$$

Determine the work equivalent of one square:

$$A_{\text{square}} = (0.5\,\text{N})(0.25\,\text{m}) = 0.125\,\text{J}$$

Estimate the number of squares under the curve between $x = 0$ and $x = 2.0$ m:

$$n \approx 22$$

Substitute for n and A_{square} in equation (1) and evaluate W:

$$W_{0\to 2\,\text{m}} = (22)(0.125\text{ J}) = 2.75\text{ J} = \boxed{2.8\text{ J}}$$

(b) Relate the kinetic energy of the object at $x = 2.0$ m, K_2, to its initial kinetic energy, K_0, and the work that was done on it between $x = 0$ and $x = 2.0$ m:

$$K_{2\,\text{m}} = K_0 + W_{0\to 2\,\text{m}}$$

Substitute numerical values and evaluate $K_{2\,\text{m}}$:

$$K_{2\,\text{m}} = \tfrac{1}{2}(3.0\text{ kg})(2.4\text{ m/s})^2 + 2.75\text{ J} = 11.4\text{ J} = \boxed{11\text{ J}}$$

(c) The speed of the object at $x = 2.0$ m is given by:

$$v_{2\,\text{m}} = \sqrt{\frac{2K_{2\,\text{m}}}{m}}$$

Substitute numerical values and evaluate $v_{2\,\text{m}}$:

$$v = \sqrt{\frac{2(11.4\text{ J})}{3.0\text{ kg}}} = \boxed{2.8\text{ m/s}}$$

(d) Estimate the number of squares under the curve between $x = 0$ and $x = 4.0$ m:

$$n \approx 26$$

Substitute numerical values and evaluate $W_{0\to 4\,\text{m}}$:

$$W_{0\to 4\,\text{m}} = 26(0.125\text{ J}) = 3.25\text{ J} = \boxed{3.3\text{ J}}$$

(e) Relate the kinetic energy of the object at $x = 4.0$ m, K_4, to its initial kinetic energy, K_0, and the work that was done on it between $x = 0$ and $x = 4.0$ m:

$$K_4 = K_0 + W_{0\to 4\,\text{m}}$$

Substitute numerical values and evaluate K_4:

$$K_4 = \tfrac{1}{2}(3.0\text{ kg})(2.4\text{ m/s})^2 + 3.25\text{ J} = 11.9\text{ J}$$

The speed of the object at $x = 4.0$ m is given by:

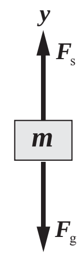
$$v_4 = \sqrt{\frac{2K_4}{m}}$$

Substitute numerical values and evaluate v_4:

$$v_4 = \sqrt{\frac{2(11.9\,\mathrm{J})}{3.0\,\mathrm{kg}}} = \boxed{2.8\,\mathrm{m/s}}$$

29 •• One end of a light spring (force constant k) is attached to the ceiling, the other end is attached to an object of mass m. The spring initially is vertical and unstressed. You then ″ease the object down″ to an equilibrium position a distance h below its initial position. Next, you repeat this experiment, but instead of easing the object down, you release it, with the result that it falls a distance H below the initial position before momentarily stopping. (a) Show that $h = mg/k$. (b) Use the work–kinetic-energy theorem to show that $H = 2h$. Try this experiment on your own.

Picture the Problem The free-body diagram shows the object at its equilibrium point. We can use Newton's second law under equilibrium conditions to show that $h = mg/k$ and the work-kinetic energy theorem to prove that $H = 2h$.



(a) Apply $\sum F_y = ma_y$ to the object in its equilibrium position:

$$F_s - F_g = 0$$

or, because $F_s = kh$ and $F_g = mg$,

$$kh - mg = 0 \Rightarrow h = \boxed{\frac{mg}{k}}$$

(b) Apply the work-kinetic energy theorem to the spring-object system to obtain:

$$W_{\text{net,ext}} = \Delta K$$

or, because the object begins and ends at rest,

$$W_{\text{net,ext}} = 0$$

The work done by the net external force is the sum of the work done by gravity and by the spring:

$$W_{\text{by gravity}} + W_{\text{by spring}} = 0$$

Because the force exerted on the object by the spring is upward and the displacement of the object is downward, $W_{\text{by spring}}$ is negative. Substituting for $W_{\text{by spring}}$ and $W_{\text{by gravity}}$ yields:

$$mgH - \tfrac{1}{2}kH^2 = 0 \Rightarrow H = \boxed{2h}$$

30 •• A force F_x acts on a particle that has a mass of 1.5 kg. The force is related to the position x of the particle by the formula $F_x = Cx^3$, where $C = 0.50$ if x is in meters and F_x is in newtons. (*a*) What are the SI units of C? (*b*) Find the work done by this force as the particle moves from x = 3.0 m to x = 1.5 m. (*c*) At x = 3.0 m, the force points opposite the direction of the particle's velocity (speed is 12.0 m/s). What is its speed at x =1.5 m? Can you tell its direction of motion at x =1.5 m using only the work-kinetic-energy theorem? Explain.

Comment: DAVID: Changes in numbers were made.

Picture the Problem The work done by this force as it displaces the particle is the area under the curve of F as a function of x. We can integrate F_x to find the work done on the particle and then use the work-kinetic energy theorem to find the particle's speed at x = 3.0 m.

(*a*) Solve the force equation for C to obtain:

$$C = \frac{F_x}{x^3} \text{ where } C \text{ has units of } \boxed{\frac{\text{N}}{\text{m}^3}}.$$

(*b*) Because F_x varies with position non-linearly, we need to express the work it does as an integral:

$$W_{\text{net, ext}} = \int \vec{\boldsymbol{F}}_x \cdot d\vec{\boldsymbol{x}} = \int -F_x dx$$

where the minus sign is a consequence of the fact that the direction of the force is opposite the displacement of the particle.

Substituting for F_x yields:

$$W_{\text{net, ext}} = \int_{3.0\,\text{m}}^{1.5\,\text{m}} -Cx^3 dx = -C\int_{3.0\,\text{m}}^{1.5\,\text{m}} x^3 dx$$

Evaluate the integral to obtain:

$$W_{\text{net, ext}} = -\left(0.50\,\frac{\text{N}}{\text{m}^3}\right)\left[\tfrac{1}{4}x^4\right]_{3.0\,\text{m}}^{1.5\,\text{m}} = -\tfrac{1}{4}\left(0.50\,\frac{\text{N}}{\text{m}^3}\right)\left[(1.5\,\text{m})^4 - (3.0\,\text{m})^4\right] = 9.492\text{ J}$$
$$= \boxed{9.5\text{ J}}$$

(*c*) Apply the work-kinetic-energy theorem to obtain:

$$W_{\text{net, ext}} = \Delta K = K_{1.5\,\text{m}} - K_{3.0\,\text{m}}$$

Substituting for $K_{1.5\,\text{m}}$ and $K_{3.0\,\text{m}}$ yields:

$$W_{\text{net, ext}} = \tfrac{1}{2}mv_{1.5\,\text{m}}^2 - \tfrac{1}{2}mv_{3.0\,\text{m}}^2$$

Solve for $v_{1.5\,\text{m}}$ to obtain:

$$v_{1.5\,\text{m}} = \sqrt{v_{3.0\,\text{m}}^2 + \frac{2W_{\text{net, ext}}}{m}} \qquad (1)$$

Substitute numerical values and evaluate $v_{3.0\,\text{m}}$:

$$v_{1.5\,\text{m}} = \sqrt{(12\ \text{m/s})^2 + \frac{2(9.49\ \text{J})}{1.5\ \text{kg}}} = \boxed{13\ \text{m/s}}$$

No. All we can conclude is that the particle gained kinetic energy.

31 •• You have a vacation cabin that has a nearby solar (black) water container used to provide a warm outdoor shower. For a few days last summer, your pump went out and you had to personally haul the water up the 4.0 m from the pond to the tank. Suppose your bucket has a mass of 5.0 kg and holds 15.0 kg of water when it is full. However, the bucket has a hole in it, and as you moved it vertically at a constant speed *v*, water leaked out at a constant rate. By the time you reached the top, only 5.0 kg of water remained. (*a*) Write an expression for the mass of the bucket plus water as a function of the height climbed above the pond surface. (*b*) Find the work done by you on the bucket for each 5.0 kg of water delivered to the tank.

Comment: DAVID: I halved all the masses because the original values were too large to be realisticl

Picture the Problem We can express the mass of the water in your bucket as the difference between its initial mass and the product of the rate at which it loses water and your position during your climb. Because you must do work against gravity in lifting and carrying the bucket, the work you do is the integral of the product of the gravitational field and the mass of the bucket as a function of its position.

(*a*) Express the mass of the bucket and the water in it as a function of its initial mass, the rate at which it is losing water, and your position, *y*, during your climb:

$$m(y) = 20.0\,\text{kg} - ry$$

Find the rate, $r = \dfrac{\Delta m}{\Delta y}$, at which your bucket loses water:

$$r = \frac{\Delta m}{\Delta y} = \frac{10.0\,\text{kg}}{4.0\,\text{m}} = 2.5\,\text{kg/m}$$

Substitute for *r* to obtain:

$$m(y) = \boxed{20.0\,\text{kg} - \left(2.5\frac{\text{kg}}{\text{m}}\right)y}$$

(b) Integrate the force you exert on the bucket, $m(y)g$, between the limits of $y = 0$ and $y = 4.0$ m:

$$W = g\int_0^{4.0\,\text{m}}\left(20.0\,\text{kg} - \left(2.5\frac{\text{kg}}{\text{m}}\right)y\right)dy$$

$$= \left(9.81\,\text{m/s}^2\right)\left[\left(20.0\,\text{kg}\right)y - \tfrac{1}{2}\left(2.5\,\text{kg/m}\right)y^2\right]_0^{4.0\,\text{m}}$$

$$= \boxed{0.59\,\text{kJ}}$$

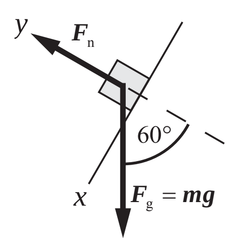
Remarks: We could also find the work you did on the bucket, at least approximately, by plotting a graph of *m*(*y*)*g* and finding the area under this curve between *y* = 0 and *y* = 4.0 m.

32 •• A 6.0-kg block slides 1.5 m down a frictionless incline that makes an angle of 60º with the horizontal. (a) Draw the free-body diagram of the block, and find the work done by each force when the block slides 1.5 m (measured along the incline). (b) What is the total work done on the block? (c) What is the speed of the block after it has slid 1.5 m, if it starts from rest? (d) What is its speed after 1.5 m, if it starts with an initial speed of 2.0 m/s?

Comment: DAVID: In 34 I changed 2.0 m to 1.5 m.

Picture the Problem We can use the definition of work as the scalar product of force and displacement to find the work done by each force acting on the block. When we have determined the work done on the block, we can use the work-kinetic energy theorem the find the speed of the block at any given location and for any initial kinetic energy it may have.

(a) The forces acting on the block are a gravitational force, exerted by Earth, that acts downward, and a normal force, exerted by the incline, that is perpendicular to the incline.



The work done by $\vec{F}_g$ is given by:

$$W_{F_g} = \vec{F}_g \cdot \vec{s} = F_g s\cos\theta$$

where $\vec{s}$ is the displacement of the block and θ is the angle between $\vec{F}_g$ and $\vec{s}$.

Substituting for F_g yields:

$$W_{F_g} = mgs\cos\theta$$

Substitute numerical values and evaluate W_{F_g}:

$$W_{F_g} = (6.0\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m})\cos 30° = 76.46\,\text{J} = \boxed{76\,\text{J}}$$

The work done by $\vec{F}_n$ is given by:

$$W_{F_n} = \vec{F}_n \cdot \vec{s} = F_n s \cos\theta$$

where $\vec{s}$ is the displacement of the block and θ is the angle between $\vec{F}_n$ and $\vec{s}$.

Substitute numerical values and evaluate W_{F_n}:

$$W_{F_n} = F_n(1.5\,\text{m})\cos 90° = \boxed{0}$$

(*b*) The total work done on the block is the sum of the work done by $\vec{F}_n$ and the work done by $\vec{F}_g$:

$$W_{\text{tot}} = W_{F_g} + W_{F_n}$$

Substitute numerical values and evaluate W_{tot}:

$$W_{\text{tot}} = 0 + 76\,\text{J} = \boxed{76\,\text{J}}$$

(*c*) Apply the work-kinetic energy theorem to the block to obtain:

$$W_{F_g} = \Delta K = K_f - K_i \qquad (1)$$

or, because $v_i = 0$,

$$W_{F_g} = K_f$$

Substituting for W_{F_g} (from Part (*b*)) and K_f yields:

$$mgs\cos\theta = \tfrac{1}{2}mv_f^2 \Rightarrow v_f = \sqrt{2gs\cos\theta}$$

Substitute numerical values and evaluate v_f for $s = 1.5$ m:

$$v_f = \sqrt{2(9.81\,\text{m/s}^2)(1.5\,\text{m})\cos 30°} = \boxed{5.0\,\text{m/s}}$$

(*d*) If $K_i \neq 0$, equation (1) becomes:

$$W_{F_g} = K_f - K_i = \tfrac{1}{2}mv_f^2 - mv_i^2$$

Solving for v_f and simplifying yields:

$$v_f = \sqrt{2gs\cos\theta + v_i^2}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{2(9.81\,\text{m/s}^2)(1.5\,\text{m})\cos 30° + (2.0\,\text{m/s})^2} = \boxed{5.4\,\text{m/s}}$$

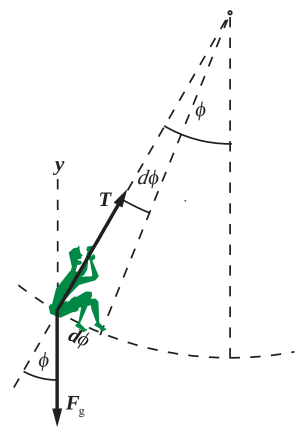
33 •• [SSM] You are designing a jungle-vine–swinging sequence for the latest Tarzan movie. To determine his speed at the low point of the swing and to make sure it does not exceed mandatory safety limits, you decide to model the system of Tarzan + vine as a pendulum. Assume your model consists of a particle (Tarzan, mass 100 kg) hanging from a light string (the vine) of length ℓ attached to a support. The angle between the vertical and the string is written as ϕ. (*a*) Draw a free-body diagram for the object on the end of the string (Tarzan on the vine). (*b*) An infinitesimal distance along the arc (along which the object travels) is $\ell d\phi$. Write an expression for the total work dW_{total} done on the particle as it traverses that distance for an arbitrary angle ϕ. (*c*) If the ℓ = 7.0 m, and if the particle starts from rest at an angle 50°, determine the particle's kinetic energy and speed at the low point of the swing using the work–kinetic-energy theorem.

Comment: DAVID: I changed *l* to ℓ and object to particle.

Comment: DAVID: I changed *l* to ℓ and object to particle.

Picture the Problem Because Tarzan's displacement is always perpendicular to the tension force exerted by the rope, this force can do no work on him. The net work done on Tarzan is the work done by the gravitational force acting on him. We can use the definition of work and the work-kinetic energy theorem to find his kinetic energy and speed at the low point of his swing.

(*a*) The forces acting on Tarzan are a gravitational force $\vec{F}_g$ (his weight) exerted by Earth and the tension force $\vec{T}$ exerted by the vine on which he is swinging.



(*b*) The work dW_{total} done by the gravitational force $\vec{F}_g$ as Tarzan swings through an arc of length ds is:

$$dW_{\text{total}} = \vec{F}_g \cdot d\vec{s} = F_g\, ds \cos(90° - \phi) = F_g\, ds \sin\phi$$

Note that, because $d\phi$ is decreasing as Tarzan swings toward his equilibrium position:

$$ds = -\ell d\phi$$

Substituting for ds and F_g yields:

$$dW_{\text{total}} = \boxed{-mg\ell \sin\phi\, d\phi}$$

(*c*) Apply the work-kinetic energy theorem to Tarzan to obtain:

$$W_{\text{total}} = \Delta K = K_{\text{f}} - K_{\text{i}}$$

or, because $K_{\text{i}} = 0$,

$$W_{\text{total}} = K_{\text{f}}$$

Substituting for W_{total} yields:

$$K_{\text{f}} = -\int_{50°}^{0°} mg\ell \sin\phi \, d\phi = -mg\ell\left[-\cos\phi\right]_{50°}^{0°}$$

Substitute numerical values and evaluate the integral to find K_{f}:

$$\begin{aligned} K_{\text{f}} &= -(100\text{ kg})(9.81\text{ m/s}^2)(7.0\text{ m})\left[-\cos\phi\right]_{50°}^{0°} \\ &= (100\text{ kg})(9.81\text{ m/s}^2)(7.0\text{ m})(1-\cos 50°) \\ &= 2.453\text{ kJ} = \boxed{2.5\text{ kJ}} \end{aligned}$$

Express K_{f} as a function of v_{f}:

$$K_{\text{f}} = \tfrac{1}{2}mv_{\text{f}}^2 \Rightarrow v_{\text{f}} = \sqrt{\frac{2K_{\text{f}}}{m}}$$

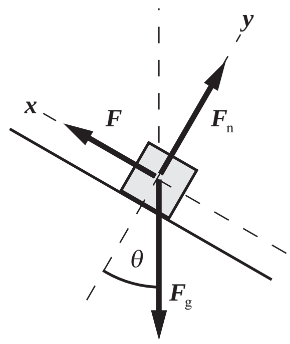
Substitute numerical values and evaluate v_{f}:

$$v_{\text{f}} = \sqrt{\frac{2(2.453\text{ kJ})}{100\text{ kg}}} = \boxed{7.0\text{ m/s}}$$

34 •• *Simple machines* are frequently used for reducing the amount of force that must be supplied to perform a task such as lifting a heavy weight. Such machines include the screw, block-and-tackle systems, and levers, but the simplest of the simple machines is the inclined plane. In Figure 6-32, you are raising a heavy box to the height of the truck bed by pushing it up an inclined plane (a ramp). (*a*) The *mechanical advantage MA* of the inclined plane is defined as the ratio of the magnitude of the force it would take to lift the block straight up (at constant speed) to the magnitude of the force it would take to push it up the ramp (at constant speed). If the plane is frictionless, show that $MA = 1/\sin\theta = L/H$, where H is the height of the truck bed and L is the length of the ramp. (*b*) Show that the work you do by moving the block into the truck is the same whether you lift it straight up or push it up the frictionless ramp.

Comment: DAVID: In 36 changed M to MA.

Picture the Problem The free-body diagram, with $\vec{F}$ representing the force required to move the block at constant speed, shows the forces acting on the block. We can apply Newton's second law to the block to relate F to its weight w and then use the definition of the mechanical advantage of an inclined plane. In the second part of the problem we'll use the definition of work.



(*a*) Express the mechanical advantage MA of the inclined plane:

$$MA = \frac{F_g}{F} \quad (1)$$

Apply $\sum F_x = ma_x$ to the block:

$$F - F_g \sin\theta = ma_x$$

or, because $a_x = 0$,

$$F - F_g \sin\theta = 0 \Rightarrow F = F_g \sin\theta$$

Substitute for F in equation (1) and simplify to obtain:

$$MA = \frac{F_g}{F_g \sin\theta} = \boxed{\frac{1}{\sin\theta}} \quad (2)$$

Refer to the figure to obtain:

$$\sin\theta = \frac{H}{L}$$

Substitute for $\sin\theta$ in equation (2) and simplify to obtain:

$$MA = \frac{1}{\frac{H}{L}} = \boxed{\frac{L}{H}}$$

(*b*) The work you do pushing the block up the ramp is given by:

$$W_{\text{ramp}} = FL$$

Because $F = F_g \sin\theta$ and $F_g = mg$:

$$W_{\text{ramp}} = (F_g \sin\theta)L = mgL\sin\theta$$

The work you do lifting the block into the truck is given by:

$$W_{\text{lifting}} = F_g H$$

Because $F_g = mg$ and $H = L\sin\theta$:

$$W_{\text{lifting}} = mgL\sin\theta \Rightarrow \boxed{W_{\text{ramp}} = W_{\text{lifting}}}$$

35 •• Particle a has mass m, is initially located on the positive x axis at $x = x_0$ and is subject to a repulsive force F_x from particle b. The location of particle b is fixed at the origin. The force F_x is inversely proportional to the square of the distance x between the particles. That is, $F_x = A/x^2$, where A is a positive constant. Particle a is released from rest and allowed to move under the influence of the

force. Find an expression for the work done by the force on a as a function of x. Find both the kinetic energy and speed of a as x approaches infinity.

Picture the Problem Because the force varies inversely with distance, we need to use the integral form of the expression for the work done on the particle. We can then apply the work-kinetic energy theorem to the particle to find its kinetic energy and speed as functions of x.

The work done by the force on the particle is given by:

$$W_{x_0\to x}=\int_{x_0}^{x}\vec{\boldsymbol{F}}\cdot dx\hat{\boldsymbol{i}}=\int_{x_0}^{x}F(\cos\theta)dx$$

where θ is the angle between $\vec{\boldsymbol{F}}$ and $dx\hat{i}$.

Substituting for F and noting that $\theta = 0$:

$$W_{x_0\to x}=\int_{x_0}^{x}\frac{A}{x^2}(\cos 0°)dx=A\int_{x_0}^{x}\frac{1}{x^2}dx$$

Evaluate this integral to obtain:

$$W_{x_0\to x}=-A\left[\frac{1}{x}\right]_{x_0}^{x}=\boxed{\frac{A}{x_0}-\frac{A}{x}}$$

Applying the work-kinetic energy theorem to the particle yields:

$$W_{x_0\to x}=\Delta K=K_x-K_{x_0}$$

or, because the particle is released from rest at $x = x_0$,

$$W_{x_0\to x}=K_x$$

Substitute for $W_{x_0\to x}$ to obtain:

$$K_x=\frac{A}{x_0}-\frac{A}{x}$$

As $x\to\infty$:

$$K_{x\to\infty}=\boxed{\frac{A}{x_0}}$$

and

$$v_{x\to\infty}=\sqrt{\frac{2K_{x\to}}{m}}=\sqrt{\frac{2\left(\frac{A}{x_0}\right)}{m}}=\boxed{\sqrt{\frac{2A}{mx_0}}}$$

36 •• You exert a force of magnitude F on the free end of the rope. If the load moves up a distance h, (*a*) through what distance does the point of application of the force move? (*b*) How much work is done by the rope on the load? (*c*) How much work do you do on the rope? (*d*) The mechanical advantage (defined in Problem 34) of this system is the ratio F/F_g, where F_g is the weight of the load. What is this mechanical advantage?

Comment: AU: Italic F OK? Or should it have an arrow over it as the F in the figure does?

Picture the Problem The object whose weight is $\vec{w}$ is supported by two portions of the rope resulting in what is known as a *mechanical advantage* of 2. The work that is done in each instance is the product of the force doing the work and the displacement of the object on which it does the work.

(*a*) If w moves through a distance h: The point of application of F moves a distance $\boxed{2h}$.

(*b*) Assuming that the kinetic energy of the weight does not change, relate the work done on the object to the change in its potential energy to obtain:

$$W = \Delta U = wh\cos\theta = \boxed{wh}$$

(*c*) Because the force you exert on the rope and its displacement are in the same direction:

$$W = F(2h)\cos\theta = F(2h)$$

Determine the tension in the ropes supporting the object:

$$\sum F_{\text{vertical}} = 2F - w = 0 \Rightarrow F = \tfrac{1}{2}w$$

Substitute for F:

$$W = F(2h) = \tfrac{1}{2}w(2h) = \boxed{wh}$$

(*d*) The mechanical advantage of the pulley-rope system is the ratio of the weight that is lifted to the force required to lift it, that is:

$$MA = \frac{w}{F} = \frac{w}{\frac{1}{2}w} = \boxed{2}$$

Remarks: Note that the mechanical advantage is also equal to the number of ropes supporting the load.

Scalar (Dot) Products

37 • What is the angle between the vectors $\vec{A}$ and $\vec{B}$ if $\vec{A}\cdot\vec{B} = -AB$?

Picture the Problem Because $\vec{A}\cdot\vec{B} = AB\cos\theta$, we can solve for $\cos\theta$ and use the fact that $\vec{A}\cdot\vec{B} = -AB$ to find θ.

Solve $\vec{A}\cdot\vec{B} = AB\cos\theta$, for θ to obtain:

$$\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)$$

Substitute for $\vec{A}\cdot\vec{B}$ and evaluate θ:

$$\theta = \cos^{-1}\left(\frac{-AB}{AB}\right) = \cos^{-1}(-1) = \boxed{180°}$$

38 • Two vectors $\vec{A}$ and $\vec{B}$ each have magnitudes of 6.0 m and the angle between their directions is 60º. Find $\vec{A}\cdot\vec{B}$.

Picture the Problem We can use the definition of the scalar product to evaluate $\vec{A}\cdot\vec{B}$.

Express the definition of the scalar product $\vec{A}\cdot\vec{B}$:

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

Substitute numerical values and evaluate $\vec{A}\cdot\vec{B}$:

$$\vec{A}\cdot\vec{B} = (6.0\,\text{m})(6.0\,\text{m})\cos 60° = \boxed{18\,\text{m}^2}$$

39 • Find $\vec{A}\cdot\vec{B}$ for the following vectors: (*a*) $\vec{A} = 3\hat{i} - 6\hat{j}$, $\vec{B} = -4\hat{i} + 2\hat{j}$; (*b*) $\vec{A} = 5\hat{i} + 5\hat{j}$, $\vec{B} = 2\hat{i} - 4\hat{j}$; and (*c*) $\vec{A} = 6\hat{i} + 4\hat{j}$, $\vec{B} = 4\hat{i} - 6\hat{j}$.

Picture the Problem The scalar product of two-dimensional vectors $\vec{A}$ and $\vec{B}$ is $A_xB_x + A_yB_y$.

(*a*) For $\vec{A} = 3\hat{i} - 6\hat{j}$ and $\vec{B} = -4\hat{i} + 2\hat{j}$:

$$\vec{A}\cdot\vec{B} = (3)(-4) + (-6)(2) = \boxed{-24}$$

(*b*) For $\vec{A} = 5\hat{i} + 5\hat{j}$ and $\vec{B} = 2\hat{i} - 4\hat{j}$:

$$\vec{A}\cdot\vec{B} = (5)(2) + (5)(-4) = \boxed{-10}$$

(*c*) For $\vec{A} = 6\hat{i} + 4\hat{j}$ and $\vec{B} = 4\hat{i} - 6\hat{j}$:

$$\vec{A}\cdot\vec{B} = (6)(4) + (4)(-6) = \boxed{0}$$

40 • Find the angles between the vectors $\vec{A}$ and $\vec{B}$ given: (*a*) $\vec{A} = 3\hat{i} - 6\hat{j}$, $\vec{B} = -4\hat{i} + 2\hat{j}$; (*b*) $\vec{A} = 5\hat{i} + 5\hat{j}$, $\vec{B} = 2\hat{i} - 4\hat{j}$; and (*c*) $\vec{A} = 6\hat{i} + 4\hat{j}$, $\vec{B} = 4\hat{i} - 6\hat{j}$.

Picture the Problem The scalar product of two-dimensional vectors $\vec{A}$ and $\vec{B}$ is $AB\cos\theta = A_xB_x + A_yB_y$. Hence the angle between vectors $\vec{A}$ and $\vec{B}$ is given by

$$\theta = \cos^{-1}\left(\frac{A_xB_x + A_yB_y}{AB}\right).$$

(*a*) For $\vec{A} = 3\hat{i} - 6\hat{j}$ and $\vec{B} = -4\hat{i} + 2\hat{j}$:

$$\theta = \cos^{-1}\left(\frac{(3)(-4)+(-6)(2)}{\sqrt{3^2+(-6)^2}\sqrt{(-4)^2+(2)^2}}\right) = \boxed{143°}$$

(*b*) For $\vec{A} = 5\hat{i} + 5\hat{j}$ and $\vec{B} = 2\hat{i} - 4\hat{j}$:

$$\theta = \cos^{-1}\left(\frac{(5)(2)+(5)(-4)}{\sqrt{5^2+5^2}\sqrt{2^2+(-4)^2}}\right) = \boxed{108°}$$

(*c*) For $\vec{A} = 6\hat{i} + 4\hat{j}$ and $\vec{B} = 4\hat{i} - 6\hat{j}$:

$$\theta = \cos^{-1}\left(\frac{(6)(4)+(4)(-6)}{\sqrt{6^2+4^2}\sqrt{4^2+(-6)^2}}\right) = \boxed{90°}$$

41 • A 2.0-kg particle is given a displacement of $\Delta\vec{r} = (3.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j} + (-2.0\text{ m})\hat{k}$. During the displacement, a constant force $\vec{F} = (2.0\text{ N})\hat{i} - (1.0\text{ N})\hat{j} + (1.0\text{ N})\hat{k}$ acts on the particle. (*a*) Find the work done by $\vec{F}$ for this displacement. (*b*) Find the component of $\vec{F}$ in the direction of this displacement.

Picture the Problem The work *W* done by a force $\vec{F}$ during a displacement $\Delta\vec{s}$ for which it is responsible is given by $\vec{F}\cdot\Delta\vec{r}$.

(*a*) Using the definitions of work and the scalar product, calculate the work done by the given force during the specified displacement:

$$\begin{aligned} W &= \vec{F}\cdot\Delta\vec{r} = \left[(2.0\text{ N})\hat{i} - (1.0\text{ N})\hat{j} + (1.0\text{ N})\hat{k}\right]\cdot\left[(3.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j} - (2.0\text{ m})\hat{k}\right] \\ &= [(2.0)(3.0) + (-1.0)(3.0) + (1.0)(-2.0)]\text{N}\cdot\text{m} = \boxed{1.0\text{ J}} \end{aligned}$$

(*b*) Use the definition of work that includes the angle between the force and displacement vectors to express the work done by $\vec{F}$:

$$W = F\Delta r\cos\theta = (F\cos\theta)\Delta r$$

Solve for the component of $\vec{F}$ in the direction of $\Delta\vec{r}$:

$$F\cos\theta = \frac{W}{\Delta r}$$

Substitute numerical values and evaluate $F\cos\theta$:

$$F\cos\theta = \frac{1.0\,\text{J}}{\sqrt{(3.0\,\text{m})^2 + (3.0\,\text{m})^2 + (-2.0\,\text{m})^2}} = \boxed{0.21\,\text{N}}$$

42 •• (*a*) Find the unit vector that is in the same direction as the vector $\vec{A} = 2.0\hat{i} - 1.0\hat{j} - 1.0\hat{k}$. (*b*) Find the component of the vector $\vec{A} = 2.0\hat{i} - 1.0\hat{j} - 1.0\hat{k}$ in the direction of the vector $\vec{B} = 3.0\hat{i} + 4.0\hat{j}$.

Picture the Problem A unit vector that is in the same direction as a given vector and is the given vector divided by its magnitude. The component of a vector that is in the direction of another vector is the scalar product of the former vector and a unit vector that is in the direction of the latter vector.

(*a*) By definition, the unit vector that is in the same direction as $\vec{A}$ is:

$$\hat{u}_A = \frac{\vec{A}}{A}$$

Substitute for $\vec{A}$ and A and evaluate $\hat{u}_A$:

$$\hat{u}_A = \frac{2.0\hat{i} - 1.0\hat{j} - 1.0\hat{k}}{\sqrt{(2.0)^2 + (-1.0)^2 + (-1.0)^2}} = \frac{2.0\hat{i} - 1.0\hat{j} - 1.0\hat{k}}{\sqrt{6}} = \boxed{0.82\hat{i} - 0.41\hat{j} - 0.41\hat{k}}$$

(*b*) The component of $\vec{A}$ in the direction of $\vec{B}$ is given by:

$$A_{\substack{\text{direction}\\ \text{of } \vec{B}}} = \vec{A}\cdot\hat{u}_B \qquad (1)$$

The unit vector in the direction of $\vec{B}$ is:

$$\hat{u}_B = \frac{\vec{B}}{B} = \frac{3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}}{\sqrt{(3.0)^2 + (4.0)^2}} = \frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$$

Substitute for $\vec{B}$ and $\hat{u}_B$ in equation (1) and evaluate $A_{\substack{\text{direction}\\ \text{of } \vec{B}}}$:

$$A_{\substack{\text{direction}\\ \text{of } \vec{B}}} = \left(2.0\hat{i} - 1.0\hat{j} - 1.0\hat{k}\right)\cdot\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right) = (2.0)\left(\frac{3}{5}\right) + (-1.0)\left(\frac{4}{5}\right) + (-1.0)(0) = \boxed{0.40}$$

43 •• **[SSM]** (*a*) Given two nonzero vectors $\vec{A}$ and $\vec{B}$, show that if $\left|\vec{A}+\vec{B}\right|=\left|\vec{A}-\vec{B}\right|$, then $\vec{A}\perp\vec{B}$. (*b*) Given a vector $\vec{A}=4\hat{i}-3\hat{j}$, find a vector in the *xy* plane that is perpendicular to $\vec{A}$ and has a magnitude of 10. Is this the only vector that satisfies the specified requirements? Explain.

Picture the Problem We can use the definitions of the magnitude of a vector and the dot product to show that if $\left|\vec{A}+\vec{B}\right|=\left|\vec{A}-\vec{B}\right|$, then $\vec{A}\perp\vec{B}$.

(*a*) Express $\left|\vec{A}+\vec{B}\right|^2$:

$$\left|\vec{A}+\vec{B}\right|^2=\left(\vec{A}+\vec{B}\right)^2$$

Express $\left|\vec{A}-\vec{B}\right|$:

$$\left|\vec{A}-\vec{B}\right|^2=\left(\vec{A}-\vec{B}\right)^2$$

Equate these expressions to obtain:

$$\left(\vec{A}+\vec{B}\right)^2=\left(\vec{A}-\vec{B}\right)^2$$

Expand both sides of the equation to obtain:

$$A^2+2\vec{A}\cdot\vec{B}+B^2=A^2-2\vec{A}\cdot\vec{B}+B^2$$

Simplify to obtain:

$$4\vec{A}\cdot\vec{B}=0 \text{ or } \vec{A}\cdot\vec{B}=0$$

From the definition of the dot product we have:

$$\vec{A}\cdot\vec{B}=AB\cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

Because neither $\vec{A}$ nor $\vec{B}$ is the zero vector:

$$\cos\theta=0 \Rightarrow \theta=90° \text{ and } \vec{A}\perp\vec{B}.$$

(*b*) Let the required vector be $\vec{B}$. The condition that $\vec{A}$ and $\vec{B}$ be perpendicular is that:

$$\vec{A}\cdot\vec{B}=0 \qquad (1)$$

Express $\vec{B}$ as:

$$\vec{B}=B_x\hat{i}+B_y\hat{j} \qquad (2)$$

Substituting for $\vec{A}$ and $\vec{B}$ in equation (1) yields:

$$\left(4\hat{i}-3\hat{j}\right)\cdot\left(B_x\hat{i}+B_y\hat{j}\right)=0$$

or

$$4B_x-3B_y=0 \qquad (3)$$

Because the magnitude of $\vec{B}$ is 10:

$$B_x^2+B_y^2=100 \qquad (4)$$

Solving equation (3) for B_x and substituting in equation (4) yields:

$$\left(\tfrac{3}{4}B_y\right)^2+B_y^2=100$$

Solve for B_y to obtain:

$$B_y = \pm 8$$

Substituting for B_y in equation (3) and solving for B_x yields:

If $B_y = +8$, then $B_x = +6$.
If $B_y = -8$, then $B_x = -6$

Substitute for B_x and B_y in equation (2) to obtain:

$$\vec{B} = \boxed{6\hat{i} + 8\hat{j}}$$

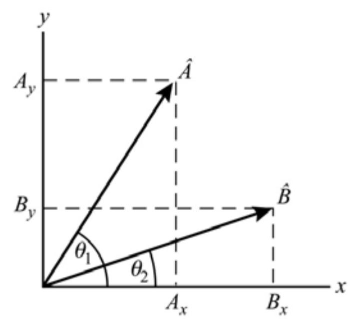
or

$$\vec{B} = \boxed{-6\hat{i} - 8\hat{j}}$$

No. There are two possibilities. The vectors point in opposite directions. One vector is in the 1st quadrant and the other vector is in the 3rd quadrant.

44 •• Unit vectors $\hat{A}$ and $\hat{B}$ are in the *xy* plane. They make angles of θ_1 and θ_2, respectively, with the +*x* axis, respectively. (*a*) Use trigonometry to find the *x* and *y* components of the two vectors directly. (Your answer should be in terms of the angles.) (*b*) By considering the scalar product of $\hat{A}$ and $\hat{B}$, show that $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

Picture the Problem The diagram shows the unit vectors $\hat{A}$ and $\hat{B}$ arbitrarily located in the 1st quadrant. We can express these vectors in terms of the unit vectors $\hat{i}$ and $\hat{j}$ and their *x* and *y* components. We can then form the scalar product of $\hat{A}$ and $\hat{B}$ to show that $\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$.



(*a*) Express $\hat{A}$ in terms of the unit vectors $\hat{i}$ and $\hat{j}$:

$$\hat{A} = A_x\hat{i} + A_y\hat{j}$$

where

$$A_x = \boxed{\cos\theta_1} \text{ and } A_y = \boxed{\sin\theta_1}$$

Proceed as above to obtain:

$$\hat{B} = B_x\hat{i} + B_y\hat{j}$$

where

$$B_x = \boxed{\cos\theta_2} \text{ and } B_y = \boxed{\sin\theta_2}$$

(*b*) Evaluate $\hat{A}\cdot\hat{B}$:

$$\begin{aligned}\hat{A}\cdot\hat{B} &= \left(\cos\theta_1\hat{i} + \sin\theta_1\hat{j}\right) \\ &\quad\cdot\left(\cos\theta_2\hat{i} + \sin\theta_2\hat{j}\right) \\ &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\end{aligned}$$

From the diagram we note that: $\hat{\boldsymbol{A}}\cdot\hat{\boldsymbol{B}}=\cos(\theta_1-\theta_2)$

Substitute for $\hat{\boldsymbol{A}}\cdot\hat{\boldsymbol{B}}$ to obtain $\boxed{\cos(\theta_1-\theta_2)=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}$.

45 •• In Chapter 8, we shall introduce a new vector for a particle, called its *linear momentum*, symbolized by $\vec{\boldsymbol{p}}$. Mathematically, it is related to the mass m and velocity $\vec{\mathbf{v}}$ of the particle by $\vec{\boldsymbol{p}}=m\vec{\mathbf{v}}$. (*a*) Show that the particle's kinetic energy K can be expressed as $K=\dfrac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}}{2m}$. (*b*) Compute the linear momentum of a particle of mass 2.5 kg that is moving at a speed of 15 m/s at an angle of 25° clockwise from the +*x* axis in the *xy* plane. (*c*) Compute its kinetic energy using both $K=\frac{1}{2}mv^2$ and $K=\dfrac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}}{2m}$ and verify that they give the same result.

Picture the Problem The scalar product of two vectors is the product of their magnitudes multiplied by the cosine of the angle between them.

(*a*) We're to show that:

$$K=\frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}}{2m}\qquad(1)$$

The scalar product of $\vec{\boldsymbol{p}}$ with itself is: $\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}=p^2$

Because $p = mv$: $\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}=m^2v^2$

Substitute in equation (1) and simplify to obtain:

$$K=\frac{m^2v^2}{2m}=\boxed{\tfrac{1}{2}mv^2}$$

(*b*) The linear momentum of the particle is given by: $\vec{\boldsymbol{p}}=m\vec{\mathbf{v}}$

Substitute numerical values for m and $\vec{\mathbf{v}}$ and simplify to obtain:

$$\begin{aligned}\vec{\boldsymbol{p}}&=(2.5\text{ kg})\left[\{(15\text{ m/s})\cos 335°\}\hat{\boldsymbol{i}}+\{(15\text{ m/s})\sin 335°\}\hat{\boldsymbol{j}}\right]\\&=\boxed{(34\text{ kg}\cdot\text{m/s})\hat{\boldsymbol{i}}+(-16\text{ kg}\cdot\text{m/s})\hat{\boldsymbol{j}}}\end{aligned}$$

(*c*) Evaluating $K=\frac{1}{2}mv^2$ yields: $K=\frac{1}{2}(2.5\text{ kg})(15\text{ m/s})^2=\boxed{0.28\text{ kJ}}$

Evaluating $K = \dfrac{\vec{p}\cdot\vec{p}}{2m}$ yields:

$$K = \frac{\left[(34\text{ m/s})\hat{i} + (-16\text{ m/s})\hat{j}\right]\cdot\left[(34\text{ m/s})\hat{i} + (-16\text{ m/s})\hat{j}\right]}{2(2.5\text{ kg})}$$
$$= \frac{(34\text{ m/s})^2 + (-16\text{ m/s})^2}{5.0\text{ kg}} = \boxed{0.28\text{ kJ}}$$

46 •• (a) Let $\vec{A}$ be a constant vector in the xy plane with its tail at the origin. Let $\vec{r} = x\hat{i} + y\hat{j}$ be a vector in the xy plane that satisfies the relation $\vec{A}\cdot\vec{r} = 1$. Show that the points with coordinates (x,y) lie on a straight line. (b) If $\vec{A} = 2\hat{i} - 3\hat{j}$, find the slope and y intercept of the line. (c) If we now let $\vec{A}$ and $\vec{r}$ be vectors in three-dimensional space, show that the relation $\vec{A}\cdot\vec{r} = 1$ specifies a plane.

Picture the Problem We can form the dot product of $\vec{A}$ and $\vec{r}$ and require that $\vec{A}\cdot\vec{r} = 1$ to show that the points at the head of all such vectors $\vec{r}$ lie on a straight line. We can use the equation of this line and the components of $\vec{A}$ to find the slope and intercept of the line.

(a) Let $\vec{A} = a_x\hat{i} + a_y\hat{j}$. Then:

$$\vec{A}\cdot\vec{r} = \left(a_x\hat{i} + a_y\hat{j}\right)\cdot\left(x\hat{i} + y\hat{j}\right)$$
$$= a_x x + a_y y = 1$$

Solve for y to obtain:

$$y = \boxed{-\frac{a_x}{a_y}x + \frac{1}{a_y}}$$

which is of the form $y = mx + b$ and hence is the equation of a straight line.

(b) Given that $\vec{A} = 2\hat{i} - 3\hat{j}$:

$$m = -\frac{a_x}{a_y} = -\frac{2}{-3} = \boxed{\frac{2}{3}}$$

and

$$b = \frac{1}{a_y} = \frac{1}{-3} = \boxed{-\frac{1}{3}}$$

(c) The equation $\vec{A}\cdot\vec{r} = 1$ specifies all vectors $\vec{r}$ whose component r_A (the component of $\vec{r}$ parallel to $\vec{A}$) has the constant magnitude A^{-1}.

$$1 = \vec{A}\cdot\vec{r} = A\hat{A}\cdot\vec{r} = A\left(\vec{r}\cdot\hat{A}\right) = Ar_A$$

so

$$\frac{1}{A} = r_A$$

Let $\vec{B}$ represent all vectors $\vec{r}$ such that $\vec{r} = A^{-1}\hat{A} + \vec{B}$. Multiplying both sides by $\hat{A}$ shows that $\vec{B}$ is perpendicular to $\vec{A}$:

$$\begin{aligned}\vec{r}\cdot\hat{A} &= \left(A^{-1}\hat{A} + \vec{B}\right)\cdot\hat{A}\\ &= A^{-1}\hat{A}\cdot\hat{A} + \vec{B}\cdot\hat{A}\\ &= A^{-1} + \vec{B}\cdot\hat{A}\end{aligned}$$

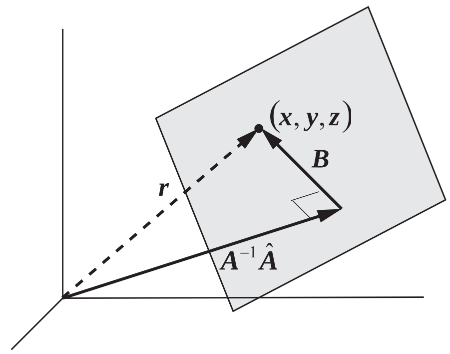
Because $\vec{r}\cdot\hat{A} = r_{\mathrm{A}} = A^{-1}$:

$$A^{-1} = A^{-1} + \vec{B}\cdot\hat{A} \Rightarrow \vec{B}\cdot\hat{A} = 0$$

Because $\vec{B}\cdot\hat{A} = 0$:

$$\vec{B} \perp \hat{A}$$

The relation $\vec{r} = A^{-1}\hat{A} + \vec{B}$ is shown graphically to the right:



From the figure we can see that the tip of $\vec{r}$ is restricted to a plane that is perpendicular to $\hat{A}$ and contains the point a distance A^{-1} from the origin in the direction of $\hat{A}$. The equation $\vec{A}\cdot\vec{r} = 1$ specifies this plane.

47 ••• [SSM] When a particle moves in a circle that is centered at the origin and the magnitude of its position vector $\vec{r}$ is constant. (*a*) Differentiate $\vec{r}\cdot\vec{r} = r^2 = \text{constant}$ with respect to time to show that $\vec{v}\cdot\vec{r} = 0$, and therefore $\vec{v}\perp\vec{r}$. (*b*) Differentiate $\vec{v}\cdot\vec{r} = 0$ with respect to time and show that $\vec{a}\cdot\vec{r} + v^2 = 0$, and therefore $a_r = -v^2/r$. (*c*) Differentiate $\vec{v}\cdot\vec{v} = v^2$ with respect to time to show that $\vec{a}\cdot\vec{v} = dv/dt$, and therefore $a_{\mathrm{t}} = dv/dt$.

Comment: DAVID: Parts (b) and (c) have been transposed, and the restrictions to constant speed have been removed throughout.

Picture the Problem The rules for the differentiation of vectors are the same as those for the differentiation of scalars and scalar multiplication is commutative.

(*a*) Differentiate $\vec{r}\cdot\vec{r} = r^2 = \text{constant}$:

$$\begin{aligned}\frac{d}{dt}\left(\vec{r}\cdot\vec{r}\right) &= \vec{r}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{r} = 2\vec{v}\cdot\vec{r}\\ &= \frac{d}{dt}\left(\text{constant}\right) = 0\end{aligned}$$

Because $\vec{v}\cdot\vec{r} = 0$:

$$\boxed{\vec{v}\perp\vec{r}}$$

(*b*) Differentiate $\vec{v}\cdot\vec{r} = 0$ with respect to time:

$$\frac{d}{dt}(\vec{v}\cdot\vec{r}) = \vec{v}\cdot\frac{d\vec{r}}{dt} + \vec{r}\cdot\frac{d\vec{v}}{dt}$$

$$= v^2 + \vec{r}\cdot\vec{a} = \frac{d}{dt}(0) = 0$$

Because $v^2 + \vec{r}\cdot\vec{a} = 0$:

$$\boxed{\vec{r}\cdot\vec{a} = -v^2} \qquad (1)$$

Express a_r in terms of θ, where θ is the angle between $\vec{r}$ and $\vec{a}$:

$$a_r = a\cos\theta$$

Express $\vec{r}\cdot\vec{a}$:

$$\vec{r}\cdot\vec{a} = ra\cos\theta = ra_r$$

Substitute in equation (1) to obtain:

$$ra_r = -v^2$$

Solving for a_r yields:

$$a_r = \boxed{-\frac{v^2}{r}}$$

(*c*) Differentiate $\vec{v}\cdot\vec{v} = v^2$ with respect to time:

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt}\cdot\vec{v} = 2\vec{a}\cdot\vec{v}$$

$$= \frac{d(v^2)}{dt} = 2v\frac{dv}{dt}$$

$2\vec{a}\cdot\vec{v}$ is also given by:

$$2\vec{a}\cdot\vec{v} = 2a_t v = 2v\frac{dv}{dt} \Rightarrow a_t = \frac{dv}{dt}$$

The results of (*a*) and (*b*) tell us that $\vec{a}$ is perpendicular to $\vec{v}$ and parallel (or antiparallel to $\vec{r}$.

Work and Power

48 • Force A does 5.0 J of work in 10 s. Force B does 3.0 J of work in 5.0 s. Which force delivers greater power, A or B? Explain.

Picture the Problem The average power delivered by a force is defined as the rate at which the force does work; i.e., $P_{av} = \frac{\Delta W}{\Delta t}$.

Calculate the average rate at which force *A* does work:

$$P_{A,\,av} = \frac{\Delta W}{\Delta t} = \frac{5.0\,\text{J}}{10\,\text{s}} = 0.50\,\text{W}$$

Calculate the rate at which force B does work:

$$P_{\mathrm{B,av}}=\frac{3.0\,\mathrm{J}}{5.0\,\mathrm{s}}=0.60\,\mathrm{W}$$

Force B delivers greater power because, although it does only 60% of the work force A does, it does it in 50% of the time required by force A.

49 • A single force of 5.0 N in the + x direction acts on an 8.0-kg object. (a) If the object starts from rest at x = 0 at time t = 0, write an expression for the power delivered by this force as a function of time. (b) What is the power delivered by this force at time t = 3.0 s?

Picture the Problem We can use Newton's second law and the definition of acceleration to express the velocity of this object as a function of time. The power input of the force accelerating the object is defined to be the rate at which it does work; that is, $P=dW/dt=\vec{F}\cdot\vec{v}$.

(a) The power of the force as a function of time is given by:

$$P(t)=\frac{dW}{dt}=\vec{F}\cdot\vec{v}=Fv(t)\cos\theta$$

or, because the force acts in the same direction as the velocity of the object,

$$P(t)=Fv(t) \qquad (1)$$

Express the velocity of the object as a function of its acceleration and time:

$$v(t)=at$$

Applying $\sum\vec{F}=m\vec{a}$ to the object yields:

$$a=\frac{F}{m}$$

Substitute for a in the expression for v to obtain:

$$v(t)=\frac{F}{m}t$$

Substituting in equation (1) yields:

$$P(t)=\frac{F^2}{m}t$$

Substitute numerical values and evaluate $P(t)$:

$$P(t)=\frac{(5.0\,\mathrm{N})^2}{8.0\,\mathrm{kg}}t=(3.125\,\mathrm{W/s})t$$

$$=\boxed{(3.1\,\mathrm{W/s})t}$$

(*b*) Evaluate *P*(3.0 s): $P(3.0\,\text{s}) = (3.125\,\text{W/s})(3.0\,\text{s}) = \boxed{9.4\,\text{W}}$

50 • Find the power delivered by a force $\vec{F}$ acting on a particle that moves with a velocity $\vec{v}$, where (*a*) $\vec{F} = (4.0\ \text{N})\hat{i} + (3.0\ \text{N})\hat{k}$ and $\vec{v} = (6.0\ \text{m/s})\hat{i}$;
(*b*) $\vec{F} = (6.0\ \text{N})\hat{i} - (5.0\ \text{N})\hat{j}$ and $\vec{v} = -(5.0\ \text{m/s})\hat{i} + (4.0\ \text{m/s})\hat{j}$; and
(*c*) $\vec{F} = (3.0\ \text{N})\hat{i} + (6.0\ \text{N})\hat{j}$ and $\vec{v} = (2.0\ \text{m/s})\hat{i} + (3.0\ \text{m/s})\hat{j}$.

Picture the Problem The power delivered by a force is defined as the rate at which the force does work; i.e., $P = \dfrac{dW}{dt} = \vec{F}\cdot\vec{v}$.

(*a*) For $\vec{F} =$ (4.0 N) $\hat{i}$ + (3.0 N) $\hat{k}$ and $\vec{v}$ = (6.0 m/s) $\hat{i}$:

$$P = \vec{F}\cdot\vec{v} = \left[(4.0\,\text{N})\hat{i} + (3.0\,\text{N})\hat{k}\right]\cdot\left[(6.0\,\text{m/s})\hat{i}\right] = \boxed{24\,\text{W}}$$

(*b*) For $\vec{F}$ = (6.0 N) $\hat{i}$ −(5.0 N) $\hat{j}$ and $\vec{v}$ = − (5.0 m/s) $\hat{i}$ +(4.0 m/s) $\hat{j}$:

$$P = \vec{F}\cdot\vec{v} = \left[(6.0\,\text{N})\hat{i} - (5.0\,\text{N})\hat{j}\right]\cdot\left[-(5.0\,\text{m/s})\hat{i} + (4.0\,\text{m/s})\hat{j}\right] = \boxed{-50\,\text{W}}$$

(*c*) For $\vec{F}$ = (3.0 N) $\hat{i}$ + (6.0 N) $\hat{j}$ and $\vec{v}$ = (2.0 m/s) $\hat{i}$ + (3.0 m/s) $\hat{j}$:

$$P = \vec{F}\cdot\vec{v} = \left[(3.0\,\text{N})\hat{i} + (6.0\,\text{N})\hat{j}\right]\cdot\left[(2.0\,\text{m/s})\hat{i} + (3.0\,\text{m/s})\hat{j}\right] = \boxed{24\,\text{W}}$$

51 • **[SSM]** You are in charge of installing a small food-service elevator (called a *dumbwaiter* in the food industry) in a campus cafeteria. The elevator is connected by a pulley system to a motor, as shown in Figure 6-34. The motor raises and lowers the dumbwaiter. The mass of the dumbwaiter is 35 kg. In operation, it moves at a speed of 0.35 m/s upward, without accelerating (except for a brief initial period, which we can neglect, just after the motor is turned on). Electric motors typically have an efficiency of 78%. If you purchase a motor with an efficiency of 78%, what minimum power rating should the motor have? Assume that the pulleys are frictionless.

Comment: DAVID: Efficiency changed to 78%.

Picture the Problem Choose a coordinate system in which upward is the positive *y* direction. We can find P_{in} from the given information that $\boldsymbol{P}_{\text{out}} = 0.78\boldsymbol{P}_{\text{in}}$. We can express P_{out} as the product of the tension in the cable *T* and the constant speed *v* of the dumbwaiter. We can apply Newton's second law to the dumbwaiter to express *T* in terms of its mass *m* and the gravitational field *g*.

Express the relationship between the motor's input and output power: $P_{\text{out}} = 0.78P_{\text{in}} \Rightarrow P_{\text{in}} = 1.282P_{\text{out}}$ (1)

Express the power required to move the dumbwaiter at a constant speed v:

$$P_{out} = Tv$$

Apply $\sum F_y = ma_y$ to the dumbwaiter:

$$T - mg = ma_y$$

or, because $a_y = 0$,

$$T = mg$$

Substitute for T in equation (1) to obtain:

$$P_{in} = 1.282Tv = 1.282mgv$$

Substitute numerical values and evaluate P_{in}:

$$P_{in} = (1.282)(35\,\text{kg})(9.81\,\text{m/s}^2)(0.35\,\text{m/s}) = \boxed{0.15\,\text{kW}}$$

52 •• A cannon placed at the edge of a cliff of height H fires a cannonball directly upward with an initial speed v_0. The cannonball rises, falls back down (missing the cannon by a small margin) and lands at the foot of the cliff. Neglecting air resistance, calculate the velocity $\vec{v}$ as a function of time, and show explicitly that the integral of $\vec{F}_{net} \cdot \vec{v}$ over the time that the cannonball spends in flight is equal to the change in the kinetic energy of the cannonball over the same time.

Comment: DAVID: Subscript "net" added.

Picture the Problem Because, in the absence of air resistance, the acceleration of the cannonball is constant, we can use a constant-acceleration equation to relate its velocity to the time it has been in flight. We can apply Newton's second law to the cannonball to find the net force acting on it and then form the dot product of $\vec{F}$ and $\vec{v}$ to express the rate at which the gravitational field does work on the cannonball. Integrating this expression over the time-of-flight T of the ball will yield the desired result.

Express the velocity of the cannonball as a function of time while it is in the air:

$$\vec{v}(t) = 0\hat{i} + (v_0 - gt)\hat{j} = \boxed{(v_0 - gt)\hat{j}}$$

Apply $\sum \vec{F} = m\vec{a}$ to the cannonball to express the force acting on it while it is in the air:

$$\vec{F}_{net} = -mg\,\hat{j}$$

Evaluate $\vec{F}_{net} \cdot \vec{v}$:

$$\vec{F}_{net} \cdot \vec{v} = -mg\,\hat{j} \cdot (v_0 - gt)\hat{j} = -mgv_0 + mg^2t$$

Relate $\vec{F}_{\text{net}} \cdot \vec{v}$ to the rate at which work is being done on the cannonball:

$$\frac{dW}{dt} = \vec{F}_{\text{net}} \cdot \vec{v} = -mgv_0 + mg^2t$$

Separate the variables and integrate over the time T that the cannonball is in the air:

$$W = \int_0^T \left(-mgv_0 + mg^2t\right) dt = \tfrac{1}{2}mg^2T^2 - mgv_0T \qquad (1)$$

Using a constant-acceleration equation, relate the time-of-flight T to the initial and impact speeds of the cannonball:

$$v = v_0 - gT \Rightarrow T = \frac{v_0 - v}{g}$$

Substitute for T in equation (1) and simplify to evaluate W:

$$W = \tfrac{1}{2}mg^2\,\frac{v_0^2 - 2vv_0 + v^2}{g^2} - mgv_0\left(\frac{v_0 - v}{g}\right) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \boxed{\Delta K}$$

53 •• A particle of mass m moves from rest at t = 0 under the influence of a single constant force $\vec{F}$. Show that the power delivered by the force at any time t is $P = F^2t/m$.

Comment: DAVID: Part (b) was deleted because it was circular.

Picture the Problem If the particle is acted on by a *single* force, that force is the *net* force acting on the particle and is responsible for its acceleration. The rate at which energy is delivered by the force is $P = \vec{F} \cdot \vec{v}$. We can apply the work-kinetic energy theorem to the particle to show that the total work done by the constant force after a time t is equal to the gain in kinetic energy of the particle.

Express the rate at which this force does work in terms of $\vec{F}$ and $\vec{v}$:

$$P = \vec{F} \cdot \vec{v} \qquad (1)$$

The velocity of the particle, in terms of its acceleration and the time that the force has acted is:

$$\vec{v} = \vec{a}t$$

Using Newton's second law, substitute for $\vec{a}$:

$$\vec{v} = \frac{\vec{F}}{m}t$$

Substitute for $\vec{v}$ in equation (1) and simplify to obtain:

$$P = \vec{F} \cdot \frac{\vec{F}}{m}t = \frac{\vec{F} \cdot \vec{F}}{m}t = \boxed{\frac{F^2}{m}t}$$

54 •• A 7.5-kg box is being lifted by means of a light rope that is threaded through a single, light, frictionless pulley that is attached to the ceiling. (*a*) If the box is being lifted at a *constant speed* of 2.0 m/s, what is the power delivered by the person pulling on the rope? (*b*) If the box is lifted, at *constant acceleration,* from rest on the floor to a height of 1.5 m above the floor in 0.42 s, what average power is delivered by the person pulling on the rope?

Picture the Problem Because, in (*a*), the box is being lifted at constant speed; the force acting on it is equal to the force due to gravity. We're given the speed and the force and thus calculation of the power is straightforward. In (*b*) we have a constant acceleration and are given the height to which the box is lifted and the time during which it is lifted. The work done in this case is the force times the displacement and the average power is the ratio of the work done to the time required. We can find the force acting on the box by applying Newton's second law and the acceleration of the box by using a constant-acceleration equation.

(*a*) The power exerted by the person pulling on the rope is given by:

$$P = \vec{T}\cdot\vec{v} = Tv\cos\theta$$

or, because $\vec{T}$ and $\vec{v}$ are in the same direction,

$$P = Tv \qquad (1)$$

Apply $\sum F_y = ma_y$ to the box to obtain:

$$T - F_g = ma_y$$

or, because $F_g = mg$ and $a_y = 0$,

$$T - mg = 0 \Rightarrow T = mg$$

Substituting for T in equation (1) yields:

$$P = mgv$$

Substitute numerical values and evaluate P:

$$P = (7.5\ \text{kg})(9.81\ \text{m/s}^2)(2.0\ \text{m/s}) = \boxed{0.15\ \text{kW}}$$

(*b*) The average power exerted by the person pulling the rope is given by:

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{F\Delta y}{\Delta t} \qquad (2)$$

Use a constant-acceleration equation to relate Δy to the interval of acceleration Δt:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or because the box starts from rest,

$$\Delta y = \tfrac{1}{2}a_y(\Delta t)^2 \Rightarrow a_y = \frac{2\Delta y}{(\Delta t)^2}$$

Apply $\sum F_y = ma_y$ to the box to obtain:

$$F = ma_y = m\frac{2\Delta y}{(\Delta t)^2}$$

Substitute for F in equation (2) to obtain:

$$P_{av} = \frac{m\dfrac{2\Delta y}{(\Delta t)^2}\Delta y}{\Delta t} = \frac{2m(\Delta y)^2}{(\Delta t)^3}$$

Substitute numerical values and evaluate P_{av}:

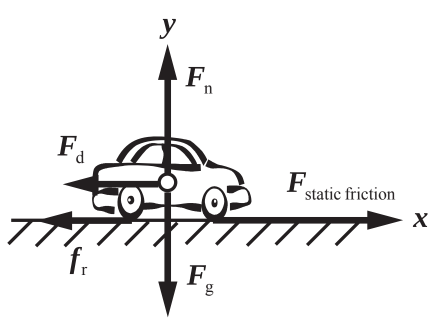
$$P_{av} = \frac{2(7.5\ \text{kg})(1.5\ \text{m})^2}{(0.42\ \text{s})^3} = \boxed{0.46\ \text{kW}}$$

Center of Mass Work and Center of Mass Translational Kinetic Energy

55 ••• You have been asked to test drive a car and study its actual performance relative to its specifications. This particular car's engine is rated at 164 hp. This value is the *peak* rating, which means that it is capable, at most, of providing energy at the rate of 164 hp to the drive wheels. You determine that the car's mass (including test equipment and driver on board) is 1220 kg. (*a*) When cruising at a constant 55.0 mi/h, your onboard engine-monitoring computer determines that the engine is producing 13.5 hp. From previous coasting experiments, it has been determined that the coefficient of rolling friction on the car is 0.0150. Assume that the drag force on the car varies as the square of the car's speed. That is, $F_d = Cv^2$. (*a*) What is the value of the constant, C? (*b*) Considering the peak power, what is the maximum speed (to the nearest 1 mi/h) that you would expect the car could attain? (This problem can be done by hand analytically, but it can be done more easily and quickly using a **graphing calculator** or **spreadsheet**.)

Comment: DAVID: In the solution the static frictional force of the road on the tires is referred to as the force exerted by the engine. This is not a good idea.

Picture the Problem When the car is moving at a constant speed, the static friction force exerted by the road on the tires must be balancing the force of rolling friction exerted by the roadway and the drag force exerted by the air. The power the engine produces is the product of this force and the velocity of the car.



(*a*) Apply $\sum F_x = ma_x$ to the car as it cruises at constant velocity:

$$F_{\substack{\text{static}\\\text{friction}}} - F_d - f = 0$$

Solve for $F_{\substack{\text{static}\\\text{friction}}}$ and substitute for F_d and f to obtain:

$$F_{\substack{\text{static}\\\text{friction}}} = Cv^2 + \mu_r F_g$$

or, because $F_n = F_g = mg$,

$$F_{\substack{\text{static}\\\text{friction}}} = Cv^2 + \mu_r mg$$

Multiplying both sides of this equation by the speed v of the car yields an expression for the power P developed by the engine:

$$F_{\substack{\text{static}\\\text{friction}}}\, v = P = \left(Cv^2 + \mu_{\text{r}} mg\right)v \quad (1)$$

Solve for C to obtain:

$$C = \frac{P}{v^3} - \frac{\mu_{\text{r}} mg}{v^2}$$

Express $P = 13.5$ hp in watts:

$$P = 13.5\ \text{hp} \times \frac{746\ \text{W}}{\text{hp}} = 10.07\ \text{kW}$$

Express $v = 55$ mi/h in m/s:

$$v = 55\,\frac{\text{mi}}{\text{h}} \times \frac{1609\ \text{m}}{1\ \text{mi}} \times \frac{1\ \text{h}}{3600\ \text{s}} = 24.58\ \text{m/s}$$

Substitute numerical values and evaluate C:

$$C = \frac{10.07\ \text{kW}}{(24.58\ \text{m/s})^3} - \frac{(0.0150)(1220\ \text{kg})(9.81\ \text{m/s}^2)}{(24.58\ \text{m/s})^2} = 0.3808\ \text{kg/m} = \boxed{0.381\ \text{kg/m}}$$

(*b*) Rewriting equation (1) with v as the variable yields the cubic equation:

$$v^3 + \frac{\mu_{\text{r}} mg}{C} v - \frac{P}{C} = 0$$

Express 164 hp in watts:

$$P = 164\ \text{hp} \times \frac{746\ \text{W}}{\text{hp}} = 122.3\ \text{kW}$$

Substitute numerical values in the cubic equation and simplify to obtain:

$$v^3 + \frac{(0.015)(1220\ \text{kg})(9.81\ \text{m/s}^2)}{0.3808\ \text{kg/m}} v - \frac{122.3\ \text{kW}}{0.3808\ \text{kg/m}} = 0$$

or

$$v^3 + \left(471.4\ \text{m}^2/\text{s}^2\right)v - 3.213\times 10^5\ \text{m}^3/\text{s}^3 = 0$$

Use your graphing calculator to solve this cubic equation and then use conversion factors from your text to convert your answer to mi/h:

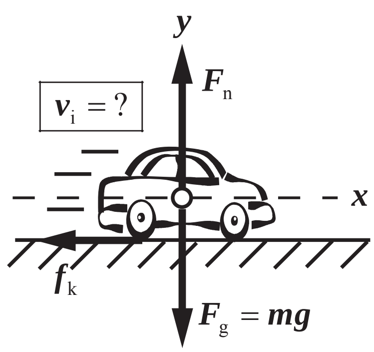
$$v = 66.20\,\frac{\text{m}}{\text{s}} \times \frac{0.6818\,\frac{\text{mi}}{\text{h}}}{0.3048\,\frac{\text{m}}{\text{s}}} = \boxed{148\ \text{mi/h}}$$

56 •• As you drive your car along a country road at night, a deer jumps out of the woods and stands in the middle of the road ahead of you. This occurs just as you are passing from a 55-mi/h zone to a 50-mi/h zone. At the 50-mi/h speed-

limit sign, you slam on the car's brakes, causing them to lock up, and skid to a stop inches from the startled deer. As you breathe a sigh of relief, you hear the sound of a police siren. The policeman proceeds to write you a ticket for driving 56 mi/h in 50-mi/h zone. Because of your preparation in physics, you are able to use the 25-m-long skid marks that your car left behind as evidence that you were not speeding. What evidence do you present? In formulating your answer, you will need to know the coefficient of kinetic friction between automobile tires and dry concrete (see Table 5-1).

Comment: DAVID: Your solution needs to refer to the table value. In addition, you need to change static to kinetic near the end of the solution

Picture the Problem We can apply the work-kinetic energy theorem to the car as it slides to a halt to relate the work done by kinetic friction to its initial speed v_i. Because the sum of the forces acting on the car in the y direction is zero, we can use the definition of the coefficient of kinetic friction to express the kinetic friction force $\vec{f}_k$ in terms of the gravitational force $\vec{F}_g = m\vec{g}$.



Apply the work-kinetic energy theorem to your car to obtain:

$$W_{total} = \Delta K = K_f - K_i$$

or, because $K_f = 0$,

$$W_{total} = -K_i \qquad (1)$$

The total work done on your car is done by the kinetic friction force that brings your car to a stop:

$$W_{total} = W_{by\ friction} = \vec{f}_k \cdot \vec{d} = f_k d \cos\theta$$

or, because $\vec{f}_k$ and $\vec{d}$ are in opposite directions ($\theta = 180°$),

$$W_{total} = -f_k d$$

Substituting for K_i and W_{total} in equation (1) yields:

$$-f_k d = -\tfrac{1}{2} m v_i^2$$

or, because $f_k = \mu_k F_n$,

$$\mu_k F_n d = \tfrac{1}{2} m v_i^2 \qquad (2)$$

Apply $\sum F_y = ma_y$ to your car to obtain:

$$F_n - F_g = 0 \Rightarrow F_n = F_g = mg$$

Substituting for F_n in equation (2) yields:

$$\mu_k mgd = \tfrac{1}{2} m v_i^2 \Rightarrow v_i = \sqrt{2\mu_k g d}$$

Referring to Table 5-1 for the coefficient of kinetic friction for rubber on dry concrete, substitute numerical values and evaluate v_i:

$$v_i = \sqrt{2(0.80)(9.81\ \text{m/s}^2)(25\ \text{m})} = 19.81\ \text{m/s}$$

Convert v_i to mi/h to obtain:

$$v = 19.81\frac{\text{m}}{\text{s}} \times \frac{0.6818\,\frac{\text{mi}}{\text{h}}}{0.3048\,\frac{\text{m}}{\text{s}}} = 44\ \text{mi/h}$$

Thus, at the time you applied your brakes, which occurred as you passed the 50 mi/h sign, you were doing 6 mi/h under the limit. Your explanation should convince the judge to void the ticket.

General Problems

57 • In February 2002, a total of 60.7 billion kW·h of electrical energy was generated by nuclear power plants in the United States. At that time, the population of the United States was about 287 million people. If the average American has a mass of 60 kg, and if 25% of the entire energy output of all nuclear power plants was diverted to supplying energy for a single giant elevator, estimate the height h to which the entire population of the country could be lifted by the elevator. In your calculations, assume that g is constant over the entire height h.

Picture the Problem 25 % of the electrical energy generated is to be diverted to do the work required to change the potential energy of the American people. We can calculate the height to which they can be lifted by equating the change in potential energy to the available energy.

Express the change in potential energy of the population of the United States in this process:

$$\Delta U = Nmgh$$

Letting E represent the total energy generated in February 2002, relate the change in potential to the energy available to operate the elevator:

$$Nmgh = 0.25E \Rightarrow h = \frac{0.25E}{Nmg}$$

Substitute numerical values and evaluate *h*:

$$h=\frac{(0.25)\left(60.7\times10^{9}\,\text{kW}\cdot\text{h}\right)\left(\dfrac{3600\,\text{s}}{1\,\text{h}}\right)}{\left(287\times10^{6}\right)(60\,\text{kg})\left(9.81\,\text{m/s}^{2}\right)}=\boxed{3.2\times10^{5}\,\text{m}}$$

58 • One of the most powerful cranes in the world operates in Switzerland. It can slowly raise a 6000-t load to a height of 12.0 m. (Note that 1 t = one tonne is sometimes called a metric ton. It is a unit of mass, not force, and is equal to 1000 kg.) (*a*) How much work is done by the crane during this lift? (*b*) If it takes 1.00 min to lift this load to this height at constant velocity, and the crane is 20 percent efficient, find the total (gross) power rating of the crane.

Picture the Problem We can use the definition of the work to find the work done by the crane in lifting this load and the definition of power to find the total power rating of the crane.

(*a*) The work done by the crane in lifting the load from position 1 to position 2 is given by:

$$W_{\text{by crane}}=\int_1^2\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{s}}$$

where $\vec{\boldsymbol{F}}$ is the force exerted by the crane.

Because $\vec{\boldsymbol{F}}$ and $d\vec{\boldsymbol{s}}$ are in the same direction:

$$W_{\text{by crane}}=\int_1^2 F\,ds=F(s_2-s_1)=Fh$$

or, because $F=F_{\text{g}}=mg$,

$$W_{\text{by crane}}=mgh$$

Substitute numerical values and evaluate $W_{\text{by crane}}$:

$$W_{\text{by crane}}=\left(6.000\times10^{6}\,\text{kg}\right)\left(9.81\,\text{m/s}^{2}\right)(12.0\,\text{m})=706.3\,\text{MJ}=\boxed{706\,\text{MJ}}$$

(*b*) If the efficiency of the crane is η, the total power rating of the crane is given by:

$$P_{\text{total}}=\frac{\Delta W}{\eta(\Delta t)}$$

where ΔW is the work we calculated in part (*a*).

Substitute numerical values and evaluate P_{total}:

$$P_{\text{total}}=\frac{706.3\,\text{MJ}}{(0.20)(60.0\,\text{s})}=\boxed{59\,\text{MW}}$$

59 • In Austria, there once was a 5.6-km-long ski lift. It took about 60 min for a gondola to travel up its length. If there were 12 gondolas going up, each with a cargo of mass 550 kg, and if there were 12 empty gondolas going down, and the angle of ascent was 30º, estimate the power *P* the engine needed to deliver in order to operate the ski lift.

Picture the Problem The power *P* of the engine needed to operate this ski lift is equal to the rate at which it does work on the gondolas. Because as many empty gondolas are descending as are ascending, we do not need to know their mass.

Express the rate at which work is done as the cars are lifted:

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{F\Delta s}{\Delta t}$$

where *F* is the force exerted by the engine on the gondola cars and Δs is the vertical displacement of the cars.

For *N* gondola cars, each of mass *M*, $F = F_g = NMg$:

$$P_{av} = \frac{NMg\Delta s}{\Delta t}$$

Relate Δs to the angle of ascent θ and the length *L* of the ski lift:

$$\Delta s = L\sin\theta$$

Substituting for Δs yields:

$$P_{av} = \frac{NMgL\sin\theta}{\Delta t}$$

Substitute numerical values and evaluate *P*:

$$P = \frac{12(550\,\text{kg})(9.81\,\text{m/s}^2)(5.6\,\text{km})\sin 30°}{(60\,\text{min})(60\,\text{s/min})} = \boxed{50\,\text{kW}}$$

60 •• To complete your master's degree in physics, your advisor has you design a small linear accelerator capable of emitting protons, each with a kinetic energy of 10.0 keV. (The mass of a single proton is 1.67×10^{-27} kg.) In addition, 1.00×10^{9} protons per second must reach the target at the end of the 1.50-m-long accelerator. (*a*) What average power must be delivered to the stream of protons? (*b*) What force (assumed constant) must be applied to each proton? (*c*) What speed does each proton attain just before it strikes the target, assuming the protons start from rest?

Picture the Problem We can use the definition of average power and the work-kinetic energy theorem to find the average power delivered to the target. A second application of the work-kinetic energy theorem and the use of the definition of

work will yield the force exerted on each proton. Finally, we can use the definition of kinetic energy to find the speed of each proton just before it hits the target.

(*a*) The average power delivered by the stream of protons is given by:

$$P_{\text{av}} = \frac{\Delta W}{\Delta t} \quad (1)$$

Apply the work-kinetic energy theorem to the proton beam to obtain:

$$W_{\text{total}} = \Delta K$$

Substituting ΔK for ΔW in equation (1) yields:

$$P_{\text{av}} = \frac{\Delta K}{\Delta t}$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = \frac{10.0\ \text{keV/particle}}{1.00\times10^{-9}\ \text{s/particle}} \times \frac{1.602\times10^{-19}\ \text{J}}{\text{eV}} = 1.602\times10^{-6}\ \text{W} = \boxed{1.60\ \mu\text{W}}$$

(*b*) Apply the work-kinetic energy theorem to a proton to obtain:

$$W_{\text{total}} = \Delta K \quad (2)$$

The work done by the accelerating force $\vec{F}$ on each proton is:

$$W_{\text{total}} = \vec{F}\cdot\vec{d} = Fd\cos\theta$$

or, because $\vec{F}$ and $\vec{d}$ are in the same direction,

$$W_{\text{total}} = Fd$$

Substituting for W_{total} in equation (2) yields:

$$Fd = \Delta K \Rightarrow F = \frac{\Delta K}{d}$$

Substitute numerical values and evaluate F:

$$F = \frac{10.0\ \text{keV}\times\dfrac{1.602\times10^{-19}\ \text{J}}{\text{eV}}}{1.50\ \text{m}} = 1.068\times10^{-15}\ \text{N} = \boxed{1.07\ \text{fN}}$$

(*c*) The kinetic energy of each proton is related to its speed:

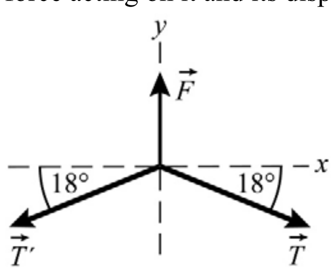
$$K_{\text{f}} = \tfrac{1}{2}mv_{\text{f}}^2 \Rightarrow v_{\text{f}} = \sqrt{\frac{2K_{\text{f}}}{m}}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{\frac{2\left(10.0\,\text{keV} \times \frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)}{1.67\times10^{-27}\ \text{kg}}} = \boxed{1.39\times10^{6}\ \text{m/s}}$$

61 ••• The four strings pass over the bridge of a violin, as shown in Figure 6-35. The strings make an angle of 72.0° with the normal to the plane of the instrument on either side of the bridge. The resulting total normal force pressing the bridge into the violin is 1.00×10^3 N. The length of the strings from the bridge to the peg to which each is attached is 32.6 cm. (*a*) Determine the tension in the strings, assuming the tension is the same for each string. (*b*) One of the strings is plucked out a distance of 4.00 mm, as shown. Make a free-body diagram showing all of the forces acting on the segment of the string in contact with the finger (not shown), and determine the force pulling the segment back to its equilibrium position. Assume the tension in the string remains constant during the pluck. (*c*) Determine the work done on the string in plucking it out that distance. Remember that the net force pulling the string back to its equilibrium position is changing as the string is being pulled out, but assume that the magnitudes of the tension forces remain constant.

Picture the Problem The free-body diagram shows the forces acting on one of the strings at the bridge. The force whose magnitude is F is one-fourth of the force (1.00×10^3 N) the bridge exerts on the strings. We can apply the condition for equilibrium in the y direction to find the tension in each string. Repeating this procedure at the site of the plucking will yield the restoring force acting on the string. We can find the work done on the string as it returns to equilibrium from the product of the average force acting on it and its displacement.



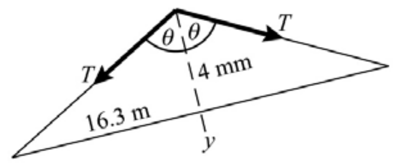
(*a*) Noting that, due to symmetry $T' = T$, apply $\sum F_y = 0$ to the string at the point of contact with the bridge:

$$F - 2T\sin 18.0° = 0 \Rightarrow T = \frac{F}{2\sin 18.0°}$$

Substitute numerical values and evaluate T:

$$T = \frac{\frac{1}{4}\left(1.00\times10^{3}\ \text{N}\right)}{2\sin18.0°} = 404.51\ \text{N}$$
$$= \boxed{405\ \text{N}}$$

(b) A free-body diagram showing the forces restoring the string to its equilibrium position just after it has been plucked is shown to the right:



Express the net force acting on the string immediately after it is released:

$$F_{\text{net}} = 2T\cos\theta \qquad (1)$$

Use trigonometry to find θ:

$$\theta = \tan^{-1}\left(\frac{16.3\,\text{cm}}{4.00\,\text{mm}}\times\frac{10\,\text{mm}}{\text{cm}}\right) = 88.594°$$

Substitute numerical values for T and θ in equation (1) and evaluate F_{net}:

$$F_{\text{net}} = 2(404.51\,\text{N})\cos88.594°$$
$$= 19.851\text{N} = \boxed{19.9\,\text{N}}$$

(c) Express the work done on the string in displacing it a distance dx':

$$dW = Fdx'$$

If we pull the string out a distance x', the magnitude of the force pulling it down is approximately:

$$F = (2T)\frac{x'}{L/2} = \frac{4T}{L}x'$$

Substitute to obtain:

$$dW = \frac{4T}{L}x'\,dx'$$

Integrate dW to obtain:

$$W = \frac{4T}{L}\int_0^x x'dx' = \frac{2T}{L}x^2$$

where x is the final displacement of the string.

Substitute numerical values and evaluate W:

$$W = \frac{2(404.51\,\text{N})}{32.6\times10^{-2}\ \text{m}}\left(4.00\times10^{-3}\ \text{m}\right)^2$$
$$= \boxed{39.7\,\text{mJ}}$$

62 •• The magnitude of the single force acting on a particle of mass m is given by $F = bx^2$, where b is a constant. The particle starts from rest. After it travels a distance L, determine its (a) kinetic energy and (b) speed.

Comment: DAVID: F_x was changed to F, and a was changed to b.

Picture the Problem We can apply the work-kinetic energy theorem and the definition of work to the particle to find its kinetic energy when it has traveled a distance L. We can then use the definition of kinetic energy to find its speed at this location.

(a) Apply the work-kinetic energy theorem to the particle to obtain:

$$W_{\text{total}} = \Delta K = K_{\text{f}} - K_{\text{i}}$$

or, because the particle is initially at rest,

$$W_{\text{total}} = K_{\text{f}} = \tfrac{1}{2}mv_{\text{f}}^2 \qquad (1)$$

The total work done on the particle is also given by:

$$dW_{\text{total}} = \vec{F}\cdot d\vec{x} = Fdx\cos\theta$$

or, because $\vec{F}$ and $d\vec{x}$ are in the same direction,

$$dW_{\text{total}} = K_{\text{f}} = Fdx$$

Substituting for F_x and going from differential to integral form yields:

$$K_{\text{f}} = \int_0^L bx^2dx = b\int_0^L x^2dx = \boxed{\tfrac{1}{3}bL^3}$$

(b) Substitute for K_{f} in equation (1) to obtain:

$$\tfrac{1}{3}bL^3 = \tfrac{1}{2}mv_{\text{f}}^2 \Rightarrow v_{\text{f}} = \boxed{\sqrt{\frac{2bL^3}{3m}}}$$

63 •• **[SSM]** A single horizontal force in the $+x$ direction acts on a cart of mass m. The cart starts from rest at $x = 0$, and the speed of the cart increases with x as $v = Cx$, where C is a constant. (a) Find the force acting on the cart as a function of x. (b) Find the work done by the force in moving the cart from $x = 0$ to $x = x_1$.

Picture the Problem We can use the definition of work to obtain an expression for the position-dependent force acting on the cart. The work done on the cart can be calculated from its change in kinetic energy.

(a) Express the force acting on the cart in terms of the work done on it:

$$F(x) = \frac{dW}{dx} \qquad (1)$$

Apply the work-kinetic energy theorem to the cart to obtain:

$$W = \Delta K = K - K_{\text{i}}$$

or, because the cart starts from rest,

$$W = K$$

Substitute for W in equation (1) to obtain:

$$F(x)=\frac{dK}{dx}=\frac{d}{dx}\left(\tfrac{1}{2}mv^2\right)=\frac{d}{dx}\left[\tfrac{1}{2}m(Cx)^2\right]=\boxed{mC^2x}$$

(*b*) The work done by this force changes the kinetic energy of the cart:

$$W=\Delta K=\tfrac{1}{2}mv_1^2-\tfrac{1}{2}mv_0^2=\tfrac{1}{2}mv_1^2-0=\tfrac{1}{2}m(Cx_1)^2=\boxed{\tfrac{1}{2}mC^2x_1^2}$$

64 •• A force $\vec{F}=\left(2.0\ \text{N/m}^2\right)x^2\hat{i}$ is applied to a particle initially at rest in the *xy* plane. Find the work done by this force on the particle, and its final speed, as it moves along a path that is (*a*) in a straight line from point (2.0 m, 2.0 m) to point (2.0 m, 7.0 m), and (*b*) in a straight line from point (2.0 m, 2.0 m) to point (5.0 m, 6.0 m). (The given force is the only force doing work on the particle.)

Picture the Problem The work done by $\vec{\boldsymbol{F}}$ on the particle depends on whether $\vec{\boldsymbol{F}}$ causes a displacement of the particle in the direction it acts. We can apply the work-kinetic energy theorem to find the final speed of the particle in each case.

(*a*) The work done on the particle is given by:

$$W=\int\vec{\boldsymbol{F}}\cdot d\vec{\ell}$$

Substitute for $\vec{\boldsymbol{F}}$ and $\boldsymbol{d}\vec{\ell}$ to obtain:

$$W=\int_{2.0\,\text{m}}^{7.0\,\text{m}}\left(2.0\ \text{N/m}^2\right)x^2\,\hat{\boldsymbol{i}}\cdot dx\,\hat{\boldsymbol{j}}=\boxed{0}$$

Apply the work-kinetic energy theorem to the particle to obtain:

$$W=\Delta K=K_\text{f}-K_\text{i}$$

or, because the particle starts from rest and $W=0$,

$$K_\text{f}=0\Rightarrow v_\text{f}=\boxed{0}$$

(*b*) Calculate the work done by $\vec{F}$ during the displacement of the particle from $x = 2.0$ m to 5.0 m:

$$W = \int_{2.0\,\text{m}}^{5.0\,\text{m}} \left(\left(2.0\,\text{N/m}^2\right)x^2\right)\hat{\boldsymbol{i}} \cdot dx\,\hat{\boldsymbol{i}}$$

$$= \left(2.0\,\text{N/m}^2\right)\int_{2.0\,\text{m}}^{5.0\,\text{m}} x^2\,dx$$

$$= \left(2.0\,\text{N/m}^2\right)\left[\frac{x^3}{3}\right]_{2.0\,\text{m}}^{5.0\,\text{m}}$$

$$= \boxed{78\,\text{J}}$$

Apply the work-kinetic energy theorem to the particle to obtain:

$$W = \Delta K = K_\text{f} - K_\text{i}$$

or, because the particle starts from rest,

$$K_\text{f} = 78\text{ J} \Rightarrow v_\text{f} = \sqrt{\frac{2(78\text{ J})}{m}} = \boxed{\frac{156\text{ J}}{m}}$$

where m is the mass of the particle.

65 •• A particle of mass m moves along the x axis. Its position varies with time according to $x = 2t^3 - 4t^2$, where x is in meters and t is in seconds. Find (*a*) the velocity and acceleration of the particle as functions of t, (*b*) the power delivered to the particle as a function of t, and (*c*) the work done by the net force from $t = 0$ to $t = t_1$.

Picture the Problem The velocity and acceleration of the particle can be found by differentiation. The power delivered to the particle can be expressed as the product of its velocity and the net force acting on it, and the work done by the force and can be found from the change in kinetic energy this work causes.

In the following, if t is in seconds and m is in kilograms, then v is in m/s, a is in m/s^2, P is in W, and W is in J.

(*a*) The velocity of the particle is given by:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(2t^3 - 4t^2\right) = \boxed{\left(6t^2 - 8t\right)}$$

The acceleration of the particle is given by:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(6t^2 - 8t\right) = \boxed{(12t - 8)}$$

(*b*) Express the rate at which energy is delivered to this particle as it accelerates:

$$P = Fv = mav$$

Substitute for a and v and simplify to obtain:

$$P = m(12t - 8)(6t^2 - 8t) = \boxed{8mt(9t^2 - 18t + 8)}$$

(c) Apply the work-kinetic energy theorem to the particle to obtain:

$$W = \Delta K = K_1 - K_0$$

Substitute for K_1 and K_0 and simplify:

$$W = \tfrac{1}{2}m\left[(v(t_1))^2 - (v(0))^2\right] = \tfrac{1}{2}m\left[(6t_1^2 - 8t_1)\right]^2 - 0 = \boxed{2mt_1^2(3t_1 - 4)^2}$$

Remarks: We could also find *W* by integrating *P*(*t*) with respect to time.

66 •• A 3.0-kg particle starts from rest at x = 0.050 m and moves along the x axis under the influence of a single force $F_x = 6.0 + 4.0x - 3.0x^2$, where F_x is in newtons and x is in meters. (a) Find the work done by the force as the particle moves from x = 0.050 m to x = 3.0 m. (b) Find the power delivered to the particle as it passes through the point x = 3.0 m.

Comment: DAVID: The initial position has been changed.

Picture the Problem We can calculate the work done by the given force from its definition. The power can be determined from $P = \vec{F}\cdot\vec{v}$ and v from the change in kinetic energy of the particle produced by the work done on it.

(a) Express the work done on the particle:

$$W = \int \vec{F}_x \cdot d\vec{s} = \int F_x\hat{i}\cdot ds\,\hat{i} = \int F_x ds$$

Substitute for F_x and ds and evaluate the definite integral:

$$W = \int_{0.50\text{ m}}^{3.0\text{ m}} (6.0 + 4.0x - 3.0x^2)dx = \left[6.0x + \frac{4.0x^2}{2} - \frac{3.0x^3}{3}\right]_{0.50\text{ m}}^{3.0\text{ m}} = 5.625\text{ J} = \boxed{5.6\text{ J}}$$

(b) Express the power delivered to the particle as it passes through the point at x = 3.0 m:

$$P = \vec{F}\cdot\vec{v} = F_{x=3.0\text{ m}}v_{x=3.0\text{ m}} \qquad (1)$$

Apply the work-kinetic energy theorem to find the work done on the particle as it moves from the point at x = 0.050 m to the point at x = 3.0 m:

$$W = \Delta K = K_{x=3.0\text{ m}} - K_{x=0.050\text{ m}}$$

or, because $K_{x=0.050\text{ m}} = 0$,

$$W = K_{x=3.0\text{ m}} = \tfrac{1}{2}mv_{x=3.0\text{ m}}^2$$

Solving for $v_{x=3.0\,\text{m}}$ gives:

$$v_{x=3.0\,\text{m}} = \sqrt{\frac{2W}{m}}$$

Substituting for $v_{x\,=\,3.0\,\text{m}}$ and $F_{x\,=\,3.0\,\text{m}}$ in equation (1) yields:

$$P_{x=3.0\,\text{m}} = \left(6.0 + 4.0x - 3.0x^2\right)_{x=3.0\,\text{m}} \sqrt{\frac{2W_{x=3.0\,\text{m}}}{m}}$$

Substitute numerical values and evaluate $P_{x\,=\,3.0\,\text{m}}$:

$$P_{x=3.0\,\text{m}} = \left[6.0 + 4.0(3.0\,\text{m}) - 3.0(3.0\,\text{m})^2\right]\sqrt{\frac{2(5.625\,\text{J})}{3.0\,\text{kg}}} = \boxed{-17\,\text{W}}$$

67 •• The initial kinetic energy imparted to a 0.0200-kg bullet is 1200 J. (*a*) Assuming it accelerated down a 1.00-m-long rifle barrel, estimate the average power delivered to it during the firing. (*b*) Neglecting air resistance, find the range of this projectile when it is fired at an angle such that the range equals the maximum height attained.

Picture the Problem We'll assume that the firing height is negligible and that the bullet lands at the same elevation from which it was fired. We can use the equation $R = \left(v_0^2/g\right)\sin 2\theta$ to find the range of the bullet and constant-acceleration equations to find its maximum height. The bullet's initial speed can be determined from its initial kinetic energy.

(*a*) The average power delivered to the bullet during firing is given by:

$$P_{\text{av}} = \frac{\Delta W}{\Delta t} \qquad (1)$$

Apply the work-kinetic energy theorem to the bullet to obtain:

$$W_{\text{total}} = \Delta K = K_{\text{f}} - K_{\text{i}}$$

or, because the initial speed of the bullet is zero,

$$W_{\text{total}} = K_{\text{f}} = \tfrac{1}{2}mv_{\text{f}}^2 \Rightarrow v_{\text{f}} = \sqrt{\frac{2K_{\text{f}}}{m}}$$

Because the acceleration of the bullet is (assumed to be) constant, its average speed in the barrel is:

$$v_{\text{av}} = \frac{v_{\text{i}} + v_{\text{f}}}{2} = \tfrac{1}{2}v_{\text{f}}$$

The time-in-the-barrel is given by:

$$\Delta t = \frac{\ell}{v_{\text{av}}} = \frac{2\ell}{v_{\text{f}}} = \frac{2\ell}{\sqrt{\dfrac{2K_{\text{f}}}{m}}}$$

Substituting for Δt in equation (1) and simplifying yields:

$$P_{\text{av}} = \frac{K_{\text{f}}}{2\ell}\sqrt{\frac{2K_{\text{f}}}{m}}$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = \frac{1200\ \text{J}}{2(1.00\ \text{m})}\sqrt{\frac{2(1200\ \text{J})}{0.0200\ \text{kg}}} = \boxed{208\ \text{kW}}$$

(*b*) Using the ″equal-height″ range equation, express the range of the bullet as a function of its firing speed and angle of firing:

$$R = \frac{v_0^2}{g}\sin 2\theta$$

Rewrite the range equation using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{2v_0^2 \sin\theta\cos\theta}{g} \quad (2)$$

Using constant-acceleration equations, express the position coordinates of the projectile along its flight path in terms of the parameter *t*:

$$x = (v_0\cos\theta)t$$

and

$$y = (v_0\sin\theta)t - \tfrac{1}{2}gt^2$$

Eliminate the parameter *t* and make use of the fact that the maximum height occurs when the projectile is at half the range to obtain:

$$h = \frac{(v_0\sin\theta)^2}{2g}$$

Equate *R* and *h* to obtain:

$$\frac{(v_0\sin\theta)^2}{2g} = \frac{2v_0^2\sin\theta\cos\theta}{g}$$

Simplifying this equation yields:

$$\tan\theta = 4 \Rightarrow \theta = \tan^{-1}(4) = 76.0°$$

Relate the bullet's kinetic energy to its mass and speed and solve for the square of its speed:

$$K = \tfrac{1}{2}mv_0^2 \Rightarrow v_0^2 = \frac{2K}{m}$$

Substitute for v_0^2 in equation (2) to obtain:

$$R = \frac{2K\sin 2\theta}{mg}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{2(1200\ \text{J})\sin[2(76.0°)]}{(0.0200\ \text{kg})(9.81\ \text{m/s}^2)} = \boxed{5.74\ \text{km}}$$

68 •• The force F_x acting on a 0.500-kg particle is shown as a function of x in Figure 6-36. (*a*) From the graph, calculate the work done by the force when the particle moves from x = 0.00 to the following values of x: –4.00, –3.00, –2.00, –1.00, +1.00, +2.00, +3.00, and +4.00 m. (*b*) If it starts with a velocity of 2.00 m/s in the +x direction, how far will the particle go in that direction before stopping?

Picture the Problem The work done on the particle is the area under the force-versus-displacement curve. Note that for negative displacements, F is positive, so W is negative for $x < 0$.

(*a*) Use either the formulas for the areas of simple geometric figures *or* counting squares and multiplying by the work represented by one square (each square is 1.00 J) to complete the following table:

x, m	–4.00	–3.00	–2.00	–1.00	1.00	2.00	3.00	4.00
W, J	–11	–10	–7.0	–3.0	1.0	0	–2.0	–3.0

(*b*) The energy that the particle must lose before it stops is its kinetic energy at $x = 0$:

$$K_{x=0} = \tfrac{1}{2} m v_{x=0}^2$$

Substitute numerical values and evaluate $K_{x=0}$:

$$K_{x=0} = \tfrac{1}{2}(0.500\text{ kg})(2.00\text{ m/s})^2 = 1.00\text{ J}$$

The particle is accelerated (recall that $F_x = -dU/dx$) between $x = 0$ and $x = 1.00$ m and gains an additional 1.00 J of kinetic energy. When it arrives at $x = 2.00$ m, however, it will have lost the kinetic energy it gained between $x = 0$ and $x = 1.00$ m and its kinetic energy will again be 1.00 J. Inspection of the graph leads us to conclude that it will have lost its remaining kinetic energy when it reaches $x = \boxed{2.50\text{ m.}}$

69 •• (*a*) Repeat Problem 68(*a*) for the force F_x shown in Figure 6-37. (*b*) If the object starts at the origin moving to the right with a kinetic energy of 25.0 J, how much kinetic energy does it have at x = 4.00 m.

Picture the Problem The work done on the particle is the area under the force-versus-displacement curve. Note that for negative displacements, F is negative, so W is positive for $x < 0$. In (*b*), we can apply the work-kinetic energy theorem to find the kinetic energy at x = 4.0 m.

(*a*) Use either the formulas for the areas of simple geometric figures *or* counting squares and multiplying by the work represented by one square (each square is 1.00 J) to complete the following table:

x, m	–4.00	–3.00	–2.00	–1.00	0	1.00	2.00	3.00	4.00
W, J	6	4	2	0.5	0	0.5	1.5	2.5	3.0

(*b*) Apply the work-kinetic energy theorem to the particle to obtain:

$$W_{\text{total}} = \Delta K = K_{x=4\,\text{m}} - K_{x=0}$$

Solving for $K_{x=4\,\text{m}}$ yields:

$$K_{x=4\,\text{m}} = W_{\text{total}} + K_{x=0} = W_{\text{total}} + 25.0\ \text{J}$$

From $x = 0$ to $x = 4$ m, the total work done by the force is:

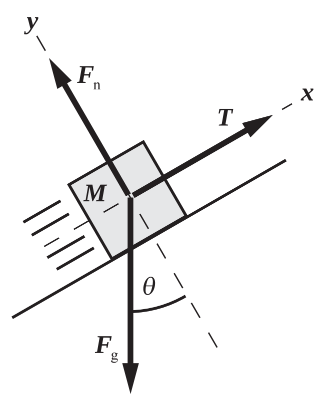
$$W_{\text{total}} = 3\ \text{square} \times \frac{1.00\ \text{J}}{\text{square}} = 3.00\ \text{J}$$

Substitute for W_{total} in the expression for $K_{x=4\,\text{m}}$ to obtain:

$$K_{x=4\,\text{m}} = 3.00\ \text{J} + 25.0\ \text{J} = \boxed{28.0\ \text{J}}$$

70 •• A box of mass M is at rest at the bottom of a frictionless inclined plane (Figure 6-38). The box is attached to a string that pulls with a constant tension T. (*a*) Find the work done by the tension T as the box moves through a distance x along the plane. (*b*) Find the speed of the box as a function of x. (*c*) Determine the power delivered by the tension in the string as a function of x.

Picture the Problem The forces acting on the block are shown in the free-body diagram. We can use the definition of work to find the work done by $\vec{T}$ in displacing the block a distance x up the incline and the work-kinetic energy theorem to express the speed of the box as a function of x and θ. Finally, we can use the definition of power to determine the power produced by the tension in terms of x and θ.



(*a*) Use the definition of work to express the work the tension *T* does moving the box a distance *x* up the incline:

$$W=\int_0^x \vec{\boldsymbol{T}}\cdot d\vec{\mathbf{x}}=\int_0^x T\,dx\cos\phi$$

or, because the angle between $\vec{\boldsymbol{T}}$ and $d\vec{\mathbf{x}}$ is zero,

$$W=T\int_0^x dx$$

Evaluate this integral to obtain:

$$W_{\text{by }T}=\boxed{Tx}$$

(*b*) Apply the work-kinetic energy theorem to the box to obtain:

$$W_{\text{total}}=\Delta K=K_{\text{f}}-K_{\text{i}}$$

or, because the box is initially at rest,

$$W_{\text{total}}=K_{\text{f}}=\tfrac{1}{2}Mv_{\text{f}}^2\Rightarrow v_{\text{f}}=\sqrt{\frac{2W_{\text{total}}}{M}}$$

The total work done on the box is the work done by the net force acting on it:

$$W_{\text{total}}=\int_0^x \vec{\boldsymbol{F}}_{\text{net}}\cdot d\vec{\mathbf{x}}=\int_0^x F_{\text{net}}\,dx\cos\phi$$

or, because the angle between $\vec{F}_{\text{net}}$ and $d\vec{x}$ is zero, $W_{\text{total}}=\int_0^x F_{\text{net}}\,dx$

Referring to the force diagram, we see that the net force acting on the block is:

$$F_{\text{net}}=T-Mg\sin\theta$$

Substitute for F_{net} and evaluate the integral to obtain:

$$W_{\text{total}}=\int_0^x (T-Mg\sin\theta)dx=(T-Mg\sin\theta)x$$

Substituting for W_{total} in the expression for v_{f} and simplifying yields:

$$v_{\text{f}}=\sqrt{\frac{2(T-Mg\sin\theta)x}{M}}=\boxed{\sqrt{2\left(\frac{T}{M}-g\sin\theta\right)x}}$$

(c) The power produced by the tension in the string is given by:

$$P=\vec{\boldsymbol{T}}\cdot\vec{\boldsymbol{v}}_{\text{f}}=Tv_{\text{f}}\cos\phi$$

or, because the angle between $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{v}}_{\text{f}}$ is zero, $P=Tv_{\text{f}}$

Substitute for v_f to obtain:

$$P = \boxed{T\sqrt{2\left(\frac{T}{M} - g\sin\theta\right)x}}$$

71 ••• [SSM] A force acting on a particle in the *xy* plane at coordinates (*x*, *y*) is given by $\vec{F} = (F_0/r)(y\hat{i} - x\hat{j})$, where F_0 is a positive constant and *r* is the distance of the particle from the origin. (*a*) Show that the magnitude of this force is F_0 and that its direction is perpendicular to $\vec{r} = x\hat{i} + y\hat{j}$. (*b*) Find the work done by this force on a particle that moves once around a circle of radius 5.0 m that is centered at the origin.

Picture the Problem (*a*) We can use the definition of the magnitude of vector to show that the magnitude of $\vec{F}$ is F_0 and the definition of the scalar product to show that its direction is perpendicular to $\vec{r}$. (*b*) The work done as the particle moves in a circular path can be found from its definition.

(*a*) Express the magnitude of $\vec{F}$:

$$\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2} = \sqrt{\left(\frac{F_0}{r}y\right)^2 + \left(-\frac{F_0}{r}x\right)^2} = \frac{F_0}{r}\sqrt{x^2 + y^2}$$

Because $r = \sqrt{x^2 + y^2}$:

$$\left|\vec{F}\right| = \frac{F_0}{r}\sqrt{x^2 + y^2} = \frac{F_0}{r}r = \boxed{F_0}$$

Form the scalar product of $\vec{F}$ and $\vec{r}$:

$$\vec{F}\cdot\vec{r} = \left(\frac{F_0}{r}\right)\left(y\,\hat{i} - x\,\hat{j}\right)\cdot\left(x\,\hat{i} + y\,\hat{j}\right) = \left(\frac{F_0}{r}\right)(xy - xy) = 0$$

Because $\vec{F}\cdot\vec{r} = 0$:

$$\boxed{\vec{F} \perp \vec{r}}$$

(*b*) The work done in an angular displacement from θ_1 to θ_2 is given by:

$$W_{1\,\text{rev}} = \int_{\theta_1}^{\theta_2} \vec{F}\cdot d\vec{s} = \int_{\theta_1}^{\theta_2}\left(\frac{F_0}{r}\right)\left(y\,\hat{i} - x\,\hat{j}\right)\cdot d\vec{s} = \frac{F_0}{r}\int_{\theta_1}^{\theta_2}\left(r\sin\theta\,\hat{i} - r\cos\theta\,\hat{j}\right)\cdot d\vec{s} = F_0\int_{\theta_1}^{\theta_2}\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right)\cdot d\vec{s}$$

Express the radial vector $\vec{r}$ in terms of its magnitude r and the angle θ it makes with the positive x axis:

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

The differential of $\vec{r}$ is tangent to the circle (you can easily convince yourself of this by forming the dot product of $\vec{r}$ and $d\vec{r}$) and so we can use it as $d\vec{s}$ in our expression for the work done by $\vec{F}$ in one revolution:

$$d\vec{r} = d\vec{s} = -r\sin\theta\,d\theta\,\hat{i} + r\cos\theta\,d\theta\,\hat{j} = \left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right)d\theta$$

Substitute for $d\vec{s}$, simplifying, and integrate to obtain:

$$W_{1\,\text{rev,ccw}} = F_0\int_0^{2\pi}\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right)\cdot\left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right)d\theta$$
$$= -rF_0\int_0^{2\pi}\left(\sin^2\theta + \cos^2\theta\right)d\theta = -rF_0\int_0^{2\pi}d\theta = -rF_0\,\theta\Big]_0^{2\pi}$$
$$= -2\pi rF_0$$

Substituting the numerical value of r yields:

$$W_{1\text{rev, ccw}} = -2\pi(5.0\text{ m})F_0 = \boxed{(-10\pi\text{ m})F_0}$$

If the rotation is clockwise, the integral becomes:

$$W_{1\,\text{rev,cw}} = -rF_0\,\theta\Big]_{2\pi}^{0} = 2\pi rF_0$$

Substituting the numerical value of r yields:

$$W_{1\text{rev, cw}} = 2\pi(5.0\text{ m})F_0 = \boxed{(10\pi\text{ m})F_0}$$

72 ••• A force acting on a 2.0-kg particle in the *xy* plane at coordinates (*x*, *y*) is given by $\vec{F} = -\left(b/r^3\right)\left(x\,\hat{i} + y\,\hat{j}\right)$, where *b* is a positive constant and *r* is the

distance from the origin. (a) Show that the magnitude of the force is inversely proportional to r^2, and that its direction is antiparallel (opposite) to the radius vector $\vec{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. ($b$) If $b = 3.0\ \text{N}\cdot\text{m}^2$, find the work done by this force as the particle moves from (2.0 m, 0.0 m), to (5.0 m, 0.0 m) along a straight-line path. (c) Find the work done by this force on a particle moving once around a circle of radius r = 7.0 m that is centered at the origin.

Comment: DAVID: Part (d) has be deleted.

Picture the Problem We can substitute for r and $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ in $\vec{F}$ to show that the magnitude of the force varies as the reciprocal of the square of the distance to the origin, and that its direction is opposite to the radius vector. We can find the work done by this force by evaluating the integral of F with respect to x from an initial position x = 2.0 m, y = 0 m to a final position x = 5.0 m, y = 0 m. Finally, we can apply Newton's second law to the particle to relate its speed to its radius, mass, and the constant b.

(a) Express the magnitude of $\vec{F}$:

$$\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2} = \sqrt{\left(-\frac{b}{r^3}x\right)^2 + \left(-\frac{b}{r^3}y\right)^2} = \frac{b}{r^3}\sqrt{x^2+y^2}$$

The magnitude of r is given by:

$$r = \sqrt{x^2+y^2}$$

Substitute for $\sqrt{x^2+y^2}$ and simplify to obtain:

$$\left|\vec{F}\right| = \frac{b}{r^3}r = \boxed{\frac{b}{r^2}}$$

That is, the magnitude of the force varies inversely with the square of the distance to the origin. Because $\vec{F} = -\left(b/r^3\right)\vec{r}$, the direction of $\vec{F}$ is opposite the direction of $\vec{r}$.

(b) Find the work done by this force by evaluating the integral of F with respect to x from an initial position x = 2.0 m, y = 0 m to a final position x = 5.0 m, y = 0 m:

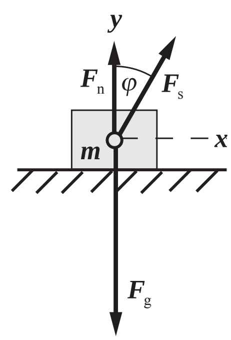
$$W = -\int_{2.0\,\text{m}}^{5.0\,\text{m}} \frac{b}{x^2}dx = b\left[\frac{1}{x}\right]_{2.0\,\text{m}}^{5.0\,\text{m}} = 3.0\ \text{N}\cdot\text{m}^2\left(\frac{1}{5.0\ \text{m}} - \frac{1}{2.0\ \text{m}}\right) = \boxed{-0.90\ \text{J}}$$

(c) No work is done as the force is perpendicular to the velocity.

73 ••• A block of mass m on a horizontal frictionless tabletop is attached by a swivel to a spring that is attached to the ceiling (Figure 6-39). The vertical distance between the top of the block and the ceiling is y_o, and the horizontal position is x. When the block is at $x = 0$, the spring, which has force constant k, is completely unstressed. (a) What is F_x, the x component of the force on the block due to the spring, as a function of x? (b) Show that F_x is proportional to x^3 for sufficiently small values of $|x|$. (c) If the block is released from rest at $x = x_0$, where $|\mathbf{x}_0| << \mathbf{y}_0$, what is its speed when it reaches $x = 0$?

Comment: DAVID: The initial position has be labeled x_0.

Picture the Problem We can use trigonometry to express the x-component of $\vec{F}_s$ and the Pythagorean theorem to write it as a function of x. Expanding on of the terms in F_x binomially reveals its cubic dependence on x. In (c), we can apply the work-kinetic energy theorem and use the definition of work to find the speed of the block when it reaches $x = 0$.



(a) F_x, the x-component of $\vec{F}_s$, is given by:

$$F_x = -F_s \sin\varphi \quad (1)$$

Express F_s when the length of the spring is ℓ:

$$F_s = k(\ell - y_0)$$

Substituting for F_s in equation (1) yields:

$$F_x = -k(\ell - y_0)\sin\varphi \quad (2)$$

Referring to the figure, note that:

$$\sin\varphi = \frac{x}{\ell}$$

Substitute for $\sin\varphi$ to obtain:

$$F_x = -k(\ell - y_0)\frac{x}{\ell} = -kx\left(1 - \frac{y_0}{\ell}\right) \quad (3)$$

Refer to the figure again to see that:

$$\ell^2 = \mathbf{x}^2 + \mathbf{y}_0^2 \Rightarrow \ell = \sqrt{\mathbf{x}^2 + \mathbf{y}_0^2}$$

Substituting for ℓ in equation (3) yields:

$$F_x = \boxed{-kx\left(1 - \frac{y_0}{\sqrt{x^2 + y_0^2}}\right)} \quad (4)$$

(b) Rewrite the second term in parentheses in equation (4) by dividing numerator and denominator by y_0 to obtain:

$$F_x = -kx\left(1 - \frac{1}{\frac{1}{y_0}\sqrt{x^2 + y_0^2}}\right) = -kx\left(1 - \frac{1}{\sqrt{1 + \frac{x^2}{y_0^2}}}\right) = -kx\left(1 - \left(1 + \frac{x^2}{y_0^2}\right)^{-\frac{1}{2}}\right)$$

Expanding $\left(1 + \frac{x^2}{y_0^2}\right)^{-\frac{1}{2}}$ binomially yields:

$$\left(1 + \frac{x^2}{y_0^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{x^2}{y_0^2} + \text{higher order terms}$$

For $x << y_0$, we can ignore the higher –order terms and :

$$\left(1 + \frac{x^2}{y_0^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{x^2}{y_0^2}$$

Substitute for $\left(1 + \frac{x^2}{y_0^2}\right)^{-\frac{1}{2}}$ in the expression for F_x and simplify to obtain:

$$F_x = -kx\left[1 - \left(1 + \frac{1}{2}\frac{x^2}{y_0^2}\right)\right] = \boxed{-\frac{k}{2y_0^2}x^3}$$

(c) Apply the work-kinetic energy theorem to the block to obtain:

$$W_{\text{total}} = \Delta K = K_{\text{f}} - K_{\text{i}}$$

or, because the block is released from rest,

$$W_{\text{total}} = K_{\text{f}} = \tfrac{1}{2}mv_{\text{f}}^2 \qquad (5)$$

Integrate F_x to obtain:

$$W_{\text{total}} = -\frac{k}{2y_0^2}\int_L^0 x^3 dx = \frac{kL^4}{8y_0^2}$$

Substitute for W_{total} in equation (5):

$$\frac{kL^4}{8y_0^2} = \tfrac{1}{2}mv_{\text{f}}^2 \Rightarrow v_{\text{f}} = \boxed{\frac{L^2}{2y_0}\sqrt{\frac{k}{m}}}$$

74 •• Two horses pull a large crate at constant speed across a barn floor by means of two light steel cables. A large box of mass 250 kg sits on the crate (Figure 6-40). As the horses pull, the cables are parallel to the horizontal floor. The coefficient of friction between the crate and the barn floor is 0.25. (a) What is the work done by each horse if the box is moved a distance of 25 m? (b) What

is the tension in each cable if the angle between each cable and the direction the crate moves is 15°?

Picture the Problem We can apply the work-kinetic energy theorem to the crate to relate the work done by the horses and the work done by the friction force. In part (*b*), we can use the definition of work and our result from Part (*a*) to find the tension in the cables.

(*a*) Apply the work-kinetic energy theorem to the crate to obtain:

$$W_{\text{total}} = W_{\text{horses}} + W_{\text{friction}} = \Delta K$$

or, because the horses pull the crate at constant speed, $\Delta K = 0$, and

$$W_{\text{horses}} + W_{\text{friction}} = 0 \qquad (1)$$

The work done by friction is given by:

$$W_{\text{friction}} = \vec{\boldsymbol{f}}_{\text{k}} \cdot \vec{\boldsymbol{d}} = f_{\text{k}} d \cos\phi$$

where ϕ is the angle between $\vec{\boldsymbol{f}}_{\text{k}}$ and $\vec{\boldsymbol{d}}$.

Because $\vec{\boldsymbol{f}}_{\text{k}}$ and $\vec{\boldsymbol{d}}$ are in opposite directions:

$$W_{\text{friction}} = -f_{\text{k}} d$$

Because the normal force acting on the crate equals the weight of the crate:

$$W_{\text{friction}} = -\mu_{\text{k}} F_{\text{n}} d = -\mu_{\text{k}} mgd$$

Substitute for W_{friction} in equation (1) to obtain:

$$W_{\text{horses}} - \mu_{\text{k}} mgd = 0$$

and

$$W_{\text{horses}} = \mu_{\text{k}} mgd$$

Because each horse does half the work:

$$W_{\text{each horse}} = \tfrac{1}{2} W_{\text{horses}} = \tfrac{1}{2} \mu_{\text{k}} mgd$$

Substitute numerical values and evaluate $W_{\text{each horse}}$:

$$W_{\text{each horse}} = \tfrac{1}{2}(0.25)(250\ \text{kg})(9.81\ \text{m/s}^2)(25\ \text{m}) = 7.664\ \text{kJ} = \boxed{7.7\ \text{kJ}}$$

(*b*) The work done by each horse as a function of the tension in each cable:

$$W_{\text{each horse}} = \vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{d}} = Td\cos\theta$$

Solving for *T* yields:

$$T = \frac{W_{\text{each horse}}}{d\cos\theta}$$

Substitute numerical values and evaluate *T*:

$$T = \frac{7.664\ \text{kJ}}{(25\ \text{m})\cos 15°} = \boxed{0.32\ \text{kN}}$$

Chapter 7
The Conservation of Energy

Conceptual Problems

1 • [SSM] Two cylinders of unequal mass are connected by a massless cord that passes over a frictionless pulley (Figure 7-34). After the system is released from rest, which of the following statements are true? (U is the gravitational potential energy and K is the kinetic energy of the system.)
(a) $\Delta U < 0$ and $\Delta K > 0$, (b) $\Delta U = 0$ and $\Delta K > 0$, (c) $\Delta U < 0$ and $\Delta K = 0$,
(d) $\Delta U = 0$ and $\Delta K = 0$, (e) $\Delta U > 0$ and $\Delta K < 0$.

Determine the Concept Because the pulley is frictionless, mechanical energy is conserved as this system evolves from one state to another. The system moves and so we know that $\Delta K > 0$. Because $\Delta K + \Delta U = constant$, $\Delta U < 0$. $\boxed{(a)}$ is correct.

2 • Two stones are simultaneously thrown with the same initial speed from the roof of a building. One stone is thrown at an angle of 30° above the horizontal, the other is thrown horizontally. (Neglect effects due to air resistance.) Which statement below is true?

(a) The stones strike the ground at the same time and with equal speeds.
(b) The stones strike the ground at the same time with different speeds.
(c) The stones strike the ground at different times with equal speeds.
(d) The stones strike the ground at different times with different speeds.

Determine the Concept Choose the zero of gravitational potential energy to be at ground level. The two stones have the same initial energy because they are thrown from the same height with the same initial speeds. Therefore, they will have the same total energy at all times during their fall. When they strike the ground, their gravitational potential energies will be zero and their kinetic energies will be equal. Thus, their speeds at impact will be equal. The stone that is thrown at an angle of 30° above the horizontal has a longer flight time due to its initial upward velocity and so they do not strike the ground at the same time. $\boxed{(c)}$ is correct.

3 • True or false:

(a) The total energy of a system cannot change.
(b) When you jump into the air, the floor does work on you increasing your mechanical energy.
(c) Work done by frictional forces must always decrease the total mechanical energy of a system.
(d) Compressing a given spring 2.0 cm from its unstressed length takes more work than stretching it 2.0 cm from its unstressed length.

(*a*) False. Forces that are external to a system can do work on the system to change its energy.

(*b*) False. In order for some object to do work, it must exert a force *over some distance*. The mechanical energy gained by the jumper comes from the internal energy of the jumper, not from the floor.

(*c*) False. The frictional force that accelerates a sprinter increases the total mechanical energy of the sprinter.

(*d*) False. Because the work required to stretch a spring a given distance varies as the square of that distance, the work is the same regardless of whether the spring is stretched or compressed.

4 • As a novice ice hockey player (assume frictionless situation), you have not mastered the art of stopping except by coasting straight for the boards of the rink (assumed to be a rigid wall). Discuss the energy changes that occur as you use the boards to slow your motion to a stop.

Determine the Concept The boards do not do any work on you. Your loss of kinetic energy is converted into thermal energy of your body.

5 • True or false (The particle in this question can move only along the *x* axis and is acted on by only one force, and $U(x)$ is the potential-energy function associated with this force.):

(*a*) The particle will be in equilibrium if it is at a location where $dU/dx = 0$.
(*b*) The particle will accelerate in the –*x* direction if it is at a location where $dU/dx > 0$.
(*c*) The particle will both be in equilibrium and have constant speed if it is at a section of the *x* axis where $dU/dx = 0$ throughout the section.
(*d*) The particle will be in stable equilibrium if it is at a location where both $dU/dx = 0$ and $d^2U/dx^2 > 0$.
(*e*) The particle will be in neutral equilibrium if it is at a location where both $dU/dx = 0$ and $d^2U/dx^2 > 0$.

Determine the Concept dU/dx is the slope of the graph of $U(x)$ and d^2U/dx^2 is the rate at which the slope is changing. The force acting on the object is given by $F = -dU/dx$.

(*a*) True. If $dU/dx = 0$, then the net force acting on the object is zero (the condition for equilibrium).

(*b*) True. If $dU/dx > 0$ at a given location, then the net force acting on the object is negative and its acceleration is to the left.

(*c*) True. If $dU/dx = 0$ over a section of the *x* axis, then the net force acting on the object is zero and its acceleration is zero.

(*d*) True. If $dU/dx = 0$ and $d^2U/dx^2 > 0$ at a given location, then $U(x)$ is concave upward at that location (the condition for stable equilibrium).

(*e*) False. If $dU/dx = 0$ and $d^2U/dx^2 > 0$ at a given location, then $U(x)$ is concave upward at that location (the condition for stable equilibrium).

6 • Two knowledge seekers decide to ascend a mountain. Sal chooses a short, steep trail, while Joe, who weighs the same as Sal, chooses a long, gently sloped trail. At the top, they get into an argument about who gained more potential energy. Which of the following is true:

(*a*) Sal gains more gravitational potential energy than Joe.
(*b*) Sal gains less gravitational potential energy than Joe.
(*c*) Sal gains the same gravitational potential energy as Joe.
(*d*) To compare the gravitational potential energies, we must know the height of the mountain.
(*e*) To compare the gravitational potential energies, we must know the length of the two trails.

Determine the Concept The change in gravitational potential energy, over elevation changes that are small enough so that the gravitational field can be considered constant, is $mg\Delta h$, where Δh is the elevation change. Because Δh is the same for both Sal and Joe, their gains in gravitational potential energy are the same. $\boxed{(c)}$ is correct.

7 • True or false:

(*a*) Only conservative forces can do work.
(*b*) If only conservative forces act on a particle, the kinetic energy of the particle can not change.
(*c*) The work done by a conservative force equals the change in the potential energy associated with that force.

(d) If, for a particle constrained to the x axis, the potential energy associated with a conservative force decreases as the particle moves to the right, then the force points to the left.
(e) If, for a particle constrained to the x axis, a conservative force points to the right, then the potential energy associated with the force increases as the particle moves to the left.

(a) False. The definition of work is not limited to displacements caused by conservative forces.

(b) False. Consider the work done by the gravitational force on an object in freefall.

(c) False. The work done may change the kinetic energy of the system.

(d) False. The direction of the force is given by $F = -dU/dx$, so if the potential energy is decreasing to the right (the slope of $U(x)$ is negative), F must be positive (that is, points to the right).

(e) True. The direction of the force is given by $F = -dU/dx$, so if F points to the right, the potential energy function must increase to the left.

8 • Figure 7-35 shows the plot of a potential-energy function U versus x. (a) At each point indicated, state whether the x component of the force associated with this function is positive, negative, or zero. (b) At which point does the force have the greatest magnitude? (c) Identify any equilibrium points, and state whether the equilibrium is stable, unstable, or neutral

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x; that is, $F_x = -dU/dx$.

(a) Examine the slopes of the curve at each of the lettered points, remembering that F_x is the negative of the slope of the potential energy graph, to complete the table:

Point	dU/dx	F_x
A	+	–
B	0	0
C	–	+
D	0	0
E	+	–
F	0	0

(b) Find the point where the slope is steepest:

$|F_x|$ is greatest at point C.

(c) If $d^2U/dx^2 < 0$, then the curve is concave downward and the equilibrium is *unstable*.

The equilibrium is unstable at point B.

If $d^2U/dx^2 > 0$, then the curve is concave upward and the equilibrium is *stable*.

The equilibrium is stable at point D.

Remarks: At point F, if $d^2U/dx^2 = 0$ while the fourth derivative is positive, then the equilibrium is *stable*.

9 • Assume that, when the brakes are applied, a constant frictional force is exerted on the wheels of a car by the road. If that is so, then which of the following are necessarily true? (a) The distance the car travels before coming to rest is proportional to the speed of the car just as the brakes are first applied, (b) the car's kinetic energy diminishes at a constant rate, (c) the kinetic energy of the car is inversely proportional to the time that has elapsed since the application of the brakes, (d) none of the above.

Picture the Problem Because the constant friction force is responsible for a constant acceleration, we can apply the constant-acceleration equations to the analysis of these statements. We can also apply the work-energy theorem with friction to obtain expressions for the kinetic energy of the car and the rate at which it is changing. Choose the system to include Earth and car and assume that the car is moving on a horizontal surface so that $\Delta U = 0$.

(a) A constant frictional force causes a constant acceleration. The stopping distance of the car is related to its speed before the brakes were applied through a constant-acceleration equation.

$$v^2 = v_0^2 + 2a\Delta s$$

or, because $v = 0$,

$$0 = v_0^2 + 2a\Delta s \Rightarrow \Delta s = \frac{-v_0^2}{2a}$$

where $a < 0$.

Thus, $\Delta s \propto v_0^2$:

Statement (a) is *false*.

(b) Apply the work-energy theorem with friction to obtain:

$$\Delta K = -W_{\text{f}} = -\mu_{\text{k}} mg\Delta s$$

Express the rate at which K is dissipated:

$$\frac{\Delta K}{\Delta t} = -\mu_{\text{k}} mg \frac{\Delta s}{\Delta t}$$

Thus, $\dfrac{\Delta K}{\Delta t} \propto v$ and therefore not constant.

Statement (*b*) is *false*.

(*c*) In Part (*b*) we saw that:

$K \propto \Delta s$

Because $\Delta s \propto \Delta t$ and $K \propto \Delta t$:

Statement (*c*) is *false*.

Because none of the above are correct:

$\boxed{(d)}$ is correct.

10 •• If a rock is attached to a massless, rigid rod and swung in a vertical circle (Figure 7-36) at a constant speed, the total mechanical energy of the rock-Earth system does not remain constant. The kinetic energy of the rock remains constant, but the gravitational potential energy is continually changing. Is the total work done on the rock equal to zero during all time intervals? Does the force by the rod on the rock ever have a nonzero tangential component?

Determine the Concept No. From the work-kinetic energy theorem, no total work is being done on the rock, as its kinetic energy is constant. However, the rod must exert a tangential force on the rock to keep the speed constant. The effect of this force is to cancel the component of the force of gravity that is tangential to the trajectory of the rock.

11 •• Use the rest energies given in Table 7-1 to answer the following questions. (*a*) Can the triton naturally decay into a helion? (*b*) Can the alpha particle naturally decay into helion plus a neutron? (*c*) Can the proton naturally decay into a neutron and a positron?

Determine the Concept
(*a*) Yes, because the triton mass is slightly more than that of the helion (^{3}He) mass.

(*b*) No, because the total of the neutron and helion masses is 3747.96 MeV which is larger than the alpha particle mass.

(*c*) No, because the neutron mass is already larger than that of the proton.

Estimation and Approximation

12 • Estimate (*a*) the change in your potential energy on taking an elevator from the ground floor to the top of the Empire State building, (*b*) the average force acting on you by the elevator to bring you to the top, and (*c*) the average power due to that force. The building is 102 stories high.

Picture the Problem You can estimate your change in potential energy due to this change in elevation from the definition of ΔU. You'll also need to estimate the height of one story of the Empire State building. We'll assume your mass is 70.0 kg and the height of one story to be 3.50 m. This approximation gives us a height of 1170 ft (357 m), a height that agrees to within 7% with the actual height of 1250 ft from the ground floor to the observation deck. We'll also assume that it takes 3 min to ride non-stop to the top floor in one of the high-speed elevators.

(*a*) Express the change in your gravitational potential energy as you ride the elevator to the 102nd floor:

$$\Delta U = mg\Delta h$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = (70.0\,\text{kg})(9.81\,\text{m/s}^2)(357\,\text{m}) = 245.2\,\text{kJ} = \boxed{245\,\text{kJ}}$$

(*b*) Ignoring the acceleration intervals at the beginning and the end of your ride, express the work done on you by the elevator in terms of the change in your gravitational potential energy:

$$W = Fh = \Delta U \Rightarrow F = \frac{\Delta U}{h}$$

Substitute numerical values and evaluate F:

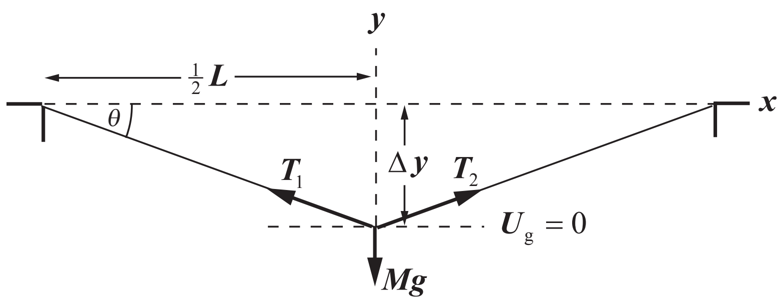
$$F = \frac{245.2\,\text{kJ}}{357\,\text{m}} = \boxed{687\,\text{N}}$$

(*c*) Assuming a 3 min ride to the top, the average power delivered to the elevator is:

$$P = \frac{\Delta U}{\Delta t} = \frac{245.2\,\text{kJ}}{(3\,\text{min})(60\,\text{s/min})} = \boxed{1.36\,\text{kW}}$$

13 • A tightrope walker whose mass is 50 kg walks across a tightrope held between two supports 10 m apart; the tension in the rope is 5000 N when she stands at the exact center of the rope. Estimate: (*a*) the sag in the tightrope when the acrobat stands in the exact center, and (*b*) the change in her gravitational potential energy from when she steps onto the tightrope to when she stands at its exact center.

Picture the Problem The diagram depicts the situation when the tightrope walker is at the center of rope and shows a coordinate system in which the +*x* direction is to the right and +*y* direction is upward. *M* represents her mass and the vertical components of tensions $\vec{T}_1$ and $\vec{T}_2$, which are equal in magnitude, support her weight. We can apply a condition for static equilibrium in the vertical direction to relate the tension in the rope to the angle θ and use trigonometry to find Δy as a function of θ.



(*a*) Use trigonometry to relate the sag Δy in the rope to its length L and θ:

$$\tan\theta = \frac{\Delta y}{\frac{1}{2}L} \Rightarrow \Delta y = \tfrac{1}{2}L\tan\theta \quad (1)$$

Apply $\sum F_y = 0$ to the tightrope walker when she is at the center of the rope to obtain:

$$2T\sin\theta - Mg = 0$$

where T is the magnitude of $\vec{T}_1$ and $\vec{T}_2$.

Solve for θ to obtain:

$$\theta = \sin^{-1}\left(\frac{Mg}{2T}\right)$$

Substituting for θ in equation (1) yields:

$$\Delta y = \tfrac{1}{2}L\tan\left[\sin^{-1}\left(\frac{Mg}{2T}\right)\right]$$

Substitute numerical values and evaluate Δy:

$$\Delta y = \tfrac{1}{2}(10\text{ m})\tan\left[\sin^{-1}\left[\frac{(50\text{ kg})(9.81\text{ m/s}^2)}{2(5000\text{ N})}\right]\right] = 0.2455\text{ m} = \boxed{25\text{ cm}}$$

(*b*) Express the change in the tightrope walker's gravitational potential energy as the rope sags:

$$\Delta U = U_{\text{at center}} - U_{\text{end}} = 0 + Mg\Delta y$$
$$= Mg\Delta y$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = (50\text{ kg})(9.81\text{ m/s}^2)(-0.2455\text{ m})$$
$$= \boxed{-0.12\text{ kJ}}$$

14 •• *The metabolic rate* is defined as the rate at which the body uses chemical energy to sustain its life functions. The average metabolic rate has been found to be proportional to the total skin surface area of the body. The surface area for a 5-ft, 10-in. male weighing 175 lb is about 2.0 m^2, and for a 5-ft, 4-in. female weighing 110 lb it is approximately 1.5 m^2. There is about a 1 percent change in surface area for every three pounds above or below the weights quoted here and a 1 percent change for every inch above or below the heights quoted. (*a*) Estimate your average metabolic rate over the course of a day using the following guide for metabolic rates (per square meter of skin area) for various

physical activities: sleeping, 40 W/m^2; sitting, 60 W/m^2; walking, 160 W/m^2; moderate physical activity, 175 W/m^2; and moderate aerobic exercise, 300 W/m^2. How do your results compare to the power of a 100-W light bulb? (*b*) Express your average metabolic rate in terms of kcal/day (1 kcal = 4.19 kJ). (A kcal is the ″food calorie″ used by nutritionists.) (*c*) An estimate used by nutritionists is that each day the ″average person″ must eat roughly 12–15 kcal of food for each pound of body weight to maintain his or her weight. From the calculations in Part (*b*), are these estimates plausible?

Picture the Problem We'll use the data for the "typical male" described above and assume that he spends 8 hours per day sleeping, 2 hours walking, 8 hours sitting, 1 hour in aerobic exercise, and 5 hours doing moderate physical activity. We can approximate his energy utilization using $E_{\text{activity}} = AP_{\text{activity}}\Delta t_{\text{activity}}$, where A is the surface area of his body, P_{activity} is the rate of energy consumption in a given activity, and $\Delta t_{\text{activity}}$ is the time spent in the given activity. His total energy consumption will be the sum of the five terms corresponding to his daily activities.

(*a*) Express the energy consumption of the hypothetical male:

$$E = E_{\text{sleeping}} + E_{\text{walking}} + E_{\text{sitting}} + E_{\text{mod. act.}} + E_{\text{aerobic act.}} \qquad (1)$$

Evaluate E_{sleeping}:

$$\begin{aligned} E_{\text{sleeping}} &= AP_{\text{sleeping}}\Delta t_{\text{sleeping}} = \left(2.0\,\text{m}^2\right)\left(40\,\text{W/m}^2\right)\left(8.0\,\text{h}\right)\left(3600\,\text{s/h}\right) \\ &= 2.30\times10^{6}\ \text{J} \end{aligned}$$

Evaluate E_{walking}:

$$\begin{aligned} E_{\text{walking}} &= AP_{\text{walking}}\Delta t_{\text{walking}} = \left(2.0\,\text{m}^2\right)\left(160\,\text{W/m}^2\right)\left(2.0\,\text{h}\right)\left(3600\,\text{s/h}\right) \\ &= 2.30\times10^{6}\ \text{J} \end{aligned}$$

Evaluate E_{sitting}

$$\begin{aligned} E_{\text{sitting}} &= AP_{\text{sitting}}\Delta t_{\text{sitting}} = \left(2.0\,\text{m}^2\right)\left(60\,\text{W/m}^2\right)\left(8.0\,\text{h}\right)\left(3600\,\text{s/h}\right) \\ &= 3.46\times10^{6}\ \text{J} \end{aligned}$$

Evaluate $E_{\text{mod. act.}}$:

$$E_{\text{mod. act.}} = AP_{\text{mod. act.}}\Delta t_{\text{mod. act.}} = \left(2.0\,\text{m}^2\right)\left(175\,\text{W/m}^2\right)\left(5.0\,\text{h}\right)\left(3600\,\text{s/h}\right)$$
$$= 6.30\times10^6\ \text{J}$$

Evaluate $E_{\text{aerobic act.}}$:

$$E_{\text{aerobic act.}} = AP_{\text{aerobic act.}}\Delta t_{\text{aerobic act.}} = \left(2.0\,\text{m}^2\right)\left(300\,\text{W/m}^2\right)\left(1.0\,\text{h}\right)\left(3600\,\text{s/h}\right)$$
$$= 2.16\times10^6\ \text{J}$$

Substitute numerical values in equation (1) and evaluate E:

$$E = 2.30\times10^6\ \text{J} + 2.30\times10^6\ \text{J} + 3.46\times10^6\ \text{J} + 6.30\times10^6\ \text{J} + 2.16\times10^6\ \text{J}$$
$$= 16.5\times10^6\ \text{J} = \boxed{17\ \text{MJ}}$$

Express the average metabolic rate represented by this energy consumption:

$$P_{\text{av}} = \frac{E}{\Delta t}$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = \frac{16.5\times10^6\ \text{J}}{\left(24\,\text{h}\right)\left(3600\,\text{s/h}\right)} = \boxed{0.19\,\text{kW}}$$

or about twice that of a 100 W light bulb.

(*b*) Express his average energy consumption in terms of kcal/day:

$$E = \frac{16.5\times10^6\ \text{J/d}}{4190\,\text{J/kcal}} = 3938\,\text{kcal/d}$$
$$\approx \boxed{3.9\,\text{Mcal/d}}$$

(*c*) $\dfrac{3940\,\text{kcal}}{175\,\text{lb}} \approx 23\dfrac{\text{kcal}}{\text{lb}}$ is higher than the estimate given in the statement of the problem. However, by adjusting the day's activities, the metabolic rate can vary by more than a factor of 2.

15 •• [SSM] Assume that your maximum metabolic rate (the maximum rate at which your body uses its chemical energy) is 1500 W (about 2.7 hp). Assuming a 40 percent efficiency for the conversion of chemical energy into mechanical energy, estimate the following: (*a*) the shortest time you could run up four flights of stairs if each flight is 3.5 m high, (*b*) the shortest time you could climb the Empire State Building (102 stories high) using your Part (*a*) result. Comment on the feasibility of you actually achieving Part (*b*) result.

Picture the Problem The rate at which you expend energy, that is do work, is defined as *power* and is the ratio of the work done to the time required to do the work.

(*a*) Relate the rate at which you can expend energy to the work done in running up the four flights of stairs:

$$\varepsilon P = \frac{\Delta W}{\Delta t} \Rightarrow \Delta t = \frac{\Delta W}{\varepsilon P}$$

where *e* is the efficiency for the conversion of chemical energy into mechanical energy.

The work you do in climbing the stairs increases your gravitational potential energy:

$$\Delta W = mgh$$

Substitute for ΔW to obtain:

$$\Delta t = \frac{mgh}{\varepsilon P} \qquad (1)$$

Assuming that your mass is 70 kg, substitute numerical values in equation (1) and evaluate Δt:

$$\Delta t = \frac{(70\,\text{kg})(9.81\,\text{m/s}^2)(4\times 3.5\,\text{m})}{(0.40)(1500\,\text{W})} \approx \boxed{16\,\text{s}}$$

(*b*) Substituting numerical values in equation (1) yields:

$$\Delta t = \frac{(70\,\text{kg})(9.81\,\text{m/s}^2)(102\times 3.5\,\text{m})}{(0.40)(1500\,\text{W})} = 409\,\text{s} \approx \boxed{6.8\,\text{min}}$$

The time of about 6.8 min is clearly not reasonable. The fallacy is that you cannot do work at the given rate of 1500 W for more than very short intervals of time.

16 •• You are in charge of determining when the uranium fuel rods in a local nuclear power plant are to be replaced with fresh ones. To make this determination you decide to estimate how much the mass of a core of a nuclear-fueled electric-generating plant is reduced per unit of electric energy produced. (*Note:* In such a generating plant the reactor core generates thermal energy, which is then transformed to electric energy by a steam turbine. It requires 3.0 J of thermal energy for each 1.0 J of electric energy produced.) What are your results for the production of (*a*) 1.0 J of thermal energy? (*b*) enough electric energy to keep a 100-W light bulb burning for 10.0 y? (*c*) electric energy at a constant rate of 1.0 GW for a year? (This is typical of modern plants.)

Picture the Problem The intrinsic rest energy in matter is related to the mass of matter through Einstein's equation $E_0 = mc^2$.

(*a*) Relate the rest mass consumed to the energy produced and solve for *m*:

$$E_0 = mc^2 \Rightarrow m = \frac{E_0}{c^2} \qquad (1)$$

Substitute numerical values and evaluate *m*:

$$m = \frac{1.0\,\text{J}}{(2.998\times10^8\ \text{m/s})^2} = \boxed{1.1\times10^{-17}\ \text{kg}}$$

(*b*) Because the reactor core must produce 3 J of thermal energy for each joule of electrical energy produced:

$$E = 3P\Delta t$$

Substitute for E_0 in equation (1) to obtain:

$$m = \frac{3P\Delta t}{c^2} \qquad (2)$$

Substitute numerical values in equation (2) and evaluate *m*:

$$m = \frac{3(100\ \text{W})(10\ \text{y})\left(\frac{365.24\,\text{d}}{\text{y}}\right)\left(\frac{24\,\text{h}}{\text{d}}\right)\left(\frac{3600\,\text{s}}{\text{h}}\right)}{(2.998\times10^8\ \text{m/s})^2} = \boxed{1.1\times10^{-6}\ \text{kg}}$$

(*c*) Substitute numerical values in equation (2) and evaluate *m*:

$$m = \frac{3(1.0\,\text{GW})(1.0\,\text{y})\left(\frac{365.24\,\text{d}}{\text{y}}\right)\left(\frac{24\,\text{h}}{\text{d}}\right)\left(\frac{3600\,\text{s}}{\text{h}}\right)}{(2.998\times10^8\ \text{m/s})^2} = \boxed{1.1\,\text{kg}}$$

17 •• [SSM] The chemical energy released by burning a gallon of gasoline is approximately 1.3×10^5 kJ. Estimate the total energy used by all of the cars in the United States during the course of one year. What fraction does this represent of the total energy use by the United States in one year (currently about 5×10^{20} J)?

Picture the Problem There are about 3×10^8 people in the United States. On the assumption that the average family has 4 people in it and that they own two cars, we have a total of 1.5×10^8 automobiles on the road (excluding those used for industry). We'll assume that each car uses about 15 gal of fuel per week.

Calculate, based on the assumptions identified above, the total annual consumption of energy derived from gasoline:

$$\left(1.5\times10^{8}\,\text{auto}\right)\left(15\frac{\text{gal}}{\text{auto}\cdot\text{week}}\right)\left(52\frac{\text{weeks}}{\boldsymbol{y}}\right)\left(1.3\times10^{5}\,\frac{\text{kJ}}{\text{gal}}\right)=\boxed{1.5\times10^{19}\ \text{J/y}}$$

Express this rate of energy use as a fraction of the total annual energy use by the United States:

$$\frac{1.5\times10^{19}\ \text{J/y}}{5\times10^{20}\ \text{J/y}}\approx\boxed{3\%}$$

18 •• The maximum efficiency of a solar-energy panel in converting solar energy into useful electrical energy is currently about 12 percent. In a region such as the southwestern United States the solar intensity reaching Earth's surface is about 1.0 kW/m^2 on average during the day. Estimate the area that would have to be covered by solar panels in order to supply the energy requirements of the United States (approximately 5×10^{20} J/y) and compare it to the area of Arizona. Assume cloudless skies.

Picture the Problem The energy consumption of the U.S. works out to an average power consumption of about 1.6×10^{13} watt. The solar constant is roughly 10^3 W/m^2 (reaching the ground), or about 120 W/m^2 of useful power with a 12% conversion efficiency. Letting P represent the daily rate of energy consumption, we can relate the power available at the surface of Earth to the required area of the solar panels using $P = IA$. Using the internet, one finds that the area of Arizona is about 114,000 mi^2 or 3.0×10^{11} m^2.

Relate the required area to the electrical energy to be generated by the solar panels:

$$A=\frac{P}{I}=\frac{\Delta E}{I\Delta t}$$

where I is the solar intensity that reaches the surface of Earth.

Substitute numerical values and evaluate A:

$$A=\frac{2\left(5\times10^{20}\ \text{J}/y\right)}{\left(120\,\text{W/m}^2\right)\left(1\ \text{y}\right)\left(3.156\times10^{7}\ \text{s/y}\right)}$$

$$=2.640\times10^{11}\ \text{m}^2=\boxed{2.6\times10^{11}\ \text{m}^2}$$

where the factor of 2 comes from the fact that the sun is only "up" for roughly half the day.

Express the ratio of A to the area of Arizona to obtain:

$$\frac{A}{A_{\text{Arizona}}}=\frac{2.640\times10^{11}\ \text{m}^2}{3.0\times10^{11}\ \text{m}^2}\approx0.88$$

That is, the required area is about 88% of the area of Arizona.

Remarks: A more realistic estimate that would include the variation of sunlight over the day and account for latitude and weather variations might very well increase the area required by an order of magnitude.

19 •• Hydroelectric power plants convert gravitational potential energy into more useful forms by flowing water downhill through a turbine system to generate electric energy. The Hoover Dam on the Colorado River is 211 m high and generates 4×10^9 kW·h/y. At what rate (in L/s) must water be flowing through the turbines to generate this power? The density of water is 1.00 kg/L. Assume a total efficiency of 90.0 percent in converting the water's potential energy into electrical energy.

Picture the Problem We can relate the energy available from the water in terms of its mass, the vertical distance it has fallen, and the efficiency of the process. Differentiation of this expression with respect to time will yield the rate at which water must pass through its turbines to generate Hoover Dam's annual energy output.

Assuming a total efficiency ε, use the expression for the gravitational potential energy near Earth's surface to express the energy available from the water when it has fallen a distance *h*:

$$E = \varepsilon mgh$$

Differentiate this expression with respect to time to obtain:

$$P = \frac{d}{dt}\left[\varepsilon mgh\right] = \varepsilon gh\frac{dm}{dt} = \varepsilon \rho gh\frac{dV}{dt}$$

Solving for dV/dt yields:

$$\frac{dV}{dt} = \frac{P}{\varepsilon\rho gh} \qquad (1)$$

Using its definition, relate the dam's annual power output to the energy produced:

$$P = \frac{\Delta E}{\Delta t}$$

Substituting for *P* in equation (1) yields:

$$\frac{dV}{dt} = \frac{\Delta E}{\varepsilon\rho gh\Delta t}$$

Substitute numerical values and evaluate dV/dt:

$$\frac{dV}{dt} = \frac{4.00\times10^9\ \text{kW}\cdot\text{h}}{(0.90)(1.00\,\text{kg/L})(9.81\,\text{m/s}^2)(211\,\text{m})\left(365.24\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\right)} = \boxed{2.4\times10^5\ \text{L/s}}$$

Force, Potential Energy, and Equilibrium

20 • Water flows over Victoria Falls, which is 128 m high, at an average rate of 1.4×10^6 kg/s. If half the potential energy of this water were converted into electric energy, how much electric power would be produced by these falls?

Picture the Problem The water going over the falls has gravitational potential energy relative to the base of the falls. As the water falls, the falling water acquires kinetic energy until, at the base of the falls; its energy is entirely kinetic. The rate at which energy is delivered to the base of the falls is given by $P = dW/dt = -dU/dt$.

Express the rate at which energy is being delivered to the base of the falls; remembering that half the potential energy of the water is converted to electric energy:

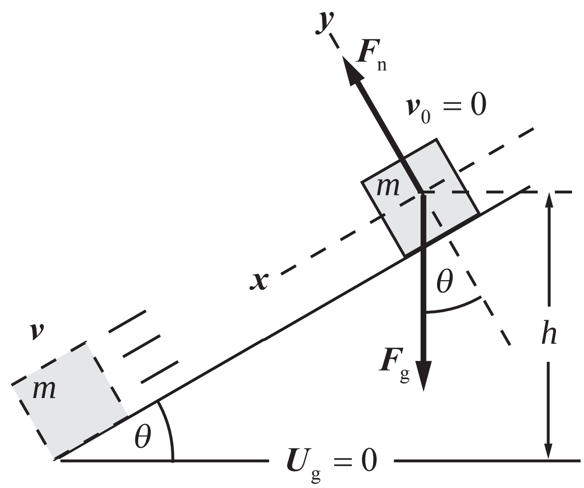
$$P = \frac{dW}{dt} = -\frac{dU}{dt} = -\tfrac{1}{2}\frac{d}{dt}(mgh)$$
$$= -\tfrac{1}{2}gh\frac{dm}{dt}$$

Substitute numerical values and evaluate P:

$$P = -\tfrac{1}{2}\left(9.81\,\text{m/s}^2\right)(-128\,\text{m})\left(1.4\times10^6\ \text{kg/s}\right) = \boxed{0.88\,\text{GW}}$$

21 • A 2.0-kg box slides down a long, frictionless incline of angle 30º. It starts from rest at time $t = 0$ at the top of the incline at a height of 20 m above the ground. (*a*) What is the potential energy of the box relative to the ground at $t = 0$? (*b*) Use Newton's laws to find the distance the box travels during the interval $0.0\text{ s} < t < 1.0\text{ s}$ and its speed at $t = 1.0$ s. (*c*) Find the potential energy and the kinetic energy of the box at $t = 1.0$ s. (*d*) Find the kinetic energy and the speed of the box just as it reaches the ground at the bottom of the incline.

Picture the Problem In the absence of friction, the sum of the potential and kinetic energies of the box remains constant as it slides down the incline. We can use the conservation of the mechanical energy of the system to calculate where the box will be and how fast it will be moving at any given time. We can also use Newton's second law to show that the acceleration of the box is constant and constant-acceleration equations to calculate where the box will be and how fast it will be moving at any given time.



(*a*) Express the gravitational potential energy of the box, relative to the ground, at the top of the incline:

$$U_i = mgh$$

Substitute numerical values and evaluate U_i:

$$U_i = (2.0\,\text{kg})(9.81\,\text{m/s}^2)(20\,\text{m}) = 392\,\text{J} = \boxed{0.39\,\text{kJ}}$$

(*b*) Using a constant-acceleration equation, relate the displacement of the box to its initial speed, acceleration and time-of-travel:

$$\Delta x = v_0 t + \tfrac{1}{2}at^2$$

or, because $v_0 = 0$,

$$\Delta x = \tfrac{1}{2}at^2 \qquad (1)$$

Apply $\sum F_x = ma_x$ to the box as it slides down the incline and solve for its acceleration:

$$F_g \sin\theta = ma$$

or, because $F_g = mg$,

$$mg\sin\theta = ma \Rightarrow a = g\sin\theta$$

Substitute for a in equation (1) to obtain:

$$\Delta x = \tfrac{1}{2}(g\sin\theta)t^2$$

Substitute numerical values and evaluate $\Delta x(t = 1.0\text{ s})$:

$$\Delta x(1.0\,\text{s}) = \tfrac{1}{2}(9.81\,\text{m/s}^2)(\sin 30°)(1.0\,\text{s})^2 = 2.45\,\text{m} = \boxed{2.5\,\text{m}}$$

Using a constant-acceleration equation, relate the speed of the box at any time to its initial speed and acceleration:

$$v = v_0 + at$$

or, because $v_0 = 0$,

$$v = at = g(\sin\theta)t$$

Substitute numerical values and evaluate v(1.0 s):

$$v(1.0\,\text{s}) = (9.81\,\text{m/s}^2)(\sin 30°)(1.0\,\text{s}) = 4.91\,\text{m/s} = \boxed{4.9\,\text{m/s}}$$

(c) The kinetic energy of the box is given by:

$$K = \tfrac{1}{2}mv^2$$

or, because $v = g(\sin\theta)t$,

$$K(t) = \tfrac{1}{2}mg^2(\sin^2\theta)t^2$$

Substitute numerical values and evaluate K(1.0 s):

$$K(1.0\,\text{s}) = \tfrac{1}{2}(2.0\,\text{kg})(9.81\,\text{m/s}^2)^2(\sin^2 30)(1.0\,\text{s})^2 = 24.1\,\text{J} = \boxed{24\,\text{J}}$$

Express the potential energy of the box after it has traveled for 1.0 s in terms of its initial potential energy and its kinetic energy:

$$U = U_\text{i} - K = 392\,\text{J} - 24\,\text{J} = \boxed{0.37\,\text{kJ}}$$

(d) Apply conservation of mechanical energy to the box-Earth system as the box as it slides down the incline:

$$W_\text{ext} = \Delta K + \Delta U = 0$$

or, because $K_\text{i} = U_\text{f} = 0$,

$$K_\text{f} - U_\text{i} = 0 \qquad (2)$$

Solving for K_f yields:

$$K_\text{f} = U_\text{i} = \boxed{0.39\,\text{kJ}}$$

From equation (2) we have:

$$\tfrac{1}{2}mv_\text{f}^2 = U_\text{i} \Rightarrow v_\text{f} = \sqrt{\frac{2U_\text{i}}{m}}$$

Substitute numerical values and evaluate v_f:

$$v_\text{f} = \sqrt{\frac{2(392\,\text{J})}{2.0\,\text{kg}}} = \boxed{20\,\text{m/s}}$$

22 • A constant force F_x = 6.0 N is in the +x direction. (a) Find the potential-energy function $U(x)$ associated with this force if $U(x_0) = 0$. (b) Find a function $U(x)$ such that U(4 .0 m) = 0. (c) Find a function $U(x)$ such that U(6 .0 m) = 14 J.

Picture the Problem The potential energy function $U(x)$ is defined by the equation $U(x) - U(x_0) = -\int_{x_0}^{x} F\,dx.$ We can use the given force function to determine $U(x)$ and then the conditions on $U(x)$ to determine the potential functions that satisfy the given conditions.

(*a*) Use the definition of the potential energy function to find the potential energy function associated with F_x:

$$U(x)=U(x_0)-\int_{x_0}^{x}F_x dx$$
$$=U(x_0)-\int_{x_0}^{x}(6\,\text{N})dx$$
$$=U(x_0)-(6.0\,\text{N})(x-x_0)$$

Because $U(x_0) = 0$:

$$U(x)=\boxed{-(6.0\,\text{N})(x-x_0)}$$

(*b*) Use the result obtained in (*a*) to find $U(x)$ that satisfies the condition that $U(4.0\text{ m}) = 0$:

$$U(4.0\,\text{m})=-(6.0\,\text{N})(4.0\,\text{m}-x_0)$$
$$=0\Rightarrow x_0=4.0\,\text{m}$$

and

$$U(x)=-(6.0\,\text{N})(x-4.0\,\text{m})$$
$$=\boxed{24\,\text{J}-(6.0\,\text{N})x}$$

(*c*) Use the result obtained in (*a*) to find $U(x)$ that satisfies the condition that U(6.0 m) = 14 J:

$$U(6.0\,\text{m})=-(6.0\,\text{N})(6.0\,\text{m}-x_0)$$
$$=14\,\text{J}\Rightarrow x_0=\frac{25}{3}\,\text{m}$$

and

$$U(x)=-(6.0\,\text{N})\left(x-\frac{25}{3.0}\,\text{m}\right)$$
$$=\boxed{50\,\text{J}-(6.0\,\text{N})x}$$

23 • A spring has a force constant of 1.0×10^4 N/m . How far must the spring be stretched for its potential energy to equal (*a*) 50 J, and (*b*) 100 J?

Picture the Problem The potential energy of a stretched or compressed ideal spring U_s is related to its force (stiffness) constant k and stretch or compression Δx by $U_s=\frac{1}{2}kx^2$.

(*a*) Relate the potential energy stored in the spring to the distance it has been stretched:

$$U_s=\tfrac{1}{2}kx^2\Rightarrow x=\sqrt{\frac{2U_s}{k}}$$

Substitute numerical values and evaluate x:

$$\boldsymbol{x}=\sqrt{\frac{2(50\,\text{J})}{1.0\times10^4\,\text{N/m}}}=\boxed{10\,\text{cm}}$$

(*b*) Proceed as in (*a*) with $U_s = 100$ J:

$$x = \sqrt{\frac{2(100\,\text{J})}{1.0\times10^4\ \text{N/m}}} = \boxed{14\ \text{cm}}$$

24 • (*a*) Find the force F_x associated with the potential-energy function $U = Ax^4$, where A is a constant. (*b*) At what value(s) of x does the force equal zero?

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, that is, $F_x = -dU/dx$. Consequently, given U as a function of x, we can find F_x by differentiating U with respect to x.

(*a*) Evaluate $F_x = -\dfrac{dU}{dx}$:

$$F_x = -\frac{d}{dx}\left(Ax^4\right) = \boxed{-4Ax^3}$$

(*b*) Set $F_x = 0$ and solve for x to obtain:

$$F_x = 0 \Rightarrow \boxed{x = 0}$$

25 •• **[SSM]** The force F_x is associated with the potential-energy function $U = C/x$, where C is a positive constant. (*a*) Find the force F_x as a function of x. (*b*) Is this force directed toward the origin or away from it in the region $x > 0$? Repeat the question for the region $x < 0$. (*c*) Does the potential energy U increase or decrease as x increases in the region $x > 0$? (*d*) Answer Parts (*b*) and (*c*) where C is a negative constant.

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, that is $F_x = -dU/dx$. Consequently, given U as a function of x, we can find F_x by differentiating U with respect to x.

(*a*) Evaluate $F_x = -\dfrac{dU}{dx}$:

$$F_x = -\frac{d}{dx}\left(\frac{C}{x}\right) = \boxed{\frac{C}{x^2}}$$

(*b*) Because $C > 0$, if $x > 0$, F_x is positive and $\vec{F}$ points away from the origin. If $x < 0$, F_x is still positive and $\vec{F}$ points toward the origin.

(*c*) Because U is inversely proportional to x and $C > 0$, $U(x)$ decreases with increasing x.

(*d*) When $C < 0$, if $x > 0$, F_x is negative and $\vec{F}$ points toward the origin. If $x < 0$, F_x is negative and $\vec{F}$ points away from the origin.

Because U is inversely proportional to x and $C < 0$, $U(x)$ becomes less negative as x increases and $U(x)$ increases with increasing x.

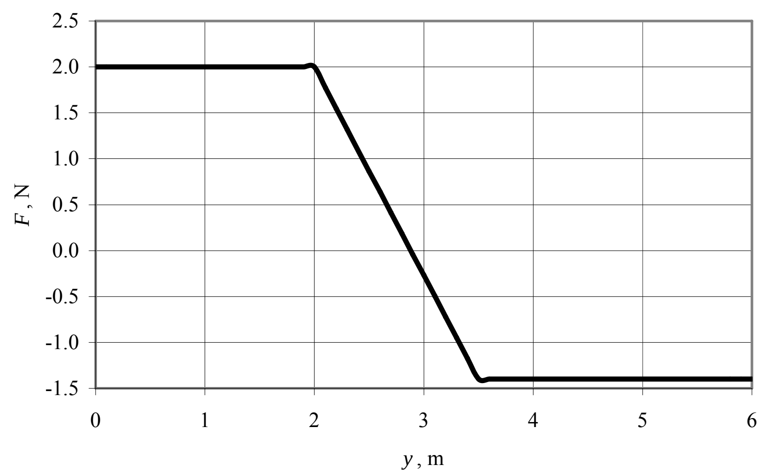
26 •• The force F_y is associated with the potential-energy function $U(y)$. On the potential-energy curve for U versus y, shown in Figure 7-37, the segments AB and CD are straight lines. Plot F_y versus y. Include numerical values, with units, on both axes. These values can be obtained from the U versus y plot.

Picture the Problem F_y is defined to be the negative of the derivative of the potential-energy function with respect to y; that is, $F_y = -dU/dy$. Consequently, we can obtain F_y by examining the slopes of the graph of U as a function of y.

The table to the right summarizes the information we can obtain from Figure 7-37:

	Slope	F_y
Interval	(N)	(N)
A→B	−2	2
B→C	transitional	$2 \rightarrow -1.4$
C→D	1.4	−1.4

The following graph shows F as a function of y:



27 •• The force acting on an object is given by $F_x = a/x^2$. At $x = 5.0$ m, the force is known to point in the $-x$ direction and have a magnitude of 25.0 N. Determine the potential energy associated with this force as a function of x, assuming we assign a reference value of −10.0 J at $x = 2.0$ m for the potential energy.

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, i.e. $F_x = -dU/dx$. Consequently, given F as a function of x, we can find U by integrating F_x with respect to x. Applying the condition on $F_{x}x$ will allow us to determine the value of a and using the fact that the potential energy is −10.0 J at $x = 2.00$ m will give us the value of U_0.

Evaluate the integral of F_x with respect to x:

$$U(x) = -\int F_x dx = -\int \frac{a}{x^2} dx = \frac{a}{x} + U_0 \quad (1)$$

Because $F_x(5.0\text{ m}) = -25.0\text{ N}$:

$$\frac{a}{(5.00\text{ m})^2} = -25.0\text{ N}$$

Solving for a yields:

$$a = -625\text{ N}\cdot\text{m}^2$$

Substitute for a in equation (1) to obtain:

$$U(x) = \frac{-625\text{ N}\cdot\text{m}^2}{x} + U_0 \quad (2)$$

Applying the condition $U(2.00\text{ m}) = -10.0\text{ J}$ yields:

$$-10.0\text{ J} = \frac{-625\text{ N}\cdot\text{m}^2}{2.00\text{ m}} + U_0$$

Solve for U_0 to obtain:

$$U_0 = 303\text{ J}$$

Substituting for U_0 in equation (2) yields:

$$U(x) = \boxed{\frac{-625\text{ N}\cdot\text{m}^2}{x} + 303\text{ J}}$$

28 •• The potential energy of an object constrained to the x axis is given by $U(x) = 3x^2 - 2x^3$, where U is in joules and x is in meters. (*a*) Determine the force F_x associated with this potential energy function. (*b*) Assuming no other forces act on the object, at what positions is this object in equilibrium? (*c*) Which of these equilibrium positions are stable and which are unstable?

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, that is, $F_x = -dU/dx$. Consequently, given U as a function of x, we can find F_x by differentiating U with respect to x. To determine whether the object is in stable or unstable equilibrium at a given point, we'll evaluate d^2U/dx^2 at the point of interest.

(*a*) Evaluate $F_x = -\dfrac{dU}{dx}$:

$$F_x = -\frac{d}{dx}\left(3x^2 - 2x^3\right) = \boxed{6x(x-1)}$$

(*b*) We know that, at equilibrium, $F_x = 0$:

When $F_x = 0$, $6x(x - 1) = 0$. Therefore, the object is in equilibrium at $\boxed{x = 0 \text{ and } x = 1\text{m.}}$

(*c*) To decide whether the equilibrium at a particular point is stable or unstable, evaluate the 2nd derivative of the potential energy function at the point of interest:

$$\frac{dU}{dx}=\frac{d}{dx}\left(3x^2-2x^3\right)=6x-6x^2$$

and

$$\frac{d^2U}{dx^2}=6-12x$$

Evaluate $\frac{d^2U}{dx^2}$ at $x = 0$:

$$\left.\frac{d^2U}{dx^2}\right|_{x=0}=6>0$$

$$\Rightarrow \boxed{\text{stable equilibrium at } x=0}$$

Evaluate $\frac{d^2U}{dx^2}$ at $x = 1$ m:

$$\left.\frac{d^2U}{dx^2}\right|_{x=1\,\text{m}}=6-12<0$$

$$\Rightarrow \boxed{\text{unstable equilibrium at } x=1\,\text{m}}$$

29 •• [SSM] The potential energy of an object constrained to the x axis is given by $U(x) = 8x^2 - x^4$, where U is in joules and x is in meters. (*a*) Determine the force F_x associated with this potential energy function. (*b*) Assuming no other forces act on the object, at what positions is this object in equilibrium? (*c*) Which of these equilibrium positions are stable and which are unstable?

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, that is $F_x=-dU/dx$. Consequently, given U as a function of x, we can find F_x by differentiating U with respect to x. To determine whether the object is in stable or unstable equilibrium at a given point, we'll evaluate d^2U/dx^2 at the point of interest.

(*a*) Evaluate the negative of the derivative of U with respect to x:

$$F_x=-\frac{dU}{dx}=-\frac{d}{dx}\left(8x^2-x^4\right)$$
$$=4x^3-16x=\boxed{4x(x+2)(x-2)}$$

(*b*) The object is in equilibrium wherever $F_{\text{net}}=F_x=0$:

$4x(x+2)(x-2)=0 \Rightarrow$ the equilibrium points are $\boxed{x=-2\,\text{m}, 0, \text{and } 2\,\text{m.}}$

(c) To decide whether the equilibrium at a particular point is stable or unstable, evaluate the 2nd derivative of the potential energy function at the point of interest:

$$\frac{d^2U}{dx^2}=\frac{d}{dx}\left(16x-4x^3\right)=16-12x^2$$

Evaluating d^2U/dx^2 at $x = -2$ m, 0 and $x = 2$ m yields the following results:

x, m	d^2U/dx^2	Equilibrium
−2	−32	Unstable
0	16	Stable
2	−32	Unstable

Remarks: You could also decide whether the equilibrium positions are stable or unstable by plotting *F*(*x*) and examining the curve at the equilibrium positions.

30 •• The net force acting on an object constrained to the x axis is given by $F_x(x) = x^3 - 4x$. (The force is in newtons and x in meters.) Locate the positions of unstable and stable equilibrium. Show that each of these positions is either stable or unstable by calculating the force one millimeter on either side of the locations.

Picture the Problem The equilibrium positions are those values of x for which $F(x) = 0$. Whether the equilibrium positions are stable or unstable depends on whether the signs of the force either side of the equilibrium position are the same (unstable equilibrium) of opposite (stable equilibrium).

Determine the equilibrium locations by setting $F_{\text{net}} = F(x) = 0$:

$F(x) = x^3 - 4x = x(x^2 - 4) = 0$
and the positions of stable and unstable equilibrium are at

$\boxed{x = -2\text{ m}, 0\text{ and }2\text{ m}}$.

Noting that we need only determine whether each value of $F(x)$ is positive or negative, evaluate $F(x)$ at x = −201 mm and x = −199 mm to determine the stability at x = −200 mm … and repeat these calculations at x = −1 mm, 1 mm and x = 199 mm, 201 mm to complete the following table:

x, mm	$F_{x-1\,\text{mm}}$	$F_{x+1\,\text{mm}}$	Equilibrium
−200	< 0	< 0	Unstable
0	> 0	< 0	Stable
200	> 0	> 0	Unstable

Remarks: You can very easily confirm these results by using your graphing calculator to plot *F*(*x*). You could also, of course, find *U*(*x*) from *F*(*x*) and examine it at the equilibrium positions.

31 •• The potential energy of a 4.0-kg object constrained to the x axis is given by $U = 3x^2 - x^3$ for $x \le 3.0$ m and $U = 0$ for $x \ge 3.0$ m, where U is in joules and x is in meters, and the only force acting on this object is the force associated with this potential energy function. (*a*) At what positions is this object in equilibrium? (*b*) Sketch a plot of U versus x. (*c*) Discuss the stability of the equilibrium for the values of x found in Part (*a*). (*d*) If the total mechanical energy of the particle is 12 J, what is its speed at x = 2.0 m?

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to x, that is $F_x = -dU/dx$. Consequently, given U as a function of x, we can find F_x by differentiating U with respect to x. To determine whether the object is in stable or unstable equilibrium at a given point, we can examine the graph of U.

(*a*) Evaluate $F_x = -\dfrac{dU}{dx}$ for $x \le 3.0$ m:

$$F_x = -\frac{d}{dx}\left(3x^2 - x^3\right) = 3x\left(2 - x\right)$$

Set $F_x = 0$ and solve for those values of x for which the 4.0-kg object is in equilibrium:

$3x(2 - x) = 0$

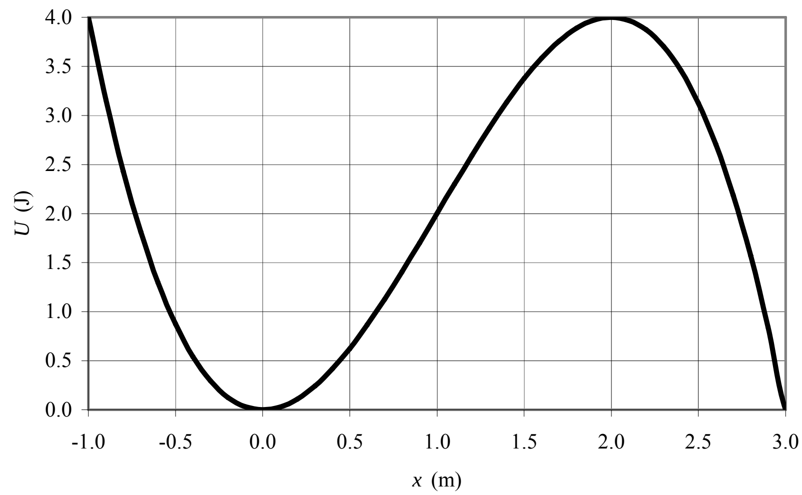
Therefore, the object is in equilibrium at $\boxed{\boldsymbol{x} = 0 \text{ and } \boldsymbol{x} = 2.0\,\text{m}.}$

Because $U = 0$:

$$F_x\left(x \ge 3\,\text{m}\right) = -\frac{dU}{dx} = 0$$

Therefore, the object is in neutral equilibrium for $x \ge 3.0$ m.

(*b*) A graph of $U(x)$ in the interval $-1.0\text{ m} \le x \le 3.0\text{ m}$ follows:



(*c*) From the graph, $U(x)$ is a minimum at $x = 0$ and so the equilibrium is stable at this point

From the graph, $U(x)$ is a maximum at $x = 2.0$ m and so the equilibrium is unstable at this point.

(*d*) Relate the kinetic energy of the object K to its total energy E and its potential energy U:

$$K = \tfrac{1}{2}mv^2 = E - U \Rightarrow v = \sqrt{\frac{2(E-U)}{m}}$$

Substitute numerical values and evaluate $v(2.0\text{ m})$:

$$v(2.0\text{ m}) = \sqrt{\frac{2\left(12\text{ J} - \left(\left(3.0\text{ J/m}^2\right)(2.0\text{ m})^2 - \left(1.0\text{ J/m}^3\right)(2.0\text{ m})^3\right)\right)}{4.0\text{ kg}}} = \boxed{2.0\text{ m/s}}$$

32 •• A force is given by $F_x = Ax^{-3}$, where $A = 8.0\text{ N}\cdot\text{m}^3$. (*a*) For positive values of *x*, does the potential energy associated with this force increase or decrease with increasing *x*? (You can determine the answer to this question by imagining what happens to a particle that is placed at rest at some point *x* and is then released.) (*b*) Find the potential-energy function *U* associated with this force such that *U* approaches zero as *x* approaches infinity. (*c*) Sketch *U* versus *x*.

Picture the Problem F_x is defined to be the negative of the derivative of the potential-energy function with respect to *x*, that is $F_x = -dU/dx$. Consequently, given *F* as a function of *x*, we can find *U* by integrating F_x with respect to *x*.

(*a*) Evaluate the negative of the integral of $F(x)$ with respect to x:

$$U(x) = -\int F(x) = -\int Ax^{-3}dx$$
$$= \frac{1}{2}\frac{A}{x^2} + U_0$$

where U_0 is a constant whose value is determined by conditions on $U(x)$.

For $x > 0$:

$$\boxed{U \text{ decreases as } x \text{ increases}}$$

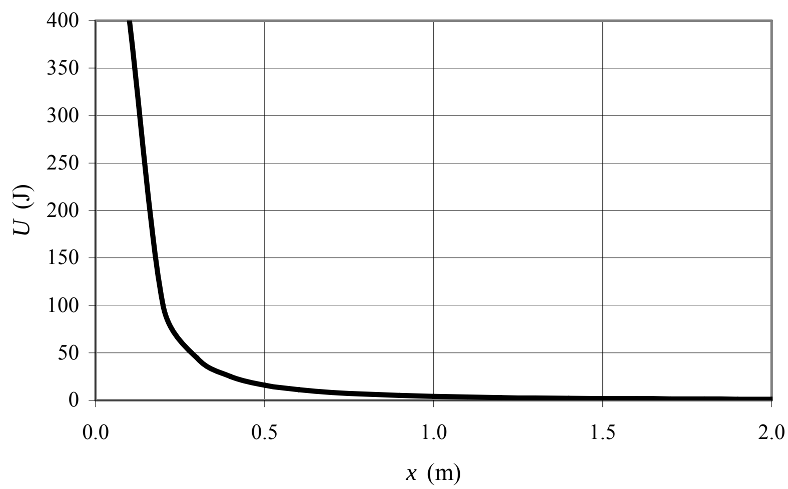
(*b*) As $x \to \infty$, $\frac{1}{2}\frac{A}{x^2} \to 0$. Hence:

$U_0 = 0$

and

$$U(x) = \frac{1}{2}\frac{A}{x^2} = \frac{1}{2}\left(\frac{8.0\,\text{N}\cdot\text{m}^3}{x^2}\right)$$
$$= \boxed{\frac{4.0}{x^2}\text{N}\cdot\text{m}^3}$$

(*c*) The graph of $U(x)$ follows:



33 •• [SSM] A straight rod of negligible mass is mounted on a frictionless pivot, as shown in Figure 7-38. Blocks have masses m_1 and m_2 are attached to the rod at distances ℓ_1 and ℓ_2. (*a*) Write an expression for the gravitational potential energy of the blocks-Earth system as a function of the angle θ made by the rod and the horizontal. (*b*) For what angle θ is this potential energy a minimum? Is the statement "systems tend to move toward a configuration of minimum potential energy" consistent with your result? (*c*) Show that if $m_1\ell_1 = m_2\ell_2$, the potential energy is the same for all values of θ. (When this holds, the system will balance at any angle θ. This result is known as *Archimedes' law of the lever*.)

Picture the Problem The gravitational potential energy of this system of two objects is the sum of their individual potential energies and is dependent on an arbitrary choice of where, or under what condition(s), the gravitational potential energy is zero. The best choice is one that simplifies the mathematical details of the expression for U. In this problem let's choose $U = 0$ where $\theta = 0$.

(*a*) Express U for the 2-object system as the sum of their gravitational potential energies; noting that because the object whose mass is m_2 is above the position we have chosen for $U = 0$, its potential energy is positive while that of the object whose mass is m_1 is negative:

$$U(\theta) = U_1 + U_2 = m_2 g\ell_2 \sin\theta - m_1 g\ell_1 \sin\theta = \boxed{(m_2\ell_2 - m_1\ell_1)g\sin\theta}$$

(*b*) Differentiate U with respect to θ and set this derivative equal to zero to identify extreme values:

$$\frac{dU}{d\theta} = (m_2\ell_2 - m_1\ell_1)g\cos\theta = 0$$

from which we can conclude that $\cos\theta = 0$ and $\theta = \cos^{-1}0$.

To be physically meaningful, $-\pi/2 \le \theta \le \pi/2$. Hence:

$$\theta = \pm\pi/2$$

Express the 2nd derivative of U with respect to θ and evaluate this derivative at $\theta = \pm\pi/2$:

$$\frac{d^2U}{d\theta^2} = -(m_2\ell_2 - m_1\ell_1)g\sin\theta$$

If we assume, in the expression for U that we derived in (*a*), that $m_2\ell_2 - m_1\ell_1 > 0$, then $U(\theta)$ is a sine function and, in the interval of interest,

$$-\pi/2 \le \theta \le \pi/2,$$

takes on its minimum value when $\theta = -\pi/2$:

$$\left.\frac{d^2U}{d\theta^2}\right|_{-\pi/2} > 0 \text{ and}$$

$$\boxed{U \text{ is a minimum at } \theta = -\pi/2}$$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\pi/2} < 0 \text{ and}$$

$$\boxed{U \text{ is a maximum at } \theta = \pi/2}$$

(c) If $m_2\ell_2 = m_1\ell_1$, then:

$$m_1\ell_1 - m_2\ell_2 = 0$$

and

$$\boxed{U = 0 \text{ independent of } \theta.}$$

Remarks: An alternative approach to establishing that *U* is a maximum at $\theta = \pi/2$ is to plot its graph and note that, in the interval of interest, *U* is concave downward with its maximum value at $\theta = \pi/2$. Similarly, it can be shown that *U* is a minimum at $\theta = -\pi/2$ (Part (*b*)).

34 •• An Atwood's machine (Figure 7-39) consists of masses m_1 and m_2, and a pulley of negligible mass and friction. Starting from rest, the speed of the two masses is 4.0 m/s at the end of 3.0 s. At that time, the kinetic energy of the system is 80 J and each mass has moved a distance of 6.0 m. Determine the values of m_1 and m_2.

Picture the Problem In a simple Atwood's machine, the only effect of the pulley is to connect the motions of the two objects on either side of it; that is, it could be replaced by a piece of polished pipe. We can relate the kinetic energy of the rising and falling objects to the mass of the system and to their common speed and relate their accelerations to the sum and difference of their masses … leading to simultaneous equations in m_1 and m_2.

Relate the kinetic energy of the system to the total mass being accelerated:

$$K = \tfrac{1}{2}(m_1 + m_2)v^2 \Rightarrow m_1 + m_2 = \frac{2K}{v^2}$$

Substitute numerical values and evaluate $m_1 + m_2$:

$$m_1 + m_2 = \frac{2(80\,\text{J})}{(4.0\,\text{m/s})^2} = 10.0\,\text{kg} \qquad (1)$$

In Chapter 4, the acceleration of the masses was shown to be:

$$a = \frac{m_1 - m_2}{m_1 + m_2}g$$

Because $v(t) = at$, we can eliminate a in the previous equation to obtain:

$$v(t) = \frac{m_1 - m_2}{m_1 + m_2}gt$$

Solving for $m_1 - m_2$ yields:

$$m_1 - m_2 = \frac{(m_1 + m_2)v(t)}{gt}$$

Substitute numerical values and evaluate $m_1 - m_2$:

$$m_1 - m_2 = \frac{(10\,\text{kg})(4.0\,\text{m/s})}{(9.81\,\text{m/s}^2)(3.0\,\text{s})} = 1.36\,\text{kg} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$m_1 = \boxed{5.7\,\text{kg}} \text{ and } m_2 = \boxed{4.3\,\text{kg}}$$

35 ••• You have designed a novelty desk clock, as shown in Figure 7-40. You are worried that it is not ready for market because the clock itself might be in an unstable equilibrium configuration. You decide to apply your knowledge of potential energies and equilibrium conditions and analyze the situation. The clock (mass *m*) is supported by two light cables running over the two frictionless pulleys of negligible diameter, which are attached to counterweights that each have mass *M*. (*a*) Find the potential energy of the system as a function of the distance *y*. (*b*) Find the value of *y* for which the potential energy of the system is a minimum. (*c*) If the potential energy is a minimum, then the system is in equilibrium. Apply Newton's second law to the clock and show that it is in equilibrium (the forces on it sum to zero) for the value of *y* obtained for Part (*b*). *(d)* Finally, determine whether you are going to be able to market this gadget: is this a point of stable or unstable equilibrium?

Picture the Problem Let *L* be the total length of one cable and the zero of gravitational potential energy be at the top of the pulleys. We can find the value of *y* for which the potential energy of the system is an extremum by differentiating *U*(*y*) with respect to *y* and setting this derivative equal to zero. We can establish that this value corresponds to a minimum by evaluating the second derivative of *U*(*y*) at the point identified by the first derivative. We can apply Newton's second law to the clock to confirm the result we obtain by examining the derivatives of *U*(*y*).

(*a*) Express the potential energy of the system as the sum of the potential energies of the clock and counterweights:

$$U(y)=U_{\text{clock}}(y)+U_{\text{weights}}(y)$$

Substitute for $U_{\text{clock}}(y)$ and $U_{\text{weights}}(y)$ to obtain:

$$U(y)=\boxed{-mgy-2Mg\left(L-\sqrt{y^2+d^2}\right)}$$

(*b*) Differentiate *U*(*y*) with respect to *y*:

$$\frac{dU(y)}{dy}=-\frac{d}{dy}\left[mgy+2Mg\left(L-\sqrt{y^2+d^2}\right)\right]=-\left[mg-2Mg\frac{y}{\sqrt{y^2+d^2}}\right]$$

For extreme values (relative maxima and minima):

$$mg-2Mg\frac{y'}{\sqrt{y'^2+d^2}}=0$$

Solve for y' to obtain:

$$y'=d\sqrt{\frac{m^2}{4M^2-m^2}}$$

Find $\frac{d^2U(y)}{dy^2}$:

$$\frac{d^2U(y)}{dy^2} = -\frac{d}{dy}\left[mg - 2Mg\frac{y}{\sqrt{y^2+d^2}}\right] = \frac{2Mgd^2}{\left(y^2+d^2\right)^{3/2}}$$

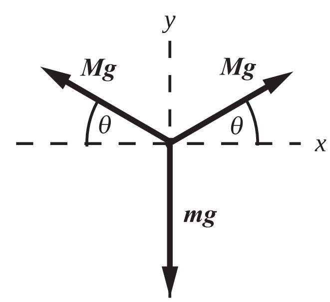
Evaluate $\frac{d^2U(y)}{dy^2}$ at $y = y'$:

$$\left.\frac{d^2U(y)}{dy^2}\right|_{y'} = \left.\frac{2Mgd^2}{\left(y^2+d^2\right)^{3/2}}\right|_{y'} = \frac{2Mgd}{\left(\frac{m^2}{4M^2-m^2}+1\right)^{3/2}} > 0$$

and the potential energy is a minimum at

$$y = \boxed{d\sqrt{\frac{m^2}{4M^2-m^2}}}$$

(*c*) The free-body diagram, showing the magnitudes of the forces acting on the support point just above the clock, is shown to the right:



Apply $\sum F_y = 0$ to this point to obtain:

$$2Mg\sin\theta - mg = 0 \Rightarrow \sin\theta = \frac{m}{2M}$$

Express $\sin\theta$ in terms of y and d:

$$\sin\theta = \frac{y}{\sqrt{y^2+d^2}}$$

Equate the two expressions for $\sin\theta$ to obtain:

$$\frac{m}{2M} = \boxed{\frac{y}{\sqrt{y^2+d^2}}}$$

which is equivalent to the first equation in Part (*b*).

(*d*) This is a point of stable equilibrium. If the clock is displaced downward, θ increases, leading to a larger upward force on the clock. Similarly, if the clock is displaced upward, the net force from the cables decreases. Because of this, the clock will be pulled back toward the equilibrium point if it is displaced away from it.

Remarks: Because we've shown that the potential energy of the system is a minimum at $y = y'$ (i.e., $U(y)$ is concave upward at that point), we can conclude that this point is one of stable equilibrium.

The Conservation of Mechanical Energy

36 • A block of mass *m* on a horizontal frictionless tabletop is pushed against a horizontal spring, compressing it a distance *x*, and the block is then released. The spring propels the block along the tabletop, giving a speed *v*. The same spring is then used to propel a second block of mass 4*m*, giving it a speed 3*v*. What distance was the spring compressed in the second case? Express your answer in terms of *x*.

Picture the Problem The work done in compressing the spring is stored in the spring as potential energy. When the block is released, the energy stored in the spring is transformed into the kinetic energy of the block. Equating these energies will give us a relationship between the compressions of the spring and the speeds of the blocks.

Let the numeral 1 refer to the first case and the numeral 2 to the second case. Relate the compression of the spring in the second case to its potential energy, which equals its initial kinetic energy when released:

$$\begin{aligned}\tfrac{1}{2}kx_2^2 &= \tfrac{1}{2}m_2v_2^2\\ &= \tfrac{1}{2}(4m_1)(3v_1)^2\\ &= 18m_1v_1^2\end{aligned}$$

Relate the compression of the spring in the first case to its potential energy, which equals its initial kinetic energy when released:

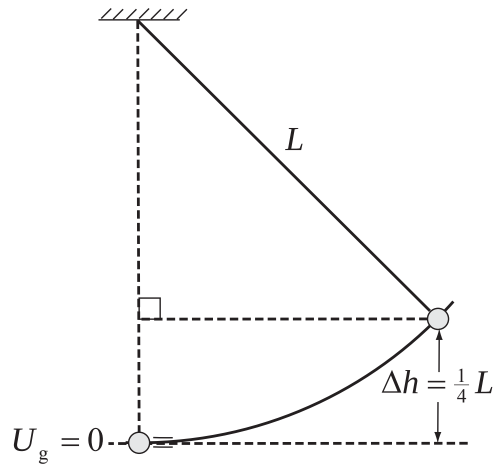
$$\tfrac{1}{2}kx_1^2 = \tfrac{1}{2}m_1v_1^2 \Rightarrow m_1v_1^2 = kx_1^2$$

Substitute for $m_1v_1^2$ to obtain:

$$\tfrac{1}{2}kx_2^2 = 18kx_1^2 \Rightarrow x_2 = \boxed{6x_1}$$

37 • A simple pendulum of length *L* with a bob of mass *m* is pulled aside until the bob is at a height *L*/4 above its equilibrium position. The bob is then released. Find the speed of the bob as it passes through the equilibrium position. Neglect any effects due to air resistance.

Picture the Problem The diagram shows the pendulum bob in its initial position. Let the zero of gravitational potential energy be at the low point of the pendulum's swing, the equilibrium position, and let the system include the pendulum and Earth. We can find the speed of the bob at it passes through the equilibrium position by applying conservation of mechanical energy to the system.



Apply conservation of mechanical energy to the system to obtain:

$W_{ext} = \Delta K + \Delta U$
or, because $W_{ext} = 0$,
$\Delta K + \Delta U = 0$

Because $K_i - U_f = 0$:

$K_f - U_i = 0$

Substituting for K_f and U_i yields:

$\frac{1}{2}mv_f^2 - mg\Delta h = 0 \Rightarrow v_f = \sqrt{2g\Delta h}$

Express Δh in terms of the length L of the pendulum:

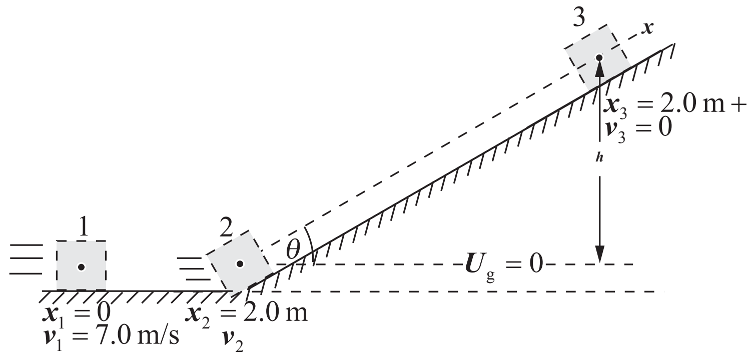
$\Delta h = \frac{L}{4}$

Substitute for Δh in the expression for v_f and simplify to obtain:

$v_f = \boxed{\sqrt{\frac{gL}{2}}}$

38 • A 3.0-kg block slides along a frictionless horizontal surface with a speed of 7.0 m/s (Figure 7-41). After sliding a distance of 2.0 m, the block makes a smooth transition to a frictionless ramp inclined at an angle of 40° to the horizontal. What distance along the ramp does the block slide before coming momentarily to rest?

Picture the Problem The pictorial representation shows the block in its initial, intermediate, and final states. It also shows a choice for $U_g = 0$. Let the system consist of the block, ramp, and Earth. Because the surfaces are frictionless, the initial kinetic energy of the system is equal to its final gravitational potential energy when the block has come to rest on the incline.



Apply conservation of mechanical energy to the system to obtain:

$$W_{ext} = \Delta K + \Delta U$$

or, because $W_{ext} = 0$,

$$\Delta K + \Delta U = 0$$

Because $K_3 = U_1 = 0$:

$$-K_1 + U_3 = 0$$

Substituting for K_1 and U_3 yields:

$$-\tfrac{1}{2}mv_1^2 + mgh = 0 \Rightarrow h = \frac{v_1^2}{2g}$$

where h is the change in elevation of the block as it slides to a momentary stop on the ramp.

Relate the height h to the displacement ℓ of the block along the ramp and the angle the ramp makes with the horizontal:

$$h = \ell \sin\theta$$

Equate the two expressions for h and solve for ℓ to obtain:

$$\ell \sin\theta = \frac{v_1^2}{2g} \Rightarrow \ell = \frac{v_1^2}{2g\sin\theta}$$

Substitute numerical values and evaluate ℓ:

$$\ell = \frac{(7.0\text{ m/s})^2}{2(9.81\text{ m/s}^2)\sin 40°} = \boxed{3.9\text{ m}}$$

39 • The 3.00-kg object in Figure 7-42 is released from rest at a height of 5.00 m on a curved frictionless ramp. At the foot of the ramp is a spring of force constant 400 N/m. The object slides down the ramp and into the spring, compressing it a distance *x* before coming momentarily to rest. (*a*) Find *x*. (*b*) Describe the motion object (if any) after the block momentarily comes to rest?

Picture the Problem Let the system consist of Earth, the block, and the spring. With this choice there are no external forces doing work to change the energy of the system. Let $U_g = 0$ at the elevation of the spring. Then the initial gravitational

potential energy of the 3.00-kg object is transformed into kinetic energy as it slides down the ramp and then, as it compresses the spring, into potential energy stored in the spring.

(*a*) Apply conservation of mechanical energy to the system to relate the distance the spring is compressed to the initial potential energy of the block:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

and, because $\Delta K = 0$,

$$-mgh + \tfrac{1}{2}kx^2 = 0 \Rightarrow x = \sqrt{\frac{2mgh}{k}}$$

Substitute numerical values and evaluate *x*:

$$x = \sqrt{\frac{2(3.00\,\text{kg})(9.81\,\text{m/s}^2)(5.00\,\text{m})}{400\,\text{N/m}}} = \boxed{0.858\,\text{m}}$$

(*b*) The energy stored in the compressed spring will accelerate the block, launching it back up the incline and the block will retrace its path, rising to a height of 5.00 m.

40 • You are designing a game for small children and want to see if the ball's maximum speed is sufficient to require the use of goggles. In your game, a 15.0-g ball is to be shot from a spring gun whose spring has a force constant of 600 N/m. The spring will be compressed 5.00 cm when in use. How fast will the ball be moving as it leaves the gun and how high will the ball go if the gun is aimed vertically upward? What would be your recommendation on the use of goggles?

Picture the Problem With U_g chosen to be zero at the uncompressed level of the spring, the ball's initial gravitational potential energy is negative. Let the system consist of the ball, the spring and gun, and Earth. The difference between the initial potential energy of the spring and the gravitational potential energy of the ball is first converted into the kinetic energy of the ball and then into gravitational potential energy as the ball rises and slows … eventually coming momentarily to rest.

Apply conservation of mechanical energy to the system as the ball leaves the gun to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U_{\text{g}} + \Delta U_{\text{s}} = 0$$

or, because $K_{\text{i}} = U_{\text{s,f}} = U_{\text{g,f}} = 0$,

$$\tfrac{1}{2}mv_{\text{f}}^2 + mgx - \tfrac{1}{2}kx^2 = 0$$

Solving for v_{f} yields:

$$v_{\text{f}} = \sqrt{\left(\frac{k}{m}x - 2g\right)x}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{\left[\left(\frac{600\text{ N/m}}{0.0150\text{ kg}}\right)(0.0500\text{ m}) - 2\left(9.81\text{ m/s}^2\right)\right](0.0500\text{ m})} = \boxed{9.95\text{ m/s}}$$

This initial speed of the ball is fast enough to warrant the use of goggles.

Apply conservation of mechanical energy to the system as the ball rises to its maximum height to obtain:

$$W_{ext} = \Delta K + \Delta U_g + \Delta U_s = 0$$

or, because $\Delta K = U_{s,f} = 0$,

$$mgh + mgx - \tfrac{1}{2}kx^2 = 0$$

where h is the maximum height of the ball.

Solving for h gives:

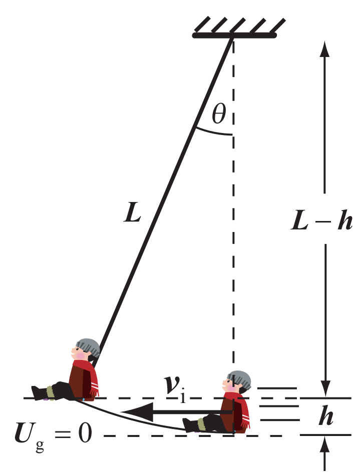
$$h = \frac{kx^2}{2mg} - x$$

Substitute numerical values and evaluate h:

$$h = \frac{(600\text{ N/m})(0.0500\text{ m})^2}{2(0.0150\text{ kg})\left(9.81\text{ m/s}^2\right)} - 0.0500\text{ m}$$
$$= \boxed{5.05\text{ m}}$$

41 • A 16-kg child on a 6.0-m-long playground swing moves with a speed of 3.4 m/s when the swing seat passes through its lowest point. What is the angle that the swing makes with the vertical when the swing is at its highest point? Assume that the effects due to air resistance are negligible, and assume that the child is not pumping the swing.

Picture the Problem Let the system consist of Earth and the child. Then $W_{ext} = 0$. Choose $U_g = 0$ at the child's lowest point as shown in the diagram to the right. Then the child's initial energy is entirely kinetic and its energy when it is at its highest point is entirely gravitational potential. We can determine h from conservation of mechanical energy and then use trigonometry to determine θ.



Using the diagram, relate θ to h and L:

$$\theta = \cos^{-1}\left(\frac{L-h}{L}\right) = \cos^{-1}\left(1 - \frac{h}{L}\right) \quad (1)$$

Apply conservation of mechanical energy to the system to obtain:

$$W_{ext} = \Delta K + \Delta U = 0$$

or, because $K_f = U_{g,i} = 0$,

$$-K_i + U_{g,f} = 0$$

Substituting for K_i and $U_{g,f}$ yields:

$$-\tfrac{1}{2}mv_i^2 + mgh = 0 \Rightarrow h = \frac{v_i^2}{2g}$$

Substitute for h in equation (1) to obtain:

$$\theta = \cos^{-1}\left(1 - \frac{v_i^2}{2gL}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left(1 - \frac{(3.4\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)(6.0\,\text{m})}\right) = \boxed{26°}$$

42 •• The system shown in Figure 7-44 is initially at rest when the lower string is cut. Find the speed of the objects when they are momentarily at the same height. The frictionless pulley has negligible mass.

Picture the Problem Let the system include the two objects and Earth. Then $W_{ext} = 0$. Choose $U_g = 0$ at the elevation at which the two objects meet. With this choice, the initial potential energy of the 3.0-kg object is positive and that of the 2.0-kg object is negative. Their sum, however, is positive. Given our choice for $U_g = 0$, this initial potential energy is transformed entirely into kinetic energy.

Apply conservation of mechanical energy to the system to obtain:

$$W_{ext} = \Delta K + \Delta U_g = 0$$

or, because $W_{ext} = 0$,

$$\Delta K = -\Delta U_g$$

Noting that m represents the sum of the masses of the objects as they are both moving in the final state, substitute for ΔK:

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = -\Delta U_g$$

or, because $v_i = 0$,

$$\tfrac{1}{2}mv_f^2 = -\Delta U_g \Rightarrow v_f = \sqrt{\frac{-2\Delta U_g}{m}}$$

ΔU_g is given by:

$$\Delta U_g = U_{g,f} - U_{g,i} = 0 - (m_3 - m_2)gh$$

Substitute for ΔU_g to obtain:

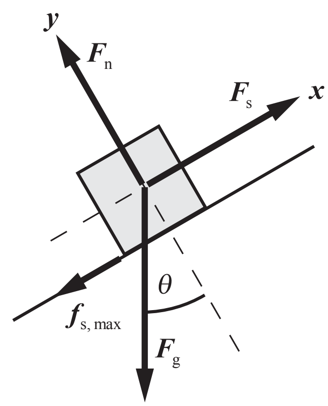
$$v_f = \sqrt{\frac{2(m_3 - m_2)gh}{m}}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{\frac{2(3.0\,\text{kg} - 2.0\,\text{kg})(0.50\,\text{m})(9.81\,\text{m/s}^2)}{3.0\,\text{kg} + 2.0\,\text{kg}}} = \boxed{1.4\,\text{m/s}}$$

43 •• A block of mass *m* rests on an inclined plane (Figure 7-44). The coefficient of static friction between the block and the plane is μ_s. A gradually increasing force is pulling down on the spring (force constant *k*). Find the potential energy *U* of the spring (in terms of the given symbols) at the moment the block begins to move.

Picture the Problem F_s is the force exerted by the spring and, because the block is on the verge of sliding, $f_s = f_{s,max}$. We can use Newton's second law, under equilibrium conditions, to express the elongation of the spring as a function of *m*, *k* and θ and then substitute in the expression for the potential energy stored in a stretched or compressed spring.



Express the potential energy of the spring when the block is about to move:

$$U = \tfrac{1}{2}kx^2$$

Apply $\sum \vec{F} = m\vec{a}$, under equilibrium conditions, to the block:

$$\sum F_x = F_s - f_{s,max} - mg\sin\theta = 0$$

and

$$\sum F_y = F_n - mg\cos\theta = 0$$

Using $f_{s,\,max} = \mu_s F_n$ and $F_s = kx$, eliminate $f_{s,\,max}$ and F_s from the *x* equation and solve for *x*:

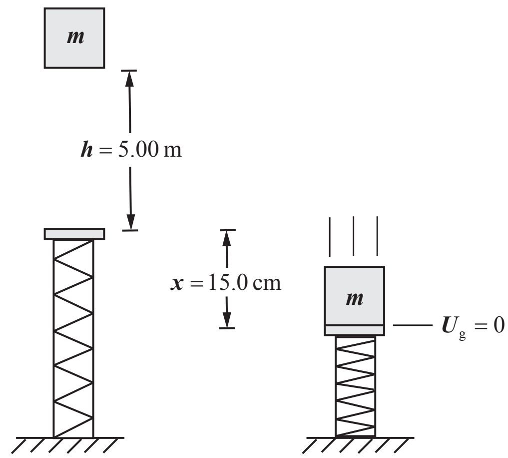
$$x = \frac{mg(\sin\theta + \mu_s\cos\theta)}{k}$$

Substitute for *x* in the expression for *U* and simplify to obtain:

$$U = \tfrac{1}{2}k\left[\frac{mg(\sin\theta + \mu_s\cos\theta)}{k}\right]^2 = \boxed{\frac{[mg(\sin\theta + \mu_s\cos\theta)]^2}{2k}}$$

44 •• A 2.40-kg block is dropped onto a spring and platform (Figure 7-45) of negligible mass. The block is released a distance of 5.00 m above the platform. When the block is momentary at rest, the spring is compressed by 25.0 cm. Find the speed of the block when the compression of the spring is only 15.0 cm

Picture the Problem Let the system include the block, the spring, and Earth. Let $U_g = 0$ where the spring is compressed 15.0 cm. Then the mechanical energy when the compression of the spring is 15.0 cm will be partially kinetic and partially stored in the spring. We can use conservation of mechanical energy to relate the initial potential energy of the system to the energy stored in the spring and the kinetic energy of block when it has compressed the spring 15.0 cm.



Apply conservation of mechanical energy to the system to obtain:

$$W_{ext} = \Delta U + \Delta K = 0$$

or

$$U_{g,f} - U_{g,i} + U_{s,f} - U_{s,i} + K_f - K_i = 0$$

Because $U_{g,f} = U_{s,i} = K_i = 0$:

$$-U_{g,i} + U_{s,f} + K_f = 0$$

Substitute to obtain:

$$-mg(h+x) + \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = 0$$

Solving for v yields:

$$v = \sqrt{2g(h+x) - \frac{kx^2}{m}}$$

The spring's force constant is given by:

$$k = \frac{w_{block}}{\Delta x} = \frac{mg}{\Delta x}$$

where Δx is the compression of the spring when the block is momentarily at rest.

Substituting for k and simplifying gives:

$$v = \sqrt{2g(h+x) - \frac{mgx^2}{m\Delta x}}$$

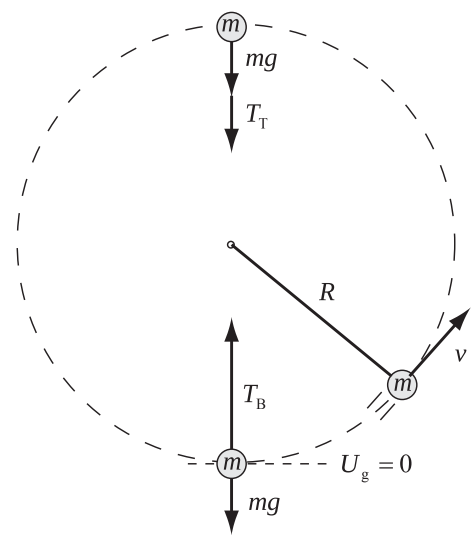
$$= \sqrt{g\left(2(h+x) - \frac{x^2}{\Delta x}\right)}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{\left(9.81\,\text{m/s}^2\right)\left[2\left(5.00\,\text{m}+0.150\,\text{m}\right)-\frac{\left(0.150\,\text{m}\right)^2}{0.250\,\text{m}}\right]}=\boxed{10.0\,\text{m/s}}$$

45 •• **[SSM]** A ball at the end of a string moves in a vertical circle with constant mechanical energy E. What is the difference between the tension at the bottom of the circle and the tension at the top?

Picture the Problem The diagram represents the ball traveling in a circular path with constant energy. U_g has been chosen to be zero at the lowest point on the circle and the superimposed free-body diagrams show the forces acting on the ball at the top (T) and bottom (B) of the circular path. We'll apply Newton's second law to the ball at the top and bottom of its path to obtain a relationship between T_T and T_B and conservation of mechanical energy to relate the speeds of the ball at these two locations.



Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the ball at the bottom of the circle and solve for T_B:

$$T_B - mg = m\frac{v_B^2}{R}$$

and

$$T_B = mg + m\frac{v_B^2}{R} \qquad (1)$$

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the ball at the top of the circle and solve for TT:

$$T_T + mg = m\frac{v_T^2}{R}$$

and

$$T_T = -mg + m\frac{v_T^2}{R} \qquad (2)$$

Subtract equation (2) from equation (1) to obtain:

$$T_B - T_T = mg + m\frac{v_B^2}{R} - \left(-mg + m\frac{v_T^2}{R}\right) = m\frac{v_B^2}{R} - m\frac{v_T^2}{R} + 2mg \qquad (3)$$

Using conservation of mechanical energy, relate the energy of the ball at the bottom of its path to its mechanical energy at the top of the circle:

$$\tfrac{1}{2}mv_B^2 = \tfrac{1}{2}mv_T^2 + mg(2R)$$

or

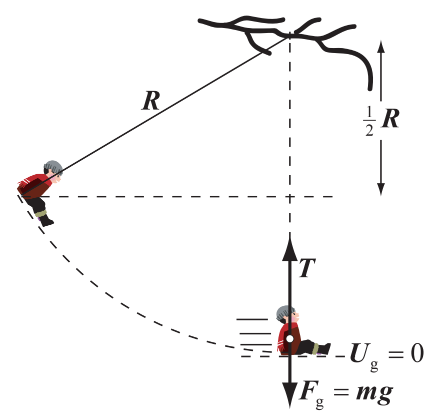
$$m\frac{v_B^2}{R} - m\frac{v_T^2}{R} = 4mg$$

Substituting in equation (3) yields:

$$T_B - T_T = \boxed{6mg}$$

46 •• A girl of mass m is taking a picnic lunch to her grandmother. She ties a rope of length R to a tree branch over a creek and starts to swing from rest at a point that is a distance $R/2$ lower than the branch. What is the minimum breaking tension for the rope if it is not to break and drop the girl into the creek?

Picture the Problem Let the system consist of the girl and Earth and let $U_g = 0$ at the lowest point in the girl's swing. We can apply conservation of mechanical energy to the system to relate the girl's speed v to R. The force diagram shows the forces acting on the girl at the low point of her swing. Applying Newton's second law to her will allow us to establish the relationship between the tension T and her speed.



Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the girl at her lowest point and solve for T:

$$T - mg = m\frac{v^2}{R}$$

and

$$T = mg + m\frac{v^2}{R} \qquad (1)$$

Apply conservation of mechanical energy to the system to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or, because $K_i = U_f = 0$,

Substituting for K_f and U_i yields:

$$K_f - U_i = 0$$

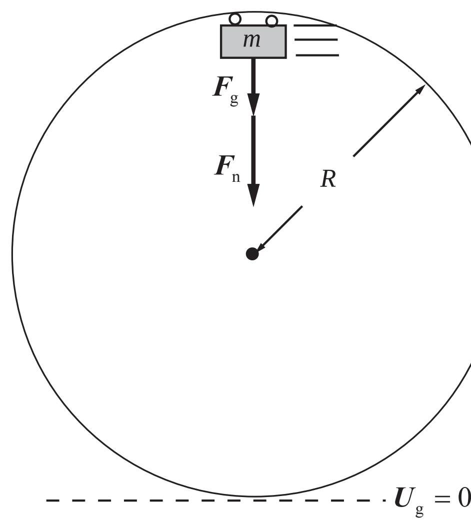
$$\tfrac{1}{2}mv^2 - mg\frac{R}{2} = 0 \Rightarrow \frac{v^2}{R} = g$$

Substitute for v^2/R in equation (1) and simplify to obtain:

$$T = mg + mg = \boxed{2mg}$$

47 •• A 1500-kg roller coaster car starts from rest a height H = 23.0 m (Figure 7-46) above the bottom of a 15.0-m-diameter loop. If friction is negligible, determine the downward force of the rails on the car when the upside-down car is at the top of the loop.

Picture the Problem Let the system include the car, the track, and Earth. The pictorial representation shows the forces acting on the car when it is upside down at the top of the loop. Choose U_g = 0 at the bottom of the loop. We can express F_n in terms of v and R by apply Newton's second law to the car and then obtain a second expression in these same variables by applying conservation of mechanical energy to the system. The simultaneous solution of these equations will yield an expression for F_n in terms of known quantities.



Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the car at the top of the circle and solve for F_n:

$$F_n + mg = m\frac{v^2}{R}$$

and

$$F_n = m\frac{v^2}{R} - mg \qquad (1)$$

Using conservation of mechanical energy, relate the energy of the car at the beginning of its motion to its energy when it is at the top of the loop:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or, because K_i = 0,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}mv^2 + mg(2R) - mgH = 0$$

Solving for $m\dfrac{v^2}{R}$ yields:

$$m\frac{v^2}{R} = 2mg\left(\frac{H}{R} - 2\right) \qquad (2)$$

Substitute equation (2) in equation (1) to obtain:

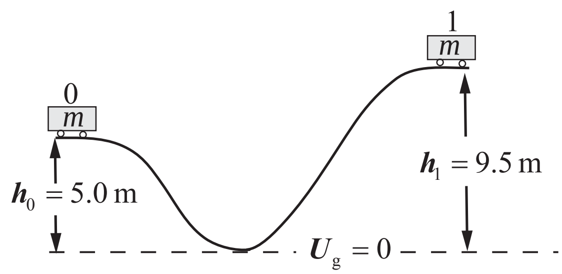
$$F_n = 2mg\left(\frac{H}{R} - 2\right) - mg$$
$$= mg\left(\frac{2H}{R} - 5\right)$$

Substitute numerical values and evaluate F_n:

$$F_n = (1500\,\text{kg})(9.81\,\text{m/s}^2)\left[\frac{2(23.0\,\text{m})}{7.50\,\text{m}} - 5\right] = \boxed{16.7\,\text{kN}}$$

48 •• A single roller-coaster car is moving with speed v_0 on the first section of track when it descends a 5.0-m-deep valley, then climbs to the top of a hill that is 4.5 m above the first section of track. Assume any effects of friction or of air resistance are negligible. (*a*) What is the minimum speed v_0 required if the car is to travel beyond the top of the hill? (*b*) Can we affect this speed by changing the depth of the valley to make the coaster pick up more speed at the bottom? Explain.

Picture the Problem Let the system include the roller coaster, the track, and Earth and denote the starting position with the numeral 0 and the top of the second hill with the numeral 1. We can use the work-energy theorem to relate the energies of the coaster at its initial and final positions. Let *m* be the mass of the roller coaster.



(*a*) Use conservation of mechanical energy to relate the work done by external forces to the change in the energy of the system:

$$W_{ext} = \Delta E_{sys} = \Delta K + \Delta U$$

Because the track is frictionless, $W_{ext} = 0$:

$$\Delta K + \Delta U = 0$$
and
$$K_1 - K_0 + U_1 - U_0 = 0$$

Substitute to obtain:

$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgh_1 - mgh_0 = 0$$

Solving for v_0 yields:

$$v_0 = \sqrt{v_1^2 + 2g(h_1 - h_0)}$$

If the coaster just makes it to the top of the second hill, $v_1 = 0$ and:

$$v_0 = \sqrt{2g(h_1 - h_0)}$$

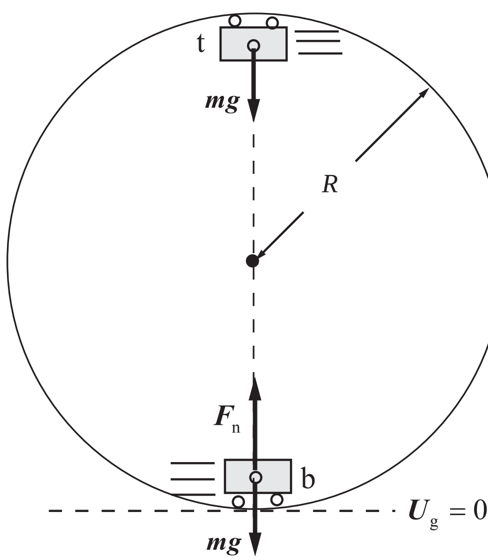
Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{2(9.81\,\text{m/s}^2)(9.5\,\text{m} - 5.0\,\text{m})} = \boxed{9.4\,\text{m/s}}$$

(*b*) No. Note that the required speed depends only on the difference in the heights of the two hills.

49 •• The Gravitron single-car roller coaster consists of a single loop-the-loop. The car is initially pushed, giving it just the right mechanical energy so the riders on the coaster will feel "weightless" when they pass through the top of the circular arc. How heavy will they feel when they pass through the bottom of the arc (that is, what is the normal force pressing up on them when they are at the bottom of the loop)? Express the answer as a multiple of *mg* (their actual weight). Assume any effects of friction or of air resistance are negligible.

Picture the Problem Let the radius of the loop be *R* and the mass of one of the riders be *m*. At the top of the loop, the centripetal force on her is her weight (the force of gravity). The two forces acting on her at the bottom of the loop are the normal force exerted by the seat of the car, pushing up, and the force of gravity, pulling down. We can apply Newton's second law to her at both the top and bottom of the loop to relate the speeds at those locations to *m* and *R* and, at *b*, to *F*, and then use conservation of mechanical energy to relate v_t and v_b.



Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the rider at the bottom of the circular arc:

$$F_n - mg = m\frac{v_b^2}{R} \Rightarrow F_n = mg + m\frac{v_b^2}{R} \quad (1)$$

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the rider at the top of the circular arc:

$$mg = m\frac{v_{\text{t}}^2}{R} \Rightarrow v_{\text{t}}^2 = gR$$

Apply conservation of mechanical energy to the system to obtain:

$$K_{\text{b}} - K_{\text{t}} + U_{\text{b}} - U_{\text{t}} = 0$$

or, because $U_{\text{b}} = 0$,

$$K_{\text{b}} - K_{\text{t}} - U_{\text{t}} = 0$$

Substitute for K_{b}, K_{t}, and U_{t} to obtain:

$$\tfrac{1}{2}mv_{\text{b}}^2 - \tfrac{1}{2}mv_{\text{t}}^2 - 2mgR = 0$$

Solving for v_{b}^2 yields:

$$v_{\text{b}}^2 = 5gR$$

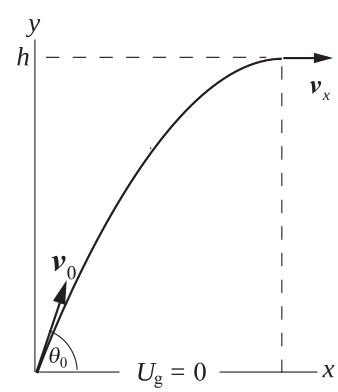
Substitute for v_{b}^2 in equation (1) and simplify to obtain:

$$F_{\text{n}} = mg + m\frac{5gR}{R} = \boxed{6mg}$$

That is, the rider will feel six times heavier than her normal weight.

50 •• A stone is thrown upward at an angle of 53° above the horizontal. Its maximum height above the release point is 24 m. What was the stone's initial speed? Assume any effects of air resistance are negligible.

Picture the Problem Let the system consist of the stone and Earth and ignore the influence of air resistance. Then $W_{\text{ext}} = 0$. Choose $U_{\text{g}} = 0$ as shown in the figure. Apply conservation of mechanical energy to describe the energy transformations as the stone rises to the highest point of its trajectory.



Apply conservation of mechanical energy to the system:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

and

$$K_1 - K_0 + U_1 - U_0 = 0$$

Because $U_0 = 0$:

$$K_1 - K_0 + U_1 = 0$$

Substitute for the kinetic and potential energies yields:

$$\tfrac{1}{2}mv_x^2 - \tfrac{1}{2}mv_0^2 + mgh = 0$$

In the absence of air resistance, the horizontal component of $\vec{v}$ is constant and equal to $v_x = v_0 \cos\theta$:

$$\tfrac{1}{2}m(v_0 \cos\theta)^2 - \tfrac{1}{2}mv_0^2 + mgh = 0$$

Solving for v_0 gives:

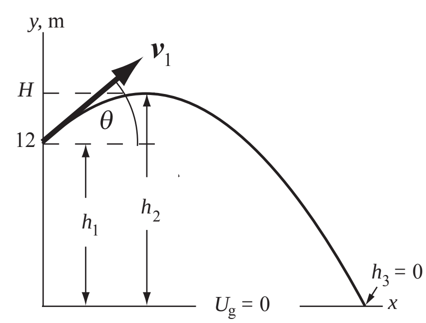
$$v_0 = \sqrt{\frac{2gh}{1-\cos^2\theta}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \sqrt{\frac{2(9.81\,\text{m/s}^2)(24\,\text{m})}{1-\cos^2 53°}} = \boxed{27\,\text{m/s}}$$

51 •• A 0.17-kg baseball is launched from the roof of a building 12 m above the ground. Its initial velocity is 30 m/s at 40° above the horizontal. Assume any effects of air resistance are negligible. (*a*) What is the maximum height above the ground the ball reaches? (*b*) What is the speed of the ball as it strikes the ground?

Picture the Problem The figure shows the ball being thrown from the roof of the building. Let the system consist of the ball and Earth. Then $W_{\text{ext}} = 0$. Choose $U_g = 0$ at ground level. We can use conservation of mechanical energy to determine the maximum height of the ball and its speed at impact with the ground.



(*a*) Apply conservation of mechanical energy to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or

$$K_2 - K_1 + U_2 - U_1 = 0$$

Substitute for the energies to obtain:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 + mgh_2 - mgh_1 = 0$$

Note that, at point 2, the ball is moving horizontally and:

$$v_2 = v_1 \cos\theta$$

Substitute for v_2 and h_2 to obtain:

$$\tfrac{1}{2}m(v_1 \cos\theta)^2 - \tfrac{1}{2}mv_1^2 + mgH - mgh_1 = 0$$

Solving for H gives:

$$H = h_1 - \frac{v^2}{2g}\left(\cos^2\theta - 1\right)$$

Substitute numerical values and evaluate H:

$$H = 12\,\text{m} - \frac{(30\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)}(\cos^2 40° - 1)$$
$$= \boxed{31\,\text{m}}$$

(*b*) Apply conservation of mechanical energy to the system to relate the initial mechanical energy of the ball to its just-before-impact energy:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or, because $U_3 = 0$,

$$K_3 - K_1 - U_1 = 0$$

Substituting for K_3, K_1, and U_1 yields:

$$\tfrac{1}{2}mv_3^2 - \tfrac{1}{2}mv_1^2 - mgh_1 = 0$$

Solve for v_3 to obtain:

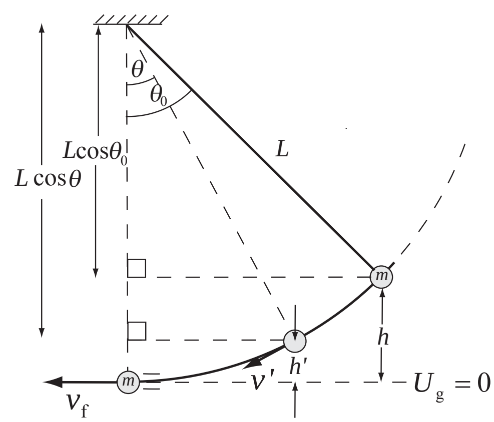
$$v_3 = \sqrt{v_1^2 + 2gh_1}$$

Substitute numerical values and evaluate v_3:

$$v_3 = \sqrt{(30\,\text{m/s})^2 + 2(9.81\,\text{m/s}^2)(12\,\text{m})}$$
$$= \boxed{34\,\text{m/s}}$$

52 •• An 80-cm-long pendulum with a 0.60-kg bob is released from rest at an initial angle of θ_0 with the vertical. At the bottom of the swing, the speed of the bob is 2.8 m/s. (*a*) What is θ_0? (*b*) What angle does the pendulum make with the vertical when the speed of the bob is 1.4 m/s? Is this angle equal to $\frac{1}{2}\theta_0$? Explain why or why not.

Picture the Problem The figure shows the pendulum bob in its release position and in the two positions in which it is in motion with the given speeds. Let the system consist of the pendulum and Earth and choose $U_g = 0$ at the low point of the swing. We can apply conservation of mechanical energy to relate the two angles of interest to the speeds of the bob at the intermediate and low points of its trajectory.



(*a*) Apply conservation of mechanical energy to the system to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or

$$K_f - K_i + U_f - U_i = 0.$$

Because $K_i = U_f = 0$:

$$K_f - U_i = 0 \qquad (1)$$

Refer to the pictorial representation to see that U_i is given by:

$$U_i = mgh = mgL(1-\cos\theta_0)$$

Substitute for K_f and U_i in equation (1) to obtain:

$$\tfrac{1}{2}mv_f^2 - mgL(1-\cos\theta_0) = 0$$

Solving for θ_0 yields:

$$\theta_0 = \cos^{-1}\left(1-\frac{v^2}{2gL}\right)$$

Substitute numerical values and evaluate θ_0:

$$\theta_0 = \cos^{-1}\left[1-\frac{(2.8\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)(0.80\,\text{m})}\right] = \boxed{60°}$$

(*b*) Letting primed quantities describe the indicated location, use conservation of mechanical energy to obtain :

$$K_f' - K_i + U_f' - U_i = 0$$

Because $K_i = 0$:

$$K_f' + U_f' - U_i = 0$$

Refer to the pictorial representation to see that U_f' is given by:

$$U_f' = mgh' = mgL(1-\cos\theta)$$

Substitute for K_f', U_f' and U_i :

$$\tfrac{1}{2}m(v_f')^2 + mgL(1-\cos\theta) - mgL(1-\cos\theta_0) = 0$$

Solving for θ yields :

$$\theta = \cos^{-1}\left[\frac{(v_f')^2}{2gL} + \cos\theta_0\right]$$

Substitute numerical values and evaluate θ:

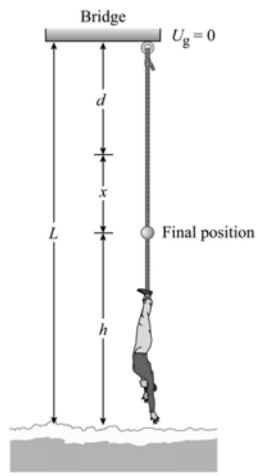
$$\theta = \cos^{-1}\left[\frac{(1.4\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)(0.80\,\text{m})} + \cos 60°\right] = \boxed{51°}$$

No. The change in gravitational potential energy is linearly dependent on the cosine of the angle rather than on the angle itself.

53 •• The Royal Gorge bridge over the Arkansas River is 310 m above the river. A 60-kg bungee jumper has an elastic cord with an unstressed length of 50 m attached to her feet. Assume that, like an ideal spring, the cord is massless and

provides a linear restoring force when stretched. The jumper leaps, and at her lowest point she barely touches the water. After numerous ascents and descents, she comes to rest at a height *h* above the water. Model the jumper as a point particle and assume that any effects of air resistance are negligible. (*a*) Find *h*. (*b*) Find the maximum speed of the jumper.

Picture the Problem Choose $U_g = 0$ at the bridge and let the system be Earth, the jumper and the bungee cord. Then $W_{ext} = 0$. We can use conservation of mechanical energy to relate her initial and final gravitational potential energies to the energy stored in the stretched bungee cord U_s. In Part (*b*), we'll use a similar strategy but include a kinetic energy term because we are interested in finding her maximum speed.



(*a*) Express her final height *h* above the water in terms of *L*, *d* and the distance *x* the bungee cord has stretched:

$$h = L - d - x \qquad (1)$$

Use conservation of mechanical energy to relate her gravitational potential energy as she just touches the water to the energy stored in the stretched bungee cord:

$$W_{ext} = \Delta K + \Delta U = 0$$

Because $\Delta K = 0$ and $\Delta U = \Delta U_g + \Delta U_s$:

$$-mgL + \tfrac{1}{2}ks^2 = 0 \Rightarrow k = \frac{2mgL}{s^2}$$

where *s* is the maximum distance the bungee cord has stretched.

Find the maximum distance the bungee cord stretches:

$s = 310\text{ m} - 50\text{ m} = 260\text{ m}$.

Substitute numerical values and evaluate *k*:

$$k = \frac{2(60\,\text{kg})(9.81\,\text{m/s}^2)(310\,\text{m})}{(260\,\text{m})^2} = 5.40\,\text{N/m}$$

Express the relationship between the forces acting on her when she has finally come to rest x:

$$F_{net} = kx - mg = 0 \Rightarrow x = \frac{mg}{k}$$

Substitute numerical values and evaluate x:

$$x = \frac{(60\,\text{kg})(9.81\,\text{m/s}^2)}{5.40\,\text{N/m}} = 109\,\text{m}$$

Substitute in equation (1) and evaluate h:

$$h = 310\,\text{m} - 50\,\text{m} - 109\,\text{m} = 151\,\text{m} = \boxed{0.15\,\text{km}}$$

(b) Using conservation of mechanical energy, express her total energy E:

$$E = K + U_g + U_s = E_i = 0$$

Because v is a maximum when K is a maximum, solve for K to obtain:

$$K = -U_g - U_s = mg(d + x) - \tfrac{1}{2}kx^2 \qquad (2)$$

Use the condition for an extreme value to obtain:

$$\frac{dK}{dx} = mg - kx = 0 \Rightarrow x = \frac{mg}{k}$$

Substitute numerical values and evaluate x:

$$\boldsymbol{x} = \frac{(60\,\text{kg})(9.81\,\text{m/s}^2)}{5.40\,\text{N/m}} = 109\,\text{m}$$

From equation (2) we have:

$$\tfrac{1}{2}mv^2 = mg(d + x) - \tfrac{1}{2}kx^2$$

Solve for v to obtain:

$$v = \sqrt{2g(d + x) - \frac{kx^2}{m}}$$

Substitute numerical values and evaluate v for x = 109 m:

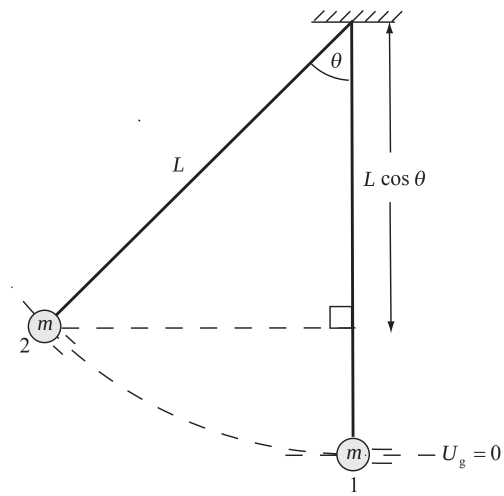
$$v = \sqrt{2(9.81\,\text{m/s}^2)(50\,\text{m} + 109\,\text{m}) - \frac{(5.4\,\text{N/m})(109\,\text{m})^2}{60\,\text{kg}}} = \boxed{45\,\text{m/s}}$$

Because $\dfrac{d^2K}{dx^2} = -k < 0$, x = 109 m corresponds to K_{max} and so v is a maximum.

54 •• A pendulum consists of a 2.0-kg bob attached to a light 3.0-m-long string. While hanging at rest with the string vertical, the bob is struck a sharp horizontal blow, giving it a horizontal velocity of 4.5 m/s. At the instant the string

makes an angle of 30° with the vertical, what is (*a*) the speed, (*b*) the gravitational potential energy (relative to its value is at the lowest point), and (*c*) the tension in the string? (*d*) What is the angle of the string with the vertical when the bob reaches its greatest height?

Picture the Problem Let the system be Earth and the pendulum bob. Then $W_{ext} = 0$. Choose $U_g = 0$ at the low point of the bob's swing and apply conservation of mechanical energy to the system. When the bob reaches the 30° position its energy will be partially kinetic and partially potential. When it reaches its maximum height, its energy will be entirely potential. Applying Newton's second law will allow us to express the tension in the string as a function of the bob's speed and its angular position.



(*a*) Apply conservation of mechanical energy to relate the energies of the bob at points 1 and 2:

$$W_{ext} = \Delta K + \Delta U = 0$$

or

$$K_2 - K_1 + U_2 - U_1 = 0$$

Because $U_1 = 0$:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 + U_2 = 0 \qquad (1)$$

The potential energy of the system when the bob is at point 2 is given by:

$$U_2 = mgL(1 - \cos\theta)$$

Substitute for U_2 in equation (1) to obtain:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 + mgL(1 - \cos\theta) = 0$$

Solving for v_2 yields:

$$v_2 = \sqrt{v_1^2 - 2gL(1 - \cos\theta)}$$

Substitute numerical values and evaluate v_2:

$$v_2 = \sqrt{(4.5\,\text{m/s})^2 - 2(9.81\,\text{m/s}^2)(3.0\,\text{m})(1 - \cos 30°)} = 3.52\,\text{m/s} = \boxed{3.5\,\text{m/s}}$$

(*b*) From (*a*) we have:

$$U_2 = mgL(1 - \cos\theta)$$

Substitute numerical values and evaluate U_2:

$$U_2 = (2.0\,\text{kg})(9.81\,\text{m/s}^2)(3.0\,\text{m})(1-\cos 30°) = \boxed{7.9\,\text{J}}$$

(*c*) Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the bob to obtain:

$$T - mg\cos\theta = m\frac{v_2^2}{L}$$

Solving for T yields:

$$T = m\left(g\cos\theta + \frac{v_2^2}{L}\right)$$

Substitute numerical values and evaluate T:

$$T = (2.0\,\text{kg})\left[(9.81\,\text{m/s}^2)\cos 30° + \frac{(3.52\,\text{m/s})^2}{3.0\,\text{m}}\right] = \boxed{25\,\text{N}}$$

(*d*) When the bob reaches its greatest height:

$$U = U_{\max} = mgL(1-\cos\theta_{\max})$$

and

$$-K_1 + U_{\max} = 0$$

Substitute for K_1 and $U_{\max}$:

$$-\tfrac{1}{2}mv_1^2 + mgL(1-\cos\theta_{\max}) = 0$$

Solve for $\theta_{\max}$ to obtain:

$$\theta_{\max} = \cos^{-1}\left(1 - \frac{v_1^2}{2gL}\right)$$

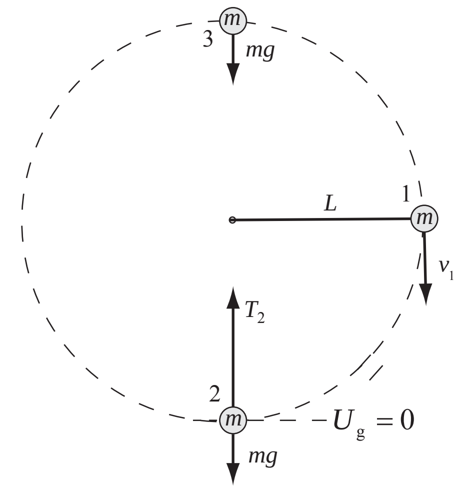
Substitute numerical values and evaluate $\theta_{\max}$:

$$\theta_{\max} = \cos^{-1}\left[1 - \frac{(4.5\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)(3.0\,\text{m})}\right] = \boxed{49°}$$

55 •• **[SSM]** A pendulum consists of a string of length L and a bob of mass m. The bob is rotated until the string is horizontal. The bob is then projected downward with the minimum initial speed needed to enable the bob to make a full revolution in the vertical plane. (*a*) What is the maximum kinetic energy of the bob? (*b*) What is the tension in the string when the kinetic energy is maximum?

Picture the Problem Let the system consist of Earth and the pendulum bob. Then $W_{\text{ext}} = 0$. Choose $U_g = 0$ at the bottom of the circle and let points 1, 2 and 3 represent the bob's initial point, lowest point and highest point, respectively. The bob will gain speed and kinetic energy until it reaches point 2 and slow down

until it reaches point 3; so it has its maximum kinetic energy when it is at point 2. We can use Newton's second law at points 2 and 3 in conjunction with conservation of mechanical energy to find the maximum kinetic energy of the bob and the tension in the string when the bob has its maximum kinetic energy.



(*a*) Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to the bob at the top of the circle and solve for v_3^2:

$$mg = m\frac{v_3^2}{L} \Rightarrow v_3^2 = gL$$

Apply conservation of mechanical energy to the system to express the relationship between K_2, K_3 and U_3:

$$K_3 - K_2 + U_3 - U_2 = 0$$

or, because $U_2 = 0$,

$$K_3 - K_2 + U_3 = 0$$

Solving for K_2 yields:

$$K_2 = K_{\max} = K_3 + U_3$$

Substituting for K_3 and U_3 yields:

$$K_{\max} = \tfrac{1}{2}mv_3^2 + mg(2L)$$

Substitute for v_3^2 and simplify to obtain:

$$K_{\max} = \tfrac{1}{2}m(gL) + 2mgL = \boxed{\tfrac{5}{2}mgL}$$

(*b*) Apply $\sum F_{\text{radial}} = ma_{\text{c}}$ to the bob at the bottom of the circle and solve for T_2:

$$F_{\text{net}} = T_2 - mg = m\frac{v_2^2}{L}$$

and

$$T_2 = mg + m\frac{v_2^2}{L} \qquad (1)$$

Use conservation of mechanical energy to relate the energies of the bob at points 2 and 3 and solve for K_2:

$K_3 - K_2 + U_3 - U_2 = 0$ where $U_2 = 0$ and

$K_2 = K_3 + U_3 = \frac{1}{2}mv_3^2 + mg(2L)$

Substitute for v_3^2 and K_2 to obtain:

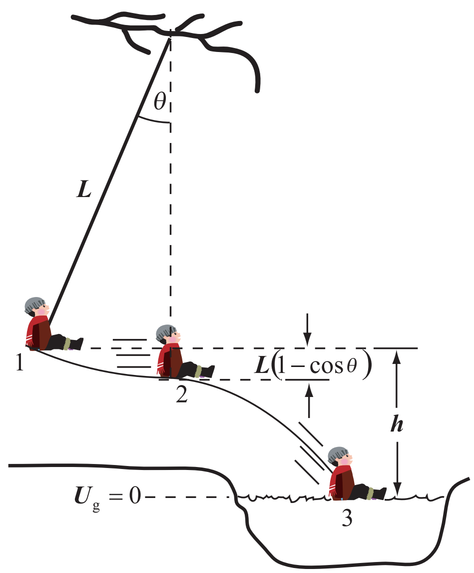
$\frac{1}{2}mv_2^2 = \frac{1}{2}m(gL) + mg(2L) \Rightarrow v_2^2 = 5gL$

Substitute for v_2^2 in equation (1) and simplify to obtain:

$$T_2 = mg + m\frac{5gL}{L} = \boxed{6mg}$$

56 •• A child whose weight is 360 N swings out over a pool of water using a rope attached to the branch of a tree at the edge of the pool. The branch is 12 m above ground level and the surface of the water is 1.8 m below ground level. The child holds onto the rope at a point 10.6 m from the branch and moves back until the angle between the rope and the vertical is 23°. When the rope is in the vertical position, the child lets go and drops into the pool. Find the speed of the child just as he impacts the surface of the water. (Model the child as a point particle attached to the rope 10.6 m from the branch.)

Picture the Problem Let the system consist of Earth and the child. Then $W_{ext} = 0$. In the figure, the child's initial position is designated with the numeral 1; the point at which the child releases the rope and begins to fall with a 2, and its point of impact with the water is identified with a 3. Choose $U_g = 0$ at the water level. While one could apply conservation of mechanical energy between points 1 and 2 and then between points 2 and 3, it is more direct to consider the energy transformations between points 1 and 3.



Apply conservation of mechanical energy between points 1 and 3:

$W_{ext} = \Delta K + \Delta U = 0$

$K_3 - K_1 + U_3 - U_1 = 0$

where U_3 and K_1 are zero.

Substitute for K_3 and U_1;

$\frac{1}{2}mv_3^2 - mg[h + L(1 - \cos\theta)] = 0$

Solving for v_3 yields:

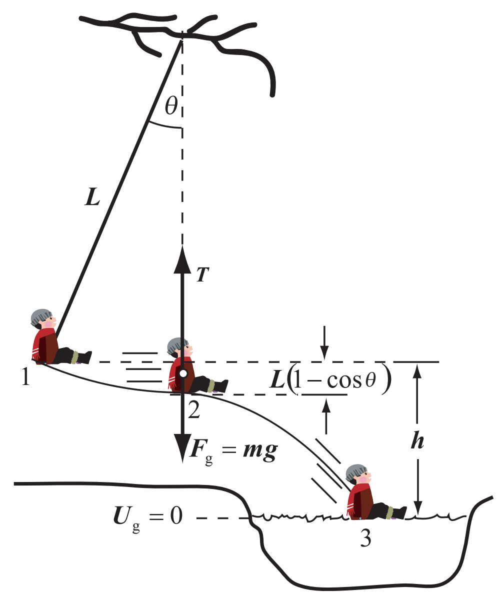
$$v_3 = \sqrt{2g[h + L(1 - \cos\theta)]}$$

Substitute numerical values and evaluate v_3:

$$v_3 = \sqrt{2\left(9.81\,\text{m/s}^2\right)\left[3.2\,\text{m} + (10.6\,\text{m})(1-\cos 23°)\right]} = \boxed{8.9\,\text{m/s}}$$

57 •• Walking by a pond, you find a rope attached to a stout tree limb that is 5.2 m above ground level. You decide to use the rope to swing out over the pond. The rope is a bit frayed, but supports your weight. You estimate that the rope might break if the tension is 80 N greater than your weight. You grab the rope at a point 4.6 m from the limb and move back to swing out over the pond. (Model yourself as a point particle attached to the rope 4.6 m from the limb.) (*a*) What is the maximum safe initial angle between the rope and the vertical at which it will not break during the swing? (*b*) If you begin at this maximum angle, and the surface of the pond is 1.2 m below the level of the ground, with what speed will you enter the water if you let go of the rope when the rope is vertical?

Picture the Problem Let the system consist of you and Earth. Then there are no external forces to do work on the system and $W_{\text{ext}} = 0$. In the figure, your initial position is designated with the numeral 1, the point at which you release the rope and begin to fall with a 2, and your point of impact with the water is identified with a 3. Choose $U_g = 0$ at the water level. We can apply Newton's second law to the forces acting on you at point 2 and apply conservation of mechanical energy between points 1 and 2 to determine the maximum angle at which you can begin your swing and then between points 1 and 3 to determine the speed with which you will hit the water.



(*a*) Use conservation of mechanical energy to relate your speed at point 2 to your potential energy there and at point 1:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or

$$K_2 - K_1 + U_2 - U_1 = 0$$

Because $K_1 = 0$:

$$\tfrac{1}{2}mv_2^2 + mgh - \left[mgL(1-\cos\theta) + mgh\right] = 0$$

Solve this equation for θ to obtain:

$$\theta = \cos^{-1}\left[1 - \frac{v_2^2}{2gL}\right] \qquad (1)$$

Apply $\sum F_{\text{radial}} = ma_{\text{radial}}$ to yourself at point 2 and solve for T:

$$T - mg = m\frac{v_2^2}{L} \text{ and } T = mg + m\frac{v_2^2}{L}$$

Because you've estimated that the rope might break if the tension in it exceeds your weight by 80 N, it must be that:

$$m\frac{v_2^2}{L} = 80\,\text{N} \Rightarrow v_2^2 = \frac{(80\,\text{N})L}{m}$$

Let's assume that your mass is 70 kg. Then:

$$v_2^2 = \frac{(80\,\text{N})(4.6\,\text{m})}{70\,\text{kg}} = 5.26\,\text{m}^2/\text{s}^2$$

Substitute numerical values in equation (1) to obtain:

$$\theta = \cos^{-1}\left[1 - \frac{5.26\,\text{m}^2/\text{s}^2}{2(9.81\,\text{m/s}^2)(4.6\,\text{m})}\right]$$
$$= 19.65° = \boxed{20°}$$

(*b*) Apply conservation of mechanical energy between points 1 and 3:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or, because $U_3 = K_1 = 0$,

$$K_3 - U_1 = 0$$

Substitute for K_3 and U_1 to obtain:

$$\tfrac{1}{2}mv_3^2 - mg[h + L(1 - \cos\theta)] = 0$$

Solving for v_3 yields:

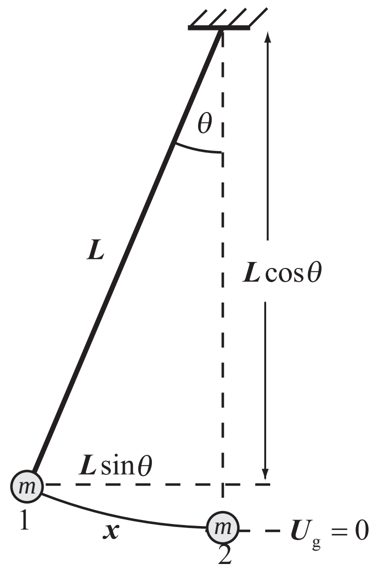
$$v_3 = \sqrt{2g[h + L(1 - \cos\theta)]}$$

Substitute numerical values and evaluate v_3:

$$v_3 = \sqrt{2(9.81\,\text{m/s}^2)[1.2\,\text{m} + (4.6\,\text{m})(1 - \cos 19.65°)]} = \boxed{5.4\,\text{m/s}}$$

58 ••• A pendulum bob of mass m is attached to a light string of length L and is also attached to a spring of force constant k. With the pendulum in the position shown in Figure 7-47, the spring is at its unstressed length. If the bob is now pulled aside so that the string makes a *small* angle θ with the vertical and released, what is the speed of the bob as it passes through the equilibrium position? *Hint: Recall the small-angle approximations: if* θ *is expressed in radians, and if* $|\theta| < 1$, *then* $\sin\theta \approx \tan\theta \approx \theta$ *and* $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

Picture the Problem Choose $U_g = 0$ at point 2, the lowest point of the bob's trajectory and let the system consist of the bob, spring, string and Earth. Given this choice, there are no external forces doing work on the system. Because $\theta \ll 1$, we can use the trigonometric series for the sine and cosine functions to approximate these functions. The bob's initial energy is partially gravitational potential and partially potential energy stored in the stretched spring. As the bob swings down to point 2 this energy is transformed into kinetic energy. By equating these energies, we can derive an expression for the speed of the bob at point 2.



Apply conservation of mechanical energy to the system as the pendulum bob swings from point 1 to point 2:

$$W_{ext} = \Delta K + \Delta U = 0$$

or, because $K_1 = U_2 = 0$,

$$K_2 - U_1 = 0$$

Substituting for K_2 and U_1 yields:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}kx^2 - mgL(1-\cos\theta) = 0$$

Note, from the figure, that when $\theta \ll 1$, $x \approx L\sin\theta$:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}k(L\sin\theta)^2 - mgL(1-\cos\theta) = 0$$

Also, when $\theta \ll 1$:

$$\sin\theta \approx \theta \text{ and } \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

Substitute for $\sin\theta$ and $\cos\theta$ to obtain:

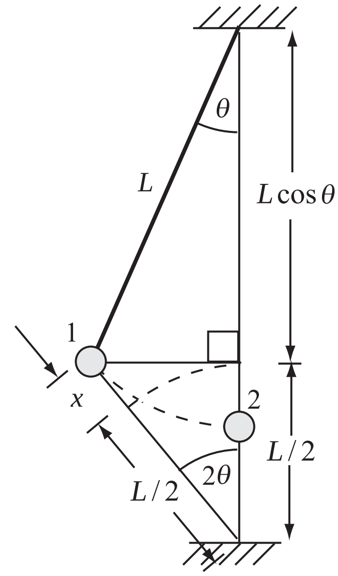
$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}k(L\theta)^2 - mgL\left[1 - \left(1 - \tfrac{1}{2}\theta^2\right)\right]$$

Solving for v_2 yields:

$$v_2 = \boxed{L\theta\sqrt{\frac{k}{m} + \frac{g}{L}}}$$

59 ••• [SSM] A pendulum is suspended from the ceiling and attached to a spring fixed to the floor directly below the pendulum support (Figure 7-48). The mass of the pendulum bob is m, the length of the pendulum is L, and the force constant is k. The unstressed length of the spring is $L/2$ and the distance between the floor and ceiling is $1.5L$. The pendulum is pulled aside so that it makes an

angle θ with the vertical and is then released from rest. Obtain an expression for the speed of the pendulum bob as the bob passes through a point directly below the pendulum support.



Picture the Problem Choose $U_g = 0$ at point 2, the lowest point of the bob's trajectory and let the system consist of Earth, the ceiling, the spring, and the pendulum bob. Given this choice, there are no external forces doing work to change the energy of the system. The bob's initial energy is partially gravitational potential and partially potential energy stored in the stretched spring. As the bob swings down to point 2 this energy is transformed into kinetic energy. By equating these energies, we can derive an expression for the speed of the bob at point 2.

Apply conservation of mechanical energy to the system as the pendulum bob swings from point 1 to point 2:

$$W_{ext} = \Delta K + \Delta U_g + \Delta U_s = 0$$

or, because $K_1 = U_{g,2} = U_{s,2} = 0$,

$$K_2 - U_{g,1} - U_{s,1} = 0$$

Substituting for K_2, $U_{g,1}$, and $U_{s,2}$ yields:

$$\tfrac{1}{2}mv_2^2 - mgL(1-\cos\theta) - \tfrac{1}{2}kx^2 = 0 \quad (1)$$

Apply the Pythagorean theorem to the lower triangle in the diagram to obtain:

$$\begin{aligned}\left(x+\tfrac{1}{2}L\right)^2 &= L^2\left[\sin^2\theta + \left(\tfrac{3}{2}-\cos\theta\right)^2\right] = L^2\left[\sin^2\theta + \tfrac{9}{4} - 3\cos\theta + \cos^2\theta\right]\\ &= L^2\left(\tfrac{13}{4} - 3\cos\theta\right)\end{aligned}$$

Take the square root of both sides of the equation to obtain:

$$x + \tfrac{1}{2}L = L\sqrt{\left(\tfrac{13}{4} - 3\cos\theta\right)}$$

Solving for x yields:

$$x = L\left[\sqrt{\left(\tfrac{13}{4} - 3\cos\theta\right)} - \tfrac{1}{2}\right]$$

Substitute for x in equation (1) to obtain:

$$\tfrac{1}{2}mv_2^2 = \tfrac{1}{2}kL^2\left[\sqrt{\left(\tfrac{13}{4} - 3\cos\theta\right)} - \tfrac{1}{2}\right]^2 + mgL(1-\cos\theta)$$

Solving for v_2 yields:

$$v_2 = \boxed{L\sqrt{2\frac{g}{L}(1-\cos\theta)+\frac{k}{m}\left(\sqrt{\tfrac{13}{4}-3\cos\theta}-\tfrac{1}{2}\right)^2}}$$

Total Energy and Non-conservative Forces

60 • In a volcanic eruption, 4.00 km^3 of mountain with an average density of 1600 kg/m^3 was raised an average height of 500 m. (*a*) What is the minimum amount of energy, in joules, that was released during this eruption? (*b*) The energy released by thermonuclear bombs is measured in megatons of TNT, where 1 megaton of TNT = 4.2×10^{15} J. Convert your answer for Part (*a*) to megatons of TNT.

Picture the Problem The energy of the eruption is initially in the form of the kinetic energy of the material it thrusts into the air. This energy is then transformed into gravitational potential energy as the material rises.

(*a*) Express the energy of the eruption in terms of the height Δh to which the debris rises:

$$E = mg\Delta h$$

Relate the mass of the material to its density and volume:

$$m = \rho V$$

Substitute for *m* to obtain:

$$E = \rho V g\Delta h$$

Substitute numerical values and evaluate *E*:

$$E = \left(1600\,\text{kg/m}^3\right)\left(4.00\,\text{km}^3\right)\left(9.81\,\text{m/s}^2\right)\left(500\,\text{m}\right) = \boxed{3.14\times10^{16}\ \text{J}}$$

(*b*) Convert 3.14×10^{16} J to megatons of TNT:

$$3.14\times10^{16}\ \text{J} = 3.14\times10^{16}\ \text{J}\times\frac{1\,\text{Mton TNT}}{4.2\times10^{15}\ \text{J}} = \boxed{7.5\,\text{Mton TNT}}$$

61 • To work off a large pepperoni pizza you ate on Friday night, on Saturday morning you climb a 120-m-high hill. (*a*) Assuming a reasonable value for your mass, determine your increase in gravitational potential energy. (*b*) Where does this energy come from? (*c*) The human body is typically 20 percent efficient. How much energy was converted into thermal energy? (*d*) How much chemical energy is expended by you during the climb? Given that oxidation

(burning) of a single slice of pepperoni pizza releases about 1.0 MJ (250 food calories) of energy, do you think one climb up the hill is enough?

Picture the Problem The work you did equals the change in your gravitational potential energy and is enabled by the transformation of metabolic energy in your muscles. Let the system consist of you and Earth and apply the conservation of mechanical energy to this system.

(*a*) Your increase in gravitational potential energy is:

$$\Delta U_{\text{g}} = mg\Delta h$$

Assuming that your mass is 70 kg, your increase in gravitational potential energy is:

$$\Delta U_{\text{g}} = (70\,\text{kg})(9.81\,\text{m/s}^2)(120\,\text{m}) = 82.4\,\text{kJ} = \boxed{82\,\text{kJ}}$$

(*b*) The energy required to do this work comes from the conversion of stored internal chemical energy into gravitational potential energy and thermal energy.

(*c*) Because only 20% of the energy you expend is converted into gravitational potential energy, four times this amount is converted into wasted thermal energy:

$$\Delta E_{\text{therm}} = 4\Delta U_{\text{g}}$$

Substitute the numerical value of ΔU_{g} and evaluate ΔE_{therm}:

$$\Delta E_{\text{therm}} = 4(82.4\,\text{kJ}) = \boxed{328\,\text{kJ}}$$

(*d*) Apply the conservation of mechanical energy to the system to obtain:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}} + \Delta E_{\text{chem}} = 0$$

or, because you begin and end your ascent at rest, $\Delta K = 0$ and,

$$\Delta U_{\text{g}} + \Delta E_{\text{therm}} + \Delta E_{\text{chem}} = 0$$

Solving for ΔE_{chem} yields:

$$\Delta E_{\text{chem}} = -\Delta U_{\text{g}} - \Delta E_{\text{therm}}$$

Substitute numerical values and evaluate ΔE_{chem}:

$$\Delta E_{\text{chem}} = -(82.4\,\text{kJ}) - (328\,\text{kJ}) = -410\,\text{kJ} = \boxed{-0.410\,\text{MJ}}$$

The climb uses up about 40% of the energy released from the pizza.

62 • A 2000-kg car moving at an initial speed of 25 m/s along a horizontal road skids to a stop in 60 m. (*a*) Find the energy dissipated by friction. (*b*) Find the coefficient of kinetic friction between the tires and the road. (*Note*: When

stopping without skidding and using conventional brakes, 100 percent of the kinetic energy is dissipated by friction within the brakes. With regenerative braking, such as that used in hybrid vehicles, only 70 percent of the kinetic energy is dissipated.)

Picture the Problem Let the car and Earth constitute the system. As the car skids to a stop on a horizontal road, its kinetic energy is transformed into internal (thermal) energy. Knowing that energy is transformed into heat by friction, we can use the definition of the coefficient of kinetic friction to calculate its value.

(*a*) The energy dissipated by friction is given by:

$$f\Delta s = \Delta E_{\text{therm}}$$

Apply the work-energy theorem for problems with kinetic friction:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}} = \Delta E_{\text{mech}} + f\Delta s$$

or, because $\Delta E_{\text{mech}} = \Delta K = -K_{\text{i}}$ and $W_{\text{ext}} = 0$,

$$0 = -\tfrac{1}{2}mv_{\text{i}}^2 + f\Delta s \Rightarrow f\Delta s = \tfrac{1}{2}mv_{\text{i}}^2$$

Substitute numerical values and evaluate $f\Delta s$:

$$f\Delta s = \tfrac{1}{2}(2000\,\text{kg})(25\,\text{m/s})^2 = 6.25\times10^5\ \text{J} = \boxed{6.3\times10^5\ \text{J}}$$

(*b*) Relate the kinetic friction force to the coefficient of kinetic friction and the weight of the car:

$$f_{\text{k}} = \mu_{\text{k}}mg \Rightarrow \mu_{\text{k}} = \frac{f_{\text{k}}}{mg} \qquad (1)$$

Express the relationship between the energy dissipated by friction and the kinetic friction force:

$$\Delta E_{\text{therm}} = f_{\text{k}}\Delta s \Rightarrow f_{\text{k}} = \frac{\Delta E_{\text{therm}}}{\Delta s}$$

Substitute for f_{k} in equation (1) to obtain:

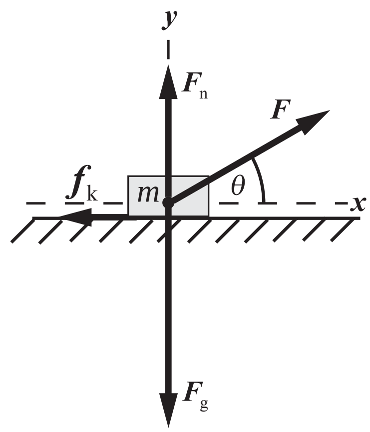
$$\mu_{\text{k}} = \frac{\Delta E_{\text{therm}}}{mg\Delta s}$$

Substitute numerical values and evaluate μ_{k}:

$$\mu_{\text{k}} = \frac{6.25\times10^5\ \text{J}}{(2000\,\text{kg})(9.81\,\text{m/s}^2)(60\,\text{m})} = \boxed{0.53}$$

63 • An 8.0-kg sled is initially at rest on a horizontal road. The coefficient of kinetic friction between the sled and the road is 0.40. The sled is pulled a distance of 3.0 m by a force of 40 N applied to the sled at an angle of 30° above the horizontal. (*a*) Find the work done by the applied force. (*b*) Find the energy dissipated by friction. (*c*) Find the change in the kinetic energy of the sled. (*d*) Find the speed of the sled after it has traveled 3.0 m.

Picture the Problem Let the system consist of the sled and Earth. Then the 40-N force is external to the system. The free-body diagram shows the forces acting on the sled as it is pulled along a horizontal road. The work done by the applied force can be found using the definition of work. To find the energy dissipated by friction, we'll use Newton's second law to determine f_k and then use it in the definition of work. The change in the kinetic energy of the sled is equal to the net work done on it. Finally, knowing the kinetic energy of the sled after it has traveled 3.0 m will allow us to solve for its speed at that location.



(*a*) The work done by the applied force is given by:

$$W_{ext} = \vec{F}\cdot\vec{s} = Fs\cos\theta$$

Substitute numerical values and evaluate W_{ext}:

$$W_{ext} = (40\,\text{N})(3.0\,\text{m})\cos 30° = 103.9\,\text{J} = \boxed{0.10\,\text{kJ}}$$

(*b*) The energy dissipated by friction as the sled is dragged along the surface is given by:

$$\Delta E_{therm} = f\Delta x = \mu_k F_n \Delta x \qquad (1)$$

Apply $\sum F_y = ma_y$ to the sled:

$$F_n + F\sin\theta - mg = 0$$

Solving for F_n yields:

$$F_n = mg - F\sin\theta$$

Substitute for F_n in equation (1) to obtain:

$$\Delta E_{therm} = \mu_k \Delta x(mg - F\sin\theta)$$

Substitute numerical values and evaluate ΔE_{therm}:

$$\Delta E_{therm} = (0.40)(3.0\,\text{m})\left[(8.0\,\text{kg})(9.81\,\text{m/s}^2) - (40\,\text{N})\sin 30°\right] = 70.2\,\text{J} = \boxed{70\,\text{J}}$$

(*c*) Apply the work-energy theorem for systems with kinetic friction:

$$W_{ext} = \Delta E_{mech} + \Delta E_{therm} = \Delta E_{mech} + f\Delta s$$

or, because $\Delta E_{mech} = \Delta K + \Delta U$ and $\Delta U = 0$,

$$W_{ext} = \Delta K + \Delta E_{therm}$$

Solving for ΔK yields:

$$\Delta K = W_{ext} - \Delta E_{therm}$$

Substitute numerical values and evaluate ΔK:

$$\Delta K = 103.9\,\text{J} - 70.2\,\text{J} = 33.7\,\text{J} = \boxed{34\,\text{J}}$$

(*d*) Because $K_i = 0$:

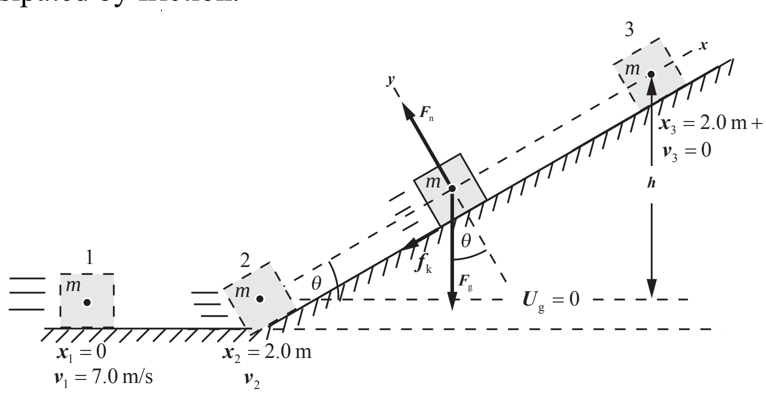
$$K_f = \Delta K = \tfrac{1}{2}mv_f^2 \Rightarrow v_f = \sqrt{\frac{2\Delta K}{m}}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{\frac{2(33.7\,\text{J})}{8.0\,\text{kg}}} = \boxed{2.9\,\text{m/s}}$$

64 •• Using Figure 7-41, suppose that the surfaces described are not frictionless and that the coefficient of kinetic friction between the block and the surfaces is 0.30. The block has an initial speed of 7.0 m/s and slides 2.0 m before reaching the ramp. Find (*a*) the speed of the block when it reaches the ramp, and (*b*) the distance that the block slides along the inclined surface before coming momentarily to rest. (Neglect any energy dissipated along the transition curve.)

Picture the Problem The pictorial representation shows the block in its initial, intermediate, and final states. It also shows a choice for $U_g = 0$. Let the system consist of the block, ramp, and Earth. Then the kinetic energy of the block at the foot of the ramp is equal to its initial kinetic energy less the energy dissipated by friction. The block's kinetic energy at the foot of the incline is partially converted to gravitational potential energy and partially converted to thermal energy (dissipated by friction) as the block slides up the incline. The free-body diagram shows the forces acting on the block as it slides up the incline. Applying Newton's second law to the block will allow us to determine f_k and express the energy dissipated by friction.



(a) Apply conservation of energy to the system while the block is moving horizontally:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}} = \Delta K + \Delta U + f\Delta s$$

or, because $\Delta U = W_{\text{ext}} = 0$,

$$0 = \Delta K + f\Delta s = K_2 - K_1 + f\Delta s$$

Solving for K_2 yields:

$$K_2 = K_1 - f\Delta s$$

Substitute for K_2, K_1, and $f\Delta s$ to obtain:

$$\tfrac{1}{2}mv_2^2 = \tfrac{1}{2}mv_1^2 - \mu_k mg\Delta x$$

Solving for v_2 yields:

$$v_2 = \sqrt{v_1^2 - 2\mu_k g\Delta x}$$

Substitute numerical values and evaluate v_2:

$$v_2 = \sqrt{(7.0\,\text{m/s})^2 - 2(0.30)(9.81\,\text{m/s}^2)(2.0\,\text{m})} = 6.10\,\text{m/s} = \boxed{6.1\,\text{m/s}}$$

(b) Apply conservation of energy to the system while the block is on the incline:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}} = \Delta K + \Delta U + f\Delta s$$

or, because $K_3 = U_2 = W_{\text{ext}} = 0$,

$$0 = -K_2 + U_3 + f\Delta s \qquad (1)$$

Apply $\sum F_y = ma_y$ to the block when it is on the incline:

$$F_n - mg\cos\theta = 0 \Rightarrow F_n = mg\cos\theta$$

Express $f\Delta s$:

$$f\Delta s = f_k\ell = \mu_k F_n\ell = \mu_k mg\ell\cos\theta$$

The final potential energy of the block is:

$$U_3 = mg\ell\sin\theta$$

Substitute for U_3, K_2, and $f\Delta s$ in equation (1) to obtain:

$$0 = -\tfrac{1}{2}mv_2^2 + mg\ell\sin\theta + \mu_k mg\ell\cos\theta$$

Solving for ℓ yields:

$$\ell = \frac{\tfrac{1}{2}v_2^2}{g(\sin\theta + \mu_k\cos\theta)}$$

Substitute numerical values and evaluate ℓ:

$$\ell = \frac{\tfrac{1}{2}(6.10\,\text{m/s})^2}{(9.81\,\text{m/s}^2)(\sin 40° + (0.30)\cos 40°)} = \boxed{2.2\,\text{m}}$$

65 •• [SSM] The 2.0-kg block in Figure 7-49 slides down a frictionless curved ramp, starting from rest at a height of 3.0 m. The block then slides 9.0 m on a rough horizontal surface before coming to rest. (*a*) What is the speed of the block at the bottom of the ramp? (*b*) What is the energy dissipated by friction? (*c*) What is the coefficient of kinetic friction between the block and the horizontal surface?

Picture the Problem Let the system include the block, the ramp and horizontal surface, and Earth. Given this choice, there are no external forces acting that will change the energy of the system. Because the curved ramp is frictionless, mechanical energy is conserved as the block slides down it. We can calculate its speed at the bottom of the ramp by using conservation of energy. The potential energy of the block at the top of the ramp or, equivalently, its kinetic energy at the bottom of the ramp is converted into thermal energy during its slide along the horizontal surface.

(*a*) Let the numeral 1 designate the initial position of the block and the numeral 2 its position at the foot of the ramp. Choose $U_g = 0$ at point 2 and use conservation of energy to relate the block's potential energy at the top of the ramp to its kinetic energy at the bottom:

$$W_{ext} = \Delta E_{mech} + \Delta E_{therm}$$

or, because $W_{ext} = K_i = U_f = \Delta E_{therm} = 0$,

$$0 = \tfrac{1}{2}mv_2^2 - mg\Delta h = 0 \Rightarrow v_2 = \sqrt{2g\Delta h}$$

Substitute numerical values and evaluate v_2:

$$v_2 = \sqrt{2(9.81\,\text{m/s}^2)(3.0\,\text{m})} = 7.67\,\text{m/s}$$
$$= \boxed{7.7\,\text{m/s}}$$

(b) The energy dissipated by friction is responsible for changing the thermal energy of the system:

$$W_f + \Delta K + \Delta U = \Delta E_{therm} + \Delta K + \Delta U$$
$$= 0$$

Because $\Delta K = 0$ for the entire slide; beginning at the top and ending when the block is at rest again:

$$W_f = -\Delta U = -(U_2 - U_1) = U_1$$

Substituting for U_1 yields:

$$W_f = mg\Delta h$$

Substitute numerical values and evaluate U_1:

$$W_f = (2.0\,\text{kg})(9.81\,\text{m/s}^2)(3.0\,\text{m}) = 58.9\,\text{J}$$
$$= \boxed{59\,\text{J}}$$

(c) The energy dissipated by friction is given by:

$$\Delta E_{\text{therm}} = f\Delta s = \mu_k mg\Delta x$$

Solving for μ_k yields:

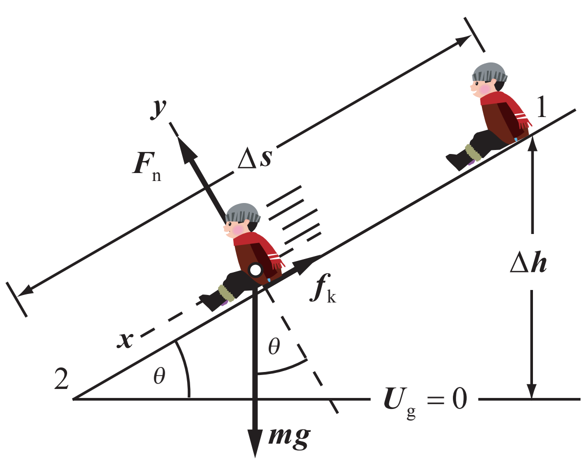
$$\mu_k = \frac{\Delta E_{\text{therm}}}{mg\Delta x}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{58.9\,\text{J}}{(2.0\,\text{kg})(9.81\,\text{m/s}^2)(9.0\,\text{m})}$$
$$= \boxed{0.33}$$

66 •• A 20-kg girl slides down a playground slide with a vertical drop of 3.2 m. When she reaches the bottom of the slide, her speed is 1.3 m/s. (*a*) How much energy was dissipated by friction? (*b*) If the slide is inclined at 20° with the horizontal, what is the coefficient of kinetic friction between the girl and the slide?

Picture the Problem Let the system consist of Earth, the girl, and the slide. Given this choice, there are no external forces doing work to change the energy of the system**.** By the time she reaches the bottom of the slide, her potential energy at the top of the slide has been converted into kinetic and thermal energy. Choose $U_g = 0$ at the bottom of the slide and denote the top and bottom of the slide as shown in the figure. We'll use the work-energy theorem with friction to relate these quantities and the forces acting on her during her slide to determine the friction force that transforms some of her initial potential energy into thermal energy.



(*a*) Express the work-energy theorem:

$$W_{ext} = \Delta K + \Delta U + \Delta E_{therm} = 0$$

Because $U_2 = K_1 = W_{ext} = 0$:

$$0 = K_2 - U_1 + \Delta E_{therm} = 0$$

or

$$\Delta E_{therm} = U_1 - K_2 = mg\Delta h - \tfrac{1}{2}mv_2^2$$

Substitute numerical values and evaluate ΔE_{therm}:

$$\Delta E_{therm} = (20\,\text{kg})(9.81\,\text{m/s}^2)(3.2\,\text{m}) - \tfrac{1}{2}(20\,\text{kg})(1.3\,\text{m/s})^2 = 611\,\text{J} = \boxed{0.61\,\text{kJ}}$$

(*b*) Relate the energy dissipated by friction to the kinetic friction force and the distance over which this force acts:

$$\Delta E_{therm} = f\Delta s = \mu_k F_n \Delta s$$

Solve for μ_k to obtain:

$$\mu_k = \frac{\Delta E_{therm}}{F_n \Delta s} \qquad (1)$$

Apply $\sum F_y = ma_y$ to the girl and solve for F_n:

$$F_n - mg\cos\theta = 0 \Rightarrow F_n = mg\cos\theta$$

Referring to the figure, relate Δh to Δs and θ:

$$\Delta s = \frac{\Delta h}{\sin\theta}$$

Substitute for Δs and F_n in equation (1) and simplify to obtain:

$$\mu_k = \frac{\Delta E_{therm}}{mg\dfrac{\Delta h}{\sin\theta}\cos\theta} = \frac{\Delta E_{therm}\tan\theta}{mg\Delta h}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{(611\,\text{J})\tan 20°}{(20\,\text{kg})(9.81\,\text{m/s}^2)(3.2\,\text{m})} = \boxed{0.35}$$

67 •• In Figure 7-50, the coefficient of kinetic friction between the 4.0-kg block and the shelf is 0.35. (*a*) Find the energy dissipated by friction when the 2.0-kg block falls a distance *y*. (*b*) Find the change in mechanical energy E_{mech} of the two-block–Earth system during the time it takes the 2.0-kg block to fall a distance *y*. (*c*) Use your result for Part (*b*) to find the speed of either block after the 2.0-kg block falls 2.0 m.

Picture the Problem Let the system consist of the two blocks, the shelf, and Earth. Given this choice, there are no external forces doing work to change the energy of the system. Due to the friction between the 4.0-kg block and the surface on which it slides, not all of the energy transformed during the fall of the 2.0-kg block is realized in the form of kinetic energy. We can find the energy dissipated by friction and then use the work-energy theorem with kinetic friction to find the speed of either block when they have moved the given distance.

(*a*) The energy dissipated by friction when the 2.0-kg block falls a distance *y* is given by:

$$\Delta E_{\text{therm}} = f\Delta s = \mu_k m_1 g y$$

Substitute numerical values and evaluate ΔE_{therm}:

$$\Delta E_{\text{therm}} = (0.35)(4.0\,\text{kg})(9.81\,\text{m/s}^2)y = (13.7\,\text{N})y = \boxed{(14\,\text{N})y}$$

(*b*) From the work-energy theorem with kinetic friction we have:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}}$$

or, because $W_{\text{ext}} = 0$ and $E_{\text{mech,i}} = 0$,

$$E_{\text{mech}} = -\Delta E_{\text{therm}} = \boxed{-(14\,\text{N})y}$$

(*c*) The total energy of the system is given by:

$$\tfrac{1}{2}(m_1 + m_2)v^2 - m_2 g y + \Delta E_{\text{therm}} = 0$$

Solving for *v* yields:

$$v = \sqrt{\frac{2(m_2 g y - \Delta E_{\text{therm}})}{m_1 + m_2}}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{\frac{2[(2.0\,\text{kg})(9.81\,\text{m/s}^2)(2.0\,\text{m}) - (13.7\,\text{N})(2.0\,\text{m})]}{4.0\,\text{kg} + 2.0\,\text{kg}}} = \boxed{2.0\,\text{m/s}}$$

68 •• A small object of mass *m* moves in a horizontal circle of radius *r* on a rough table. It is attached to a horizontal string fixed at the center of the circle. The speed of the object is initially v_0. After completing one full trip around the circle, the speed of the object is $0.5v_0$. (*a*) Find the energy dissipated by friction during that one revolution in terms of *m*, v_0, and *r*. (*b*) What is the coefficient of kinetic friction? (*c*) How many more revolutions will the object make before coming to rest?

Picture the Problem Let the system consist of the particle, the table, and Earth. Then $W_{\text{ext}} = 0$ and the energy dissipated by friction during one revolution is the change in the thermal energy of the system.

(*a*) Apply the work-energy theorem with kinetic friction to obtain:

$$W_{ext} = \Delta K + \Delta U + \Delta E_{therm}$$

or, because $\Delta U = W_{ext} = 0$,

$$0 = \Delta K + \Delta E_{therm}$$

Substitute for ΔK_f and simplify to obtain:

$$\begin{aligned}\Delta E_{therm} &= -\left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right)\\ &= -\left[\tfrac{1}{2}m\left(\tfrac{1}{2}v_0\right)^2 - \tfrac{1}{2}m\left(v_0\right)^2\right]\\ &= \boxed{\tfrac{3}{8}mv_0^2}\end{aligned}$$

(*b*) Relate the energy dissipated by friction to the distance traveled and the coefficient of kinetic friction:

$$\Delta E_{therm} = f\Delta s = \mu_k mg\Delta s = \mu_k mg(2\pi r)$$

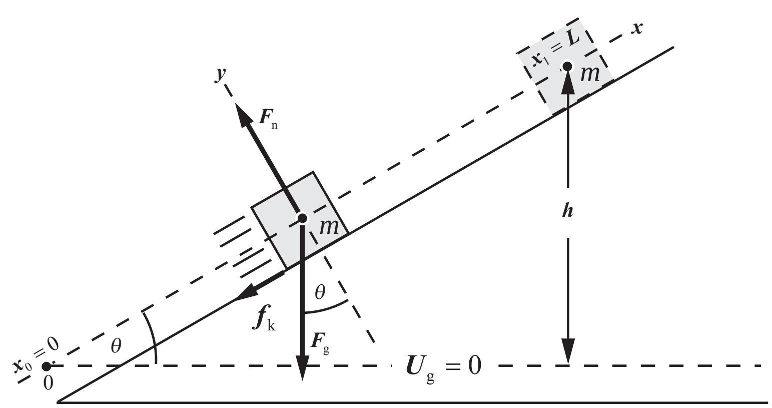
Substitute for ΔE and solve for μ_k to obtain:

$$\mu_k = \frac{\Delta E_{therm}}{2\pi mgr} = \frac{\frac{3}{8}mv_0^2}{2\pi mgr} = \boxed{\frac{3v_0^2}{16\pi gr}}$$

(*c*) Because the object lost $\frac{3}{4}K_i$ in one revolution, it will only require another 1/3 revolution to lose the remaining $\frac{1}{4}K_i$

69 •• [SSM] The initial speed of a 2.4-kg box traveling up a plane inclined 37° to the horizontal is 3.8 m/s. The coefficient of kinetic friction between the box and the plane is 0.30. (*a*) How far along the incline does the box travel before coming to a stop? (*b*) What is its speed when it has traveled half the distance found in Part (*a*)?

Picture the Problem The box will slow down and stop due to the dissipation of thermal energy. Let the system be Earth, the box, and the inclined plane and apply the work-energy theorem with friction. With this choice of the system, there are no external forces doing work to change the energy of the system. The pictorial representation shows the forces acting on the box when it is moving up the incline.



(*a*) Apply the work-energy theorem with friction to the system:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}} = \Delta K + \Delta U + \Delta E_{\text{therm}}$$

Substitute for ΔK, ΔU, and ΔE_{therm} to obtain:

$$0 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgh + \mu_k F_n L \quad (1)$$

Referring to the free-body diagram, relate the normal force to the weight of the box and the angle of the incline:

$$F_n = mg\cos\theta$$

Relate h to the distance L along the incline:

$$h = L\sin\theta$$

Substitute in equation (1) to obtain:

$$\mu_k mgL\cos\theta + \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgL\sin\theta = 0 \qquad (2)$$

Setting $v_1 = 0$ and solving equation (2) for L yields:

$$L = \frac{v_0^2}{2g(\mu_k\cos\theta + \sin\theta)}$$

Substitute numerical values and evaluate L:

$$L = \frac{(3.8\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)[(0.30)\cos 37° + \sin 37°]} = 0.8747\,\text{m} = \boxed{0.87\,\text{m}}$$

(*b*) Let $v_{\frac{1}{2}L}$ represent the box's speed when it is halfway up the incline. Then equation (2) becomes:

$$\mu_k mg\left(\tfrac{1}{2}L\right)\cos\theta + \tfrac{1}{2}mv_{\frac{1}{2}L}^2 - \tfrac{1}{2}mv_0^2 + mg\left(\tfrac{1}{2}L\right)\sin\theta = 0$$

Solving for $v_{\frac{1}{2}L}$ yields :

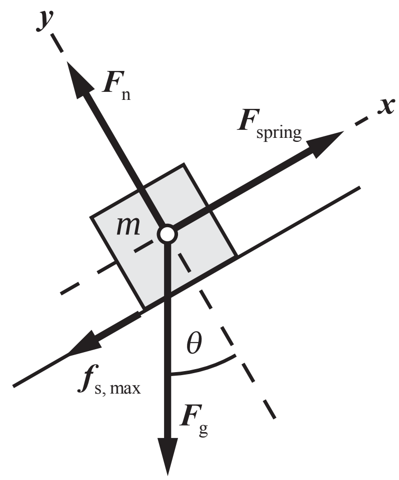
$$v_{\frac{1}{2}L} = \sqrt{v_0^2 - gL(\sin\theta + \mu_k\cos\theta)}$$

Substitute numerical values and evaluate $v_{\frac{1}{2}L}$:

$$v_f = \sqrt{(3.8\,\text{m/s})^2 - (9.81\,\text{m/s}^2)(0.8747\,\text{m})[\sin 37° + (0.30)\cos 37°]} = \boxed{2.7\,\text{m/s}}$$

70 ••• A block of mass m rests on a plane inclined an angle θ with the horizontal (Figure 7-51). A spring with force constant k is attached to the block. The coefficient of static friction between the block and plane is μ_s. The spring is pulled upward along the plane very slowly. (*a*) What is the extension of the spring the instant the block begins to move? (*b*) The block stops moving just as the extension of the contracting spring reaches zero. Express μ_k (the kinetic coefficient of friction) in terms of μ_s and θ.

Picture the Problem Let the system consist of Earth, the block, the incline, and the spring. With this choice of the system, there are no external forces doing work to change the energy of the system. The free-body diagram shows the forces acting on the block just before it begins to move. We can apply Newton's second law to the block to obtain an expression for the extension of the spring at this instant. We'll apply the work-energy theorem with friction to the second part of the problem.



(*a*) Apply $\sum \vec{F} = m\vec{a}$ to the block when it is on the verge of sliding:

$$\sum F_x = F_{spring} - f_{s,max} - mg\sin\theta = 0$$

and

$$\sum F_y = F_n - mg\cos\theta = 0$$

Eliminate F_n, $f_{s,max}$, and F_{spring} between the two equations to obtain:

$$kd - \mu_s mg\cos\theta - mg\sin\theta = 0$$

Solving for d yields:

$$d = \boxed{\frac{mg}{k}\left(\sin\theta + \mu_s\cos\theta\right)}$$

(*b*) Begin with the work-energy theorem with friction and no work being done by an external force:

$$\begin{aligned} W_{ext} &= \Delta E_{mech} + \Delta E_{therm} \\ &= \Delta K + \Delta U_g + \Delta U_s + \Delta E_{therm} \end{aligned}$$

Because the block is at rest in both its initial and final states, $\Delta K = 0$ and:

$$\Delta U_g + \Delta U_s + \Delta E_{therm} = 0 \qquad (1)$$

Let $U_g = 0$ at the initial position of the block. Then:

$$\Delta U_g = U_{g,\text{final}} - U_{g,\text{initial}} = mgh - 0 = mgd\sin\theta$$

Express the change in the energy stored in the spring as it relaxes to its unstretched length:

$$\Delta U_s = U_{s,\text{final}} - U_{s,\text{initial}} = 0 - \tfrac{1}{2}kd^2 = -\tfrac{1}{2}kd^2$$

The energy dissipated by friction is:

$$\Delta E_{\text{therm}} = f\Delta s = f_k d = \mu_k F_n d = \mu_k mgd\cos\theta$$

Substitute in equation (1) to obtain:

$$mgd\sin\theta - \tfrac{1}{2}kd^2 + \mu_k mgd\cos\theta = 0$$

or

$$mg\sin\theta - \tfrac{1}{2}kd + \mu_k mg\cos\theta = 0$$

Substituting for d (from Part (a)) yields:

$$mg\sin\theta - \tfrac{1}{2}k\left[\frac{mg}{k}\left(\sin\theta + \mu_s\cos\theta\right)\right] + \mu_k mg\cos\theta = 0$$

Finally, solve for μ_k to obtain:

$$\mu_k = \boxed{\tfrac{1}{2}\left(\mu_s - \tan\theta\right)}$$

Mass and Energy

71 • (a) Calculate the rest energy of 1.0 g of dirt. (b) If you could convert this energy completely into electrical energy and sell it for \$0.10/kW·h, how much money would you take in? (c) If you could power a 100-W light bulb with this energy, for how long could you keep the bulb lit?

Picture the Problem The intrinsic rest energy in matter is related to the mass of matter through Einstein's equation $E_0 = mc^2$.

(a) The rest energy of the dirt is given by:

$$E_0 = mc^2$$

Substitute numerical values and evaluate E_0:

$$E_0 = \left(1.0\times10^{-3}\text{ kg}\right)\left(2.998\times10^{8}\text{ m/s}\right)^2 = 8.988\times10^{13}\text{ J} = \boxed{9.0\times10^{13}\text{ J}}$$

(*b*) Express kW·h in joules:

$$1\,\text{kW}\cdot\text{h} = \left(1\times10^{3}\ \text{J/s}\right)\left(1\,\text{h}\times\frac{3600\ \text{s}}{\text{h}}\right)$$
$$= 3.60\times10^{6}\ \text{J}$$

Convert 8.988 × 10^13 J to kW·h:

$$8.988\times10^{13}\ \text{J} = \left(8.988\times10^{13}\ \text{J}\right)\times\left(\frac{1\,\text{kW}\cdot\text{h}}{3.60\times10^{6}\ \text{J}}\right)$$
$$= 2.50\times10^{7}\ \text{kW}\cdot\text{h}$$

Determine the price of the electrical energy:

$$\text{Price} = \left(2.50\times10^{7}\ \text{kW}\cdot\text{h}\right)\left(\frac{\$0.10}{\text{kW}\cdot\text{h}}\right)$$
$$= \boxed{\$2.5\times10^{6}}$$

(*c*) Relate the energy consumed to its rate of consumption and the time:

$$\Delta E = P\Delta t \Rightarrow \Delta t = \frac{\Delta E}{P}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{8.988\times10^{13}\ \text{J}}{100\ \text{W}} = 8.988\times10^{11}\ \text{s}$$
$$= \boxed{9.0\times10^{11}\ \text{s}}$$
$$= 8.988\times10^{11}\ \text{s}\times\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}}$$
$$= \boxed{2.8\times10^{4}\ \text{y}}$$

72 • One kiloton of TNT, when detonated, yields an explosive energy of roughly 4 × 10^12 J. How much less is the total mass of the bomb remnants after the explosion than before? If you could find and reassemble the pieces, would this loss of mass be noticeable?

Picture the Problem We can use the equation expressing the equivalence of energy and matter, $E = \Delta mc^2$, to find the reduction in the mass of the bomb due to the explosion.

Solve $E = \Delta mc^2$ for Δm:

$$\Delta m = \frac{E}{c^2}$$

Substitute numerical values and evaluate Δm:

$$\Delta m = \frac{4\times10^{12}\ \text{J}}{\left(2.998\times10^{8}\ \text{m/s}\right)^2}$$
$$= 4.45\times10^{-5}\ \text{kg} \approx \boxed{4\times10^{-5}\ \text{kg}}$$

Express the ratio of Δm to the mass of the bomb before its explosion:

$$\frac{\Delta m}{m_{\text{bomb}}}=\frac{4.45\times10^{-5}\text{ kg}}{1\text{ kton}\times\frac{2000\text{ lb}}{\text{ton}}\times\frac{1\text{ kg}}{2.2046\text{ lb}}}\approx 5\times10^{-11}$$

No, not noticeable! The mass change, compared to the mass of the bomb, is negligible.

73 • Calculate your rest energy in both mega electron-volts and joules. If that energy could be converted completely to the kinetic energy of your car, estimate its speed. Use the nonrelativistic expression for kinetic energy and comment on whether or not your answer justifies using the nonrelativistic expression for kinetic energy.

Picture the Problem Your rest energy is given by $E_0 = mc^2$.

Your rest energy if given by:

$$E_0 = mc^2$$

Assuming that your mass is 70 kg, substitute numerical values and evaluate E_0:

$$E_0=(70\text{ kg})(2.998\times10^8\text{ m/s})^2 = 6.292\times10^{18}\text{ J} = \boxed{6.3\times10^{18}\text{ J}}$$

Convert E_0 to MeV to obtain:

$$E_0 = 6.292\times10^{18}\text{ J}\times\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}} = \boxed{3.9\times10^{31}\text{ MeV}}$$

The nonrelativistic expression for the kinetic energy of your car is:

$$K=\tfrac{1}{2}mv^2 \Rightarrow v=\sqrt{\frac{2K}{m}}$$

Assuming the mass of your car to be 1.4×10^3 kg (approximately 3000 lb), substitute numerical values and evaluate v:

$$v=\sqrt{\frac{2(6.292\times10^{18}\text{ J})}{1.4\times10^3\text{ kg}}}\approx\boxed{9.5\times10^7\text{ m/s}}$$

As expected, this result (4.2×10^8 m/s) is greater than the speed of light (and thus incorrect). Use of the nonrelativistic expression for kinetic energy is not justified.

74 • If a black hole and a "normal" star orbit each other, gases from the normal star falling into the black hole can have their temperature increased by millions of degrees due to frictional heating. When the gases are heated that much, they begin to radiate light in the X-ray region of the electromagnetic spectrum (high-energy light photons). Cygnus X-1, the second strongest known X-ray source in the sky, is thought to be one such binary system; it radiates at an

estimated power of 4×10^{31} W. If we assume that 1.0 percent of the in-falling mass escapes as X ray energy, at what rate is the black hole gaining mass?

Picture the Problem We can differentiate the mass-energy equation to obtain an expression for the rate at which the black hole gains energy.

Using the mass-energy relationship, express the energy radiated by the black hole:

$$E = (0.010)mc^2$$

Differentiate this expression to obtain an expression for the rate at which the black hole is radiating energy:

$$\frac{dE}{dt} = \frac{d}{dt}\left[(0.010)mc^2\right] = (0.010)c^2\,\frac{dm}{dt}$$

Solving for dm/dt yields:

$$\frac{dm}{dt} = \frac{dE/dt}{(0.010)c^2}$$

Substitute numerical values and evaluate dm/dt:

$$\frac{dm}{dt} = \frac{4\times10^{31}\ \text{watt}}{(0.010)\left(2.998\times10^{8}\ \text{m/s}\right)^2} \approx \boxed{4\times10^{16}\ \text{kg/s}}$$

75 • [SSM] You are designing the fuel requirements for a small fusion electric-generating plant. Assume 33% conversion to electric energy. For the deuterium–tritium (D–T) fusion reaction in Example 7-18, calculate the number of reactions per second that are necessary to generate 1.00 kW of electric power.

Picture the Problem The number of reactions per second is given by the ratio of the power generated to the energy released per reaction. The number of reactions that must take place to produce a given amount of energy is the ratio of the energy per second (power) to the energy released per second.

In Example 7-18 it is shown that the energy per reaction is 17.59 MeV. Convert this energy to joules:

$$17.59\,\text{MeV} = (17.59\,\text{MeV})\left(1.602\times10^{-19}\ \text{J/eV}\right) = 28.18\times10^{-13}\ \text{J}$$

Assuming 33% conversion to electric energy, the number of reactions per second is:

$$\frac{1000\,\text{J/s}}{(0.33)\left(28.18\times10^{-13}\ \text{J/reaction}\right)} \approx \boxed{1.1\times10^{15}\ \text{reactions/s}}$$

76 • Use Table 7-1 to calculate the energy needed to remove one neutron from a stationary alpha particle, leaving a stationary helion plus a neutron with a kinetic energy of 1.5 MeV.

Picture the Problem The energy required for this reaction is the sum of 1.5 MeV and the difference between the rest energy of ^{4}He and the sum of the rest energies of a helion (^{3}He) and a neutron.

The required energy is the sum of the energy required to remove the neutron and the kinetic energy of the neutron:

$$E_{\text{total}} = E + K_{\text{n}} \quad (1)$$

Express the reaction:

$$^4\text{He} \rightarrow {}^3\text{He} + \text{n}$$

From Table 7-1, the rest energy of a neutron is: 939.565 MeV

From Table 7-1, the rest energy of ^{4}He is: 3727.379 MeV

From Table 7-1, the rest energy of ^{3}He is: 2808.391 MeV

The energy required to remove the neutron is the difference between the sum of the rest energies of ^{3}He and the neutron and the rest energy of the alpha particle:

$$E = [(2808.391 + 939.565) - 3727.379]\ \text{MeV} = 20.577\ \text{MeV}$$

Substitute numerical values in equation (1) and evaluate E_{total}:

$$E_{\text{total}} = 20.577\ \text{MeV} + 1.5\ \text{MeV} = \boxed{22.1\ \text{MeV}}$$

77 • A free neutron can decay into a proton plus an electron and an electron antineutrino [an electron antineutrino (symbol $\overline{\nu}_{\text{e}}$) is a nearly massless elementary particle]: $\text{n} \rightarrow \text{p} + \text{e}^- + \overline{\nu}_{\text{e}}$. Use Table 7-1 to calculate the energy released during this reaction.

Picture the Problem The energy released during this reaction is the difference between the rest energy of a neutron and the sum of the rest energies of a proton and an electron.

The rest energy of a proton is: 938.272 MeV

The rest energy of an electron (Table 7-1) is: 0.511 MeV

The rest energy of a neutron (Table 7-1) is: 939.565 MeV

Substitute numerical values to find the difference in the rest energy of a neutron and the sum of the rest energies of a positron and an electron:

$$E = [939.565 - (938.272 + 0.511)]\text{ MeV} = \boxed{0.782\text{ MeV}}$$

78 •• During one type of nuclear fusion reaction, two deuterons combine to produce an alpha particle. (*a*) How much energy is released during this reaction? (*b*) How many such reactions must take place per second to produce 1 kW of power?

Picture the Problem The reaction is $^2\text{H} + ^2\text{H} \rightarrow ^4\text{He} + E$. The energy released in this reaction is the difference between twice the rest energy of ^{2}H and the rest energy of ^{4}He. The number of reactions that must take place to produce a given amount of energy is the ratio of the energy per second (power) to the energy released per reaction. See Table 7-1 for the rest energies of the particles.

(*a*) The rest energy of ^{4}He is: 3727.379 MeV

The rest energy of a deuteron, ^{2}H, is: 1875.613 MeV

The energy released in the reaction is:

$$E = [2(1875.613) - 3727.379]\text{ MeV} = 23.847\text{ MeV} \times \frac{1.6022\times10^{-19}\text{ J}}{\text{eV}} = 3.821\times10^{-12}\text{ J} = \boxed{3.82\times10^{-12}\text{ J}}$$

(*b*) The number of reactions per second is:

$$\frac{1.00\times10^{3}\text{ J/s}}{3.820\times10^{-12}\text{ J/reaction}} = \boxed{2.62\times10^{14}\text{ reactions/s}}$$

79 •• A large nuclear power plant produces 1000 MW of electrical power by nuclear fission. (*a*) By how many kilograms does the mass of the nuclear fuel decrease by in one year? (Assume an efficiency of 33 percent for a nuclear power plant.) (*b*) In a coal-burning power plant, each kilogram of coal releases 31 MJ of

thermal energy when burned. How many kilograms of coal are needed each year for a 1000-MW coal-burning power plant? (Assume an efficiency of 38 percent for a coal-burning power plant.)

Picture the Problem The annual consumption of matter by the fission plant is the ratio of its annual energy output to the square of the speed of light. The annual consumption of coal in a coal-burning power plant is the ratio of its annual energy output to energy per unit mass of the coal.

(*a*) The yearly consumption of matter is given by:

$$\Delta m = \frac{E}{\varepsilon c^2}$$

where E is the energy to be generated and ε is the efficiency of the plant.

Because the energy to be generated is the product of the power output of the plant and the elapsed time:

$$\Delta m = \frac{P\Delta t}{\varepsilon c^2} \qquad (1)$$

Substitute numerical values and evaluate Δm:

$$\Delta m = \frac{(1000\text{ MW})\left(1\text{ y}\times\dfrac{3.156\times10^7\text{ s}}{\text{y}}\right)}{(0.33)(2.998\times10^8\text{ m/s})^2} = \boxed{1.1\text{ kg}}$$

(*b*) For a coal-burning power plant, equation (1) becomes:

$$\Delta m_{\text{coal}} = \frac{P\Delta t}{\varepsilon\left(\dfrac{\text{energy released}}{\text{unit mass}}\right)}$$

Substitute numerical values and evaluate Δm_{coal}:

$$\Delta m_{\text{coal}} = \frac{(1000\text{ MW})\left(1\text{ y}\times\dfrac{3.156\times10^7\text{ s}}{\text{y}}\right)}{(0.38)(31\text{ MJ/kg})} = \boxed{2.7\times10^9\text{ kg}}$$

Remarks: 2.7×10^9 kg is approximately 3 million tons!

Quantization of Energy

80 •• A mass on the end of a spring with a force constant of 1000 N/kg oscillates at a frequency of 2.5 oscillations per second. (*a*) Determine the quantum number, *n*, of the state it is in if it has a total energy of 10 J. (*b*) What is its ground state energy?

Picture the Problem The energy number n of a state whose energy is E is given by $E = \left(n + \tfrac{1}{2}\right)hf$ where h is Planck's constant and f is the frequency of the state.

(*a*) The energy of the vibrational state is given by:

$$E = \left(n + \tfrac{1}{2}\right)hf$$

or, because we expect n to be very large,

$$E \approx nhf \Rightarrow n = \frac{E}{hf}$$

Substitute numerical values and evaluate n:

$$n = \frac{10\ \text{J}}{\left(6.63 \times 10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.5\ \text{s}^{-1}\right)} = \boxed{6.0 \times 10^{33}}$$

(*b*) The ground state energy of the oscillator is the energy of the system when $n = 0$:

$$E_0 = \tfrac{1}{2}hf$$

Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}\left(6.63 \times 10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.5\ \text{s}^{-1}\right) = \boxed{8.3 \times 10^{-34}\ \text{J}}$$

81 •• Repeat Problem 80, but consider instead an atom in a solid vibrating at a frequency of 1.00×10^{14} oscillations per second and having a total energy of 2.7 eV.

Picture the Problem The energy number n of a state whose energy is E is given by $E = \left(n + \tfrac{1}{2}\right)hf$ where h is Planck's constant and f is the frequency of the state.

(*a*) The energy of the vibrational state is given by:

$$E = \left(n + \tfrac{1}{2}\right)hf \Rightarrow n = \frac{E}{hf} - \frac{1}{2}$$

Substitute numerical values and evaluate n:

$$n = \frac{2.7\ \text{eV} \times \dfrac{1.602 \times 10^{-19}\ \text{J}}{\text{eV}}}{\left(6.63 \times 10^{-34}\ \text{J}\cdot\text{s}\right)\left(10^{14}\ \text{s}^{-1}\right)} - \frac{1}{2} = \boxed{6}$$

(*b*) The ground state energy of the oscillator is the energy of the system when $n = 0$:

$$E_0 = \tfrac{1}{2}hf$$

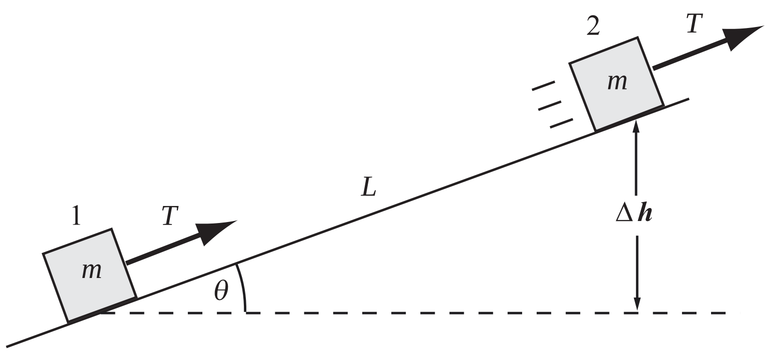
Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}\left(6.63\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(10^{14}\ \text{s}^{-1}\right)$$
$$= 3.315\times10^{-20}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}}$$
$$= \boxed{0.21\,\text{eV}}$$

General Problems

82 • A block of mass *m*, starting from rest, is pulled up a frictionless inclined plane that makes an angle θ with the horizontal by a string parallel to the plane. The tension in the string is *T*. After traveling a distance *L*, the speed of the block is v_f. Derive an expression for work done by the tension force.

Picture the Problem Let the system consist of the block, Earth, and the incline. Then the tension in the string is an external force that will do work to change the energy of the system. Because the incline is frictionless; the work done by the tension in the string as it displaces the block on the incline is equal to the sum of the changes in the kinetic and gravitational potential energies.



Relate the work done on the block by the tension force to the changes in the kinetic and gravitational potential energies of the block:

$$W_{\substack{\text{tension}\\\text{force}}} = W_{\text{ext}} = \Delta U + \Delta K \qquad (1)$$

Referring to the figure, express the change in the potential energy of the block as it moves from position 1 to position 2:

$$\Delta U = mg\Delta h = mgL\sin\theta$$

Because the block starts from rest:

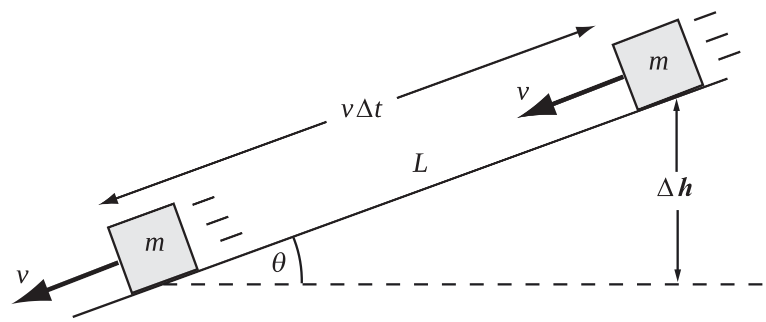
$$\Delta K = K_2 = \tfrac{1}{2}mv_f^2$$

Substitute for ΔU and ΔK in equation (1) to obtain:

$$W_{\substack{\text{tension}\\\text{force}}} = \boxed{mgL\sin\theta + \tfrac{1}{2}mv_f^2}$$

83 • A block of mass *m* slides with constant speed *v* down a plane inclined at angle θ with the horizontal. Derive an expression for the energy dissipated by friction during the time interval Δt.

Picture the Problem Let the system include Earth, the block, and the inclined plane. Then there are no external forces to do work on the system and $W_{\text{ext}} = 0$. Apply the work-energy theorem with friction to find an expression for the energy dissipated by friction.



Apply the work-energy theorem with friction to the block:

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{therm}} = 0$$

Because the velocity of the block is constant, $\Delta K = 0$ and:

$$\Delta E_{\text{therm}} = -\Delta U = -mg\Delta h$$

In time Δt the block slides a distance $v\Delta t$. From the figure:

$$\Delta h = -v\Delta t \sin\theta$$

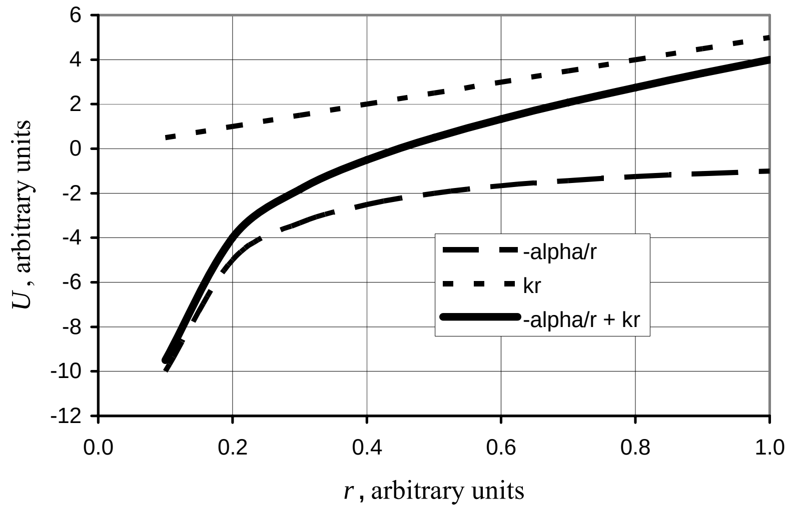
Substitute for Δh to obtain:

$$\Delta E_{\text{therm}} = \boxed{mgv\Delta t \sin\theta}$$

84 • In particle physics, the potential energy associated with a pair of quarks bound together by the strong nuclear force is in one particular theoretical model written as the following function: $U(r) = -(\alpha/r) + kr$, where k and α are positive constants, and r is the distance of separation between the two quarks. (*a*) Sketch the general shape of the potential-energy function. (*b*) What is a general form for the force each quark exerts on the other? (*c*) At the two extremes of very small and very large values of *r*, what does the force simplify to?

Picture the Problem The force between the two quarks is given by $F = -\dfrac{dU(r)}{dr}$.

(*a*) The following graph was plotted using a spreadsheet program. α was set equal to 1 and k was set equal to 5.



(*b*) F is given by:

$$F = -\frac{dU(r)}{dr}$$

Substitute for $U(r)$ and evaluate F to obtain:

$$F = -\frac{d}{dr}\left(-\frac{\alpha}{r} + kr\right) = \boxed{-\frac{\alpha}{r^2} + k}$$

(*c*) For $r >> 1$:

$$F_{r>>1} \to \boxed{k}$$

For $r << 1$:

$$F_{r<<1} \to \boxed{-\frac{\alpha}{r^2}}$$

85 • **[SSM]** You are in charge of ″solar-energizing″ your grandfather's farm. At the farm's location, an average of 1.0 kW/m^2 reaches the surface during the daylight hours on a clear day. If this could be converted at 25% efficiency to electric energy, how large a collection area would you need to run a 4.0-hp irrigation water pump during the daylight hours?

Picture the Problem The solar constant is the average energy per unit area and per unit time reaching the upper atmosphere. This physical quantity can be thought of as the power per unit area and is known as *intensity*.

Letting I_{surface} represent the intensity of the solar radiation at the surface of Earth, express I_{surface} as a function of power and the area on which this energy is incident:

$$\varepsilon I_{\text{surface}} = \frac{P}{A} \Rightarrow A = \frac{P}{\varepsilon I_{\text{surface}}}$$

where ε is the efficiency of conversion to electric energy.

Substitute numerical values and evaluate A:

$$A = \frac{4.0\ \text{hp} \times \dfrac{746\ \text{W}}{\text{hp}}}{(0.25)\left(1.0\ \text{kW/m}^2\right)} = \boxed{12\ \text{m}^2}$$

86 •• The radiant energy from the Sun that reaches Earth's orbit is $1.35\ \text{kW/m}^2$. (*a*) Even when the Sun is directly overhead and under dry desert conditions, 25% of this energy is absorbed and/or reflected by the atmosphere before it reaches Earth's surface. If the average frequency of the electromagnetic radiation from the Sun is $5.5 \times 10^{14}\ \text{Hz}$, how many photons per second would be incident upon a 1.0-m^2 solar panel? (*b*) Suppose the efficiency of the panels for converting the radiant energy to electrical energy and delivering it is a highly efficient 10.0%. How large a solar panel is needed to supply the needs of a 5.0-hp solar-powered car (assuming the car runs directly off the solar panel and not batteries) during a race in Cairo at noon on March 21? (*c*) Assuming a more-realistic efficiency of 3.3% and panels capable of rotating to be always perpendicular to the sunlight, how large an array of solar panels is needed to supply the power needs of the International Space Station (ISS)? The ISS requires about 110 kW of continuous electric power.

Picture the Problem The number of photons n incident on a solar panel is related to the energy E of the incident radiation ($E = nhf$) and the intensity of the solar radiation is the rate at which it delivers energy per unit area.

(*a*) The number of photons n incident on the solar panel is related to the energy E of the radiation:

$$E = nhf \Rightarrow n = \frac{E}{hf} \qquad (1)$$

The intensity I of the radiation is given by:

$$I = \frac{P}{A} = \frac{E}{A\Delta t} \Rightarrow E = IA\Delta t$$

Substituting for E in equation (1) yields:

$$n = \frac{IA\Delta t}{hf}$$

The number of photons arriving per unit time is given by:

$$\frac{n}{\Delta t} = \frac{IA}{hf} \text{ or } \frac{n}{\Delta t} = \frac{\varepsilon I'A}{hf}$$

where I' is the unreduced solar constant and ε is the percentage of the energy absorbed.

Substitute numerical values and evaluate the number of photons arriving per unit time:

$$\frac{n}{\Delta t} = \frac{(0.75)\left(1.35\ \text{kW/m}^2\right)\left(1.0\ \text{m}^2\right)}{\left(6.63 \times 10^{-34}\ \text{J}\cdot\text{s}\right)\left(5.5 \times 10^{14}\ \text{s}^{-1}\right)} = \boxed{2.8 \times 10^{21}\ \text{s}^{-1}}$$

(*b*) The effective intensity of the radiation is given by:

$$I=\frac{P}{\varepsilon A}\Rightarrow A=\frac{P}{\varepsilon I} \qquad (2)$$

where ε is the efficiency of energy conversion.

Substitute numerical values and evaluate *A*:

$$A=\frac{5.0\,\text{hp}\times\dfrac{746\,\text{W}}{\text{hp}}}{(0.10)(1.35\,\text{kW/m}^2)}=\boxed{28\,\text{m}^2}$$

(*c*) Substitute numerical values in equation (2) to obtain:

$$A=\frac{110\,\text{kW}}{(0.033)(1.35\,\text{kW/m}^2)}=\boxed{2.5\times10^3\,\text{m}^2}$$

87 •• In 1964, after the 1250-kg jet-powered car *Spirit of America* lost its parachute and went out of control during a run at Bonneville Salt Flats, Utah, it left skid marks about 8.00 km long. (This earned a place in the Guinness Book of World Records for longest skid marks.) (*a*) If the car was moving initially at a speed of about 800 km/h, and was still going at about 300 km/h when it crashed into a brine pond, estimate the coefficient of kinetic friction μ_k. (*b*) What was the kinetic energy of the car 60 s after the skid began?

Picture the Problem Let the system include Earth and the *Spirit of America*. Then there are no external forces to do work on the car and $W_{ext}=0$. We can use the work-energy theorem for problems with kinetic friction to relate the coefficient of kinetic friction to the given information. A constant-acceleration equation will yield the car's velocity when 60 s have elapsed.

(*a*) Apply the work-energy theorem with friction to relate the coefficient of kinetic friction μ_k to the initial and final kinetic energies of the car:

$$W_{\text{ext}}=\Delta K+\Delta U+\Delta E_{\text{therm}}=0$$

or

$$\tfrac{1}{2}mv_f^2-\tfrac{1}{2}mv_i^2+\mu_k mg\Delta s=0$$

Solving for μ_k yields:

$$\mu_k=\frac{v_i^2-v_f^2}{2g\Delta s}$$

Substitute numerical values and evaluate μ_k:

$$\mu_k=\frac{\left[\left(800\dfrac{\text{km}}{\text{h}}\right)^2-\left(300\dfrac{\text{km}}{\text{h}}\right)^2\right]\times\left(\dfrac{1\,\text{h}}{3600\,\text{s}}\right)^2}{2(9.81\,\text{m/s}^2)(8.00\,\text{km})}=\boxed{0.270}$$

(*b*) The kinetic energy of the car as a function of its speed is:

$$K = \tfrac{1}{2}mv^2 \quad (1)$$

Using a constant-acceleration equation, relate the speed of the car to its acceleration, initial speed, and the elapsed time:

$$v = v_0 + at \quad (2)$$

Express the braking force acting on the car:

$$F_{\text{net}} = -f_{\text{k}} = -\mu_{\text{k}}mg = ma$$

Solving for *a* yields:

$$a = -\mu_{\text{k}}g$$

Substitute for *a* in equation (2) to obtain:

$$v = v_0 - \mu_{\text{k}}gt$$

Substituting for *v* in equation (1) yields an expression for the kinetic energy of car as a function of the time it has been skidding:

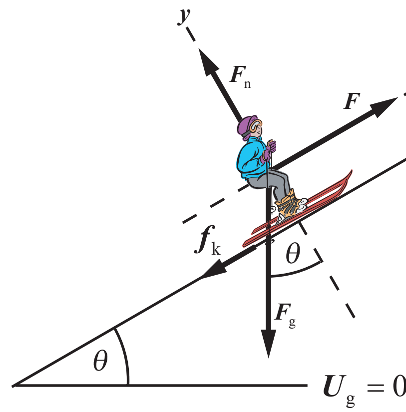
$$K(t) = \tfrac{1}{2}m(v_0 - \mu_{\text{k}}gt)^2$$

Substitute numerical values and evaluate $K(60\text{ s})$:

$$K(60\,\text{s}) = \tfrac{1}{2}(1250\,\text{kg})\left[800\frac{\text{km}}{\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}} - (0.270)(9.81\,\text{m/s}^2)(60\,\text{s})\right]^2 = \boxed{2.5\,\text{MJ}}$$

88 •• A T-bar tow is planned in a new ski area. At any one time, it will be required, to pull a maximum of 80 skiers up a 600-m slope inclined at 15° above the horizontal at a speed of 2.50 m/s. The coefficient of kinetic friction between the skiers skis and the snow is typically 0.060. As the manager of the facility, what motor power should you request of the construction contractor if the mass of the average skier is 75.0 kg. Assume you want to be ready for any emergency and will order a motor whose power rating is 50% larger than the bare minimum.

Picture the Problem The free-body diagram shows the forces acting on a skier as he/she is towed up the slope at constant speed. We can apply the work-energy theorem to find the minimum rate at which the motor will have to supply energy to tow the skiers up an incline whose length is ℓ.



Apply the work-energy theorem to the skiers:

$$W_\text{ext} = \Delta E_\text{mech} + \Delta E_\text{therm} = \Delta K + \Delta U_\text{g} + \Delta E_\text{therm}$$

Because $\Delta K = 0$, $\Delta E_\text{therm} = f_\text{k}\ell$, and $\Delta U_\text{g} = m_\text{tot} g\ell\sin\theta$:

$$W_\text{ext} = m_\text{tot} g\ell\sin\theta + f_\text{k}\ell \qquad (1)$$

The external work done by the electric motor is given by:

$$W_\text{ext} = P_\text{min}\Delta t = P_\text{min}\frac{\ell}{v}$$

where v is the speed with which the skiers are towed up the incline.

The kinetic friction force is given by:

$$f_\text{k} = \mu_\text{k} F_\text{n} = \mu_\text{k} m_\text{tot} g\cos\theta$$

Substituting for W_ext and f_k in equation (1) yields:

$$P_\text{min}\frac{\ell}{v} = m_\text{tot} g\ell\sin\theta + \mu_\text{k} m_\text{tot} g\cos\theta$$

Solve for P_min to obtain:

$$P_\text{min} = m_\text{tot} gv(\sin\theta + \mu_\text{k}\cos\theta)$$

Because you want a safety factor of 50%, the power output of the motor you should order should be 150% of P_min:

$$P = 1.5 m_\text{tot} gv(\sin\theta + \mu_\text{k}\cos\theta)$$

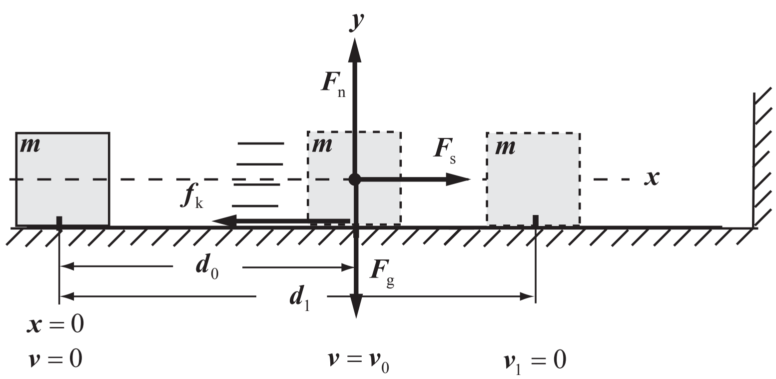
Substitute numerical values and evaluate P:

$$P = (1.5)(80)(75.0\,\text{kg})(9.81\,\text{m/s}^2)(2.50\,\text{m/s})[\sin 15.0° + (0.060)\cos 15.0°]$$
$$= \boxed{70\,\text{kW}}$$

Remarks: We could have solved this problem using Newton's second law.

89 •• A box of mass m on the floor is connected to a horizontal spring of force constant k (Figure 7-52). The coefficient of kinetic friction between the box and the floor is μ_k. The other end of the spring is connected to a wall. The spring is initially unstressed. If the box is pulled away from the wall a distance d_0 and released, the box slides toward the wall. Assume the box does not slide so far that the coils of the spring touch. (*a*) Obtain an expression for the distance d_1 the box slides before it first comes to a stop, (*b*) Assuming $d_1 > d_0$, obtain an expression for the speed of the box when it has slid a distance d_0 following the release. (*c*) Obtain the special value of μ_k such that $d_1 = d_0$.

Picture the Problem Let the system include Earth, the box, and the surface on which the box slides and apply the work-energy theorem for problems with kinetic friction to the box to derive the expressions for distance the box slides and the speed of the box when it first reaches its equilibrium position. The pictorial representation summarizes the salient features of this problem.



(*a*) Apply the work-energy theorem for problems with kinetic friction to the box as it moves from $x = 0$ to $x = d_1$ to obtain:

$$W_{\text{ext}} = \Delta E_{\text{sys}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}}$$

or, because $W_{\text{ext}} = \Delta K = \Delta U_g = 0$, $\Delta E_{\text{therm}} = f_k \Delta x$, and $\Delta E_{\text{mech}} = \Delta U_s$,

$$\Delta U_s + f_k \Delta x = 0$$

Substitute for ΔU_s and Δx to obtain:

$$\tfrac{1}{2}k(d_1 - d_0)^2 - \tfrac{1}{2}kd_0^2 + f_k d_1 = 0 \qquad (1)$$

Apply $\sum F_y = 0$ to the box to obtain: $F_n - F_g = 0 \Rightarrow F_n = F_g = mg$

f_k is given by: $f_k = \mu_k F_n = \mu_k mg$

Substituting for f_k in equation (1) yields: $\frac{1}{2}k(d_1 - d_0)^2 - \frac{1}{2}kd_0^2 + \mu_k mgd_1 = 0$

Solve for d_1 to obtain: $d_1 = \boxed{2d_0 - \frac{2\mu_k mg}{k}}$

(*b*) Apply the work-energy theorem to the box as it moves from $x = 0$ to $x = d_0$ to obtain: $W_{ext} = \Delta E_{sys} = \Delta E_{mech} + \Delta E_{therm}$
or, because $W_{ext} = \Delta U_g = 0$,
$\Delta K + \Delta U_s + \Delta E_{therm} = 0$

Noting that $K_0 = U_{s,f} = 0$, substitute for ΔK, ΔU_s, and ΔE_{therm} to obtain: $\frac{1}{2}mv_0^2 - \frac{1}{2}kd_0^2 + f_k d_0 = 0$

Substituting the expression for f_k obtained in (*a*) yields: $\frac{1}{2}mv_0^2 - \frac{1}{2}kd_0^2 + \mu_k mgd_0 = 0$

Solving for $\boldsymbol{v}_0$ yields: $v_0 = \boxed{\sqrt{\frac{k}{m}d_0^2 - 2\mu_k gd_0}}$

(*c*) Let $d_1 = d_0$ in the expression for d_1 derived in (*a*) to obtain: $d_0 = 2d_0 - \frac{2\mu_k mg}{k} \Rightarrow \mu_k = \boxed{\frac{kd_0}{2mg}}$

Remarks: You can obtain the same Part (*c*) result by setting $\boldsymbol{v}_0 = 0$ in the expression derived in Part (*b*).

90 •• You operate a small grain elevator near Champaign, Illinois. One of your silos uses a bucket elevator that carries a full load of 800 kg through a vertical distance of 40 m. (A bucket elevator works with a continuous belt, like a conveyor belt.) (*a*) What is the power provided by the electric motor powering the bucket elevator when the bucket elevator ascends with a full load at a speed of 2.3 m/s? (*b*) Assuming the motor is 85% efficient, how much does it cost you to run this elevator, per day, assuming it runs 60 percent of the time between 7:00 A.M. and 7:00 P.M. with and average load of 85 percent of a full load? Assume the cost of electric energy in your location is 15 cents per kilowatt hour.

Picture the Problem The power provided by a motor that is delivering sufficient energy to exert a force $\vec{F}$ on a load which it is moving at a speed $\vec{v}$ is $\vec{F} \cdot \vec{v}$.

(*a*) The power provided by the motor is given by:

$P = \vec{F} \cdot \vec{v} = Fv\cos\theta$

or, because $\vec{F}$ and $\vec{v}$ are in the same direction, $P = Fv$ (1)

Because the elevator is ascending with constant speed, the required force is:

$F = m_{\text{load}}g$

Substitute for F in equation (1) to obtain:

$P = m_{\text{load}}gv$

Substitute numerical values and evaluate P:

$$P = (800\,\text{kg})(9.81\,\text{m/s}^2)(2.3\,\text{m/s}) = 18.05\,\text{kW} = \boxed{18\,\text{kW}}$$

(*b*) The daily cost of operating the elevator is given by:

$C_{\text{daily}} = E_{\text{used}}c$ (2)

where c is the per unit cost of the energy.

The energy used by the motor is:

$$E_{\text{used}} = \frac{P_{\text{motor}}}{\varepsilon}\Delta t$$

where ε is the efficiency of the motor and Δt is the number of hours the elevator operates daily.

Substituting for E_{used} in equation (2) yields:

$$C_{\text{daily}} = \frac{P_{\text{motor}}\Delta tc}{\varepsilon}$$

Substitute numerical values and evaluate C_{daily}:

$$C_{\text{daily}} = \frac{(18.05\,\text{kW})(12\,\text{h} \times 0.60)\left(\frac{\$0.15}{\text{kWh}}\right)}{0.85} = \boxed{\$22.93}$$

91 •• To reduce the power requirement of elevator motors, elevators are counterbalanced with weights connected to the elevator by a cable that runs over a pulley at the top of the elevator shaft. Neglect any effects of friction in the pulley. If a 1200-kg elevator that carries a maximum load of 800 kg is counterbalanced with a mass of 1500 kg, (*a*) what is the power provided by the motor when the elevator ascends fully loaded at a speed of 2.3 m/s? (*b*) How

much power is provided by the motor when the elevator ascends at 2.3 m/s without a load?

Picture the Problem The power provided by a motor that is delivering sufficient energy to exert a force $\vec{F}$ on a load which it is moving at a speed $\vec{v}$ is $\vec{F}\cdot\vec{v}$.The counterweight does negative work and the power of the motor is reduced from that required with no counterbalance.

(*a*) The power provided by the motor is given by:

$$P = \vec{F}\cdot\vec{v} = Fv\cos\theta$$

or, because $\vec{F}$ and $\vec{v}$ are in the same direction,

$$P = Fv \qquad (1)$$

Because the elevator is counterbalanced and ascending with constant speed, the tension in the support cable(s) is:

$$F = (m_{\text{elev}} + m_{\text{load}} - m_{\text{cw}})g$$

Substitute for F in equation (1) to obtain:

$$P = (m_{\text{elev}} + m_{\text{load}} - m_{\text{cw}})gv$$

Substitute numerical values and evaluate P:

$$P = (1200\,\text{kg} + 800\,\text{kg} - 1500\,\text{kg})(9.81\,\text{m/s}^2)(2.3\,\text{m/s}) = 11.28\,\text{kW} = \boxed{11\,\text{kW}}$$

(*b*) Without a load:

$$F = (m_{\text{elev}} - m_{\text{cw}})g$$

and

$$P = Fv = (m_{\text{elev}} - m_{\text{cw}})gv$$

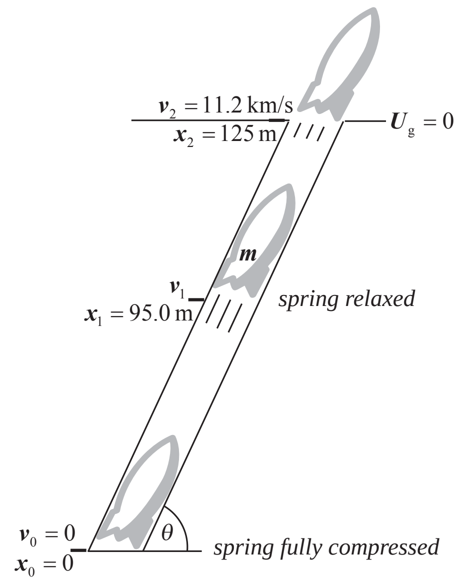
Substitute numerical values and evaluate P:

$$P = (1200\,\text{kg} - 1500\,\text{kg})(9.81\,\text{m/s}^2)(2.3\,\text{m/s}) = -6.77\,\text{kW} = \boxed{-6.8\,\text{kW}}$$

92 •• In old science fiction movies, writers attempted to come up with novel ways of launching spacecraft toward the moon. In one hypothetical case, a screenwriter envisioned launching a moon probe from a deep, smooth tunnel, inclined at 65.0° above the horizontal. At the bottom of the tunnel a very stiff spring designed to launch the craft was anchored. The top of the spring, when the spring is unstressed, is 30.0 m from the upper end of the table. The screenwriter knew from his research that to reach the moon, the 318-kg probe should have a speed of at least 11.2 km/s when it exits the tunnel. If the spring is compressed by

95.0 m just before launch, what is the minimum value for its force constant to achieve a successful launch? Neglect friction with the tunnel walls and floor.

Picture the Problem Let the system consist of Earth, the spring, the tunnel, and the spacecraft and the zero of gravitational potential energy be at the surface of Earth. Then there are no external forces to do work on the system and $W_{ext} = 0$. We can use conservation of mechanical energy to find the minimum value of the force constant that will result in a successful launch. The pictorial representation summarizes the details of the launch. Note that the spacecraft slows somewhat over the last 30 m of its launch.



(*a*) Apply conservation of mechanical energy to the spacecraft as it moves from $x = x_0$ to $x = x_2$ to obtain:

$$W_{ext} = \Delta E_{mech}$$

or, because $W_{ext} = 0$,

$$\Delta E_{mech} = 0 \qquad (1)$$

The change in the mechanical energy of the system is:

$$\begin{aligned}\Delta E_{mech} &= \Delta K + \Delta U_g + \Delta U_s \\ &= K_2 - K_0 + U_{g,2} - U_{g,0} \\ &\quad + U_{s,2} - U_{s,0}\end{aligned}$$

Because $K_0 = U_{g,2} = U_{s,2} = 0$:

$$\Delta E_{mech} = K_2 - U_{g,0} - U_{s,0}$$

Substituting for K_2, $U_{g,0}$, and $U_{s,0}$ yields:

$$\begin{aligned}\Delta E_{mech} &= \tfrac{1}{2}mv_2^2 - (-mgx_2 \sin\theta) - \tfrac{1}{2}kx_1^2 \\ &= \tfrac{1}{2}mv_2^2 + mgx_2 \sin\theta - \tfrac{1}{2}kx_1^2\end{aligned}$$

Substituting for ΔE_{mech} in equation (1) yields:

$$\tfrac{1}{2}mv_2^2 + mgx_2 \sin\theta - \tfrac{1}{2}kx_1^2 = 0$$

Solving for k yields:

$$k = \frac{mv_2^2 + 2mgx_2 \sin\theta}{x_1^2}$$

Substitute numerical values and evaluate k:

$$k = \frac{(318\,\text{kg})(11.2\,\text{km/s})^2 + 2(318\,\text{kg})(9.81\,\text{m/s}^2)(125\,\text{m})\sin 65.0°}{(95.0\,\text{m})^2}$$

$$= \boxed{4.42\times10^3\ \text{kN/m}}$$

93 •• **[SSM]** In a volcanic eruption, a 2-kg piece of porous volcanic rock is thrown straight upward with an initial speed of 40 m/s. It travels upward a distance of 50 m before it begins to fall back to Earth. (*a*) What is the initial kinetic energy of the rock? (*b*) What is the increase in thermal energy due to air resistance during ascent? (*c*) If the increase in thermal energy due to air resistance on the way down is 70% of that on the way up, what is the speed of the rock when it returns to its initial position?

Picture the Problem Let the system consist of Earth, the rock and the air. Given this choice, there are no external forces to do work on the system and $W_{\text{ext}} = 0$. Choose $U_g = 0$ to be where the rock begins its upward motion. The initial kinetic energy of the rock is partially transformed into potential energy and partially dissipated by air resistance as the rock ascends. During its descent, its potential energy is partially transformed into kinetic energy and partially dissipated by air resistance.

(*a*) The initial kinetic energy of the rock is given by:

$$K_i = \tfrac{1}{2}mv_i^2$$

Substitute numerical values and evaluate K_i:

$$K_i = \tfrac{1}{2}(2.0\,\text{kg})(40\,\text{m/s})^2 = \boxed{1.6\,\text{kJ}}$$

(*b*) Apply the work-energy theorem with friction to relate the energies of the system as the rock ascends:

$$\Delta K + \Delta U + \Delta E_{\text{therm}} = 0$$

or, because $K_f = 0$,

$$-K_i + \Delta U + \Delta E_{\text{therm}} = 0$$

Solving for ΔE_{therm} yields:

$$\Delta E_{\text{therm}} = K_i - \Delta U$$

Substitute numerical values and evaluate ΔE_{therm}:

$$\Delta E_{\text{therm}} = 1.6\,\text{kJ} - (2.0\,\text{kg})(9.81\,\text{m/s}^2)(50\,\text{m}) = 0.619\,\text{kJ} = \boxed{0.6\,\text{kJ}}$$

(*c*) Apply the work-energy theorem with friction to relate the energies of the system as the rock descends:

$$\Delta K + \Delta U + 0.70\Delta E_{\text{therm}} = 0$$

Because $K_i = U_f = 0$:

$$K_f - U_i + 0.70\Delta E_{\text{therm}} = 0$$

Substitute for the energies to obtain:

$$\tfrac{1}{2}mv_f^2 - mgh + 0.70\Delta E_{\text{therm}} = 0$$

Solve for v_f to obtain:

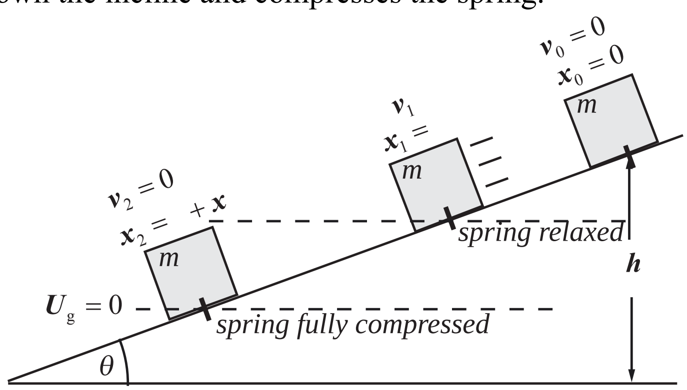
$$v_f = \sqrt{2gh - \frac{1.40\Delta E_{\text{therm}}}{m}}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{2\left(9.81\,\text{m/s}^2\right)\left(50\,\text{m}\right) - \frac{1.40\left(0.619\,\text{kJ}\right)}{2.0\,\text{kg}}} = \boxed{23\,\text{m/s}}$$

94 •• A block of mass *m* starts from rest at a height *h* and slides down a frictionless plane inclined at angle θ with the horizontal, as shown in Figure 7-53. The block strikes a spring of force constant *k*. Find the distance the spring is compressed when the block momentarily stops.

Picture the Problem Let the distance the block slides before striking the spring be ℓ. The pictorial representation shows the block at the top of the incline ($x_0 = 0$), just as it strikes the spring ($x_1 = \ell$), and the block against the fully compressed spring ($x_2 = \ell + x$). Let the block, spring, and Earth comprise the system. Then $W_{\text{ext}} = 0$. Let $U_g = 0$ where the spring is at maximum compression. We can apply the work-energy theorem to the block to relate the energies of the system as the block slides down the incline and compresses the spring.



Apply the work-energy theorem to the block from x_0 to x_2:

$$\Delta K + \Delta U_g + \Delta U_s = 0$$

or

$$\Delta K + U_{g,2} - U_{g,0} + U_{s,2} - U_{s,0} = 0$$

Because $\Delta K = U_{g,2} = U_{s,0} = 0$:

$$-U_{g,0} + U_{s,2} = 0$$

Substitute for each of these energy terms to obtain:

$$-mgh_0 + \tfrac{1}{2}kx^2 = 0 \qquad (1)$$

where x is the distance the spring compresses.

h_0 is given by:

$$h_0 = x_2 \sin\theta = (\ell + x)\sin\theta$$

Substitute for h_0 in equation (1) to obtain:

$$-mg(\ell + x)\sin\theta + \tfrac{1}{2}kx^2 = 0$$

Rewrite this equation explicitly as a quadratic equation to obtain:

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2mg\ell\sin\theta}{k} = 0$$

Solving for x yields:

$$x = \boxed{\frac{mg}{k}\sin\theta + \sqrt{\left(\frac{mg}{k}\right)^2 \sin^2\theta + \frac{2mg\ell}{k}\sin\theta}}$$

Note that the negative sign between the two terms leads to a non-physical solution and has been ignored.

95 •• **[SSM]** A block of mass m is suspended from a wall bracket by a spring and is free to move vertically (Figure 7-54). The $+y$ direction is downward and the origin is at the position of the block when the spring is unstressed. (a) Show that the potential energy as a function of position may be expressed as $U = \tfrac{1}{2}ky^2 - mgy$, ($b$) Using a **spreadsheet** program or graphing calculator, make a graph of U as a function of y with k = 2 N/m and mg = 1 N. (c) Explain how this graph shows that there is a position of stable equilibrium for a positive value of y. Using the Part (a) expression for U, determine (symbolically) the value of y when the block is at its equilibrium position. (d) From the expression for U, find the net force acting on m at any position y. (e) The block is released from rest with the spring unstressed; if there is no friction, what is the maximum value of y that will be reached by the mass? Indicate y_{max} on your graph/spreadsheet.

Picture the Problem Let the system include Earth, the block and the spring. Given the potential energy function as a function of y, we can find the net force acting on the given system using $F = -dU/dy$. The maximum extension of the

spring; that is, the lowest position of the mass on its end, can be found by applying the work-energy theorem. The equilibrium position of the system can be found by applying the work-energy theorem with friction … as can the amount of thermal energy produced as the system oscillates to its equilibrium position. In Part (*c*), setting dU/dy equal to zero and solving the resulting equation for y will yield the value of y when the block is in its equilibrium position

(*a*) The potential energy of the oscillator is the sum of the gravitational potential energy of block and the energy stored in the stretched spring:

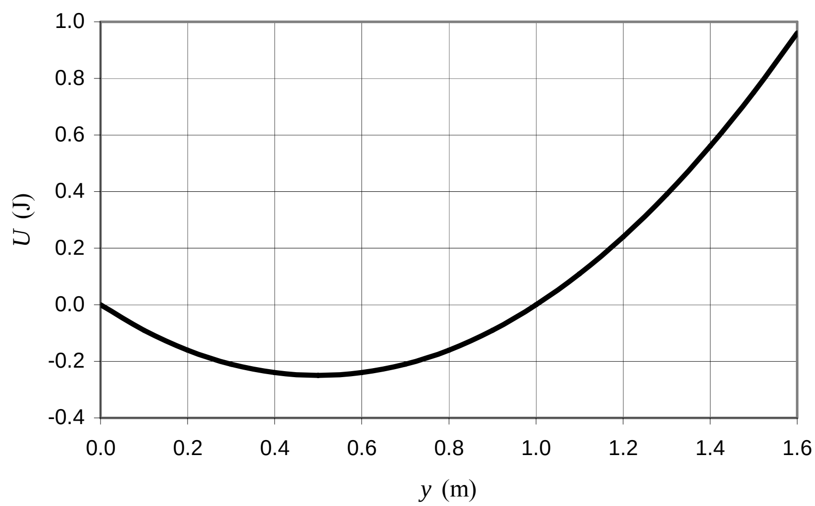
$$U = U_g + U_s$$

Letting the zero of gravitational potential energy be at the oscillator's equilibrium position yields:

$$U = \boxed{\tfrac{1}{2}ky^2 - mgy}$$

where y is the distance the spring is stretched.

(*b*) A graph of U as a function of y follows. Because k and m are not specified, k has been set equal to 2 and mg to 1.



(*c*) The fact that U is a minimum near $y = 0.5$ m tells us that this is a position of stable equilibrium.

Differentiate U with respect to y to obtain:

$$\frac{dU}{dy} = \frac{d}{dy}\left(\tfrac{1}{2}ky^2 - mgy\right) = ky - mg$$

Setting this expression equal to zero for extrema yields:

$$ky - mg = 0 \Rightarrow y = \boxed{\frac{mg}{k}}$$

(*d*) Evaluate the negative of the derivative of *U* with respect to *y*:

$$F = -\frac{dU}{dy} = -\frac{d}{dy}\left(\tfrac{1}{2}ky^2 - mgy\right)$$

$$= \boxed{-ky + mg}$$

(*e*) Apply conservation of energy to the movement of the mass from $y = 0$ to $y = y_{max}$:

$$\Delta K + \Delta U + \Delta E_{therm} = 0$$

Because $\Delta K = 0$ (the object starts from rest and is momentarily at rest at $y = y_{max}$) and (no friction), it follows that:

$$\Delta U = U(y_{max}) - U(0) = 0$$

Because $U(0) = 0$:

$$U(y_{max}) = 0 \Rightarrow \tfrac{1}{2}ky_{max}^2 - mgy_{max} = 0$$

Solve for y_{max} to obtain:

$$y_{max} = \boxed{\frac{2mg}{k}}$$

On the graph, y_{max} is at (1.0, 0.0).

96 •• A spring-loaded gun is cocked by compressing a short, strong spring by a distance *d*. It fires a signal flare of mass *m* directly upward. The flare has speed v_0 as it leaves the spring and is observed to rise to a maximum height *h* above the point where it leaves the spring. After it leaves the spring, effects of drag force by the air on the flare are significant. (Express answers in terms of *m*, v_0, *d*, *h*, and *g*.) (*a*) How much work is done on the spring during the compression? (*b*) What is the value of the force constant *k*? (*c*) Between the time of firing and the time at which maximum elevation is reached, how much mechanical energy is dissipated into thermal energy?

Picture the Problem The energy stored in the compressed spring is initially transformed into the kinetic energy of the signal flare and then into gravitational potential energy and thermal energy as the flare climbs to its maximum height. Let the system contain Earth, the air, and the flare so that $W_{ext} = 0$. We can use the work-energy theorem with friction in the analysis of the energy transformations during the motion of the flare.

(*a*) The work done on the spring in compressing it is equal to the kinetic energy of the flare at launch:

$$W_s = K_{i,\,flare} = \boxed{\tfrac{1}{2}mv_0^2}$$

(*b*) Ignoring changes in gravitational potential energy (that is, assume that the compression of the spring is small compared to the maximum elevation of the flare), apply the conservation of mechanical energy to the transformation that takes place as the spring decompresses and gives the flare its launch speed:

$$\Delta K + \Delta U_s = 0$$

or

$$K_f - K_i + U_{s,f} - U_{s,i} = 0$$

Because $K_i = U_{s,f} = 0$:

$$K_f - U_{s,i} = 0$$

Substitute for K_f and $U_{s,i}$ to obtain:

$$\tfrac{1}{2}mv_0^2 - \tfrac{1}{2}kd^2 = 0 \Rightarrow k = \boxed{\frac{mv_0^2}{d^2}}$$

(c) Apply the work-energy theorem with friction to the upward trajectory of the flare:

$$\Delta K + \Delta U_g + \Delta E_{therm} = 0$$

Solve for ΔE_{therm}:

$$\Delta E_{therm} = -\Delta K - \Delta U_g$$
$$= K_i - K_f + U_i - U_f$$

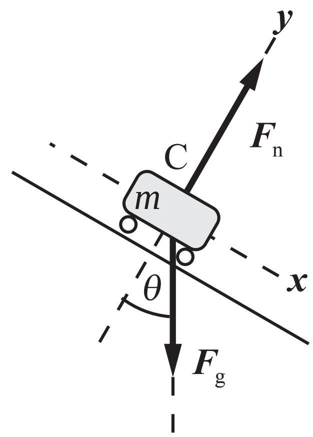
Because $K_f = U_i = 0$:

$$\Delta E_{therm} = \boxed{\tfrac{1}{2}mv_0^2 - mgh}$$

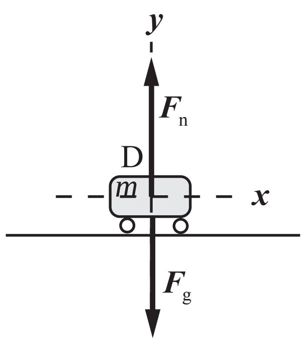
97 •• Your firm is designing a new roller-coaster ride. The permit process requires the calculation of forces and accelerations at various important locations on the ride. Each roller-coaster car will have a total mass (including passengers) of 500 kg and will travel freely along the winding frictionless track shown in Figure 7-55. Points A, E, and G are on horizontal straight sections, all at the same height of 10 m above the ground. Point C is at a height of 10 m above the ground on an inclined section of slope angle 30°. Point B is at the crest of a hill, while point D is at ground level at the bottom of a valley; the radius of curvature at both of these points is 20 m. Point F is at the middle of a banked horizontal curve with a radius of curvature of 30 m, and at the same height as points A, E, and G. At point A the speed of the car is 12 m/s. (*a*) If the car is just barely to make it over the hill at point B, what must be the height of point B above the ground? (*b*) If the car is to just barely make it over the hill at point B, what should be the magnitude of the force exerted by the track on the car at that point? (*c*) What will be the acceleration of the car at point C? (*d*) What will be the magnitude and direction of the force exerted by the track on the car at point D? (*e*) What will be the magnitude and direction of the force exerted by the track on the car at point F? (*f*) At point G a constant braking force is to be applied to the

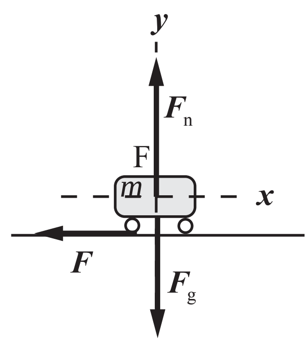
car, bringing it to a halt in a distance of 25 m. What is the magnitude of this required braking force?

Picture the Problem Let $U_D = 0$. Choose the system to include Earth, the track and the car. Then there are no external forces to do work on the system and change its energy and we can use Newton's second law and the work-energy theorem to describe the system's energy transformations to point G … and then the work-energy theorem with friction to determine the braking force that brings the car to a stop. The free-body diagram for point C is shown above.



The free-body diagrams for the rollercoaster cars at points D and F are shown below.





(*a*) Apply the work-energy theorem to the system's energy transformations between A and B:

$$\Delta K + \Delta U = 0$$

or

$$K_B - K_A + U_B - U_A = 0$$

If we assume that the car arrives at point B with $v_B = 0$, then:

$$-\tfrac{1}{2}mv_A^2 + mg\Delta h = 0 \Rightarrow \Delta h = \frac{v_A^2}{2g}$$

where Δh is the difference in elevation between A and B.

The height above the ground is given by:

$$h + \Delta h = h + \frac{v_A^2}{2g}$$

Substitute numerical values and evaluate $h + \Delta h$:

$$h+\Delta h = 10\,\text{m} + \frac{(12\,\text{m/s})^2}{2(9.81\,\text{m/s}^2)} = 17.3\,\text{m}$$
$$= \boxed{17\,\text{m}}$$

(*b*) If the car just makes it to point B; i.e., if it gets there with $v_B = 0$, then the force exerted by the track on the car will be the normal force:

$$F_{\substack{\text{track}\\ \text{on car}}} = F_n = mg$$

Substitute numerical values and evaluate $F_{\substack{\text{track}\\ \text{on car}}}$:

$$F_{\substack{\text{track}\\ \text{on car}}} = (500\,\text{kg})(9.81\,\text{m/s}^2)$$
$$= \boxed{4.91\,\text{kN}}$$

(*c*) Apply $\sum F_x = ma_x$ to the car at point C (see the FBD) and solve for a:

$$mg\sin\theta = ma \Rightarrow a = g\sin\theta$$

Substitute numerical values and evaluate a:

$$a = (9.81\,\text{m/s}^2)\sin 30° = \boxed{4.9\,\text{m/s}^2}$$

(*d*) Apply $\sum F_y = ma_y$ to the car at point D (see the FBD) and solve for F_n:

$$F_n - mg = m\frac{v_D^2}{R} \Rightarrow F_n = mg + m\frac{v_D^2}{\text{R}}$$

Apply the work-energy theorem to the system's energy transformations between B and D:

$$\Delta K + \Delta U = 0$$
or
$$K_D - K_B + U_D - U_B = 0$$

Because $K_B = U_D = 0$:

$$K_D - U_B = 0$$

Substitute to obtain:

$$\tfrac{1}{2}mv_D^2 - mg(h+\Delta h) = 0$$

Solving for v_D^2 yields:

$$v_D^2 = 2g(h+\Delta h)$$

Substitute for v_D^2 in the expression for F_n and simplify to obtain:

$$F_n = mg + m\frac{v_D^2}{R} = mg + m\frac{2g(h+\Delta h)}{R}$$
$$= mg\left[1 + \frac{2(h+\Delta h)}{R}\right]$$

Substitute numerical values and evaluate F_n:

$$F_n = (500\,\text{kg})(9.81\,\text{m/s}^2)\left[1 + \frac{2(17.3\,\text{m})}{20\,\text{m}}\right] = \boxed{13\,\text{kN, directed upward.}}$$

(*e*) $\vec{F}$ has two components at point F; one horizontal (the inward force that the track exerts) and the other vertical (the normal force). Apply $\sum \vec{F} = m\vec{a}$ to the car at point F:

$$\sum F_y = F_n - mg = 0 \Rightarrow F_n = mg$$

and

$$\sum F_x = F_c = m\frac{v_F^2}{R}$$

Express the resultant of these two forces:

$$F = \sqrt{F_c^2 + F_n^2} = \sqrt{\left(m\frac{v_F^2}{R}\right)^2 + (mg)^2} = m\sqrt{\frac{v_F^4}{R^2} + g^2}$$

Substitute numerical values and evaluate F:

$$F = (500\,\text{kg})\sqrt{\frac{(12\,\text{m/s})^4}{(30\,\text{m})^2} + (9.81\,\text{m/s}^2)^2} = \boxed{5.5\,\text{kN}}$$

The angle the resultant makes with the x axis is given by:

$$\theta = \tan^{-1}\left(\frac{F_n}{F_c}\right) = \tan^{-1}\left(\frac{gR}{v_F^2}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{(9.81\,\text{m/s}^2)(30\,\text{m})}{(12\,\text{m/s})^2}\right] = 63.9° = \boxed{64°}$$

(*f*) Apply the work-energy theorem with friction to the system's energy transformations between F and the car's stopping position:

$$-K_G + \Delta E_{\text{therm}} = 0$$

and

$$\Delta E_{\text{therm}} = K_G = \tfrac{1}{2}mv_G^2$$

The work done by friction is also given by:

$$\Delta E_{\text{therm}} = f\Delta s = F_{\text{brake}}d$$

where d is the stopping distance.

Equate the two expressions for ΔE_{therm} and solve for F_{brake}:

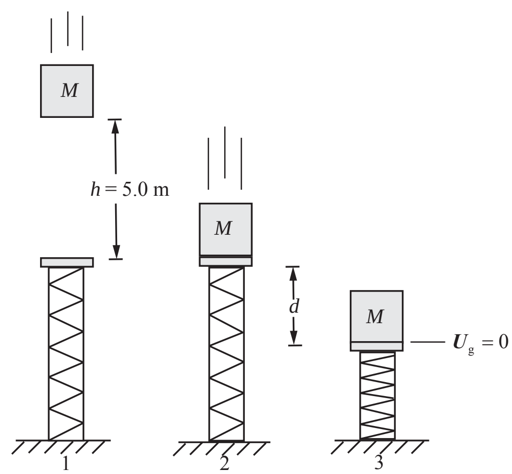
$$F_{\text{brake}} = \frac{mv_F^2}{2d}$$

Substitute numerical values and evaluate F_{brake}:

$$F_{\text{brake}} = \frac{(500\,\text{kg})(12\,\text{m/s})^2}{2(25\,\text{m})} = \boxed{1.4\,\text{kN}}$$

98 •• The cable of a 2000-kg elevator has broken, and the elevator is moving downward at a steady speed of 1.5 m/s. A safety braking system that works on friction prevents the downward speed from increasing. (*a*) At what rate is the braking system converting mechanical energy to thermal energy? (*b*) While the elevator is moving downward at 1.5 m/s, the braking system fails and the elevator is in free-fall for a distance of 5.0 m before hitting the top of a large safety spring with force constant of 1.5×10^4 N/m. After the elevator hits the top of the spring, find the distance *d* that the spring is compressed before the elevator is brought to rest.

Picture the Problem The rate of conversion of mechanical energy can be determined from $P = \vec{F} \cdot \vec{v}$. The pictorial representation shows the elevator moving downward just as it goes into freefall as state 1. In state 2 the elevator is moving faster and is about to strike the relaxed spring. The momentarily at rest elevator on the compressed spring is shown as state 3. Let $U_g = 0$ where the spring has its maximum compression and the system consist of Earth, the elevator and the spring. Then $W_{\text{ext}} = 0$ and we can apply conservation of mechanical energy to the analysis of the falling elevator and compressing spring.



(*a*) Express the rate of conversion of mechanical energy to thermal energy as a function of the speed of the elevator and braking force acting on it:

$$P = F_{\text{braking}} v_0$$

Because the elevator is moving with constant speed, the net force acting on it is zero and:

$$F_{\text{braking}} = Mg$$

Substitute for F_{braking} to obtain:

$$P = Mgv_0$$

Substitute numerical values and evaluate P:

$$P = (2000\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m/s}) = \boxed{29\,\text{kW}}$$

(b) Apply the conservation of mechanical energy to the falling elevator and compressing spring:

$$\Delta K + \Delta U_{\text{g}} + \Delta U_{\text{s}} = 0$$

or

$$K_3 - K_1 + U_{\text{g},3} - U_{\text{g},1} + U_{\text{s},3} - U_{\text{s},1} = 0$$

Because $K_3 = U_{\text{g},3} = U_{\text{s},1} = 0$:

$$-\tfrac{1}{2}Mv_0^2 - Mg(h+d) + \tfrac{1}{2}kd^2 = 0$$

Rewrite this equation as a quadratic equation in d, the maximum compression of the spring:

$$d^2 - \left(\frac{2Mg}{k}\right)d - \frac{M}{k}\left(2gh + v_0^2\right) = 0$$

Solve for d to obtain:

$$d = \frac{Mg}{k} \pm \sqrt{\frac{M^2g^2}{k^2} + \frac{M}{k}\left(2gh + v_0^2\right)}$$

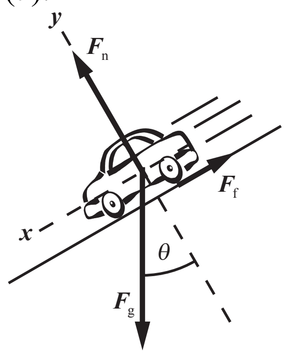
Substitute numerical values and evaluate d:

$$d = \frac{(2000\,\text{kg})(9.81\,\text{m/s}^2)}{1.5\times10^4\ \text{N/m}} + \sqrt{\frac{(2000\,\text{kg})^2(9.81\,\text{m/s}^2)^2}{(1.5\times10^4\ \text{N/m})^2} + \frac{2000\,\text{kg}}{1.5\times10^4\ \text{N/m}}\left[2(9.81\,\text{m/s}^2)(5.0\,\text{m}) + (1.5\,\text{m/s})^2\right]} = \boxed{5.2\,\text{m}}$$

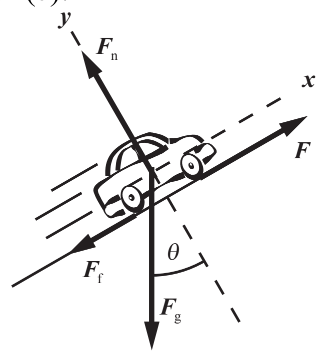
99 ••• [SSM] To measure the combined force of friction (rolling friction plus air drag) on a moving car, an automotive engineering team you are on turns off the engine and allows the car to coast down hills of known steepness. The team collects the following data: (1) On a 2.87° hill, the car can coast at a steady 20 m/s. (2) On a 5.74° hill, the steady coasting speed is 30 m/s. The total mass of the car is 1000 kg. (a) What is the magnitude of the combined force of friction at 20 m/s (F_{20}) and at 30 m/s (F_{30})? (b) How much power must the engine deliver to drive the car on a level road at steady speeds of 20 m/s (P_{20}) and 30 m/s (P_{30})? (c) The maximum power the engine can deliver is 40 kW. What is the angle of the steepest incline up which the car can maintain a steady 20 m/s? (d) Assume that the engine delivers the same total useful work from each liter of gas, no matter what the speed. At 20 m/s on a level road, the car gets 12.7 km/L. How many kilometers per liter does it get if it goes 30 m/s instead?

Picture the Problem We can use Newton's second law to determine the force of friction as a function of the angle of the hill for a given constant speed. The power output of the engine is given by $P = \vec{F}_f \cdot \vec{v}$.

FBD for (*a*):



FBD for (*c*):



(*a*) Apply $\sum F_x = ma_x$ to the car:

$$mg\sin\theta - F = 0 \Rightarrow F = mg\sin\theta$$

Evaluate F for the two speeds:

$$F_{20} = (1000\,\text{kg})(9.81\,\text{m/s}^2)\sin(2.87°) = \boxed{491\,\text{N}}$$

and

$$F_{30} = (1000\,\text{kg})(9.81\,\text{m/s}^2)\sin(5.74°) = \boxed{981\,\text{N}}$$

(*b*) The power an engine must deliver on a level road in order to overcome friction loss is given by:

$$P = F_f v$$

Evaluate this expression for v = 20 m/s and 30 m/s:

$$P_{20} = (491\,\text{N})(20\,\text{m/s}) = \boxed{9.8\,\text{kW}}$$

and

$$P_{30} = (981\,\text{N})(30\,\text{m/s}) = \boxed{29\,\text{kW}}$$

(*c*) Apply $\sum F_x = ma_x$ to the car:

$$\sum F_x = F - mg\sin\theta - F_f = 0$$

Solving for F yields:

$$F = mg\sin\theta + F_f$$

Relate F to the power output of the engine and the speed of the car:

$$F = \frac{P}{v}$$

Equate these expressions for F to obtain:

$$\frac{P}{v} = mg\sin\theta + F_{\text{f}}$$

Solving for θ yields:

$$\theta = \sin^{-1}\left[\frac{\frac{P}{v} - F_{\text{f}}}{mg}\right]$$

Substitute numerical values and evaluate θ for $F_{\text{f}} = F_{20}$:

$$\theta = \sin^{-1}\left[\frac{\frac{40\,\text{kW}}{20\,\text{m/s}} - 491\,\text{N}}{(1000\,\text{kg})(9.81\,\text{m/s}^2)}\right] = \boxed{8.8°}$$

(*d*) Express the equivalence of the work done by the engine in driving the car at the two speeds:

$$W_{\text{engine}} = F_{20}(\Delta s)_{20} = F_{30}(\Delta s)_{30}$$

Let ΔV represent the volume of fuel consumed by the engine driving the car on a level road and divide both sides of the work equation by ΔV to obtain:

$$F_{20}\frac{(\Delta s)_{20}}{\Delta V} = F_{30}\frac{(\Delta s)_{30}}{\Delta V}$$

Solve for $\frac{(\Delta s)_{30}}{\Delta V}$:

$$\frac{(\Delta s)_{30}}{\Delta V} = \frac{F_{20}}{F_{30}}\frac{(\Delta s)_{20}}{\Delta V}$$

Substitute numerical values and evaluate $\frac{(\Delta s)_{30}}{\Delta V}$:

$$\frac{(\Delta s)_{30}}{\Delta V} = \frac{491\,\text{N}}{981\,\text{N}}(12.7\,\text{km/L}) = \boxed{6.36\,\text{km/L}}$$

100 •• (*a*) Calculate the kinetic energy of a 1200-kg car moving at 50 km/h. (*b*) If friction (rolling friction and air drag) results in a retarding force of 300 N at a speed of 50 km/h, what is the minimum energy needed to move the car a distance of 300 m at a constant speed of 50 km/h?

Picture the Problem While on a horizontal surface, the work done by an automobile engine changes the kinetic energy of the car and does work against friction. These energy transformations are described by the work-energy theorem with friction. Let the system include Earth, the roadway, and the car *but not the car's engine*.

(*a*) The kinetic energy of the car is:

$$K = \tfrac{1}{2}(1200\,\text{kg})\left(50\frac{\text{km}}{\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2 = \boxed{0.12\,\text{MJ}}$$

(*b*) The required energy equals the energy dissipated by friction:

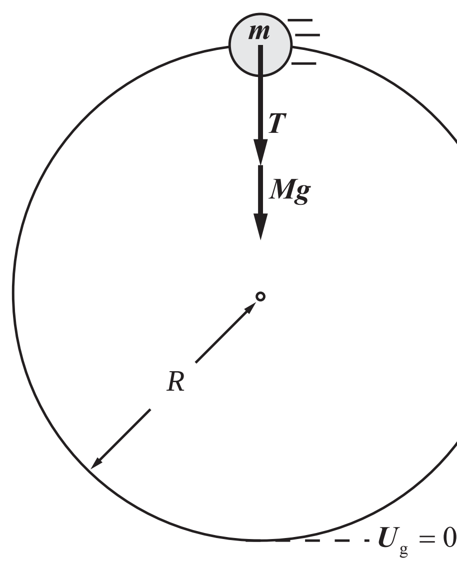
$$\Delta E_{\text{therm}} = f\Delta s$$

Substitute numerical values and evaluate ΔE_{therm}:

$$\Delta E_{\text{therm}} = (300\,\text{N})(300\,\text{m}) = \boxed{90.0\,\text{kJ}}$$

101 ••• A pendulum consists of a string of length *L* with a small bob of mass *m*. The bob is held to the side with the string horizontal (see Figure 7-56). Then the bob is released from rest. At the lowest point of the swing, the string catches on a thin peg a distance *R* above the lowest point. Show that *R* must be smaller than 2*L*/5 if the string is to remain taut as the bob swings around the peg in a full circle.

Picture the Problem Assume that the bob is moving with speed *v* as it passes the top vertical point when looping around the peg. There are two forces acting on the bob: the tension in the string (if any) and the force of gravity, *Mg*; both point downward when the ball is in the topmost position. The minimum possible speed for the bob to pass the vertical occurs when the tension is 0; from this, gravity must supply the centripetal force required to keep the ball moving in a circle. We can use conservation of mechanical energy to relate *v* to *L* and *R*.



Express the condition that the bob swings around the peg in a full circle:

$$M\frac{v^2}{R} > Mg \Rightarrow \frac{v^2}{R} > g \qquad (1)$$

Use conservation of mechanical energy to relate the kinetic energy of the bob at the bottom of the loop to its potential energy at the top of its swing:

$$\tfrac{1}{2}Mv^2 = Mg(L-2R)$$

Solving for v^2 yields:

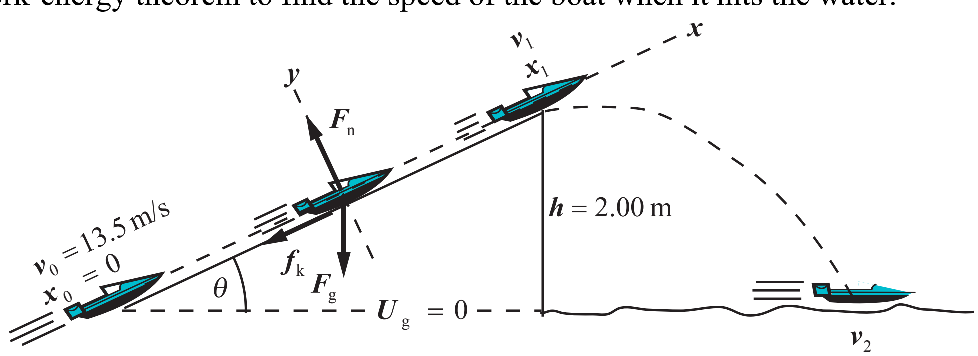
$$v^2 = 2g(L-2R)$$

Substitute for v^2 in equation (1) to obtain:

$$\frac{2g(L-2R)}{R} > g \Rightarrow R < \boxed{\frac{2}{5}L}$$

102 •• A 285-kg stunt boat is driven on the surface of a lake at a constant speed of 13.5 m/s toward a ramp, which is angled at 25.0° above the horizontal. The coefficient of friction between the boat bottom and the ramp's surface is 0.150, and the raised end of the ramp is 2.00 m above the water surface. (*a*) Assuming the engines are cut off when the boat hits the ramp, what is the speed of the boat as it leaves the ramp? (*b*) What is the speed of the boat when it strikes the water again? Neglect any effects due to air resistance.

Picture the Problem The pictorial representation summarizes the details of the problem. Let the system consist of Earth, the boat, and the ramp. Then no external forces do work on the system. We can use the work-energy theorem for problems with kinetic friction to find the speed of the boat at the top of the ramp and the work-energy theorem to find the speed of the boat when it hits the water.



(*a*) Apply the work-energy theorem to the boat as it slides up the ramp to obtain:

$$W_{ext} = \Delta E_{mech} + \Delta E_{therm}$$

or, because $W_{ext} = 0$,

$$\Delta E_{mech} + \Delta E_{therm} = 0 \qquad (1)$$

ΔE_{mech} is given by:

$$\Delta E_{mech} = \Delta K + \Delta U_g = K_1 - K_0 + U_{g,1} - U_{g,0}$$

or, because $U_{g,0} = 0$,

$$\Delta E_{mech} = K_1 - K_0 + U_{g,1}$$

Substituting for K_1, K_0, and $U_{g,1}$ yields:

$$\Delta E_{\text{mech}} = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgh$$

ΔE_{therm} is given by:

$$\Delta E_{\text{therm}} = f_k x_1 = \mu_k F_n x_1$$

Because $F_n = mg\cos\theta$:

$$\Delta E_{\text{therm}} = \mu_k mgx_1\cos\theta$$

Substituting for ΔE_{mech} and ΔE_{therm} in equation (1) yields:

$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgh + \mu_k mgx_1\cos\theta = 0$$

Referring to the pictorial representation, express x_1 in terms of h to obtain:

$$x_1 = \frac{h}{\sin\theta}$$

Substituting for x_1 yields:

$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 + mgh + \frac{\mu_k mgh\cos\theta}{\sin\theta} = 0$$

Solve for v_1 to obtain:

$$v_1 = \sqrt{v_0^2 - 2gh(1+\mu_k\cot\theta)}$$

Substitute numerical values and evaluate v_1:

$$\begin{aligned} v_1 &= \sqrt{(13.5\text{ m/s})^2 - 2(9.81\text{ m/s}^2)(2.00\text{ m})[1+(0.150)\cot(25.0°)]} \\ &= 11.42\text{ m/s} = \boxed{11.4\text{ m/s}} \end{aligned}$$

(*b*) Apply the work-kinetic energy theorem to the boat while it is airborne:

$$W_{\text{ext}} = \Delta K + \Delta U_g$$

or, because $W_{\text{ext}} = 0$,

$$\Delta K + \Delta U_g = 0$$

Substitute for ΔK and ΔU_g to obtain:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 - mgh = 0$$

Solving for v_2 yields:

$$v_2 = \sqrt{v_1^2 + 2gh}$$

Substitute numerical values and evaluate v_2:

$$v_2 = \sqrt{(11.42\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(2.00\text{ m})} = \boxed{13.0\text{ m/s}}$$

103 •• A standard introductory-physics lab-experiment to examine the conservation of energy and Newton's laws is shown in Figure 7-57. A glider is set up on a linear air track and is attached by a string over a massless-frictionless pulley to a hanging weight. The mass of the glider is *M*, while the mass of the hanging weight is *m*. When the air supply to the air track is turned on, the track

becomes essentially frictionless. You then release the hanging weight and measure the speed of the glider after the weight has fallen a given distance (y). To show that the measured speed is the speed predicted by theory; (a) apply conservation of mechanical energy and calculate the speed as a function of y. To verify this calculation; (b) apply Newton's second and third laws directly by sketching a free-body diagram for each of the two masses and applying Newton's laws to find their accelerations. Then use kinematics to calculate the speed of the glider as a function of y.

Picture the Problem For Part (a), we'll let the system include the glider, track, weight, and Earth. The speeds of the glider and the falling weight will be the same while they are in motion. Let their common speed when they have moved a distance Y be v and let the zero of potential energy be at the elevation of the weight when it has fallen the distance Y. We can use conservation of mechanical energy to relate the speed of the glider (and the weight) to the distance the weight has fallen. In Part (b), we'll let the direction of motion be the x direction, the tension in the connecting string be T, and apply Newton's second law to the glider and the weight to find their common acceleration. Because this acceleration is constant, we can use a constant-acceleration equation to find their common speed when they have moved a distance Y.

(a) Apply the work-energy theorem to the system to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U = 0$$

or, because $W_{\text{ext}} = 0$,

$$K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} = 0$$

Because the system starts from rest and $U_{\text{f}} = 0$:

$$K_{\text{f}} - U_{\text{i}} = 0$$

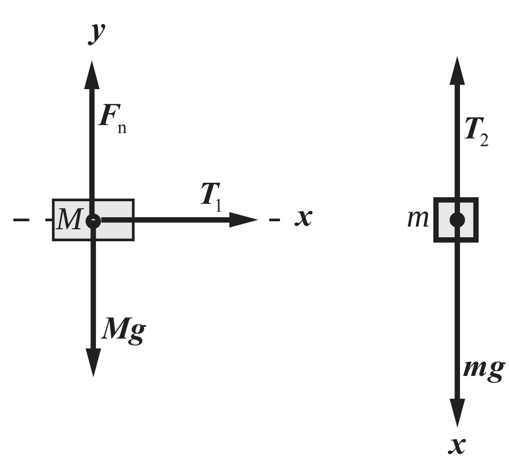
Substitute for K_{f} and U_{i} to obtain:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}Mv^2 - mgY = 0$$

Solving for v yields:

$$v = \boxed{\sqrt{\frac{2mgY}{M+m}}}$$

(b) The free-body diagrams for the glider and the weight are shown to the right:



Apply Newton's 3rd law to obtain: $\left|\vec{T}_1\right| = \left|\vec{T}_2\right| = T$

Apply $\sum F_x = ma$ to the glider: $T = Ma$

Apply $\sum F_x = ma$ to the hanging weight: $mg - T = ma$

Add these equations to eliminate T and obtain: $mg = Ma + ma \Rightarrow a = g\dfrac{m}{m+M}$

Using a constant-acceleration equation, relate the speed of the glider to its initial speed and to the distance that the weight has fallen: $v^2 = v_0^2 + 2aY$
or, because $v_0 = 0$,
$v^2 = 2aY$

Substitute for a and solve for v to obtain: $v = \boxed{\sqrt{\dfrac{2mgY}{M+m}}}$, the same result we obtained in Part (a).

104 •• In one model of a person jogging, the energy expended is assumed to go into accelerating and decelerating the feet and the lower portions of the legs. If the jogging speed is v then the maximum speed of the foot and lower leg is about $2v$. (From the moment a foot leaves the ground, to the moment it next contacts the ground, the foot travels nearly twice as far as the torso, so it must be going, on average, nearly twice as fast as the torso.) If the mass of the foot and lower portion of a leg is m, the energy needed to accelerate the foot and lower portion of a leg from rest to speed $2v$ is $\frac{1}{2}m(2v)^2 = 2mv^2$, and the same energy is needed to decelerate this mass back to rest for the next stride. Assume that the mass of the foot and lower portion of a man's leg is 5.0 kg and that he jogs at a speed of 3.0 m/s with 1.0 m between one footfall and the next. The energy he must provide to each leg in each 2.0 m of travel is $2mv^2$, so the energy he must provide to both legs during each second of jogging is $6mv^2$. Calculate the rate of the man's energy expenditure using this model, assuming that his muscles have an efficiency of 20 percent.

Picture the Problem We're given $P = dW/dt$ and are asked to evaluate it under the assumed conditions.

We're given that the rate of energy expenditure by the man is: $P = 6mv^2$

Substitute numerical values and evaluate P: $P = 6(10\,\text{kg})(3.0\,\text{m/s})^2 = 540\,\text{W}$

Express the rate of energy expenditure P' assuming that his muscles have an efficiency of 20%:

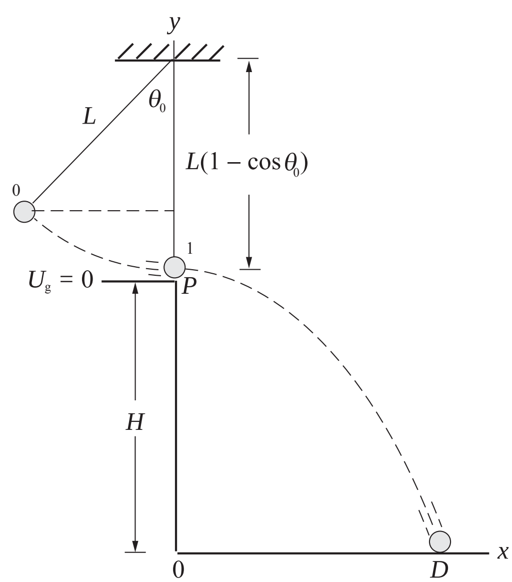
$$P = \tfrac{1}{5}P' \Rightarrow P' = 5P$$

Substitute numerical values and evaluate P':

$$P' = 5(540\ \text{W}) = \boxed{2.7\ \text{kW}}$$

105 •• **[SSM]** A high school teacher once suggested measuring the magnitude of free-fall acceleration by the following method: Hang a mass on a very fine thread (length L) to make a pendulum with the mass a height H above the floor when at its lowest point P. Pull the pendulum back so that the thread makes an angle θ_0 with the vertical. Just above point P, place a razor blade that is positioned to cut through the thread as the mass swings through point P. Once the thread is cut, the mass is projected horizontally, and hits the floor a horizontal distance D from point P. The idea was that the measurement of D as a function of θ_0 should somehow determine g. Apart from some obvious experimental difficulties, the experiment had one fatal flaw: D does not depend on g! Show that this is true, and that D depends only on the angle θ_0.

Picture the Problem The pictorial representation shows the bob swinging through an angle θ_0 before the thread is cut and the ball is launched horizontally. Let its speed at position 1 be v_1. We can use conservation of mechanical energy to relate v_1 to the change in the potential energy of the bob as it swings through the angle θ_0. We can find its flight time Δt from a constant-acceleration equation and then express D as the product of v_1 and Δt.



Relate the distance D traveled horizontally by the bob to its launch speed v_1 and time of flight Δt:

$$D = v_1\Delta t \qquad (1)$$

Use conservation of mechanical energy to relate its energies at positions 0 and 1:

$$K_1 - K_0 + U_1 - U_0 = 0$$

or, because $U_1 = K_0 = 0$,

$$K_1 - U_0 = 0$$

Substitute for K_1 and U_0 to obtain:

$$\tfrac{1}{2}mv_1^2 - mgL(1-\cos\theta_0) = 0$$

Solving for v_1 yields:

$$v_1 = \sqrt{2gL(1-\cos\theta_0)}$$

In the absence of air resistance, the horizontal and vertical motions of the bob are independent of each other and we can use a constant-acceleration equation to express the time of flight (the time to fall a distance H):

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $\Delta y = -H$, $a_y = -g$, and $v_{0y} = 0$,

$$-H = -\tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2H}{g}}$$

Substitute in equation (1) and simplify to obtain:

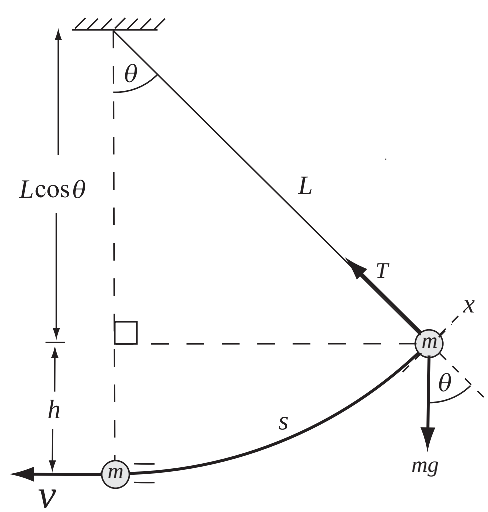
$$D = \sqrt{2gL(1-\cos\theta_0)}\sqrt{\frac{2H}{g}}$$

$$= \boxed{2\sqrt{HL(1-\cos\theta_0)}}$$

which shows that, while D depends on θ_0, it is independent of g.

106 ••• The bob of a pendulum of length L is pulled aside so that the string makes an angle θ_0 with the vertical, and the bob is then released. In Example 7-5, the conservation of energy was used to obtain the speed of the bob at the bottom of its swing. In this problem, you are to obtain the same result using Newton's second law. (*a*) Show that the tangential component of Newton's second law gives $dv/dt = -g \sin\theta$, where v is the speed and θ is the angle between the string and the vertical. (*b*) Show that v can be written $v = L\, d\theta/dt$. (*c*) Use this result and the chain rule for derivatives to obtain $\frac{dv}{dt} = \frac{dv}{d\theta}\frac{v}{L}$. (*d*) Combine the results of Parts (*a*) and (*c*) to obtain $v\, dv = -gL \sin\theta\, d\theta$. (*e*) Integrate the left side of the equation in Part (*d*) from $v = 0$ to the final speed v and the right side from $\theta = \theta_0$ to $\theta = 0$, and show that the result is equivalent to $v = \sqrt{2gh}$, where h is the original height of the bob above the bottom of its swing.

Picture the Problem The free-body diagram shows the forces acting on the pendulum bob. The application of Newton's second law leads directly to the required expression for the tangential acceleration. Recall that, provided θ is in radian measure, $s = L\theta$. Differentiation with respect to time produces the result called for in (*b*). The remaining parts of the problem simply require following



the directions for each part.

(a) Apply $\sum F_x = ma_x$ to the bob:

$$F_{\tan} = -mg\sin\theta = ma_{\tan}$$

Solving for $a_{\tan}$ yields:

$$a_{\tan} = dv/dt = \boxed{-g\sin\theta}$$

(b) Relate the arc distance s to the length of the pendulum L and the angle θ:

$$s = L\theta$$

Differentiate s with respect to time:

$$ds/dt = \boxed{v = Ld\theta/dt}$$

(c) Multiply $\dfrac{dv}{dt}$ by $\dfrac{d\theta}{d\theta}$ and substitute for $\dfrac{d\theta}{dt}$ from Part (b):

$$\frac{dv}{dt} = \frac{dv}{dt}\frac{d\theta}{d\theta} = \frac{dv}{d\theta}\frac{d\theta}{dt} = \boxed{\frac{dv}{d\theta}\left(\frac{v}{L}\right)}$$

(d) Equate the expressions for dv/dt from (a) and (c) to obtain:

$$\frac{dv}{d\theta}\left(\frac{v}{L}\right) = -g\sin\theta$$

Separating the variables yields:

$$vdv = \boxed{-gL\sin\theta\, d\theta}$$

(e) Integrate the left side of the equation in Part (d) from $v = 0$ to the final speed v and the right side from $\theta = \theta_0$ to $\theta = 0$:

$$\int_0^v vdv = \int_{\theta_0}^0 -gL\sin\theta\; d\theta$$

Integrate both sides of the equation to obtain:

$$\tfrac{1}{2}v^2 = gL(1-\cos\theta_0)$$

Note, from the figure, that $h = L(1-\cos\theta_0)$. Substitute to obtain:

$$\tfrac{1}{2}v^2 = gh \Rightarrow v = \boxed{\sqrt{2gh}}$$

107 ••• A rock climber is rappelling down the face of a cliff when his hold slips and he slides down over the rock face, supported only by the bungee cord he attached to the top of the cliff. The cliff face is in the form of a smooth quarter-cylinder with height (and radius) $H = 300$ m (Figure 7-58). Treat the bungee cord as a spring with force constant $k = 5.00$ N/m and unstressed length $L = 60.0$ m. The climber's mass is 85.0 kg. (a) Using a **spreadsheet** program, make a graph of the rock climber's potential energy as a function of s, his distance from the top of

the cliff *measured along the curved surface*. Use values of s between 60.0 m and 200 m. (*b*) His fall began when he was a distance s_i = 60.0 m from the top of the cliff, and ended when he was a distance s_f = 110 m from the top. Determine how much energy is dissipated by friction between the time he initially slipped and the time when he came to a stop.

Picture the Problem The potential energy of the climber is the sum of his gravitational potential energy and the potential energy stored in the spring-like bungee cord. Let θ be the angle which the position of the rock climber on the cliff face makes with a vertical axis and choose the zero of gravitational potential energy to be at the bottom of the cliff. We can use the definitions of U_g and U_{spring} to express the climber's total potential energy and the work-energy theorem for problems with friction to determine how energy is dissipated by friction between the time he initially slipped and finally came to a stop.

(*a*) The total potential energy of the climber is the sum of $U_{\substack{\text{bungee}\\\text{cord}}}$ and U_g:

$$U(s) = U_{\substack{\text{bungee}\\\text{cord}}} + U_g \qquad (1)$$

$U_{\substack{\text{bungee}\\\text{cord}}}$ is given by:

$$U_{\substack{\text{bungee}\\\text{cord}}} = \tfrac{1}{2}k(s-L)^2$$

U_g is given by:

$$U_g = Mgy = MgH\cos\theta = MgH\cos\left(\frac{s}{H}\right)$$

Substitute for $U_{\substack{\text{bungee}\\\text{cord}}}$ and U_g in equation (1) to obtain:

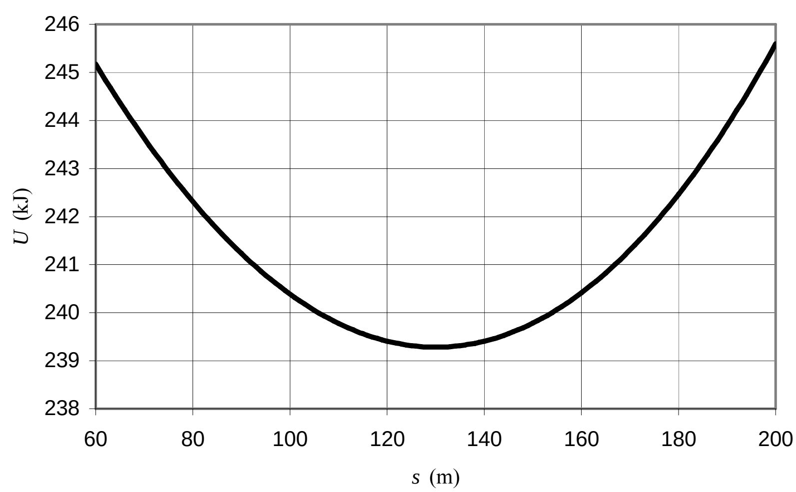
$$U(s) = \tfrac{1}{2}k(s-L)^2 + MgH\cos\left(\frac{s}{H}\right)$$

A spreadsheet solution is shown below. The constants used in the potential energy function and the formulas used to calculate the potential energy are as follows:

Cell	Content/Formula	Algebraic Form
B3	300	H
B4	5.00	k
B5	60.0	L
B6	85.0	M
B7	9.81	g
D11	60.0	s
D12	D11+1	$s+1$
E11	0.5*$B$4*(D11−$B$5)^2 +$B$6*$B$7*$B$3*(cos(D11/$B$3))	$\tfrac{1}{2}k(s-L)^2 + MgH\cos\left(\frac{s}{H}\right)$

	A	B	C	D	E
1					
2					
3	H =	300	m		
4	k =	5.00	N/m		
5	L =	60.0	m		
6	m =	85.0	kg		
7	g =	9.81	m/s^2		
8					
9				s	$U(s)$
10				(m)	(J)
11				60	2.452E+05
12				61	2.450E+05
13				62	2.448E+05
14				63	2.447E+05
15				64	2.445E+05
59				108	2.399E+05
60				109	2.398E+05
61				110	2.398E+05
62				111	2.397E+05
63				112	2.397E+05

The following graph was plotted using the data from columns D (s) and E ($U(s)$).



(b) Apply the work-kinetic energy theorem for problems with friction to the climber to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U(s) + \Delta E_{\text{therm}}$$

or, because $W_{\text{ext}} = \Delta K = 0$,

$$\Delta U(s) + \Delta E_{\text{therm}} = 0$$

Solve for the energy dissipated by friction to obtain:

$$\Delta E_{\text{therm}} = -\Delta U(s) = -\left(U(110\text{ m}) - U(60.0\text{ m})\right) = -U(110\text{ m}) + U(60.0\text{ m})$$

Substituting for U(110 m) and U(60.0 m) and simplifying yields:

$$\begin{aligned}\Delta E_{\text{therm}} &= -\left[\tfrac{1}{2}k(110\text{ m} - L)^2 + MgH\cos\left(\frac{110\text{ m}}{H}\right)\right] \\ &\quad + \left[\tfrac{1}{2}k(60.0\text{ m} - L)^2 + MgH\cos\left(\frac{60.0\text{ m}}{H}\right)\right] \\ &= -\tfrac{1}{2}k(110\text{ m} - L)^2 - MgH\cos\left(\frac{110\text{ m}}{H}\right) + \tfrac{1}{2}k(60.0\text{ m} - L)^2 \\ &\quad + MgH\cos\left(\frac{60.0\text{ m}}{H}\right)\end{aligned}$$

Because L = 60.0 m, the third term is zero. Simplifying yields:

$$\begin{aligned}\Delta E_{\text{therm}} &= -\tfrac{1}{2}k(110\text{ m} - L)^2 - MgH\cos\left(\frac{110\text{ m}}{H}\right) + MgH\cos\left(\frac{60.0\text{ m}}{H}\right) \\ &= -\tfrac{1}{2}k(110\text{ m} - L)^2 - MgH\left[\cos\left(\frac{110\text{ m}}{H}\right) - \cos\left(\frac{60.0\text{ m}}{H}\right)\right]\end{aligned}$$

Substitute numerical values and evaluate ΔE_{therm}:

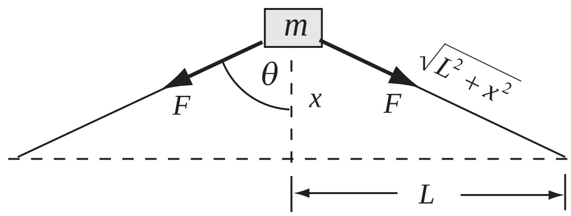
$$\begin{aligned}\Delta E_{\text{therm}} &= -\tfrac{1}{2}(5.00\text{ N/m})(110\text{ m} - 60.0\text{ m})^2 \\ &\quad - (85.0\text{ kg})(9.81\text{ m/s}^2)(300\text{ m})\left[\cos\left(\frac{110\text{ m}}{(300\text{ m})}\right) - \cos\left(\frac{60.0\text{ m}}{(300\text{ m})}\right)\right] \\ &\approx \boxed{5.4\text{ kJ}}\end{aligned}$$

Remarks: You can obtain this same result by examining the partial spreadsheet printout or the graph shown above.

108 ••• A block of wood (mass m) is connected to two massless springs, as shown in Figure 7-59. Each spring has unstressed length L and force constant k. (a) If the block is displaced a distance x, as shown, what is the change in the potential energy stored in the springs? (b) What is the magnitude of the force pulling the block back toward the equilibrium position? (c) Using a **spreadsheet** program or graphing calculator, make a graph of the potential energy U as a function of x for $0 \le x \le 0.20$ m. Assume k = 1.0 N/m, L = 0.10 m, and m = 1.0 kg. (d) If the block is displaced a distance x = 0.10 m and released, what is its speed as

it passes through the equilibrium point? Assume that the block is resting on a frictionless surface.

Picture the Problem The diagram shows the forces the springs exert on the block. Because the block is resting on a horizontal surface and they have no role in the motion of the block, the gravitational force and the normal force are not shown. The change in the potential energy stored in the springs is due to the elongation of both springs when the block is displaced a distance x from its equilibrium position and we can find ΔU using $\frac{1}{2}k(\Delta L)^2$. We can find the magnitude of the force pulling the block back toward its equilibrium position by finding the sum of the magnitudes of the y components of the forces exerted by the springs. In Part (d) we can use conservation of mechanical energy to find the speed of the block as it passes through its equilibrium position.



(a) Express the change in the potential energy stored in the springs when the block is displaced a distance x:

$$\Delta U = 2\left[\tfrac{1}{2}k(\Delta L)^2\right] = k(\Delta L)^2$$

where ΔL is the change in length of either spring.

Use the diagram to express ΔL:

$$\Delta L = \sqrt{L^2 + x^2} - L$$

Substitute for ΔL to obtain:

$$\Delta U = \boxed{k\left(\sqrt{L^2 + x^2} - L\right)^2}$$

(b) Sum the forces acting on the block to express $F_{\text{restoring}}$:

$$F_{\text{restoring}} = 2F\cos\theta = 2k\Delta L\cos\theta$$

$$= 2k\Delta L\frac{x}{\sqrt{L^2 + x^2}}$$

Substitute for ΔL to obtain:

$$F_{\text{restoring}} = 2k\left(\sqrt{L^2 + x^2} - L\right)\frac{x}{\sqrt{L^2 + x^2}}$$

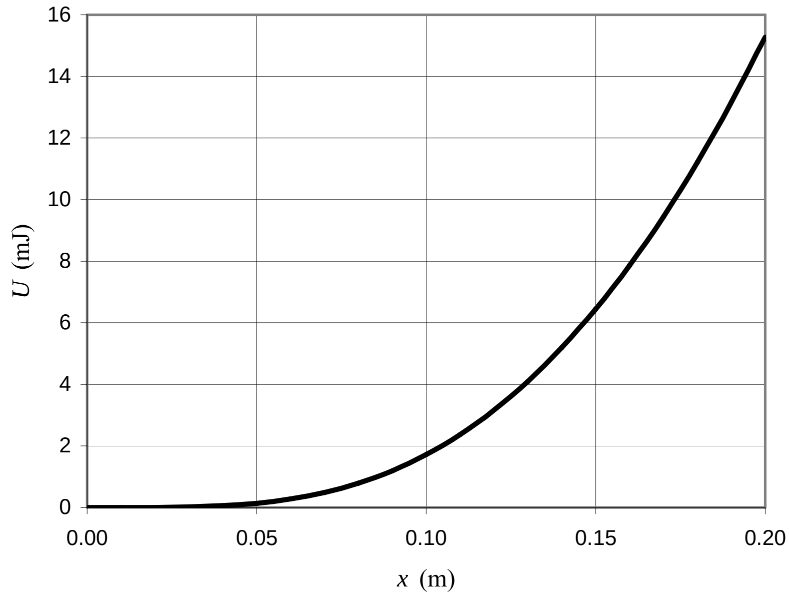
$$= \boxed{2kx\left(1 - \frac{L}{\sqrt{L^2 + x^2}}\right)}$$

(*c*) A spreadsheet program to calculate $U(x)$ is shown below. The constants used in the potential energy function and the formulas used to calculate the potential energy are as follows:

Cell	Content/Formula	Algebraic Form
B1	0.1	L
B2	1.0	k
B3	1.0	M
C8	C7+0.01	x
D7	\$B\$2*((C7^2+\$B\$1^2)^0.5−\$B\$1)^2	$U(x)$

	A	B	C	D
1	L =	0.1	m	
2	k =	1.0	N/m	
3	m =	1.0	kg	
4				
5			x	$U(x)$
6			(m)	(J)
7			0	0
8			0.01	2.49E−07
9			0.02	3.92E−06
10			0.03	1.94E−05
11			0.04	5.93E−05
12			0.05	1.39E−04
23			0.16	7.86E−03
24			0.17	9.45E−03
25			0.18	1.12E−02
26			0.19	1.32E−02
27			0.20	1.53E−02

The following graph was plotted using the data from columns C (x) and D ($U(x)$).



(d) Use conservation of mechanical energy to relate the kinetic energy of the block as it passes through the equilibrium position to the change in its potential energy as it returns to its equilibrium position:

$$K_{\text{equilibrium}} = \Delta U$$

or

$$\tfrac{1}{2}mv^2 = \Delta U \Rightarrow v = \sqrt{\frac{2\Delta U}{m}}$$

Substitute for ΔU and simplify to obtain:

$$v = \sqrt{\frac{2k\left(\sqrt{L^2 + x^2} - L\right)^2}{m}}$$

$$= \left(\sqrt{L^2 + x^2} - L\right)\sqrt{\frac{2k}{m}}$$

Substitute numerical values and evaluate v:

$$v = \left(\sqrt{(0.10\,\text{m})^2 + (0.10\,\text{m})^2} - 0.10\,\text{m}\right)\sqrt{\frac{2(1.0\,\text{N/m})}{1.0\,\text{kg}}} = \boxed{5.9\,\text{cm/s}}$$

Chapter 8
Conservation of Linear Momentum

Conceptual Problems

1 • **[SSM]** Show that if two particles have equal kinetic energies, the magnitudes of their momenta are equal only if they have the same mass.

Determine the Concept The kinetic energy of a particle, as a function of its momentum, is given by $K = p^2/2m$.

The kinetic energy of the particles is given by:

$$K_1 = \frac{p_1^2}{2m_1} \text{ and } K_2 = \frac{p_2^2}{2m_2}$$

Equate these kinetic energies to obtain:

$$\frac{p_1^2}{2m_1} = \frac{p_2^2}{2m_2}$$

Because the magnitudes of their momenta are equal:

$$\frac{1}{m_1} = \frac{1}{m_2} \text{ and } m_1 = \boxed{m_2}$$

2 • Particle A has twice the (magnitude) momentum and four times the kinetic energy of particle B. A also has four times the kinetic energy of B. What is the ratio of their masses (the mass of particle A to that of particle B)? Explain your reasoning.

Determine the Concept The kinetic energy of a particle, as a function of its momentum, is given by $K = p^2/2m$.

The kinetic energy of particle A is given by:

$$K_A = \frac{p_A^2}{2m_A} \Rightarrow m_A = \frac{p_A^2}{2K_A}$$

The kinetic energy of particle B is given by:

$$K_B = \frac{p_B^2}{2m_B} \Rightarrow m_B = \frac{p_B^2}{2K_B}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{m_A}{m_B} = \frac{\dfrac{p_A^2}{2K_A}}{\dfrac{p_B^2}{2K_B}} = \frac{K_B p_A^2}{K_A p_B^2} = \frac{K_B}{K_A}\left(\frac{p_A}{p_B}\right)^2$$

Because particle A has twice the (magnitude) momentum of particle B and four times as much kinetic energy:

$$\frac{m_A}{m_B} = \frac{K_B}{4K_B}\left(\frac{2p_B}{p_B}\right)^2 = \boxed{1}$$

3 • Using SI units, show that the units of momentum squared divided by those of mass is equivalent to the joule.

Determine the Concept The SI units of momentum are kg·m/s.

Express the ratio of the square of the units of momentum to the units of mass:

$$\frac{\left(\text{kg}\cdot\dfrac{\text{m}}{\text{s}}\right)^2}{\text{kg}}$$

Simplify to obtain:

$$\frac{\left(\text{kg}\cdot\dfrac{\text{m}}{\text{s}}\right)^2}{\text{kg}} = \frac{\text{kg}^2\cdot\dfrac{\text{m}^2}{\text{s}^2}}{\text{kg}} = \text{kg}\cdot\frac{\text{m}^2}{\text{s}^2} = \left(\text{kg}\cdot\frac{\text{m}}{\text{s}^2}\right)\cdot\text{m} = \text{N}\cdot\text{m} = \boxed{\text{J}}$$

4 • True or false:

(*a*) The total linear momentum of a system may be conserved even when the mechanical energy of the system is not.
(*b*) For the total linear momentum of a system to be conserved, there must be no external forces acting on the system.
(*c*) The velocity of the center of mass of a system changes only when there is a net external force on the system.

(*a*) True. Consider the collision of two objects of equal mass traveling in opposite directions with the same speed. Assume that they collide inelastically. The mechanical energy of the system is not conserved (it is transformed into other forms of energy), but the momentum of the system is the same after the collision as before the collision; that is, zero. Therefore, for any inelastic collision, the momentum of a system may be conserved even when mechanical energy is not.

(*b*) False. The *net* external force must be zero if the linear momentum of the system is to be conserved.

(*c*) True. This non-zero net force accelerates the center of mass. Hence its velocity changes.

5 • If a bullet is fired due west, explain how conservation of linear momentum enables you to predict that the recoil of the rifle be exactly due east. Is kinetic energy conserved here?

Determine the Concept The momentum of the bullet-gun system is initially zero. After firing, the bullet's momentum is directed west. Momentum conservation requires that the system's total momentum does not change, so the gun's momentum must be directed east. Kinetic energy is not conserved. Some of the initial chemical energy of the gun powder, for example, is transformed into thermal energy and sound energy.

6 • A child jumps from a small boat to a dock. Why does she have to jump with more effort than she would need if she were jumping through an identical displacement, but from a boulder to a tree stump?

Determine the Concept When she jumps from a boat to a dock, she must, in order for momentum to be conserved, give the boat a recoil momentum, that is, her forward momentum must be the same as the boat's backward momentum. When she jumps through an identical displacement from a boulder to a tree stump, the mass of the boulder plus Earth is so large that the momentum she imparts to them is essentially zero.

7 •• **[SSM]** Much early research in rocket motion was done by Robert Goddard, physics professor at Clark College in Worcester, Massachusetts. A quotation from a 1920 editorial in the *New York Times* illustrates the public opinion of his work: "That Professor Goddard with his 'chair' at Clark College and the countenance of the Smithsonian Institution does not know the relation between action and reaction, and the need to have something better than a vacuum against which to react—to say that would be absurd. Of course, he only seems to lack the knowledge ladled out daily in high schools." The belief that a rocket needs something to push against was a prevalent misconception before rockets in space were commonplace. Explain why that belief is wrong.

Determine the Concept In a way, the rocket does need something to push upon. It pushes the exhaust in one direction, and the exhaust pushes it in the opposite direction. However, the rocket does not push against the air. Another way to look at this is conservation of momentum. The momentum of the exhaust is equal and opposite to the momentum of the rocket, so momentum is conserved.

8 • Two identical bowling balls are moving with the same center-of-mass velocity, but one just slides down the alley without rotating, whereas the other rolls down the alley. Which ball has more kinetic energy?

Determine the Concept The kinetic energy of the sliding ball is $\frac{1}{2}mv_{\text{cm}}^2$. The kinetic energy of the rolling ball is $\frac{1}{2}mv_{\text{cm}}^2 + K_{\text{rel}}$, where K_{rel} is its kinetic energy

relative to its center of mass. Because the bowling balls are identical and have the same velocity, the rolling ball has more kinetic energy. There is no problem here because the relationship $K = p^2/2m$ is between the center of mass kinetic energy of the ball and its linear momentum.

9 • A philosopher tells you, ″Changing motion of objects is impossible. Forces always come in equal but opposite pairs. Therefore, all forces cancel out. Because forces cancel, the momenta of objects can never be changed.″ Answer his argument.

Determine the Concept Think of someone pushing a box across a floor. Her push on the box is equal but opposite to the push of the box on her, but the action and reaction forces act on *different objects*. Newton's second law is that the sum of the forces acting on the box equals the rate of change of momentum of the box. This sum does not include the force of the box on her.

10 • A moving object collides with a stationary object. Is it possible for both objects to be at rest immediately after the collision? (Assume any external forces acting on this two-object system are negligibly small.) Is it possible for one object to be at rest immediately after the collision? Explain.

Determine the Concept It's not possible for both to remain at rest after the collision, as that wouldn't satisfy the requirement that momentum is conserved. It is possible for one to remain at rest. This is what happens for a one-dimensional collision of two identical particles colliding elastically.

11 • Several researchers in physics education claim that part of the cause of physical misconceptions amongst students comes from special effects they observe in cartoons and movies. Using the conservation of linear momentum, how would you explain to a class of high school physics students what is conceptually wrong with a superhero hovering at rest in midair while tossing massive objects such as cars at villains? Does this action violate conservation of energy as well? Explain.

Determine the Concept Hovering in midair while tossing objects violates the conservation of linear momentum! To throw something forward requires being pushed backward. Superheroes are not depicted as experiencing this backward motion that is predicted by conservation of linear momentum. This action does not violate the conservation of energy.

12 •• A struggling physics student asks ″If only external forces can cause the center of mass of a system of particles to accelerate, how can a car move? Doesn't the car's engine supply the force needed to accelerate the car? ″ Explain what external agent produces the force that accelerates the car, and explain how the engine makes that agent do so.

Determine the Concept There is only one force which can cause the car to move forward–the friction of the road! The car's engine causes the tires to rotate, but if the road were frictionless (as is closely approximated by icy conditions) the wheels would simply spin without the car moving anywhere. Because of friction, the car's tire pushes backwards against the road and the frictional force acting on the tire pushes it forward. This may seem odd, as we tend to think of friction as being a retarding force only, but it is true.

13 •• When we push on the brake pedal to slow down a car, a brake pad is pressed against the rotor so that the friction of the pad slows the rotation of the rotor and thus the rotation of the wheel. However, the friction of the pad against the rotor can't be the force that slows the car down, because it is an internal force—both the rotor and the wheel are parts of the car, so any forces between them are internal, not external, forces. What external agent exerts the force that slows down the car? Give a detailed explanation of how this force operates.

Determine the Concept The frictional force by the road on the tire causes the car to slow. Normally the wheel is rotating at just the right speed so both the road and the tread in contact with the road are moving backward at the same speed relative to the car. By stepping on the brake pedal, you slow the rotation rate of the wheel. The tread in contact with the road is no longer moving as fast, relative to the car, as the road. To oppose the tendency to skid, the tread exerts a forward frictional force on the road and the road exerts an equal and opposite force on the tread.

14 • Explain why a circus performer falling into a safety net can survive unharmed, while a circus performer falling from the same height onto the hard concrete floor suffers serious injury or death. Base your explanation on the impulse-momentum theorem.

Determine the Concept Because $\Delta p = F\Delta t$ is constant, the safety net reduces the force acting on the performer by increasing the time Δt during which the slowing force acts.

15 •• **[SSM]** In Problem 14, estimate the ratio of the collision time with the safety net to the collision time with the concrete for the performer falling from a height of 25 m. *Hint: Use the procedure outlined in Step 4 of the Problem-Solving Strategy located in Section 8-3.*

Determine the Concept The stopping time for the performer is the ratio of the distance traveled during stopping to the average speed during stopping.

Letting d_{net} be the distance the net gives on impact, d_{concrete} the distance the concrete gives, and $v_{\text{av, with net}}$ and $v_{\text{av,without net}}$ the average speeds during stopping, express the ratio of the impact times:

$$r=\frac{\Delta t_{\text{net}}}{\Delta t_{\text{concrete}}}=\frac{\dfrac{d_{\text{net}}}{v_{\text{av, with net}}}}{\dfrac{d_{\text{concrete}}}{v_{\text{av, without net}}}} \qquad (1)$$

Assuming constant acceleration, the average speed of the performer during stopping is given by:

$$v_{\text{av}}=\frac{v_{\text{f}}+v}{2}$$

or, because $v_{\text{f}}=0$ in both cases,

$$v_{\text{av}}=\tfrac{1}{2}v$$

where v is the impact speed.

Substituting in equation (1) and simplifying yields:

$$r=\frac{\dfrac{d_{\text{net}}}{\frac{1}{2}v}}{\dfrac{d_{\text{concrete}}}{\frac{1}{2}v}}=\frac{d_{\text{net}}}{d_{\text{concrete}}}$$

Assuming that the net gives about 1 m and concrete about 0.1 mm yields:

$$r=\frac{1\,\text{m}}{0.1\,\text{mm}}\approx\boxed{10^{4}}$$

16 •• (*a*) Why does a drinking glass survive a fall onto a carpet but not onto a concrete floor? (*b*) On many race tracks, dangerous curves are surrounded by massive bails of hay. Explain how this setup reduces the chances of car damage and driver injury.

Determine the Concept In both (*a*) and (*b*), longer impulse times (Impulse = $F_{\text{av}}\Delta t$) are the result of collisions with a carpet and bails of hay. The average force on a drinking glass or a car is reduced (nothing can be done about the impulse, or change in linear momentum, during a collision but increasing the impulse time decreases the average force acting on an object) and the likelihood of breakage, damage or injury is reduced.

17 • True or false:

(*a*) Following any perfectly inelastic collision, the kinetic energy of the system is zero after the collision in all inertial reference frames.
(*b*) For a head-on elastic collision, the relative speed of recession equals the relative speed of approach.
(*c*) During a perfectly inelastic head-on collision with one object initially at rest, only some of the system's kinetic energy is dissipated.

(*d*) After a perfectly inelastic head-on collision along the east-west direction, the two objects are observed to be moving west. The initial total system momentum was, therefore, to the west.

(*a*) False. Following a perfectly inelastic collision, the colliding bodies stick together but may or may not continue moving, depending on the momentum each brings to the collision.

(*b*) True. For a head-on elastic collision both kinetic energy and momentum are conserved and the relative speeds of approach and recession are equal.

(*c*) True. This is the definition of an inelastic collision.

(*d*) True. The linear momentum of the system before the collision must be in the same direction as the linear momentum of the system after the collision.

18 •• Under what conditions can all the initial kinetic energy of an isolated system consisting of two colliding objects be lost in a collision? Explain how this result can be, and yet the momentum of the system can be conserved.

Determine the Concept If the collision is perfectly inelastic, the bodies stick together and neither will be moving after the collision. Therefore, the final kinetic energy will be zero and all of it will have been lost (that is, transformed into some other form of energy). Momentum is conserved because in an isolated system the net external force is zero.

19 •• Consider a perfectly inelastic collision of two objects of equal mass. (*a*) Is the loss of kinetic energy greater if the two objects are moving in opposite directions, each moving at speed $v/2$, or if one of the two objects is initially at rest and the other has an initial speed of v? (*b*) In which of these situations is the percentage loss in kinetic energy the greatest?

Determine the Concept We can find the loss of kinetic energy in these two collisions by finding the initial and final kinetic energies. We'll use conservation of momentum to find the final velocities of the two masses in each perfectly elastic collision.

(*a*) Letting V represent the velocity of the masses after their perfectly inelastic collision, use conservation of momentum to determine V:

$$p_{\text{before}} = p_{\text{after}}$$

or

$$mv - mv = 2mV \Rightarrow V = 0$$

Express the loss of kinetic energy for the case in which the two objects have oppositely directed velocities of magnitude $v/2$:

$$\Delta K = K_f - K_i = 0 - 2\left(\tfrac{1}{2}m\left(\frac{v}{2}\right)^2\right)$$
$$= -\tfrac{1}{4}mv^2$$

Letting V represent the velocity of the masses after their perfectly inelastic collision, use conservation of momentum to determine V:

$$p_{\text{before}} = p_{\text{after}}$$
or
$$mv = 2mV \Rightarrow V = \tfrac{1}{2}v$$

Express the loss of kinetic energy for the case in which the one object is initially at rest and the other has an initial velocity v:

$$\Delta K = K_f - K_i$$
$$= \tfrac{1}{2}(2m)\left(\frac{v}{2}\right)^2 - \tfrac{1}{2}mv^2 = -\tfrac{1}{4}mv^2$$

The loss of kinetic energy is the same in both cases.

(b) Express the percentage loss for the case in which the two objects have oppositely directed velocities of magnitude $v/2$:

$$\frac{\Delta K}{K_{\text{before}}} = \frac{\frac{1}{4}mv^2}{\frac{1}{4}mv^2} = 100\%$$

Express the percentage loss for the case in which the one object is initially at rest and the other has an initial velocity v:

$$\frac{\Delta K}{K_{\text{before}}} = \frac{\frac{1}{4}mv^2}{\frac{1}{2}mv^2} = 50\%$$

The percentage loss is greatest for the case in which the two objects have oppositely directed velocities of magnitude $v/2$.

20 •• A double-barreled pea shooter is shown in Figure 8-41. Air is blown into the left end of the pea shooter, and identical peas A and B are positioned inside each straw as shown. If the pea shooter is held horizontally while the peas are shot off, which pea, A or B, will travel farther after leaving the straw? Explain. Base your explanation on the impulse–momentum theorem.

Determine the Concept Pea A will travel farther. Both peas are acted on by the same force, but pea A is acted on by that force for a longer time. By the impulse-momentum theorem, its momentum (and, hence, speed) will be higher than pea B's speed on leaving the shooter.

21 •• A particle of mass m_1 traveling with a speed v makes a head-on elastic collision with a stationary particle of mass m_2. In which scenario will the largest amount of energy be imparted to the particle of mass m_2? (*a*) $m_2 < m_1$, (*b*) $m_2 = m_1$, (*c*) $m_2 > m_1$, (*d*) None of the above.

Determine the Concept Refer to the particles as particle 1 and particle 2. Let the direction particle 1 is moving before the collision be the positive x direction. We'll use both conservation of momentum and conservation of mechanical energy to obtain an expression for the velocity of particle 2 after the collision. Finally, we'll examine the ratio of the final kinetic energy of particle 2 to that of particle 1 to determine the condition under which there is maximum energy transfer from particle 1 to particle 2.

Use conservation of momentum to obtain one relation for the final velocities:

$$m_1 v_{1,\text{i}} = m_1 v_{1,\text{f}} + m_2 v_{2,\text{f}} \qquad (1)$$

Use conservation of mechanical energy to set the velocity of recession equal to the negative of the velocity of approach:

$$v_{2,\text{f}} - v_{1,\text{f}} = -\left(v_{2,\text{i}} - v_{1,\text{i}}\right) = v_{1,\text{i}} \qquad (2)$$

To eliminate $v_{1,\text{f}}$, solve equation (2) for $v_{1,\text{f}}$, and substitute the result in equation (1):

$$v_{1,\text{f}} = v_{2,\text{f}} + v_{1,\text{i}}$$

$$m_1 v_{1,\text{i}} = m_1\left(v_{2,\text{f}} - v_{1,\text{i}}\right) + m_2 v_{2,\text{f}}$$

Solve for $v_{2,\text{f}}$ to obtain:

$$v_{2,\text{f}} = \frac{2m_1}{m_1 + m_2} v_{1,\text{i}}$$

Express the ratio R of $K_{2,\text{f}}$ to $K_{1,\text{i}}$ in terms of m_1 and m_2:

$$R = \frac{K_{2,\text{f}}}{K_{1,\text{i}}} = \frac{\frac{1}{2} m_2 \left(\dfrac{2m_1}{m_1 + m_2}\right)^2 v_{1,\text{i}}^2}{\frac{1}{2} m_1 v_{1,\text{i}}^2}$$

$$= \frac{m_2}{m_1} \frac{4m_1^2}{\left(m_1 + m_2\right)^2}$$

Differentiate this ratio with respect to m_2, set the derivative equal to zero, and obtain the quadratic equation:

$$-\frac{m_2^2}{m_1^2} + 1 = 0$$

Solve this equation for m_2 to determine its value for maximum energy transfer:

$$m_2 = m_1$$

$\boxed{(b)}$ is correct because all of particle 1's kinetic energy is transferred to particle 2 when $m_2 = m_1$.

22 •• Suppose you are in charge of an accident-reconstruction team which has reconstructed an accident in which a car was ″rear-ended″ causing the two cars to lock bumpers and skid to a halt. During the trial, you are on the stand as an expert witness for the prosecution and the defense lawyer claims that you wrongly neglected friction and the force of gravity during the fraction of a second while the cars collided. Defend your report. Why were you correct in ignoring these forces? You did not ignore these two forces in your skid analysis both before and after the collision. Can you explain to the jury why you did not ignore these two forces during the pre- and post-collision skids?

Determine the Concept You only used conservation of linear momentum for a fraction of a second of actual contact between the cars. Over that short time, friction and other external forces can be neglected. In the long run, over the duration of the accident, they cannot.

23 •• Nozzles for a garden hose are often made with a right-angle shape as shown in Figure 8-42. If you open the nozzle and spray water out, you will find that the nozzle presses against your hand with a pretty strong force—much stronger than if you used a nozzle not bent into a right angle. Why is this situation true?

Determine the Concept The water is changing direction when it rounds the corner in the nozzle. Therefore, the nozzle must exert a force on the stream of water to change its momentum, and from Newton's third law, the water exerts an equal but opposite force on the nozzle. This requires a net force in the direction of the momentum change.

Conceptual Problems from Optional Sections

24 •• Describe a perfectly inelastic head-on collision between two stunt cars as viewed in the center-of-mass reference frame.

Determine the Concept In the center-of-mass reference frame the two objects approach with equal but opposite momenta and remain at rest after the collision.

25 •• One air-hockey puck is initially at rest. An identical air-hockey puck collides with it, striking it with a glancing blow. Assume the collision was elastic and neglect any rotational motion of the pucks. Describe the collision in the center-of-mass frame of the pucks.

Determine the Concept In the center-of-mass frame the two velocities are equal and opposite, both before and after the collision. In addition, the speed of each puck is the same before and after the collision. The direction of the velocity of each puck changes by some angle during the collision.

26 •• A baton with one end more massive than the other is tossed at an angle into the air. (*a*) Describe the trajectory of the center of mass of the baton in the reference frame of the ground. (*b*) Describe the motion of the two ends of the baton in the center-of-mass frame of the baton.

Determine the Concept
(*a*) In the center-of-mass frame of the ground, the center of mass moves in a parabolic arc.

(*b*) Relative to the center of mass, each end of the baton would describe a circular path. The more massive end of the baton would travel in the circle with the smaller radius because it is closer to the location of the center of mass.

27 •• Describe the forces acting on a descending Lunar lander as it fires its retrorockets to slow it down for a safe landing. (Assume the lander's mass loss during the rocket firing is not negligible.)

Determine the Concept The forces acting on a descending Lunar lander are the downward force of lunar gravity and the upward thrust provided by the rocket engines.

28 •• A railroad car rolling along by itself is passing by a grain elevator, which is dumping grain into it at a constant rate. (*a*) Does momentum conservation imply that the railroad car should be slowing down as it passes the grain elevator? Assume that the track is frictionless and perfectly level and that the grain is falling vertically. (*b*) If the car is slowing down, this situation implies that there is some external force acting on the car to slow it down. Where does this force come from? (*c*) After passing the elevator, the railroad car springs a leak, and grain starts leaking out of a vertical hole in its floor at a constant rate. Should the car speed up as it loses mass?

Determine the Concept We can apply conservation of linear momentum and Newton's laws of motion to each of these scenarios.

(*a*) Yes, the car should slow down. An easy way of seeing this is to imagine a "packet" of grain being dumped into the car all at once: This is a completely inelastic collision, with the packet having an initial horizontal velocity of 0. After

the collision, it is moving with the same horizontal velocity that the car does, so the car must slow down.

(*b*) When the packet of grain lands in the car, it initially has a horizontal velocity of 0, so it must be accelerated to come to the same speed as the car of the train. Therefore, the train must exert a force on it to accelerate it. By Newton's third law, the grain exerts an equal but opposite force on the car, slowing it down. In general, this is a frictional force which causes the grain to come to the same speed as the car.

(*c*) No it dos not speed up. Imagine a packet of grain being "dumped" out of the railroad car. This can be treated as a collision, too. It has the same horizontal speed as the railroad car when it leaks out, so the train car doesn't have to speed up or slow down to conserve momentum.

29 ••• [SSM] To show that even really intelligent people can make mistakes, consider the following problem which was asked of a freshman class at Caltech on an exam (paraphrased): *A sailboat is sitting in the water on a windless day. In order to make the boat move, a misguided sailor sets up a fan in the back of the boat to blow into the sails to make the boat move forward. Explain why the boat won't move.* The idea was that the net force of the wind pushing the sail forward would be counteracted by the force pushing the fan back (Newton's third law). However, as one of the students pointed out to his professor, the sailboat *could* in fact move forward. Why is that?

Determine the Concept Think of the sail facing the fan (like the sail on a square rigger might), and think of the stream of air molecules hitting the sail. Imagine that they bounce off the sail elastically−their net change in momentum is then roughly twice the change in momentum that they experienced going through the fan. Thus the change in momentum of the air is backward, so to conserve momentum of the air-fan-boat system, the change in momentum of the fan-boat system will be forward.

Estimation and Approximation

30 •• A 2000-kg car traveling at 90 km/h crashes into an immovable concrete wall. (*a*) Estimate the time of collision, assuming that the center of the car travels halfway to the wall with constant acceleration. (Use any plausible length for the car.) (*b*) Estimate the average force exerted by the wall on the car.

Picture the Problem We can estimate the time of collision from the average speed of the car and the distance traveled by the center of the car during the collision. We'll assume a car length of 6.0 m. We can calculate the average force exerted by the wall on the car from the car's change in momentum and its stopping time.

(*a*) Relate the stopping time to the assumption that the center of the car travels halfway to the wall with constant deceleration:

$$\Delta t=\frac{d_{\text{stopping}}}{v_{\text{av}}}=\frac{\frac{1}{2}\left(\frac{1}{2}L_{\text{car}}\right)}{v_{\text{av}}}=\frac{\frac{1}{4}L_{\text{car}}}{v_{\text{av}}}\quad(1)$$

Because a is constant, the average speed of the car is given by:

$$v_{\text{av}}=\frac{v_{\text{i}}+v_{\text{f}}}{2}$$

Substitute numerical values and evaluate v_{av}:

$$v_{\text{av}}=\frac{0+90\dfrac{\text{km}}{\text{h}}\times\dfrac{1\,\text{h}}{3600\,\text{s}}\times\dfrac{1000\,\text{m}}{\text{km}}}{2}$$
$$=12.5\,\text{m/s}$$

Substitute numerical values in equation (1) and evaluate Δt:

$$\Delta t=\frac{\frac{1}{4}(6.0\,\text{m})}{12.5\,\text{m/s}}=0.120\,\text{s}=\boxed{0.12\,\text{s}}$$

(*b*) Relate the average force exerted by the wall on the car to the car’s change in momentum:

$$F_{\text{av}}=\frac{\Delta p}{\Delta t}=\frac{(2000\,\text{kg})\left(90\dfrac{\text{km}}{\text{h}}\times\dfrac{1\,\text{h}}{3600\,\text{s}}\times\dfrac{1000\,\text{m}}{\text{km}}\right)}{0.120\,\text{s}}=\boxed{4.2\times10^{5}\,\text{N}}$$

31 •• In hand-pumped railcar races, a speed of 32.0 km/h has been achieved by teams of four people. A car that has a mass equal to 350 kg is moving at that speed toward a river when Carlos, the chief pumper, notices that the bridge ahead is out. All four people (each with a mass of 75.0 kg) simultaneously jump backward off the car with a velocity that has a horizontal component of 4.00 m/s relative to the car. The car proceeds off the bank and falls into the water a horizontal distance of 25.0 m from the bank. (*a*) Estimate the time of the fall of the railcar. (*b*) What is the horizontal component of the velocity of the pumpers when they hit the ground?

Picture the Problem Let the direction the railcar is moving be the +*x* direction and the system include Earth, the pumpers, and the railcar. We’ll also denote the railcar with the letter c and the pumpers with the letter p. We’ll use conservation of linear momentum to relate the center of mass frame velocities of the car and the pumpers and then transform to the Earth frame of reference to find the time of fall of the car.

(*a*) Relate the time of fall of the railcar to the horizontal distance it travels and its horizontal speed as it leaves the bank:

$$\Delta t = \frac{\Delta x}{v_c} \qquad (1)$$

Use conservation of momentum to find the speed of the car relative to the speed of its center of mass:

$$\vec{p}_i = \vec{p}_f$$

or

$$m_c u_c + m_p u_p = 0$$

Relate u_c to u_p and solve for u_c:

$$u_p - u_c = -4.00\text{ m/s}$$

and

$$u_p = u_c - 4.00\text{ m/s}$$

Substitute for u_p to obtain:

$$m_c u_c + m_p(u_c - 4.00\text{ m/s}) = 0$$

Solving for u_c yields:

$$u_c = \frac{4.00\text{ m/s}}{1 + \frac{m_c}{m_p}}$$

Substitute numerical values and evaluate u_c:

$$u_c = \frac{4.00\text{ m/s}}{1 + \frac{350\text{ kg}}{4(75.0\text{ kg})}} = 1.846\text{ m/s}$$

Relate the speed of the car to its speed relative to the center of mass of the system:

$$v_c = u_c + v_{cm}$$

Substitute numerical values and evaluate v_c:

$$v_c = 1.846\frac{\text{m}}{\text{s}} + \left(32.0\frac{\text{km}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(\frac{1000\text{ m}}{\text{km}}\right) = 10.73\text{ m/s}$$

Substitute numerical values in equation (1) and evaluate Δt:

$$\Delta t = \frac{25.0\text{ m}}{10.73\text{ m/s}} = \boxed{2.33\text{ s}}$$

(*b*) The horizontal velocity of the pumpers when they hit the ground is:

$$v_p = v_c - u_p = 10.73\text{ m/s} - 4.00\text{ m/s} = \boxed{6.7\text{ m/s}}$$

32 •• A wooden block and a gun are firmly fixed to opposite ends of a long glider mounted on a frictionless air track (Figure 8-43). The block and gun are a distance L apart. The system is initially at rest. The gun is fired and the bullet leaves the gun with a velocity v_b and impacts the block, becoming imbedded in it. The mass of the bullet is m_b and the mass of the gun–glider–block system is m_p. (*a*) What is the velocity of the glider immediately after the bullet leaves the gun? (*b*) What is the velocity of the glider immediately after the bullet comes to rest in the block? (*c*) How far does the glider move while the bullet is in transit between the gun and the block?

Picture the Problem Let the system include Earth, platform, gun, bullet, and block. Then $\vec{F}_{\text{net,ext}} = 0$ and momentum is conserved within the system. Choose a coordinate system in which the $+x$ direction is the direction of the bullet and let b and p denote the bullet and platform, respectively.

(*a*) Apply conservation of linear momentum to the system just before and just after the bullet leaves the gun:

$$\vec{p}_{\text{before}} = \vec{p}_{\text{after}}$$

or

$$0 = \vec{p}_{\text{bullet}} + \vec{p}_{\text{glider}}$$

Substitute for $\vec{p}_{\text{bullet}}$ and $\vec{p}_{\text{glider}}$ to obtain:

$$0 = m_b v_b \hat{\boldsymbol{i}} + m_p \vec{v}_p$$

Solving for $\vec{v}_p$ yields:

$$\vec{v}_p = \boxed{-\frac{m_b}{m_p} v_b \hat{\boldsymbol{i}}}$$

(*b*) Apply conservation of momentum to the system just before the bullet leaves the gun and just after it comes to rest in the block:

$$\vec{p}_{\text{before}} = \vec{p}_{\text{after}}$$

or

$$0 = \vec{p}_{\text{glider}} \Rightarrow \vec{v}_{\text{glider}} = \boxed{0}$$

(*c*) Express the distance Δs traveled by the glider:

$$\Delta s = v_p \Delta t$$

Express the velocity of the bullet relative to the glider:

$$v_{\text{rel}} = v_b - v_p = v_b + \frac{m_b}{m_p} v_b$$

$$= \left(1 + \frac{m_b}{m_p}\right) v_b = \frac{m_p + m_b}{m_p} v_b$$

Relate the time of flight Δt to L and v_{rel}:

$$\Delta t = \frac{L}{v_{rel}}$$

Substitute and simplify to find the distance Δs moved by the glider in time Δt:

$$\Delta s = v_p \Delta t = \left(\frac{m_b}{m_p} v_b\right)\left(\frac{L}{v_{rel}}\right) = \left(\frac{m_b}{m_p} v_b\right)\left(\frac{L}{\dfrac{m_p + m_b}{m_p} v_b}\right) = \boxed{\left(\frac{m_b}{m_p + m_b}\right)L}$$

Conservation of Linear Momentum

33 • **[SSM]** Tyrone, an 85-kg teenager, runs off the end of a horizontal pier and lands on a free-floating 150-kg raft that was initially at rest. After he lands on the raft, the raft, with him on it, moves away from the pier at 2.0 m/s. What was Tyrone's speed as he ran off the end of the pier?

Picture the Problem Let the system include Earth, the raft, and Tyrone and apply conservation of linear momentum to find Tyrone's speed when he ran off the end of the pier.

Apply conservation of linear momentum to the system consisting of the raft and Tyrone to obtain:

$$\Delta\vec{p}_{system} = \Delta\vec{p}_{Tyrone} + \Delta\vec{p}_{raft} = 0$$

or, because the motion is one-dimensional,

$$p_{f,Tyrone} - p_{i,\,Tyrone} + p_{f,raft} - p_{i,raft} = 0$$

Because the raft is initially at rest:

$$p_{f,Tyrone} - p_{i,\,Tyrone} + p_{f,raft} = 0$$

Use the definition of linear momentum to obtain:

$$m_{Tyrone} v_{f,Tyrone} - m_{Tyrone} v_{i,Tyrone} + m_{raft} v_{f,raft} = 0$$

Solve for $v_{i,Tyrone}$ to obtain:

$$v_{i,Tyrone} = \frac{m_{raft}}{m_{Tyrone}} v_{f,raft} + v_{f,Tyrone}$$

Letting v represent the common final speed of the raft and Tyrone yields:

$$v_{i,Tyrone} = \left(1 + \frac{m_{raft}}{m_{Tyrone}}\right)v$$

Substitute numerical values and evaluate $v_{\text{i,Tyrone}}$:

$$v_{\text{i, Tyrone}} = \left(1 + \frac{150\text{ kg}}{85\text{ kg}}\right)(2.0\text{ m/s}) = \boxed{5.5\text{ m/s}}$$

34 •• A 55-kg woman contestant on a reality television show is at rest at the south end of a horizontal 150-kg raft that is floating in crocodile-infested waters. She and the raft are initially at rest. She needs to jump from the raft to a platform that is several meters off the north end of the raft. She takes a running start. When she reaches the north end of the raft she is running at 5.0 m/s relative to the raft. At that instant, what is her velocity relative to the water?

Picture the Problem Let the system include the woman, the raft, and Earth. Then the *net* external force is zero and linear momentum is conserved as she jumps off the raft. Let the direction the woman is running be the +*x* direction.

Apply conservation of linear momentum to the system:

$$\sum m_i \vec{v}_i = m_{\text{woman}}\vec{v}_{\text{woman}} + m_{\text{raft}}\vec{v}_{\text{raft}} = 0$$

Solving for $\vec{v}_{\text{raft}}$ yields:

$$\vec{v}_{\text{raft}} = -\frac{m_{\text{woman}}\vec{v}_{\text{woman}}}{m_{\text{raft}}}$$

Substituting numerical values gives:

$$\vec{v}_{\text{raft}} = -\left(\frac{55\text{ kg}}{150\text{ kg}}\right)\vec{v}_{\text{woman}} = -\left(\frac{55}{150}\right)\vec{v}_{\text{woman}}$$

It is given that:

$$\vec{v}_{\text{woman}} - \vec{v}_{\text{raft}} = (5.0\text{ m/s})\hat{\boldsymbol{i}}$$

Substituting for $\vec{v}_{\text{raft}}$ yields:

$$\vec{v}_{\text{woman}} + \left(\frac{55}{150}\right)\vec{v}_{\text{woman}} = (5.0\text{ m/s})\hat{\boldsymbol{i}}$$

Solve for $\vec{v}_{\text{woman}}$ to obtain:

$$\vec{v}_{\text{woman}} = \left(\frac{5.0\text{ m/s}}{1 + \dfrac{55}{150}}\right)\hat{\boldsymbol{i}} = \boxed{(3.7\text{ m/s})\hat{\boldsymbol{i}}}$$

35 • A 5.0-kg object and a 10-kg object, both resting on a frictionless table, are connected by a massless compressed spring. The spring is released and the objects fly off in opposite directions. The 5.0-kg object has a velocity of 8.0 m/s to the left. What is the velocity of the 10-kg object?

Picture the Problem If we include Earth in our system, then the net external force is zero and linear momentum is conserved as the spring delivers its energy to the two objects. Choose a coordinate system in which the +*x* direction is to the right.

Apply conservation of linear momentum to the system:

$$\sum m_i\vec{v}_i = m_5\vec{v}_5 + m_{10}\vec{v}_{10} = 0$$

Solving for $\vec{v}_{10}$ yields:

$$\vec{v}_{10} = \left(-\frac{m_5}{m_{10}}\right)\vec{v}_5$$

Substitute numerical values and evaluate $\vec{v}_{10}$:

$$\vec{v}_{10} = \left[-\frac{(5.0\,\text{kg})(-8.0\,\text{m/s})}{10\,\text{kg}}\right]\hat{i}$$
$$= \boxed{(4.0\,\text{m/s})\hat{i}}$$

or 4.0 m/s to the right.

36 • Figure 8-44 shows the behavior of a projectile just after it has broken up into three pieces. What was the speed of the projectile the instant before it broke up? (*a*) v_3. (*b*) $v_3/3$. (*c*) $v_3/4$. (*d*) $4v_3$. (*e*) $(v_1 + v_2 + v_3)/4$.

Picture the Problem This is an explosion-like event in which linear momentum is conserved. Thus we can equate the initial and final momenta in the *x* direction and the initial and final momenta in the *y* direction. Choose a coordinate system in the +*x* direction is to the right and the +*y* direction is upward.

Equate the momenta in the *y* direction before and after the explosion:

$$\sum p_{y,i} = \sum p_{y,f} = mv_2 - 2mv_1$$
$$= m(2v_1) - 2mv_1 = 0$$

We can conclude that the momentum was entirely in the *x* direction before the particle exploded.

Equate the momenta in the *x* direction before and after the explosion:

$$\sum p_{x,i} = \sum p_{x,f}$$

or

$$4mv_{\text{projectile}} = mv_3$$

Solving for $v_{\text{projectile}}$ yields:

$v_{\text{projectile}} = \frac{1}{4}v_3$ and $\boxed{(c)}$ is correct.

37 • A shell of mass m and speed v explodes into two identical fragments. If the shell was moving horizontally with respect to Earth, and one of the fragments is subsequently moving vertically with speed v, find the velocity $\vec{v}'$ of the other fragment immediately following the explosion.

Picture the Problem Choose the direction the shell is moving just before the explosion to be the positive x direction and apply conservation of momentum.

Use conservation of momentum to relate the masses of the fragments to their velocities:

$$\vec{p}_{\text{i}} = \vec{p}_{\text{f}}$$

or

$$mv\hat{i} = \tfrac{1}{2}mv\hat{j} + \tfrac{1}{2}m\vec{v}' \Rightarrow \vec{v}' = \boxed{2v\hat{i} - v\hat{j}}$$

38 •• During this week's physics lab, the experimental setup consists of two gliders on a horizontal frictionless air track (see Figure 8-45). Each glider supports a strong magnet centered on top of it, and the magnets are oriented so they attract each other. The mass of glider 1 and its magnet is 0.100 kg and the mass of glider 2 and its magnet is 0.200 kg. You and your lab partners take the origin to be at the left end of the track and to center glider 1 at x_1 = 0.100 m and glider 2 at x_2 = 1.600 m. Glider 1 is 10.0 cm long, while glider 2 is 20.0 cm long and each glider has its center of mass at its geometric center. When the two are released from rest, they will move toward each other and stick. (*a*) Predict the position of the center of each glider when they first touch. (*b*) Predict the velocity the two gliders will continue to move with after they stick. Explain the reasoning behind this prediction for your lab partners.

Picture the Problem Because no external forces act on either glider, the center of mass of the two-glider system can't move. We can use the data concerning the masses and separation of the gliders initially to calculate its location and then apply the definition of the center of mass a second time to relate the positions x_1 and x_2 of the centers of the carts when they first touch. We can also use the separation of the centers of the gliders when they touch to obtain a second equation in x_1 and x_2 that we can solve simultaneously with the equation obtained from the location of the center of mass.

(*a*) The x coordinate of the center of mass of the 2-glider system is given by:

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Substitute numerical values and evaluate x_{cm}:

$$x_{\text{cm}} = \frac{(0.100\,\text{kg})(0.100\,\text{m}) + (0.200\,\text{kg})(1.600\,\text{m})}{0.100\,\text{kg} + 0.200\,\text{kg}} = 1.10\,\text{m}$$

from the left end of the air track.

Because the location of the center of mass has not moved when two gliders first touch:

$$1.10\,\text{m} = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}$$

Substitute numerical values and simplify to obtain:

$$1.10\,\text{m} = \tfrac{1}{3} X_1 + \tfrac{2}{3} X_2$$

Also, when they first touch, their centers are separated by half their combined lengths:

$$X_2 - X_1 = \tfrac{1}{2}(10.0\,\text{cm} + 20.0\,\text{cm}) = 0.150\,\text{m}$$

Thus we have:

$$\tfrac{1}{3} X_1 + \tfrac{2}{3} X_2 = 1.10\,\text{m}$$

and

$$X_2 - X_1 = 0.150\,\text{m}$$

Solving these equations simultaneously yields:

$$X_1 = \boxed{1.00\,\text{m}} \text{ and } X_2 = \boxed{1.15\,\text{m}}$$

(*b*) Because the momentum of the system was zero initially, it must be zero just before the collision and after the collision in which the gliders stick together. Hence their velocity after the collision must be $\boxed{0}$.

39 •• Bored, a boy shoots his pellet gun at a piece of cheese that sits on a massive block of ice. On one particular shot, his 1.2 g pellet gets stuck in the cheese, causing it to slide 25 cm before coming to a stop. If the muzzle velocity of the gun is known to be 65 m/s, and the cheese has a mass of 120 g, what is the coefficient of friction between the cheese and ice?

Picture the Problem Let the system consist of the pellet and the cheese. Then we can apply the conservation of linear momentum and the conservation of energy with friction to this inelastic collision to find the coefficient of friction between the cheese and the ice.

Apply conservation of linear momentum to the system to obtain:

$$\Delta\vec{\boldsymbol{p}}_{\text{system}} = \Delta\vec{\boldsymbol{p}}_{\text{pellet}} + \Delta\vec{\boldsymbol{p}}_{\text{cheese}} = 0$$

or, because the motion is one-dimensional,

$$p_{\text{f,pellet}} - p_{\text{i,pellet}} + p_{\text{f,cheese}} - p_{\text{i,cheese}} = 0$$

Because the cheese is initially at rest:

$$p_{\text{f,pellet}} - p_{\text{i,pellet}} + p_{\text{f,cheeset}} = 0$$

Letting m represent the mass of the pellet, M the mass of the cheese, and v the common final speed of the pellet and the cheese, use the definition of linear momentum to obtain:

$$mv - mv_{\text{i,pellet}} + Mv = 0$$

Solving for v yields:

$$v = \frac{m}{m+M} v_{\text{i,pellet}} \qquad (1)$$

Apply the conservation of energy with friction to the system to obtain:

$$W_{\text{ext}} = \Delta E_{\text{mech}} + \Delta E_{\text{therm}}$$

or, because $W_{\text{ext}} = 0$,

$$\Delta K + \Delta U_{\text{g}} + \Delta E_{\text{therm}} = 0$$

Because $\Delta U_{\text{g}} = K_{\text{f}} = 0$, and $\Delta E_{\text{therm}} = f\Delta s$ (where Δs is the distance the cheese slides on the ice):

$$-\tfrac{1}{2}(m+M)v^2 + f\Delta s = 0$$

f is given by:

$$f = \mu_{\text{k}}(m+M)g$$

Substituting for f yields:

$$-\tfrac{1}{2}(m+M)v^2 + \mu_{\text{k}}(m+M)g\Delta s = 0$$

Substitute for v from equation (1) to obtain:

$$-\tfrac{1}{2}(m+M)\left(\frac{m}{m+M} v_{\text{i,pellet}}\right)^2 + \mu_{\text{k}}(m+M)g\Delta s = 0$$

Solving for μ_{k} yields:

$$\mu_{\text{k}} = \frac{1}{2g\Delta s}\left(\frac{mv_{\text{i,pellet}}}{m+M}\right)^2$$

Substitute numerical values and evaluate μ_{k}:

$$\mu_{\text{k}} = \frac{1}{2(9.81\ \text{m/s}^2)(0.25\ \text{m})}\left(\frac{(0.0012\ \text{kg})(65\ \text{m/s})}{0.0012\ \text{kg} + 0.120\ \text{kg}}\right)^2 = \boxed{0.084}$$

40 ••• A wedge of mass M, as shown in Figure 8-46, is placed on a frictionless, horizontal surface, and a block of mass m is placed on the wedge, whose surface is also frictionless. The center of mass of the block moves

downward a distance h, as the block slides from its initial position to the horizontal floor. (*a*) What are the speeds of the block and of the wedge, as they separate from each other and each go their own way? (*b*) Check your calculation plausibility by considering the limiting case when $M >> m$.

Picture the Problem Let the system include Earth, the block, and the wedge and apply conservation of energy and conservation of linear momentum.

(*a*) Apply conservation of energy with no frictional forces to the system to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U$$

or, because $W_{\text{ext}} = 0$,

$$\Delta K + \Delta U = 0$$

Substituting for ΔK and ΔU yields:

$$K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} = 0$$

Because $K_{\text{i}} = U_{\text{f}} = 0$:

$$K_{\text{f}} - U_{\text{i}} = 0$$

Letting "b" refer to the block and "w" to the wedge yields:

$$K_{\text{b,f}} + K_{\text{w,f}} - U_{\text{b,i}} = 0$$

Substitute for $K_{\text{b,f}}$, $K_{\text{w,f}}$, and $U_{\text{b,i}}$ to obtain:

$$\tfrac{1}{2}mv_{\text{b}}^2 + \tfrac{1}{2}Mv_{\text{w}}^2 - mgh = 0 \qquad (1)$$

Applying conservation of linear momentum to the system yields:

$$\Delta\vec{\boldsymbol{p}}_{\text{sys}} = \Delta\vec{\boldsymbol{p}}_{\text{b}} + \Delta\vec{\boldsymbol{p}}_{\text{w}} = 0$$

or

$$\vec{\boldsymbol{p}}_{\text{b,f}} - \vec{\boldsymbol{p}}_{\text{b,i}} + \vec{\boldsymbol{p}}_{\text{w,f}} - \vec{\boldsymbol{p}}_{\text{w,i}} = 0$$

Because $\vec{\boldsymbol{p}}_{\text{b,i}} = \vec{\boldsymbol{p}}_{\text{w,i}} = 0$:

$$\vec{\boldsymbol{p}}_{\text{b,f}} + \vec{\boldsymbol{p}}_{\text{w,f}} = 0$$

Substituting for $\vec{\boldsymbol{p}}_{\text{b,f}}$ and $\vec{\boldsymbol{p}}_{\text{w,f}}$ yields:

$$-mv_{\text{b}}\hat{\boldsymbol{i}} + Mv_{\text{w}}\hat{\boldsymbol{i}} = 0$$

or

$$-mv_{\text{b}} + Mv_{\text{w}} = 0$$

Solve for v_{w} to obtain:

$$v_{\text{w}} = \frac{m}{M}v_{\text{b}} \qquad (2)$$

Substituting for v_{w} in equation (1) yields:

$$\tfrac{1}{2}mv_{\text{b}}^2 + \tfrac{1}{2}M\left(\frac{m}{M}v_{\text{b}}\right)^2 - mgh = 0$$

Solve for v_{b} to obtain:

$$v_{\text{b}} = \boxed{\sqrt{\frac{2ghM}{M+m}}} \qquad (3)$$

Substitute for v_b in equation (2) and simplify to obtain:

$$v_w = \frac{m}{M}\sqrt{\frac{2ghM}{M+m}} = \boxed{\sqrt{\frac{2ghm^2}{M(M+m)}}} \quad (4)$$

(*b*) Rewriting equation (3) by dividing the numerator and denominator of the radicand by *M* yields:

$$v_b = \sqrt{\frac{2gh}{1+\frac{m}{M}}}$$

When $M >> m$:

$$v_b = \sqrt{2gh}$$

Rewriting equation (4) by dividing the numerator and denominator of the radicand by *M* yields:

$$v_w = \sqrt{\frac{2gh\left(\frac{m}{M}\right)^2}{1+\frac{m}{M}}}$$

When $M >> m$:

$$v_w = 0$$

These results are exactly what we would expect in this case: The physics is that of a block sliding down a fixed wedge incline with no movement of the incline.

Kinetic Energy of a System of Particles

41 •• [SSM] A 3.0-kg block is traveling to the right (the +*x* direction) at 5.0 m/s, and a second 3.0-kg block is traveling to the left at 2.0 m/s. (*a*) Find the total kinetic energy of the two blocks. (*b*) Find the velocity of the center of mass of the two-block system. (*c*) Find the velocity of each block relative to the center of mass. (*d*) Find the kinetic energy of the blocks relative to the center of mass. (*e*) Show that your answer for Part (*a*) is greater than your answer for Part (*d*) by an amount equal to the kinetic energy associated with the motion of the center of mass.

Picture the Problem Choose a coordinate system in which the +*x* direction is to the right. Use the expression for the total momentum of a system to find the velocity of the center of mass and the definition of relative velocity to express the sum of the kinetic energies relative to the center of mass.

(*a*) The total kinetic energy is the sum of the kinetic energies of the blocks:

$$K = K_1 + K_2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$

Substitute numerical values and evaluate K:

$$K = \tfrac{1}{2}(3.0\,\text{kg})(5.0\,\text{m/s})^2 + \tfrac{1}{2}(3.0\,\text{kg})(2.0\,\text{m/s})^2 = 43.5\,\text{J} = \boxed{44\,\text{J}}$$

(*b*) Relate the velocity of the center of mass of the system to its total momentum:

$$M\vec{\boldsymbol{v}}_{\text{cm}} = m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2$$

Solving for $\vec{v}_{\text{cm}}$ yields:

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2}{m_1 + m_2}$$

Substitute numerical values and evaluate $\vec{\boldsymbol{v}}_{\text{cm}}$:

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{(3.0\,\text{kg})(5.0\,\text{m/s})\hat{\boldsymbol{i}} + (3.0\,\text{kg})(-2.0\,\text{m/s})\hat{\boldsymbol{i}}}{3.0\,\text{kg} + 3.0\,\text{kg}} = \boxed{(1.5\,\text{m/s})\hat{\boldsymbol{i}}}$$

(*c*) The velocity of an object relative to the center of mass is given by:

$$\vec{\boldsymbol{v}}_{\text{rel}} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{\text{cm}}$$

Substitute numerical values to obtain:

$$\vec{\boldsymbol{v}}_{1,\text{rel}} = (5.0\,\text{m/s})\hat{\boldsymbol{i}} - (1.5\,\text{m/s})\hat{\boldsymbol{i}} = \boxed{(3.5\,\text{m/s})\hat{\boldsymbol{i}}}$$

$$\vec{\boldsymbol{v}}_{2,\text{rel}} = (-2.0\,\text{m/s})\hat{\boldsymbol{i}} - (1.5\,\text{m/s})\hat{\boldsymbol{i}} = \boxed{(-3.5\,\text{m/s})\hat{\boldsymbol{i}}}$$

(*d*) Express the sum of the kinetic energies relative to the center of mass:

$$K_{\text{rel}} = K_{1,\text{rel}} + K_{2,\text{rel}} = \tfrac{1}{2}m_1v_{1,\text{rel}}^2 + \tfrac{1}{2}m_2v_{2,\text{rel}}^2$$

Substitute numerical values and evaluate K_{rel}:

$$K_{\text{rel}} = \tfrac{1}{2}(3.0\,\text{kg})(3.5\,\text{m/s})^2 + \tfrac{1}{2}(3.0\,\text{kg})(-3.5\,\text{m/s})^2 = \boxed{37\,\text{J}}$$

(*e*) K_{cm} is given by:

$$K_{\text{cm}} = \tfrac{1}{2}m_{\text{tot}}v_{\text{cm}}^2$$

Substitute numerical values and evaluate K_{cm}:

$$K_{cm} = \tfrac{1}{2}(6.0\,\text{kg})(1.5\,\text{m/s})^2 = 6.75\,\text{J} = 43.5\,\text{J} - 36.75\,\text{J} = \boxed{K - K_{rel}}$$

42 •• Repeat Problem 41 with the second 3.0-kg block replaced by a 5.0-kg block moving to the right at 3.0 m/s.

Picture the Problem Choose a coordinate system in which the +*x* direction is to the right. Use the expression for the total momentum of a system to find the velocity of the center of mass and the definition of relative velocity to express the sum of the kinetic energies relative to the center of mass.

(*a*) The total kinetic energy is the sum of the kinetic energies of the blocks:

$$K = K_1 + K_2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$

Substitute numerical values and evaluate *K*:

$$K = \tfrac{1}{2}(3.0\,\text{kg})(5.0\,\text{m/s})^2 + \tfrac{1}{2}(5.0\,\text{kg})(3.0\,\text{m/s})^2 = 60.0\,\text{J} = \boxed{60\,\text{J}}$$

(*b*) Relate the velocity of the center of mass of the system to its total momentum:

$$M\vec{\boldsymbol{v}}_{cm} = m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2$$

Solving for $\vec{\boldsymbol{v}}_{cm}$ yields:

$$\vec{\boldsymbol{v}}_{cm} = \frac{m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2}{m_1 + m_2}$$

Substitute numerical values and evaluate $\vec{\boldsymbol{v}}_{cm}$:

$$\vec{\boldsymbol{v}}_{cm} = \frac{(3.0\,\text{kg})(5.0\,\text{m/s})\hat{\boldsymbol{i}} + (5.0\,\text{kg})(3.0\,\text{m/s})\hat{\boldsymbol{i}}}{3.0\,\text{kg} + 5.0\,\text{kg}} = (3.75\,\text{m/s})\hat{\boldsymbol{i}} = \boxed{(3.8\,\text{m/s})\hat{\boldsymbol{i}}}$$

(*c*) The velocity of an object relative to the center of mass is given by:

$$\vec{\boldsymbol{v}}_{rel} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{cm}$$

Substitute numerical values to obtain:

$$\vec{v}_{1,\text{rel}} = (5.0\,\text{m/s})\hat{i} - (3.75\,\text{m/s})\hat{i} = \boxed{(1.3\,\text{m/s})\hat{i}}$$

$$\vec{v}_{2,\text{rel}} = (3.0\,\text{m/s})\hat{i} - (3.75\,\text{m/s})\hat{i} = (-0.75\,\text{m/s})\hat{i} = \boxed{(-0.8\,\text{m/s})\hat{i}}$$

(*d*) Express the sum of the kinetic energies relative to the center of mass:

$$K_{\text{rel}} = K_{1,\text{rel}} + K_{2,\text{rel}} = \tfrac{1}{2}m_1v_{1,\text{rel}}^2 + \tfrac{1}{2}m_2v_{2,\text{rel}}^2$$

Substitute numerical values and evaluate K_{rel}:

$$K_{\text{rel}} = \tfrac{1}{2}(3.0\,\text{kg})(1.25\,\text{m/s})^2 + \tfrac{1}{2}(5.0\,\text{kg})(-0.75\,\text{m/s})^2 = 3.75\text{ J} = \boxed{4\,\text{J}}$$

(*e*) K_{cm} is given by:

$$K_{\text{cm}} = \tfrac{1}{2}m_{\text{tot}}v_{\text{cm}}^2$$

Substitute numerical values and evaluate K_{cm}:

$$K_{\text{cm}} = \tfrac{1}{2}(8.0\,\text{kg})(3.75\,\text{m/s})^2 \approx 56.3\,\text{J} \approx \boxed{K - K_{\text{rel}}}$$

Impulse and Average Force

43 • [SSM] You kick a soccer ball whose mass is 0.43 kg. The ball leaves your foot with an initial speed of 25 m/s. (*a*) What is the magnitude of the impulse associated with the force of your foot on the ball? (*b*) If your foot is in contact with the ball for 8.0 ms, what is the magnitude of the average force exerted by your foot on the ball?

Picture the Problem The impulse imparted to the ball by the kicker equals the *change* in the ball's momentum. The impulse is also the product of the average force exerted on the ball by the kicker and the time during which the average force acts.

(*a*) Relate the magnitude of the impulse delivered to the ball to its change in momentum:

$$I = |\Delta\vec{p}| = p_{\text{f}} - p_{\text{i}}$$

or, because $v_{\text{i}} = 0$,

$$I = mv_{\text{f}}$$

Substitute numerical values and evaluate *I*:

$$I = (0.43\,\text{kg})(25\,\text{m/s}) = 10.8\,\text{N}\cdot\text{s} = \boxed{11\,\text{N}\cdot\text{s}}$$

(*b*) The impulse delivered to the ball as a function of the average force acting on it is given by:

$$I = F_{av}\Delta t \Rightarrow F_{av} = \frac{I}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{10.8\,\text{N}\cdot\text{s}}{0.0080\,\text{s}} = \boxed{1.3\,\text{kN}}$$

44 • A 0.30-kg brick is dropped from a height of 8.0 m. It hits the ground and comes to rest. (*a*) What is the impulse exerted by the ground on the brick during the collision? (*b*) If it takes 0.0013 s from the time the brick first touches the ground until it comes to rest, what is the average force exerted by the ground on the brick during impact?

Picture the Problem The impulse exerted by the ground on the brick equals the *change* in momentum of the brick and is also the product of the average force exerted by the ground on the brick and the time during which the average force acts.

(*a*) Express the magnitude of the impulse exerted by the ground on the brick:

$$I = \left|\Delta p_{\text{brick}}\right| = \left|p_{\text{f,brick}} - p_{\text{i,brick}}\right|$$

Because $p_{\text{f,brick}} = 0$:

$$I = p_{\text{i,brick}} = m_{\text{brick}}v \qquad (1)$$

Use conservation of energy to determine the speed of the brick at impact:

$$\Delta K + \Delta U = 0$$

or

$$K_f - K_i + U_f - U_i = 0$$

Because $U_f = K_i = 0$:

$$K_f - U_i = 0$$

and

$$\tfrac{1}{2}m_{\text{brick}}v^2 - m_{\text{brick}}gh = 0 \Rightarrow v = \sqrt{2gh}$$

Substitute in equation (1) to obtain:

$$I = m_{\text{brick}}\sqrt{2gh}$$

Substitute numerical values and evaluate *I*:

$$I = (0.30\,\text{kg})\sqrt{2(9.81\,\text{m/s}^2)(8.0\,\text{m})}$$
$$= 3.76\,\text{N}\cdot\text{s} = \boxed{3.8\,\text{N}\cdot\text{s}}$$

(*c*) The average force acting on the brick is:

$$F_{av} = \frac{I}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{3.76\,\text{N}\cdot\text{s}}{0.0013\,\text{s}} = \boxed{2.9\,\text{kN}}$$

45 • A meteorite that has a mass equal to 30.8 tonne (1 tonne = 1000 kg) is exhibited in the American Museum of Natural History in New York City. Suppose that the kinetic energy of the meteorite as it hit the ground was 617 MJ. Find the magnitude of the impulse *I* experienced by the meteorite up to the time its kinetic energy was halved (which took about t = 3.0 s). Find also the average force *F* exerted on the meteorite during this time interval.

Picture the Problem The impulse exerted by the ground on the meteorite equals the *change* in momentum of the meteorite and is also the product of the average force exerted by the ground on the meteorite and the time during which the average force acts.

Express the magnitude of the impulse exerted by the ground on the meteorite:

$$I = \left|\Delta\vec{\boldsymbol{p}}_{\text{meteorite}}\right| = p_f - p_i$$

Relate the kinetic energy of the meteorite to its initial momentum and solve for its initial momentum:

$$K_i = \frac{p_i^2}{2m} \Rightarrow p_i = \sqrt{2mK_i}$$

Express the ratio of the initial and final kinetic energies of the meteorite:

$$\frac{K_i}{K_f} = \frac{\dfrac{p_i^2}{2m}}{\dfrac{p_f^2}{2\text{m}}} = \frac{p_i^2}{p_f^2} = 2 \Rightarrow p_f = \frac{p_i}{\sqrt{2}}$$

Substitute in our expression for *I* and simplify:

$$I = \frac{p_i}{\sqrt{2}} - p_i = p_i\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \sqrt{2mK_i}\left(\frac{1}{\sqrt{2}} - 1\right)$$

Because our interest is in its magnitude, substitute numerical values and evaluate the absolute value of *I*:

$$I = \left|\sqrt{2\left(30.8\times10^3\text{ kg}\right)\left(617\times10^6\text{ J}\right)}\left(\frac{1}{\sqrt{2}} - 1\right)\right| = \boxed{1.81\,\text{MN}\cdot\text{s}}$$

The average force acting on the meteorite is:

$$F_{av} = \frac{I}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{1.81\,\text{MN}\cdot\text{s}}{3.0\,\text{s}} = \boxed{0.60\,\text{MN}}$$

46 •• A 0.15-kg baseball traveling horizontally is hit by a bat and its direction exactly reversed. Its velocity changes from +20 m/s to –20 m/s. (*a*) What is the magnitude of the impulse delivered by the bat to the ball? (*b*) If the baseball is in contact with the bat for 1.3 ms, what is the average force exerted by the bat on the ball?

Picture the Problem The impulse exerted by the bat on the ball equals the *change* in momentum of the ball and is also the product of the average force exerted by the bat on the ball and the time during which the bat and ball were in contact.

(*a*) Express the impulse exerted by the bat on the ball in terms of the change in momentum of the ball:

$$\vec{I} = \Delta\vec{p}_{\text{ball}} = \vec{p}_{\text{f}} - \vec{p}_{\text{i}} = mv_{\text{f}}\hat{i} - \left(-mv_{\text{i}}\hat{i}\right) = 2mv\,\hat{i}$$

where $v = v_{\text{f}} = v_{\text{i}}$

Substitute for m and v and evaluate $|\vec{I}|$:

$$I = 2(0.15\,\text{kg})(20\,\text{m/s}) = 6.00\,\text{N}\cdot\text{s} = \boxed{6.0\,\text{N}\cdot\text{s}}$$

(*b*) The average force acting on the ball is:

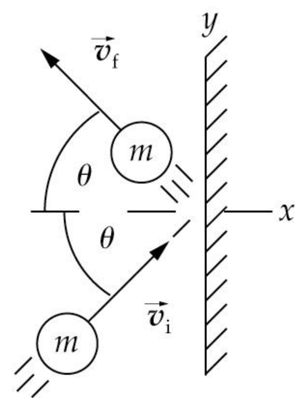
$$F_{av} = \frac{I}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{6.00\,\text{N}\cdot\text{s}}{1.3\,\text{ms}} = \boxed{4.6\,\text{kN}}$$

47 •• A 60-g handball moving with a speed of 5.0 m/s strikes the wall at an angle of 40° with the normal, and then bounces off with the same speed at the same angle with the normal. It is in contact with the wall for 2.0 ms. What is the average force exerted by the ball on the wall?

Picture the Problem The figure shows the handball just before and immediately after its collision with the wall. Choose a coordinate system in which the +*x* direction is to the right. The wall changes the momentum of the ball by exerting a force on it during the ball's collision with it. The reaction to this force is the force the ball exerts on the wall. Because these action and reaction forces are equal in magnitude, we can find the average force exerted on the ball by finding the change in momentum of the ball.



Using Newton's 3rd law, relate the average force exerted by the ball on the wall to the average force exerted by the wall on the ball:

$$\vec{F}_{\text{av on wall}} = -\vec{F}_{\text{av on ball}}$$
and
$$F_{\text{av on wall}} = F_{\text{av on ball}} \quad (1)$$

Relate the average force exerted by the wall on the ball to its change in momentum:

$$\vec{F}_{\text{av on ball}} = \frac{\Delta\vec{p}}{\Delta t} = \frac{m\Delta\vec{v}}{\Delta t}$$

Express $\Delta\vec{v}$ in terms of its components:

$$\Delta\vec{v} = \Delta\vec{v}_x + \Delta\vec{v}_y$$
or, because $\Delta\vec{v}_y = v_{\text{f},y}\hat{j} - v_{\text{i},y}\hat{j}$ and $v_{\text{f},y} = v_{\text{i},y}$, $\Delta\vec{v} = \Delta\vec{v}_x$

Express $\Delta\vec{v}_x$ for the ball:

$$\Delta\vec{v}_x = v_{\text{f},x}\hat{i} - v_{\text{i},x}\hat{i}$$
or, because $v_{\text{i},x} = v\cos\theta$ and $v_{\text{f},x} = -v\cos\theta$,
$$\Delta\vec{v}_x = -v\cos\theta\,\hat{i} - v\cos\theta\,\hat{i} = -2v\cos\theta\,\hat{i}$$

Substituting in the expression for $\vec{F}_{\text{av on ball}}$ yields:

$$\vec{F}_{\text{av on ball}} = \frac{m\Delta\vec{v}}{\Delta t} = -\frac{2mv\cos\theta}{\Delta t}\hat{i}$$

The magnitude of $\vec{F}_{\text{av on ball}}$ is:

$$F_{\text{av on ball}} = \frac{2mv\cos\theta}{\Delta t}$$

Substitute numerical values and evaluate $F_{\text{av on ball}}$:

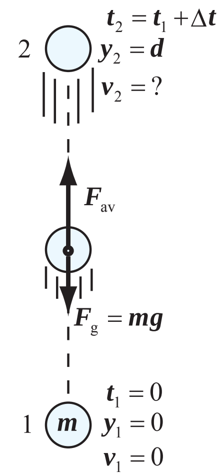
$$F_{\text{av on ball}} = \frac{2(0.060\,\text{kg})(5.0\,\text{m/s})\cos 40°}{2.0\,\text{ms}} = 0.23\,\text{kN}$$

Substitute in equation (1) to obtain:

$$F_{\text{av on wall}} = \boxed{0.23\,\text{kN}}$$

48 •• You throw a 150-g ball straight up to a height of 40.0 m. (*a*) Use a reasonable value for the displacement of the ball while it is in your hand to estimate the time the ball is in your hand while you are throwing it. (*b*) Calculate the average force exerted by your hand while you are throwing it. (Is it OK to neglect the gravitational force on the ball while it is being thrown?)

Picture the Problem The pictorial representation shows the ball during the interval of time during which you are exerting a force on it to accelerate it upward. The average force you exert can be determined from the change in momentum of the ball. The change in the velocity of the ball can be found by applying conservation of mechanical energy to the ball-Earth system once it has left your hand.



(*a*) Relate the time the ball is in your hand to its average speed while it is in your hand and the displacement of your hand:

$$\Delta t = \frac{\Delta y}{v_{\text{av, in your hand}}}$$

Letting $U_g = 0$ at the initial elevation of your hand, use conservation of mechanical energy to relate the initial kinetic energy of the ball to its potential energy when it is at its highest point:

$$\Delta K + \Delta U = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_f *and* U_i and solve for v_2:

$$-\tfrac{1}{2}mv_2^2 + mgh = 0 \Rightarrow v_2 = \sqrt{2gh}$$

Because $v_{\text{av, in your hand}} = \tfrac{1}{2}v_2$:

$$\Delta t = \frac{\Delta y}{\tfrac{1}{2}v_2} = \frac{2\Delta y}{\sqrt{2gh}}$$

Assuming the displacement of your hand is 0.70 m as you throw the ball straight up, substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(0.70\text{ m})}{\sqrt{2(9.81\text{ m/s}^2)(40\text{ m})}} = 50.0\text{ ms}$$
$$= \boxed{50\text{ ms}}$$

(*b*) Relate the average force exerted by your hand on the ball to the change in momentum of the ball:

$$F_{\text{av}} = \frac{\Delta p}{\Delta t} = \frac{p_2 - p_1}{\Delta t}$$

or, because $v_1 = p_1 = 0$,

$$F_{\text{av}} = \frac{mv_2}{\Delta t}$$

Substitute for v_2 to obtain:

$$F_{\text{av}} = \frac{m\sqrt{2gh}}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{\text{av}} = \frac{(0.15\text{ kg})\sqrt{2(9.81\text{ m/s}^2)(40\text{ m})}}{50.0\text{ ms}}$$
$$= 84.04\text{ N} = \boxed{84\text{ N}}$$

Express the ratio of the gravitational force on the ball to the average force acting on it:

$$\frac{F_{\text{g}}}{F_{\text{av}}} = \frac{mg}{F_{\text{av}}}$$

Substitute numerical values and evaluate $F_{\text{g}}/F_{\text{av}}$:

$$\frac{F_{\text{g}}}{F_{\text{av}}} = \frac{(0.15\text{ kg})(9.81\text{ m/s}^2)}{84.04\text{ N}} < 2\%$$

Because the gravitational force acting on the ball is less than 2% of the average force exerted by your hand on the ball, it is reasonable to have neglected the gravitational force.

49 •• A 0.060-kg handball is thrown straight toward a wall with a speed of 10 m/s. It rebounds straight backward at a speed of 8.0 m/s. (*a*) What impulse is exerted on the wall? (*b*) If the ball is in contact with the wall for 3.0 ms, what average force is exerted on the wall by the ball? (*c*) The rebounding ball is caught by a player who brings it to rest. During the process, her hand moves back 0.50 m. What is the impulse received by the player? (*d*) What average force was exerted on the player by the ball?

Picture the Problem Choose a coordinate system in which the direction the ball is moving *after* its collision with the wall is the +*x* direction. The impulse delivered to the wall or received by the player equals the change in the momentum of the ball during these two collisions. We can find the average forces from the rate of change in the momentum of the ball.

(*a*) The impulse delivered to the wall is the change in momentum of the handball:

$$\vec{I} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$$

Substitute numerical values and evaluate $\vec{I}$:

$$\begin{aligned}\vec{I} &= (0.060\,\text{kg})(8.0\,\text{m/s})\hat{i} \\ &\quad -\left[-(0.060\,\text{kg})(10\,\text{m/s})\hat{i}\right] \\ &= (1.08\,\text{N}\cdot\text{s})\hat{i} = \boxed{(1.1\,\text{N}\cdot\text{s})\hat{i}}\end{aligned}$$

or 1.1 N·s directed into the wall.

(*b*) F_{av} is the rate of change of the ball's momentum:

$$F_{av} = \frac{\Delta p}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{1.08\,\text{N}\cdot\text{s}}{0.0030\,\text{s}} = 360\,\text{N}$$
$$= \boxed{0.36\,\text{kN, into the wall.}}$$

(*c*) The impulse received by the player from the change in momentum of the ball is given by:

$$I = \Delta p_{ball} = m\Delta v$$

Substitute numerical values and evaluate I:

$$I = (0.060\,\text{kg})(8.0\,\text{m/s}) = 0.480\,\text{N}\cdot\text{s}$$
$$= \boxed{0.48\,\text{N}\cdot\text{s, away from the wall.}}$$

(*d*) Relate F_{av} to the change in the ball's momentum:

$$F_{av} = \frac{\Delta p_{ball}}{\Delta t}$$

Express the stopping time in terms of the average speed v_{av} of the ball and its stopping distance d:

$$\Delta t = \frac{d}{v_{av}}$$

Substitute for Δt and simplify to obtain:

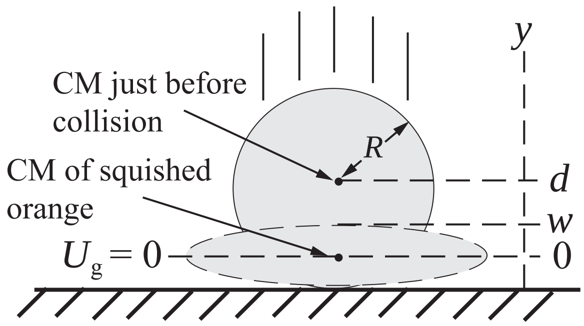
$$F_{av} = \frac{v_{av}\Delta p_{ball}}{d}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = \frac{(4.0\,\text{m/s})(0.480\,\text{N}\cdot\text{s})}{0.50\,\text{m}}$$
$$= \boxed{3.8\,\text{N, away from the wall.}}$$

50 •• A spherical 0.34-kg orange, 2.0 cm in radius, is dropped from the top of a 35 m-tall building. After striking the pavement, the shape of the orange is a 0.50 cm thick pancake. Neglect air resistance and assume that the collision is completely inelastic. (*a*) How much time did the orange take to completely "squish" to a stop? (*b*) What average force did the pavement exert on the orange during the collision?

Picture the Problem The following pictorial representation shows the orange moving with speed *v*, just before impact, after falling from a height of 35 m. Let the system be the orange and let the zero of gravitational potential energy be at the center of mass of the squished orange. The external forces are gravity, acting on the orange throughout its fall, and the normal force exerted by the ground that acts on the orange as it is squished. We can find the squishing time from the displacement of the center-of-mass of the orange as it stops and its average speed during this period of (assumed) constant acceleration. We can use the impulse-momentum theorem to find the average force exerted by the ground on the orange as it slowed to a stop.



(*a*) Express the stopping time for the orange in terms of its average speed and the distance traveled by its center of mass:

$$\Delta t = \frac{d}{v_{av}} = \frac{R - \frac{1}{2}w}{v_{av}} \quad (1)$$

where *w* is the thickness of the pancaked orange.

In order to find v_{av}, apply the conservation of mechanical energy to the free-fall portion of the orange's motion:

$$K_f - K_i + U_{g,f} - U_{g,i} = 0$$

or, because $K_i = 0$,

$$K_f + U_{g,f} - U_{g,i} = 0$$

Substituting for K_f, $U_{g,f}$, and $U_{g,i}$ yields:

$$\tfrac{1}{2}mv^2 + mgd - mg(h-d) = 0$$

Solving for *v* yields:

$$v = \sqrt{2gh\left(1 - 2\frac{d}{h}\right)}$$

or, because $d \ll h$,

$$v \approx \sqrt{2gh}$$

Assuming constant acceleration as the orange squishes:

$$v_{av}=\tfrac{1}{2}v=\tfrac{1}{2}\sqrt{2gh}$$

Substituting for v_{av} in equation (1) and simplifying yields:

$$\Delta t=\frac{2\left(R-\frac{1}{2}w\right)}{\sqrt{2gh}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t=\frac{2\left[2.0\text{ cm}-\frac{1}{2}(0.50\text{ cm})\right]}{\sqrt{2\left(9.81\text{ m/s}^2\right)(35\text{ m})}}=1.336\times10^{-3}\text{ s}=\boxed{1.3\text{ ms}}$$

(*b*) Apply the impulse-momentum theorem to the squishing orange to obtain:

$$\left|\vec{\boldsymbol{F}}_{av}\Delta \boldsymbol{t}\right|=\left|\Delta\vec{\boldsymbol{p}}\right|=\left|\vec{\boldsymbol{p}}_f-\vec{\boldsymbol{p}}_i\right|$$

or, because $p_f = 0$,

$$F_{av}\Delta t=p_i\Rightarrow F_{av}=\frac{p_i}{\Delta t}=\frac{mv_i}{\Delta t}$$

In Part (*a*) we showed that $v_i=v\approx\sqrt{2gh}$. Therefore:

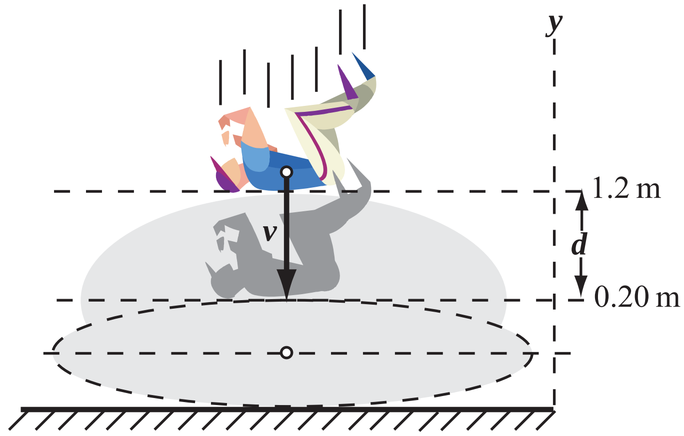
$$F_{av}=\frac{m\sqrt{2gh}}{\Delta t}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av}=\frac{(0.34\text{ kg})\sqrt{2\left(9.81\text{ m/s}^2\right)(35\text{ m})}}{1.336\times10^{-3}\text{ s}}\approx\boxed{6.7\text{ kN}}$$

51 •• The pole-vault landing pad at an Olympic competition contains what is essentially a bag of air that compresses from its "resting" height of 1.2 m down to 0.20 m as the vaulter is slowed to a stop. (*a*) What is the time interval during which a vaulter who has just cleared a height of 6.40 m slows to a stop? (*b*) What is the time interval if instead the vaulter is brought to rest by a 20 cm layer of sawdust that compresses to 5.0 cm when he lands? (*c*) Qualitatively discuss the difference in the average force the vaulter experiences from the two different landing pads. That is, which landing pad exerts the least force on the vaulter and why?

Picture the Problem The pictorial representation shows the vaulter just before impact on the landing pad after falling from a height of 6.40 m. In order to determine the time interval during which the vaulter stops, we have to know his momentum change and the average net force acting on him. With knowledge of these quantities, we can use the impulse-momentum equation, $F_{net}\Delta t=\Delta p$. We can determine the average force by noting that as the vaulter comes to a stop on the landing pad, work is done on him by the airbag.



(*a*) Use the impulse-momentum theorem to relate the stopping time to the average force acting on the vaulter:

$$F_{\text{net}}\Delta t = \Delta p$$

Use the work-kinetic energy theorem to obtain:

$$W_{\text{net}} = F_{\text{net}}d = \left(F_{\text{airbag}} - mg\right)d = \Delta K$$

where d is the distance the vaulter moves while being decelerated.

Substituting for F_{net} in equation (1) yields:

$$\left(F_{\text{airbag}} - mg\right)\Delta t = \Delta p$$

or

$$\frac{\Delta K}{d}\Delta t = \Delta p$$

Solve for Δt to obtain:

$$\Delta t = \frac{d\Delta p}{\Delta K} = \frac{d\left(p_{\text{f}} - p_{\text{i}}\right)}{K_{\text{f}} - K_{\text{i}}}$$

or, because $K_{\text{f}} = p_{\text{f}} = 0$,

$$\Delta t = \frac{d\left(-p_{\text{i}}\right)}{-K_{\text{i}}} = \frac{p_{\text{i}}d}{K_{\text{i}}}$$

Use $K = \dfrac{p^2}{2m}$ to obtain:

$$\Delta t = \frac{2mp_{\text{i}}d}{p_{\text{i}}^2} = \frac{2md}{p_{\text{i}}} = \frac{2md}{\sqrt{2mK_{\text{i}}}}$$

Rationalizing the denominator of this expression and simplifying yields:

$$\Delta t = d\sqrt{\frac{2m}{K_{\text{i}}}} \qquad (1)$$

K_i is equal to the change in the gravitational potential energy of the vaulter as he falls a distance Δy before hitting the airbag:

$$K_i = mg\Delta y$$

Substituting for K_i in equation (1) and simplifying yields:

$$\Delta t = d\sqrt{\frac{2}{g\Delta y}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = (1.2\text{ m} - 0.2\text{ m})\sqrt{\frac{2}{(9.81\text{ m/s}^2)(6.4\text{ m} - 1.2\text{ m})}} = 0.198\text{ s} = \boxed{0.20\text{ s}}$$

(*b*) In this case, d_1 = 20 cm, d_2 = 5.0 cm. Substitute numerical values and evaluate Δt:

$$\Delta t = (0.20\text{ m} - 0.05\text{ m})\sqrt{\frac{2}{(9.81\text{ m/s}^2)(6.4\text{ m} - 0.20\text{ m})}} = 27.2\text{ ms} = \boxed{27\text{ ms}}$$

(*c*) Because the collision time is much shorter for the sawdust landing, the average force exerted on the vaulter by the airbag is much less than the average force the sawdust exerts on him.

52 ••• Great limestone caverns have been formed by dripping water. (*a*) If water droplets of 0.030 mL fall from a height of 5.0 m at a rate of 10 droplets per minute, what is the average force exerted on the limestone floor by the droplets of water during a 1.0-min period? (Assume the water does not accumulate on the floor.) (*b*) Compare this force to the weight of one water droplet.

Picture the Problem The average force exerted on the limestone by the droplets of water equals the rate at which momentum is being delivered to the floor. We're given the number of droplets that arrive per minute and can use conservation of mechanical energy to determine their velocity as they reach the floor.

(*a*) Letting N represent the number of droplets that fall in a time interval Δt, relate F_{av} to the change in the droplet's momentum:

$$F_{av} = \frac{\Delta p_{\text{droplets}}}{\Delta t} = \frac{N}{\Delta t}mv \qquad (1)$$

where v is their speed after falling 5.0 m from rest.

The mass of the droplets is the product of their density and volume:

$$m = \rho V$$

Letting $U_g = 0$ at the point of impact of the droplets, use conservation of mechanical energy to relate their speed at impact to their fall distance:

$$\Delta K + \Delta U = 0$$

or

$$K_f - K_i + U_f - U_i = 0$$

Because $K_i = U_f = 0$:

$$\tfrac{1}{2}mv^2 - mgh = 0 \Rightarrow v = \sqrt{2gh}$$

Substitute for m and v in equation (1) to obtain:

$$F_{av} = \frac{N}{\Delta t}\rho V\sqrt{2gh}$$

Substitute numerical values and evaluate F_{av}:

$$F_{av} = 4.95\times10^{-5}\text{ N} = \boxed{50\ \mu\text{N}}$$

(*b*) Express the ratio of the weight of a droplet to F_{av}:

$$\frac{w}{F_{av}} = \frac{mg}{F_{av}}$$

Substitute numerical values and evaluate w/F_{av}:

$$\frac{w}{F_{av}} = \frac{(3\times10^{-5}\text{ kg})(9.81\text{m/s}^2)}{4.95\times10^{-5}\text{ N}} \approx \boxed{6}$$

Collisions in One Dimension

53 • [SSM] A 2000-kg car traveling to the right at 30 m/s is chasing a second car of the same mass that is traveling in the same direction at 10 m/s. (*a*) If the two cars collide and stick together, what is their speed just after the collision? (*b*) What fraction of the initial kinetic energy of the cars is lost during this collision? Where does it go?

Picture the Problem We can apply conservation of linear momentum to this perfectly inelastic collision to find the after-collision speed of the two cars. The ratio of the transformed kinetic energy to kinetic energy before the collision is the fraction of kinetic energy lost in the collision.

(*a*) Letting V be the velocity of the two cars after their collision, apply conservation of linear momentum to their perfectly inelastic collision:

$$p_{\text{initial}} = p_{\text{final}}$$

or

$$mv_1 + mv_2 = (m+m)V \Rightarrow V = \frac{v_1 + v_2}{2}$$

Substitute numerical values and evaluate V:

$$V=\frac{30\,\text{m/s}+10\,\text{m/s}}{2}=\boxed{20\,\text{m/s}}$$

(*b*) The ratio of the kinetic energy that is lost to the kinetic energy of the two cars before the collision is:

$$\frac{\Delta K}{K_{\text{initial}}}=\frac{K_{\text{final}}-K_{\text{initial}}}{K_{\text{initial}}}=\frac{K_{\text{final}}}{K_{\text{initial}}}-1$$

Substitute for the kinetic energies and simplify to obtain:

$$\frac{\Delta K}{K_{\text{initial}}}=\frac{\frac{1}{2}(2m)V^2}{\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2}-1$$
$$=\frac{2V^2}{v_1^2+v_2^2}-1$$

Substitute numerical values and evaluate $\Delta K/K_{\text{initial}}$:

$$\frac{\Delta K}{K_{\text{initial}}}=\frac{2(20\,\text{m/s})^2}{(30\,\text{m/s})^2+(10\,\text{m/s})^2}-1$$
$$=-0.20$$

Twenty percent of the initial kinetic energy is transformed into thermal energy, acoustical energy, and the deformation of the materials from which the car is constructed.

54 • An 85-kg running back moving at 7.0 m/s makes a perfectly inelastic head-on collision with a 105-kg linebacker who is initially at rest. What is the speed of the players just after their collision?

Picture the Problem We can apply conservation of linear momentum to this perfectly inelastic collision to find the after-collision speed of the two players.

Letting the subscript 1 refer to the running back and the subscript 2 refer to the linebacker, apply conservation of momentum to their perfectly inelastic collision:

$$p_{\text{i}}=p_{\text{f}}$$

or

$$m_1v_1=(m_1+m_2)V\Rightarrow V=\frac{m_1}{m_1+m_2}v_1$$

Substitute numerical values and evaluate V:

$$V=\frac{85\,\text{kg}}{85\,\text{kg}+105\,\text{kg}}(7.0\,\text{m/s})=\boxed{3.1\,\text{m/s}}$$

55 • A 5.0-kg object with a speed of 4.0 m/s collides head-on with a 10-kg object moving toward it with a speed of 3.0 m/s. The 10-kg object stops dead after the collision. (*a*) What is the post-collision speed of the 5.0-kg object? (*b*) Is the collision elastic?

Picture the Problem We can apply conservation of linear momentum to this collision to find the post-collision speed of the 5.0-kg object. Let the direction the 5.0-kg object is moving before the collision be the positive direction. We can decide whether the collision was elastic by examining the initial and final kinetic energies of the system.

(*a*) Letting the subscript 5 refer to the 5.0-kg object and the subscript 10 refer to the 10-kg object, apply conservation of momentum to obtain:

$$p_{\mathrm{i}} = p_{\mathrm{f}}$$

or

$$m_5 v_{\mathrm{i},5} - m_{10} v_{\mathrm{i},10} = m_5 v_{\mathrm{f},5}$$

Solve for $v_{\mathrm{f},5}$:

$$v_{\mathrm{f},5} = \frac{m_5 v_{\mathrm{i},5} - m_{10} v_{\mathrm{i},10}}{m_5}$$

Substitute numerical values and evaluate $v_{\mathrm{f},5}$:

$$v_{\mathrm{f},5} = \frac{(5.0\,\mathrm{kg})(4.0\,\mathrm{m/s}) - (10\,\mathrm{kg})(3.0\,\mathrm{m/s})}{5\,\mathrm{kg}} = \boxed{-2.0\,\mathrm{m/s}}$$

where the minus sign means that the 5.0-kg object is moving to the left after the collision.

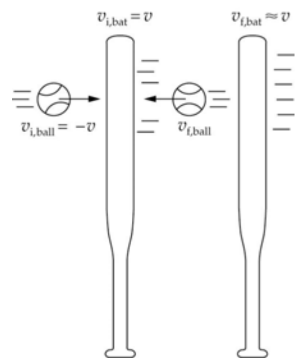
(*b*) Evaluate ΔK for the collision:

$$\Delta K = K_{\mathrm{f}} - K_{\mathrm{i}} = \tfrac{1}{2}(5.0\,\mathrm{kg})(2.0\,\mathrm{m/s})^2 - \left[\tfrac{1}{2}(5.0\,\mathrm{kg})(4.0\,\mathrm{m/s})^2 + \tfrac{1}{2}(10\,\mathrm{kg})(3.0\,\mathrm{m/s})^2\right]$$
$$= -75\,\mathrm{J}$$

Because $\Delta K \neq 0$, the collision was not elastic.

56 • A small superball of mass *m* moves with speed *v* to the right toward a much more massive bat that is moving to the left with speed *v*. Find the speed of the ball after it makes an elastic head-on collision with the bat.

Picture the Problem The pictorial representation shows the ball and bat just before and just after their collision. Take the direction the bat is moving to be the positive direction. Because the collision is elastic, we can equate the speeds of recession and approach, with the approximation that $v_{\text{i,bat}} \approx v_{\text{f,bat}}$ to find $v_{\text{f,ball}}$.



Express the speed of approach of the bat and ball:

$$v_{\text{f,bat}} - v_{\text{f,ball}} = -(v_{\text{i,bat}} - v_{\text{i,ball}})$$

Because the mass of the bat is much greater than that of the ball:

$$v_{\text{i,bat}} \approx v_{\text{f,bat}}$$

Substitute to obtain:

$$v_{\text{f,bat}} - v_{\text{f,ball}} = -(v_{\text{f,bat}} - v_{\text{i,ball}})$$

Solve for and evaluate $v_{\text{f,ball}}$:

$$\begin{aligned} v_{\text{f,ball}} &= v_{\text{f,bat}} + (v_{\text{f,bat}} - v_{\text{i,ball}}) \\ &= -v_{\text{i,ball}} + 2v_{\text{f,bat}} = v + 2v \\ &= \boxed{3v} \end{aligned}$$

57 •• A proton that has a mass m and is moving at 300 m/s undergoes a head-on elastic collision with a stationary carbon nucleus of mass $12m$. Find the velocity of the proton and the carbon nucleus after the collision.

Picture the Problem Let the direction the proton is moving before the collision be the $+x$ direction. We can use both conservation of momentum and conservation of mechanical energy to obtain an expression for velocities of the proton and the carbon nucleus after the collision.

Use conservation of linear momentum to obtain one relation for the final velocities:

$$m_{\text{p}} v_{\text{p,i}} = m_{\text{p}} v_{\text{p,f}} + m_{\text{nuc}} v_{\text{nuc,f}} \quad (1)$$

Use conservation of mechanical energy to set the velocity of recession equal to the negative of the velocity of approach:

$$v_{\text{nuc,f}} - v_{\text{p,f}} = -(v_{\text{nuc,i}} - v_{\text{p,i}}) = v_{\text{p,i}} \quad (2)$$

To eliminate $v_{nuc,f}$, solve equation (2) for $v_{nuc,f}$, and substitute the result in equation (1):

$$v_{nuc,f} = v_{p,i} + v_{p,f}$$

and

$$m_p v_{p,i} = m_p v_{p,f} + m_{nuc}(v_{p,i} + v_{p,f})$$

Solving for $v_{p,f}$ yields:

$$v_{p,f} = \frac{m_p - m_{nuc}}{m_p + m_{nuc}} v_{p,i}$$

Substituting for m_p and m_{nuc} and simplifying yields:

$$v_{p,f} = \frac{m - 12m}{m + 12m} v_{p,i} = -\frac{11}{13} v_{p,i}$$

Substitute the numerical value of $v_{p,i}$ and evaluate $v_{p,f}$:

$$v_{p,f} = -\frac{11}{13}(300\,\text{m/s}) = \boxed{-254\,\text{m/s}}$$

where the minus sign tells us that the velocity of the proton was reversed in the collision.

Solving equation (2) for $v_{nuc,f}$ yields:

$$v_{nuc,f} = v_{p,i} + v_{p,f}$$

Substitute numerical values and evaluate $v_{nuc,f}$:

$$v_{nuc,f} = 300\text{ m/s} - 254\text{ m/s} = \boxed{46\text{ m/s, forward}}$$

58 •• A 3.0-kg block moving at 4.0 m/s makes a head-on elastic collision with a stationary block of mass 2.0 kg. Use conservation of momentum and the fact that the relative speed of recession equals the relative speed of approach to find the velocity of each block after the collision. Check your answer by calculating the initial and final kinetic energies of each block.

Picture the Problem We can use conservation of momentum and the definition of an elastic collision to obtain two equations in v_{2f} and v_{3f} that we can solve simultaneously.

Use conservation of momentum to obtain one relation for the final velocities:

$$m_3 v_{3i} = m_3 v_{3f} + m_2 v_{2f} \quad (1)$$

Use conservation of mechanical energy to set the speed of recession equal to the negative of the speed of approach:

$$v_{2f} - v_{3f} = -(v_{2i} - v_{3i}) = v_{3i} \quad (2)$$

Solve equation (2) for v_{3f}, substitute in equation (1) to eliminate v_{3f}, and solve for v_{2f} to obtain:

$$v_{2f} = \frac{2m_3 v_{3i}}{m_2 + m_3}$$

Substitute numerical values and evaluate v_{2f}:

$$v_{2f} = \frac{2(3.0\,\text{kg})(4.0\,\text{m/s})}{2.0\,\text{kg} + 3.0\,\text{kg}} = 4.80\,\text{m/s}$$
$$= \boxed{4.8\,\text{m/s}}$$

Use equation (2) to find v_{3f}:

$$v_{3f} = v_{2f} - v_{3i} = 4.8\,\text{m/s} - 4.0\,\text{m/s}$$
$$= \boxed{0.8\,\text{m/s}}$$

Evaluate K_i and K_f:

$$K_i = K_{3i} = \tfrac{1}{2} m_3 v_{3i}^2 = \tfrac{1}{2}(3.0\,\text{kg})(4.0\,\text{m/s})^2 = 24\,\text{J}$$

and

$$K_f = K_{3f} + K_{2f} = \tfrac{1}{2} m_3 v_{3f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$
$$= \tfrac{1}{2}(3.0\,\text{kg})(0.8\,\text{m/s})^2 + \tfrac{1}{2}(2.0\,\text{kg})(4.8\,\text{m/s})^2 = 24\,\text{J}$$

Because $K_i = K_f$, we can conclude that the values obtained for v_{2f} and v_{3f} are consistent with the collision having been elastic.

59 •• A block of mass m_1 = 2.0 kg slides along a frictionless table with a speed of 10 m/s. Directly in front of it, and moving in the same direction with a speed of 3.0 m/s, is a block of mass m_2 = 5.0 kg. A massless spring that has a force constant k = 1120 N/m is attached to the second block as in Figure 8-47. (*a*) What is the velocity of the center of mass of the system? (*b*) During the collision, the spring is compressed by a maximum amount Δx. What is the value of Δx? (*c*) The blocks will eventually separate again. What are the final velocities of the two blocks measured in the reference frame of the table, after they separate?

Picture the Problem We can find the velocity of the center of mass from the definition of the total momentum of the system. We'll use conservation of energy to find the maximum compression of the spring and express the initial (i.e., before collision) and final (i.e., at separation) velocities. Finally, we'll transform the velocities from the center-of-mass frame of reference to the table frame of reference. Let the direction the block of mass m_1 is moving be the +x direction.

(*a*) Use the definition of the total momentum of a system to relate the initial momenta to the velocity of the center of mass:

$$\vec{P} = \sum_i m_i \vec{v}_i = M\vec{v}_{\text{cm}}$$

or

$$m_1\vec{v}_{1\text{i}} + m_2\vec{v}_{2\text{i}} = (m_1 + m_2)\vec{v}_{\text{cm}}$$

Solving for v_{cm} gives:

$$\vec{v}_{\text{cm}} = \frac{m_1}{m_1 + m_2}\vec{v}_{1\text{i}} + \frac{m_2}{m_1 + m_2}\vec{v}_{2\text{i}}$$

Substitute numerical values and evaluate $\vec{v}_{\text{cm}}$:

$$\vec{v}_{\text{cm}} = \left(\frac{2.0\text{ kg}}{2.0\text{ kg} + 5.0\text{ kg}}\right)(10\text{ m/s})\hat{i} + \left(\frac{3.0\text{ kg}}{2.0\text{ kg} + 5.0\text{ kg}}\right)(3.0\text{ m/s})\hat{i}$$
$$= (5.00\text{ m/s})\hat{i} = \boxed{(5.0\text{ m/s})\hat{i}}$$

(*b*) Find the kinetic energy of the system at maximum compression ($u_1 = u_2 = 0$):

$$K = K_{\text{cm}} = \tfrac{1}{2}Mv_{\text{cm}}^2$$
$$= \tfrac{1}{2}(7.0\text{ kg})(5.00\text{ m/s})^2 = 87.5\text{ J}$$

Use conservation of mechanical energy to relate the kinetic energy of the system to the potential energy stored in the spring at maximum compression:

$$\Delta K + \Delta U_{\text{s}} = 0$$

or

$$K_{\text{f}} - K_{\text{i}} + U_{\text{sf}} - U_{\text{si}} = 0$$

Because $K_{\text{f}} = K_{\text{cm}}$ and $U_{\text{si}} = 0$:

$$K_{\text{cm}} - K_{\text{i}} + \tfrac{1}{2}k(\Delta x)^2 = 0$$

Solving for Δx yields:

$$\Delta x = \sqrt{\frac{2(K_{\text{i}} - K_{\text{cm}})}{k}} = \sqrt{\frac{2\left[\frac{1}{2}m_1v_{1\text{i}}^2 + \frac{1}{2}m_2v_{2\text{i}}^2 - K_{\text{cm}}\right]}{k}} = \sqrt{\frac{m_1v_{1\text{i}}^2 + m_2v_{2\text{i}}^2 - 2K_{\text{cm}}}{k}}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \sqrt{\frac{(2.0\text{ kg})(10\text{ m/s})^2 + (5.0\text{ kg})(3.0\text{ m/s})^2}{1120\text{ N/m}} - \frac{2(87.5\text{ J})}{1120\text{ N/m}}\Bigg]} = \boxed{0.25\text{ m}}$$

(*c*) Find u_{1i}, u_{2i}, and u_{1f} for this elastic collision:

$$u_{1i} = v_{1i} - v_{cm} = 10\,\text{m/s} - 5\,\text{m/s} = 5\,\text{m/s},$$
$$u_{2i} = v_{2i} - v_{cm} = 3\,\text{m/s} - 5\,\text{m/s} = -2\,\text{m/s},$$
and
$$u_{1f} = v_{1f} - v_{cm} = 0 - 5\,\text{m/s} = -5\,\text{m/s}$$

Use conservation of mechanical energy to set the speed of recession equal to the negative of the speed of approach and solve for u_{2f}:

$$u_{2f} - u_{1f} = -(u_{2i} - u_{1i})$$
and
$$u_{2f} = -u_{2i} + u_{1i} + u_{1f}$$

Substitute numerical values and evaluate u_{2f}:

$$u_{2f} = -(-2.0\,\text{m/s}) + 5.0\,\text{m/s} - 5.0\,\text{m/s} = 2.0\,\text{m/s}$$

Transform u_{1f} and u_{2f} to the table frame of reference:

$$v_{1f} = u_{1f} + v_{cm} = -5.0\,\text{m/s} + 5.0\,\text{m/s} = 0$$
and
$$v_{2f} = u_{2f} + v_{cm} = 2.0\,\text{m/s} + 5.0\,\text{m/s} = 7.0\,\text{m/s}$$

Express the velocities of the blocks to obtain:

$$\vec{v}_{1f} = \boxed{0} \text{ and } \vec{v}_{2f} = \boxed{(7.0\,\text{m/s})\hat{i}}$$

60 •• A bullet of mass *m* is fired vertically from below into a thin horizontal sheet of plywood of mass *M* that is initially at rest, supported by a thin sheet of paper (Figure 8-48). The bullet punches through the plywood, which rises to a height *H* above the paper before falling back down. The bullet continues rising to a height *h* above the paper. (*a*) Express the upward velocity of the bullet and the plywood immediately after the bullet exits the plywood in terms of *h* and *H*. (*b*) What is the speed of the bullet? (*c*) What is the mechanical energy of the system before and after the inelastic collision? (*d*) How much mechanical energy is dissipated during the collision?

Picture the Problem Let the system include Earth, the bullet, and the sheet of plywood. Then $W_{ext} = 0$. Choose the zero of gravitational potential energy to be where the bullet enters the plywood. We can apply both conservation of energy and conservation of momentum to obtain the various physical quantities called for in this problem.

(*a*) Use conservation of mechanical energy after the bullet exits the sheet of plywood to relate its exit speed to the height to which it rises:

$$\Delta K + \Delta U = 0$$
or, because $K_f = U_i = 0$,
$$-\tfrac{1}{2}mv_m^2 + mgh = 0 \Rightarrow v_m = \boxed{\sqrt{2gh}}$$

Proceed similarly to relate the initial velocity of the plywood to the height to which it rises:

$$v_M = \boxed{\sqrt{2gH}}$$

(*b*) Apply conservation of momentum to the collision of the bullet and the sheet of plywood:

$$\vec{p}_i = \vec{p}_f$$

or

$$mv_{mi} = mv_m + Mv_M$$

Substitute for v_m and v_M and solve for v_{mi}:

$$v_{mi} = \boxed{\sqrt{2gh} + \frac{M}{m}\sqrt{2gH}}$$

(*c*) Express the initial mechanical energy of the system (i.e., just before the collision):

$$E_i = \tfrac{1}{2}mv_{mi}^2$$

$$= \boxed{mg\left[h + \frac{2M}{m}\sqrt{hH} + \left(\frac{M}{m}\right)^2 H\right]}$$

Express the final mechanical energy of the system (that is, when the bullet and block have reached their maximum heights):

$$E_f = mgh + MgH = \boxed{g(mh + MH)}$$

(*d*) Use the work-energy theorem with $W_{ext} = 0$ to find the energy dissipated by friction in the inelastic collision:

$$E_f - E_i + W_{friction} = 0$$

and

$$W_{friction} = E_i - E_f$$

$$= \boxed{gMH\left[2\sqrt{\frac{h}{H}} + \frac{M}{m} - 1\right]}$$

61 •• A proton of mass *m* is moving with initial speed v_0 directly toward the center of an α particle of mass 4*m*, which is initially at rest. Both particles carry positive charge, so they repel each other. (The repulsive forces are sufficient to prevent the two particles from coming into direct contact.) Find the speed v_α of the α particle (*a*) when the distance between the two particles is a minimum, and (*b*) later when the two particles are far apart.

Picture the Problem We can find the speed of the center of mass from the definition of the total momentum of the system. We'll use conservation of energy to find the speeds of the particles when their separation is a minimum and when they are far apart.

(*a*) Noting that when the distance between the two particles is a minimum, both move at the same speed, namely v_{cm}, use the definition of the total momentum of a system to relate the initial momenta to the speed of the center of mass:

$$\vec{P} = \sum_i m_i \vec{v}_i = M\vec{v}_{cm}$$

or

$$m_p v_p = (m_p + m_\alpha)v_{cm}\,.$$

Solve for v_{cm} to obtain:

$$v_{cm} = v_\alpha = \frac{m_p v_p + m_\alpha v_\alpha}{m_p + m_\alpha}$$

Further simplification yields:

$$v_{cm} = v_\alpha = \frac{mv_0 + 0}{m + 4m} = \boxed{0.2\,v_0}$$

(*b*) Use conservation of linear momentum to obtain one relation for the final speeds:

$$m_p v_0 = m_p v_{pf} + m_\alpha v_{\alpha f} \qquad (1)$$

Use conservation of mechanical energy to set the speed of recession equal to the negative of the speed of approach:

$$v_{pf} - v_{\alpha f} = -(v_{pi} - v_{\alpha i}) = -v_{pi} \qquad (2)$$

Solve equation (2) for v_{pf}, substitute in equation (1) to eliminate v_{pf}, and solve for $v_{\alpha f}$:

$$v_{\alpha f} = \frac{2m_p v_0}{m_p + m_\alpha}$$

Simplifying further yields:

$$v_{\alpha f} = \frac{2mv_0}{m + 4m} = \boxed{0.4\,v_0}$$

62 •• An electron collides elastically with a hydrogen atom initially at rest. Assume all the motion occurs along a straight line. What fraction of the electron's initial kinetic energy is transferred to the atom? (Take the mass of the hydrogen atom to be 1840 times the mass of an electron.)

Picture the Problem Let the numeral 1 denote the electron and the numeral 2 the hydrogen atom. We can find the final velocity of the electron and, hence, the fraction of its initial kinetic energy that is transferred to the atom, by transforming to the center-of-mass reference frame, calculating the post-collision velocity of the electron, and then transforming back to the laboratory frame of reference.

Express f, the fraction of the electron's initial kinetic energy that is transferred to the atom:

$$f = \frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{\frac{1}{2}m_1 v_{1f}^2}{\frac{1}{2}m_1 v_{1i}^2} = 1 - \left(\frac{v_{1f}}{v_{1i}}\right)^2 \quad (1)$$

Find the velocity of the center of mass:

$$v_{cm} = \frac{m_1 v_{1i}}{m_1 + m_2}$$

or, because $m_2 = 1840m_1$,

$$v_{cm} = \frac{m_1 v_{1i}}{m_1 + 1840m_1} = \frac{1}{1841}v_{1i}$$

Find the initial velocity of the electron in the center-of-mass reference frame:

$$u_{1i} = v_{1i} - v_{cm} = v_{1i} - \frac{1}{1841}v_{1i} = \left(1 - \frac{1}{1841}\right)v_{1i}$$

Find the post-collision velocity of the electron in the center-of-mass reference frame by reversing its velocity:

$$u_{1f} = -u_{1i} = \left(\frac{1}{1841} - 1\right)v_{1i}$$

To find the final velocity of the electron in the original frame, add v_{cm} to its final velocity in the center-of-mass reference frame:

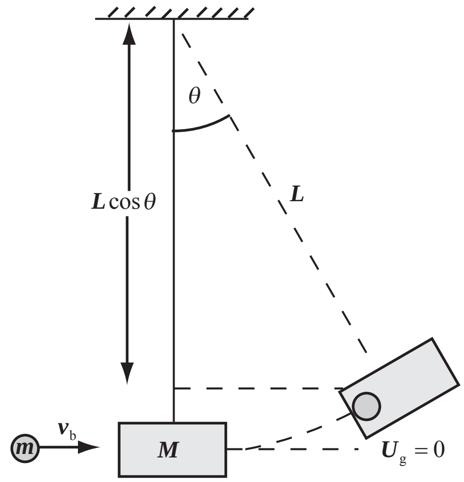
$$v_{1f} = u_{1f} + v_{cm} = \left(\frac{2}{1841} - 1\right)v_{1i}$$

Substituting in equation (1) and simplifying yields:

$$f = 1 - \left(\frac{2}{1841} - 1\right)^2 = \boxed{0.217\%}$$

63 •• [SSM] A16-g bullet is fired into the bob of a 1.5-kg ballistic pendulum (Figure 8-18). When the bob is at its maximum height, the strings make an angle of 60° with the vertical. The pendulum strings are 2.3 m long. Find the speed of the bullet prior to impact.

Picture the Problem The pictorial representation shows the bullet about to imbed itself in the bob of the ballistic pendulum and then, later, when the bob plus bullet have risen to their maximum height. We can use conservation of momentum during the collision to relate the speed of the bullet to the initial speed of the bob plus bullet (V). The initial kinetic energy of the bob plus bullet is transformed into gravitational potential energy when they reach their maximum height. Hence we apply conservation of mechanical energy to relate V to the angle through which the bullet plus bob swings and then solve the momentum and energy equations simultaneously for the speed of the bullet.



Use conservation of momentum to relate the speed of the bullet just before impact to the initial speed of the bob-bullet:

$$mv_b = (m+M)V \Rightarrow v_b = \left(1+\frac{M}{m}\right)V \quad (1)$$

Use conservation of energy to relate the initial kinetic energy of the bob-bullet to their final potential energy:

$$\Delta K + \Delta U = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}(m+M)V^2 + (m+M)gL(1-\cos\theta) = 0$$

Solving for V yields:

$$V = \sqrt{2gL(1-\cos\theta)}$$

Substitute for V in equation (1) to obtain:

$$v_b = \left(1+\frac{M}{m}\right)\sqrt{2gL(1-\cos\theta)}$$

Substitute numerical values and evaluate v_b:

$$v_b = \left(1+\frac{1.5\,\text{kg}}{0.016\,\text{kg}}\right)\sqrt{2(9.81\,\text{m/s}^2)(2.3\,\text{m})(1-\cos 60°)} = \boxed{0.45\,\text{km/s}}$$

64 •• Show that in a one-dimensional elastic collision, if the mass and velocity of object 1 are m_1 and v_{1i}, and if the mass and velocity of object 2 are m_2 and v_{2i}, then their final velocities v_{1f} and v_{2f} are given by

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i} + \frac{2m_2}{m_1 + m_2}v_{2i} \text{ and } v_{2f} = \frac{2m_1}{m_1 + m_2}v_{1i} + \frac{m_2 - m_1}{m_1 + m_2}v_{2i}.$$

Picture the Problem We can apply conservation of linear momentum and the definition of an elastic collision to obtain equations relating the initial and final velocities of the colliding objects that we can solve for v_{1f} and v_{2f}.

Apply conservation of momentum to the elastic collision of the particles to obtain:

$$m_1v_{1f} + m_2v_{2f} = m_1v_{1i} + m_2v_{2i} \quad (1)$$

Use conservation of mechanical energy to set the speed of recession equal to the negative of the speed of approach:

$$v_{2f} - v_{1f} = -(v_{2i} - v_{1i})$$

or

$$v_{1f} - v_{2f} = v_{2i} - v_{1i} \quad (2)$$

Multiply equation (2) by m_2 and add it to equation (1) to obtain:

$$(m_1 + m_2)v_{1f} = (m_1 - m_2)v_{1i} + 2m_2v_{2i}$$

Solve for v_{1f} to obtain:

$$v_{1f} = \boxed{\frac{m_1 - m_2}{m_1 + m_2}v_{1i} + \frac{2m_2}{m_1 + m_2}v_{2i}}$$

Multiply equation (2) by m_1 and subtract it from equation (1) to obtain:

$$(m_1 + m_2)v_{2f} = (m_2 - m_1)v_{2i} + 2m_1v_{1i}$$

Solve for v_{2f} to obtain:

$$v_{2f} = \boxed{\frac{2m_1}{m_1 + m_2}v_{1i} + \frac{m_2 - m_1}{m_1 + m_2}v_{2i}}$$

Remarks: Note that the velocities satisfy the condition that $\boldsymbol{v_{2f} - v_{1f} = -(v_{2i} - v_{1i})}$. This verifies that the speed of recession equals the speed of approach.

65 •• Investigate the plausibility of the results of Problem 64 by calculating the final velocities in the following limits: (*a*) When the two masses are equal, show that the particles "swap" velocities: $v_{1f} = v_{2i}$ i and $v_{2f} = v_{1i}$ (*b*) If $m_2 >> m_1$, and $v_{2i} = 0$, show that $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$. (*c*) If $m_1 >> m_2$, and $v_{2i} = 0$, show that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$.

Picture the Problem As in this problem, Problem 74 involves an elastic, one-dimensional collision between two objects. Both solutions involve using the conservation of momentum equation $m_1v_{1f} + m_2v_{2f} = m_1v_{1i} + m_2v_{2i}$ and the elastic collision equation $v_{1f} - v_{2f} = v_{2i} - v_{1i}$. In Part (*a*) we can simply set the masses equal to each other and substitute in the equations in Problem 64 to show that the particles "swap" velocities. In Part (*b*) we can divide the numerator and denominator of the equations in Problem 64 by m_2 and use the condition that $m_2 >> m_1$ to show that $v_{1f} \approx -v_{1i} + 2v_{2i}$ and $v_{2f} \approx v_{2i}$.

(*a*) From Problem 64 we have:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \quad (1)$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \quad (2)$$

Set $m_1 = m_2 = m$ to obtain:

$$v_{1f} = \frac{2m}{m+m} v_{2i} = \boxed{v_{2i}}$$

and

$$v_{2f} = \frac{2m}{m+m} v_{1i} = \boxed{v_{1i}}$$

(*b*) Divide the numerator and denominator of both terms in equation (1) by m_2 to obtain:

$$v_{1f} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} v_{1i} + \frac{2}{\frac{m_1}{m_2} + 1} v_{2i}$$

If $m_2 >> m_1$ and $v_{2i} = 0$:

$$v_{1f} \approx \boxed{-v_{1i}}$$

Divide the numerator and denominator of both terms in equation (2) by m_2 to obtain:

$$v_{2f} = \frac{2\frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1} v_{1i} + \frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1} v_{2i}$$

If $m_2 >> m_1$:

$$v_{2f} \approx \boxed{v_{2i}}$$

(*c*) Divide the numerator and denominator of equation (1) by m_1 to obtain:

$$v_{1f} = \frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}} v_{1i} + \frac{2\frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}} v_{2i}$$

If $m_1 >> m_2$ and $v_{2i} = 0$:

$$v_{1f} \approx \boxed{v_{1i}}$$

Divide the numerator and denominator of equation (2) by m_1 to obtain:

$$v_{2f} = \frac{2}{1 + \frac{m_2}{m_1}} v_{1i} + \frac{\frac{m_2}{m_1} - 1}{1 + \frac{m_2}{m_1}} v_{2i}$$

If $m_1 >> m_2$ and $v_{2i} = 0$:

$$v_{2f} \approx \boxed{2v_{1i}}$$

Remarks: Note that, in both parts of this problem, the velocities satisfy the condition that $v_{2f} - v_{1f} = -(v_{2i} - v_{1i})$. This verifies that the speed of recession equals the speed of approach.

66 •• A bullet of mass m_1 is fired horizontally with a speed v_0 into the bob of a ballistic pendulum of mass m_2. The pendulum consists of a bob attached to one end of a very light rod of length L. The rod is free to rotate about a horizontal axis through its other end. The bullet is stopped in the bob. Find the minimum v_0 such that the bob will swing through a complete circle.

Picture the Problem Choose $U_g = 0$ at the bob's equilibrium position. Momentum is conserved in the collision of the bullet with bob and the initial kinetic energy of the bob plus bullet is transformed into gravitational potential energy as it swings up to the top of the circle. If the bullet plus bob just makes it to the top of the circle with zero speed, it will swing through a complete circle.

Use conservation of momentum to relate the speed of the bullet just before impact to the initial speed of the bob plus bullet:

$$m_1 v_0 - (m_1 + m_2)V = 0$$

Solve for the speed of the bullet to obtain:

$$v_0 = \left(1 + \frac{m_2}{m_1}\right)V \qquad (1)$$

Use conservation of mechanical energy to relate the initial kinetic energy of the bob plus bullet to their potential energy at the top of the circle:

$$\Delta K + \Delta U = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f:

$$-\tfrac{1}{2}(m_1 + m_2)V^2 + (m_1 + m_2)g(2L) = 0$$

Solving for V yields:

$$V = 2\sqrt{gL}$$

Substitute for V in equation (1) and simplify to obtain:

$$v_0 = \boxed{2\left(1 + \frac{m_2}{m_1}\right)\sqrt{gL}}$$

67 •• A bullet of mass m_1 is fired horizontally with a speed v into the bob of a ballistic pendulum of mass m_2 (Figure 8-19). Find the maximum height h attained by the bob if the bullet passes through the bob and emerges with a speed $v/3$.

Picture the Problem Choose $U_g = 0$ at the equilibrium position of the ballistic pendulum. Momentum is conserved in the collision of the bullet with the bob and kinetic energy is transformed into gravitational potential energy as the bob swings up to its maximum height.

Letting V represent the initial speed of the bob as it begins its upward swing, use conservation of momentum to relate this speed to the speeds of the bullet just before and after its collision with the bob:

$$m_1 v = m_1\left(\tfrac{1}{3}v\right) + m_2 V \Rightarrow V = \frac{2m_1}{3m_2}v$$

Use conservation of energy to relate the initial kinetic energy of the bob to its potential energy at its maximum height:

$$\Delta K + \Delta U = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f:

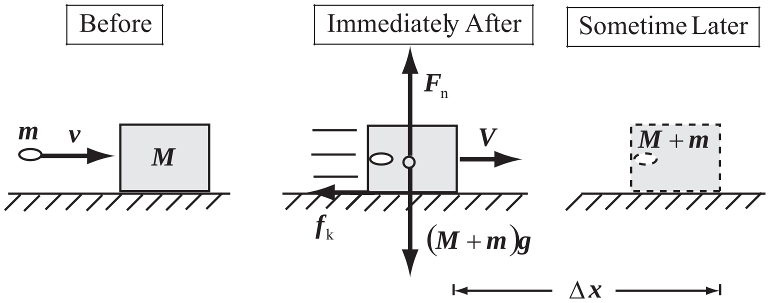
$$-\tfrac{1}{2}m_2 V^2 + m_2 g h = 0 \Rightarrow h = \frac{V^2}{2g}$$

Substitute for V in the expression for h and simplify to obtain:

$$h = \frac{\left(\dfrac{2m_1}{3m_2}v\right)^2}{2g} = \boxed{\frac{2m_1^2 v^2}{9m_2^2 g}}$$

68 •• A heavy wooden block rests on a flat table and a high-speed bullet is fired horizontally into the block, the bullet stopping in it. How far will the block slide before coming to a stop? The mass of the bullet is 10.5 g, the mass of the block is 10.5 kg, the bullet's impact speed is 750 m/s, and the coefficient of kinetic friction between the block and the table is 0.220. (Assume that the bullet does not cause the block to spin.)

Picture the Problem Let the mass of the bullet be m, that of the wooden block M, the pre-collision speed of the bullet v, and the post-collision speed of the block+bullet be V. We can use conservation of momentum to find the speed of the block with the bullet imbedded in it immediately after their perfectly inelastic collision. We can use Newton's second law to find the acceleration of the sliding block and a constant-acceleration equation to find the distance the block slides.



Using a constant-acceleration equation, relate the speed of the block+bullet just after their collision to their acceleration and displacement before stopping:

$$0 = V^2 + 2a\Delta x \Rightarrow \Delta x = -\frac{V^2}{2a}$$

because the final speed of the block+bullet is zero.

Use conservation of momentum to relate the pre-collision speed of the bullet to the post-collision speed of the block+bullet:

$$mv = (m+M)V \Rightarrow V = \frac{m}{m+M}v$$

Substitute for V in the expression for Δx to obtain:

$$\Delta x = -\frac{1}{2a}\left(\frac{m}{m+M}v\right)^2$$

Apply $\sum \vec{F} = m\vec{a}$ to the block+bullet (see the force diagram above):

$$\sum F_x = -f_k = (m+M)a \qquad (1)$$

and

$$\sum F_y = F_n - (m+M)g = 0 \qquad (2)$$

Use the definition of the coefficient of kinetic friction and equation (2) to obtain:

$$f_k = \mu_k F_n = \mu_k (m+M)g$$

Substituting for f_k in equation (1) yields:

$$-\mu_k(m+M)g = (m+M)a$$

Solve for a to obtain:

$$a = -\mu_k g$$

Substituting for a in the expression for Δx yields:

$$\Delta x = \frac{1}{2\mu_k g}\left(\frac{m}{m+M}v\right)^2$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{1}{2(0.220)(9.81\,\text{m/s}^2)}\left(\frac{0.0105\,\text{kg}}{0.0105\,\text{kg}+10.5\,\text{kg}}(750\,\text{m/s})\right)^2 = \boxed{13.0\,\text{cm}}$$

69 •• **[SSM]** A 0.425-kg ball with a speed of 1.30 m/s rolls across a level surface toward an open 0.327-kg box that is resting on its side. The ball enters the box, and the box (with the ball inside slides across the surface a distance of x = 0.520 m. What is the coefficient of kinetic friction between the box and the table?

Picture the Problem The collision of the ball with the box is perfectly inelastic and we can find the speed of the box-and-ball immediately after their collision by applying conservation of momentum. If we assume that the kinetic friction force is constant, we can use a constant-acceleration equation to find the acceleration of the box and ball combination and the definition of μ_k to find its value.

Using its definition, express the coefficient of kinetic friction of the table:

$$\mu_k = \frac{f_k}{F_n} = \frac{(M+m)|a|}{(M+m)g} = \frac{|a|}{g} \qquad (1)$$

Use conservation of momentum to relate the speed of the ball just before the collision to the speed of the ball+box immediately after the collision:

$$MV = (m+M)v \Rightarrow v = \frac{MV}{m+M} \qquad (2)$$

Use a constant-acceleration equation to relate the sliding distance of the ball+box to its initial and final velocities and its acceleration:

$$v_f^2 = v_i^2 + 2a\Delta x$$

or, because $v_f = 0$ and $v_i = v$,

$$0 = v^2 + 2a\Delta x \Rightarrow a = -\frac{v^2}{2\Delta x}$$

Substitute for a in equation (1) to obtain:

$$\mu_k = \frac{v^2}{2g\Delta x}$$

Use equation (2) to eliminate v:

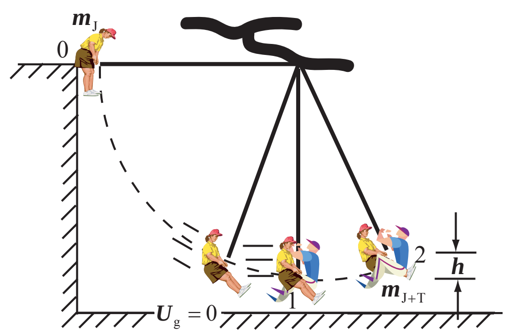
$$\mu_k = \frac{1}{2g\Delta x}\left(\frac{MV}{m+M}\right)^2 = \frac{1}{2g\Delta x}\left(\frac{V}{\frac{m}{M}+1}\right)^2$$

Substitute numerical values and evaluate μ_k:

$$\mu_k = \frac{1}{2(9.81\,\text{m/s}^2)(0.520\,\text{m})}\left(\frac{1.30\,\text{m/s}}{\frac{0.327\,\text{kg}}{0.425\,\text{kg}}+1}\right)^2 = \boxed{0.0529}$$

70 •• Tarzan is in the path of a pack of stampeding elephants when Jane swings in to the rescue on a rope vine, hauling him off to safety. The length of the vine is 25 m, and Jane starts her swing with the rope horizontal. If Jane's mass is 54 kg, and Tarzan's is 82 kg, to what height above the ground will the pair swing after she rescues him? (Assume that the rope is vertical when she grabs him.)

Picture the Problem Jane's collision with Tarzan is a perfectly inelastic collision. We can find her speed v_1 just before she grabs Tarzan from conservation of energy and their speed V just after she grabs him from conservation of momentum. Their kinetic energy just after their collision will be transformed into gravitational potential energy when they have reached their greatest height h.



Use conservation of energy to relate the potential energy of Jane and Tarzan at their highest point (2) to their kinetic energy immediately after Jane grabbed Tarzan:

$$U_2 = K_1$$

or

$$m_{J+T}gh = \tfrac{1}{2}m_{J+T}V^2 \Rightarrow h = \frac{V^2}{2g} \qquad (1)$$

Apply conservation of linear momentum to relate Jane's velocity just before she collides with Tarzan to their velocity just after their perfectly inelastic collision:

$$m_J v_1 - m_{J+T}V = 0 \Rightarrow V = \frac{m_J}{m_{J+T}}v_1 \quad (2)$$

Apply conservation of mechanical energy to relate Jane's kinetic energy at 1 to her potential energy at 0:

$$K_1 = U_0$$

or

$$\tfrac{1}{2}m_J v_1^2 = m_J gL \Rightarrow v_1 = \sqrt{2gL}$$

Substitute for v_1 in equation (2) to obtain:

$$V = \frac{m_J}{m_{J+T}}\sqrt{2gL}$$

Substitute for V in equation (1) and simplify:

$$h = \frac{1}{2g}\left(\frac{m_J}{m_{J+T}}\right)^2 2gL = \left(\frac{m_J}{m_{J+T}}\right)^2 L$$

Substitute numerical values and evaluate h:

$$h = \left(\frac{54\,\text{kg}}{54\,\text{kg} + 82\,\text{kg}}\right)^2 (25\,\text{m}) = \boxed{3.9\,\text{m}}$$

71 •• [SSM] Scientists estimate that the meteorite responsible for the creation of Barringer Meteorite Crater in Arizona weighed roughly 2.72×10^5 tonne (1 tonne = 1000 kg) and was traveling at a speed of 17.9 km/s. Take Earth's orbital speed to be about 30.0 km/s. (*a*) What should the direction of impact be if Earth's orbital speed is to be changed by the maximum possible amount? (*b*) Assuming the condition of collision in Part (*a*), estimate the maximum percentage change in Earth's orbital speed as a result of this collision. (*c*) What mass asteroid, having a speed equal to Earth's orbital speed, would be necessary to change Earth's orbital speed by 1.00%?

Picture the Problem Let the system include Earth and the asteroid. Choose a coordinate system in which the direction of Earth's orbital speed is the +*x* direction. We can apply conservation of linear momentum to the perfectly inelastic collision of Earth and the asteroid to find the percentage change in Earth's orbital speed as well as the mass of an asteroid that would change Earth's orbital speed by 1.00%. Note that the following solution neglects the increase in Earth's orbital speed due to the gravitational pull of the asteroid during descent.

(*a*) The meteorite should impact Earth along a line exactly opposite Earth's orbital velocity vector.

(*b*) Express the percentage change in Earth's orbital speed as a result of the collision:

$$\left|\frac{\Delta v}{v_{\text{Earth}}}\right| = \left|\frac{v_{\text{Earth}} - v_{\text{f}}}{v_{\text{Earth}}}\right| = \left|1 - \frac{v_{\text{f}}}{v_{\text{Earth}}}\right| \qquad (1)$$

where v_{f} is Earth's orbital speed after the collision.

Apply the conservation of linear momentum to the system to obtain:

$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_{\text{f}} - \vec{\boldsymbol{p}}_{\text{i}} = 0$$

or, because the asteroid and Earth are moving horizontally,

$$p_{\text{f},x} - p_{\text{i},x} = 0$$

Because the collision is perfectly inelastic:

$$(m_{\text{Earth}} + m_{\text{asteroid}})v_{\text{f}} - (m_{\text{Earth}}v_{\text{Earth}} - m_{\text{asteroid}}v_{\text{asteroid}}) = 0$$

Solving for v_f yields:

$$v_f = \frac{m_{\text{Earth}}v_{\text{Earth}} - m_{\text{asteroid}}v_{\text{asteroid}}}{m_{\text{Earth}} + m_{\text{asteroid}}} = \frac{m_{\text{Earth}}v_{\text{Earth}}}{m_{\text{Earth}} + m_{\text{asteroid}}} - \frac{m_{\text{asteroid}}v_{\text{asteroid}}}{m_{\text{Earth}} + m_{\text{asteroid}}}$$

$$= \frac{v_{\text{Earth}}}{1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}} - \frac{\frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}}{1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}}$$

Because $m_{\text{asteroid}} << m_{\text{Earth}}$:

$$v_f \approx v_{\text{Earth}} - \frac{\frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}}{1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}} = v_{\text{Earth}} - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}\left(1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}\right)^{-1}$$

Expanding $\left(1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}\right)^{-1}$ binomially yields:

$$\left(1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}\right)^{-1} = 1 - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}} + \text{higher order terms}$$

Substitute for $\left(1 + \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}\right)^{-1}$ in the expression for v_f to obtain:

$$v_f \approx v_{\text{Earth}} - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}\left(1 - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}\right)$$
$$\approx v_{\text{Earth}} - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}$$

Substitute for v_f in equation (1) to obtain:

$$\left|\frac{\Delta v}{v_{\text{Earth}}}\right| = \left|1 - \frac{v_{\text{Earth}} - \frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}}{v_{\text{Earth}}}\right| = \left|\frac{\frac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}}{v_{\text{Earth}}}\right|$$

Using data found in the appendices of your text or given in the problem statement, substitute numerical values and evaluate $\left|\frac{\Delta v}{v_{\text{Earth}}}\right|$:

$$\left|\frac{\Delta v}{v_{\text{Earth}}}\right| = \left|\frac{\dfrac{\left(2.72\times10^{5}\ \text{tonne}\times\dfrac{10^{3}\ \text{kg}}{1\ \text{tonne}}\right)(17.9\ \text{km/s})}{5.98\times10^{24}\ \text{kg}}}{30.0\ \text{km/s}}\right| = \boxed{2.71\times10^{-15}\%}$$

(*c*) If the asteroid is to change Earth's orbital speed by one percent:

$$\frac{\dfrac{m_{\text{asteroid}}}{m_{\text{Earth}}}v_{\text{asteroid}}}{v_{\text{Earth}}} = \frac{1}{100}$$

Solve for m_{asteroid} to obtain:

$$m_{\text{asteroid}} = \frac{v_{\text{Earth}}m_{\text{Earth}}}{100v_{\text{asteroid}}}$$

Substitute numerical values and evaluate m_{asteroid}:

$$m_{\text{asteroid}} = \frac{(30.0\ \text{km/s})(5.98\times10^{24}\ \text{kg})}{100(17.9\ \text{km/s})} = \boxed{1.00\times10^{23}\ \text{kg}}$$

Remarks: The mass of this asteroid is approximately that of the moon!

72 ••• William Tell shoots an apple from his son's head. The speed of the 125-g arrow just before it strikes the apple is 25.0 m/s, and at the time of impact it is traveling horizontally. If the arrow sticks in the apple and the arrow/apple combination strikes the ground 8.50 m behind the son's feet, how massive was the apple? Assume the son is 1.85 m tall.

Picture the Problem Let the system include Earth, the apple, and the arrow. Choose a coordinate system in which the direction the arrow is traveling before imbedding itself in the apple is the +*x* direction. We can apply conservation of linear momentum to express the mass of the apple in terms of the speed of the arrow-apple combination just after the collision and then use constant-acceleration equations to find this post-collision speed.

Apply conservation of linear momentum to the system to obtain:

$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_\mathrm{f} - \vec{\boldsymbol{p}}_\mathrm{i} = 0$$

or, because the arrow and apple are moving horizontally,

$$p_{\mathrm{f},x} - p_{\mathrm{i},x} = 0$$

Because the collision is perfectly inelastic (the arrow is imbedded in the apple):

$$(m_\text{arrow} + m_\text{apple})v_x - m_\text{arrow}v_\text{i,arrow} = 0$$

Solving for m_apple yields:

$$m_\text{apple} = m_\text{arrow}\left(\frac{v_\text{i,arrow}}{v_x} - 1\right) \qquad (1)$$

Using constant-acceleration equations, express the horizontal and vertical displacements of the apple-arrow after their collision:

$$\Delta x = v_x \Delta t \qquad (2)$$

and

$$\Delta y = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta y}{g}}$$

Substituting for Δt in equation (2) yields:

$$\Delta x = v_x\sqrt{\frac{2\Delta y}{g}} \Rightarrow v_x = \Delta x\sqrt{\frac{g}{2\Delta y}}$$

Substitute for v_x in equation (1) to obtain:

$$m_\text{apple} = m_\text{arrow}\left(\frac{v_\text{i,arrow}}{\Delta x\sqrt{\dfrac{g}{2\Delta y}}} - 1\right)$$

Substitute numerical values and evaluate m_apple:

$$m_\text{apple} = (0.125\ \text{kg})\left(\frac{25.0\ \text{m/s}}{(8.50\ \text{m})\sqrt{\dfrac{9.81\ \text{m/s}^2}{2(1.85\ \text{m})}}} - 1\right) = \boxed{101\ \text{g}}$$

Explosions and Radioactive Decay

73 •• **[SSM]** The beryllium isotope ^{8}Be is unstable and decays into two α particles ($m_\alpha = 6.64 \times 10^{-27}$ kg) and releases 1.5×10^{-14} J of energy. Determine the velocities of the two α particles that arise from the decay of a ^{8}Be nucleus at rest, assuming that all the energy appears as kinetic energy of the particles.

Picture the Problem This nuclear reaction is $^{4}\text{Be} \rightarrow 2\alpha + 1.5 \times 10^{-14}$ J. In order to conserve momentum, the alpha particles will have move in opposite directions with the same velocities. We'll use conservation of energy to find their speeds.

Letting E represent the energy released in the reaction, express conservation of energy for this process:

$$2K_\alpha = 2\left(\tfrac{1}{2}m_\alpha v_\alpha^2\right) = E \Rightarrow v_\alpha = \sqrt{\frac{E}{m_\alpha}}$$

Substitute numerical values and evaluate v_α:

$$v_\alpha = \sqrt{\frac{1.5\times10^{-14}\ \text{J}}{6.64\times10^{-27}\ \text{kg}}} = \boxed{1.5\times10^{6}\ \text{m/s}}$$

74 •• The light isotope of lithium, ^{5}Li, is unstable and breaks up spontaneously into a proton and an α particle. During this process, 3.15×10^{-13} J of energy are released, appearing as the kinetic energy of the two decay products. Determine the velocities of the proton and the α particle that arise from the decay of a ^{5}Li nucleus at rest. (*Note*: The masses of the proton and alpha particle are $m_\text{p} = 1.67 \times 10^{-27}$ kg and $m_\alpha = 4m_\text{p} = 6.64 \times 10^{-27}$ kg.)

Picture the Problem This nuclear reaction is $^{5}\text{Li} \rightarrow \alpha + \text{p} + 3.15 \times 10^{-13}$ J. To conserve momentum, the alpha particle and proton must move in opposite directions. We'll apply both conservation of energy and conservation of momentum to find the speeds of the proton and alpha particle.

Use conservation of momentum in this process to express the alpha particle's speed in terms of the proton's:

$$p_\text{i} = p_\text{f} = 0$$

and

$$0 = m_\text{p}v_\text{p} - m_\alpha v_\alpha$$

Solve for v_α and substitute for m_α to obtain:

$$v_\alpha = \frac{m_\text{p}}{m_\alpha}v_\text{p} = \frac{m_\text{p}}{4m_\text{p}}v_\text{p} = \tfrac{1}{4}v_\text{p}$$

Letting E represent the energy released in the reaction, apply conservation of energy to the process:

$$K_\text{p} + K_\alpha = E$$

or

$$\tfrac{1}{2}m_\text{p}v_\text{p}^2 + \tfrac{1}{2}m_\alpha v_\alpha^2 = E$$

Substitute for v_α:

$$\tfrac{1}{2}m_\text{p}v_\text{p}^2 + \tfrac{1}{2}m_\alpha\left(\tfrac{1}{4}v_\text{p}\right)^2 = E$$

Solve for v_p and substitute for m_α to obtain:

$$v_\text{p} = \sqrt{\frac{32E}{16m_\text{p} + m_\alpha}}$$

Substitute numerical values and evaluate v_p:

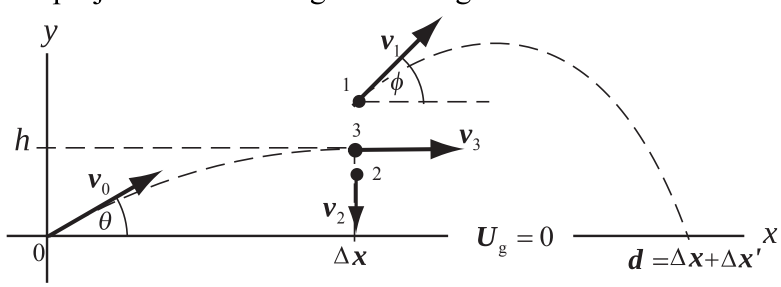
$$v_p = \sqrt{\frac{32(3.15\times10^{-13}\text{ J})}{16(1.67\times10^{-27}\text{ kg}) + 6.64\times10^{-27}\text{ kg}}} = \boxed{1.74\times10^{7}\text{ m/s}}$$

Use the relationship between v_p and v_α to obtain v_α:

$$v_\alpha = \tfrac{1}{4}v_p = \tfrac{1}{4}(1.74\times10^{7}\text{ m/s}) = \boxed{4.34\times10^{6}\text{ m/s}}$$

75 ••• A 3.00-kg projectile is fired with an initial speed of 120 m/s at an angle of 30.0º with the horizontal. At the top of its trajectory, the projectile explodes into two fragments of masses 1.00 kg and 2.00 kg. At 3.60 s after the explosion the 2.00-kg fragment lands on the ground directly below the point of explosion. (*a*) Determine the velocity of the 1.00-kg fragment immediately after the explosion. (*b*) Find the distance between the point of firing and the point at which the 1.00-kg fragment strikes the ground. (*c*) Determine the energy released in the explosion.

Picture the Problem The pictorial representation shows the projectile at its maximum elevation and is moving horizontally. It also shows the two fragments resulting from the explosion. We'll choose the system to include the projectile and Earth so that no external forces act to change the momentum of the system during the explosion. With this choice of system we can also use conservation of energy to determine the elevation of the projectile when it explodes. We'll also find it useful to use constant-acceleration equations in our description of the motion of the projectile and its fragments. Neglect air resistance.



(*a*) Use conservation of linear momentum to relate the velocity of the projectile before its explosion to the velocities of its two parts after the explosion:

$$\vec{p}_i = \vec{p}_f$$

$$m_3\vec{v}_3 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$m_3v_3\hat{i} = m_1v_{x1}\hat{i} + m_1v_{y1}\hat{j} - m_2v_{y2}\hat{j}$$

The only way this equality can hold is if the x and y components are equal:

$m_3 v_3 = m_1 v_{x1}$
and
$m_1 v_{y1} = m_2 v_{y2}$

Express v_3 in terms of v_0 and substitute for the masses to obtain:

$$v_{x1} = 3v_3 = 3v_0 \cos\theta = 3(120\text{ m/s})\cos 30.0° = 312\text{ m/s}$$
and
$$v_{y1} = 2v_{y2} \qquad (1)$$

Using a constant-acceleration equation with the downward direction positive, relate v_{y2} to the time it takes the 2.00-kg fragment to hit the ground:

$$h = v_{y2}\Delta t + \tfrac{1}{2}g(\Delta t)^2$$

Solving for v_{y2} gives:

$$v_{y2} = \frac{h - \frac{1}{2}g(\Delta t)^2}{\Delta t} \qquad (2)$$

With $U_g = 0$ at the launch site, apply conservation of energy to the climb of the projectile to its maximum elevation h:

$\Delta K + \Delta U = 0$
Because $K_f = U_i = 0$, $-K_i + U_f = 0$
or
$-\frac{1}{2}m_3 v_{y0}^2 + m_3 g h = 0$

Solving for h yields:

$$h = \frac{v_{y0}^2}{2g} = \frac{(v_0 \sin\theta)^2}{2g}$$

Substitute numerical values and evaluate h:

$$h = \frac{[(120\text{ m/s})\sin 30.0°]^2}{2(9.81\text{ m/s}^2)} = 183.5\text{ m}$$

Substitute numerical values in equation (2) and evaluate v_{y2}:

$$v_{y2} = \frac{183.5\text{ m} - \frac{1}{2}(9.81\text{ m/s}^2)(3.60\text{ s})^2}{3.60\text{ s}} = 33.31\text{ m/s}$$

Substitute numerical values in equation (1) and evaluate v_{y1}:

$$v_{y1} = 2(33.31\text{ m/s}) = 66.62\text{ m/s}$$

Express $\vec{v}_1$ in vector form:

$$\vec{v}_1 = v_{x1}\hat{i} + v_{y1}\hat{j} = \boxed{(312\text{ m/s})\hat{i} + (66.6\text{ m/s})\hat{j}}$$

(*b*) Express the total distance *d* traveled by the 1.00-kg fragment:

$$d = \Delta x + \Delta x' \qquad (3)$$

Relate Δx to v_0 and the time-to-explosion:

$$\Delta x = (v_0 \cos\theta)(\Delta t_{\text{exp}}) \qquad (4)$$

Using a constant-acceleration equation, express Δt_{exp}:

$$\Delta t_{\text{exp}} = \frac{v_{y0}}{g} = \frac{v_0 \sin\theta}{g}$$

Substitute numerical values and evaluate Δt_{exp}:

$$\Delta t_{\text{exp}} = \frac{(120\,\text{m/s})\sin 30.0°}{9.81\,\text{m/s}^2} = 6.116\,\text{s}$$

Substitute in equation (4) and evaluate Δx:

$$\Delta x = (120\,\text{m/s})(\cos 30.0°)(6.116\,\text{s}) = 635.6\,\text{m}$$

Relate the distance traveled by the 1.00-kg fragment after the explosion to the time it takes it to reach the ground:

$$\Delta x' = v_{x1}\Delta t'$$

Using a constant-acceleration equation, relate the time $\Delta t'$ for the 1.00-kg fragment to reach the ground to its initial speed in the *y* direction and the distance to the ground:

$$\Delta y = v_{y1}\Delta t' - \tfrac{1}{2}g(\Delta t')^2$$

Substitute to obtain the quadratic equation:

$$(\Delta t')^2 - (13.6\,\text{s})\Delta t' - 37.4\,\text{s}^2 = 0$$

Solve the quadratic equation to find $\Delta t'$:

$$\Delta t' = 15.945\text{ s}$$

Substitute in equation (3) and evaluate *d*:

$$\begin{aligned} d &= \Delta x + \Delta x' = \Delta x + v_{x1}\Delta t' \\ &= 635.6\,\text{m} + (312\,\text{m/s})(15.945\,\text{s}) \\ &= \boxed{5.6\,\text{km}} \end{aligned}$$

(*c*) Express the energy released in the explosion:

$$E_{\text{exp}} = \Delta K = K_{\text{f}} - K_{\text{i}} \qquad (5)$$

Find the kinetic energy of the fragments after the explosion:

$$K_f = K_1 + K_2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$
$$= \tfrac{1}{2}(1.00\,\text{kg})\left[(312\,\text{m/s})^2 + (66.6\,\text{m/s})^2\right]$$
$$+ \tfrac{1}{2}(2.00\,\text{kg})(33.3\,\text{m/s})^2$$
$$= 52.0\,\text{kJ}$$

Find the kinetic energy of the projectile before the explosion:

$$K_i = \tfrac{1}{2}m_3v_3^2 = \tfrac{1}{2}m_3(v_0\cos\theta)^2$$
$$= \tfrac{1}{2}(3.00\,\text{kg})\left[(120\,\text{m/s})\cos 30°\right]^2$$
$$= 16.2\,\text{kJ}$$

Substitute in equation (5) to determine the energy released in the explosion:

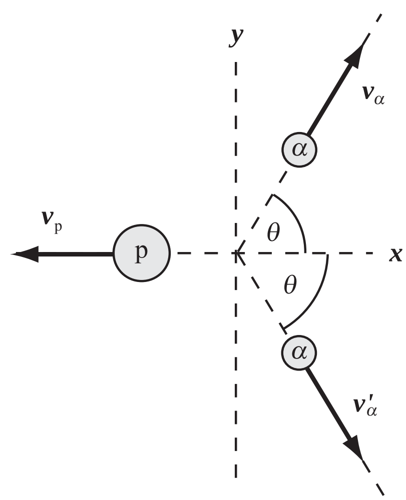
$$E_{\text{exp}} = K_f - K_i = 52.0\,\text{kJ} - 16.2\,\text{kJ}$$
$$= \boxed{35.8\,\text{kJ}}$$

76 ••• The boron isotope ^{9}B is unstable and disintegrates into a proton and two α particles. The total energy released as kinetic energy of the decay products is 4.4×10^{-14} J. After one such event, with the ^{9}B nucleus at rest prior to decay, the velocity of the proton is measured as 6.0×10^6 m/s. If the two α particles have equal energies, find the magnitude and the direction of their velocities with respect to the direction of the proton.

Picture the Problem This nuclear reaction is

$$^9\text{B} \rightarrow 2\alpha + \text{p} + 4.4\times10^{-14}\ \text{J}.$$

Assume that the proton moves in the $-x$ direction as shown in the diagram. The sum of the kinetic energies of the decay products equals the energy released in the decay. We'll use conservation of momentum to find the angle between the velocities of the proton and the alpha particles. Note that $v_\alpha = v'_\alpha$.



Express the energy released to the kinetic energies of the decay products:

$$K_p + 2K_\alpha = E_{\text{rel}}$$

or

$$\tfrac{1}{2}m_pv_p^2 + 2\left(\tfrac{1}{2}m_\alpha v_\alpha^2\right) = E_{\text{rel}}$$

Solving for v_α yields:

$$v_\alpha = \sqrt{\frac{E_{\text{rel}} - \frac{1}{2}m_p v_p^2}{m_\alpha}}$$

Substitute numerical values and evaluate v_α:

$$v_\alpha = \sqrt{\frac{4.4\times10^{-14}\ \text{J}}{6.64\times10^{-27}\ \text{kg}} - \frac{\frac{1}{2}(1.67\times10^{-27}\ \text{kg})(6.0\times10^{6}\ \text{m/s})^2}{6.64\times10^{-27}\ \text{kg}}} = 1.44\times10^{6}\ \text{m/s}$$
$$= \boxed{1.4\times10^{6}\ \text{m/s}}$$

Given that the boron isotope was at rest prior to the decay, use conservation of momentum to relate the momenta of the decay products:

$\vec{\boldsymbol{p}}_\text{f} = \vec{\boldsymbol{p}}_\text{i} = 0$
or, because $p_{x\text{f}} = 0$,
$2(m_\alpha v_\alpha \cos\theta) - m_p v_p = 0$

Substituting for m_α to obtain:

$$2(4m_p v_\alpha \cos\theta) - m_p v_p = 0$$

Solving for θ yields:

$$\theta = \cos^{-1}\left[\frac{v_p}{8v_\alpha}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left[\frac{6.0\times10^{6}\ \text{m/s}}{8(1.44\times10^{6}\ \text{m/s})}\right] = \pm59°$$

Let θ' equal the angle the velocities of the alpha particles make with that of the proton:

$$\theta' = \pm(180° - 59°) = \boxed{\pm121°}$$

Coefficient of Restitution

77 • [SSM] You are in charge of measuring the coefficient of restitution for a new alloy of steel. You convince your engineering team to accomplish this task by simply dropping a small ball onto a plate, both the ball and plate made from the experimental alloy. If the ball is dropped from a height of 3.0 m and rebounds to a height of 2.5 m, what is the coefficient of restitution?

Picture the Problem The coefficient of restitution is defined as the ratio of the speed of recession to the speed of approach. These speeds can be determined from the height from which the ball was dropped and the height to which it rebounded by using conservation of mechanical energy.

Use its definition to relate the coefficient of restitution to the speeds of approach and recession:

$$e = \frac{v_{rec}}{v_{app}}$$

Letting $U_g = 0$ at the surface of the steel plate, apply conservation of energy to obtain:

$$\Delta K + \Delta U = 0$$
or, because $K_i = U_f = 0$,
$$K_f - U_i = 0$$

Substituting for K_f and U_i yields:

$$\tfrac{1}{2}mv_{app}^2 - mgh_{app} = 0$$

Solving for v_{app} yields:

$$v_{app} = \sqrt{2gh_{app}}$$

In like manner, show that:

$$v_{rec} = \sqrt{2gh_{rec}}$$

Substitute in the equation for e to obtain:

$$e = \frac{\sqrt{2gh_{rec}}}{\sqrt{2gh_{app}}} = \sqrt{\frac{h_{rec}}{h_{app}}}$$

Substitute numerical values and evaluate e:

$$e = \sqrt{\frac{2.5\,\text{m}}{3.0\,\text{m}}} = \boxed{0.91}$$

78 • According to the official racquetball rules, to be acceptable for tournament play, a ball must bounce to a height of between 173 and 183 cm when dropped from a height of 254 cm at room temperature. What is the acceptable range of values for the coefficient of restitution for the racquetball–floor system?

Picture the Problem The coefficient of restitution is defined as the ratio of the speed of recession to the speed of approach. These speeds can be determined from the heights from which the ball was dropped and the height to which it rebounded by using conservation of mechanical energy.

Use its definition to relate the coefficient of restitution to the speeds of approach and recession:

$$e = \frac{v_{rec}}{v_{app}}$$

Letting $U_g = 0$ at the surface of the steel plate, the mechanical energy of the ball-Earth system is:

$$\Delta K + \Delta U = 0$$
or, because $K_i = U_f = 0$,
$$K_f - U_i = 0$$

Substituting for K_f and U_i yields:

$$\tfrac{1}{2}mv_{app}^2 - mgh_{app} = 0 \Rightarrow v_{app} = \sqrt{2gh_{app}}$$

In like manner, show that: $v_{\text{rec}} = \sqrt{2gh_{\text{rec}}}$

Substitute in the equation for e to obtain:

$$e = \frac{\sqrt{2gh_{\text{rec}}}}{\sqrt{2gh_{\text{app}}}} = \sqrt{\frac{h_{\text{rec}}}{h_{\text{app}}}}$$

Substitute numerical values and evaluate e_{min}:

$$e_{\text{min}} = \sqrt{\frac{173\,\text{cm}}{254\,\text{cm}}} = 0.825$$

Substitute numerical values and evaluate e_{max}:

$$e_{\text{max}} = \sqrt{\frac{183\,\text{cm}}{254\,\text{cm}}} = 0.849$$

and $\boxed{0.825 \le e \le 0.849}$

79 •• A ball bounces to 80 percent of its original height. (*a*) What fraction of its mechanical energy is lost each time it bounces? (*b*) What is the coefficient of restitution of the ball–floor system?

Picture the Problem Because the rebound kinetic energy is proportional to the rebound height, the percentage of mechanical energy lost in one bounce can be inferred from knowledge of the rebound height. The coefficient of restitution is defined as the ratio of the speed of recession to the speed of approach. These speeds can be determined from the heights from which an object was dropped and the height to which it rebounded by using conservation of mechanical energy.

(*a*) We know, because the mechanical energy of the ball-Earth system is constant, that the kinetic energy of an object dropped from a given height h is proportional to h. If, for each bounce of the ball, $h_{\text{rec}} = 0.80h_{\text{app}}$, 20% of its mechanical energy is lost.

(*b*) Use its definition to relate the coefficient of restitution to the speeds of approach and recession:

$$e = \frac{v_{\text{rec}}}{v_{\text{app}}} \qquad (1)$$

Letting $U_g = 0$ at the surface from which the ball is rebounding, the mechanical energy of the ball is:

$\Delta K + \Delta U = 0$
or, because $K_i = U_f = 0$,
$K_f - U_i = 0$

Substituting for K_f and U_i yields:

$$\tfrac{1}{2}mv_{\text{app}}^2 - mgh_{\text{app}} = 0 \Rightarrow v_{\text{app}} = \sqrt{2gh_{\text{app}}}$$

In like manner, show that: $v_{\text{rec}} = \sqrt{2gh_{\text{rec}}}$

Substitute for v_{rec} and v_{app} in equation (1) to obtain:

$$e = \frac{\sqrt{2gh_{rec}}}{\sqrt{2gh_{app}}} = \sqrt{\frac{h_{rec}}{h_{app}}}$$

Substitute for $\frac{h_{rec}}{h_{app}}$ to obtain:

$$e = \sqrt{0.80} = \boxed{0.89}$$

80 •• A 2.0-kg object moving to the right at 6.0 m/s collides head-on with a 4.0-kg object that is initially at rest. After the collision, the 2.0-kg object is moving to the left at 1.0 m/s. (*a*) Find the velocity of the 4.0-kg object after the collision. (*b*) Find the energy lost in the collision. (*c*) What is the coefficient of restitution for these objects?

Picture the Problem Let the numerals 2 and 4 refer, respectively, to the 2.0-kg object and the 4.0-kg object. Choose a coordinate system in which the direction the 2.0-kg object is moving before the collision is the +*x* direction and let the system consist of Earth, the surface on which the objects slide, and the objects. Then we can use conservation of momentum to find the velocity of the recoiling 4.0-kg object. We can find the energy transformed in the collision by calculating the difference between the pre- and post-collision kinetic energies and find the coefficient of restitution from its definition.

(*a*) Use conservation of linear momentum in one dimension to relate the initial and final momenta of the participants in the collision:

$$\vec{\boldsymbol{p}}_i = \vec{\boldsymbol{p}}_f$$

or

$$m_2v_{2i} = m_4v_{4f} - m_2v_{2f}$$

Solve for the final velocity of the 4.0-kg object:

$$v_{4f} = \frac{m_2v_{2i} + m_2v_{2f}}{m_4}$$

Substitute numerical values and evaluate v_{4f}:

$$v_{4f} = \frac{(2.0\,\text{kg})(6.0\,\text{m/s} + 1.0\,\text{m/s})}{4.0\,\text{kg}} = 3.50\,\text{m/s} = \boxed{3.5\,\text{m/s}}$$

(*b*) Express the energy lost in terms of the kinetic energies before and after the collision:

$$\begin{aligned} E_{lost} &= K_i - K_f \\ &= \tfrac{1}{2}m_2v_{2i}^2 - \left(\tfrac{1}{2}m_2v_{2f}^2 + \tfrac{1}{2}m_4v_{4f}^2\right) \\ &= \tfrac{1}{2}\left[m_2\left(v_{2i}^2 - v_{2f}^2\right) - m_4v_{4f}^2\right] \end{aligned}$$

Substitute numerical values and evaluate E_{lost}:

$$E_{lost} = \tfrac{1}{2}\left[\left((2.0\,\text{kg})\left\{(6.0\,\text{m/s})^2 - (1.0\,\text{m/s})^2\right\}\right) - (4.0\,\text{kg})(3.50\,\text{m/s})^2\right] = \boxed{11\,\text{J}}$$

(*c*) From the definition of the coefficient of restitution we have:

$$e = \frac{v_{\text{rec}}}{v_{\text{app}}} = \frac{v_{4\text{f}} - v_{2\text{f}}}{v_{2\text{i}}}$$

Substitute numerical values and evaluate *e*:

$$e = \frac{3.50\,\text{m/s} - (-1.0\,\text{m/s})}{6.0\,\text{m/s}} = \boxed{0.75}$$

81 •• A 2.0-kg block moving to the right with speed of 5.0 m/s collides with a 3.0-kg block that is moving in the same direction at 2.0 m/s, as in Figure 8-49. After the collision, the 3.0-kg block moves to the right at 4.2 m/s. Find (*a*) the velocity of the 2.0-kg block after the collision and (*b*) the coefficient of restitution between the two blocks.

Picture the Problem Let the numeral 2 refer to the 2.0-kg block and the numeral 3 to the 3.0-kg block. Choose a coordinate system in which the direction the blocks are moving before the collision is the +*x* direction and let the system consist of Earth, the surface on which the blocks move, and the blocks. Then we can use conservation of momentum to find the speed of the 2.0-kg block after the collision. We can find the coefficient of restitution from its definition.

(*a*) Use conservation of linear momentum in one dimension to relate the initial and final momenta of the participants in the collision:

$$\vec{p}_{\text{i}} = \vec{p}_{\text{f}}$$

or

$$m_2 v_{2\text{i}} + m_3 v_{3\text{i}} = m_2 v_{2\text{f}} + m_3 v_{3\text{f}}$$

Solve for the final speed of the 2.0-kg object:

$$v_{2\text{f}} = \frac{m_2 v_{2\text{i}} + m_3 v_{3\text{i}} - m_3 v_{3\text{f}}}{m_2}$$

Substitute numerical values and evaluate $v_{2\text{f}}$:

$$v_{2\text{f}} = \frac{(2.0\,\text{kg})(5.0\,\text{m/s}) + (3.0\,\text{kg})(2.0\,\text{m/s} - 4.2\,\text{m/s})}{2.0\,\text{kg}} = 1.70\,\text{m/s} = \boxed{1.7\,\text{m/s}}$$

(*b*) From the definition of the coefficient of restitution we have:

$$e = \frac{v_{\text{rec}}}{v_{\text{app}}} = \frac{v_{3\text{f}} - v_{2\text{f}}}{v_{2\text{i}} - v_{3\text{i}}}$$

Substitute numerical values and evaluate *e*:

$$e = \frac{4.2\,\text{m/s} - 1.7\,\text{m/s}}{5.0\,\text{m/s} - 2.0\,\text{m/s}} = \boxed{0.83}$$

82 ••• To keep homerun records and distances consistent from year to year, organized baseball randomly checks the coefficient of restitution between new baseballs and wooden surfaces similar to that of an average bat. Suppose you are in charge of making sure that no "juiced" baseballs are produced. (*a*) In a random

test, you find one that when dropped from 2.0 m rebounds 0.25 m. What is the coefficient of restitution for this ball? (*b*) What is the maximum distance home run shot you would expect from this ball, neglecting any effects due to air resistance and making reasonable assumption for bat speeds and incoming pitch speeds? Is this a ″juiced″ ball, a ″normal″ ball, or a ″dead″ ball?

Picture the Problem The coefficient of restitution is defined as the ratio of the speed of recession to the speed of approach. These speeds can be determined from the heights from which the ball was dropped and the height to which it rebounded by using conservation of mechanical energy. We can use the same elevation range equation to find the maximum home run you would expect from the ball with the experimental coefficient of restitution.

(*a*) The coefficient of restitution is the ratio of the speeds of approach and recession:

$$e=\frac{v_{rec}}{v_{app}} \qquad (1)$$

Letting $U_g = 0$ at the surface of from which the ball rebounds, the mechanical energy of the ball-Earth system is:

$$\Delta K+\Delta U=0$$

or, because $K_i = U_f = 0$,

$$K_f-U_i=0$$

Substituting for K_f and U_i yields:

$$\tfrac{1}{2}mv_{app}^2-mgh_{app}=0\Rightarrow v_{app}=\sqrt{2gh_{app}}$$

In like manner, show that:

$$v_{rec}=\sqrt{2gh_{rec}}$$

Substitute for v_{rec} and v_{app} in equation (1) and simplify to obtain:

$$e=\frac{\sqrt{2gh_{rec}}}{\sqrt{2gh_{app}}}=\sqrt{\frac{h_{rec}}{h_{app}}}$$

Substitute numerical values and evaluate e:

$$e=\sqrt{\frac{0.25\,\text{m}}{2.0\,\text{m}}}=0.3536=\boxed{0.35}$$

(*b*) The ″same-elevation″ range equation is:

$$R=\frac{v_{rec}^2\sin 2\theta}{g}=\frac{e^2v_{app}^2\sin 2\theta}{g} \qquad (2)$$

v_{app} is the sum of the speed of the ball and the speed of the bat:

$$v_{app}=v_{ball}+v_{bat}$$

Assuming that the bat travels about 1 m in 0.2 s yields:

$$v_{bat}=\frac{1\,\text{m}}{0.2\,\text{s}}=5\text{ m/s}$$

Assuming that the speed of the baseball thrown by the pitcher is close to 100 mi/h yields:

$$v_{\text{ball}} \approx 45\ \text{m/s}$$

Evaluate v_{app} to obtain:

$$v_{\text{app}} = 45\ \text{m/s} + 5\ \text{m/s} = 50\ \text{m/s}$$

Assuming a 45° launch angle, substitute numerical values in equation (2) and evaluate R:

$$R = \frac{(0.3536)^2(50\ \text{m/s})^2 \sin 2(45°)}{9.81\ \text{m/s}^2} \approx \boxed{32\ \text{m}}$$

Because home runs must travel at least 100 m in modern major league ballparks, this is a ″dead″ ball and should be tossed out.

83 •• [SSM] To make puck handling easy, hockey pucks are kept frozen until they are used in the game. (*a*) Explain why room temperature pucks would be more difficult to handle on the end of a stick than a frozen puck. (*Hint: Hockey pucks are made of rubber*.) (*b*) A room-temperature puck rebounds 15 cm when dropped onto a wooden surface from 100 cm. If a frozen puck has only half the coefficient of restitution of a room-temperature one, predict how high the frozen puck would rebound under the same conditions.

Picture the Problem The coefficient of restitution is defined as the ratio of the speed of recession to the speed of approach. These speeds can be determined from the heights from which the ball was dropped and the height to which it rebounded by using conservation of mechanical energy.

(*a*) At room-temperature rubber will bounce more when it hits a stick than it will at freezing temperatures.

(*b*) The mechanical energy of the rebounding puck is constant:

$$\Delta K + \Delta U = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

If the puck's speed of recession is v_{rec} and it rebounds to a height h, then:

$$-\tfrac{1}{2}mv_{\text{rec}}^2 + mgh = 0 \Rightarrow h = \frac{v_{\text{rec}}^2}{2g}$$

The coefficient of restitution is the ratio of the speeds of approach and recession:

$$e = \frac{v_{\text{rec}}}{v_{\text{app}}} \Rightarrow v_{\text{rec}} = ev_{\text{app}} \qquad (1)$$

Substitute for v_{rec} to obtain: $$h = \frac{e^2 v_{app}^2}{2g} \qquad (2)$$

Letting $U_g = 0$ at the surface of from which the puck rebounds, the mechanical energy of the puck-Earth system is: $\Delta K + \Delta U = 0$

Because $K_i = U_f = 0$,

$K_f - U_i = 0$

Substituting for K_f and U_i yields: $$\tfrac{1}{2} m v_{app}^2 - mgh_{app} = 0 \Rightarrow v_{app} = \sqrt{2gh_{app}}$$

In like manner, show that: $$v_{rec} = \sqrt{2gh_{rec}}$$

Substitute for v_{rec} and v_{app} in equation (1) and simplify to obtain: $$e = \frac{\sqrt{2gh_{rec}}}{\sqrt{2gh_{app}}} = \sqrt{\frac{h_{rec}}{h_{app}}}$$

Substitute numerical values and evaluate $e_{room\ temp}$: $$e_{room\ temp} = \sqrt{\frac{15\,\text{cm}}{100\,\text{cm}}} = 0.3873$$

For the falling puck, v_{app} is given by: $$v_{app} = \sqrt{2gH}$$

where H is the height from which the puck was dropped.

Substituting for v_{app} in equation (2) and simplifying yields: $$h = \frac{2e^2 gH}{2g} = e^2 H \qquad (3)$$

For the room-temperature puck: $$e_{frozen} = \tfrac{1}{2} e_{room\ temp}$$

Substituting for e in equation (3) yields: $$h = \tfrac{1}{4} e_{room\ temp}^2 H$$

Substitute numerical values and evaluate h: $$h = \tfrac{1}{4}(0.3873)^2(100\ \text{cm}) = \boxed{3.8\,\text{cm}}$$

Remarks: The puck that rebounds only 3.8 cm is a much ″deader″ and, therefore, much better puck.

Collisions in More Than One Dimension

84 •• In Section 8-3 it was proven by using geometry that when a particle elastically collides with another particle of equal mass that is initially at rest, the

two post-collision velocities are perpendicular. Here we examine another way of proving this result that illustrates the power of vector notation. (*a*) Given that $\vec{\boldsymbol{A}} = \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}$, square both sides of this equation (obtain the scalar product of each side with itself) to show that $A^2 = B^2 + C^2 + 2\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}}$. (*b*) Let the momentum of the initially moving particle be $\vec{\boldsymbol{P}}$ and the momenta of the particles after the collision be $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$. Write the vector equation for the conservation of linear momentum and square both sides (obtain the dot product of each side with itself). Compare it to the equation gotten from the elastic-collision condition (kinetic energy is conserved) and finally show that these two equations imply that $\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 = 0$.

Picture the Problem We can use the definition of the magnitude of a vector and the definition of the scalar product to establish the result called for in (*a*). In Part (*b*) we can use the result of Part (*a*), the conservation of momentum, and the definition of an elastic collision (kinetic energy is conserved) to show that the particles separate at right angles.

(*a*) Find the dot product of $\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}$ with itself:

$$\left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)\cdot\left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)= B^2 + C^2 + 2\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}}$$

Because $\vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}$:

$$\boldsymbol{A}^2 = \left|\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right|^2 = \left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)\cdot\left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)$$

Substitute for $\left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)\cdot\left(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}\right)$ to obtain:

$$\boxed{A^2 = B^2 + C^2 + 2\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}}}$$

(*b*) Apply conservation of momentum to the collision of the particles:

$$\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 = \vec{\boldsymbol{p}}$$

Form the scalar product of each side of this equation with itself to obtain:

$$\left(\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2\right)\cdot\left(\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2\right) = \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}$$

or

$$p_1^2 + p_2^2 + 2\vec{\boldsymbol{p}}_1\cdot\vec{\boldsymbol{p}}_2 = p^2 \qquad (1)$$

Use the definition of an elastic collision to obtain:

$$\frac{p_1^2}{2m} + \frac{p_2^2}{2m} = \frac{p^2}{2m}$$

or

$$p_1^2 + p_2^2 = p^2 \qquad (2)$$

Subtract equation (1) from equation (2) to obtain:

$$2\vec{\boldsymbol{p}}_1\cdot\vec{\boldsymbol{p}}_2 = 0 \text{ or } \boxed{\vec{\boldsymbol{p}}_1\cdot\vec{\boldsymbol{p}}_2 = 0}$$

That is, the particles move apart along paths that are at right angles to each other.

85 •• During a pool game, the cue ball, which has an initial speed of 5.0 m/s, makes an elastic collision with the eight ball, which is initially at rest. After the collision, the eight ball moves at an angle of 30º to the right of the original direction of the cue ball. Assume that the balls have equal masses. (*a*) Find the direction of motion of the cue ball immediately after the collision. (*b*) Find the speed of each ball immediately after the collision.

Picture the Problem Let the initial direction of motion of the cue ball be the +*x* direction. We can apply conservation of energy to determine the angle the cue ball makes with the +*x* direction and the conservation of momentum to find the final velocities of the cue ball and the eight ball.

(*a*) Use conservation of energy to relate the velocities of the collision participants before and after the collision:

$$\tfrac{1}{2}mv_{\text{ci}}^2 = \tfrac{1}{2}mv_{\text{cf}}^2 + \tfrac{1}{2}mv_8^2$$

or

$$v_{\text{ci}}^2 = v_{\text{cf}}^2 + v_8^2$$

This Pythagorean relationship tells us that $\vec{v}_{\text{ci}}$, $\vec{v}_{\text{cf}}$, and $\vec{v}_8$ form a right triangle. Hence:

$$\theta_{\text{cf}} + \theta_8 = 90°$$

and

$$\theta_{\text{cf}} = \boxed{60°}$$

(*b*) Use conservation of momentum in the *x* direction to relate the velocities of the collision participants before and after the collision:

$$\vec{p}_{\text{xi}} = \vec{p}_{\text{xf}}$$

or

$$mv_{\text{ci}} = mv_{\text{cf}}\cos\theta_{\text{cf}} + mv_8\cos\theta_8$$

Use conservation of momentum in the *y* direction to obtain a second equation relating the velocities of the collision participants before and after the collision:

$$\vec{p}_{\text{yi}} = \vec{p}_{\text{yf}}$$

or

$$0 = mv_{\text{cf}}\sin\theta_{\text{cf}} + mv_8\sin\theta_8$$

Solve these equations simultaneously to obtain:

$$v_{\text{cf}} = \boxed{2.50\,\text{m/s}} \text{ and } v_8 = \boxed{4.33\,\text{m/s}}$$

86 •• Object A, which has a mass *m* and a velocity $v_0\hat{\boldsymbol{i}}$ collides with object B, which has a mass 2*m* and a velocity $\tfrac{1}{2}v_0\hat{\boldsymbol{j}}$. Following the collision, object B has a velocity of $\tfrac{1}{4}v_0\hat{\boldsymbol{i}}$. (*a*) Determine the velocity of object A after the collision. (*b*) Is the collision elastic? If not, express the change in the kinetic energy in terms of *m* and v_0.

Picture the Problem We can find the final velocity of the object whose mass is *m* by using the conservation of momentum. Whether the collision was elastic can be decided by examining the difference between the initial and final kinetic energy of the interacting objects.

(*a*) Use conservation of linear momentum to relate the initial and final velocities of the two objects:

$$\vec{p}_\text{i} = \vec{p}_\text{f}$$
or
$$mv_0\hat{i} + 2m\left(\tfrac{1}{2}v_0\hat{j}\right) = 2m\left(\tfrac{1}{4}v_0\hat{i}\right) + m\vec{v}_{1\text{f}}$$

Simplify to obtain:

$$v_0\hat{i} + v_0\hat{j} = \tfrac{1}{2}v_0\hat{i} + \vec{v}_{1\text{f}}$$

Solving for $\vec{v}_{1\text{f}}$ yields:

$$\vec{v}_{1\text{f}} = \boxed{\tfrac{1}{2}v_0\hat{i} + v_0\hat{j}}$$

(*b*) Express the difference between the kinetic energy of the system before the collision and its kinetic energy after the collision:

$$\Delta E = K_\text{i} - K_\text{f} = K_{1\text{i}} + K_{2\text{i}} - \left(K_{1\text{f}} + K_{2\text{f}}\right)$$

Substituting for the kinetic energies yields:

$$\Delta E = \tfrac{1}{2}\left(mv_{1\text{i}}^2 + 2mv_{2\text{i}}^2 - mv_{1\text{f}}^2 - 2mv_{2\text{f}}^2\right)$$

Substitute for speeds and simplify to obtain:

$$\Delta E = \tfrac{1}{2}m\left[v_0^2 + 2\left(\tfrac{1}{4}v_0^2\right) - \tfrac{5}{4}v_0^2 - 2\left(\tfrac{1}{16}v_0^2\right)\right] = \boxed{\tfrac{1}{16}mv_0^2}$$

Because $\Delta E \neq 0$, the collision is inelastic.

87 •• **[SSM]** A puck of mass 5.0 kg moving at 2.0 m/s approaches an identical puck that is stationary on frictionless ice. After the collision, the first puck leaves with a speed v_1 at 30º to the original line of motion; the second puck leaves with speed v_2 at 60º, as in Figure 8-50. (*a*) Calculate v_1 and v_2. (*b*) Was the collision elastic?

Picture the Problem Let the direction of motion of the puck that is moving before the collision be the +*x* direction. Applying conservation of momentum to the collision in both the *x* and *y* directions will lead us to two equations in the unknowns v_1 and v_2 that we can solve simultaneously. We can decide whether the collision was elastic by either calculating the system's kinetic energy before and after the collision or by determining whether the angle between the final velocities is 90°.

(*a*) Use conservation of linear momentum in the *x* direction to obtain:

$p_{xi} = p_{xf}$

or

$$mv = mv_1 \cos 30° + mv_2 \cos 60°$$

Simplify further to obtain:

$$v = v_1 \cos 30° + v_2 \cos 60° \qquad (1)$$

Use conservation of momentum in the *y* direction to obtain a second equation relating the velocities of the collision participants before and after the collision:

$p_{yi} = p_{yf}$

or

$$0 = mv_1 \sin 30° - mv_2 \sin 60°$$

Simplifying further yields:

$$0 = v_1 \sin 30° - v_2 \sin 60° \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$v_1 = \boxed{1.7\,\text{m/s}} \text{ and } v_2 = \boxed{1.0\,\text{m/s}}$$

(*b*) Because the angle between $\vec{v}_1$ and $\vec{v}_2$ is 90°, the collision was elastic.

88 •• Figure 8-51 shows the result of a collision between two objects of unequal mass. (*a*) Find the speed v_2 of the larger mass after the collision and the angle θ_2. (*b*) Show that the collision is elastic.

Picture the Problem Let the direction of motion of the object that is moving before the collision be the +*x* direction. Applying conservation of momentum to the motion in both the *x* and *y* directions will lead us to two equations in the unknowns v_2 and θ_2 that we can solve simultaneously. We can show that the collision was elastic by showing that the system's kinetic energy before and after the collision is the same.

(*a*) Use conservation of linear momentum in the *x* direction to relate the velocities of the collision participants before and after the collision:

$p_{xi} = p_{xf}$

or

$$3mv_0 = \sqrt{5}mv_0 \cos\theta_1 + 2mv_2 \cos\theta_2$$

or

$$3v_0 = \sqrt{5}v_0 \cos\theta_1 + 2v_2 \cos\theta_2$$

Use conservation of linear momentum in the *y* direction to obtain a second equation relating the velocities of the collision participants before and after the collision:

$p_{yi} = p_{yf}$

or

$$0 = \sqrt{5}mv_0 \sin\theta_1 - 2mv_2 \sin\theta_2$$

Simplifying further yields:

$$0=\sqrt{5}v_0\sin\theta_1-2v_2\sin\theta_2$$

Note that if $\tan\theta_1 = 2$, then:

$$\cos\theta_1=\frac{1}{\sqrt{5}}\text{ and }\sin\theta_1=\frac{2}{\sqrt{5}}$$

Substitute in the x-direction momentum equation and simplify to obtain:

$$3v_0=\sqrt{5}v_0\frac{1}{\sqrt{5}}+2v_2\cos\theta_2$$

or

$$v_0=v_2\cos\theta_2 \qquad (1)$$

Substitute in the y-direction momentum equation and simplify to obtain:

$$0=\sqrt{5}v_0\frac{2}{\sqrt{5}}-2v_2\sin\theta_2$$

or

$$0=v_0-v_2\sin\theta_2 \qquad (2)$$

Solve equations (1) and (2) simultaneously for θ_2 :

$$\theta_2=\tan^{-1}(1)=\boxed{45.0°}$$

Substitute in equation (1) to find v_2:

$$v_2=\frac{v_0}{\cos\theta_2}=\frac{v_0}{\cos 45°}=\boxed{\sqrt{2}v_0}$$

(b) To show that the collision was elastic, find the before-collision and after-collision kinetic energies:

$$K_\text{i}=\tfrac{1}{2}m(3v_0)^2=4.5mv_0^2$$

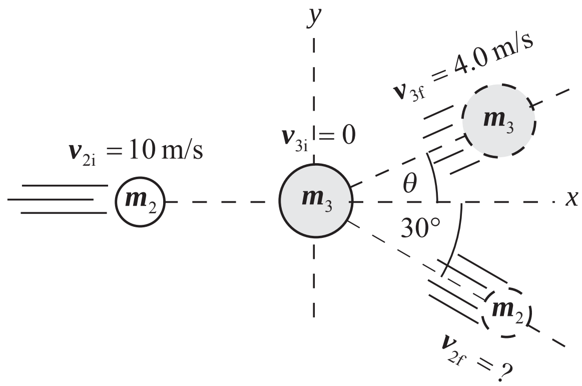
and

$$K_\text{f}=\tfrac{1}{2}m\left(\sqrt{5}v_0\right)^2+\tfrac{1}{2}(2m)\left(\sqrt{2}v_0\right)^2$$
$$=4.5mv_0^2$$

Because $K_\text{i} = K_\text{f}$, the collision is elastic.

89 •• A 2.0-kg ball moving at 10 m/s makes an off-center collision with a 3.0-kg ball that is initially at rest. After the collision, the 2.0-kg ball is deflected at an angle of 30º from its original direction of motion and the 3.0-kg ball is moving at 4.0 m/s. Find the speed of the 2.0-kg ball and the direction of the 3.0-kg ball after the collision. *Hint*: $\sin^2\alpha+\cos^2\alpha=1$.

Picture the Problem Let the direction of motion of the ball that is moving before the collision be the +x direction and use the subscripts 2 and 3 to designate the 2.0-kg and 3.0-kg balls, respectively. Applying conservation of momentum to the collision in both the x and y directions will lead us to two equations in the unknowns $\boldsymbol{v}_{2\text{f}}$ and θ that we can solve simultaneously.



Use conservation of momentum in the x and y directions to relate the speeds and directions of the balls before and after the collision:

$$m_2v_{2i} = m_2v_{2f}\cos 30° + m_3v_{3f}\sin\theta$$

and

$$0 = m_3v_{3f}\sin\theta - m_2v_{2f}\sin 30°$$

Solve the first of these equations for $\cos\theta$ to obtain:

$$\cos\theta = \frac{m_2v_{2i} - m_2v_{2f}\cos 30°}{m_3v_{3f}} \quad (1)$$

Solve the second of these equations for $\sin\theta$ to obtain:

$$\sin\theta = \frac{m_2v_{2f}\sin 30°}{m_3v_{3f}} \quad (2)$$

Using the hint given in the problem statement, square and add equations (1) and (2) and simplify the result to obtain the quadratic equation:

$$v_{2f}^2\sin^2 30° + \left(v_{2i} - v_{2f}\cos 30°\right)^2 = \frac{m_3^2v_{3f}^2}{m_2^2}$$

Substituting numerical values and simplifying yields:

$$v_{2f}^2 + \left(10\text{ m/s} - 0.866v_{2f}\right)^2 = 144\text{ m}^2/\text{s}^2$$

Use the quadratic formula or your graphing calculator to obtain:

$$v_{2f} = 5.344\text{ m/s or } 11.977\text{ m/s}$$

Because the larger of these values corresponds to there being more kinetic energy in the system after the collision than there was before the collision:

$$v_{2f} = \boxed{5.3\text{ m/s}}$$

Solving equation (2) for θ yields:

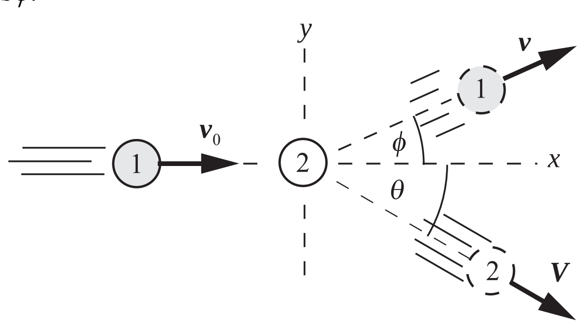
$$\theta = \sin^{-1}\left[\frac{m_2 v_{2f}\sin 30°}{m_3 v_{3f}}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{(2.0\ \text{kg})(5.344\ \text{m/s})\sin 30°}{(3.0\ \text{kg})(4.0\ \text{m/s})}\right] = \boxed{26°}$$

90 •• A particle has initial speed v_0. It collides with a second particle with the same mass that is initially at rest. The first particle is deflected through an angle ϕ. Its speed after the collision is v. The second particle recoils, and its velocity makes an angle θ with the initial direction of the first particle. (*a*) Show that $\tan\theta = \dfrac{v\sin\phi}{(v_0 - v\cos\phi)}$. (*b*) Show that if the collision is elastic, then $v = v_0\cos\phi$.

Picture the Problem Choose the coordinate system shown in the following diagram with the +*x* direction the direction of the initial approach of the projectile particle. Call *V* the speed of the target particle after the collision. In Part (*a*) we can apply conservation of momentum in the *x* and *y* directions to obtain two equations that we can solve simultaneously for $\tan\theta$. In Part (*b*) we can use conservation of momentum in vector form and the elastic-collision equation to show that $v = v_0\cos\phi$.



(*a*) Apply conservation of linear momentum in the *x* direction to obtain:

$$v_0 = v\cos\phi + V\cos\theta \qquad (1)$$

Apply conservation of linear momentum in the *y* direction to obtain:

$$v\sin\phi = V\sin\theta \qquad (2)$$

Solve equation (1) for $V\cos\theta$:

$$V\cos\theta = v_0 - v\cos\phi \qquad (3)$$

Divide equation (2) by equation (3) to obtain:

$$\frac{V\sin\theta}{V\cos\theta}=\frac{v\sin\phi}{v_0-v\cos\phi}$$

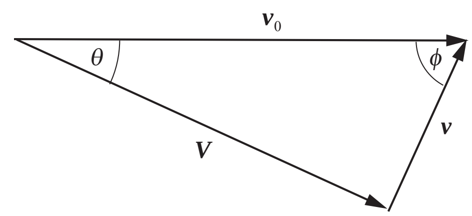
or

$$\tan\theta=\boxed{\frac{v\sin\phi}{v_0-v\cos\phi}}$$

(*b*) Noting that the masses of the particles are equal, apply conservation of linear momentum to obtain:

$$\vec{\boldsymbol{v}}_0=\vec{\boldsymbol{v}}+\vec{\boldsymbol{V}}$$

Draw the vector diagram representing this equation:



Use the definition of an elastic collision to obtain:

$$v_0^2=v^2+V^2$$

If this Pythagorean condition is to hold, the third angle of the triangle must be a right angle and, using the definition of the cosine function:

$$v=\boxed{v_0\cos\phi}$$

*Center-of-Mass Reference Frame

91 •• In the center-of-mass reference frame a particle with mass m_1 and momentum p_1 makes an elastic head-on collision with a second particle of mass m_2 and momentum $p_2 = -p_1$. After the collision its momentum is p_1'. Write the total kinetic energy in terms of m_1, m_2, and p_1 and the total final energy in terms of m_1, m_2, and p_1', and show that $p_1'=\pm p_1$. If $p_1'=-p_1$, the particle is merely turned around by the collision and leaves with the speed it had initially. What is the situation for the $p_1'=+p_1$ solution?

Picture the Problem The total kinetic energy of a system of particles is the sum of the kinetic energy of the center of mass and the kinetic energy relative to the center of mass. The kinetic energy of a particle of mass m is related to its momentum according to $K = p^2/2m$.

Express the total kinetic energy of the system:

$$K=K_{\text{rel}}+K_{\text{cm}} \qquad (1)$$

Relate the kinetic energy relative to the center of mass to the momenta of the two particles:

$$K_{\text{rel}} = \frac{p_1^2}{2m_1} + \frac{p_1^2}{2m_2} = \frac{p_1^2(m_1 + m_2)}{2m_1m_2}$$

Express the kinetic energy of the center of mass of the two particles:

$$K_{\text{cm}} = \frac{(2p_1)^2}{2(m_1 + m_2)} = \frac{2p_1^2}{m_1 + m_2}$$

Substitute in equation (1) and simplify to obtain:

$$K = \frac{p_1^2(m_1 + m_2)}{2m_1m_2} + \frac{2p_1^2}{m_1 + m_2}$$
$$= \frac{p_1^2}{2}\left[\frac{m_1^2 + 6m_1m_2 + m_2^2}{m_1^2m_2 + m_1m_2^2}\right]$$

In an elastic collision:

$$K_{\text{i}} = K_{\text{f}}$$
$$= \boxed{\frac{p_1^2}{2}\left[\frac{m_1^2 + 6m_1m_2 + m_2^2}{m_1^2m_2 + m_1m_2^2}\right]}$$
$$= \boxed{\frac{p_1'^2}{2}\left[\frac{m_1^2 + 6m_1m_2 + m_2^2}{m_1^2m_2 + m_1m_2^2}\right]}$$

Simplify to obtain:

$$(p_1')^2 = (p_1)^2 \Rightarrow \boxed{p_1' = \pm p_1}$$

and if $p_1' = +p_1$, the particles do not collide.

92 •• A 3.0-kg block is traveling in the $-x$ direction at 5.0 m/s, and a 1.0-kg block is traveling in the $+x$ direction at 3.0 m/s. (*a*) Find the velocity v_{cm} of the center of mass. (*b*) Subtract v_{cm} from the velocity of each block to find the velocity of each block in the center-of-mass reference frame. (*c*) After they make a head-on elastic collision, the velocity of each block is reversed (in the center-of-mass frame). Find the velocity of each block in the center-of-mass frame after the collision. (*d*) Transform back into the original frame by adding v_{cm} to the velocity of each block. (*e*) Check your result by finding the initial and final kinetic energies of the blocks in the original frame and comparing them.

Picture the Problem Let the numerals 3 and 1 denote the blocks whose masses are 3.0 kg and 1.0 kg respectively. We can use $\sum_i m_i\vec{v}_i = M\vec{v}_{\text{cm}}$ to find the velocity of the center-of-mass of the system and simply follow the directions in the problem step by step.

(*a*) Express the total momentum of this two-particle system in terms of the velocity of its center of mass:

$$\vec{P} = \sum_i m_i\vec{v}_i = m_1\vec{v}_1 + m_3\vec{v}_3$$
$$= M\vec{v}_{\text{cm}} = (m_1 + m_3)\vec{v}_{\text{cm}}$$

Solve for $\vec{v}_{\text{cm}}$:

$$\vec{v}_{\text{cm}} = \frac{m_3\vec{v}_3 + m_1\vec{v}_1}{m_3 + m_1}$$

Substitute numerical values and evaluate $\vec{v}_{\text{cm}}$:

$$\vec{v}_{\text{cm}} = \frac{(3.0\,\text{kg})(-5.0\,\text{m/s})\hat{i} + (1.0\,\text{kg})(3.0\,\text{m/s})\hat{i}}{3.0\,\text{kg} + 1.0\,\text{kg}} = \boxed{(-3.0\,\text{m/s})\hat{i}}$$

(*b*) Find the velocity of the 3-kg block in the center of mass reference frame:

$$\vec{u}_3 = \vec{v}_3 - \vec{v}_{\text{cm}}$$
$$= (-5.0\,\text{m/s})\hat{i} - (-3.0\,\text{m/s})\hat{i}$$
$$= \boxed{(-2.0\,\text{m/s})\hat{i}}$$

Find the velocity of the 1-kg block in the center of mass reference frame:

$$\vec{u}_1 = \vec{v}_1 - \vec{v}_{\text{cm}}$$
$$= (3.0\,\text{m/s})\hat{i} - (-3.0\,\text{m/s})\hat{i}$$
$$= \boxed{(6.0\,\text{m/s})\hat{i}}$$

(*c*) Express the after-collision velocities of both blocks in the center of mass reference frame:

$$\vec{u}_3' = \boxed{(2.0\,\text{m/s})\hat{i}}$$

and

$$\vec{u}_1' = \boxed{(-6.0\,\text{m/s})\hat{i}}$$

(*d*) Transform the after-collision velocity of the 3-kg block from the center of mass reference frame to the original reference frame:

$$\vec{v}_3' = \vec{u}_3' + \vec{v}_{\text{cm}}$$
$$= (2.0\,\text{m/s})\hat{i} + (-3.0\,\text{m/s})\hat{i}$$
$$= \boxed{(-1.0\,\text{m/s})\hat{i}}$$

Transform the after-collision velocity of the 1-kg block from the center of mass reference frame to the original reference frame:

$$\vec{v}_1' = \vec{u}_1' + \vec{v}_{\text{cm}}$$
$$= (-6.0\,\text{m/s})\hat{i} + (-3.0\,\text{m/s})\hat{i}$$
$$= \boxed{(-9.0\,\text{m/s})\hat{i}}$$

(*e*) Express K_{i} in the original frame of reference:

$$K_{\text{i}} = \tfrac{1}{2}m_3v_3^2 + \tfrac{1}{2}m_1v_1^2$$

Substitute numerical values and evaluate K_i:

$$K_\text{i} = \tfrac{1}{2}\left[(3.0\,\text{kg})(5.0\,\text{m/s})^2 + (1.0\,\text{kg})(3.0\,\text{m/s})^2\right] = \boxed{42\,\text{J}}$$

Express K_i in the original frame of reference:

$$K_\text{f} = \tfrac{1}{2}m_3v'^2_3 + \tfrac{1}{2}m_1v'^2_1$$

Substitute numerical values and evaluate K_f:

$$K_\text{f} = \tfrac{1}{2}\left[(3.0\,\text{kg})(1.0\,\text{m/s})^2 + (1.0\,\text{kg})(9.0\,\text{m/s})^2\right] = \boxed{42\,\text{J}}$$

93 •• [SSM] Repeat Problem 92 with the second block having a mass of 5.0 kg and moving to the right at 3.0 m/s.

Picture the Problem Let the numerals 3 and 5 denote the blocks whose masses are 3.0 kg and 5.0 kg respectively. We can use $\sum_i m_i\vec{v}_i = M\vec{v}_\text{cm}$ to find the velocity of the center-of-mass of the system and simply follow the directions in the problem step by step.

(*a*) Express the total momentum of this two-particle system in terms of the velocity of its center of mass:

$$\vec{P} = \sum_i m_i\vec{v}_i = m_3\vec{v}_3 + m_5\vec{v}_5 = M\vec{v}_\text{cm} = (m_3 + m_5)\vec{v}_\text{cm}$$

Solve for $\vec{v}_\text{cm}$:

$$\vec{v}_\text{cm} = \frac{m_3\vec{v}_3 + m_5\vec{v}_5}{m_3 + m_5}$$

Substitute numerical values and evaluate $\vec{v}_\text{cm}$:

$$\vec{v}_\text{cm} = \frac{(3.0\,\text{kg})(-5.0\,\text{m/s})\hat{i} + (5.0\,\text{kg})(3.0\,\text{m/s})\hat{i}}{3.0\,\text{kg} + 5.0\,\text{kg}} = \boxed{0}$$

(*b*) Find the velocity of the 3.0-kg block in the center of mass reference frame:

$$\vec{u}_3 = \vec{v}_3 - \vec{v}_\text{cm} = (-5.0\,\text{m/s})\,\hat{i} - 0 = \boxed{(-5.0\,\text{m/s})\,\hat{i}}$$

Find the velocity of the 5.0-kg block in the center of mass reference frame:

$$\vec{u}_5 = \vec{v}_5 - \vec{v}_\text{cm} = (3.0\,\text{m/s})\hat{i} - 0 = \boxed{(3.0\,\text{m/s})\hat{i}}$$

(*c*) Express the after-collision velocities of both blocks in the center of mass reference frame:

$$\vec{u}_3' = \boxed{(5.0\,\text{m/s})\hat{\boldsymbol{i}}}$$
and
$$\vec{u}_5' = \boxed{(-3.0\,\text{m/s})\hat{\boldsymbol{i}}}$$

(*d*) Transform the after-collision velocity of the 3.0-kg block from the center of mass reference frame to the original reference frame:

$$\vec{v}_3' = \vec{u}_3' + \vec{v}_{\text{cm}} = (5.0\,\text{m/s})\hat{\boldsymbol{i}} + 0$$
$$= \boxed{(5.0\,\text{m/s})\hat{\boldsymbol{i}}}$$

Transform the after-collision velocity of the 5.0-kg block from the center of mass reference frame to the original reference frame:

$$\vec{v}_5' = \vec{u}_5' + \vec{v}_{\text{cm}} = (-3.0\,\text{m/s})\hat{\boldsymbol{i}} + 0$$
$$= \boxed{(-3.0\,\text{m/s})\hat{\boldsymbol{i}}}$$

(*e*) Express K_{i} in the original frame of reference:

$$K_{\text{i}} = \tfrac{1}{2}m_3v_3^2 + \tfrac{1}{2}m_5v_5^2$$

Substitute numerical values and evaluate K_{i}:

$$K_{\text{i}} = \tfrac{1}{2}\left[(3.0\,\text{kg})(5.0\,\text{m/s})^2 + (5.0\,\text{kg})(3.0\,\text{m/s})^2\right] = \boxed{60\,\text{J}}$$

Express K_{f} in the original frame of reference:

$$K_{\text{f}} = \tfrac{1}{2}m_3v'^2_3 + \tfrac{1}{2}m_5v'^2_5$$

Substitute numerical values and evaluate K_{f}:

$$K_{\text{f}} = \tfrac{1}{2}\left[(3.0\,\text{kg})(5.0\,\text{m/s})^2 + (5.0\,\text{kg})(3.0\,\text{m/s})^2\right] = \boxed{60\,\text{J}}$$

*Systems With Continuously Varying Mass: Rocket Propulsion

94 • A rocket burns fuel at a rate of 200 kg/s and exhausts the gas at a relative speed of 6.00 km/s relative to the rocket. Find the magnitude of the thrust of the rocket.

Picture the Problem The thrust of a rocket F_{th} depends on the burn rate of its fuel *dm/dt* and the relative speed of its exhaust gases u_{ex} according to $F_{\text{th}} = |dm/dt|u_{\text{ex}}$.

Using its definition, relate the rocket's thrust to the relative speed of its exhaust gases:

$$F_{th} = \left|\frac{dm}{dt}\right| u_{ex}$$

Substitute numerical values and evaluate F_{th}:

$$F_{th} = (200\,\text{kg/s})(6.00\,\text{km/s})$$
$$= \boxed{1.20\,\text{MN}}$$

95 •• A rocket has an initial mass of 30,000 kg, of which 80 percent is the fuel. It burns fuel at a rate of 200 kg/s and exhausts its gas at a relative speed of 1.80 km/s. Find (*a*) the thrust of the rocket, (*b*) the time until burnout, and (*c*) its speed at burnout assuming it moves straight upward near the surface of Earth. Assume that *g* is constant and neglect any effects of air resistance.

Picture the Problem The thrust of a rocket F_{th} depends on the burn rate of its fuel *dm/dt* and the relative speed of its exhaust gases u_{ex} according to $F_{th} = |dm/dt| u_{ex}$. The final velocity v_f of a rocket depends on the relative speed of its exhaust gases u_{ex}, its payload to initial mass ratio m_f/m_0 and its burn time according to $v_f = -u_{ex} \ln(m_f/m_0) - gt_b$.

(*a*) Using its definition, relate the rocket's thrust to the relative speed of its exhaust gases:

$$F_{th} = \left|\frac{dm}{dt}\right| u_{ex}$$

Substitute numerical values and evaluate F_{th}:

$$F_{th} = (200\,\text{kg/s})(1.80\,\text{km/s})$$
$$= \boxed{360\,\text{kN}}$$

(*b*) Relate the time to burnout to the mass of the fuel and its burn rate:

$$t_b = \frac{m_{fuel}}{dm/dt} = \frac{0.8m_0}{dm/dt}$$

Substitute numerical values and evaluate t_b:

$$t_b = \frac{(0.80)(30{,}000\,\text{kg})}{200\,\text{kg/s}} = \boxed{120\,\text{s}}$$

(*c*) Relate the final velocity of a rocket to its initial mass, exhaust velocity, and burn time:

$$v_f = -u_{ex} \ln\left(\frac{m_f}{m_0}\right) - gt_b$$

Substitute numerical values and evaluate v_f:

$$v_f = -(1.80\,\text{km/s})\ln\left(\frac{1}{5}\right) - (9.81\,\text{m/s}^2)(120\,\text{s}) = \boxed{1.72\,\text{km/s}}$$

96 •• The *specific impulse* of a rocket propellant is defined as $I_{sp} = F_{th}/(Rg)$, where F_{th} is the thrust of the propellant, g the magnitude of free-fall acceleration, and R the rate at which the propellant is burned. The rate depends predominantly on the type and exact mixture of the propellant. (*a*) Show that the specific impulse has the dimension of time. (*b*) Show that $u_{ex} = gI_{sp}$, where u_{ex} is the relative speed of the exhaust. (*c*) What is the specific impulse (in seconds) of the propellant used in the Saturn V rocket of Example 8-16.

Picture the Problem We can use the dimensions of thrust, burn rate, and acceleration to show that the dimension of specific impulse is time. Combining the definitions of rocket thrust and specific impulse will lead us to $u_{ex} = gI_{sp}$.

(*a*) Express the dimension of specific impulse in terms of the dimensions of F_{th}, R, and g:

$$[I_{sp}] = \frac{[F_{th}]}{[R][g]} = \frac{\frac{\mathrm{M \cdot L}}{\mathrm{T}^2}}{\frac{\mathrm{M}}{\mathrm{T}} \cdot \frac{\mathrm{L}}{\mathrm{T}^2}} = \boxed{\mathrm{T}}$$

(*b*) From the definition of rocket thrust we have:

$$F_{th} = Ru_{ex} \Rightarrow u_{ex} = \frac{F_{th}}{R}$$

Substitute for F_{th} to obtain:

$$u_{ex} = \frac{RgI_{sp}}{R} = \boxed{gI_{sp}} \qquad (1)$$

(*c*) Solve equation (1) for I_{sp} and substitute for u_{ex} to obtain:

$$I_{sp} = \frac{F_{th}}{Rg}$$

From Example 8-21 we have:

$R = 1.384\times10^4$ kg/s
and
$F_{th} = 3.4\times10^6$ N

Substitute numerical values and evaluate I_{sp}:

$$I_{sp} = \frac{3.4\times10^6\ \mathrm{N}}{(1.384\times10^4\ \mathrm{kg/s})(9.81\,\mathrm{m/s^2})} = \boxed{25\,\mathrm{s}}$$

97 ••• [SSM] The initial *thrust-to-weight ratio* τ_0 of a rocket is $\tau_0 = F_{th}/(m_0 g)$, where F_{th} is the rocket's thrust and m_0 the initial mass of the rocket, including the propellant. (*a*) For a rocket launched straight up from Earth's surface, show that $\tau_0 = 1 + (a_0/g)$, where a_0 is the initial acceleration of the rocket. For manned rocket flight, τ_0 cannot be made much larger than 4 for the comfort and safety of the astronauts. (The astronauts will feel that their weight as the rocket lifts off is equal to τ_0 times their normal weight.) (*b*) Show that the final velocity of a rocket launched from Earth's surface, in terms of τ_0 and I_{sp} (see Problem 96) can be written as

$$v_{\mathrm{f}} = gI_{\mathrm{sp}}\left[1n\left(\frac{m_0}{m_{\mathrm{f}}}\right) - \frac{1}{\tau_0}\left(1 - \frac{m_{\mathrm{f}}}{m_0}\right)\right]$$

where m_{f} is the mass of the rocket (not including the spent propellant). (*c*) Using a **spreadsheet** program or **graphing calculator**, graph v_{f} as a function of the mass ratio m_0/m_{f} for I_{sp} = 250 s and τ_0 = 2 for values of the mass ratio from 2 to 10. (Note that the mass ratio cannot be less than 1.) (*d*) To lift a rocket into orbit, a final velocity after burnout of v_{f} = 7.0 km/s is needed. Calculate the mass ratio required of a single stage rocket to do this, using the values of specific impulse and thrust ratio given in Part (*b*). For engineering reasons, it is difficult to make a rocket with a mass ratio much greater than 10. Can you see why multistage rockets are usually used to put payloads into orbit around Earth?

Picture the Problem We can use the rocket equation and the definition of rocket thrust to show that $\tau_0 = 1 + a_0/g$. In Part (*b*) we can express the burn time t_{b} in terms of the initial and final masses of the rocket and the rate at which the fuel burns, and then use this equation to express the rocket's final velocity in terms of I_{sp}, τ_0, and the mass ratio m_0/m_{f}. In Part (*d*) we'll need to use trial-and-error methods or a graphing calculator to solve the transcendental equation giving v_{f} as a function of m_0/m_{f}.

(*a*) The rocket equation is:

$$-mg + Ru_{\mathrm{ex}} = ma$$

From the definition of rocket thrust we have:

$$F_{\mathrm{th}} = Ru_{\mathrm{ex}}$$

Substitute for Ru_{ex} to obtain:

$$-mg + F_{\mathrm{th}} = ma$$

Solve for F_{th} at takeoff:

$$F_{\mathrm{th}} = m_0 g + m_0 a_0$$

Divide both sides of this equation by $m_0 g$ to obtain:

$$\frac{F_{\mathrm{th}}}{m_0 g} = 1 + \frac{a_0}{g}$$

Because $\tau_0 = F_{\mathrm{th}}/(m_0 g)$:

$$\tau_0 = \boxed{1 + \frac{a_0}{g}}$$

(*b*) Use Equation 8-39 to express the final speed of a rocket that starts from rest with mass m_0:

$$v_{\mathrm{f}} = u_{\mathrm{ex}} \ln\left(\frac{m_0}{m_{\mathrm{f}}}\right) - gt_{\mathrm{b}}, \qquad (1)$$

where t_{b} is the burn time.

Express the burn time in terms of the burn rate R (assumed constant):

$$t_{\mathrm{b}} = \frac{m_0 - m_{\mathrm{f}}}{R} = \frac{m_0}{R}\left(1 - \frac{m_{\mathrm{f}}}{m_0}\right)$$

Multiply t_b by one in the form $\frac{gF_{th}}{gF_{th}}$ and simplify to obtain:

$$t_b = \frac{gF_{th}}{gF_{th}}\frac{m_0}{R}\left(1-\frac{m_f}{m_0}\right) = \frac{gm_0}{F_{th}}\frac{F_{th}}{gR}\left(1-\frac{m_f}{m_0}\right) = \frac{I_{sp}}{\tau_0}\left(1-\frac{m_f}{m_0}\right)$$

Substitute in equation (1):

$$v_f = u_{ex}\ln\left(\frac{m_0}{m_f}\right) - \frac{gI_{sp}}{\tau_0}\left(1-\frac{m_f}{m_0}\right)$$

From Problem 96 we have:

$$u_{ex} = gI_{sp},$$

where u_{ex} is the exhaust velocity of the propellant.

Substitute for u_{ex} and factor to obtain:

$$v_f = gI_{sp}\ln\left(\frac{m_0}{m_f}\right) - \frac{gI_{sp}}{\tau_0}\left(1-\frac{m_f}{m_0}\right) = \boxed{gI_{sp}\left[\ln\left(\frac{m_0}{m_f}\right) - \frac{1}{\tau_0}\left(1-\frac{m_f}{m_0}\right)\right]}$$

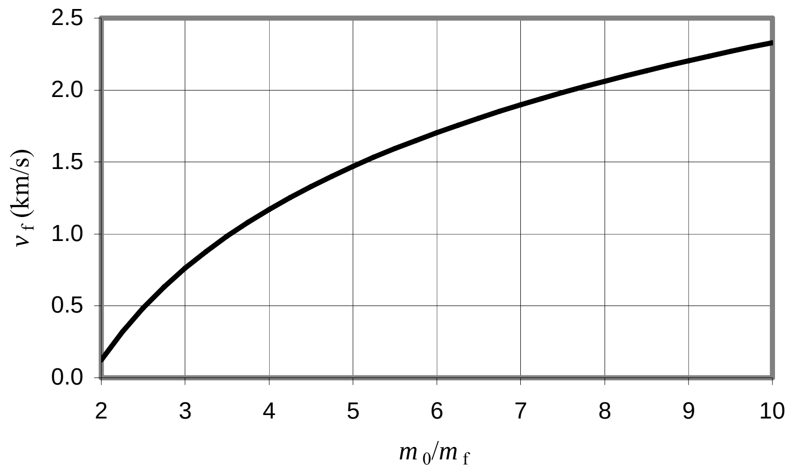
(*c*) A spreadsheet program to calculate the final velocity of the rocket as a function of the mass ratio m_0/m_f is shown below. The constants used in the velocity function and the formulas used to calculate the final velocity are as follows:

Cell	Content/Formula	Algebraic Form
B1	250	I_{sp}
B2	9.81	g
B3	2	τ_0
D9	D8 + 0.25	m_0/m_f
E8	\$B\$2*\$B\$1*(LOG(D8) – (1/\$B\$3)*(1/D8))	$gI_{sp}\left[\ln\left(\frac{m_0}{m_f}\right)-\frac{1}{\tau_0}\left(1-\frac{m_f}{m_0}\right)\right]$

	A	B	C	D	E
1	I_{sp} =	250	s		
2	g =	9.81	m/s^2		
3	τ_0=	2			
4					
5				mass ratio	v_f
6				2.00	1.252E+02
7				2.25	3.187E+02

8				2.50	4.854E+02
9				2.75	6.316E+02
10				3.00	7.614E+02
34				9.00	2.204E+03
35				9.25	2.237E+03
36				9.50	2.269E+03
37				9.75	2.300E+03
38				10.00	2.330E+03
39				725.00	7.013E+03

A graph of final velocity as a function of mass ratio follows.



(*d*) Substitute the data given in part (*c*) in the equation derived in Part (*b*) to obtain:

$$7.00\,\text{km/s} = \left(9.81\,\text{m/s}^2\right)\left(250\,\text{s}\right)\left(\ln\left(\frac{m_0}{m_f}\right) - \frac{1}{2}\left(1 - \frac{m_f}{m_0}\right)\right)$$

or

$$2.854 = \ln x - 0.5 + \frac{0.5}{x} \quad \text{where } x = m_0/m_f.$$

Use trial-and-error methods or a graphing calculator to solve this transcendental equation for the root greater than 1:

$\boldsymbol{x} \approx \boxed{28}$, a value considerably larger than the practical limit of 10 for single-stage rockets.

98 •• The height that a model rocket launched from Earth's surface can reach can be estimated by assuming that the burn time is short compared to the total flight time; the rocket is therefore in free-fall for most of the flight. (This estimate neglects the burn time in calculations of both time and displacement.) For a model rocket with specific impulse I_{sp} = 100 s, mass ratio m_0/m_f = 1.20, and

initial thrust-to-weight ratio τ_0 = 5.00 (these parameters are defined in Problems 96 and 97), estimate (*a*) the height the rocket can reach, and (*b*) the total flight time. (*c*) Justify the assumption used in the estimates by comparing the flight time from Part (*b*) to the time it takes to consume the fuel.

Picture the Problem We can use the velocity-at-burnout equation from Problem 96 to find v_f and constant-acceleration equations to approximate the maximum height the rocket will reach and its total flight time.

(*a*) Assuming constant acceleration, relate the maximum height reached by the model rocket to its time-to-top-of-trajectory:

$$h = \tfrac{1}{2} g t_{\text{top}}^2 \qquad (1)$$

From Problem 96 we have:

$$v_{\text{f}} = gI_{\text{sp}}\left(\ln\left(\frac{m_0}{m_{\text{f}}}\right) - \frac{1}{\tau}\left(1 - \frac{m_{\text{f}}}{m_0}\right)\right)$$

Evaluate the velocity at burnout v_f for I_{sp} = 100 s, m_0/m_f = 1.2, and τ = 5:

$$v_{\text{f}} = \left(9.81\,\text{m/s}^2\right)\left(100\,\text{s}\right)\left[\ln(1.2) - \frac{1}{5}\left(1 - \frac{1}{1.2}\right)\right] = 146\,\text{m/s}$$

Assuming that the time for the fuel to burn up is short compared to the total flight time, find the time to the top of the trajectory:

$$t_{\text{top}} = \frac{v_{\text{f}}}{g} = \frac{146\,\text{m/s}}{9.81\,\text{m/s}^2} = 14.9\,\text{s}$$

Substitute in equation (1) and evaluate *h*:

$$h = \tfrac{1}{2}\left(9.81\,\text{m/s}^2\right)\left(14.9\,\text{s}\right)^2 = \boxed{1.09\,\text{km}}$$

(*b*) Find the total flight time from the time it took the rocket to reach its maximum height:

$$t_{\text{flight}} = 2t_{\text{top}} = 2\left(14.9\,\text{s}\right) = \boxed{29.8\,\text{s}}$$

(*c*) The fuel burn time t_b is:

$$t_{\text{b}} = \frac{I_{sp}}{\tau}\left(1 - \frac{m_f}{m_0}\right) = \frac{100\,\text{s}}{5}\left(1 - \frac{1}{1.2}\right) = 3.33\,\text{s}$$

Because this burn time is approximately one-fifth of the total flight time, we can't expect the answer we obtain in Part (*b*) to be very accurate. It should, however, be good to about thirty percent accuracy, as the maximum distance the model rocket could possibly move in this time is $\frac{1}{2}v_f t_b = 244\text{ m}$, assuming constant acceleration until burnout.

General Problems

99 • [SSM] A 250-g model-train car traveling at 0.50 m/s links up with a 400-g car that is initially at rest. What is the speed of the cars immediately after they link up? Find the pre- and post-collision kinetic energies of the two-car system.

Picture the Problem Let the direction the 250-g car is moving before the collision be the +*x* direction. Let the numeral 1 refer to the 250-kg car, the numeral 2 refer to the 400-kg car, and *V* represent the velocity of the linked cars. Let the system include Earth and the cars. We can use conservation of momentum to find their speed after they have linked together and the definition of kinetic energy to find their pre- and post-collision kinetic energies.

Use conservation of momentum to relate the speeds of the cars immediately before and immediately after their collision:

$$p_{ix} = p_{fx}$$

or

$$m_1 v_1 = (m_1 + m_2)V \Rightarrow V = \frac{m_1 v_1}{m_1 + m_2}$$

Substitute numerical values and evaluate *V*:

$$V = \frac{(0.250\text{ kg})(0.50\text{ m/s})}{0.250\text{ kg} + 0.400\text{ kg}} = 0.192\text{ m/s}$$

$$= \boxed{0.19\text{ m/s}}$$

Find the pre-collision kinetic energy of the cars:

$$K_{\text{pre}} = \tfrac{1}{2}m_1 v_1^2 = \tfrac{1}{2}(0.250\text{ kg})(0.50\text{ m/s})^2$$

$$= \boxed{31\text{ mJ}}$$

Find the post-collision kinetic energy of the coupled cars:

$$K_{\text{post}} = \tfrac{1}{2}(m_1 + m_2)V^2$$

$$= \tfrac{1}{2}(0.250\text{ kg} + 0.400\text{ kg})(0.192\text{ m/s})^2$$

$$= \boxed{12\text{ mJ}}$$

100 • A 250-g model train car traveling at 0.50 m/s heads toward a 400-g car that is initially at rest. (*a*) Find the kinetic energy of the two-car system. (*b*) Find the velocity of each car in the center-of-mass reference frame, and use these velocities to calculate the kinetic energy of the two-car system in the center-of-

mass reference. (*c*) Find the kinetic energy associated with the motion of the center of mass of the system. (*d*) Compare your answer for Part (*a*) with the sum of your answers for Parts (*b*) and (*c*).

Picture the Problem Let the direction the 250-g car is moving before the collision be the +*x* direction. Let the numeral 1 refer to the 250-kg car and the numeral 2 refer to the 400-g car and the system include Earth and the cars. We can use conservation of momentum to find their speed after they have linked together and the definition of kinetic energy to find their pre- and post-collision kinetic energies.

(*a*) The pre-collision kinetic energy of the two-car system is:

$$K_{\text{pre}} = \tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}(0.250\,\text{kg})(0.50\,\text{m/s})^2 = 31.3\,\text{mJ} = \boxed{31\,\text{mJ}}$$

(*b*) Relate the velocity of the center of mass to the total momentum of the system:

$$\vec{\boldsymbol{P}}_{\text{sys}} = \sum_i m_i\vec{\boldsymbol{v}}_i = m\vec{\boldsymbol{v}}_{\text{cm}}$$

Solve for v_{cm}:

$$v_{\text{cm}} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

Substitute numerical values and evaluate v_{cm}:

$$v_{\text{cm}} = \frac{(0.250\,\text{kg})(0.50\,\text{m/s})}{0.250\,\text{kg} + 0.400\,\text{kg}} = 0.192\,\text{m/s}$$

Find the initial velocity of the 250-g car relative to the velocity of the center of mass:

$$u_1 = v_1 - v_{\text{cm}} = 0.50\,\text{m/s} - 0.192\,\text{m/s} = \boxed{0.31\,\text{m/s}}$$

Find the initial velocity of the 400-g car relative to the velocity of the center of mass:

$$u_2 = v_2 - v_{\text{cm}} = 0\,\text{m/s} - 0.192\,\text{m/s} = \boxed{-0.19\,\text{m/s}}$$

Express the pre-collision kinetic energy of the system relative to the center of mass:

$$K_{\text{pre,rel}} = \tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2$$

Substitute numerical values and evaluate $K_{\text{pre,rel}}$:

$$K_{\text{pre,rel}} = \tfrac{1}{2}(0.250\,\text{kg})(0.308\,\text{m/s})^2 + \tfrac{1}{2}(0.400\,\text{kg})(-0.192\,\text{m/s})^2 = \boxed{19\,\text{mJ}}$$

(*c*) Express the kinetic energy of the center of mass:

$$K_{\rm cm} = \tfrac{1}{2}Mv_{\rm cm}^2$$

Substitute numerical values and evaluate $K_{\rm cm}$:

$$K_{\rm cm} = \tfrac{1}{2}(0.650\,\text{kg})(0.192\,\text{m/s})^2 = \boxed{12\,\text{mJ}}$$

(*d*) Relate the pre-collision kinetic energy of the system to its pre-collision kinetic energy relative to the center of mass and the kinetic energy of the center of mass:

$$K_{\rm i} = K_{\rm i,rel} + K_{\rm cm} = 19.2\,\text{mJ} + 12.0\,\text{mJ} = 31.2\,\text{mJ}$$

and

$$K_{\rm i} = \boxed{K_{\rm i,rel} + K_{\rm cm}}$$

101 •• A 1500-kg car traveling north at 70 km/h collides at an intersection with a 2000-kg car traveling west at 55 km/h. The two cars stick together. (*a*) What is the total momentum of the system before the collision? (*b*) Find the magnitude and direction of the velocity of the wreckage just after the collision.

Picture the Problem Let east be the positive *x* direction and north the positive *y* direction. Include both cars and Earth in the system and let the numeral 1 denote the 1500-kg car and the numeral 2 the 2000-kg car. Because the net external force acting on the system is zero, momentum is conserved in this perfectly inelastic collision.

(*a*) Express the total momentum of the system:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 = m_1v_1\hat{j} - m_2v_2\hat{i}$$

Substitute numerical values and evaluate $\vec{p}$:

$$\begin{aligned}\vec{p} &= (1500\,\text{kg})(70\,\text{km/h})\hat{j} - (2000\,\text{kg})(55\,\text{km/h})\hat{i}\\ &= -(1.10\times10^5\,\text{kg}\cdot\text{km/h})\,\hat{i} + (1.05\times10^5\,\text{kg}\cdot\text{km/h})\hat{j}\\ &= \boxed{-(1.1\times10^5\,\text{kg}\cdot\text{km/h})\,\hat{i} + (1.1\times10^5\,\text{kg}\cdot\text{km/h})\hat{j}}\end{aligned}$$

(*b*) The velocity of the wreckage in terms of the total momentum of the system is given by:

$$\vec{v}_{\rm f} = \vec{v}_{\rm cm} = \frac{\vec{p}}{M}$$

Substitute numerical values and evaluate $\vec{v}_\text{f}$:

$$\vec{v}_\text{f} = \frac{-\left(1.10\times10^5\ \text{kg}\cdot\text{km/h}\right)\hat{i}}{1500\,\text{kg}+2000\,\text{kg}} + \frac{\left(1.05\times10^5\ \text{kg}\cdot\text{km/h}\right)\hat{j}}{1500\,\text{kg}+2000\,\text{kg}}$$
$$= -\left(31.4\,\text{km/h}\right)\hat{i} + \left(30.0\,\text{km/h}\right)\hat{j}$$

Find the magnitude of the velocity of the wreckage:

$$v_\text{f} = \sqrt{\left(31.4\,\text{km/h}\right)^2 + \left(30.0\,\text{km/h}\right)^2} = \boxed{43\,\text{km/h}}$$

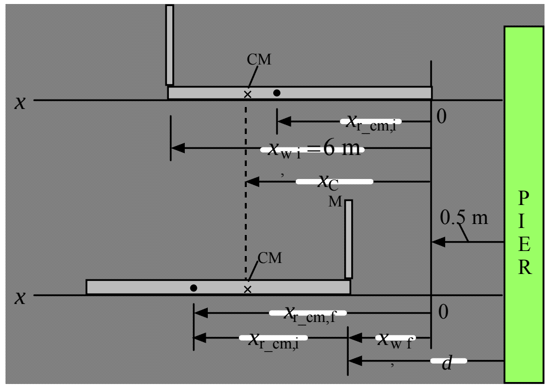
Find the direction the wreckage moves:

$$\theta = \tan^{-1}\left[\frac{30.0\,\text{km/h}}{-31.4\,\text{km/h}}\right] = -43.7°$$

The direction of the wreckage is 46° west of north.

102 •• A 60-kg woman stands on the back of a 6.0-m-long, 120-kg raft that is floating at rest in still water. The raft is 0.50 m from a fixed pier, as shown in Figure 8-52. (*a*) The woman walks to the front of the raft and stops. How far is the raft from the pier now? (*b*) While the woman walks, she maintains a constant speed of 3.0 m/s relative to the raft. Find the total kinetic energy of the system (woman plus raft), and compare with the kinetic energy if the woman walked at 3.0 m/s on a raft tied to the pier. (*c*) Where do these kinetic energies come from, and where do they go when the woman stops at the front of the raft? (*d*) On land, the woman puts a lead shot 6.0 m. She stands at the back of the raft, aims forward, and puts the shot so that just after it leaves her hand, it has the same velocity relative to her as it did when she threw it from the ground. Approximately, where does her shot land?

Picture the Problem Take the origin to be at the initial position of the right-hand end of raft and let the positive *x* direction be to the left. Let "*w*" denote the woman and "*r*" the raft, *d* be the distance of the end of the raft from the pier after the woman has walked to its front. The raft moves to the left as the woman moves to the right; with the center of mass of the woman-raft system remaining fixed (because $F_\text{ext,net} = 0$). The diagram shows the initial ($x_\text{w,i}$) and final ($x_\text{w,f}$) positions of the woman as well as the initial ($x_\text{r\_cm,i}$) and final ($x_\text{r\_cm,f}$) positions of the center of mass of the raft both before and after the woman has walked to the front of the raft.



(*a*) Express the distance of the raft from the pier after the woman has walked to the front of the raft:

$$d = 0.50\,\text{m} + x_{\text{w, f}} \qquad (1)$$

Express x_{cm} before the woman has walked to the front of the raft:

$$x_{\text{cm}} = \frac{m_{\text{w}} x_{\text{w,i}} + m_{\text{r}} x_{\text{r\_cm, i}}}{m_{\text{w}} + m_{\text{r}}}$$

Express x_{cm} after the woman has walked to the front of the raft:

$$x_{\text{cm}} = \frac{m_{\text{w}} x_{\text{w,f}} + m_{\text{r}} x_{\text{r\_cm,f}}}{m_{\text{w}} + m_{\text{r}}}$$

Because $F_{\text{ext,net}} = 0$, the center of mass remains fixed and we can equate these two expressions for x_{cm} to obtain:

$$m_{\text{w}} x_{\text{w,i}} + m_{\text{r}} x_{\text{r\_cm,i}} = m_{\text{w}} x_{\text{w,f}} + m_{\text{r}} x_{\text{r\_cm,f}}$$

Solve for $\boldsymbol{x}_{\text{w,f}}$:

$$x_{\text{w,f}} = x_{\text{w,i}} - \frac{m_{\text{r}}}{m_{\text{w}}}\left(x_{\text{r\_cm,f}} - x_{\text{r\_cm,i}}\right)$$

From the figure it can be seen that $\boldsymbol{x}_{\text{r\_cm,f}} - \boldsymbol{x}_{\text{r\_cm,i}} = \boldsymbol{x}_{\text{w,f}}$. Substitute $\boldsymbol{x}_{\text{w,f}}$ for $\boldsymbol{x}_{\text{r\_cm,f}} - \boldsymbol{x}_{\text{r\_cm,i}}$ to obtain:

$$x_{\text{w,f}} = \frac{m_{\text{w}} x_{\text{w,i}}}{m_{\text{w}} + m_{\text{r}}}$$

Substitute numerical values and evaluate $\boldsymbol{x}_{\text{w,f}}$:

$$\boldsymbol{x}_{\text{w,f}} = \frac{(60\,\text{kg})(6.0\,\text{m})}{60\,\text{kg} + 120\,\text{kg}} = 2.0\,\text{m}$$

Substitute in equation (1) to obtain:

$$d = 2.0\,\text{m} + 0.50\,\text{m} = \boxed{2.5\,\text{m}}$$

(*b*) Express the total kinetic energy of the system:

$$K_{\text{tot}} = \tfrac{1}{2} m_{\text{w}} v_{\text{w}}^2 + \tfrac{1}{2} m_{\text{r}} v_{\text{r}}^2$$

Noting that the elapsed time is 2.0 s, find v_w and v_r:

$$v_w = \frac{x_{w,f} - x_{w,i}}{\Delta t} = \frac{2.0\,\text{m} - 6.0\,\text{m}}{2.0\,\text{s}} = -2.0\,\text{m/s}$$

relative to the dock, and

$$v_r = \frac{x_{r,f} - x_{r,i}}{\Delta t} = \frac{2.50\,\text{m} - 0.50\,\text{m}}{2.0\,\text{s}} = 1.0\,\text{m/s}$$

also relative to the dock.

Substitute numerical values and evaluate K_{tot}:

$$K_{tot} = \tfrac{1}{2}(60\,\text{kg})(-2.0\,\text{m/s})^2 + \tfrac{1}{2}(120\,\text{kg})(1.0\,\text{m/s})^2 = \boxed{0.18\,\text{kJ}}$$

Evaluate K with the raft tied to the pier:

$$K_{tot} = \tfrac{1}{2}m_w v_w^2 = \tfrac{1}{2}(60\,\text{kg})(3.0\,\text{m/s})^2 = \boxed{0.27\,\text{kJ}}$$

(*c*) All the kinetic energy derives from the chemical energy of the woman and, assuming she stops via static friction, the kinetic energy is transformed into her internal energy.

(*d*) After the shot leaves the woman's hand, the raft-woman system constitutes an inertial reference frame. In that frame, the shot has the same initial velocity as did the shot that had a range of 6.0 m in the reference frame of the land. Thus, in the raft-woman frame, the shot also has a range of 6.0 m and lands at the front of the raft.

103 •• A 1.0-kg steel ball and a 2.0-m cord of negligible mass make up a simple pendulum that can pivot without friction about the point *O*, as in Figure 8-53. This pendulum is released from rest in a horizontal position and when the ball is at its lowest point it strikes a 1.0-kg block sitting at rest on a shelf. Assume that the collision is perfectly elastic and take the coefficient of kinetic friction between the block and shelf to be 0.10. (*a*) What is the velocity of the block just after impact? (*b*) How far does the block slide before coming to rest (assuming the shelf is long enough)?

Picture the Problem Let the zero of gravitational potential energy be at the elevation of the 1.0-kg block. We can use conservation of energy to find the speed of the bob just before its perfectly elastic collision with the block and

conservation of momentum to find the speed of the block immediately after the collision. We'll apply Newton's second law to find the acceleration of the sliding block and use a constant-acceleration equation to find how far it slides before coming to rest.

(*a*) Use conservation of energy to find the speed of the bob just before its collision with the block:

$$\Delta K + \Delta U = 0$$
or
$$K_f - K_i + U_f - U_i = 0$$

Because $K_i = U_f = 0$:

$$\tfrac{1}{2} m_{ball} v_{ball}^2 + m_{ball} g\Delta h = 0$$
and
$$v_{ball} = \sqrt{2g\Delta h}$$

Substitute numerical values and evaluate v_{ball}:

$$v_{ball} = \sqrt{2(9.81\,\text{m/s}^2)(2.0\,\text{m})} = 6.26\text{ m/s}$$

Because the collision is perfectly elastic and the ball and block have the same mass:

$$v_{block} = v_{ball} = \boxed{6.3\,\text{m/s}}$$

(*b*) Using a constant-acceleration equation, relate the displacement of the block to its acceleration and initial speed:

$$v_f^2 = v_i^2 + 2a_{block}\Delta x$$
or, because $v_f = 0$,
$$0 = v_i^2 + 2a_{block}\Delta x$$

Solving for Δx yields:

$$\Delta x = \frac{-v_i^2}{2a_{block}} = \frac{-v_{block}^2}{2a_{block}}$$

Apply $\sum \vec{F} = m\vec{a}$ to the sliding block:

$$\sum F_x = -f_k = ma_{block}$$
and
$$\sum F_y = F_n - m_{block} g = 0$$

Using the definition of f_k ($=\mu_k F_n$) eliminate f_k and F_n between the two equations and solve for a_{block}:

$$a_{block} = -\mu_k g$$

Substitute for a_{block} to obtain:

$$\Delta x = \frac{-v_{block}^2}{-2\mu_k g} = \frac{v_{block}^2}{2\mu_k g}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{(6.26\,\text{m/s})^2}{2(0.10)(9.81\,\text{m/s}^2)} = \boxed{20\,\text{m}}$$

104 •• Figure 8-54 shows a World War I cannon mounted on a railcar so that it will project a shell at an angle of 30º. With the car initially at rest, the cannon fires a 200-kg projectile at 125 m/s. (All values are for the frame of reference of the track.) Now consider a system composed of a cannon, shell, and railcar, all on the frictionless track. (*a*) Will the total vector momentum of that system be the same just before and just after the shell is fired? Explain your answer. (*b*) If the mass of the railcar plus cannon is 5000 kg, what will be the recoil velocity of the car along the track after the firing? (*c*) The shell is observed to rise to a maximum height of 180 m as it moves through its trajectory. At this point, its speed is 80.0 m/s. On the basis of this information, calculate the amount of thermal energy produced by air friction on the shell on its way from firing to this maximum height.

Picture the Problem We can use conservation of momentum in the horizontal direction to find the recoil velocity of the car along the track after the firing. Because the shell will neither rise as high nor be moving as fast at the top of its trajectory as it would be in the absence of air friction, we can apply the work-energy theorem to find the amount of thermal energy produced by the air friction.

(*a*) No. The vertical reaction force of the rails is an external force and so the momentum of the system will not be conserved.

(*b*) Use conservation of momentum in the horizontal (x) direction to obtain:

$$\Delta p_x = 0$$

or

$$mv\cos 30° - Mv_{\text{recoil}} = 0$$

Solving for v_{recoil} yields:

$$v_{\text{recoil}} = \frac{mv\cos 30°}{M}$$

Substitute numerical values and evaluate v_{recoil}:

$$v_{\text{recoil}} = \frac{(200\,\text{kg})(125\,\text{m/s})\cos 30°}{5000\,\text{kg}} = \boxed{4.3\,\text{m/s}}$$

(*c*) Using the work-energy theorem, relate the thermal energy produced by air friction to the change in the energy of the system:

$$W_{\text{ext}} = W_{\text{f}} = \Delta E_{\text{sys}} = \Delta U + \Delta K$$

Substitute for ΔU and ΔK to obtain:

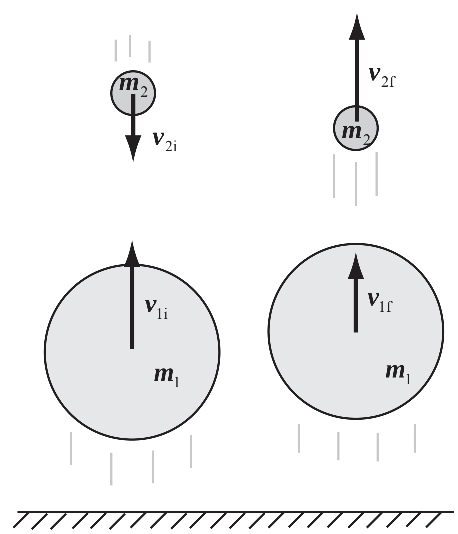
$$W_{\text{ext}} = mgy_{\text{f}} - mgy_{\text{i}} + \tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_i^2$$
$$= mg\left(y_{\text{f}} - y_{\text{i}}\right) + \tfrac{1}{2}m\left(v_{\text{f}}^2 - v_i^2\right)$$

Substitute numerical values and evaluate W_{ext}:

$$W_{\text{ext}} = (200\,\text{kg})\left(9.81\,\text{m/s}^2\right)(180\,\text{m}) + \tfrac{1}{2}(200\,\text{kg})\left[(80.0\,\text{m/s})^2 - (125\,\text{m/s})^2\right]$$
$$= \boxed{-569\,\text{kJ}}$$

105 ••• [SSM] One popular, if dangerous, classroom demonstration involves holding a baseball an inch or so directly above a basketball, holding the basketball a few feet above a hard floor, and dropping the two balls simultaneously. The two balls will collide just after the basketball bounces from the floor; the baseball will then rocket off into the ceiling tiles with a hard "thud" while the basketball will stop in midair. (The author of this problem once broke a light doing this.)
(*a*) Assuming that the collision of the basketball with the floor is elastic, what is the relation between the velocities of the balls just before they collide?
(*b*) Assuming the collision between the two balls is elastic, use the result of Part (*a*) and the conservation of momentum and energy to show that, if the basketball is three times as heavy as the baseball, the final velocity of the basketball will be zero. (This is approximately the true mass ratio, which is why the demonstration is so dramatic.) (*c*) If the speed of the baseball is *v* just before the collision, what is its speed just after the collision?

Picture the Problem Let the numeral 1 refer to the basketball and the numeral 2 to the baseball. The left-hand side of the diagram shows the balls after the basketball's elastic collision with the floor and just before they collide. The right-hand side of the diagram shows the balls just after their collision. We can apply conservation of momentum and the definition of an elastic collision to obtain equations relating the initial and final velocities of the colliding balls that we can solve for $v_{1\text{f}}$ and $v_{2\text{f}}$.



(*a*) Because both balls are in free-fall, and both are in the air for the same amount of time, they have the same velocity just before the basketball rebounds. After the basketball rebounds elastically, its velocity will have the same magnitude, but the opposite direction than just before it hit the ground. The velocity of the basketball will be equal in magnitude but opposite in direction to the velocity of the baseball.

(*b*) Apply conservation of linear momentum to the collision of the balls to obtain:

$$m_1v_{1\text{f}} + m_2v_{2\text{f}} = m_1v_{1\text{i}} + m_2v_{2\text{i}} \quad (1)$$

Use conservation of mechanical energy to set the speed of recession equal to the negative of the speed of approach:

$$v_{2\text{f}} - v_{1\text{f}} = -(v_{1\text{i}} - v_{2\text{i}})$$
or
$$v_{1\text{f}} - v_{2\text{f}} = v_{2\text{i}} - v_{1\text{i}} \quad (2)$$

Multiply equation (2) by m_2 and add it to equation (1) to obtain:

$$(m_1 + m_2)v_{1\text{f}} = (m_1 - m_2)v_{1\text{i}} + 2m_2v_{2\text{i}}$$

Solve for $v_{1\text{f}}$ to obtain:

$$v_{1\text{f}} = \frac{m_1 - m_2}{m_1 + m_2}v_{1\text{i}} + \frac{2m_2}{m_1 + m_2}v_{2\text{i}}$$

or, because $v_{2\text{i}} = -v_{1\text{i}}$,

$$v_{1\text{f}} = \frac{m_1 - m_2}{m_1 + m_2}v_{1\text{i}} - \frac{2m_2}{m_1 + m_2}v_{1\text{i}} = \frac{m_1 - 3m_2}{m_1 + m_2}v_{1\text{i}}$$

For $m_1 = 3m_2$ and $v_{1\text{i}} = v$:

$$v_{1\text{f}} = \frac{3m_2 - 3m_2}{3m_2 + m_2}v = \boxed{0}$$

(*c*) Multiply equation (2) by m_1 and subtract it from equation (1) to obtain:

$$(m_1 + m_2)v_{2\text{f}} = (m_2 - m_1)v_{2\text{i}} + 2m_1v_{1\text{i}}$$

Solve for $v_{2\text{f}}$ to obtain:

$$v_{2\text{f}} = \frac{2m_1}{m_1 + m_2}v_{1\text{i}} + \frac{m_2 - m_1}{m_1 + m_2}v_{2\text{i}}$$

or, because $v_{2\text{i}} = -v_{1\text{i}}$,

$$v_{2\text{f}} = \frac{2m_1}{m_1 + m_2}v_{1\text{i}} - \frac{m_2 - m_1}{m_1 + m_2}v_{1\text{i}} = \frac{3m_1 - m_2}{m_1 + m_2}v_{1\text{i}}$$

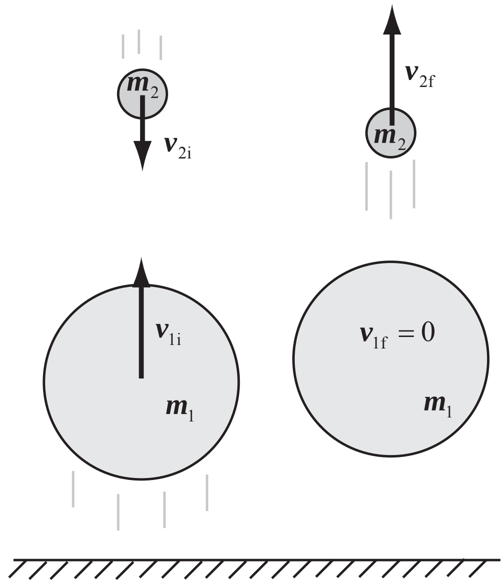
For $m_1 = 3m_2$ and $v_{1\text{i}} = v$:

$$v_{2\text{f}} = \frac{3(3m_2) - m_2}{3m_2 + m_2}v = \boxed{2v}$$

106 ••• (*a*) Referring to Problem 105, if we held a third ball above the baseball and basketball, and wanted both the basketball and baseball to stop in mid-air, what should the ratio of the mass of the top ball to the mass of the baseball be?

(*b*) If the speed of the top ball is *v* just before the collision, what is its speed just after the collision?

Picture the Problem In Problem 105 only two balls are dropped. They collide head on, each moving at speed *v*, and the collision is elastic. In this problem, as it did in Problem 105, the solution involves using the conservation of momentum equation $m_1v_{1f} + m_2v_{2f} = m_1v_{1i} + m_2v_{2i}$ and the elastic collision equation $v_{1f} - v_{2f} = v_{2i} - v_{1i}$ where the numeral 1 refers to the baseball, and the numeral 2 to the top ball. The diagram shows the balls just before and just after their collision. From Problem 105 we know that $v_{1i} = 2v$ and $v_{2i} = -v$.



(*a*) Express the final speed v_{1f} of the baseball as a function of its initial speed v_{1i} and the initial speed of the top ball v_{2i} (see Problem 64):

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i} + \frac{2m_2}{m_1 + m_2}v_{2i}$$

Substitute for v_{1i} and , v_{2i} to obtain:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}(2v) + \frac{2m_2}{m_1 + m_2}(-v)$$

Divide the numerator and denominator of each term by m_2 to introduce the mass ratio of the upper ball to the lower ball:

$$v_{1f} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1}(2v) + \frac{2}{\frac{m_1}{m_2} + 1}(-v)$$

Set the final speed of the baseball v_{1f} equal to zero and let *x* represent the mass ratio m_1/m_2 to obtain:

$$0 = \frac{x-1}{x+1}(2v) + \frac{2}{x+1}(-v)$$

Solving for *x* yields:

$$x = \frac{m_1}{m_2} = \boxed{\tfrac{1}{2}}$$

(*b*) Apply the second of the two equations in Problem 64 to the collision between the top ball and the baseball:

$$v_{2f} = \frac{2m_1}{m_1 + m_2}v_{1i} + \frac{m_2 - m_1}{m_1 + m_2}v_{2i}$$

Substitute $v_{1i} = 2v$ and $v_{2i} = -v$ to obtain:

$$v_{2f} = \frac{2m_1}{m_1 + m_2}(2v) + \frac{m_2 - m_1}{m_1 + m_2}(-v)$$

In part (*a*) we showed that $m_2 = 2m_1$. Substitute and simplify to obtain:

$$v_{3f} = \frac{2(2m_1)}{m_1 + 2m_1}(2v) - \frac{2m_1 - m_1}{m_1 + 2m_1}v = \boxed{\tfrac{7}{3}v}$$

107 ••• [SSM] In the "slingshot effect," the transfer of energy in an elastic collision is used to boost the energy of a space probe so that it can escape from the solar system. All speeds are relative to an inertial frame in which the center of the sun remains at rest. Figure 8-55 shows a space probe moving at 10.4 km/s toward Saturn, which is moving at 9.6 km/s toward the probe. Because of the gravitational attraction between Saturn and the probe, the probe swings around Saturn and heads back in the opposite direction with speed v_f. (*a*) Assuming this collision to be a one-dimensional elastic collision with the mass of Saturn much greater than that of the probe, find v_f. (*b*) By what factor is the kinetic energy of the probe increased? Where does this energy come from?

Picture the Problem Let the direction the probe is moving after its elastic collision with Saturn be the positive direction. The probe gains kinetic energy at the expense of the kinetic energy of Saturn. We'll relate the speed of approach relative to the center of mass to u_{rec} and then to v. The $+x$ direction is also the direction of the motion of Saturn.

(*a*) Relate the speed of recession to the speed of recession relative to the center of mass:

$$v = u_{rec} + v_{cm} \quad (1)$$

Find the speed of approach:

$$u_{app} = -9.6\,\text{km/s} - 10.4\,\text{km/s} = -20.0\,\text{km/s}$$

Relate the relative speed of approach to the relative speed of recession for an elastic collision:

$$u_{rec} = -u_{app} = 20.0\,\text{km/s}$$

Because Saturn is so much more massive than the space probe:

$$v_{cm} = v_{Saturn} = 9.6\,\text{km/s}$$

Substitute numerical values in equation (1) and evaluate v:

$$v = 20\,\text{km/s} + 9.6\,\text{km/s} = \boxed{30\,\text{km/s}}$$

(*b*) Express the ratio of the final kinetic energy to the initial kinetic energy and simplify:

$$\frac{K_f}{K_i}=\frac{\frac{1}{2}Mv_{rec}^2}{\frac{1}{2}Mv_i^2}=\left(\frac{v_{rec}}{v_i}\right)^2$$

Substitute numerical values and evaluate K_f/K_i:

$$\frac{K_f}{K_i}=\left(\frac{29.6\,\text{km/s}}{10.4\,\text{km/s}}\right)^2=\boxed{8.1}$$

The energy comes from an immeasurably small slowing of Saturn.

108 •• A 13-kg block is at rest on a level floor. A 400-g glob of putty is thrown at the block so that the putty travels horizontally, hits the block, and sticks to it. The block and putty slide 15 cm along the floor. If the coefficient of kinetic friction is 0.40, what is the initial speed of the putty?

Picture the Problem Let the system include the block, the putty, and Earth. Then $F_{ext,net}$ = 0 and momentum is conserved in this perfectly inelastic collision. We'll use conservation of momentum to relate the after-collision velocity of the block plus blob and conservation of energy to find their after-collision velocity.

Noting that, because this is a perfectly elastic collision, the final velocity of the block plus blob is the velocity of the center of mass, use conservation of momentum to relate the velocity of the center of mass to the velocity of the glob before the collision:

$$p_i=p_f$$

or

$$m_{gl}v_{gl}=Mv_{cm}\Rightarrow v_{gl}=\left(\frac{M}{m_{gl}}\right)v_{cm}\quad(1)$$

where $M=m_{gl}+m_{bl}$.

Use conservation of energy to find the initial energy of the block plus glob:

$$\Delta K+\Delta U+W_f=0$$

Because $\Delta U=K_f=0$,

$$-\tfrac{1}{2}Mv_{cm}^2+f_k\Delta x=0$$

Because $f_k=\mu_k Mg$:

$$-\tfrac{1}{2}Mv_{cm}^2+\mu_k Mg\Delta x=0$$

Solve for v_{cm} to obtain:

$$v_{cm}=\sqrt{2\mu_k g\Delta x}$$

Substitute numerical values and evaluate v_{cm}:

$$v_{cm}=\sqrt{2(0.40)(9.81\,\text{m/s}^2)(0.15\,\text{m})}=1.08\,\text{m/s}$$

Substitute numerical values in equation (1) and evaluate v_{gl}:

$$v_{gl} = \left(\frac{13\,\text{kg} + 0.400\,\text{kg}}{0.400\,\text{kg}}\right)(1.08\,\text{m/s}) = \boxed{36\,\text{m/s}}$$

109 ••• [SSM] Your accident reconstruction team has been hired by the local police to analyze the following accident. A careless driver rear-ended a car that was halted at a stop sign. Just before impact, the driver slammed on his brakes, locking the wheels. The driver of the struck car had his foot solidly on the brake pedal, locking his brakes. The mass of the struck car was 900 kg, and that of the initially moving vehicle was 1200 kg. On collision, the bumpers of the two cars meshed. Police determine from the skid marks that after the collision the two cars moved 0.76 m together. Tests revealed that the coefficient of kinetic friction between the tires and pavement was 0.92. The driver of the moving car claims that he was traveling at less than 15 km/h as he approached the intersection. Is he telling the truth?

Picture the Problem Let the direction the moving car was traveling before the collision be the +*x* direction. Let the numeral 1 denote this car and the numeral 2 the car that is stopped at the stop sign and the system include both cars and Earth. We can use conservation of momentum to relate the speed of the initially-moving car to the speed of the meshed cars immediately after their perfectly inelastic collision and conservation of energy to find the initial speed of the meshed cars.

Using conservation of momentum, relate the before-collision speed to the after-collision speed of the meshed cars:

$$p_i = p_f$$

or

$$m_1v_1 = (m_1 + m_2)V$$

Solving for v_1 and simplifying yields:

$$v_1 = \frac{m_1 + m_2}{m_1}V = \left(1 + \frac{m_2}{m_1}\right)V \qquad (1)$$

Using conservation of energy, relate the initial kinetic energy of the meshed cars to the work done by friction in bringing them to a stop:

$$\Delta K + \Delta E_{\text{thermal}} = 0$$

or, because $K_f = 0$ and $\Delta E_{\text{thermal}} = f\Delta s$,

$$-K_i + f_k\Delta s = 0$$

Substitute for K_i and, using $f_k = \mu_k F_n = \mu_k Mg$, eliminate f_k to obtain:

$$-\tfrac{1}{2}MV^2 + \mu_k Mg\Delta x = 0$$

Solving for V yields:

$$V = \sqrt{2\mu_k g\Delta x}$$

Substitute for V in equation (1) to obtain:

$$v_1 = \left(1 + \frac{m_2}{m_1}\right)\sqrt{2\mu_k g\Delta x}$$

Substitute numerical values and evaluate v_1:

$$v_1 = \left(1 + \frac{900\,\text{kg}}{1200\,\text{kg}}\right)\sqrt{2(0.92)(9.81\,\text{m/s}^2)(0.76\,\text{m})} = 6.48\,\text{m/s} = 23\,\text{km/h}$$

The driver was not telling the truth. He was traveling at 23 km/h.

110 •• A pendulum consists of a compact 0.40-kg bob attached to a string of length 1.6 m. A block of mass *m* rests on a horizontal frictionless surface. The pendulum is released from rest at an angle of 53º with the vertical. The bob collides elastically with the block at the lowest point in its arc. Following the collision, the maximum angle of the pendulum with the vertical is 5.73º. Determine the mass *m*.

Picture the Problem Let the zero of gravitational potential energy be at the lowest point of the bob's swing and note that the bob can swing either forward or backward after the collision. We'll use both conservation of momentum and conservation of energy to relate the velocities of the bob and the block before and after their collision. Choose the $+x$ direction to be in the direction of the motion of the block.

Express the kinetic energy of the block in terms of its after-collision momentum:

$$K_\text{m} = \frac{p_m^2}{2m} \Rightarrow m = \frac{p_m^2}{2K_m} \qquad (1)$$

Use conservation of energy to relate K_m to the change in the potential energy of the bob:

$$\Delta K + \Delta U = 0$$

or, because $K_\text{i} = 0$,

$$K_m + U_\text{f} - U_\text{i} = 0$$

Solve for K_m, substitute for U_f and U_i and simplify to obtain:

$$\begin{aligned} K_m &= -U_\text{f} + U_\text{i} \\ &= m_\text{bob} g\left[L(1-\cos\theta_\text{i}) - L(1-\cos\theta_\text{f})\right] \\ &= m_\text{bob} gL\left[\cos\theta_\text{f} - \cos\theta_\text{i}\right] \end{aligned}$$

Substitute numerical values and evaluate K_m:

$$K_m = (0.40\,\text{kg})(9.81\,\text{m/s}^2)(1.6\,\text{m})[\cos 5.73° - \cos 53°] = 2.47\,\text{J}$$

Use conservation of energy to find the velocity of the bob just before its collision with the block:	$\Delta K + \Delta U = 0$ or, because $K_i = U_f = 0$, $K_f - U_i = 0$
Substitute for K_f and U_i to obtain:	$\frac{1}{2} m_{bob} v^2 - m_{bob} gL(1-\cos\theta_i) = 0$
Solving for v yields:	$v = \sqrt{2gL(1-\cos\theta_i)}$
Substitute numerical values and evaluate v:	$v = \sqrt{2(9.81\,\text{m/s}^2)(1.6\,\text{m})(1-\cos 53°)} = 3.536\,\text{m/s}$
Use conservation of energy to find the velocity of the bob just after its collision with the block:	$\Delta K + \Delta U = 0$ or, because $K_f = U_i = 0$, $-K_i + U_f = 0$
Substitute for K_i and U_f to obtain:	$-\frac{1}{2} m_{bob} v'^2 + m_{bob} gL(1-\cos\theta_f) = 0$
Solve for v':	$v' = \sqrt{2gL(1-\cos\theta_f)}$
Substitute numerical values and evaluate v':	$v' = \sqrt{2(9.81\,\text{m/s}^2)(1.6\,\text{m})(1-\cos 5.73°)} = 0.396\,\text{m/s}$
Use conservation of momentum to relate p_m after the collision to the momentum of the bob just before and just after the collision:	$p_i = p_f$ or $m_{bob} v = \pm m_{bob} v' + p_m$
Solve for and evaluate p_m:	$p_m = m_{bob} v \pm m_{bob} v' = (0.40\,\text{kg})(3.536\,\text{m/s} \pm 0.396\,\text{m/s}) = 1.414\,\text{kg}\cdot\text{m/s} \pm 0.158\,\text{kg}\cdot\text{m/s}$
Find the larger value for p_m:	$p_m = 1.414\,\text{kg}\cdot\text{m/s} + 0.158\,\text{kg}\cdot\text{m/s} = 1.573\,\text{kg}\cdot\text{m/s}$
Find the smaller value for p_m:	$p_m = 1.414\,\text{kg}\cdot\text{m/s} - 0.158\,\text{kg}\cdot\text{m/s} = 1.256\,\text{kg}\cdot\text{m/s}$

Substitute numerical values in equation (1) to determine the two values for m:

$$m = \frac{(1.573\,\text{kg}\cdot\text{m/s})^2}{2(2.47\,\text{J})} = \boxed{0.50\,\text{kg}}$$

or

$$m = \frac{(1.256\,\text{kg}\cdot\text{m/s})^2}{2(2.47\,\text{J})} = \boxed{0.32\,\text{kg}}$$

111 ••• [SSM] A 1.00-kg block and a second block of mass M are both initially at rest on a frictionless inclined plane (Figure 8-56) Mass M rests against a spring that has a force constant of 11.0 kN/m. The distance along the plane between the two blocks is 4.00 m. The 1.00-kg block is released, making an elastic collision with the unknown block. The 1.00-kg block then rebounds a distance of 2.56 m back up the inclined plane. The block of mass M comes momentarily comes to rest 4.00 cm from its initial position. Find M.

Picture the Problem Choose the zero of gravitational potential energy at the location of the spring's maximum compression. Let the system include the spring, the blocks, and Earth. Then the net external force is zero as is work done against friction. We can use conservation of energy to relate the energy transformations taking place during the evolution of this system.

Apply conservation of energy to the system:

$$\Delta K + \Delta U_g + \Delta U_s = 0$$

Because $\Delta K = 0$:

$$\Delta U_g + \Delta U_s = 0$$

Express the change in the gravitational potential energy:

$$\Delta U_g = -mg\Delta h - Mgx\sin\theta$$

Express the change in the potential energy of the spring:

$$\Delta U_s = \tfrac{1}{2}kx^2$$

Substitute to obtain:

$$-mg\Delta h - Mgx\sin\theta + \tfrac{1}{2}kx^2 = 0$$

Solving for M and simplifying yields:

$$M = \frac{\tfrac{1}{2}kx^2 - mg\Delta h}{gx\sin 30°} = \frac{kx}{g} - \frac{2m\Delta h}{x}$$

Relate Δh to the initial and rebound positions of the block whose mass is m:

$$\Delta h = (4.00\,\text{m} - 2.56\,\text{m})\sin 30° = 0.72\,\text{m}$$

Substitute numerical values and evaluate M:

$$M = \frac{\left(11.0\times10^{3}\ \text{N/m}\right)\left(0.0400\,\text{m}\right)}{9.81\,\text{m/s}^2} - \frac{2\left(1.00\,\text{kg}\right)\left(0.72\,\text{m}\right)}{0.0400\,\text{m}} = \boxed{8.9\,\text{kg}}$$

112 ••• A neutron of mass m makes an elastic head-on collision with a stationary nucleus of mass M. (*a*) Show that the kinetic energy of the nucleus after the collision is given by $K_{\text{nucleus}} = [4mM/(m + M)^2]K_{\text{n}}$, where K_{n} is the initial kinetic energy of the neutron. (*b*) Show that the *fractional* change in the kinetic energy of the neutron is given by

$$\frac{\Delta K_{\text{n}}}{K_{\text{n}}} = -\frac{4(m/M)}{\left(1+[m/M]\right)^2}\,.$$

(*c*) Show that this expression gives plausible results both if $m << M$ and $m = M$. What is the best stationary nucleus for the neutron to collide head-on with if the objective is to produce a maximum loss in the kinetic energy of the neutron?

Picture the Problem In this elastic head-on collision, the kinetic energy of recoiling nucleus is the difference between the initial and final kinetic energies of the neutron. We can derive the indicated results by using both conservation of energy and conservation of momentum and writing the kinetic energies in terms of the momenta of the particles before and after the collision.

(*a*) Use conservation of energy to relate the kinetic energies of the particles before and after the collision:

$$\frac{p_{\text{ni}}^2}{2m} = \frac{p_{\text{nf}}^2}{2m} + \frac{p_{\text{nucleus}}^2}{2M} \qquad (1)$$

Apply conservation of momentum to obtain a second relationship between the initial and final momenta:

$$p_{\text{ni}} = p_{\text{nf}} + p_{\text{nucleus}} \qquad (2)$$

Eliminate p_{nf} in equation (1) using equation (2):

$$\frac{p_{\text{nucleus}}}{2M} + \frac{p_{\text{nucleus}}}{2m} - \frac{p_{\text{ni}}}{m} = 0 \qquad (3)$$

Use equation (3) to write $p_{\text{ni}}^2/2m$ in terms of p_{nucleus}:

$$\frac{p_{\text{ni}}^2}{2m} = K_{\text{n}} = \frac{p_{\text{nucleus}}^2\left(M+m\right)^2}{8M^2m} \qquad (4)$$

Use equation (4) to express $K_{\text{nucleus}} = p_{\text{nucleus}}^2/2M$ in terms of K_{n}:

$$K_{\text{nucleus}} = \boxed{K_{\text{n}}\left[\frac{4Mm}{\left(M+m\right)^2}\right]} \qquad (5)$$

(*b*) Relate the *change* in the kinetic energy of the neutron to the after-collision kinetic energy of the nucleus:

$$\Delta K_{\text{n}} = -K_{\text{nucleus}}$$

Using equation (5), express the fraction of the energy lost in the collision:

$$\frac{\Delta K_{\text{n}}}{K_{\text{n}}} = -\frac{4Mm}{(M+m)^2} = \boxed{-\frac{4(m/M)}{(1+(m/M))^2}}$$

(*c*) If $m << M$:

$$\frac{\Delta K_{\text{n}}}{K_{\text{n}}} \to \boxed{0} \text{ as expected.}$$

If $m = M$:

$$\frac{\Delta K_{\text{n}}}{K_{\text{n}}} = -\frac{4}{(1+1)^2} = \boxed{-1} \text{ as expected.}$$

113 ••• The mass of a carbon nucleus is approximately 12 times the mass of a neutron. (*a*) Use the results of Problem 112 to show that after *N* head-on collisions of a neutron with carbon nuclei at rest, the kinetic energy of the neutron is approximately $0.716^N K_0$, where K_0 is its initial kinetic energy. (*b*) Neutrons emitted during the fission of a uranium nucleus have kinetic energies of about 2.0 MeV. For such a neutron to cause the fission of another uranium nucleus in a reactor, its kinetic energy must be reduced to about 0.020 eV. How many head-on collisions are needed to reduce the kinetic energy of a neutron from 2.0 MeV to 0.020 eV, assuming elastic head-on collisions with stationary carbon nuclei?

Picture the Problem Problem 112 (*b*) provides an expression for the fractional loss of kinetic energy per collision.

(*a*) Using the result of Problem 112 (*b*), express the fractional loss of energy per collision:

$$\frac{K_{\text{nf}}}{K_{\text{ni}}} = \frac{K_{\text{ni}} - \Delta K_{\text{n}}}{E_0} = \frac{(M-m)^2}{(M+m)^2}$$

Evaluate this fraction to obtain:

$$\frac{K_{\text{nf}}}{E_0} = \frac{(12m-m)^2}{(12m+m)^2} = 0.716$$

Express the kinetic energy of one neutron after *N* collisions:

$$K_{\text{nf}} = \boxed{0.716^N E_0}$$

(*b*) Substitute for K_{nf} and E_0 to obtain:

$$0.716^N = 10^{-8}$$

Take the logarithm of both sides of the equation and solve for N:

$$N = \frac{-8}{\log 0.716} \approx \boxed{55}$$

114 ••• On average, a neutron actually loses only 63 percent of its energy in an elastic collision with a hydrogen atom (not 100 percent) and 11 percent of its energy during an elastic collision with a carbon atom (not 28 percent). (These numbers are an average over all types of collisions, not just head-on ones. Thus the results are lower than the ones determined from analyses like that in Problem 113 because most collisions are not head-on.) Calculate the actual number of collisions, on average, needed to reduce the energy of a neutron from 2.0 MeV to 0.020 eV if the neutron collides with stationary (*a*) hydrogen atoms and (*b*) carbon atoms.

Picture the Problem We can relate the number of collisions needed to reduce the energy of a neutron from 2 MeV to 0.02 eV to the fractional energy loss per collision and solve the resulting exponential equation for N.

(*a*) Using the result of Problem 113 (*b*), express the fractional loss of energy per collision:

$$\frac{K_{\text{nf}}}{K_{\text{ni}}} = \frac{K_{\text{ni}} - \Delta K_{\text{n}}}{E_0} = \frac{K_{\text{ni}} - 0.63K_{\text{ni}}}{K_{\text{ni}}} = 0.37$$

Express the kinetic energy of one neutron after N collisions:

$$K_{\text{nf}} = 0.37^N K_0$$

Substitute for K_{nf} and K_0 to obtain:

$$0.37^N = 10^{-8}$$

Take the logarithm of both sides of the equation and solve for N:

$$N = \frac{-8}{\log 0.37} \approx \boxed{19}$$

(*b*) Proceed as in (*a*) to obtain:

$$\frac{K_{\text{nf}}}{K_{\text{ni}}} = \frac{K_{\text{ni}} - \Delta K_{\text{n}}}{E_0} = \frac{K_{\text{ni}} - 0.11K_{\text{ni}}}{K_{\text{ni}}} = 0.89$$

Express the kinetic energy of one neutron after N collisions:

$$K_{\text{nf}} = 0.89^N K_0$$

Substitute for K_{nf} and K_0 to obtain:

$$0.89^N = 10^{-8}$$

Take the logarithm of both sides of the equation and solve for N:

$$N = \frac{-8}{\log 0.89} \approx \boxed{158}$$

115 ••• **[SSM]** Two astronauts at rest face each other in space. One, with mass m_1, throws a ball of mass m_b to the other, whose mass is m_2. She catches the ball and throws it back to the first astronaut. Following each throw the ball has a speed of v relative to the thrower. After each has made one throw and one catch, (*a*) How fast are the astronauts moving? (*b*) How much has the two-astronaut system's kinetic energy changed and where did this energy come from?

Picture the Problem Let the direction that astronaut 1 first throws the ball be the positive direction and let v_b be the initial speed of the ball in the laboratory frame. Note that each collision is perfectly inelastic. We can apply conservation of momentum and the definition of the speed of the ball relative to the thrower to each of the perfectly inelastic collisions to express the final speeds of each astronaut after one throw and one catch.

(*a*) Use conservation of linear momentum to relate the speeds of astronaut 1 and the ball after the first throw:

$$m_1v_1 + m_bv_b = 0 \tag{1}$$

Relate the speed of the ball in the laboratory frame to its speed relative to astronaut 1:

$$v = v_b - v_1 \tag{2}$$

Eliminate v_b between equations (1) and (2) and solve for v_1:

$$v_1 = -\frac{m_b}{m_1 + m_b}v \tag{3}$$

Substitute equation (3) in equation (2) and solve for v_b:

$$v_b = \frac{m_1}{m_1 + m_b}v \tag{4}$$

Apply conservation of linear momentum to express the speed of astronaut 2 and the ball after the first catch:

$$0 = m_bv_b = (m_2 + m_b)v_2 \tag{5}$$

Solving for v_2 yields:

$$v_2 = \frac{m_b}{m_2 + m_b}v_b \tag{6}$$

Express v_2 in terms of v by substituting equation (4) in equation (6):

$$\begin{aligned} v_2 &= \frac{m_b}{m_2 + m_b}\frac{m_1}{m_1 + m_b}v \\ &= \left[\frac{m_bm_1}{(m_2 + m_b)(m_1 + m_b)}\right]v \end{aligned} \tag{7}$$

Use conservation of momentum to express the speed of astronaut 2 and the ball after she throws the ball:

$$(m_2 + m_b)v_2 = m_b v_{bf} + m_2 v_{2f} \qquad (8)$$

Relate the speed of the ball in the laboratory frame to its speed relative to astronaut 2:

$$v = v_{2f} - v_{bf} \qquad (9)$$

Eliminate v_{bf} between equations (8) and (9) and solve for v_{2f}:

$$v_{2f} = \boxed{\left(\frac{m_b}{m_2 + m_b}\right)\left(1 + \frac{m_1}{m_1 + m_b}\right)v} \qquad (10)$$

Substitute equation (10) in equation (9) and solve for v_{bf}:

$$v_{bf} = \left[\frac{m_b}{m_2 + m_b} - 1\right]\left[1 + \frac{m_1}{m_1 + m_b}\right]v \qquad (11)$$

Apply conservation of momentum to express the speed of astronaut 1 and the ball after she catches the ball:

$$(m_1 + m_b)v_{1f} = m_b v_{bf} + m_1 v_1 \qquad (12)$$

Using equations (3) and (11), eliminate v_{bf} and v_1 in equation (12) and solve for v_{1f}:

$$v_{1f} = \boxed{-\frac{m_2 m_b (2m_1 + m_b)}{(m_1 + m_b)^2 (m_2 + m_b)} v}$$

(*b*) The change in the kinetic energy of the system is:

$$\Delta K = K_f - K_i$$

or, because $K_i = 0$,

$$\Delta K = K_f = K_{1f} + K_{2f} = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

Substitute for v_{1f} and v_{2f} to obtain:

$$\Delta K = \tfrac{1}{2} m_1 \left(-\frac{m_2 m_b (2m_1 + m_b)}{(m_1 + m_b)^2 (m_2 + m_b)}\right)^2 v^2 + \tfrac{1}{2} m_2 \left(\frac{m_b}{m_2 + m_b}\right)^2 \left(1 + \frac{m_1}{m_1 + m_b}\right)^2 v^2$$

Simplify to obtain:

$$\Delta K = \boxed{\tfrac{1}{2} \frac{m_2 m_b^2 (2m_1 + m_b)^2}{(m_2 + m_b)^2 (m_1 + m_b)^2} \left(1 + \frac{m_1 m_2}{(m_1 + m_b)^2}\right) v^2}$$

This additional energy came from chemical energy in the astronaut’s bodies.

116 ••• A stream of elastic glass beads, each with a mass of 0.50 g, comes out of a horizontal tube at a rate of 100 per second (see Figure 8-57). The beads fall a distance of 0.50 m to a balance pan and bounce back to their original height. How much mass must be placed in the other pan of the balance to keep the pointer at zero?

Picture the Problem Take the zero of gravitational potential energy to be at the elevation of the pan and let the system include the balance, the beads, and Earth. We can use conservation of energy to find the vertical component of the velocity of the beads as they hit the pan and then calculate the net downward force on the pan from Newton's second law. Let the +*y* direction be upward.

Use conservation of energy to relate the *y* component of the bead's velocity as it hits the pan to its height of fall:

$$\Delta K + \Delta U = 0$$

or, because $K_i = U_f = 0$,

$$\tfrac{1}{2}mv_y^2 - mgh = 0 \Rightarrow v_y = \sqrt{2gh}$$

Substitute numerical values and evaluate v_y:

$$v_y = \sqrt{2\left(9.81\,\text{m/s}^2\right)\left(0.50\,\text{m}\right)} = 3.13\,\text{m/s}$$

Express the change in momentum in the *y* direction per bead:

$$\Delta p_y = p_{yf} - p_{yi} = mv_y - \left(-mv_y\right) = 2mv_y$$

Use Newton's second law to express the net force in the *y* direction exerted on the pan by the beads:

$$F_{\text{net},y} = -N\frac{\Delta p_y}{\Delta t}$$

Letting *M* represent the mass to be placed on the other pan, equate its weight to the net force exerted by the beads, substitute for Δp_y, and solve for *M*:

$$-Mg = -N\frac{\Delta p_y}{\Delta t}$$

and

$$M = \frac{N}{\Delta t}\left(\frac{2mv_y}{g}\right)$$

Substitute numerical values and evaluate *M*:

$$M = \left(100/\text{s}\right)\frac{\left[2\left(0.00050\,\text{kg}\right)\left(3.13\,\text{m/s}\right)\right]}{9.81\,\text{m/s}^2} = \boxed{32\,\text{g}}$$

117 ••• A dumbbell, consisting of two balls of mass *m* connected by a massless 1.00-m-long rod, rests on a frictionless floor against a frictionless wall with one ball directly above the other. The center-to-center distance between the balls is equal to 1.00. The dumbbell then begins to slide down the wall as in Figure 8-58. Find the speed of the bottom ball at the moment when it equals the speed of the top ball.

Picture the Problem Assume that the connecting rod goes halfway through both balls, i.e., the centers of mass of the balls are separated by 1.00 m. Let the system include the dumbbell, the wall and floor, and Earth. Let the zero of gravitational potential be at the center of mass of the lower ball and use conservation of energy to relate the speeds of the balls to the potential energy of the system. By symmetry, the speeds will be equal when the angle with the vertical is 45°.

Use conservation of energy to express the relationship between the initial and final energies of the system:

$$E_i = E_f$$

Express the initial energy of the system:

$$E_i = mgL$$

where L is the length of the rod.

Express the energy of the system when the angle with the vertical is 45°:

$$E_f = mgL\sin 45° + \tfrac{1}{2}(2m)v^2$$

Substitute to obtain:

$$gL = gL\left(\frac{1}{\sqrt{2}}\right) + v^2$$

Solving for v yields:

$$v = \sqrt{gL\left(1 - \frac{1}{\sqrt{2}}\right)}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(9.81\,\text{m/s}^2)(1.00\,\text{m})\left(1 - \frac{1}{\sqrt{2}}\right)}$$

$$= \boxed{1.70\ \text{m/s}}$$

Chapter 9
Rotation

Conceptual Problems

1 • Two points are on a disk that is turning about a fixed-axis through its center, perpendicular to the disk and through its center, at increasing angular velocity. One point is on the rim and the other point is halfway between the rim and the center. (*a*) Which point moves the greater distance in a given time? (*b*) Which point turns through the greater angle? (*c*) Which point has the greater speed? (*d*) Which point has the greater angular speed? (*e*) Which point has the greater tangential acceleration? (*f*) Which point has the greater angular acceleration? (*g*) Which point has the greater centripetal acceleration?

Determine the Concept (*a*) Because r is greater for the point on the rim, it moves the greater distance. (*b*) Both points turn through the same angle. (*c*) Because r is greater for the point on the rim, it has the greater speed. (*d*) Both points have the same angular speed. (*e*) Both points have zero tangential acceleration. (*f*) Both have zero angular acceleration. (*g*) Because r is greater for the point on the rim, it has the greater centripetal acceleration.

2 • True or false:

(*a*) Angular speed and linear speed have the same dimensions.
(*b*) All parts of a wheel rotating about a fixed axis must have the same angular speed.
(*c*) All parts of a wheel rotating about a fixed axis must have the same angular acceleration.
(*d*) All parts of a wheel rotating about a fixed axis must have the same centripetal acceleration.

(*a*) False. Angular speed has the dimensions $\left[1/\mathrm{T}\right]$whereas linear speed has dimensions$\left[\mathrm{L}/\mathrm{T}\right]$.

(*b*) True. The angular speed of all points on a wheel is $d\theta/dt$.

(*c*) True. The angular acceleration of all points on the wheel is $d\omega/dt$.

(*d*) False. The centripetal acceleration at a point on a rotating wheel is directly proportional to its distance from the center of the wheel

3 • Starting from rest and rotating at constant angular acceleration, a disk takes 10 revolutions to reach an angular speed ω. How many additional revolutions at the same angular acceleration are required for it to reach an angular speed of 2ω? (*a*) 10 rev, (*b*) 20 rev, (*c*) 30 rev, (*d*) 40 rev, (*e*) 50 rev?

Picture the Problem The constant-acceleration equation that relates the given variables is $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$. We can set up a proportion to determine the number of revolutions required to double ω and then subtract to find the number of additional revolutions to accelerate the disk to an angular speed of 2ω.

Using a constant-acceleration equation, relate the initial and final angular velocities to the angular acceleration:

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

or, because $\omega_0^2 = 0$,

$$\omega^2 = 2\alpha\Delta\theta$$

Let $\Delta\theta_{10}$ represent the number of revolutions required to reach an angular speed ω:

$$\omega^2 = 2\alpha\Delta\theta_{10} \qquad (1)$$

Let $\Delta\theta_{2\omega}$ represent the number of revolutions required to reach an angular speed ω:

$$(2\omega)^2 = 2\alpha\Delta\theta_{2\omega} \qquad (2)$$

Divide equation (2) by equation (1) and solve for $\Delta\theta_{2\omega}$:

$$\Delta\theta_{2\omega} = \frac{(2\omega)^2}{\omega^2}\Delta\theta_{10} = 4\Delta\theta_{10}$$

The number of *additional* revolutions is:

$$4\Delta\theta_{10} - \Delta\theta_{10} = 3\Delta\theta_{10} = 3(10\,\text{rev}) = 30\,\text{rev}$$

and $\boxed{(c)}$ is correct.

4 • You are looking down from above at a merry-go-round, and observe that is rotating counterclockwise and its rotation rate is slowing. If we designate counterclockwise as positive, what is the sign of the angular acceleration?

Determine the Concept Because the merry-go-round is slowing, the sign of its angular acceleration is $\boxed{\text{negative.}}$

5 • Chad and Tara go for a ride on a merry-go-round. Chad sits on a pony that is 2.0 m from the rotation axis, and Tara sits on a pony 4.0 m from the axis. The merry-go-round is traveling counterclockwise and is speeding up. Does Chad or Tara have (*a*) the larger linear speed? (*b*) the larger centripetal acceleration? (*c*) the larger tangential acceleration?

Determine the Concept

(*a*) The linear speed of all points on the merry-go-round is given by $v = r\omega$. Because she is farther from the rotation axis, Tara has the larger linear speed.

(*b*) The centripetal acceleration of all points on the merry-go-round is given by $a_c = r\omega^2$. Because she is farther from the rotation axis, Tara has the larger centripetal acceleration.

(*c*) The tangential acceleration of all points on the merry-go-round is given by $a_t = r\alpha$. Because the angular acceleration is the same for all points on the merry-go-round and she is farther from the rotation axis, Tara has the larger tangential acceleration.

6 • Disk B was identical to disk A before a hole was drilled though the center of disk B. Which disk has the largest moment of inertia about its symmetry axis center? Explain your answer.

Determine the Concept Because its mass is now greater than that of disk B, disk $\boxed{\text{A}}$ has the larger moment of inertia about its axis of symmetry.

7 • **[SSM]** The pitcher in a baseball game has a blazing fastball. You have not been able to swing the bat in time to hit the ball. You are now just trying to make the bat connect with the ball, hit the ball foul, and avoid a strikeout. So you decide to take your coach's advice and grip the bat high rather than at the very end. This change should increase bat speed; thus, you will be able to swing the bat quicker and increase your chances of hitting the ball. Explain how this theory works in terms of the moment of inertia, angular acceleration, and torque of the bat.

Determine the Concept By choking up, you are rotating the bat about an axis closer to the center of mass, thus reducing the bat's moment of inertia. The smaller the moment of inertia the larger the angular acceleration (a quicker bat) for the same torque.

8 • (*a*) Is the direction of an object's angular velocity necessarily the same as the direction of the net torque on it? Explain. (*b*) If the net torque and angular velocity are in opposite directions, what does that tell you about the angular speed? (*c*) Can the angular velocity be zero even if the net torque is not zero? If your answer is yes, give an example.

(*a*) No. If the object is slowing down, they are oppositely directed.

(*b*) The angular speed of the object will decrease.

(*c*) Yes. At the instant the object is stopping and turning around (angularly), such as a pendulum at its turnaround (highest) point, the net torque is not zero and the angular velocity is zero.

9 • A disk is free to rotate about a fixed axis. A tangential force applied a distance d from the axis causes an angular acceleration α. What angular acceleration is produced if the same force is applied a distance $2d$ from the axis? (*a*) α, (*b*) 2α, (*c*) $\alpha/2$, (*d*) 4α, (*e*) $\alpha/4$?

Determine the Concept The angular acceleration of a rotating object is proportional to the *net* torque acting on it. The net torque is the product of the tangential force and its lever arm.

Express the angular acceleration of the disk as a function of the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{Fd}{I} = \frac{F}{I}d \Rightarrow \alpha \propto d$$

Because $\alpha \propto d$, doubling d will double the angular acceleration. $\boxed{(b)}$ is correct.

10 • The moment of inertia of an object about an axis that does not pass through its center of mass is ______ the moment of inertia about a parallel axis through its center of mass. (*a*) always less than, (*b*) sometimes less than, (*c*) sometimes equal to, (*d*) always greater than.

Determine the Concept From the parallel-axis theorem we know that $I = I_{\text{cm}} + Mh^2$, where I_{cm} is the moment of inertia of the object with respect to an axis through its center of mass, M is the mass of the object, and h is the distance between the parallel axes. Therefore, I is always greater than I_{cm} by Mh^2. $\boxed{(d)}$ is correct.

11 • **[SSM]** The motor of a merry-go-round exerts a constant torque on it. As it speeds up from rest, the power output of the motor (*a*) is constant, (*b*) increases linearly with the angular speed of the merry-go-round, (*c*) is zero. (*d*) None of the above.

Determine the Concept The power delivered by the constant torque is the product of the torque and the angular speed of the merry-go-round. Because the constant torque causes the merry-go-round to accelerate, the power output increases linearly with the angular speed of the merry-go-round. $\boxed{(b)}$ is correct.

12 • A constant net torque acts on a merry-go-round from startup until it reaches its operating speed. During this time, the merry-go-round's rotational kinetic energy (*a*) is constant, (*b*) increases linearly with angular speed, (*c*) increases quadratically as the square of the angular speed, (*d*) none of the above.

Determine the Concept The work done by the net torque increases the rotational kinetic energy of the merry-go-round. Because $K_{\text{rot}} = \frac{1}{2}I\omega^2$, $\boxed{(c)}$ is correct.

13 • Most doors knobs are designed so the knob is located on the side opposite the hinges (rather than in the center of the door, for example). Explain why this practice makes doors easier to open.

Determine the Concept One reason is to maximize the moment arm about the line through the hinge pins for the force exerted by someone pulling or pushing on the knob.

14 • A wheel of radius R and angular speed ω is rolling without slipping toward the north on a flat, stationary surface. The velocity of the point on the rim that is (momentarily) in contact with the surface is (*a*) equal in magnitude to $R\omega$ and directed toward the north, (*b*) equal to in magnitude $R\omega$ and directed toward the south, (*c*) zero, (*d*) equal to the speed of the center of mass and directed toward the north, (*e*) equal to the speed of the center of mass and directed toward the south.

Determine the Concept If the wheel is rolling without slipping, a point at the top of the wheel moves with a speed twice that of the center of mass of the wheel, but the bottom of the wheel is momentarily at rest. $\boxed{(c)}$ is correct.

15 • A uniform solid cylinder and a uniform solid sphere have equal masses. Both roll without slipping on a horizontal surface without slipping. If their total kinetic energies are the same, then (*a*) the translational speed of the cylinder is greater than the translational speed of the sphere, (*b*) the translational speed of the cylinder is less than the translational speed of the sphere, (*c*) the translational speeds of the two objects are the same, (*d*) (*a*), (*b*), or (*c*) could be correct depending on the radii of the objects.

Picture the Problem The kinetic energies of both objects is the sum of their translational and rotational kinetic energies. Their speed dependence will differ due to the differences in their moments of inertia. We can express the total kinetic energy of both objects and equate them to decide which of their translational speeds is greater.

Express the kinetic energy of the cylinder:

$$\begin{aligned} K_{\text{cyl}} &= \tfrac{1}{2}I_{\text{cyl}}\omega_{\text{cyl}}^2 + \tfrac{1}{2}mv_{\text{cyl}}^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\frac{v_{\text{cyl}}^2}{r^2} + \tfrac{1}{2}mv_{\text{cyl}}^2 \\ &= \tfrac{3}{4}mv_{\text{cyl}}^2 \end{aligned}$$

Express the kinetic energy of the sphere:

$$K_{\text{sph}} = \tfrac{1}{2} I_{\text{sph}} \omega_{\text{sphl}}^2 + \tfrac{1}{2} m v_{\text{sph}}^2 = \tfrac{1}{2}\left(\tfrac{2}{5} m r^2\right)\frac{v_{\text{sph}}^2}{r^2} + \tfrac{1}{2} m v_{\text{sph}}^2 = \tfrac{7}{10} m v_{\text{sph}}^2$$

Equate the kinetic energies and simplify to obtain:

$v_{\text{cyl}} = \sqrt{\tfrac{14}{15}}\, v_{\text{sph}} < v_{\text{sph}}$ and $\boxed{(b)}$ is correct.

16 • Two identical-looking 1.0-m-long pipes are each plugged with 10 kg of lead. In the first pipe, the lead is concentrated at the middle of the pipe, while in the second the lead is divided into two 5-kg masses placed at opposite ends of the pipe. The ends of the pipes are then sealed using four identical caps. Without opening either pipe, how could you determine which pipe has the lead at both ends?

Determine the Concept You could spin the pipes about their center. The one which is easier to spin has its mass concentrated closer to the center of mass and, hence, has a smaller moment of inertia.

17 •• Starting simultaneously from rest, a coin and a hoop roll without slipping down an incline. Which of the following statements is true? (*a*) The hoop reaches the bottom first. (*b*) The coin reaches the bottom first. (*c*) The coin and hoop arrive at the bottom simultaneously. (*d*) The race to the bottom depends on their relative masses. (*e*) The race to the bottom depends on their relative diameters.

Picture the Problem The object moving the fastest when it reaches the bottom of the incline will arrive there first. Because the coin and the hoop begin from the same elevation, they will have the same kinetic energy at the bottom of the incline. The kinetic energies of both objects is the sum of their translational and rotational kinetic energies. Their speed dependence will differ due to the differences in their moments of inertia. We can express the total kinetic of both objects and equate them to their common potential energy loss to decide which of their translational speeds is greater at the bottom of the incline.

Express the kinetic energy of the coin at the bottom of the incline:

$$K_{\text{coin}} = \tfrac{1}{2} I_{\text{cyl}} \omega_{\text{coin}}^2 + \tfrac{1}{2} m_{\text{coin}} v_{\text{coin}}^2 = \tfrac{1}{2}\left(\tfrac{1}{2} m_{\text{coin}} r^2\right)\frac{v_{\text{coin}}^2}{r^2} + \tfrac{1}{2} m_{\text{coin}} v_{\text{coin}}^2 = \tfrac{3}{4} m_{\text{coin}} v_{\text{coin}}^2$$

Express the kinetic energy of the hoop at the bottom of the incline:

$$K_{\text{hoop}} = \tfrac{1}{2} I_{\text{hoop}} \omega_{\text{hoop}}^2 + \tfrac{1}{2} m_{\text{hoop}} v_{\text{hoop}}^2 = \tfrac{1}{2}\left(m_{\text{hoop}} r^2\right)\frac{v_{\text{hoop}}^2}{r^2} + \tfrac{1}{2} m_{\text{hoop}} v_{\text{hoop}}^2 = m_{\text{hoop}} v_{\text{hoop}}^2$$

Equate the kinetic energy of the coin to its change in potential energy as it rolled down the incline and solve for v_{coin}:

$$\tfrac{3}{4} m_{\text{coin}} v_{\text{coin}}^2 = m_{\text{coin}} gh \Rightarrow v_{\text{coin}} = \sqrt{\tfrac{4}{3} gh}$$

Equate the kinetic energy of the hoop to its change in potential energy as it rolled down the incline and solve for v_{hoop}:

$$m_{\text{hoop}} v_{\text{hoop}}^2 = m_{\text{hoop}} gh \Rightarrow v_{\text{hoop}} = \sqrt{gh}$$

Express the ratio of these speeds to obtain:

$$\frac{v_{\text{coin}}}{v_{\text{hoop}}} = \frac{\sqrt{\tfrac{4}{3} gh}}{\sqrt{gh}} = \sqrt{\tfrac{4}{3}} \Rightarrow v_{\text{coin}} > v_{\text{hoop}}$$

and $\boxed{(b)}$ is correct.

18 •• For a hoop of mass *M* and radius *R* that is rolling without slipping, which is larger, its translational kinetic energy or its rotational kinetic energy? (*a*) Its translational kinetic energy is larger. (*b*) Its rotational kinetic energy is larger. (*c*) Both energies have the same magnitude. (*d*) The answer depends on the radius of the hoop. (*e*) The answer depends on the mass of the hoop.

Picture the Problem We can use the definitions of the translational and rotational kinetic energies of the hoop and the moment of inertia of a hoop (ring) to express and compare the kinetic energies.

Express the ratio of the translational kinetic energy of the hoop to its rotational kinetic energy and simplify to obtain:

$$\frac{K_{\text{trans}}}{K_{\text{rot}}} = \frac{\tfrac{1}{2} mv^2}{\tfrac{1}{2} I_{\text{hoop}} \omega^2} = \frac{mv^2}{\left(mr^2\right)\frac{v^2}{r^2}} = 1$$

Therefore, the translational and rotational kinetic energies are the same and $\boxed{(c)}$ is correct.

19 •• For a disk of mass M and radius R that is rolling without slipping, which is larger, its translational kinetic energy or its rotational kinetic energy? (a) Its translational kinetic energy is larger. (b) Its rotational kinetic energy is larger. (c) Both energies have the same magnitude. (d) The answer depends on the radius of the disk. (e) The answer depends on the mass of the disk.

Picture the Problem We can use the definitions of the translational and rotational kinetic energies of the disk and the moment of inertia of a disk (cylinder) to express and compare the kinetic energies.

Express the ratio of the translational kinetic energy of the disk to its rotational kinetic energy:

$$\frac{K_{\text{trans}}}{K_{\text{rot}}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}I_{\text{disk}}\omega^2} = \frac{mv^2}{\left(\frac{1}{2}mr^2\right)\frac{v^2}{r^2}} = 2$$

Therefore, the translational kinetic energy is greater by a factor of two and $\boxed{(a)}$ is correct.

20 •• A perfectly rigid ball rolls without slipping along a perfectly rigid horizontal plane. Show that the frictional force acting on the ball must be zero. *Hint: Consider a possible direction for the action of the frictional force and what effects such a force would have on the velocity of the center of mass and on the angular velocity.*

Picture the Problem Let us assume that $f \neq 0$ and acts along the direction of motion. Now consider the acceleration of the center of mass and the angular acceleration about the point of contact with the plane. Because $F_{\text{net}} \neq 0$, $a_{\text{cm}} \neq 0$. However, $\tau = 0$ because $\ell = 0$, and so $\alpha = 0$. But $\alpha = 0$ is not consistent with $a_{\text{cm}} \neq 0$. Consequently, $f = 0$.

21 •• **[SSM]** A spool is free to rotate about a fixed axis, and a string wrapped around the axle of the spool causes the spool to rotate in a counterclockwise direction (see Figure 9-42a). However, if the spool is set on a horizontal tabletop, the spool instead (given sufficient frictional force between the table and the spool) rotates in a clockwise direction, and rolls to the right (Figure 9-42b). By considering torque about the appropriate axes, show that these conclusions are consistent with Newton's second law for rotations.

Determine the Concept First, visualize the situation. The string pulling to the right exerts a torque on the spool with a moment arm equal in length to the radius of the inner portion of the spool. When the spool is freely rotating about that axis, then the torque due to the pulling string causes a counter clockwise rotation. Second, in the situation in which the spool is resting on the horizontal tabletop, one should (for ease of understanding) consider torques not about the central axle of the spool, but about the point of contact with the tabletop. In this situation,

there is only one force that can produce a torque – the applied force. The motion of the spool can then be understood in terms of the force applied by the string and the moment arm equal to the difference between the outer radius and the inner radius. This torque will cause a clockwise rotation about the point of contact between spool and table – and thus the spool rolls to the right (whereas we might have thought the spool would rotate in a counter-clockwise sense, and thus move left).

22 • You want to locate the center of gravity of an arbitrarily shaped flat object. You are told to suspend the object from a point, and to suspend a plumb line from the same point. Then draw a vertical line on the object to represent the plumb line. Next, you repeat the process using a different suspension point. The center of gravity will be at the intersection of the drawn lines. Explain the principle(s) behind this process.

Determine the Concept You are finding positions at which gravity exerts no torque on the object, so the gravitational force (weight) passes through the center of mass. Thus you are triangulating, and in theory, two such lines should intersect at the center of mass. In practice, several lines do a better and more accurate job.

Estimation and Approximation

23 •• A baseball is thrown at 88 mi/h, and with a spin rate of 1500 rev/min. If the distance between the pitcher's point of release and the catcher's glove is about 61 feet, estimate how many revolutions the ball makes between release and catch. Neglect any effects of gravity or air resistance on the ball's flight.

Picture the Problem The number of revolutions made by the ball is the ratio of the angle through which it rotates to 2π rad/rev.

The number of revolutions N the ball makes between release and catch is given by:

$$N = \frac{\Delta\theta}{2\pi\ \text{rad/rev}} \quad (1)$$

where $\Delta\theta$ is the angular displacement of the ball as it travels from the pitcher to the catcher.

Because $\omega = 2\pi f$, $\Delta\theta$ is given by:

$$\Delta\theta = \omega\Delta t = 2\pi f\Delta t$$

Substituting for $\Delta\theta$ in equation (1) yields:

$$N = \frac{2\pi f\Delta t}{2\pi\ \text{rad/rev}} = \frac{f\Delta t}{\text{rad/rev}} \quad (2)$$

Express the time-of-flight of the ball:

$$\Delta t = \frac{d}{v}$$

where d is the distance from the release point to the catcher's glove and v is the speed of the ball.

Substituting for Δt in equation (2) yields:

$$N = \frac{fd}{v \text{ rad/rev}}$$

Substitute numerical values and evaluate N:

$$N = \frac{\left(1500\,\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ s}}\right)(61\text{ ft})}{\left(88\,\frac{\text{mi}}{\text{h}} \times \frac{5280\text{ ft}}{\text{mi}} \times \frac{1\text{ h}}{3600\text{ s}}\right)\frac{\text{rad}}{\text{rev}}} \approx \boxed{12\text{ rev}}$$

24 •• Consider the Crab Pulsar, discussed on page 293. Justify the statement that the loss in rotational energy is equivalent to the power output of 100 000 stars. The total power radiated by the Sun is about 4×10^{26} W. Assume that the pulsar has a mass that is 2×10^{30} kg, has a radius that is 20 km, is rotating at about 30 rev/s, and has a rotational period that is increasing at 10^{-5} s/y.

Picture the Problem The power dissipated in the loss of rotational kinetic energy is the rate at which the rotational kinetic energy of the Crab Pulsar is decreasing.

Express the rate at which the rotational kinetic energy of the Crab Pulsar is changing:

$$P_{\text{Crab Pulsar}} = \frac{\Delta K}{\Delta t} = \frac{K_\text{f} - K_\text{i}}{\Delta t}$$

Substitute for K_f and K_i and simplify to obtain:

$$P_{\text{Crab Pulsar}} = \frac{\frac{1}{2}I\omega_\text{f}^2 - \frac{1}{2}I\omega_\text{i}^2}{\Delta t} = \frac{\frac{1}{2}I\left(\omega_\text{f}^2 - \omega_\text{i}^2\right)}{\Delta t}$$

Letting $\omega_\text{f} = \omega_\text{i} - \delta\omega$ yields:

$$P_{\text{Crab Pulsar}} = \frac{\frac{1}{2}I\left[(\omega_\text{i} - \delta\omega)^2 - \omega_\text{i}^2\right]}{\Delta t}$$

Expand the binomial expression and simplify to obtain:

$$P_{\text{Crab Pulsar}} = \frac{\frac{1}{2}I\left[(\delta\omega)^2 - 2\omega_\text{i}\delta\omega\right]}{\Delta t} \quad (1)$$

Express $\delta\omega$ in terms of ω_f and ω_i:

$$\delta\omega = \omega_\text{f} - \omega_\text{i} = \frac{2\pi}{T_\text{f}} - \omega_\text{i}$$

Substituting for T_f and ω_i yields:

$$\delta\omega = \frac{2\pi}{T_i + 10^{-5}\ \text{s}} - 60\pi\ \text{rad/s}$$

Substitute for T_i and simplify to obtain:

$$\delta\omega = \frac{2\pi}{\frac{2\pi}{60\pi}\text{s} + 10^{-5}\ \text{s}} - 60\pi\ \text{rad/s}$$
$$= -0.0565\ \text{s}^{-1}$$

Because $\delta\omega << \omega_i$:

$$(\delta\omega)^2 - 2\omega_i\delta\omega \approx -2\omega_i\delta\omega$$

With this substitution, equation (1) becomes:

$$P_{\text{Crab Pulsar}} = \frac{\frac{1}{2}I[-2\omega_i\delta\omega]}{\Delta t} = \frac{-I\omega_i\delta\omega}{\Delta t} \qquad (2)$$

The moment of inertia of a sphere of mass M and radius R is:

$$I = \tfrac{2}{5}MR^2$$

Substitute for I in equation (2) to obtain:

$$P_{\text{Crab Pulsar}} = \frac{-\frac{2}{5}MR^2\omega_i\delta\omega}{\Delta t} \qquad (3)$$

Dividing both sides of equation (3) by the power radiated by the Sun yields:

$$\frac{P_{\text{Crab Pulsar}}}{P_{\text{Sun}}} = \frac{-\frac{2}{5}MR^2\omega_i\delta\omega}{P_{\text{sun}}}$$

Substitute numerical values and evaluate $P_{\text{Crab Pulsar}}/P_{\text{Sun}}$:

$$\frac{P_{\text{Crab Pulsar}}}{P_{\text{Sun}}} = \frac{-\frac{2}{5}\left(2\times10^{30}\ \text{kg}\right)(20\ \text{km})^2\left(60\pi\ \frac{\text{rad}}{\text{s}}\right)\left(-0.0565\ \text{s}^{-1}\right)}{\left(4\times10^{26}\ \text{W}\right)\left(1\ \text{y}\times\frac{3.156\times10^7\ \text{s}}{\text{y}}\right)} \approx \boxed{10^5}$$

25 •• A 14-kg bicycle has 1.2-m-diameter wheels, each with a mass of 3.0 kg. The mass of the rider is 38 kg. Estimate the fraction of the total kinetic energy of the rider-bicycle system that is associated with rotation of the wheels.

Picture the Problem Assume the wheels are hoops. That is, neglect the mass of the spokes, and express the total kinetic energy of the bicycle and rider. Let M represent the mass of the rider, m the mass of the bicycle, m_w the mass of each bicycle wheel, and r the radius of the wheels.

Express the ratio of the kinetic energy associated with the rotation of the wheels to that associated with the total kinetic energy of the bicycle and rider:

$$\frac{K_{\text{rot}}}{K_{\text{tot}}}=\frac{K_{\text{rot}}}{K_{\text{trans}}+K_{\text{rot}}} \quad (1)$$

Express the translational kinetic energy of the bicycle and rider:

$$K_{\text{trans}}=K_{\text{bicycle}}+K_{\text{rider}}=\tfrac{1}{2}mv^2+\tfrac{1}{2}Mv^2$$

Express the rotational kinetic energy of the bicycle wheels:

$$K_{\text{rot}}=2K_{\text{rot, 1 wheel}}=2\left(\tfrac{1}{2}I_{\text{w}}\omega^2\right)=\left(m_{\text{w}}r^2\right)\frac{v^2}{r^2}=m_{\text{w}}v^2$$

Substitute for K_{rot} and K_{trans} in equation (1) and simplify to obtain:

$$\frac{K_{\text{rot}}}{K_{\text{tot}}}=\frac{m_{\text{w}}v^2}{\frac{1}{2}mv^2+\frac{1}{2}Mv^2+m_{\text{w}}v^2}=\frac{m_{\text{w}}}{\frac{1}{2}m+\frac{1}{2}M+m_{\text{w}}}=\frac{2}{2+\dfrac{m+M}{m_{\text{w}}}}$$

Substitute numerical values and evaluate $K_{\text{rot}}/K_{\text{trans}}$:

$$\frac{K_{\text{rot}}}{K_{\text{tot}}}=\frac{2}{2+\dfrac{14\,\text{kg}+38\,\text{kg}}{3.0\,\text{kg}}}=\boxed{10\%}$$

26 •• Why does toast falling off a table always land jelly-side down? The question may sound silly, but it has been a subject of serious scientific enquiry. The analysis is too complicated to reproduce here, but R. D. Edge and Darryl Steinert showed that a piece of toast, pushed gently over the edge of a table until it tilts off, typically falls off the table when it makes an angle of about 30° with the horizontal (Figure 9-43) and at that instant has an angular speed of $\omega=0.956\sqrt{g/\ell}$, where ℓ is the length of one edge of the piece of toast (assumed to be square). Assuming that a piece of toast is jelly-side up, what side will it land on if it falls from a 0.500-m-high table? If it falls from a 1.00-m-high table? Assume that ℓ = 10.0 cm. Ignore any forces due to air resistance.

Picture the Problem We can apply the definition of angular speed to find the angular orientation of the slice of toast when it has fallen a distance of 0.500 m (or 1.00 m) from the edge of the table. We can then interpret the orientation of the toast to decide whether it lands jelly-side up or down.

Relate the angular orientation θ of the toast to its initial angular orientation, its angular speed ω, and time of fall Δt:

$$\theta=\theta_0+\omega\Delta t$$

Substituting the expression given for ω in the problem statement to obtain:

$$\theta = \theta_0 + 0.956\sqrt{\frac{g}{\ell}}\Delta t \qquad (1)$$

Using a constant-acceleration equation, relate the distance the toast falls Δy to its time of fall Δt:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

or, because $v_{0y} = 0$ and $a_y = g$,

$$\Delta y = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta y}{g}}$$

Substitute for Δt in equation (1) and simplify to obtain:

$$\theta = \theta_0 + 0.956\sqrt{\frac{g}{\ell}}\sqrt{\frac{2\Delta y}{g}}$$
$$= \theta_0 + 0.956\sqrt{\frac{2\Delta y}{\ell}}$$

Substitute numerical values and evaluate θ for $\Delta y = 0.500$ m:

$$\theta_{0.50\,\text{m}} = \frac{\pi}{6} + 0.956\sqrt{\frac{2(0.500\text{ m})}{0.100\text{ m}}}$$
$$= 3.547\text{ rad} \times \frac{180°}{\pi\text{ rad}}$$
$$= \boxed{203°}$$

Substitute numerical values and evaluate θ for $\Delta y = 1.00$ m:

$$\theta_{1.0\,\text{m}} = \frac{\pi}{6} + 0.956\sqrt{\frac{2(1.00\text{ m})}{0.100\text{ m}}}$$
$$= 4.799\text{ rad} \times \frac{180°}{\pi\text{ rad}}$$
$$= \boxed{275°}$$

The orientation of the slice of toast will therefore be at angles of 203° and 275° with respect to ground; that is, with the jelly-side down.

27 •• Consider your moment of inertia about a vertical axis through the center of your body, both when you are standing straight up with your arms flat against your sides, and when you are standing straight up holding yours arms straight out to the side. Estimate the ratio of the moment of inertia with your arms straight out to the moment of inertia with your arms flat against your sides.

Picture the Problem Assume that the mass of an average adult male is about 80 kg, and that we can model his body when he is standing straight up with his arms at his sides as a cylinder. From experience in men's clothing stores, a man's average waist circumference seems to be about 34 inches, and the average chest circumference about 42 inches. We'll also assume that about 20% of your body's mass is in your two arms, and that each has a length $L = 1$ m, so that each arm has a mass of about $m = 8$ kg.

Letting I_{out} represent his moment of inertia with his arms straight out and I_{in} his moment of inertia with his arms at his side, the ratio of these two moments of inertia is:

$$\frac{I_{out}}{I_{in}} = \frac{I_{body} + I_{arms}}{I_{in}} \qquad (1)$$

Express the moment of inertia of the ″man as a cylinder″:

$$I_{in} = \tfrac{1}{2} MR^2$$

Express the moment of inertia of his arms:

$$I_{arms} = 2\left(\tfrac{1}{3}\right) mL^2$$

Express the moment of inertia of his body-less-arms:

$$I_{body} = \tfrac{1}{2}(M - m)R^2$$

Substitute in equation (1) to obtain:

$$\frac{I_{out}}{I_{in}} = \frac{\tfrac{1}{2}(M - m)R^2 + 2\left(\tfrac{1}{3}\right)mL^2}{\tfrac{1}{2} MR^2}$$

Assume the circumference of the cylinder to be the average of the average waist circumference and the average chest circumference:

$$c_{av} = \frac{34\,\text{in} + 42\,\text{in}}{2} = 38\,\text{in}$$

Find the radius of a circle whose circumference is 38 in:

$$R = \frac{c_{av}}{2\pi} = \frac{38\,\text{in} \times \dfrac{2.54\,\text{cm}}{\text{in}} \times \dfrac{1\,\text{m}}{100\,\text{cm}}}{2\pi} = 0.154\,\text{m}$$

Substitute numerical values and evaluate I_{out}/I_{in}:

$$\frac{I_{out}}{I_{in}} = \frac{\tfrac{1}{2}(80\,\text{kg} - 16\,\text{kg})(0.154\,\text{m})^2 + \tfrac{2}{3}(8\,\text{kg})(1\,\text{m})^2}{\tfrac{1}{2}(80\,\text{kg})(0.154\,\text{m})^2} \approx \boxed{6}$$

Angular Velocity, Angular Speed and Angular Acceleration

28 • A particle moves with a constant speed of 25 m/s in a 90-m-radius circle. (*a*) What is its angular speed, in radians per second, about the center of the circle? (*b*) How many revolutions does it make in 30 s?

Picture the Problem The tangential and angular velocities of a particle moving in a circle are directly proportional. The number of revolutions made by the particle in a given time interval is proportional to both the time interval and its angular speed.

(*a*) Relate the angular speed of the particle to its speed along the circumference of the circle:

$$\omega = \frac{v}{r}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{25\,\text{m/s}}{90\,\text{m}} = 0.278\,\text{rad/s} = \boxed{0.28\,\text{rad/s}}$$

(*b*) Using a constant-acceleration equation, relate the number of revolutions made by the particle in a given time interval to its angular speed:

$$\Delta\theta = \omega\Delta t = \left(0.278\frac{\text{rad}}{\text{s}}\right)(30\,\text{s})\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) = \boxed{1.3\,\text{rev}}$$

29 • **[SSM]** A wheel, released from rest, is rotating with constant angular acceleration of 2.6 rad/s^2. At 6.0 s after its release: (*a*) What is its angular speed? (*b*) Through what angle has the wheel turned? (*c*) How many revolutions has it completed? (*d*) What is the linear speed and what is the magnitude of the linear acceleration of a point 0.30 m from the axis of rotation?

Picture the Problem Because the angular acceleration is constant, we can find the various physical quantities called for in this problem by using constant-acceleration equations.

(*a*) Using a constant-acceleration equation, relate the angular speed of the wheel to its angular acceleration:

$$\omega = \omega_0 + \alpha\Delta t$$

or, when $\omega_0 = 0$,

$$\omega = \alpha\Delta t$$

Evaluate ω when Δt = 6.0 s:

$$\omega = (2.6\,\text{rad/s}^2)(6.0\,\text{s}) = 15.6\,\text{rad/s} = \boxed{16\,\text{rad/s}}$$

(*b*) Using another constant-acceleration equation, relate the angular displacement to the wheel's angular acceleration and the time it has been accelerating:

$$\Delta\theta = \omega_0\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

or, when $\omega_0 = 0$,

$$\Delta\theta = \tfrac{1}{2}\alpha(\Delta t)^2$$

Evaluate $\Delta\theta$ when Δt = 6.0 s:

$$\Delta\theta(6.0\,\text{s}) = \tfrac{1}{2}(2.6\,\text{rad/s}^2)(6.0\,\text{s})^2 = 46.8\,\text{rad} = \boxed{47\,\text{rad}}$$

(*c*) Convert $\Delta\theta(6.0\,\text{s})$ from radians to revolutions:

$$\Delta\theta(6.0\,\text{s}) = 46.8\,\text{rad} \times \frac{1\,\text{rev}}{2\pi\,\text{rad}} = \boxed{7.4\,\text{rev}}$$

(*d*) Relate the angular speed of the particle to its tangential speed and evaluate the latter when $\Delta t = 6.0$ s:

$$v = r\omega = (0.30\,\text{m})(15.6\,\text{rad/s})$$
$$= \boxed{4.7\,\text{m/s}}$$

Relate the resultant acceleration of the point to its tangential and centripetal accelerations when $\Delta t = 6.0$ s:

$$a = \sqrt{a_\text{t}^2 + a_\text{c}^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2}$$
$$= r\sqrt{\alpha^2 + \omega^4}$$

Substitute numerical values and evaluate *a*:

$$a = (0.30\,\text{m})\sqrt{(2.6\,\text{rad/s}^2)^2 + (15.6\,\text{rad/s})^4} = \boxed{73\,\text{m/s}^2}$$

30 • When a turntable rotating at 33 rev/min is shut off, it comes to rest in 26 s. Assuming constant angular acceleration, find (*a*) the angular acceleration. During the 26 s, find (*b*) the average angular speed, and (*c*) the angular displacement, in revolutions.

Picture the Problem Because we're assuming constant angular acceleration; we can find the various physical quantities called for in this problem by using constant-acceleration equations for rotational motion.

(*a*) The angular acceleration of the turntable is given by:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{\Delta t}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{0 - 33\dfrac{\text{rev}}{\text{min}} \times \dfrac{2\pi\,\text{rad}}{\text{rev}} \times \dfrac{1\,\text{min}}{60\,\text{s}}}{26\,\text{s}}$$
$$= \boxed{0.13\,\text{rad/s}^2}$$

(*b*) Because the angular acceleration is constant, the average angular speed is given by:

$$\omega_\text{av} = \frac{\omega_0 + \omega}{2}$$

Substitute numerical values and evaluate ω_{av}:

$$\omega_{av} = \frac{33\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}}{2} = 1.73\,\text{rad/s} = \boxed{1.7\,\text{rad/s}}$$

(*c*) Using the definition of ω_{av}, find the angular displacement of the turntable as it slows to a stop:

$$\Delta\theta = \omega_{av}\Delta t = (1.73\,\text{rad/s})(26\,\text{s}) = 44.9\,\text{rad}\times\frac{1\,\text{rev}}{2\pi\,\text{rad}} = \boxed{7.2\,\text{rev}}$$

31 • A 12-cm-radius disk that begins to rotate about its axis at $t = 0$, rotates with a constant angular acceleration of 8.0 rad/s^2. At $t = 5.0$ s, (*a*) what is the angular speed of the disk, and (*b*) what are the tangential and centripetal components of the acceleration of a point on the edge of the disk?

Picture the Problem Because the angular acceleration of the disk is constant, we can use a constant-acceleration equation to relate its angular speed to its acceleration and the time it has been accelerating. We can find the tangential and centripetal accelerations from their relationships to the angular speed and angular acceleration of the disk.

(*a*) Using a constant-acceleration equation, relate the angular speed of the disk to its angular acceleration and time during which it has been accelerating:

$$\omega = \omega_0 + \alpha\,\Delta t$$

or, because $\omega_0 = 0$,

$$\omega = \alpha\,\Delta t$$

Evaluate ω when $t = 5.0$ s:

$$\omega(5.0\,\text{s}) = (8.0\,\text{rad/s}^2)(5.0\,\text{s}) = \boxed{40\,\text{rad/s}}$$

(*b*) Express a_t in terms of α:

$$a_t = r\alpha$$

Evaluate a_t when $t = 5.0$ s:

$$a_t(5.0\,\text{s}) = (0.12\,\text{m})(8.0\,\text{rad/s}^2) = \boxed{0.96\,\text{m/s}^2}$$

Express a_c in terms of ω:

$$a_c = r\omega^2$$

Evaluate a_c when $t = 5.0$ s:

$$a_c(5.0\,\text{s}) = (0.12\,\text{m})(40\,\text{rad/s})^2 = \boxed{0.19\,\text{km/s}^2}$$

32 • A 12-m-radius Ferris wheel rotates once each 27 s. (*a*) What is its angular speed (in radians per second)? (*b*) What is the linear speed of a passenger? (*c*) What is the acceleration of a passenger?

Picture the Problem We can find the angular speed of the Ferris wheel from its definition and the linear speed and centripetal acceleration of the passenger from the relationships between those quantities and the angular speed of the Ferris wheel.

(*a*) Find ω from its definition:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\text{rad}}{27\,\text{s}} = 0.233\,\text{rad/s}$$
$$= \boxed{0.23\,\text{rad/s}}$$

(*b*) Find the linear speed of the passenger from his/her angular speed:

$$v = r\omega = (12\,\text{m})(0.233\,\text{rad/s})$$
$$= \boxed{2.8\,\text{m/s}}$$

Find the passenger's centripetal acceleration from his/her angular speed:

$$a_\text{c} = r\omega^2 = (12\,\text{m})(0.233\,\text{rad/s})^2$$
$$= \boxed{0.65\,\text{m/s}^2}$$

33 • A cyclist accelerates uniformly from rest. After 8.0 s, the wheels have rotated 3.0 rev. (*a*) What is the angular acceleration of the wheels? (*b*) What is the angular speed of the wheels at the end of the 8.0 s?

Picture the Problem Because the angular acceleration of the wheels is constant, we can use constant-acceleration equations in rotational form to find their angular acceleration and their angular speed at any given time.

(*a*) Using a constant-acceleration equation, relate the angular displacement of the wheel to its angular acceleration and the time it has been accelerating:

$$\Delta\theta = \omega_0\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

or, because $\omega_0 = 0$,

$$\Delta\theta = \tfrac{1}{2}\alpha(\Delta t)^2 \Rightarrow \alpha = \frac{2\Delta\theta}{(\Delta t)^2}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{2(3.0\,\text{rev})\left(\dfrac{2\pi\,\text{rad}}{\text{rev}}\right)}{(8.0\,\text{s})^2}$$
$$= 0.589\,\text{rad/s}^2 = \boxed{0.59\,\text{rad/s}^2}$$

(*b*) Using a constant-acceleration equation, relate the angular speed of the wheel to its angular acceleration and the time it has been accelerating:

$\omega = \omega_0 + \alpha\Delta t$

or, when $\omega_0 = 0$,

$\omega = \alpha\Delta t$

Evaluate ω when $\Delta t = 8.0$ s:

$$\omega(8.0\,\text{s}) = (0.589\,\text{rad/s}^2)(8.0\,\text{s}) = \boxed{4.7\,\text{rad/s}}$$

34 • What is the angular speed of Earth, in radians per second, as it rotates about its axis?

Picture the Problem Earth rotates through 2π radians every 24 hours.

Apply the definition of angular speed to obtain:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\text{rad}}{24\,\text{h}\times\frac{3600\,\text{s}}{\text{h}}} = \boxed{73\,\mu\text{rad/s}}$$

35 • A wheel rotates through 5.0 rad in 2.8 s as it is brought to rest with constant angular acceleration. Determine the wheel's initial angular speed before braking began.

Picture the Problem When the angular acceleration of a wheel is constant, its average angular speed is the average of its initial and final angular velocities. We can combine this relationship with the always applicable definition of angular speed to find the initial angular velocity of the wheel.

Express the average angular speed of the wheel in terms of its initial and final angular speeds:

$$\omega_{av} = \frac{\omega_0 + \omega}{2}$$

or, because $\omega = 0$,

$$\omega_{av} = \tfrac{1}{2}\omega_0$$

The average angular speed of the wheel is also given by:

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

Equate these two expressions for ω_{av} and solve for ω_0 to obtain:

$$\omega_0 = \frac{2\Delta\theta}{\Delta t}$$

Substitute numerical values and evaluate ω_0:

$$\omega_0 = \frac{2(5.0\,\text{rad})}{2.8\,\text{s}} = \boxed{3.6\,\text{rad/s}}$$

36 • A bicycle has 0.750-m-diameter wheels. The bicyclist accelerates from rest with constant acceleration to 24.0 km/h in 14.0 s. What is the angular acceleration of the wheels?

Picture the Problem The tangential and angular accelerations of the wheel are directly proportional to each other with the radius of the wheel as the proportionality constant. Provided there is no slippage, the acceleration of a point on the rim of the wheel is the same as the acceleration of the bicycle. We can use its defining equation to determine the acceleration of the bicycle.

Relate the tangential acceleration of a point on the wheel (equal to the acceleration of the bicycle) to the wheel's angular acceleration and solve for its angular acceleration:

$$a = a_{\mathrm{t}} = r\alpha \Rightarrow \alpha = \frac{a}{r}$$

Use its definition to express the acceleration of the wheel:

$$a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{\Delta t}$$

or, because $v_0 = 0$,

$$a = \frac{v}{\Delta t}$$

Substitute in the expression for α to obtain:

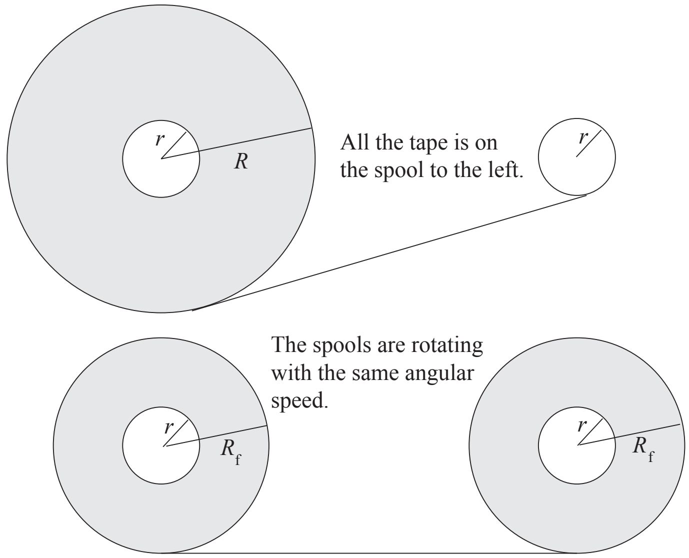
$$\alpha = \frac{v}{r\Delta t} = \frac{v}{\frac{1}{2}d\Delta t} = \frac{2v}{d\Delta t}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{2\left(24.0\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\left(\frac{1000\,\text{m}}{\text{km}}\right)}{(0.750\,\text{m})(14.0\,\text{s})} = \boxed{1.3\,\text{rad/s}^2}$$

37 •• **[SSM]** The tape in a standard VHS videotape cassette has a total length of 246 m, which is enough for the tape to play for 2.0 h (Figure 9-44). As the tape starts, the full reel has a 45-mm outer radius and a 12-mm inner radius. At some point during the play, both reels have the same angular speed. Calculate this angular speed in radians per second and in revolutions per minute. (*Hint: Between the two reels the tape moves at constant speed.*)

Picture the Problem The two tapes will have the same tangential and angular velocities when the two reels are the same size, i.e., have the same area. We can calculate the tangential speed of the tape from its length and running time and relate the angular speed to the constant tangential speed and the radii of the reels when they are turning with the same angular speed.



Relate the angular speed of a tape to its tangential speed when the tapes have the same angular speed:

$$\omega_f = \frac{v}{R_f} \qquad (1)$$

At the instant both reels have the same area:

$$\pi R_f^2 - \pi r^2 = \tfrac{1}{2}\left(\pi R^2 - \pi r^2\right)$$

Solving for R_f yields:

$$R_f = \sqrt{\frac{R^2 + r^2}{2}}$$

Express the tangential speed of the tape in terms of its length L and running time T:

$$v = \frac{L}{T}$$

Substituting or R_f and v in equation (1) gives:

$$\omega_f = \frac{\frac{L}{T}}{\sqrt{\frac{R^2 + r^2}{2}}} = \frac{L}{T}\sqrt{\frac{2}{R^2 + r^2}}$$

Substitute numerical values and evaluate R_f:

$$\omega_f = \frac{246\ \text{m}}{2.0\ \text{h} \times \frac{3600\ \text{s}}{\text{h}}}\sqrt{\frac{2}{(45\ \text{mm})^2 + (12\ \text{mm})^2}} = 1.04\ \text{rad/s} = \boxed{1.0\ \text{rad/s}}$$

$$= 1.04\ \frac{\text{rad}}{\text{s}} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} \times \frac{60\ \text{s}}{\text{min}} = \boxed{9.9\ \text{rev/min}}$$

38 •• To start a lawn mower, you must pull on a rope wound around the perimeter of a flywheel. After you pull the rope for 0.95 s, the flywheel is rotating at 4.5 revolutions per second, at which point the rope disengages. This attempt at starting the mower does not work, however, and the flywheel slows, coming to rest 0.24 s after the disengagement. Assume constant acceleration during both spin up and spin down. (*a*) Determine the average angular acceleration during the 4.5-s spin-up and again during the 0.24-s spin-down. (*b*) What is the maximum angular speed reached by the flywheel? (*c*) Determine the ratio of the number of revolutions made during spin-up to the number made during spin-down.

Picture the Problem The average angular acceleration of the starter is the ratio of the change in angular speed to the time during which the starter either speeds up slows down. The number of revolutions through which the starter turns is the product of its average angular speed and the elapsed time.

(*a*) The average angular acceleration of the starter is given by:

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

Evaluate α_{av} for $\omega_i = 0$ and $\omega_f = 4.5$ rev/s:

$$\alpha_{av} = \frac{4.5\,\frac{\text{rev}}{\text{s}}\times\frac{2\pi\ \text{rad}}{\text{rev}} - 0}{0.95\ \text{s}} = 29.8\ \text{rad/s}^2$$

$$= \boxed{30\ \text{rad/s}^2}$$

Evaluate α_{av} for $\omega_i = 4.5$ rev/s and $\omega_f = 0$:

$$\alpha_{av} = \frac{0 - 4.5\,\frac{\text{rev}}{\text{s}}\times\frac{2\pi\ \text{rad}}{\text{rev}}}{0.24\ \text{s}}$$

$$= -118\ \text{rad/s}^2 = \boxed{-1.2\times10^2\ \text{rad/s}^2}$$

(*b*) The maximum angular speed reached by the starter is the angular speed it had when the rope comes off:

$$\omega_{max} = 4.5\,\frac{\text{rev}}{\text{s}}\times\frac{2\pi\ \text{rad}}{\text{rev}} = \boxed{28\ \text{rad/s}}$$

(*c*) Use a constant-acceleration equation to express the ratio of the number of revolutions during startup to the number of revolutions during slowdown:

$$\frac{\Delta\theta_{\text{spin up}}}{\Delta\theta_{\text{spin down}}} = \frac{\omega_{\text{av,spin up}}(\Delta t)_{\text{spin up}}}{\omega_{\text{av,spin down}}(\Delta t)_{\text{spin down}}}$$

or, because $\omega_{\text{av, spin up}} = \omega_{\text{av, spin down}}$,

$$\frac{\Delta\theta_{\text{spin up}}}{\Delta\theta_{\text{spin down}}} = \frac{(\Delta t)_{\text{spin up}}}{(\Delta t)_{\text{spin down}}}$$

Substitute numerical values and evaluate $\Delta\theta_{\text{spin up}}/\Delta\theta_{\text{spin down}}$:

$$\frac{\Delta\theta_{\text{spin up}}}{\Delta\theta_{\text{spin down}}}=\frac{0.95\text{ s}}{0.24\text{ s}}=\boxed{4.0}$$

39 ••• Mars orbits the Sun at a mean orbital radius of 228 Gm (1 Gm = 10^9 m) and has an orbital period of 687 d. Earth orbits the Sun at a mean orbital radius of 149.6 Gm. (*a*) The Earth-Sun line sweeps out an angle of 360° during one Earth year. Approximately what angle is swept out by the Mars-Sun line during one Earth-year? (*b*) How frequently are Mars and the Sun in opposition (on diametrically opposite sides of Earth)?

Picture the Problem Two solutions for Part (*a*), both based on the definition of average angular speed ($\omega=\Delta\theta/\Delta t$) are given below.

(*a*) **First solution** The angle between the Earth-Sun line and the Mars-Sun line is given by:

$$\theta=\left(\frac{\Delta\theta}{\Delta t}\right)_{\text{Mars}}\Delta t$$

where $\left(\frac{\Delta\theta}{\Delta t}\right)_{\text{Mars}}=\frac{2\pi\text{ rad}}{687\text{ d}}$ is the rate at which the angle between the Earth-Sun line and the Mars-Sun line is swept out.

For Δt = 365.24 d (one Earth year):

$$\theta=\left(\frac{2\pi\text{ rad}}{687\text{ d}}\right)(365.24\text{ d})=\boxed{3.34\text{ rad}}$$

A second solution This solution assumes that the question is "If Earth and Mars have the same angular coordinate at t = 0, what will be the angle between the Earth-Sun and Mars-Sun lines when t = 1.0 y (Earth year)?" Let θ_E = 0 and θ_M = 0 at t = 0 and let T_E = orbital period of Earth = 365.24 d.

The angles through which the Mars-Sun and Earth-Sun lines rotate in time Δt are given by:

$$\theta_M=\left(\frac{\Delta\theta}{\Delta t}\right)_M\Delta t$$

and

$$\theta_E=\left(\frac{\Delta\theta}{\Delta t}\right)_E\Delta t$$

Subtracting the first equation from the second equation gives:

$$\theta_E-\theta_M=\left(\frac{\Delta\theta}{\Delta t}\right)_E\Delta t-\left(\frac{\Delta\theta}{\Delta t}\right)_M\Delta t$$

At the end of one Earth year $\left(\frac{\Delta\theta}{\Delta t}\right)_E\Delta t=2\pi$ and $\theta_M=\left(\frac{\Delta\theta}{\Delta t}\right)_M T_E$:

$$\theta_E-\theta_M=2\pi-\left(\frac{\Delta\theta}{\Delta t}\right)_M T_E$$

Substitute numerical values for $\Delta\theta$, Δt and T_E and evaluate $\theta_E - \theta_M$:

$$\theta_E - \theta_M = 2\pi - \left(\frac{2\pi}{687\text{ d}}\right)(365.24\text{ d})$$
$$= 2\pi\left(1 - \frac{365.24}{687}\right)$$
$$= \boxed{2.94\text{ rad}}$$

(*b*) The second alignment of Earth and Mars will occur when both planets have the same angular displacement from their initial alignment-with Earth having made one full revolution more than Mars:

$$\theta_{\text{Earth}} = \omega_{\text{Earth}}\Delta t = 2\pi + \theta_{\text{Mars}} \quad (1)$$

The angular position of Mars at this time is:

$$\theta_{\text{Mars}} = \omega_{\text{Mars}}\Delta t \quad (2)$$

Substituting for θ_{Mars} in equation (1) yields:

$$\omega_{\text{Earth}}\Delta t = 2\pi + \omega_{\text{Mars}}\Delta t$$

Solve for Δt to obtain:

$$\Delta t = \frac{2\pi}{\omega_{\text{Earth}} - \omega_{\text{Mars}}} \quad (3)$$

The angular speeds of Earth and Mars are related to their periods:

$$\omega_{\text{Earth}} = \frac{2\pi}{T_{\text{Earth}}} \text{ and } \omega_{\text{Mars}} = \frac{2\pi}{T_{\text{Mars}}}$$

Substituting for ω_{Earth} and ω_{Mars} in equation (3) and simplifying yields:

$$\Delta t = \frac{2\pi}{\frac{2\pi}{T_{\text{Earth}}} - \frac{2\pi}{T_{\text{Mars}}}} = \frac{T_{\text{Earth}}T_{\text{Mars}}}{T_{\text{Mars}} - T_{\text{Earth}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{(365.24\text{ d})(687\text{ d})}{687\text{ d} - 365.24\text{ d}} = \boxed{780\text{ d}}$$

Calculating the Moment of Inertia

40 • A tennis ball has a mass of 57 g and a diameter of 7.0 cm. Find the moment of inertia about its diameter. Model the ball as a thin spherical shell.

Picture the Problem One can find the formula for the moment of inertia of a thin spherical shell in Table 9-1.

The moment of inertia of a thin spherical shell about its diameter is:

$$I = \tfrac{2}{3}MR^2$$

Substitute numerical values and evaluate I:

$$I = \tfrac{2}{3}(0.057\,\text{kg})(0.035\,\text{m})^2$$
$$= \boxed{4.7\times10^{-5}\,\text{kg}\cdot\text{m}^2}$$

41 • **[SSM]** Four particles, one at each of the four corners of a square with 2.0-m long edges, are connected by massless rods (Figure 9-45). The masses of the particles are $m_1 = m_3 = 3.0$ kg and $m_2 = m_4 = 4.0$ kg. Find the moment of inertia of the system about the z axis.

Picture the Problem The moment of inertia of a system of particles with respect to a given axis is the sum of the products of the mass of each particle and the square of its distance from the given axis.

Use the definition of the moment of inertia of a system of four particles to obtain:

$$I = \sum_{\text{i}} m_{\text{i}} r_{\text{i}}^2$$
$$= m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$$

Substitute numerical values and evaluate I_z:

$$I_z = (3.0\,\text{kg})(2.0\,\text{m})^2 + (4.0\,\text{kg})(0)^2 + (3.0\,\text{kg})(2.0\,\text{m})^2 + (4.0\,\text{kg})\left(2\sqrt{2}\,\text{m}\right)^2$$
$$= \boxed{56\,\text{kg}\cdot\text{m}^2}$$

42 •• Use the parallel-axis theorem and the result for Problem 41 to find the moment of inertia of the four-particle system in Figure 9-45 about an axis that passes through the center of mass and is parallel with the z axis. Check your result by direct computation.

Picture the Problem According to the parallel-axis theorem, $I = I_{\text{cm}} + Mh^2$, where I_{cm} is the moment of inertia of the object with respect to an axis through its center of mass, M is the mass of the object, and h is the distance between the parallel axes. Note that the center of mass of the system is at the intersection of the diagonals connecting the four masses.

Use the parallel-axis theorem to obtain an expression for I_{cm}:

$$I_z = I_{\text{cm}} + Mh^2 \Rightarrow I_{\text{cm}} = I_z - Mh^2$$

By symmetry:

$$x_{\text{cm}} = 1.0\,\text{m and } y_{\text{cm}} = 1.0\,\text{m}$$

Find the square of the distance from the center of mass to the z axis (note that this distance is also the distance from the center of mass to the center of the object whose mass is m_2):

$$h^2 = r_2^2 = (1.0\,\text{m} - 0)^2 + (1.0\,\text{m} - 0)^2 = 2.0\,\text{m}^2$$

Substitute numerical values in equation (1) and evaluate I_{cm}:

$$I_{\text{cm}} = 56\,\text{kg}\cdot\text{m}^2 - (14\,\text{kg})(2.0\,\text{m}^2) = \boxed{28\,\text{kg}\cdot\text{m}^2}$$

I_{cm} is also given by:

$$I_{\text{cm}} = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$$

Because $m_1 = m_3$, $m_2 = m_4$, $r_1^2 = r_3^2$ and $r_2^2 = r_4^2$, the expression for I_{cm} becomes:

$$I_{\text{cm}} = 2\left(m_1r_1^2 + m_2r_2^2\right)$$

Substitute numerical values and evaluate I_{cm}:

$$I_{\text{cm}} = 2\left[(3.0\,\text{kg})\left(\sqrt{2.0\,\text{m}}\right)^2 + (4.0\,\text{kg})\left(\sqrt{2.0\,\text{m}}\right)^2\right] = \boxed{28\,\text{kg}\cdot\text{m}^2}$$

43 • For the four-particle system of Figure 9-45, (*a*) find the moment of inertia I_x about the x axis, which passes through m_2 and m_3, and (*b*) find the moment of inertia I_y about the y axis, which passes through m_1 and m_2.

Picture the Problem The moment of inertia of a system of particles with respect to a given axis is the sum of the products of the mass of each particle and the square of its distance from the given axis.

(*a*) Apply the definition of the moment of inertia of a system of particles to express I_x:

$$I_x = \sum_i m_i r_i^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$$

Substitute numerical values and evaluate I_x:

$$I_x = (3.0\,\text{kg})(2.0\,\text{m})^2 + (4.0\,\text{kg})(0)^2 + (3.0\,\text{kg})(0)^2 + (4.0\,\text{kg})(2.0\,\text{m})^2 = \boxed{28\,\text{kg}\cdot\text{m}^2}$$

(*b*) Apply the definition of the moment of inertia of a system of particles to express I_y:

$$I_y = \sum_i m_i r_i^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$$

Substitute numerical values and evaluate I_y:

$$I_y = (3.0\,\text{kg})(0)^2 + (4.0\,\text{kg})(0)^2 + (3.0\,\text{kg})(2.0\,\text{m})^2 + (4.0\,\text{kg})(2.0\,\text{m})^2 = \boxed{28\,\text{kg}\cdot\text{m}^2}$$

44 • Determine the moment of inertia of a uniform solid sphere of mass M and radius R about an axis that is tangent to the surface of the sphere (Figure 9-46).

Picture the Problem According to the parallel-axis theorem, $I = I_{\text{cm}} + Mh^2$, where I_{cm} is the moment of inertia of the object with respect to an axis through its center of mass, M is the mass of the object, and h is the distance between the parallel axes.

The moment of inertia of a solid sphere of mass M and radius R about an axis that is tangent to the sphere is given by:

$$I = I_{\text{cm}} + Mh^2 \quad (1)$$

Use Table 9-1 to find the moment of inertia of a sphere with respect to an axis through its center of mass:

$$I_{\text{cm}} = \tfrac{2}{5}MR^2$$

Substitute for I_{cm} and h in equation (1) and simplify to obtain:

$$I = \tfrac{2}{5}MR^2 + MR^2 = \boxed{\tfrac{7}{5}MR^2}$$

45 •• A 1.00-m-diameter wagon wheel consists of a thin rim having a mass of 8.00 kg and 6 spokes, each with a mass of 1.20 kg. Determine the moment of inertia of the wagon wheel about its axis.

Picture the Problem The moment of inertia of the wagon wheel is the sum of the moments of inertia of the rim and the six spokes.

Express the moment of inertia of the wagon wheel as the sum of the moments of inertia of the rim and the spokes:

$$I_{\text{wheel}} = I_{\text{rim}} + I_{\text{spokes}} \quad (1)$$

Using Table 9-1, find formulas for the moments of inertia of the rim and spokes:

$$I_{\text{rim}} = M_{\text{rim}}R^2$$

and

$$I_{\text{spoke}} = \tfrac{1}{3}M_{\text{spoke}}L^2$$

Substitute for I_{rim} and I_{spoke} in equation (1) to obtain:

$$I_{\text{wheel}} = M_{\text{rim}}R^2 + 6\left(\tfrac{1}{3}M_{\text{spoke}}L^2\right) = M_{\text{rim}}R^2 + 2M_{\text{spoke}}L^2$$

Substitute numerical values and evaluate I_{wheel}:

$$I_{\text{wheel}} = (8.00\,\text{kg})(0.50\,\text{m})^2 + 2(1.20\,\text{kg})(0.50\,\text{m})^2 = \boxed{2.6\,\text{kg}\cdot\text{m}^2}$$

46 •• Two point masses m_1 and m_2 are separated by a massless rod of length L. (*a*) Write an expression for the moment of inertia I about an axis perpendicular to the rod and passing through it a distance x from mass m_1. (*b*) Calculate dI/dx and show that I is at a minimum when the axis passes through the center of mass of the system.

Picture the Problem The moment of inertia of a system of particles depends on the axis with respect to which it is calculated. Once this choice is made, the moment of inertia is the sum of the products of the mass of each particle and the square of its distance from the chosen axis.

(*a*) Apply the definition of the moment of inertia of a system of particles:

$$I = \sum_i m_i r_i^2 = \boxed{m_1x^2 + m_2(L-x)^2}$$

(*b*) Set the derivative of I with respect to x equal to zero in order to identify values for x that correspond to either maxima or minima:

$$\frac{dI}{dx} = 2m_1x + 2m_2(L-x)(-1) = 2(m_1x + m_2x - m_2L) = 0 \text{ for extrema}$$

If $\frac{dI}{dx} = 0$, then:

$$m_1x + m_2x - m_2L = 0$$

Solving for x yields:

$$x = \frac{m_2L}{m_1 + m_2}$$

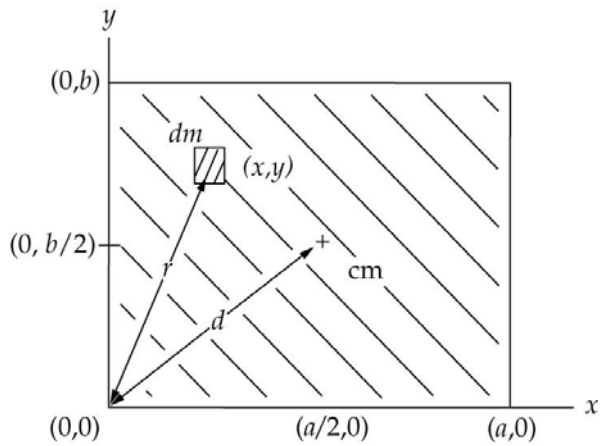
Convince yourself that you've found a minimum by showing that d^2I/dx^2 is positive at this point.

$\boxed{x = \frac{m_2L}{m_1+m_2} \text{ is, by definition, the distance of the center of mass from } m_1.}$

47 •• A uniform rectangular plate has mass m and edges of lengths a and b. (*a*) Show by integration that the moment of inertia of the plate about an axis that is perpendicular to the plate and passes through one corner is $m(a^2 + b^2)/3$.

(*b*) What is the moment of inertia about an axis that is perpendicular to the plate and passes through its center of mass?

Picture the Problem Let σ be the mass per unit area of the uniform rectangular plate. Then the elemental unit has mass $dm = \sigma\, dxdy$. Let the corner of the plate through which the axis runs be the origin. The distance of the element whose mass is dm from the corner r is related to the coordinates of dm through the Pythagorean relationship $r^2 = x^2 + y^2$.



(*a*) Express the moment of inertia of the element whose mass is dm with respect to an axis perpendicular to it and passing through one of the corners of the uniform rectangular plate:

$$dI = \sigma\left(x^2 + y^2\right)dxdy$$

Integrate this expression to find I:

$$I = \sigma\int_0^a\int_0^b\left(x^2 + y^2\right)dxdy = \tfrac{1}{3}\sigma\left(a^3b + ab^3\right)$$

$$= \boxed{\tfrac{1}{3}m\left(a^2 + b^2\right)}$$

(*b*) Letting d represent the distance from the origin to the center of mass of the plate, use the parallel axis theorem to relate the moment of inertia found in (*a*) to the moment of inertia with respect to an axis through the center of mass:

$$I = I_{\text{cm}} + md^2$$

or

$$I_{\text{cm}} = I - md^2 = \tfrac{1}{3}m\left(a^2 + b^2\right) - md^2$$

Using the Pythagorean theorem, relate the distance d to the center of mass to the lengths of the sides of the plate:

$$d^2 = \left(\tfrac{1}{2}a\right)^2 + \left(\tfrac{1}{2}b\right)^2 = \tfrac{1}{4}\left(a^2 + b^2\right)$$

Substitute for d^2 in the expression for I_{cm} and simplify to obtain:

$$I_{\text{cm}} = \tfrac{1}{3}m\left(a^2 + b^2\right) - \tfrac{1}{4}m\left(a^2 + b^2\right)^2$$
$$= \boxed{\tfrac{1}{12}m\left(a^2 + b^2\right)}$$

48 •• In attempting to ensure a spot on the pep squad, you and your friend Corey research baton-twirling. Each of you is using "The Beast" as a model baton: two uniform spheres, each of mass 500 g and radius 5.00 cm, mounted at the ends of a 30.0-cm uniform rod of mass 60.0 g (Figure 9-47). You want to determine the moment of inertia I of "The Beast" about an axis perpendicular to the rod and passing through its center. Corey uses the approximation that the two spheres can be treated as point particles that are 20.0 cm from the axis of rotation, and that the mass of the rod is negligible. You, however, decide to do an exact calculation. (*a*) Compare the two results. (Give the percentage difference between them). (*b*) Suppose the spheres were replaced by two thin spherical shells, each of the same mass as the original solid spheres. Give a conceptual argument explaining how this replacement would, or would not, change the value of I.

Picture the Problem Corey will use the point-particle relationship $I_{\text{app}} = \sum_{\text{i}} m_{\text{i}} r_{\text{i}}^2 = m_1 r_1^2 + m_2 r_2^2$ for his calculation whereas your calculation will take into account not only the rod but also the fact that the spheres are not point particles.

(*a*) Using the point-mass approximation and the definition of the moment of inertia of a system of particles, express I_{app}:

$$I_{\text{app}} = \sum_{\text{i}} m_{\text{i}} r_{\text{i}}^2 = m_1 r_1^2 + m_2 r_2^2$$

Substitute numerical values and evaluate I_{app}:

$$I_{\text{app}} = (0.500\,\text{kg})(0.200\,\text{m})^2 + (0.500\,\text{kg})(0.200\,\text{m})^2 = 0.0400\,\text{kg}\cdot\text{m}^2$$

Express the moment of inertia of the two spheres and connecting rod system:

$$I = I_{\text{spheres}} + I_{\text{rod}}$$

Use Table 9-1 to find the moments of inertia of a sphere (with respect to its center of mass) and a rod (with respect to an axis through its center of mass):

$I_{\text{sphere}} = \frac{2}{5} M_{\text{sphere}} R^2$

and

$I_{\text{rod}} = \frac{1}{12} M_{\text{rod}} L^2$

Because the spheres are not on the axis of rotation, use the parallel axis theorem to express their moment of inertia with respect to the axis of rotation:

$I_{\text{sphere}} = \frac{2}{5} M_{\text{sphere}} R^2 + M_{\text{sphere}} h^2$

where h is the distance from the center of mass of a sphere to the axis of rotation.

Substitute to obtain:

$I = 2\left[\frac{2}{5} M_{\text{sphere}} R^2 + M_{\text{sphere}} h^2\right] + \frac{1}{12} M_{\text{rod}} L^2$

Substitute numerical values and evaluate I:

$$I = 2\left[\tfrac{2}{5}(0.500\,\text{kg})(0.0500\,\text{m})^2 + (0.500\,\text{kg})(0.200\,\text{m})^2\right] + \tfrac{1}{12}(0.0600\,\text{kg})(0.300\,\text{m})^2$$
$$= 0.0415\,\text{kg}\cdot\text{m}^2$$

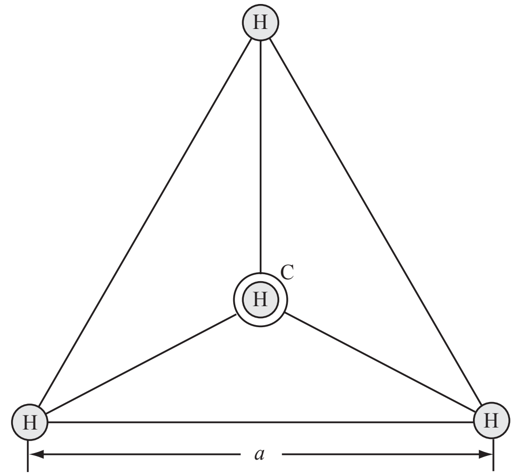
The percent difference between I and I_{app} is:

$$\frac{I - I_{\text{app}}}{I} = \frac{0.0415\,\text{kg}\cdot\text{m}^2 - 0.0400\,\text{kg}\cdot\text{m}^2}{0.0415\,\text{kg}\cdot\text{m}^2} = \boxed{3.6\ \%}$$

(*b*) The rotational inertia would increase because I_{cm} of a hollow sphere is greater than I_{cm} of a solid sphere.

49 •• The methane molecule (CH_4) has four hydrogen atoms located at the vertices of a regular tetrahedron of edge length 0.18 nm, with the carbon atom at the center of the tetrahedron (Figure 9-48). Find the moment of inertia of this molecule for rotation about an axis that passes through the centers of the carbon atom and one of the hydrogen atoms.

Picture the Problem The axis of rotation passes through the center of the base of the tetrahedron. The carbon atom and the hydrogen atom at the apex of the tetrahedron do not contribute to I because the distance of their nuclei from the axis of rotation is zero. From the geometry, the distance of the three H nuclei from the rotation axis is $a/\sqrt{3}$, where a is the length of a side of the tetrahedron.



Apply the definition of the moment of inertia for a system of particles to obtain:

$$I = \sum_i m_i r_i^2 = m_H r_1^2 + m_H r_2^2 + m_H r_3^2$$

$$= 3m_H\left(\frac{a}{\sqrt{3}}\right)^2 = m_H a^2$$

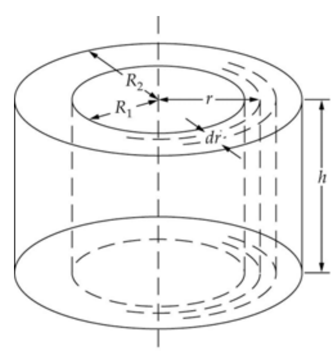
Substitute numerical values and evaluate I:

$$I = \left(1.67\times10^{-27}\ \text{kg}\right)\left(0.18\times10^{-9}\ \text{m}\right)^2$$

$$= \boxed{5.4\times10^{-47}\ \text{kg}\cdot\text{m}^2}$$

50 •• A hollow cylinder has mass m, an outside radius R_2, and an inside radius R_1. Use integration to show that the moment of inertia about its axis is given by $I = \frac{1}{2}m\left(R_2^2 + R_1^2\right)$. *Hint: Review Section 9-3, where the moment of inertia is calculated for a solid cylinder by direct integration.*

Picture the Problem Let the mass of the element of volume dV be $dm = \rho dV = 2\pi\rho hrdr$ where h is the height of the cylinder. We'll begin by expressing the moment of inertia dI for the element of volume and then integrating it between R_1 and R_2.



Express the moment of inertia of an element of mass dm:

$$dI = r^2 dm = 2\pi\rho\, hr^3 dr$$

Integrate dI from R_1 to R_2 to obtain:

$$I = 2\pi\rho h\int_{R_1}^{R_2} r^3 dr = \tfrac{1}{2}\pi\rho h\left(R_2^4 - R_1^4\right)$$
$$= \tfrac{1}{2}\pi\rho h\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)$$

The mass of the hollow cylinder is $m = \pi\rho h\left(R_2^2 - R_1^2\right)$, so:

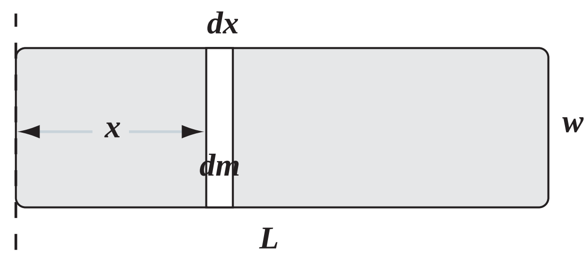
$$\rho = \frac{m}{\pi h\left(R_2^2 - R_1^2\right)}$$

Substitute for ρ and simplify to obtain:

$$I = \tfrac{1}{2}\pi\left(\frac{m}{\pi h\left(R_2^2 - R_1^2\right)}\right)h\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right) = \boxed{\tfrac{1}{2}m\left(R_2^2 + R_1^2\right)}$$

51 •• While slapping the water's surface with his tail to communicate danger, a beaver must rotate it about one of its narrow ends. Let us model the tail as a rectangle of uniform thickness and density (Figure 9-49). Estimate its moment of inertia about the line passing through its narrow end (dashed line). Assume the tail measures 15 by 30 cm with a thickness of 1.0 cm and that the flesh has the density of water.

Picture the Problem The pictorial representation shows our model of the beaver tail pivoted about the dashed line shown to the left. We can apply $I = \int x^2 dm$ to this configuration to derive an expression for the moment of inertia of the beaver tail.



The moment of inertia, about an axis through the short side of the rectangular object, is:

$$I = \int x^2 dm$$

Letting the density of the beaver's tail be represented by ρ:

$$dm = \rho dV = \rho wtdx$$

Substituting for dm yields:

$$I = \int \rho wtx^2 dx = \rho wt\int x^2 dx$$

Integrating over the length L of the beaver's tail yields:

$$I = \rho wt\int_0^L x^2 dx = \tfrac{1}{3}\rho wtL^3$$

Substitute numerical values and evaluate I:

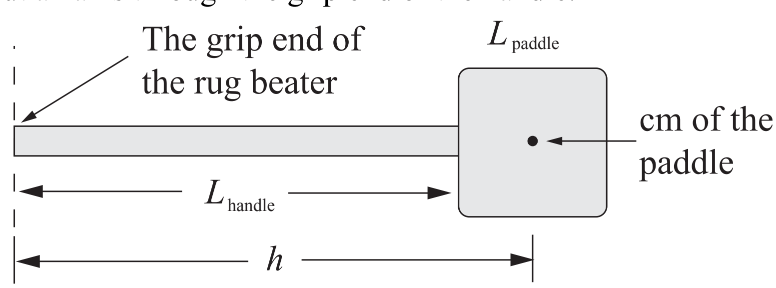
$$I = \tfrac{1}{3}\left(1.0\,\frac{\text{g}}{\text{cm}^3}\right)(15\text{ cm})(1.0\text{ cm})(30\text{ cm})^3$$

$$= \boxed{0.045\text{ kg}\cdot\text{m}^2}$$

Remarks: Had we substituted $m = \rho wtL$ in the expression for I, we would have obtained $I = \frac{1}{3}mL^2$ as the expression for the moment of inertia, about an axis through its short side, of a rectangular plate of uniform thickness.

52 •• To prevent damage to her shoulders, your elderly grandmother wants to purchase the rug beater (Figure 9-50) with the lowest moment of inertia about its grip end. Knowing you are taking physics, she asks your advice. There are two models to choose from. Model A has a 1.0-m-long handle on a 40-cm-edge-length square, where the masses of the handle and square are 1.0 kg and 0.50 kg, respectively. Model B has a 0.75-m-long handle and a 30-cm-edge-length square, where the masses of the handle and square are 1.5 kg and 0.60 kg, respectively. Which model should you recommend? Determine which beater is easiest to swing from the very end by computing the moment of inertia for both beaters.

Picture the Problem We can model the paddles attached to the handles of the rug beaters as point masses. The moment of inertia of the rug beaters can then be expressed as the sum of the moment of inertia of their handles and the moment of inertia of the point masses. Let the subscript A refer to Model A and the subscript B to Model B and find the moments of inertia of the two models. The rug beater that is easiest for your grandmother to use is the one with the smaller moment of inertia about an axis through the grip end of the handle.



The moment of inertia of Model A is:

$$I_{\text{A}} = I_{\text{handle, A}} + I_{\text{paddle, A}}$$

From Table 9-1:

$$I_{\text{handle, A}} = \tfrac{1}{3}m_{\text{handle, A}}L_{\text{handle, A}}^2$$

The moment of inertia of the point-mass paddle is given by:

$$I_{\text{paddle, A}} = m_{\text{paddle, A}} h_{\text{A}}^2$$

where h_A is the sum of L_A and half the width of the paddle.

Substituting for $I_{\text{handle, A}}$ and $I_{\text{paddle, A}}$ gives:

$$I_{\text{A}} = \tfrac{1}{3} m_{\text{handle, A}} L_{\text{handle, A}}^2 + m_{\text{paddle, A}} h_{\text{A}}^2$$

Substitute numerical values and evaluate I_A:

$$I_{\text{A}} = \tfrac{1}{3}(1.0\ \text{kg})(1.0\ \text{m})^2 + (0.50\ \text{kg})(1.2\ \text{m})^2 = 1.05\ \text{kg}\cdot\text{m}^2$$

Similarly, for Model B:

$$I_{\text{B}} = \tfrac{1}{3} m_{\text{handle, B}} L_{\text{handle, B}}^2 + m_{\text{paddle, B}} h_{\text{B}}^2$$

Substitute numerical values and evaluate I_B:

$$I_{\text{B}} = \tfrac{1}{3}(1.5\ \text{kg})(0.75\ \text{m})^2 + (0.60\ \text{kg})(0.81\ \text{m})^2 = 0.77\ \text{kg}\cdot\text{m}^2$$

Because its moment of inertia is smaller that that of the less massive rug beater, the more massive rug beater (Model B) will be easier to use.

Remarks: These point-mass approximation results agree to within 0.7% with those obtained modeling the paddles as rectangles.

53 ••• [SSM] Use integration to show that the moment of inertia of a thin spherical shell of radius R and mass m about an axis through its center is $2mR^2/3$.

Picture the Problem We can derive the given expression for the moment of inertia of a spherical shell by following the procedure outlined in the problem statement.

Find the moment of inertia of a sphere, with respect to an axis through a diameter, in Table 9-1:

$$I = \tfrac{2}{5} mR^2$$

Express the mass of the sphere as a function of its density and radius:

$$m = \tfrac{4}{3}\pi\rho R^3$$

Substitute for m to obtain:

$$I = \tfrac{8}{15}\pi\rho R^5$$

Express the differential of this expression:

$$dI = \tfrac{8}{3}\pi\rho R^4 dR \qquad (1)$$

Express the increase in mass dm as the radius of the sphere increases by dR:

$$dm = 4\pi\,\rho R^2 dR \qquad (2)$$

Eliminate dR between equations (1) and (2) to obtain:

$$dI = \tfrac{2}{3}R^2 dm$$

Integrate over the mass of the spherical shell to obtain:

$$I_{\text{spherical shell}} = \boxed{\tfrac{2}{3}mR^2}$$

54 ••• According to one model, the density of Earth varies with the distance r from the center of Earth as $\rho = C\,[1.22 - (r/R)]$, where R is the radius of Earth and C is a constant. (a) Find C in terms of the total mass M and the radius R. (b) According to this model, what is the moment of inertia of Earth about an axis through its center. (See Problem 53.)

Picture the Problem We can find C in terms of M and R by integrating a spherical shell of mass dm with the given density function to find the mass of Earth as a function of M and then solving for C. In Part (b), we'll start with the moment of inertia of the same spherical shell, substitute Earth's density function, and integrate from 0 to R. Let the axis of rotation be Earth's axis.

(a) Express the mass of Earth using the given density function:

$$M = \int dm = \int_0^R 4\pi\,\rho r^2 dr$$
$$= 4\pi C\int_0^R 1.22r^2 dr - \frac{4\pi C}{R}\int_0^R r^3 dr$$
$$= \frac{4\pi}{3}1.22CR^3 - \pi CR^3$$

Solve for C as a function of M and R to obtain:

$$C = \boxed{0.508\frac{M}{R^3}}$$

(b) From Problem 9-53 we have:

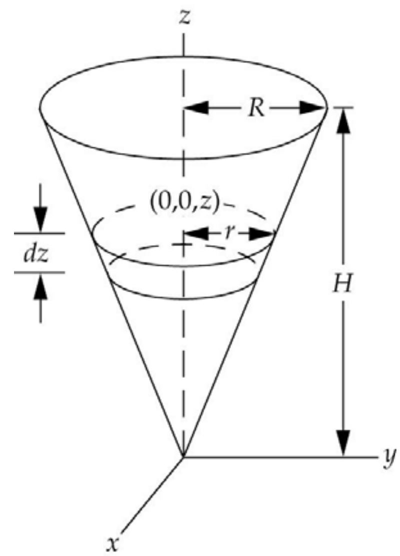
$$dI = \tfrac{8}{3}\pi\,\rho r^4 dr$$

Integrate and simplify to obtain:

$$I = \tfrac{8}{3}\pi\int_0^R \rho r^4 dr = \frac{8\pi(0.508)M}{3R^3}\left[\int_0^R 1.22r^4 dr - \frac{1}{R}\int_0^R r^5 dr\right]$$
$$= \frac{4.26M}{R^3}\left[\frac{1.22}{5}R^5 - \frac{1}{6}R^5\right] = \boxed{0.329MR^2}$$

55 ••• Use integration to determine the moment of inertia about its axis of a uniform right circular cone of height H, base radius R, and mass M.

Picture the Problem Let the origin be at the apex of the cone, with the z axis along the cone's symmetry axis. Then the radius of the elemental ring, at a distance z from the apex, can be obtained from the proportion $\frac{r}{z}=\frac{R}{H}$. The mass dm of the elemental disk is $\rho dV = \rho\pi r^2 dz$. We'll integrate $r^2 dm$ to find the moment of inertia of the disk in terms of R and H and then integrate dm to obtain a second equation in R and H that we can use to eliminate H in our expression for I.



Express the moment of inertia of the cone in terms of the moment of inertia of the elemental disk:

$$I = \tfrac{1}{2}\int r^2 dm$$
$$= \tfrac{1}{2}\int_0^H \frac{R^2}{H^2}z^2\left(\rho\pi\frac{R^2}{H^2}z^2\right)dz$$
$$= \frac{\pi\rho R^4}{2H^4}\int_0^H z^4 dz = \frac{\pi\rho R^4 H}{10}$$

Express the total mass of the cone in terms of the mass of the elemental disk:

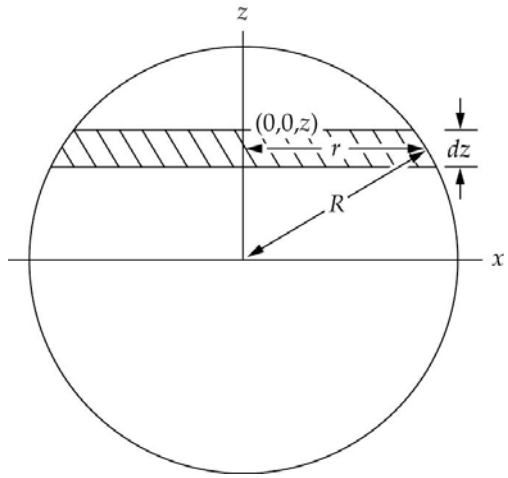
$$M = \pi\rho\int_0^H r^2 dz = \pi\rho\int_0^H \frac{R^2}{H^2}z^2 dz$$
$$= \tfrac{1}{3}\pi\rho R^2 H$$

Divide I by M, simplify, and solve for I to obtain:

$$I = \boxed{\tfrac{3}{10}MR^2}$$

56 ••• Use integration to determine the moment of inertia of a thin uniform disk of mass M and radius R about an axis in the plane of the disk and passing through its center. Check your answer by referring to Table 9-1.

Picture the Problem Let the axis of rotation be the x axis. The radius r of the elemental area is $\sqrt{R^2-z^2}$ and its mass, dm, is $\sigma\, dA = 2\sigma\sqrt{R^2-z^2}\,dz$. We'll integrate $z^2\, dm$ to determine I in terms of σ and then divide this result by M in order to eliminate σ and express I in terms of M and R.



Express the moment of inertia about the x axis: $I = \int z^2 dm = \int z^2 \sigma \, dA$

Substitute for dm to obtain: $$I = \int_{-R}^{R} z^2 \left(2\sigma\sqrt{R^2 - z^2}\, dz\right)$$

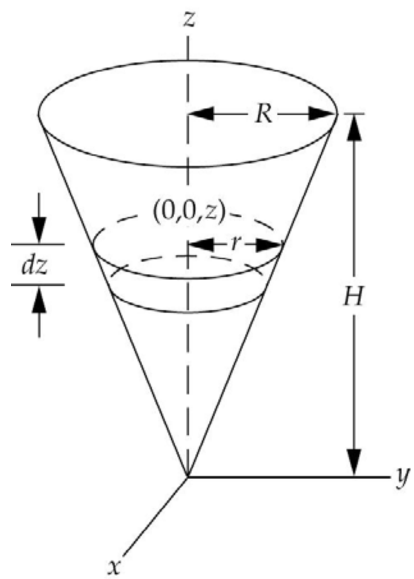
Carrying out the integration yields: $I = \frac{1}{4}\sigma\pi R^4$

The mass of the thin uniform disk is: $M = \sigma\pi R^2$

Divide I by M, simplify, and solve for I to obtain: $I = \boxed{\frac{1}{4}MR^2}$, a result in agreement with the expression given in Table 9-1 for a cylinder of length $L = 0$.

57 ••• An advertising firm has contacted your engineering firm to create a new advertisement for a local ice-cream stand. The owner of this stand wants to add rotating solid cones (painted to look like ice-cream cones, of course) to catch the eye of travelers. Each cone will rotate about an axis parallel to its base and passing through its apex. The actual size of the cones is to be decided upon, and the owner wonders if it would be more energy-efficient to rotate smaller cones than larger ones. He asks your firm to write a report showing the determination of the moment of inertia of a homogeneous right circular cone of height H, base radius R, and mass M. What is the result of your report?

Picture the Problem Choose the coordinate system shown in the diagram below. Then the radius of the elemental disk, at a distance z from the apex, can be obtained from the proportion $\frac{r}{z}=\frac{R}{H}$. The mass dm of the elemental disk is $\rho dV = \rho\pi r^2dz$. Each elemental disk rotates about an axis that is parallel to its diameter but removed from it by a distance z. We can use the result from Problem 9-56 for the moment of inertia of the elemental disk with respect to a diameter and then use the parallel axis theorem to express the moment of inertia of the cone with respect to the x axis.



Using the parallel axis theorem, express the moment of inertia of the elemental disk with respect to the x axis:

$$dI_x = dI_{\text{disk}} + dm\,z^2 \qquad (1)$$

where

$$dm = \rho\,dV = \rho\pi\,r^2dz$$

In Problem 9-56 it was established that the moment of inertia of a thin uniform disk of mass M and radius R rotating about a diameter is $\frac{1}{4}MR^2$. Express this result in terms of our elemental disk:

$$dI_{\text{disk}} = \tfrac{1}{4}\left(\rho\pi\,r^2dz\right)r^2 = \tfrac{1}{4}\rho\pi\left(\frac{R^2}{H^2}z^2\right)^2 dz$$

Substituting for dI_{disk} in equation (1) yields:

$$dI_x = \pi\rho\left[\frac{1}{4}\left(\frac{R^2}{H^2}z^2\right)^2\right]dz + \left(\pi\rho\left(\frac{R}{H}z\right)^2dz\right)z^2$$

Integrate from 0 to H to obtain:

$$I_x = \pi\rho\int_0^H\left[\frac{1}{4}\left(\frac{R^2}{H^2}z^2\right)^2 + \frac{R^2}{H^2}z^4\right]dz$$

$$= \pi\rho\left(\frac{R^4H}{20} + \frac{R^2H^3}{5}\right)$$

Express the total mass of the cone in terms of the mass of the elemental disk:

$$M = \pi\rho\int_0^H r^2 dz = \pi\rho\int_0^H \frac{R^2}{H^2} z^2 dz$$
$$= \tfrac{1}{3}\pi\rho R^2 H$$

Divide I_x by M, simplify, and solve for I_x to obtain:

$$I_x = \boxed{3M\left(\frac{H^2}{5} + \frac{R^2}{20}\right)}$$

Remarks: Because both *H* and *R* appear in the numerator, the larger the cones are, the greater their moment of inertia and the greater the energy consumption required to set them into motion.

Torque, Moment of Inertia, and Newton's Second Law for Rotation

58 • A firm wants to determine the amount of frictional torque in their current line of grindstones, so they can redesign them to be more energy efficient. To do this, they ask you to test the best-selling model, which is basically a disk-shaped grindstone of mass 1.70 kg and radius 8.00 cm which operates at 730 rev/min. When the power is shut off, you time the grindstone and find it takes 31.2 s for it to stop rotating. (*a*) What is the angular acceleration of the grindstone? (Assume constant angular acceleration.) (*b*) What is the frictional torque exerted on the grindstone?

Picture the Problem (*a*) We can use the definition of angular acceleration to find the angular acceleration of the grindstone. (*b*) Applying Newton's second law in rotational form will allow us to find the torque exerted by the friction force acting on the grindstone.

(*a*) From the definition of angular acceleration we have:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{\Delta t}$$

or, because $\omega = 0$,

$$\alpha = \frac{-\omega_0}{\Delta t}$$

Substitute numerical values and evaluate α:

$$\alpha = -\frac{730\frac{\text{rev}}{\text{min}} \times \frac{2\pi\,\text{rad}}{\text{rev}} \times \frac{1\,\text{min}}{60\,\text{s}}}{31.2\,\text{s}}$$
$$= \boxed{-2.45\,\text{rad/s}^2}$$

where the minus sign means that the grindstone is slowing down.

(*b*) Use Newton's second law in rotational form to relate the angular acceleration of the grindstone to the frictional torque slowing it:

$$\tau_{\text{frictional}} = -I\alpha$$

Express the moment of inertia of disk with respect to its axis of rotation:

$$I = \tfrac{1}{2}MR^2$$

Substitute for I to obtain:

$$\tau_{\text{frictional}} = -\tfrac{1}{2}MR^2\alpha$$

Substitute numerical values and evaluate $\tau_{\text{frictional}}$:

$$\tau_{\text{frictional}} = -\tfrac{1}{2}(1.70\,\text{kg})(0.0800\,\text{m})^2\left(-2.45\,\text{rad/s}^2\right) = \boxed{0.0133\,\text{N}\cdot\text{m}}$$

59 • **[SSM]** A 2.5-kg 11-cm-radius cylinder, initially at rest, is free to rotate about the axis of the cylinder. A rope of negligible mass is wrapped around it and pulled with a force of 17 N. Assuming that the rope does not slip, find (*a*) the torque exerted on the cylinder by the rope, (*b*) the angular acceleration of the cylinder, and (*c*) the angular speed of the cylinder after 0.50 s.

Picture the Problem We can find the torque exerted by the 17-N force from the definition of torque. The angular acceleration resulting from this torque is related to the torque through Newton's second law in rotational form. Once we know the angular acceleration, we can find the angular speed of the cylinder as a function of time.

(*a*) The torque exerted by the rope is:

$$\tau = F\ell = (17\,\text{N})(0.11\,\text{m}) = 1.87\,\text{N}\cdot\text{m} = \boxed{1.9\,\text{N}\cdot\text{m}}$$

(*b*) Use Newton's second law in rotational form to relate the acceleration resulting from this torque to the torque:

$$\alpha = \frac{\tau}{I}$$

Express the moment of inertia of the cylinder with respect to its axis of rotation:

$$I = \tfrac{1}{2}MR^2$$

Substitute for I and simplify to obtain:

$$\alpha = \frac{2\tau}{MR^2}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{2(1.87\,\text{N}\cdot\text{m})}{(2.5\,\text{kg})(0.11\,\text{m})^2} = 124\,\text{rad/s}^2$$
$$= \boxed{1.2\times10^2\ \text{rad/s}^2}$$

(*c*) Using a constant-acceleration equation, express the angular speed of the cylinder as a function of time:

$$\omega = \omega_0 + \alpha t$$

or, because $\omega_0 = 0$,

$$\omega = \alpha t$$

Substitute numerical values and evaluate ω (0.50 s):

$$\omega(0.50\,\text{s}) = (124\,\text{rad/s}^2)(0.50\,\text{s})$$
$$= \boxed{62\ \text{rad/s}}$$

60 •• A grinding wheel is initially at rest. A constant external torque of 50.0 N·m is applied to the wheel for 20.0 s, giving the wheel an angular speed of 600 rev/min. The external torque is then removed, and the wheel comes to rest 120 s later. Find (*a*) the moment of inertia of the wheel, and (*b*) the frictional torque, which is assumed to be constant.

Picture the Problem We can apply Newton's second law in rotational form to both the speeding up and slowing down motions of the wheel to obtain two equations in *I,* the moment of inertia of the wheel, and τ_{fr}, the frictional torque, that we can solve simultaneously for *I* and τ_{fr}. We'll assume that both the speeding-up and slowing-down of the wheel took place under constant-acceleration conditions.

Apply $\sum\tau = I\alpha$ to the wheel during the speeding-up portion of its motion:

$$\tau_{\text{ext}} - \tau_{\text{fr}} = I\alpha_{\text{speeding up}} \qquad (1)$$

Apply $\sum\tau = I\alpha$ to the wheel during the slowing-down portion of its motion:

$$\tau_{\text{fr}} = I\alpha_{\text{slowing down}} \qquad (2)$$

(*a*) Eliminating τ_{fr} between equations (1) and (2) and solving for *I* yields:

$$I = \frac{\tau_{\text{ext}}}{\alpha_{\text{speeding up}} + \alpha_{\text{slowing down}}}$$

Substituting for $\alpha_{\text{speeding up}}$ and $\alpha_{\text{slowing down}}$ yields:

$$I = \frac{\tau_{\text{ext}}}{\dfrac{\Delta\omega_{\text{speeding up}}}{\Delta t_{\text{speeding up}}} + \dfrac{\Delta\omega_{\text{slowing down}}}{\Delta t_{\text{slowing down}}}}$$

Substitute numerical values and evaluate I:

$$I = \frac{50.0\ \text{N}\cdot\text{m}}{\dfrac{600\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\pi\ \text{rad}}{\text{rev}}\times\dfrac{1\ \text{min}}{60\ \text{s}}}{20.0\ \text{s}} + \dfrac{-600\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\pi\ \text{rad}}{\text{rev}}\times\dfrac{1\ \text{min}}{60\ \text{s}}}{120\ \text{s}}} = 19.10\ \text{kg}\cdot\text{m}^2$$

$$= \boxed{19.1\ \text{kg}\cdot\text{m}^2}$$

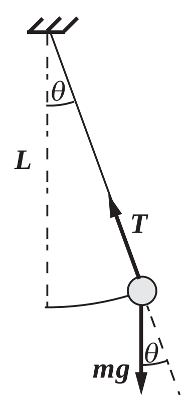
(b) Substitute numerical values in equation (2) and evaluate τ_{fr}:

$$\tau_{\text{fr}} = \left(19.10\ \text{kg}\cdot\text{m}^2\right)\left(\frac{-600\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\pi\ \text{rad}}{\text{rev}}\times\dfrac{1\ \text{min}}{60\ \text{s}}}{120\ \text{s}}\right) = \boxed{-10.0\ \text{N}\cdot\text{m}}$$

where the minus sign is a consequence of the fact that the frictional torque opposes the motion of the wheel.

61 •• A pendulum consisting of a string of length L attached to a bob of mass m swings in a vertical plane. When the string is at an angle θ to the vertical, (a) calculate the tangential acceleration of the bob using $\sum F_{\text{t}} = ma_{\text{t}}$. ($b$) What is the torque exerted about the pivot point? (c) Show that $\sum\tau = I\alpha$ with $a_{\text{t}} = L\alpha$ gives the same tangential acceleration as found in Part (a).

Picture the Problem The pendulum and the forces acting on it are shown in the free-body diagram. Note that the tension in the string is radial, and so exerts no tangential force on the ball. We can use Newton's second law in both translational and rotational form to find the tangential component of the acceleration of the bob.



(a) Referring to the free-body diagram, express the component of $m\vec{g}$ that is tangent to the circular path of the bob:

$$F_{\text{t}} = mg\sin\theta$$

Use Newton's second law to express the tangential acceleration of the bob:

$$a_{\text{t}} = \frac{F_{\text{t}}}{m} = \frac{mg\sin\theta}{m} = \boxed{g\sin\theta}$$

(*b*) Noting that, because the line-of-action of the tension passes through the pendulum's pivot point, its lever arm is zero and the net torque is due to the weight of the bob, sum the torques about the pivot point to obtain:

$$\sum \tau_{\text{pivot point}} = \boxed{mgL\sin\theta}$$

(*c*) Use Newton's second law in rotational form to relate the angular acceleration of the pendulum to the net torque acting on it:

$$\tau_{\text{net}} = mgL\sin\theta = I\alpha$$

Solve for α to obtain:

$$\alpha = \frac{mgL\sin\theta}{I}$$

Express the moment of inertia of the bob with respect to the pivot point:

$$I = mL^2$$

Substitute for I and simplify to obtain:

$$\alpha = \frac{mgL\sin\theta}{mL^2} = \frac{g\sin\theta}{L}$$

Relate α to a_t, substitute for α, and simplify to obtain:

$$a_t = r\alpha = L\left(\frac{g\sin\theta}{L}\right) = \boxed{g\sin\theta}$$

62 **•••** A uniform rod of mass M and length L is pivoted at one end and hangs as in Figure 9-51 so that it is free to rotate without friction about its pivot. It is struck a sharp horizontal blow a distance x below the pivot, as shown. (*a*) Show that, just after the rod is struck, the speed of the center of mass of the rod is given by $v_0 = 3xF_0\Delta t/(2ML)$, where F_0 and Δt are the average force and duration, respectively, of the blow. (*b*) Find the horizontal component of the force exerted by the pivot on the rod, and show that this force component is zero if $x = \frac{2}{3}L$. This point (the point of impact when the horizontal component of the pivot force is zero) is called the *center of percussion* of the rod-pivot system.

Picture the Problem We can express the velocity of the center of mass of the rod in terms of its distance from the pivot point and the angular speed of the rod. We can find the angular speed of the rod by using Newton's second law to find its angular acceleration and then a constant-acceleration equation that relates ω to α. We'll use the impulse-momentum relationship to derive the expression for the force delivered to the rod by the pivot. Finally, the location of the *center of percussion* of the rod will be verified by setting the force exerted by the pivot to zero.

(*a*) Relate the velocity of the center of mass to its distance from the pivot point:

$$v_{cm} = \frac{L}{2}\omega \qquad (1)$$

Express the torque due to F_0:

$$\tau = F_0 x = I_{pivot}\alpha \Rightarrow \alpha = \frac{F_0 x}{I_{pivot}}$$

Express the moment of inertia of the rod with respect to an axis through its pivot point:

$$I_{pivot} = \tfrac{1}{3}ML^2$$

Substitute for I_{pivot} and simplify to obtain:

$$\alpha = \frac{3F_0 x}{ML^2}$$

Express the angular speed of the rod in terms of its angular acceleration:

$$\omega = \alpha\,\Delta t = \frac{3F_0 x\Delta t}{ML^2}$$

Substitute in equation (1) to obtain:

$$v_{cm} = \boxed{\frac{3F_0 x\Delta t}{2ML}}$$

(*b*) Let I_P be the impulse exerted by the pivot on the rod. Then the total impulse (equal to the change in momentum of the rod) exerted on the rod is:

$$I_P + F_0\Delta t = Mv_{cm}$$

and

$$I_P = Mv_{cm} - F_0\Delta t$$

Substitute your result from (*a*) to obtain:

$$I_P = \frac{3F_0 x\Delta t}{2L} - F_0\Delta t = F_0\Delta t\left(\frac{3x}{2L} - 1\right)$$

Because $I_P = F_P\Delta t$:

$$F_P = \boxed{F_0\left(\frac{3x}{2L} - 1\right)}$$

If F_P is zero, then:

$$\frac{3x}{2L} - 1 = 0 \Rightarrow x = \boxed{\frac{2L}{3}}$$

63 ••• A uniform horizontal disk of mass M and radius R is spinning about the vertical axis through its center with an angular speed ω. When the spinning disk is dropped onto a horizontal tabletop, kinetic-frictional forces on the disk oppose its spinning motion. Let μ_k be the coefficient of kinetic friction between the disk and the tabletop. (*a*) Find the torque $d\tau$ exerted by the force of friction

on a circular element of radius r and width dr. (b) Find the total torque exerted by friction on the disk. (c) Find the time required for the disk to stop rotating.

Picture the Problem We'll first express the torque exerted by the force of friction on the elemental disk and then integrate this expression to find the torque on the entire disk. We'll use Newton's second law to relate this torque to the angular acceleration of the disk and then to the stopping time for the disk.

(a) Express the torque exerted on the elemental disk in terms of the friction force and the distance to the elemental disk:

$$d\tau_{\mathrm{f}} = r\,df_{\mathrm{k}} \quad (1)$$

Using the definition of the coefficient of friction, relate the force of friction to μ_{k} and the weight of the circular element:

$$df_{\mathrm{k}} = \mu_{\mathrm{k}}\,g\,dm \quad (2)$$

Letting σ represent the mass per unit area of the disk, express the mass of the circular element:

$$dm = 2\pi r\sigma\,dr \quad (3)$$

Substitute equations (2) and (3) in (1) to obtain:

$$d\tau_{\mathrm{f}} = 2\pi\mu_{\mathrm{k}}\sigma g r^2 dr \quad (4)$$

Because $\sigma = \dfrac{M}{\pi R^2}$:

$$d\tau_{\mathrm{f}} = \boxed{\frac{2\mu_{\mathrm{k}} M g}{R^2} r^2 dr}$$

(b) Integrate $d\tau_{\mathrm{f}}$ to obtain the total torque on the elemental disk:

$$\tau_{\mathrm{f}} = \frac{2\mu_{\mathrm{k}} M g}{R^2}\int_0^R r^2 dr = \boxed{\tfrac{2}{3} MR\mu_{\mathrm{k}} g}$$

(c) Relate the disk's stopping time to its angular speed and acceleration:

$$\Delta t = \frac{\omega}{\alpha}$$

Using Newton's second law, express α in terms of the net torque acting on the disk:

$$\alpha = \frac{\tau_{\mathrm{f}}}{I}$$

Substituting for α yields:

$$\Delta t = \frac{\omega}{\frac{\tau_{\mathrm{f}}}{I}} = \frac{I\omega}{\tau_{\mathrm{f}}}$$

The moment of inertia of the disk, with respect to its axis of rotation, is:

$$I = \tfrac{1}{2}MR^2$$

Substitute for I and τ_f (from Part (a)) and simplify to obtain:

$$\Delta t = \frac{\tfrac{1}{2}MR^2\omega}{\tfrac{2}{3}MR\mu_k g} = \boxed{\frac{3R\omega}{4\mu_k g}}$$

Energy Methods Including Rotational Kinetic Energy

64 • The particles in Figure 9-52 are connected by a very light rod. They rotate about the y axis at 2.0 rad/s. (a) Find the speed of each particle, and use it to calculate the kinetic energy of this system directly from $\sum \tfrac{1}{2}m_i v_i^2$. (b) Find the moment of inertia about the y axis, calculate the kinetic energy from $K = \tfrac{1}{2}I\omega^2$, and compare your result with your Part-(a) result.

Picture the Problem The kinetic energy of this rotating system of particles can be calculated either by finding the tangential velocities of the particles and using these values to find the kinetic energy or by finding the moment of inertia of the system and using the expression for the rotational kinetic energy of a system.

(a) Use the relationship between v and ω to find the speed of each particle:

$$v_3 = r_3\omega = (0.20\,\text{m})(2.0\,\text{rad/s}) = 0.40\,\text{m/s}$$

and

$$v_1 = r_1\omega = (0.40\,\text{m})(2.0\,\text{rad/s}) = 0.80\,\text{m/s}$$

The kinetic energy of the system is:

$$K = 2K_3 + 2K_1 = m_3v_3^2 + m_1v_1^2$$

Substitute numerical values and evaluate K:

$$K = (3.0\,\text{kg})(0.40\,\text{m/s})^2 + (1.0\,\text{kg})(0.80\,\text{m/s})^2 = \boxed{1.1\,\text{J}}$$

(b) Use the definition of the moment of inertia of a system of particles to obtain:

$$I = \sum_i m_i r_i^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$$

Substitute numerical values and evaluate I:

$$I = (1.0\,\text{kg})(0.40\,\text{m})^2 + (3.0\,\text{kg})(0.20\,\text{m})^2 + (1.0\,\text{kg})(0.40\,\text{m})^2 + (3.0\,\text{kg})(0.20\,\text{m})^2 = 0.560\,\text{kg}\cdot\text{m}^2$$

The kinetic energy of the system of particles is given by:

$$K = \tfrac{1}{2}I\omega^2$$

Substitute numerical values and evaluate K:

$$K = \tfrac{1}{2}\left(0.560\,\text{kg}\cdot\text{m}^2\right)\left(2.0\,\text{rad/s}\right)^2 = \boxed{1.1\,\text{J}}$$

in agreement with our Part (*a*) result.

65 • [SSM] A 1.4-kg 15-cm-diameter solid sphere is rotating about its diameter at 70 rev/min. (*a*) What is its kinetic energy? (*b*) If an additional 5.0 mJ of energy are added to the kinetic energy, what is the new angular speed of the sphere?

Picture the Problem We can find the kinetic energy of this rotating ball from its angular speed and its moment of inertia. In Part (*b*) we can use the work-kinetic energy theorem to find the angular speed of the sphere when additional kinetic energy has been added to the sphere.

(*a*) The initial rotational kinetic energy of the ball is:

$$K_\text{i} = \tfrac{1}{2}I\omega_\text{i}^2$$

Express the moment of inertia of the ball with respect to its diameter:

$$I = \tfrac{2}{5}MR^2$$

Substitute for I to obtain:

$$K_\text{i} = \tfrac{1}{5}MR^2\omega_\text{i}^2$$

Substitute numerical values and evaluate K:

$$K_\text{i} = \tfrac{1}{5}(1.4\,\text{kg})(0.075\,\text{m})^2\left(70\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)^2 = 84.6\,\text{mJ} = \boxed{85\,\text{mJ}}$$

(*b*) Apply the work-kinetic energy theorem to the sphere to obtain:

$$W = \Delta K = K_\text{f} - K_\text{i}$$

or

$$W = \tfrac{1}{2}I\omega_\text{f}^2 - K_\text{i} \Rightarrow \omega_\text{f} = \sqrt{\frac{2(W+K_\text{i})}{I}}$$

Substitute for I and simplify to obtain:

$$\omega_\text{f} = \sqrt{\frac{2(W+K_\text{i})}{\frac{2}{5}MR^2}} = \sqrt{\frac{5(W+K_\text{i})}{MR^2}}$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \sqrt{\frac{5(84.6\text{ mJ} + 5.0\text{ mJ})}{(1.4\text{ kg})(7.5\text{ cm})^2}}$$
$$= 7.542\frac{\text{rad}}{\text{s}} \times \frac{1\text{ rev}}{2\pi\text{ rad}} \times \frac{60\text{ s}}{\text{min}}$$
$$= \boxed{72\text{ rev/min}}$$

66 •• Calculate the kinetic energy of Earth due to its spinning about its axis, and compare your answer with the kinetic energy of the orbital motion of Earth's center of mass about the Sun. Assume Earth to be a homogeneous sphere of mass 6.0×10^{24} kg and radius 6.4×10^6 m. The radius of Earth's orbit is 1.5×10^{11} m.

Picture the Problem Earth's rotational kinetic energy is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2$ where I is its moment of inertia with respect to its axis of rotation. The center of mass of the Earth-Sun system is so close to the center of the Sun and the Earth-Sun distance so large that we can use the Earth-Sun distance as the separation of their centers of mass and assume each to be a point mass.

Express the rotational kinetic energy of Earth:

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2 \quad (1)$$

Find the angular speed of Earth's rotation using the definition of ω:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\text{ rad}}{24\text{ h} \times \frac{3600\text{ s}}{\text{h}}}$$
$$= 7.27 \times 10^{-5}\text{ rad/s}$$

From Table 9-1, for the moment of inertia of a homogeneous sphere, we find:

$$I = \tfrac{2}{5}MR^2$$
$$= \tfrac{2}{5}(6.0 \times 10^{24}\text{ kg})(6.4 \times 10^6\text{ m})^2$$
$$= 9.83 \times 10^{37}\text{ kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) to obtain:

$$K_{\text{rot}} = \tfrac{1}{2}(9.83 \times 10^{37}\text{ kg}\cdot\text{m}^2)$$
$$\times (7.27 \times 10^{-5}\text{ rad/s})^2$$
$$= 2.60 \times 10^{29}\text{ J} = \boxed{2.6 \times 10^{29}\text{ J}}$$

Earth's orbital kinetic energy is:

$$K_{\text{orb}} = \tfrac{1}{2}I\omega_{\text{orb}}^2 \quad (2)$$

Find the angular speed of the center of mass of the Earth-Sun system:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\text{rad}}{365.24\,\text{days} \times 24\,\frac{\text{h}}{\text{day}} \times \frac{3600\,\text{s}}{\text{h}}} = 1.99 \times 10^{-7}\ \text{rad/s}$$

The orbital moment of inertia of Earth is:

$$I = M_E R_{\text{orb}}^2 = \left(6.0 \times 10^{24}\ \text{kg}\right)\left(1.50 \times 10^{11}\ \text{m}\right)^2 = 1.35 \times 10^{47}\,\text{kg}\cdot\text{m}^2$$

Substitute for I in equation (2) and evaluate K_{orb}:

$$K_{\text{orb}} = \tfrac{1}{2}\left(1.35 \times 10^{47}\ \text{kg}\cdot\text{m}^2\right) \times \left(1.99 \times 10^{-7}\ \text{rad/s}\right)^2 = 2.68 \times 10^{33}\ \text{J}$$

Evaluate the ratio $\frac{K_{\text{orb}}}{K_{\text{rot}}}$:

$$\frac{K_{\text{orb}}}{K_{\text{rot}}} = \frac{2.68 \times 10^{33}\ \text{J}}{2.60 \times 10^{29}\ \text{J}} \approx \boxed{10^4}$$

67 •• [SSM] A 2000-kg block is lifted at a constant speed of 8.0 cm/s by a steel cable that passes over a massless pulley to a motor-driven winch (Figure 9-53). The radius of the winch drum is 30 cm. (*a*) What is the tension in the cable? (*b*) What torque does the cable exert on the winch drum? (*c*) What is the angular speed of the winch drum? (*d*) What power must be developed by the motor to drive the winch drum?

Picture the Problem Because the load is not being accelerated, the tension in the cable equals the weight of the load. The role of the massless pulley is to change the direction the force (tension) in the cable acts.

(*a*) Because the block is lifted at constant speed:

$$T = mg = \left(2000\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right) = \boxed{19.6\,\text{kN}}$$

(*b*) Apply the definition of torque of the winch drum:

$$\tau = Tr = \left(19.6\,\text{kN}\right)\left(0.30\,\text{m}\right) = \boxed{5.9\,\text{kN}\cdot\text{m}}$$

(*c*) Relate the angular speed of the winch drum to the rate at which the load is being lifted (the tangential speed of the cable on the drum):

$$\omega = \frac{v}{r} = \frac{0.080\,\text{m/s}}{0.30\,\text{m}} = \boxed{0.27\,\text{rad/s}}$$

(*d*) The power developed by the motor in terms is the product of the tension in the cable and the speed with which the load is being lifted:

$$P = Tv = (19.6\,\text{kN})(0.080\,\text{m/s}) = \boxed{1.6\,\text{kW}}$$

68 •• A uniform disk that has a mass *M* and a radius *R* can rotate freely about a fixed horizontal axis that passes through its center and is perpendicular to the plane of the disk. A small particle that has a mass *m* is attached to the rim of the disk at the top, directly above the pivot. The system is gently nudged, and the disk begins to rotate. As the particle passes through its lowest point, (*a*) what is the angular speed of the disk, and (*b*) what force is exerted by the disk on the particle?

Picture the Problem Let the zero of gravitational potential energy be at the lowest point of the small particle. We can use conservation of energy to find the angular speed of the disk when the particle is at its lowest point and Newton's second law to find the force the disk will have to exert on the particle to keep it from falling off.

(*a*) Use conservation of energy to relate the initial potential energy of the system to its rotational kinetic energy when the small particle is at its lowest point:

$$\Delta K + \Delta U = 0$$

or, because $U_f = K_i = 0$,

$$\tfrac{1}{2}\left(I_{\text{disk}} + I_{\text{particle}}\right)\omega_f^2 - mg\Delta h = 0$$

Solving for ω_f yields:

$$\omega_f = \sqrt{\frac{2mg\Delta h}{I_{\text{disk}} + I_{\text{particle}}}}$$

Substitute for I_{disk}, I_{particle}, and Δh and simplify to obtain:

$$\omega_f = \sqrt{\frac{2mg(2R)}{\frac{1}{2}MR^2 + mR^2}} = \boxed{\sqrt{\frac{8mg}{R(2m+M)}}}$$

(*b*) The mass is in uniform circular motion at the bottom of the disk, so the sum of the force *F* exerted by the disk and the gravitational force must be the centripetal force:

$$F - mg = mR\omega_f^2 \Rightarrow F = mg + mR\omega_f^2$$

Substituting for ω_f^2 and simplifying yields::

$$F = mg + mR\left(\frac{8mg}{R(2m+M)}\right)$$
$$= \boxed{mg\left(1+\frac{8m}{2m+M}\right)}$$

69 •• A uniform 1.5-m-diameter ring is pivoted at a point on its perimeter so that it is free to rotate about a horizontal axis that is perpendicular to the plane of the ring. The ring is released with the center of the ring at the same height as the axis (Figure 9-54). (*a*) If the ring was released from rest, what was its maximum angular speed? (*b*) What minimum angular speed must it be given at release if it is to rotate a full 360°?

Picture the Problem Let the zero of gravitational potential energy be at the center of mass of the ring when it is directly below the point of support. We'll use conservation of energy to relate the maximum angular speed and the initial angular speed required for a complete revolution to the changes in the potential energy of the ring.

(*a*) Use conservation of energy to relate the initial potential energy of the ring to its rotational kinetic energy when its center of mass is directly below the point of support:

$$\Delta K + \Delta U = 0$$

or, because $U_f = K_i = 0$,

$$\tfrac{1}{2}I_P\omega_{max}^2 - mg\Delta h = 0 \qquad (1)$$

Use the parallel axis theorem and Table 9-1 to express the moment of inertia of the ring with respect to its pivot point P:

$$I_P = I_{cm} + mR^2$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}\left(mR^2 + mR^2\right)\omega_{max}^2 - mgR = 0$$

Solving for ω_{max} yields:

$$\omega_{max} = \sqrt{\frac{g}{R}}$$

Substitute numerical values and evaluate ω_{max}:

$$\omega_{max} = \sqrt{\frac{9.81\,\text{m/s}^2}{0.75\,\text{m}}} = \boxed{3.6\,\text{rad/s}}$$

(*b*) Use conservation of energy to relate the final potential energy of the ring to its initial rotational kinetic energy:

$$\Delta K + \Delta U = 0$$

or, because $U_i = K_f = 0$,

$$-\tfrac{1}{2}I_P\omega_i^2 + mg\Delta h = 0$$

Noting that the center of mass must rise a distance R if the ring is to make a complete revolution, substitute for I_P and Δh to obtain:

$$-\tfrac{1}{2}\left(mR^2 + mR^2\right)\omega_i^2 + mgR = 0$$

Solving for ω_i yields:

$$\omega_i = \sqrt{\frac{g}{R}}$$

Substitute numerical values and evaluate ω_i:

$$\omega_i = \sqrt{\frac{9.81\,\text{m/s}^2}{0.75\,\text{m}}} = \boxed{3.6\,\text{rad/s}}$$

70 •• You set out to design a car that uses the energy stored in a flywheel consisting of a uniform 100-kg cylinder of radius R that has a maximum angular speed of 400 rev/s. The flywheel must deliver an average of 2.00 MJ of energy for each kilometer of distance. Find the smallest value of R for which the car can travel 300 km without the flywheel needing to be reenergized.

Picture the Problem We can find the energy that must be stored in the flywheel and relate this energy to the radius of the wheel and use the definition of rotational kinetic energy to find the wheel's radius.

Relate the kinetic energy of the flywheel to the energy it must deliver:

$$K_{\text{rot}} = \tfrac{1}{2} I_{\text{cyl}}\omega^2$$

Express the moment of inertia of the flywheel:

$$I_{\text{cyl}} = \tfrac{1}{2} MR^2$$

Substitute for I_{cyl} to obtain:

$$K_{\text{rot}} = \tfrac{1}{2}\left(\tfrac{1}{2} MR^2\right)\omega^2 \Rightarrow R = \frac{2}{\omega}\sqrt{\frac{K_{\text{rot}}}{M}}$$

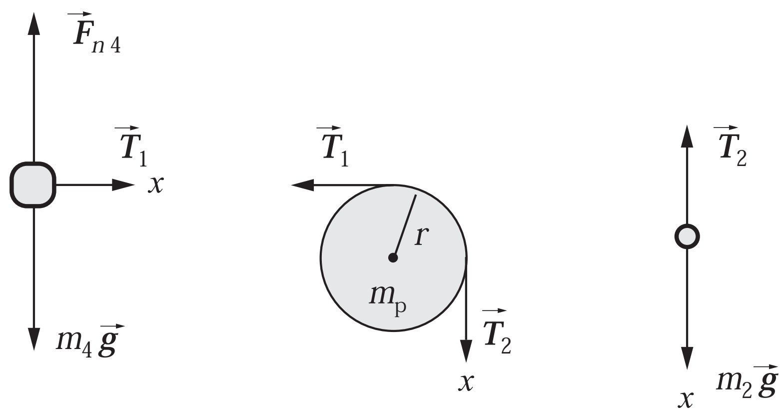
Substitute numerical values and evaluate R:

$$R = \frac{2}{400\,\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}}\sqrt{\frac{\left(2.00\,\frac{\text{MJ}}{\text{km}}\right)(300\,\text{km})}{100\,\text{kg}}} = \boxed{1.95\,\text{m}}$$

Pulleys, Yo-Yos, and Hanging Things

71 •• [SSM] The system shown in Figure 9-55consists of a 4.0-kg block resting on a frictionless horizontal ledge. This block is attached to a string that passes over a pulley, and the other end of the string is attached to a hanging 2.0-kg block. The pulley is a uniform disk of radius 8.0 cm and mass 0.60 kg. Find the acceleration of each block and the tension in the string.

Picture the Problem The diagrams show the forces acting on each of the masses and the pulley. We can apply Newton's second law to the two blocks and the pulley to obtain three equations in the unknowns T_1, T_2, and a.



Apply Newton's second law to the two blocks and the pulley:

$$\sum F_x = T_1 = m_4 a\,, \qquad (1)$$

$$\sum \tau_p = (T_2 - T_1)r = I_p\alpha\,, \qquad (2)$$

and

$$\sum F_x = m_2 g - T_2 = m_2 a \qquad (3)$$

Substitute for I_p and α in equation (2) to obtain:

$$T_2 - T_1 = \tfrac{1}{2}M_p a \qquad (4)$$

Eliminate T_1 and T_2 between equations (1), (3) and (4) and solve for a:

$$a = \frac{m_2 g}{m_2 + m_4 + \frac{1}{2}M_p}$$

Substitute numerical values and evaluate a:

$$a = \frac{(2.0\,\text{kg})(9.81\,\text{m/s}^2)}{2.0\,\text{kg} + 4.0\,\text{kg} + \frac{1}{2}(0.60\,\text{kg})} = 3.11\,\text{m/s}^2 = \boxed{3.1\,\text{m/s}^2}$$

Using equation (1), evaluate T_1:

$$T_1 = (4.0\,\text{kg})(3.11\,\text{m/s}^2) = \boxed{12\,\text{N}}$$

Solve equation (3) for T_2:

$$T_2 = m_2(g - a)$$

Substitute numerical values and evaluate T_2:

$$T_2 = (2.0\,\text{kg})(9.81\,\text{m/s}^2 - 3.11\,\text{m/s}^2) = \boxed{13\,\text{N}}$$

72 •• For the system in Problem 71, the 2.0-kg block is released from rest. (*a*) Find the speed of the block after it falls a distance of 2.5 m. (*b*) What is the angular speed of the pulley at this instant?

Picture the Problem We'll solve this problem for the general case in which the mass of the block on the ledge is M, the mass of the hanging block is m, and the mass of the pulley is M_p, and R is the radius of the pulley. Let the zero of gravitational potential energy be 2.5 m below the initial position of the 2.0-kg block and R represent the radius of the pulley. Let the system include both blocks, the shelf and pulley, and Earth. The initial potential energy of the 2.0-kg block will be transformed into the translational kinetic energy of both blocks plus rotational kinetic energy of the pulley.

(*a*) Use energy conservation to relate the speed of the 2 kg block when it has fallen a distance Δh to its initial potential energy and the kinetic energy of the system:

$$\Delta K + \Delta U = 0$$

or, because $K_\text{i} = U_\text{f} = 0$,

$$\tfrac{1}{2}(m+M)v^2 + \tfrac{1}{2}I_\text{pulley}\omega^2 - mgh = 0$$

Substitute for I_pulley and ω to obtain:

$$\tfrac{1}{2}(m+M)v^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\frac{v^2}{R^2} - mgh = 0$$

Solving for v yields:

$$v = \sqrt{\frac{2mgh}{M + m + \frac{1}{2}M_p}}$$

Substitute numerical values and evaluate v:

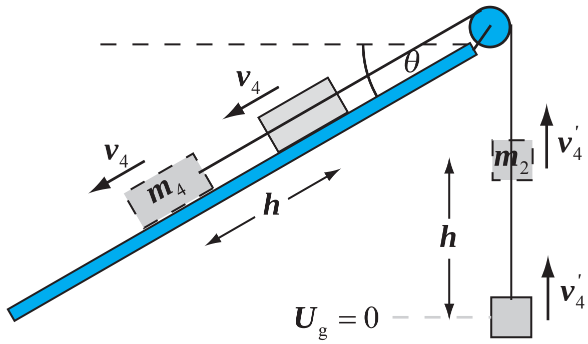
$$v = \sqrt{\frac{2(2.0\,\text{kg})(9.81\,\text{m/s}^2)(2.5\,\text{m})}{4.0\,\text{kg} + 2.0\,\text{kg} + \frac{1}{2}(0.60\,\text{kg})}} = 3.946\,\text{m/s} = \boxed{3.9\,\text{m/s}}$$

(*b*) The angular speed of the pulley is the ratio of its tangential speed to its radius:

$$\omega = \frac{v}{R} = \frac{3.946\,\text{m/s}}{0.080\,\text{m}} = \boxed{49\,\text{rad/s}}$$

73 •• For the system in Problem 71, if the (frictionless) ledge were adjustable in angle, at what angle would it have to be tilted upward so that once the system is set into motion the blocks will continue to move at constant speed?

Picture the Problem The pictorial representation shows the ledge with its left end lowered and the 4.0-kg block moving with a constant speed v_4. As this block (whose mass is m_4) slides down the frictionless incline, the block whose mass is m_2 rises. Let the system include Earth, the ledge, the pulley and both blocks and apply conservation of mechanical energy to determine the angle of inclination of the ledge.



Apply conservation of mechanical energy to the two moving objects to obtain:

$$\Delta K_2 + \Delta K_4 + \Delta K_{\text{pulley}} + \Delta U_2 + \Delta U_4 = 0$$

Because the objects are moving at constant speed, $\Delta K_2 = \Delta K_4 = \Delta K_{\text{pulley}} = 0$ and:

$$\Delta U_2 + \Delta U_4 = 0$$

Substituting for ΔU_2 and ΔU_4 yields:

$$m_2 gh - m_4 gh \sin\theta = 0$$

Solving for θ yields:

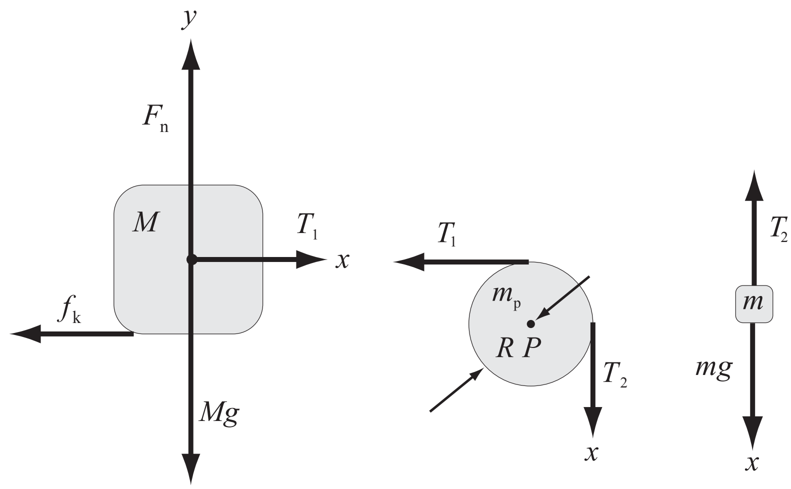
$$\theta = \sin^{-1}\left(\frac{m_2}{m_4}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left(\frac{2.0\,\text{kg}}{4.0\,\text{kg}}\right) = \boxed{30^\circ}$$

74 •• In the system shown in Figure 9-55, there is a 4.0-kg block resting on a horizontal ledge. The coefficient of kinetic friction between the ledge and the block is 0.25. The block is attached to a string that passes over a pulley and the other end of the string is attached to a hanging 2.0-kg block. The pulley is a uniform disk of radius 8.0 cm and mass 0.60 kg. Find the acceleration of each block and the tensions in the segments of string attached to each block.

Picture the Problem Assuming that the string does not stretch or slip on the pulley, we can apply Newton's second law in translational form to the both the 4.0-kg block and the 2.0-kg block and Newton's second law in rotational form to the pulley to obtain simultaneous equations in the tensions T_1 and T_2 in the string and the common acceleration a of the blocks.



Apply $\sum\vec{F} = m\vec{a}$ to the block whose mass is M:

$$\sum F_x = T_1 - f_k = Ma \qquad (1)$$

and

$$\sum F_y = F_n - Mg = 0 \Rightarrow F_n = Mg$$

Because $f_k = \mu_k F_n$ and $F_n = Mg$, equation (1) becomes:

$$T_1 - \mu_k Mg = Ma$$

or

$$T_1 = Ma + \mu_k Mg = M(a + \mu_k g) \qquad (2)$$

Apply $\sum\vec{F} = m\vec{a}$ to the block whose mass is m:

$$\sum F_x = mg - T_2 = ma$$

or

$$T_2 = mg - ma = m(g - a) \qquad (3)$$

Apply $\sum\tau = I_P\alpha$ to the pulley:

$$T_2R - T_1R = I_P\alpha$$

or, because $\alpha = a/R$,

$$T_2R - T_1R = I_P\frac{a}{R}$$

Substituting for I_p (see Table 9-1) and simplifying gives:

$$T_2R - T_1R = \tfrac{1}{2}m_P R^2\left(\frac{a}{R}\right)$$

and

$$T_2 - T_1 = \tfrac{1}{2}m_P a$$

Substituting for T_2 (from equation (3)) and T_1 (from equation (2)) yields:

$$mg - ma - Ma - \mu_k Mg = \tfrac{1}{2} m_p a$$

Solving for a yields:

$$a = \frac{(m - \mu_k M)g}{m + M + \frac{1}{2} m_p}$$

Substitute numerical values and evaluate a:

$$a = \frac{(2.0\text{ kg} - (0.25)(4.0\text{ kg}))(9.81\text{ m/s}^2)}{2.0\text{ kg} + 4.0\text{ kg} + \frac{1}{2}(0.60\text{ kg})}$$
$$= 1.56\text{ m/s}^2 = \boxed{1.6\text{ m/s}^2}$$

Substitute numerical values in equation (2) and evaluate T_1:

$$T_1 = (4.0\text{ kg})\left[1.56\text{ m/s}^2 + (0.25)(9.81\text{ m/s}^2)\right] = 16.04\text{ N} = \boxed{16\text{ N}}$$

Substitute numerical values in equation (3) and evaluate T_2:

$$T_2 = (2.0\text{ kg})(9.81\text{ m/s}^2 - 1.56\text{ m/s}^2) = \boxed{17\text{ N}}$$

75 •• A 1200-kg car is being raised over water by a winch. At the moment the car is 5.0 m above the water (Figure 9-56), the gearbox breaks—allowing the winch drum to spin freely as the car falls. During the car's fall, there is no slipping between the (massless) rope, the pulley wheel, and the winch drum. The moment of inertia of the winch drum is 320 kg·m^2, and the moment of inertia of the pulley wheel is 4.00 kg·m^2. The radius of the winch drum is 0.800 m, and the radius of the pulley is 0.300 m. Assume the car starts to fall from rest. Find the speed of the car as it hits the water.

Picture the Problem Let the zero of gravitational potential energy be at the water's surface and let the system include the winch, the car, and Earth. We'll apply conservation of energy to relate the car's speed as it hits the water to its initial potential energy. Note that some of the car's initial potential energy will be transformed into rotational kinetic energy of the winch and pulley.

Use conservation of mechanical energy to relate the car's speed as it hits the water to its initial potential energy:

$$\Delta K + \Delta U = 0$$
or, because $K_i = U_f = 0$,
$$\tfrac{1}{2} mv^2 + \tfrac{1}{2} I_w \omega_w^2 + \tfrac{1}{2} I_p \omega_p^2 - mg\Delta h = 0$$

Express ω_w^2 and ω_p^2 in terms of the speed v of the rope, which is the same throughout the system:

$$\omega_w^2=\frac{v^2}{r_w^2}\text{ and }\omega_p^2=\frac{v^2}{r_p^2}$$

Substitute for ω_w^2 and ω_p^2 to obtain:

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}I_w\frac{v^2}{r_w^2}+\tfrac{1}{2}I_p\frac{v^2}{r_p^2}-mg\Delta h=0$$

Solving for v yields:

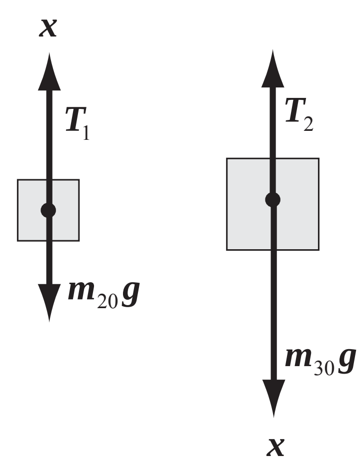
$$v=\sqrt{\frac{2mg\Delta h}{m+\dfrac{I_w}{r_w^2}+\dfrac{I_p}{r_p^2}}}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{\frac{2(1200\,\text{kg})(9.81\,\text{m/s}^2)(5.00\,\text{m})}{1200\,\text{kg}+\dfrac{320\,\text{kg}\cdot\text{m}^2}{(0.800\,\text{m})^2}+\dfrac{4.00\,\text{kg}\cdot\text{m}^2}{(0.300\,\text{m})^2}}}=\boxed{8.21\,\text{m/s}}$$

76 •• The system in Figure 9-57 is released from rest when the 30-kg block is 2.0 m above the ledge. The pulley is a uniform 5.0-kg disk with a radius of 10 cm. Just before the 30-kg block hits the ledge, find (*a*) its speed, (*b*) the angular speed of the pulley, and (*c*) the tensions in the strings. (*d*) Find the time of descent for the 30-kg block. Assume that the string does not slip on the pulley.

Picture the Problem Let the system include the blocks, the pulley and Earth. Choose the zero of gravitational potential energy to be at the ledge and apply energy conservation to relate the impact speed of the 30-kg block to the initial potential energy of the system. We can use a constant-acceleration equations and Newton's second law to find the tensions in the strings and the descent time.



(*a*) Use conservation of mechanical energy to relate the impact speed of the 30-kg block to the initial potential energy of the system:

$$\Delta K+\Delta U=0$$

or, because $K_i = U_f = 0$,

$$\tfrac{1}{2}m_{30}v^2+\tfrac{1}{2}m_{20}v^2+\tfrac{1}{2}I_p\omega_p^2+m_{20}g\Delta h-m_{30}g\Delta h=0$$

Substitute for ω_p and I_p to obtain:

$$\tfrac{1}{2}m_{30}v^2+\tfrac{1}{2}m_{20}v^2+\tfrac{1}{2}\left(\tfrac{1}{2}M_pr^2\right)\left(\frac{v^2}{r^2}\right)+m_{20}g\Delta h-m_{30}g\Delta h=0$$

Solving for v yields:

$$v=\sqrt{\frac{2g\Delta h(m_{30}-m_{20})}{m_{20}+m_{30}+\tfrac{1}{2}M_p}}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{\frac{2(9.81\,\text{m/s}^2)(2.0\,\text{m})(30\,\text{kg}-20\,\text{kg})}{20\,\text{kg}+30\,\text{kg}+\tfrac{1}{2}(5.0\,\text{kg})}}=2.73\,\text{m/s}=\boxed{2.7\,\text{m/s}}$$

(*b*) Find the angular speed at impact from the tangential speed at impact and the radius of the pulley:

$$\omega=\frac{v}{r}=\frac{2.73\,\text{m/s}}{0.10\,\text{m}}=\boxed{27\,\text{rad/s}}$$

(*c*) Apply Newton's second law to the blocks:

$$\sum F_x=T_1-m_{20}g=m_{20}a \quad (1)$$

and

$$\sum F_x=m_{30}g-T_2=m_{30}a \quad (2)$$

Using a constant-acceleration equation, relate the speed at impact to the fall distance and the acceleration:

$$v^2=v_0^2+2a\Delta h$$

or, because $v_0=0$,

$$v^2=2a\Delta h\Rightarrow a=\frac{v^2}{2\Delta h}$$

Substitute numerical values and evaluate a:

$$a=\frac{(2.73\,\text{m/s})^2}{2(2.0\,\text{m})}=1.87\,\text{m/s}^2$$

Solve equation (1) for T_1 to obtain:

$$T_1=m_{20}(g+a)$$

Substitute numerical values and evaluate T_1:

$$T_1=(20\,\text{kg})(9.81\,\text{m/s}^2+1.87\,\text{m/s}^2)=\boxed{0.23\,\text{kN}}$$

Solve equation (2) for T_2 to obtain:

$$T_2=m_{30}(g-a)$$

Substitute numerical values and evaluate T_2:

$$T_2=(30\,\text{kg})(9.81\,\text{m/s}^2-1.87\,\text{m/s}^2)=\boxed{0.24\,\text{kN}}$$

(*d*) Noting that the initial speed of the 30-kg block is zero, express the time-of-fall in terms of the fall distance and the block's average speed:

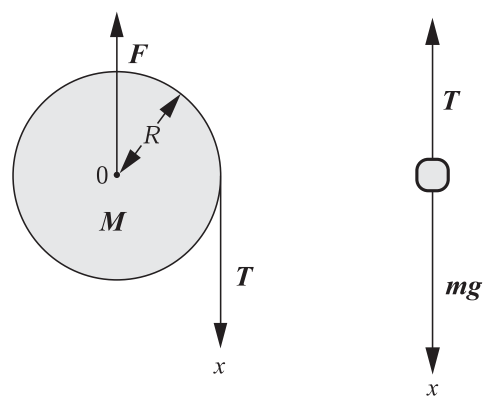
$$\Delta t = \frac{\Delta h}{v_{\text{av}}} = \frac{\Delta h}{\frac{1}{2}v} = \frac{2\Delta h}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(2.0\,\text{m})}{2.73\,\text{m/s}} = \boxed{1.5\,\text{s}}$$

77 •• A uniform solid sphere of mass *M* and radius *R* is free to rotate about a horizontal axis through its center. A string is wrapped around the sphere and is attached to an object of mass *m* (Figure 9-58). Assume that the string does not slip on the sphere. Find (*a*) the acceleration of the object, and (*b*) the tension in the string.

Picture the Problem The force diagram shows the forces acting on the sphere and the hanging object. The tension in the string is responsible for the angular acceleration of the sphere and the difference between the weight of the object and the tension is the net force acting on the hanging object. We can use Newton's second law to obtain two equations in *a* and *T* that we can solve simultaneously.



(*a*) Noting that $T = T'$, apply Newton's second law to the sphere and the hanging object:

$$\sum \tau_0 = TR = I_{\text{sphere}}\alpha \qquad (1)$$

and

$$\sum F_x = mg - T = ma \qquad (2)$$

Substitute for I_{sphere} and α in equation (1) to obtain:

$$TR = \left(\tfrac{2}{5}MR^2\right)\frac{a}{R} \qquad (3)$$

Eliminate *T* between equations (2) and (3) and solve for *a* to obtain:

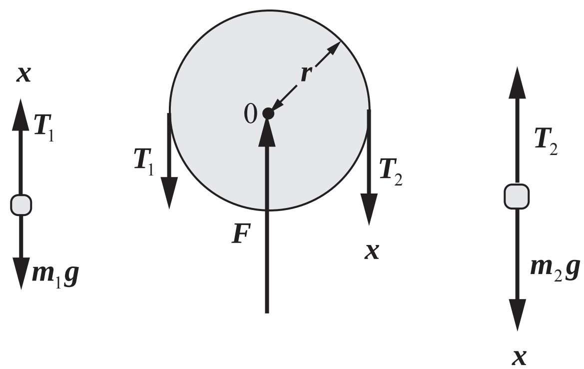
$$a = \boxed{\frac{g}{1 + \dfrac{2M}{5m}}}$$

(*b*) Substitute for a in equation (2) and solve for T to obtain:

$$T = \boxed{\frac{2mMg}{5m + 2M}}$$

78 •• Two objects, of masses m_1= 500 g and m_2 = 510 g, are connected by a string of negligible mass that passes over a pulley with frictionless bearings (Figure 9-59). The pulley is a uniform 50.0-g disk with a radius of 4.00 cm. The string does not slip on the pulley. (*a*) Find the accelerations of the objects. (*b*) What is the tension in the string between the 500-g block and the pulley? What is the tension in the string between the 510-g block and the pulley? By how much do these tensions differ? (*c*) What would your answers be if you neglected the mass of the pulley?

Picture the Problem The diagram shows the forces acting on both objects and the pulley. The direction of motion has been chosen to be the +*x* direction. By applying Newton's second law of motion, we can obtain a system of three equations in the unknowns T_1, T_2, and a that we can solve simultaneously.



(*a*) Apply Newton's second law to the pulley and the two objects:

$$\sum F_x = T_1 - m_1 g = m_1 a, \quad (1)$$

$$\sum \tau_0 = (T_2 - T_1)r = I_0 \alpha, \quad (2)$$

and

$$\sum F_x = m_2 g - T_2 = m_2 a \quad (3)$$

Substitute for $I_0 = I_{\text{pulley}}$ and α in equation (2) to obtain:

$$(T_2 - T_1)r = \left(\tfrac{1}{2}mr^2\right)\frac{a}{r} \quad (4)$$

Eliminate T_1 and T_2 between equations (1), (3) and (4) and solve for a to obtain:

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{1}{2}m}$$

Substitute numerical values and evaluate a:

$$a = \frac{(510\text{ g} - 500\text{ g})(9.81\,\text{m/s}^2)}{500\text{ g} + 510\text{ g} + \frac{1}{2}(50.0\,\text{g})}$$

$$= 9.478\,\text{cm/s}^2 = \boxed{9.48\text{ cm/s}^2}$$

(*b*) Substitute for a in equation (1) and solve for T_1 to obtain:

$$T_1 = m_1(g+a) = (0.500\,\text{kg})(9.81\,\text{m/s}^2 + 0.09478\,\text{m/s}^2) = 4.952\ \text{N} = \boxed{4.95\ \text{N}}$$

Substitute for a in equation (3) and solve for T_2 to obtain:

$$T_2 = m_1(g-a) = (0.510\,\text{kg})(9.81\,\text{m/s}^2 - 0.09478\,\text{m/s}^2) = 4.955\ \text{N} = \boxed{4.96\ \text{N}}$$

ΔT is the difference between T_2 and T_1:

$$\Delta T = T_2 - T_1 = 4.96\ \text{N} - 4.95\,\text{N} = \boxed{0.01\,\text{N}}$$

(*c*) If we ignore the mass of the pulley, our acceleration equation becomes:

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2}$$

Substitute numerical values and evaluate a:

$$a = \frac{(510\,\text{g} - 500\,\text{g})(9.81\,\text{m/s}^2)}{510\,\text{g} + 500\,\text{g}} = 9.713\,\text{cm/s}^2 = \boxed{9.71\,\text{cm/s}^2}$$

Substitute for a in equation (1) and solve for T_1 to obtain:

$$T_1 = m_1(g+a)$$

Substitute numerical values and evaluate T_1:

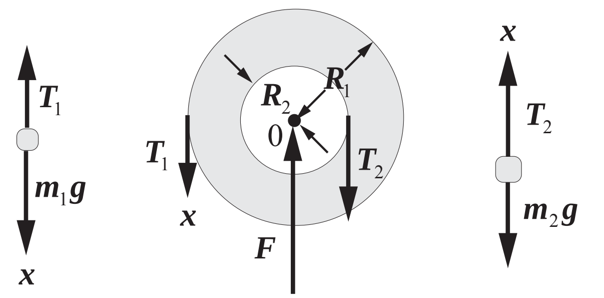
$$T_1 = (0.500\,\text{kg})(9.81\,\text{m/s}^2 + 0.09713\,\text{m/s}^2) = \boxed{4.95\,\text{N}}$$

From equation (4), if $m = 0$:

$$T_2 = T_1 = \boxed{4.95\,\text{N}}$$

79 •• [SSM] Two objects are attached to ropes that are attached to two wheels on a common axle, as shown in Figure 9-60. The two wheels are attached together so that they form a single rigid object. The moment of inertia of the rigid object is 40 kg·m^2. The radii of the wheels are R_1 = 1.2 m and R_2 = 0.40 m. (*a*) If m_1= 24 kg, find m_2 such that there is no angular acceleration of the wheels. (*b*) If 12 kg is placed on top of m_1, find the angular acceleration of the wheels and the tensions in the ropes.

Picture the Problem The following diagram shows the forces acting on both objects and the pulley for the conditions of Part (*a*). By applying Newton's second law of motion, we can obtain a system of three equations in the unknowns T_1, T_2, and α that we can solve simultaneously.



(*a*) When the system does not accelerate, $T_1 = m_1 g$ and $T_2 = m_2 g$. Under these conditions:

$$\sum \tau_0 = m_1 g R_1 - m_2 g R_2 = 0$$

Solving for m_2 yields:

$$m_2 = m_1 \frac{R_1}{R_2}$$

Substitute numerical values and evaluate m_2:

$$m_2 = (24\,\text{kg})\frac{1.2\,\text{m}}{0.40\,\text{m}} = \boxed{72\,\text{kg}}$$

(*b*) Apply Newton's second law to the objects and the pulley:

$$\sum F_x = m_1 g - T_1 = m_1 a, \qquad (1)$$

$$\sum \tau_0 = T_1 R_1 - T_2 R_2 = I_0 \alpha, \qquad (2)$$

and

$$\sum F_x = T_2 - m_2 g = m_2 a \qquad (3)$$

Eliminate a in favor of α in equations (1) and (3) and solve for T_1 and T_2:

$$T_1 = m_1 (g - R_1 \alpha) \qquad (4)$$

and

$$T_2 = m_2 (g + R_2 \alpha) \qquad (5)$$

Substitute for T_1 and T_2 in equation (2) and solve for α to obtain:

$$\alpha = \frac{(m_1 R_1 - m_2 R_2) g}{m_1 R_1^2 + m_2 R_2^2 + I_0}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{[(36\,\text{kg})(1.2\,\text{m}) - (72\,\text{kg})(0.40\,\text{m})](9.81\,\text{m/s}^2)}{(36\,\text{kg})(1.2\,\text{m})^2 + (72\,\text{kg})(0.40\,\text{m})^2 + 40\,\text{kg}\cdot\text{m}^2} = 1.37\,\text{rad/s}^2 = \boxed{1.4\,\text{rad/s}^2}$$

Substitute numerical values in equation (4) to find T_1:

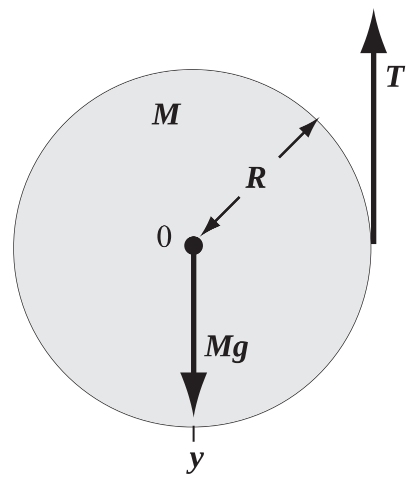
$$T_1 = (36\,\text{kg})[9.81\,\text{m/s}^2 - (1.2\,\text{m})(1.37\,\text{rad/s}^2)] = \boxed{0.29\,\text{kN}}$$

Substitute numerical values in equation (5) to find T_2:

$$T_2 = (72\,\text{kg})\left[9.81\,\text{m/s}^2 + (0.40\,\text{m})(1.37\,\text{rad/s}^2)\right] = \boxed{0.75\,\text{kN}}$$

80 •• The upper end of the string wrapped around the cylinder in Figure 9-61 is held by a hand that is accelerated upward so that the center of mass of the cylinder does not move as the cylinder spins up. Find (*a*) the tension in the string, (*b*) the angular acceleration of the cylinder, and (*c*) the acceleration of the hand.

Picture the Problem By applying Newton's second law of motion, we can obtain a system of two equations in the unknowns T and a. In (*b*) we can use the torque equation from (*a*) and our value for T to find α. In (*c*) we use the condition that the acceleration of a point on the rim of the cylinder is the same as the acceleration of the hand, together with the angular acceleration of the cylinder, to find the acceleration of the hand.



(*a*) Apply $\sum \tau_0 = I\alpha$ about an axis through its center of mass and $\sum F_y = 0$ to the cylinder to obtain:

$$\sum \tau_0 = TR = I_0\alpha \qquad (1)$$

and

$$\sum F_y = Mg - T = 0 \qquad (2)$$

Solving equation (2) for T yields:

$$T = \boxed{Mg}$$

(*b*) Solving equation (1) for α yields:

$$\alpha = \frac{TR}{I_0}$$

From Table 9-1, the moment of inertia of a cylinder about an axis through 0 and perpendicular to the end of the cylinder is:

$$I_0 = \tfrac{1}{2}MR^2$$

Substitute for T and I_0 and simplify to obtain:

$$\alpha = \frac{MgR}{\frac{1}{2}MR^2} = \boxed{\frac{2g}{R}}$$

(*c*) Relate the acceleration a of the hand to the angular acceleration of the cylinder:

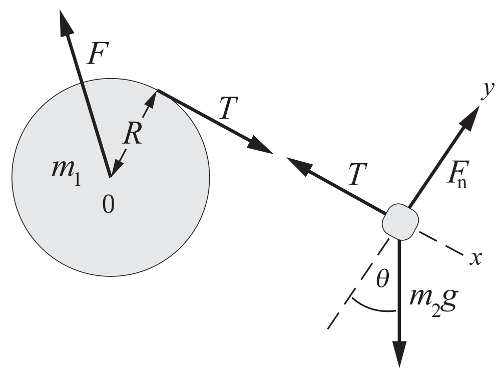
$$a = R\alpha$$

Substitute for α and simplify to obtain:

$$a = R\left(\frac{2g}{R}\right) = \boxed{2g}$$

81 •• [SSM] A uniform cylinder of mass m_1 and radius R is pivoted on frictionless bearings. A massless string wrapped around the cylinder is connected to a block of mass m_2 that is on a frictionless incline of angle θ, as shown in Figure 9-62. The system is released from rest with the block a vertical distance h above the bottom of the incline. (*a*) What is the acceleration of the block? (*b*) What is the tension in the string? (*c*) What is the speed of the block as it reaches the bottom of the incline? (*d*) Evaluate your answers for the special case where $\theta = 90°$ and $m_1 = 0$. Are your answers what you would expect for this special case? Explain.

Picture the Problem Let the zero of gravitational potential energy be at the bottom of the incline. By applying Newton's second law to the cylinder and the block we can obtain simultaneous equations in a, T, and α from which we can express a and T. By applying the conservation of energy, we can derive an expression for the speed of the block when it reaches the bottom of the incline.



(*a*) Apply Newton's second law to the cylinder and the block to obtain:

$$\sum \tau_0 = TR = I_0\alpha \qquad (1)$$

and

$$\sum F_x = m_2 g\sin\theta - T = m_2 a \qquad (2)$$

Substitute for α and I_0 in equation (1), solve for T, and substitute in equation (2) and solve for a to obtain:

$$a = \boxed{\frac{g\sin\theta}{1+\dfrac{m_1}{2m_2}}}$$

(*b*) Substituting for a in equation (2) and solve for T yields:

$$T = \boxed{\frac{\frac{1}{2}m_1 g\sin\theta}{1+\dfrac{m_1}{2m_2}}}$$

(*c*) Express the total energy of the system when the block is at the bottom of the incline in terms of its kinetic energies:

$$E_{\text{bottom}} = K_{\text{tran}} + K_{\text{rot}} = \tfrac{1}{2}m_2v^2 + \tfrac{1}{2}I_0\omega^2$$

Substitute for ω and I_0 to obtain:

$$\tfrac{1}{2}m_2v^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m_1r^2\right)\frac{v^2}{r^2} = m_2gh$$

Solving for v yields:

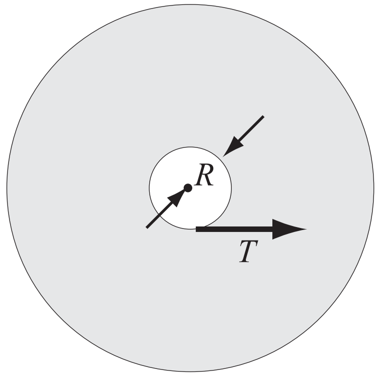
$$v = \boxed{\sqrt{\frac{2gh}{1+\dfrac{m_1}{2m_2}}}}$$

(*d*) For $\theta = 90°$ and $m_1 = 0$:

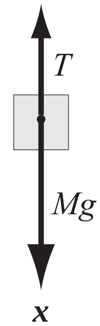
$$a = \boxed{g},\ T = \boxed{0},\ \text{and}\ v = \boxed{\sqrt{2gh}}$$

82 •• A device for measuring the moment of inertia of an object is shown in Figure 9-63. The circular platform is attached to a concentric drum of radius R, and the platform and the drum are free to rotate about a frictionless vertical axis. The string that is wound around the drum passes over a frictionless and massless pulley to a block of mass M. The block is released from rest, and the time t_1 it takes for it to drop a distance D is measured. The system is then rewound, the object whose moment of inertia I we wish to measure is placed on the platform, and the system is again released from rest. The time t_2 required for the block to drop the same distance D then provides the data needed to calculate I. Using $R = 10$ cm, $M = 2.5$ kg, $D = 1.8$ m and $t_1 = 4.2$ s and $t_2 = 6.8$ s, (*a*) find the moment of inertia of the platform-drum combination, (*b*) Find the moment of inertia of the platform-drum-object combination. (*c*) Use your results for Parts (*a*) and (*b*) to find the moment of inertia of the object.

Picture the Problem Let r be the radius of the concentric drum (10 cm) and let I_0 be the moment of inertia of the drum plus platform. We can use Newton's second law in both translational and rotational forms to express I_0 in terms of a and a constant-acceleration equation to express a and then find I_0. We can use the same equation to find the total moment of inertia when the object is placed on the platform and then subtract to find its moment of inertia.



Top view of platform



Side view of falling object

(*a*) Apply Newton's second law to the platform and the weight:

$$\sum \tau_0 = TR = I_0\alpha \qquad (1)$$

and

$$\sum F_x = Mg - T = Ma \qquad (2)$$

Substitute a/R for α in equation (1) and solve for T:

$$T = \frac{I_0}{R^2}a$$

Substitute for T in equation (2) and solve for I_0 to obtain:

$$I_0 = \frac{MR^2(g-a)}{a} \qquad (3)$$

Using a constant-acceleration equation, relate the distance of fall to the acceleration of the weight and the time of fall and solve for the acceleration:

$$\Delta x = v_0\Delta t + \tfrac{1}{2}a(\Delta t)^2$$

or, because $v_0 = 0$ and $\Delta x = D$,

$$a = \frac{2D}{(\Delta t)^2}$$

Substitute for a in equation (3) to obtain:

$$I_0 = MR^2\left(\frac{g}{a} - 1\right) = MR^2\left(\frac{g(\Delta t)^2}{2D} - 1\right)$$

Substitute numerical values and evaluate I_0:

$$I_0 = (2.5\,\text{kg})(0.10\,\text{m})^2\left[\frac{(9.81\,\text{m/s}^2)(4.2\,\text{s})^2}{2(1.8\,\text{m})} - 1\right] = 1.177\,\text{kg}\cdot\text{m}^2 = \boxed{1.2\,\text{kg}\cdot\text{m}^2}$$

(*b*) Relate the moments of inertia of the platform, drum, shaft, and pulley (I_0) to the moment of inertia of the object and the total moment of inertia:

$$I_{\text{tot}} = I_0 + I = MR^2\left(\frac{g}{a} - 1\right)$$

$$= MR^2\left(\frac{g(\Delta t)^2}{2D} - 1\right)$$

Substitute numerical values and evaluate I_{tot}:

$$I_{tot} = (2.5\,\text{kg})(0.10\,\text{m})^2\left[\frac{(9.81\,\text{m/s}^2)(6.8\,\text{s})^2}{2(1.8\,\text{m})} - 1\right] = 3.125\,\text{kg}\cdot\text{m}^2 = \boxed{3.1\,\text{kg}\cdot\text{m}^2}$$

I is the difference between I_{tot} and I_0: $\quad I = I_{tot} - I_0$

Substitute numerical values and evaluate I:

$$I = 3.125\,\text{kg}\cdot\text{m}^2 - 1.177\,\text{kg}\cdot\text{m}^2 = \boxed{1.9\,\text{kg}\cdot\text{m}^2}$$

Objects Rotating and Rolling Without Slipping

83 • A homogeneous 60-kg cylinder of radius 18 cm is rolling without slipping along a horizontal floor at a speed of 15 m/s. What is the minimum amount of work that was required to give it this motion?

Picture the Problem Any work done on the cylinder by a net force will change its kinetic energy. Therefore, the work needed to give the cylinder this motion is equal to its kinetic energy.

Express the relationship between the work needed to give the cylinder this motion:

$$|W| = |\Delta K| = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

or, because $v = r\omega$ (the cylinder is rolling without slipping),

$$|W| = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\left(\frac{v}{r}\right)^2$$

Substitute for I (see Table 9-1) and simplify to obtain:

$$|W| = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\frac{v^2}{r^2} = \tfrac{3}{4}mv^2$$

Substitute numerical values and evaluate $|W|$:

$$|W| = \tfrac{3}{4}(60\,\text{kg})(15\,\text{m/s})^2 = \boxed{10\,\text{kJ}}$$

84 • An object is rolling without slipping. What percentage of its total kinetic energy is its translational kinetic energy if the object is (*a*) a uniform sphere, (*b*) a uniform cylinder, or (*c*) a hoop.

Picture the Problem The total kinetic energy of any object that is rolling without slipping is given by $K = K_{trans} + K_{rot}$. We can find the percentages associated with each motion by expressing the moment of inertia of the objects as kmr^2 and deriving a general expression for the ratios of rotational kinetic energy to total

kinetic energy and translational kinetic energy to total kinetic energy and substituting the appropriate values of k.

Express the total kinetic energy associated with a rotating and translating object:

$$K = K_{\text{trans}} + K_{\text{rot}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$
$$= \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(kmr^2\right)\frac{v^2}{r^2}$$
$$= \tfrac{1}{2}mv^2 + \tfrac{1}{2}kmv^2 = \tfrac{1}{2}mv^2(1+k)$$

Express the ratio $\frac{K_{\text{trans}}}{K}$:

$$\frac{K_{\text{trans}}}{K} = \frac{\tfrac{1}{2}mv^2}{\tfrac{1}{2}mv^2(1+k)} = \frac{1}{1+k}$$

(a) Substitute $k = \frac{2}{5}$ for a uniform sphere to obtain:

$$\left.\frac{K_{\text{trans}}}{K}\right|_{\text{sphere}} = \frac{1}{1+0.4} = 0.714 = \boxed{71.4\%}$$

(b) Substitute $k = \frac{1}{2}$ for a uniform cylinder to obtain:

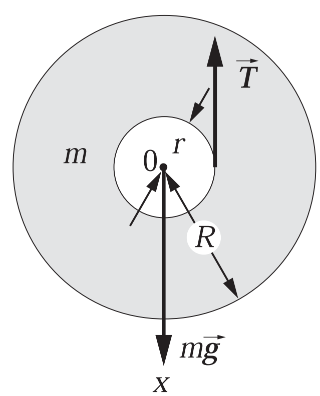
$$\left.\frac{K_{\text{trans}}}{K}\right|_{\text{cylinder}} = \frac{1}{1+0.5} = 0.667 = \boxed{66.7\%}$$

(c) Substitute $k = 1$ for a hoop to obtain:

$$\left.\frac{K_{\text{trans}}}{K}\right|_{\text{hoop}} = \frac{1}{1+1} = 0.500 = \boxed{50.0\%}$$

85 •• [SSM] In 1993 a giant 400-kg yo-yo with a radius of 1.5 m was dropped from a crane at a height of 57 m. One end of the string was tied to the top of the crane, so the yo-yo unwound as it descended. Assuming that the axle of the yo-yo had a radius of 0.10 m, estimate its linear speed at the end of the fall.

Picture the Problem The forces acting on the yo-yo are shown in the figure. We can use a constant-acceleration equation to relate the velocity of descent at the end of the fall to the yo-yo's acceleration and Newton's second law in both translational and rotational form to find the yo-yo's acceleration.



Using a constant-acceleration equation, relate the yo-yo's final speed to its acceleration and fall distance:

$$v^2 = v_0^2 + 2a\Delta h$$

or, because $v_0 = 0$,

$$v = \sqrt{2a\Delta h} \qquad (1)$$

Use Newton's second law to relate the forces that act on the yo-yo to its acceleration:

$$\sum F_x = mg - T = ma \qquad (2)$$

and

$$\sum \tau_0 = Tr = I_0\alpha \qquad (3)$$

Use $a = r\alpha$ to eliminate α in equation (3)

$$Tr = I_0 \frac{a}{r} \qquad (4)$$

Eliminate T between equations (2) and (4) to obtain:

$$mg - \frac{I_0}{r^2} a = ma \qquad (5)$$

Substitute $\frac{1}{2}mR^2$ for I_0 in equation (5):

$$mg - \frac{\frac{1}{2}mR^2}{r^2} a = ma \Rightarrow a = \frac{g}{1 + \dfrac{R^2}{2r^2}}$$

Substitute numerical values and evaluate a:

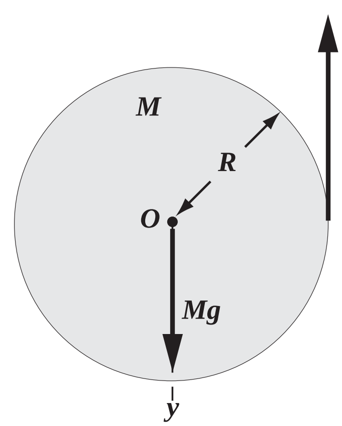
$$a = \frac{9.81\,\text{m/s}^2}{1 + \dfrac{(1.5\,\text{m})^2}{2(0.10\,\text{m})^2}} = 0.0864\,\text{m/s}^2$$

Substitute in equation (1) and evaluate v:

$$v = \sqrt{2(0.0864\,\text{m/s}^2)(57\,\text{m})} = \boxed{3.1\,\text{m/s}}$$

86 •• A uniform cylinder of mass M and radius R has a string wrapped around it. The string is held fixed, and the cylinder falls vertically as shown in Figure 9-61. (*a*) Show that the acceleration of the cylinder is downward with a magnitude $a = 2g/3$. (*b*) Find the tension in the string.

Picture the Problem The forces acting on the cylinder are shown in the diagram. Choose a coordinate system in which the +y directions is downward. By applying Newton's second law of motion, we can obtain a system of two equations in the unknowns T, a and α that we can solve simultaneously.



(*a*) Apply Newton's second law to the cylinder:

$$\sum \tau_0 = TR = I_0\alpha \qquad (1)$$

and

$$\sum F_y = Mg - T = Ma \qquad (2)$$

Substitute for α and I_0 (see Table 9-1) in equation (1) to obtain:

$$TR = \left(\tfrac{1}{2}MR^2\right)\left(\frac{a}{R}\right) \Rightarrow T = \tfrac{1}{2}Ma \quad (3)$$

Substitute for T in equation (2) to obtain:

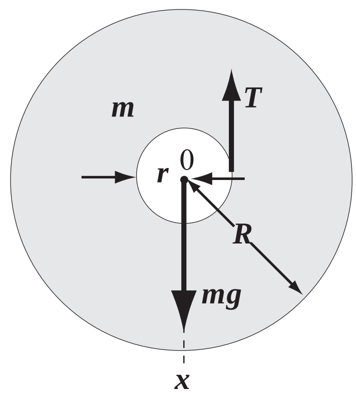
$$Mg - \tfrac{1}{2}Ma = Ma \Rightarrow a = \boxed{\tfrac{2}{3}g}$$

(*b*) Substitute for a in equation (3) and simplify to obtain:

$$T = \tfrac{1}{2}M\left(\tfrac{2}{3}g\right) = \boxed{\tfrac{1}{3}Mg}$$

87 •• A 0.10-kg yo-yo consists of two solid disks, each of radius 10 cm, is joined by a massless rod of radius 1.0 cm. A string is wrapped around the rod. One end of the string is held fixed and is under tension as the yo-yo is released. The yo-yo rotates as it descends vertically. Find (*a*) the acceleration of the yo-yo, and (*b*) the tension *T*.

Picture the Problem The forces acting on the yo-yo are shown in the figure. Choose a coordinate system in which the $+x$ direction is downward. Apply Newton's second law in both translational and rotational form to obtain simultaneous equations in T, a, and α from which we can eliminate α and solve for T and a.



Apply Newton's second law to the yo-yo:

$$\sum F_x = mg - T = ma \quad (1)$$

and

$$\sum \tau_0 = Tr = I_0\alpha \quad (2)$$

Use $a = r\alpha$ to eliminate α in equation (2)

$$Tr = I_0\frac{a}{r} \quad (3)$$

Eliminate T between equations (1) and (3) to obtain:

$$mg - \frac{I_0}{r^2}a = ma \quad (4)$$

Substitute $\tfrac{1}{2}mR^2$ for I_0 (see Table 9-1) in equation (4):

$$mg - \frac{\tfrac{1}{2}mR^2}{r^2}a = ma \Rightarrow a = \frac{g}{1 + \dfrac{R^2}{2r^2}}$$

Substitute numerical values and evaluate a:

$$a=\frac{9.81\,\text{m/s}^2}{1+\dfrac{(0.10\,\text{m})^2}{2(0.010\,\text{m})^2}}=0.192\,\text{m/s}^2$$
$$=\boxed{0.19\,\text{m/s}^2}$$

Solving equation (1) for T yields:

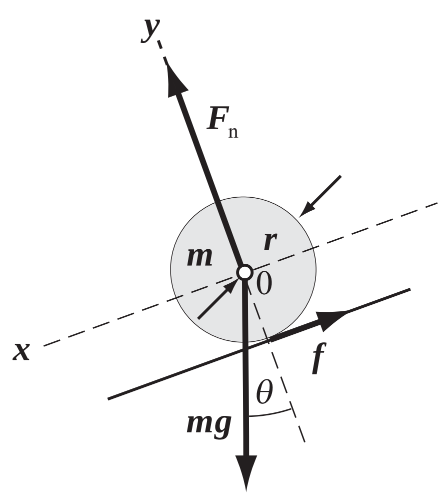
$$T=m(g-a)$$

Substitute numerical values and evaluate T:

$$T=(0.10\,\text{kg})\left(9.81\,\text{m/s}^2-0.192\,\text{m/s}^2\right)$$
$$=\boxed{0.96\,\text{N}}$$

88 •• A uniform solid sphere rolls down an incline without slipping. If the linear acceleration of the center of mass of the sphere is $0.2g$, then what is the angle the incline makes with the horizontal?

Picture the Problem From Newton's second law, the acceleration of the center of mass of the uniform solid sphere equals the net force acting on the sphere divided by its mass. The forces acting on the sphere are its weight $m\vec{g}$ downward, the normal force $\vec{F}_n$ that balances the normal component of the weight, and the force of friction $\vec{f}$ acting up the incline. As the sphere accelerates down the incline, the angular speed of rotation must increase to maintain the nonslip condition. We can apply Newton's second law for rotation about a horizontal axis through the center of mass of the sphere to find α, which is related to the acceleration by the nonslip condition. The only torque about the center of mass is due to $\vec{f}$ because both $m\vec{g}$ and $\vec{F}_n$ act through the center of mass. Choose the $+x$ direction to be down the incline.



Apply $\sum\vec{F}=m\vec{a}$ to the sphere: $$mg\sin\theta-f=ma_{cm} \quad (1)$$

Apply $\sum\tau=I_{cm}\alpha$ to the sphere: $$fr=I_{cm}\alpha$$

Use the nonslip condition to eliminate α and solve for f:

$$fr = I_{\text{cm}}\frac{a_{\text{cm}}}{r} \Rightarrow f = \frac{I_{\text{cm}}}{r^2}a_{\text{cm}}$$

Substitute this result for f in equation (1) to obtain:

$$mg\sin\theta - \frac{I_{\text{cm}}}{r^2}a_{\text{cm}} = ma_{\text{cm}} \qquad (2)$$

From Table 9-1 we have, for a solid sphere:

$$I_{\text{cm}} = \tfrac{2}{5}mr^2$$

Substitute for I_{cm} in equation (1) and simplify to obtain:

$$mg\sin\theta - \tfrac{2}{5}ma_{\text{cm}} = ma_{\text{cm}}$$

Solving for θ yields:

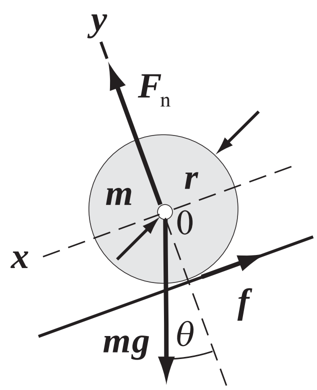
$$\theta = \sin^{-1}\left(\frac{7a_{\text{cm}}}{5g}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{7(0.2g)}{5g}\right] = \boxed{16°}$$

89 •• A thin spherical shell rolls down an incline without slipping. If the linear acceleration of the center of mass of the shell is 0.20*g,* then what is the angle the incline makes with the horizontal?

Picture the Problem From Newton's second law, the acceleration of the center of mass of the thin spherical shell equals the net force acting on the spherical shell divided by its mass. The forces acting on the thin spherical shell are its weight $m\vec{g}$ downward, the normal force $\vec{F}_{\text{n}}$ that balances the normal component of the weight, and the force of friction $\vec{f}$ acting up the incline. As the spherical shell accelerates down the incline, the angular speed of rotation must increase to maintain the nonslip condition. We can apply Newton's second law for rotation about a horizontal axis through the center of mass of the shell to find α, which is related to the acceleration by the nonslip condition. The only torque about the center of mass is due to $\vec{f}$ because both $m\vec{g}$ and $\vec{F}_{\text{n}}$ act through the center of mass. Choose the $+x$ direction to be down the incline.



Apply $\sum \vec{F} = m\vec{a}$ to the thin spherical shell:

$$mg\sin\theta - f = ma_{\text{cm}} \qquad (1)$$

Apply $\sum \tau = I_{\text{cm}}\alpha$ to the thin spherical shell:

$$fr = I_{\text{cm}}\alpha$$

Use the nonslip condition to eliminate α and solve for f:

$$fr = I_{\text{cm}}\frac{a_{\text{cm}}}{r} \text{ and } f = \frac{I_{\text{cm}}}{r^2}a_{\text{cm}}$$

Substitute this result for f in equation (1) to obtain:

$$mg\sin\theta - \frac{I_{\text{cm}}}{r^2}a_{\text{cm}} = ma_{\text{cm}} \qquad (2)$$

From Table 9-1 we have, for a thin spherical shell:

$$I_{\text{cm}} = \tfrac{2}{3}mr^2$$

Substitute for I_{cm} in equation (1) and simplify to obtain:

$$mg\sin\theta - \tfrac{2}{3}ma_{\text{cm}} = ma_{\text{cm}}$$

Solving for θ yields:

$$\theta = \sin^{-1}\left(\frac{5a_{\text{cm}}}{3g}\right)$$

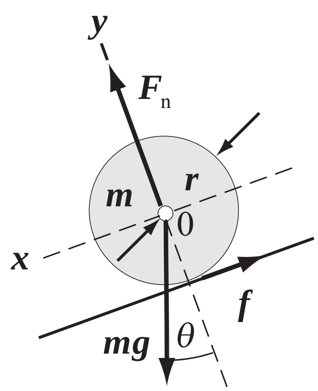
Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{5(0.20g)}{3g}\right] = \boxed{20°}$$

Remarks: This larger angle makes sense, as the moment of inertia for a given mass is larger for a thin spherical shell than for a solid one.

90 •• A basketball rolls without slipping down an incline of angle θ. The coefficient of static friction is μ_s. Model the ball as a thin spherical shell. Find (*a*) the acceleration of the center of mass of the ball, (*b*) the frictional force acting on the ball, and (*c*) the maximum angle of the incline for which the ball will roll without slipping.

Picture the Problem The three forces acting on the basketball are the weight of the ball, the normal force, and the force of friction. Because the weight can be assumed to be acting at the center of mass, and the normal force acts through the center of mass, the only force which exerts a torque about the center of mass is the frictional force. Let the mass of the basketball be m and apply Newton's second law to find a system of simultaneous equations that we can solve for the quantities called for in the problem statement.



(a) Apply Newton's second law in both translational and rotational form to the ball:

$$\sum F_x = mg\sin\theta - f_s = ma\,, \qquad (1)$$

$$\sum F_y = F_n - mg\cos\theta = 0 \qquad (2)$$

and

$$\sum \tau_0 = f_s r = I_0\alpha \qquad (3)$$

Because the basketball is rolling without slipping we know that:

$$\alpha = \frac{a}{r}$$

Substitute in equation (3) to obtain:

$$f_s r = I_0\frac{a}{r} \qquad (4)$$

From Table 9-1 we have:

$$I_0 = \tfrac{2}{3}mr^2$$

Substitute for I_0 and α in equation (4) and solve for f_s:

$$f_s r = \left(\tfrac{2}{3}mr^2\right)\frac{a}{r} \Rightarrow f_s = \tfrac{2}{3}ma \qquad (5)$$

Substitute for f_s in equation (1) and solve for a:

$$a = \boxed{\tfrac{3}{5}g\sin\theta}$$

(b) Find f_s using equation (5):

$$f_s = \tfrac{2}{3}m\left(\tfrac{3}{5}g\sin\theta\right) = \boxed{\tfrac{2}{5}mg\sin\theta}$$

(c) Solve equation (2) for F_n:

$$F_n = mg\cos\theta$$

Use the definition of $f_{s,max}$ to obtain:

$$f_{s,max} = \mu_s F_n = \mu_s mg\cos\theta_{max}$$

Use the result of Part (b) to obtain:

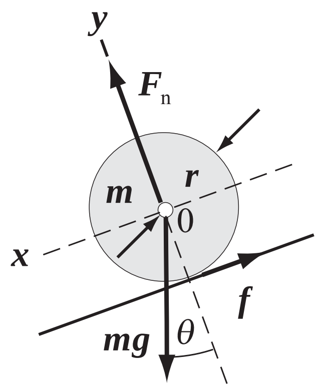
$$\tfrac{2}{5}mg\sin\theta_{max} = \mu_s mg\cos\theta_{max}$$

Solve for θ_{max}:

$$\theta_{max} = \boxed{\tan^{-1}\left(\tfrac{5}{2}\mu_s\right)}$$

91 •• A uniform solid cylinder of wood rolls without slipping down an incline of angle θ. The coefficient of static friction is μ_s. Find (*a*) the acceleration of the center of mass of the cylinder, (*b*) the frictional force acting on the cylinder, and (*c*) the maximum angle of the incline for which the cylinder will roll without slipping.

Picture the Problem The three forces acting on the cylinder are the weight of the cylinder, the normal force, and the force of friction. Because the weight can be assumed to be acting at the center of mass, and the normal force acts through the center of mass, the only force which exerts a torque about the center of mass is the frictional force. Let the mass of the cylinder by m and use Newton's second law to find a system of simultaneous equations that we can solve for the quantities called for in the problem statement.



(*a*) Apply Newton's second law in both translational and rotational form to the cylinder:

$$\sum F_x = mg\sin\theta - f_s = ma\,, \quad (1)$$

$$\sum F_y = F_n - mg\cos\theta = 0 \quad (2)$$

and

$$\sum \tau_0 = f_s r = I_0\alpha \quad (3)$$

Because the cylinder is rolling without slipping we know that:

$$\alpha = \frac{a}{r}$$

Substitute in equation (3) to obtain:

$$f_s r = I_0\frac{a}{r} \quad (4)$$

From Table 9-1 we have:

$$I_0 = \tfrac{1}{2}mr^2$$

Substitute for I_0 and α in equation (4) and solve for f_s:

$$f_s r = \left(\tfrac{1}{2}mr^2\right)\frac{a}{r} \Rightarrow f_s = \tfrac{1}{2}ma \quad (5)$$

Substitute for f_s in equation (1) and solve for a:

$$a = \boxed{\tfrac{2}{3}g\sin\theta}$$

(*b*) Find f_s using equation (5): $f_s = \frac{1}{2}m\left(\frac{2}{3}g\sin\theta\right) = \boxed{\frac{1}{3}mg\sin\theta}$

(*c*) Solve equation (2) for F_n: $F_n = mg\cos\theta$

Use the definition of $f_{s,max}$ to obtain: $f_{s,max} = \mu_s F_n = \mu_s mg\cos\theta_{max}$

Use the result of part (*b*) to obtain: $\frac{1}{3}mg\sin\theta_{max} = \mu_s mg\cos\theta_{max}$

Solving for θ_{max} gives: $\theta_{max} = \boxed{\tan^{-1}(3\mu_s)}$

92 •• Released from rest at the same height, a thin spherical shell and solid sphere of the same mass m and radius R roll without slipping down an incline through the same vertical drop H (Figure 9-64). Each is moving horizontally as it leaves the ramp. The spherical shell hits the ground a horizontal distance L from the end of the ramp and the solid sphere hits the ground a distance L' from the end of the ramp. Find the ratio L'/L.

Picture the Problem Let the zero of gravitational potential energy be at the elevation where the spheres leave the ramp. The distances the spheres will travel are directly proportional to their speeds when they leave the ramp.

Express the ratio of the distances traveled by the two spheres in terms of their speeds when they leave the ramp:

$$\frac{L'}{L} = \frac{v_{solid}\Delta t}{v_{shell}\Delta t} = \frac{v_{solid}}{v_{shell}} \qquad (1)$$

Use conservation of mechanical energy to find the speed of the spheres when they leave the ramp:

$$\Delta K + \Delta U = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0 \qquad (2)$$

Express K_f for the spheres and simplify to obtain (Note that $k = 2/3$ for the spherical shell and 2/5 for the uniform sphere):

$$\begin{aligned} K_f &= K_{trans} + K_{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I_{cm}\omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\left(kmR^2\right)\frac{v^2}{R^2} \\ &= \frac{1}{2}mv^2 + \frac{1}{2}kmv^2 = (1+k)\frac{1}{2}mv^2 \end{aligned}$$

Substitute for K_f in equation (2) to obtain:

$$(1+k)\frac{1}{2}mv^2 = mgH \Rightarrow v = \sqrt{\frac{2gH}{1+k}}$$

Substitute for v_{shell} and v_{solid} in equation (1) and simplify to obtain:

$$\frac{L'}{L} = \sqrt{\frac{1+k_{shell}}{1+k_{solid}}} = \sqrt{\frac{1+\frac{2}{3}}{1+\frac{2}{5}}} = \boxed{1.09}$$

93 •• [SSM] A uniform, thin cylindrical shell and a solid cylinder roll horizontally without slipping. The speed of the cylindrical shell is v. The solid cylinder and the hollow cylinder encounter an incline that they climb without slipping. If the maximum height they reach is the same, find the initial speed v' of the solid cylinder.

Picture the Problem Let the subscripts s and c refer to the thin cylindrical shell and solid cylinder, respectively. Because the cylinders climb to the same height, their kinetic energies at the bottom of the incline must be equal.

Express the total kinetic energy of the thin cylindrical shell at the bottom of the inclined plane:

$$K_s = K_{trans} + K_{rot} = \tfrac{1}{2}m_s v^2 + \tfrac{1}{2}I_s\omega^2 = \tfrac{1}{2}m_s v^2 + \tfrac{1}{2}\left(m_s r^2\right)\frac{v^2}{r^2} = m_s v^2$$

Express the total kinetic energy of the solid cylinder at the bottom of the inclined plane:

$$K_c = K_{trans} + K_{rot} = \tfrac{1}{2}m_c v'^2 + \tfrac{1}{2}I_c\omega'^2 = \tfrac{1}{2}m_c v'^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m_c r^2\right)\frac{v'^2}{r^2} = \tfrac{3}{4}m_c v'^2$$

Because the cylinders climb to the same height:

$$\tfrac{3}{4}m_s v'^2 = m_s gh \Rightarrow \tfrac{3}{4}v'^2 = gh$$

and

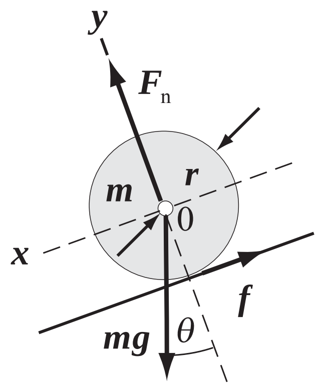
$$m_c v^2 = m_c gh \Rightarrow v^2 = gh$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{\tfrac{3}{4}v'^2}{v^2} = \frac{gh}{gh} \Rightarrow v' = \boxed{\sqrt{\frac{4}{3}}v}$$

94 •• A thin cylindrical shell and a solid sphere start from rest and roll without slipping down a 3.0-m-long inclined plane. The cylinder arrives at the bottom of the incline 2.4 s after the sphere does. Determine the angle the incline makes with the horizontal.

Picture the Problem Let the subscripts s and c refer to the solid sphere and thin-walled cylinder, respectively. Because the cylinder and sphere descend from the same height, their kinetic energies at the bottom of the incline must be equal. The force diagram shows the forces acting on the solid sphere. We'll use Newton's second law to relate the accelerations to the angle of the incline and use a constant acceleration to relate the accelerations to the distances traveled down the incline.



Apply Newton's second law to the sphere:	$\sum F_x = mg\sin\theta - f_s = ma_s$, (1) $\sum F_y = F_n - mg\cos\theta = 0$, (2) and $\sum \tau_0 = f_s r = I_0\alpha$ (3)
Substitute for I_0 and α in equation (3) and solve for f_s:	$f_s r = \left(\frac{2}{5}mr^2\right)\frac{a}{r} \Rightarrow f_s = \frac{2}{5}ma_s$
Substitute for f_s in equation (1) and solve for a_s:	$a_s = \frac{5}{7}g\sin\theta$
Proceed as above for the thin cylindrical shell to obtain:	$a_c = \frac{1}{2}g\sin\theta$
Using a constant-acceleration equation, relate the distance traveled down the incline to the acceleration and elapsed time:	$\Delta s = v_0 t + \frac{1}{2}at^2$ or, because $v_0 = 0$, $\Delta s = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2\Delta s}{a}}$ (4)
Substituting for a_s and a_c in equation (4) gives:	$t_s = \sqrt{\frac{2\Delta s}{\frac{5}{7}g\sin\theta}}$ and $t_c = \sqrt{\frac{2\Delta s}{\frac{1}{2}g\sin\theta}}$
Because $t_c - t_s = 2.4\text{ s}$:	$\sqrt{\frac{2\Delta s}{\frac{1}{2}g\sin\theta}} - \sqrt{\frac{2\Delta s}{\frac{5}{7}g\sin\theta}} = 2.4\text{ s}$

Solving for θ gives:

$$\theta = \sin^{-1}\left[\frac{\sqrt{\frac{4\Delta s}{g}} - \sqrt{\frac{14\Delta s}{5g}}}{2.4\,\text{s}}\right]^2$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{\sqrt{\frac{4(3.0\,\text{m})}{9.81\,\text{m/s}^2}} - \sqrt{\frac{14(3.0\,\text{m})}{5(9.81\,\text{m/s}^2)}}}{2.4\,\text{s}}\right]^2 = \boxed{0.058°}$$

95 •• A wheel has a thin 3.0-kg rim and four spokes, each of mass 1.2 kg. Find the kinetic energy of the wheel when it is rolling at 6.0 m/s on a horizontal surface.

Picture the Problem The kinetic energy of the wheel is the sum of its translational and rotational kinetic energies. Because the wheel is a composite object, we can model its moment of inertia by treating the rim as a cylindrical shell and the spokes as rods.

Express the kinetic energy of the wheel:

$$K = K_{\text{trans}} + K_{\text{rot}} = \tfrac{1}{2}M_{\text{tot}}v^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 = \tfrac{1}{2}M_{\text{tot}}v^2 + \tfrac{1}{2}I_{\text{cm}}\frac{v^2}{R^2}$$

where $M_{\text{tot}} = M_{\text{rim}} + 4M_{\text{spoke}}$.

The moment of inertia of the wheel is the sum of the moments of inertia of the rim and spokes:

$$I_{\text{cm}} = I_{\text{rim}} + I_{\text{spokes}} = M_{\text{rim}}R^2 + 4\left(\tfrac{1}{3}M_{\text{spoke}}R^2\right) = \left(M_{\text{rim}} + \tfrac{4}{3}M_{\text{spoke}}\right)R^2$$

Substitute for I_{cm} in the equation for K:

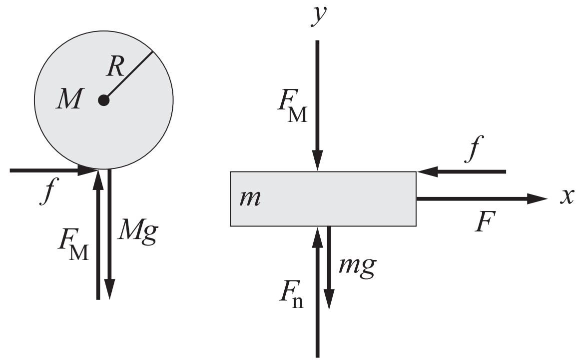
$$K = \tfrac{1}{2}M_{\text{tot}}v^2 + \tfrac{1}{2}\left[\left(M_{\text{rim}} + \tfrac{4}{3}M_{\text{spoke}}\right)R^2\right]\frac{v^2}{R^2} = \left[\tfrac{1}{2}\left(M_{\text{tot}} + M_{\text{rim}}\right) + \tfrac{2}{3}M_{\text{spoke}}\right]v^2$$

Substitute numerical values and evaluate K:

$$K = \left[\tfrac{1}{2}(7.8\,\text{kg} + 3.0\,\text{kg}) + \tfrac{2}{3}(1.2\,\text{kg})\right](6.0\,\text{m/s})^2 = \boxed{0.22\,\text{kJ}}$$

96 ••• A uniform solid cylinder of mass M and radius R is at rest on a slab of mass m, which in turn rests on a horizontal, frictionless table (Figure 9-65). If a horizontal force $\vec{F}$ is applied to the slab, it accelerates and the cylinder rolls without slipping. Find the acceleration of the slab in terms of M, R, and F.

Picture the Problem Let the letter S identify the slab and the letter C the cylinder. We can find the accelerations of the slab and cylinder by applying Newton's second law and solving the resulting equations simultaneously.



Apply $\sum F_x = ma_x$ to the slab: $F - f = ma_S$ (1)

Apply $\sum F_x = ma_x$ to the cylinder: $f = Ma_C$, (2)

Apply $\sum \tau_{CM} = I_{CM}\alpha$ to the cylinder: $fR = I_{CM}\alpha$ (3)

Substitute for I_{CM} in equation (3) and solve for f to obtain: $f = \frac{1}{2}MR\alpha$ (4)

Relate the acceleration of the slab to the acceleration of the cylinder:

$a_C = a_S + a_{CS}$

or, because $a_{CS} = -R\alpha$ is the acceleration of the cylinder relative to the slab,

$a_C = a_S - R\alpha \Rightarrow R\alpha = a_S - a_C$ (5)

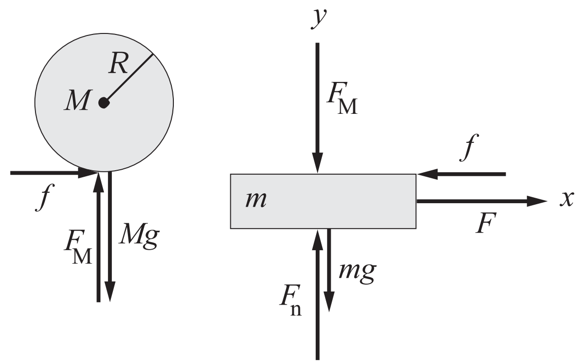
Equate equations (2) and (4) and substitute from (5) to obtain: $a_S = 3a_C$

Substitute equation (4) in equation (1) and substitute for a_C to obtain:

$$F - \tfrac{1}{3}Ma_S = ma_S \Rightarrow a_S = \boxed{\frac{3F}{M + 3m}}$$

97 ••• (*a*) Find the angular acceleration of the cylinder in Problem 96. Is the cylinder rotating clockwise or counterclockwise? (*b*) What is the cylinder's linear acceleration (magnitude and direction) relative to the table? (*c*) What is the magnitude and direction of the linear acceleration of the center of mass of the cylinder relative to the slab?

Picture the Problem Let the letter S identify the slab and the letter C the cylinder. In this problem, as in Problem 96, we can find the accelerations of the slab and cylinder by applying Newton's second law and solving the resulting equations simultaneously.



(*a*) Because the cylinder rolls without slipping, its angular acceleration is the quotient of its linear acceleration and its radius:

$$\alpha = \frac{a_c}{R} \qquad (1)$$

Apply $\sum F_x = ma_x$ to the slab:

$$F - f = ma_S \qquad (2)$$

Apply $\sum F_x = ma_x$ to the cylinder:

$$f = Ma_C\,, \qquad (3)$$

Apply $\sum \tau_{CM} = I_{CM}\alpha$ to the cylinder:

$$fR = I_{CM}\alpha \qquad (4)$$

Substitute for I_{CM} in equation (4) and solve for f to obtain:

$$f = \tfrac{1}{2}MR\alpha \qquad (5)$$

Relate the acceleration of the slab to the acceleration of the cylinder:

$$a_C = a_S + a_{CS}$$

or, because $a_{CS} = -R\alpha$,

$$a_C = a_S - R\alpha \Rightarrow \alpha = \frac{a_S - a_C}{R} \qquad (6)$$

Equate equations (3) and (5) and substitute from (6) to obtain:

$$a_S = 3a_C$$

Substituting for f and a_s in equation (1) gives:

$$F - Ma_c = 3ma_c \Rightarrow a_c = \frac{F}{3m+M}$$

Substitute for a_c in equation (1) and simplify to obtain:

$$\alpha = \boxed{\frac{F}{R(3m+M)}, \text{ counterclockwise}}$$

(*b*) From equations (2) and (5) we have:

$$F - \tfrac{1}{3}Ma_S = ma_S \Rightarrow a_S = \frac{3F}{M+3m}$$

Because $a_S = 3a_C$:

$$a_C = \tfrac{1}{3}a_S = \boxed{\frac{F}{M+3m}}$$

in the direction of $\vec{F}$.

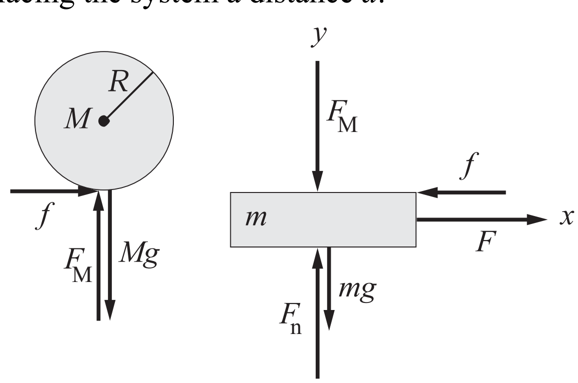
(*c*) The acceleration of the cylinder relative to the slab is the difference between the acceleration of the cylinder and the acceleration of the slab:

$$a_{CS} = a_C - a_S = a_C - 3a_C = -2a_C$$
$$= \boxed{-\frac{2F}{M+3m}}$$

opposite the direction of $\vec{F}$.

98 ••• If the force in Problem 96 acts over a distance *d*, in terms of the symbols given, find (*a*) the kinetic energy of the slab, and (*b*) the total kinetic energy of the cylinder. (*c*) Show that the total kinetic energy of the slab-cylinder system is equal to the work done by the force.

Picture the Problem Let the system include Earth, the cylinder (C), and the slab (S). Then $\vec{F}$ is an external force that changes the energy of the system by doing work on it. We can find the kinetic energy of the slab from its speed when it has traveled a distance *d*. We can find the kinetic energy of the cylinder from the sum of its translational and rotational kinetic energies. In Part (*c*) we can add the kinetic energies of the slab and the cylinder to show that their sum is the work done by $\vec{F}$ in displacing the system a distance *d*.



(*a*) Express the kinetic energy of the slab: $K_S = W_{\text{on slab}} = \frac{1}{2}mv_S^2$

Using a constant-acceleration equation, relate the velocity of the slab to its acceleration and the distance traveled: $v_S^2 = v_0^2 + 2a_S d$

or, because the slab starts from rest,

$v_S^2 = 2a_S d$

Substitute for v_S^2 to obtain: $K_S = \frac{1}{2}m(2a_S d) = ma_S d \quad (1)$

Apply $\sum F_x = ma_x$ to the slab: $F - f = ma_S \quad (2)$

Apply $\sum F_x = ma_x$ to the cylinder: $f = Ma_C\,, \quad (3)$

Apply $\sum \tau_{CM} = I_{CM}\alpha$ to the cylinder: $fR = I_{CM}\alpha \quad (4)$

Substitute for I_{CM} in equation (4) and solve for f: $f = \frac{1}{2}MR\alpha \quad (5)$

Relate the acceleration of the slab to the acceleration of the cylinder: $a_C = a_S + a_{CS}$

or, because $a_{CS} = -R\alpha$,

$a_C = a_S - R\alpha$

Solving for $R\alpha$ yields: $R\alpha = a_S - a_C \quad (6)$

Equate equations (3) and (5) and substitute in (6) to obtain: $a_S = 3a_C$

Substitute equation (5) in equation (2) and use $a_S = 3a_C$ to obtain: $F - Ma_C = ma_S$

or

$F - \frac{1}{3}Ma_S = ma_S$

Solving for a_S yields:

$$a_S = \frac{F}{m + \frac{1}{3}M}$$

Substitute for a_S in equation (1) to obtain:

$$K_S = \boxed{\frac{mFd}{m + \frac{1}{3}M}}$$

(*b*) Express the total kinetic energy of the cylinder:

$$K_C = K_{trans} + K_{rot} = \tfrac{1}{2}Mv_C^2 + \tfrac{1}{2}I_{CM}\omega^2 = \tfrac{1}{2}Mv_C^2 + \tfrac{1}{2}I_{CM}\frac{v_{CB}^2}{R^2} \qquad (7)$$

where $v_{CS} = v_C - v_S$.

In Part (*a*) it was established that:

$$a_S = 3a_C$$

Integrate both sides of $a_S = 3a_C$ with respect to time to obtain:

$$v_S = 3v_C + \text{constant}$$

where the constant of integration is determined by the initial conditions that $v_C = 0$ when $v_S = 0$.

Substitute the initial conditions to obtain:

$$\text{constant} = 0 \text{ and } v_S = 3v_C$$

Substitute in the expression for v_{CS} to obtain:

$$v_{CS} = v_C - v_S = v_C - 3v_C = -2v_C$$

Substitute for I_{CM} and v_{CS} in equation (7) to obtain:

$$K_C = \tfrac{1}{2}Mv_C^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\frac{(-2v_C)^2}{R^2} = \tfrac{3}{2}Mv_C^2 \qquad (8)$$

Because $v_C = \tfrac{1}{3}v_S$:

$$v_C^2 = \tfrac{1}{9}v_S^2$$

It Part (*a*) it was established that:

$$v_S^2 = 2a_S d \text{ and } a_S = \frac{F}{m + \tfrac{1}{3}M}$$

Substitute to obtain:

$$v_C^2 = \tfrac{1}{9}(2a_S d) = \tfrac{2}{9}\left(\frac{F}{m + \tfrac{1}{3}M}\right)d = \frac{2Fd}{9\left(m + \tfrac{1}{3}M\right)}$$

Substituting in equation (8) and simplifying yields:

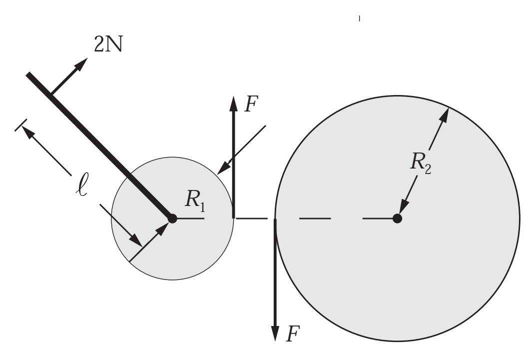
$$K_C = \tfrac{3}{2}M\left(\frac{2Fd}{9\left(m + \tfrac{1}{3}M\right)}\right) = \frac{MFd}{3\left(m + \tfrac{1}{3}M\right)} = \boxed{\frac{MFd}{3m + M}}$$

(*c*) Express the total kinetic energy of the system and simplify to obtain:

$$K_{\text{tot}} = K_{\text{S}} + K_{\text{C}} = \frac{mFd}{m+\frac{1}{3}M} + \frac{MFd}{3\left(m+\frac{1}{3}M\right)} = \frac{(3m+M)}{3\left(m+\frac{1}{3}M\right)}Fd = \boxed{Fd}$$

99 ••• [SSM] Two large gears that are being designed as part of a large machine and are shown in Figure 9-66; each is free to rotate about a fixed axis through its center. The radius and moment of inertia of the smaller gear are 0.50 m and 1.0 kg·m^2, respectively, and the radius and moment of inertia of the larger gear are 1.0 m and 16 kg·m^2, respectively. The lever attached to the smaller gear is 1.0 m long and has a negligible mass. (*a*) If a worker will typically apply a force of 2.0 N to the end of the lever, as shown, what will be the angular accelerations of the two gears? (*b*) Another part of the machine (not shown) will apply a force tangentially to the outer edge of the larger gear to temporarily keep the gear system from rotating. What should the magnitude and direction of this force (clockwise or counterclockwise) be?

Picture the Problem The forces responsible for the rotation of the gears are shown in the diagram to the right. The forces acting through the centers of mass of the two gears have been omitted because they produce no torque. We can apply Newton's second law in rotational form to obtain the equations of motion of the gears and the not slipping condition to relate their angular accelerations.



(*a*) Apply $\sum\tau = I\alpha$ to the gears to obtain their equations of motion:

$$2.0\ \text{N}\cdot\text{m} - FR_1 = I_1\alpha_1 \qquad (1)$$

and

$$FR_2 = I_2\alpha_2 \qquad (2)$$

where F is the force keeping the gears from slipping with respect to each other.

Because the gears do not slip relative to each other, the tangential accelerations of the points where they are in contact must be the same:

$$R_1\alpha_1 = R_2\alpha_2$$

or

$$\alpha_2 = \frac{R_1}{R_2}\alpha_1 = \tfrac{1}{2}\alpha_1 \qquad (3)$$

Divide equation (1) by R_1 to obtain:

$$\frac{2.0\ \text{N}\cdot\text{m}}{R_1} - F = \frac{I_1}{R_1}\alpha_1$$

Divide equation (2) by R_2 to obtain:

$$F = \frac{I_2}{R_2}\alpha_2$$

Adding these equations yields:

$$\frac{2.0\ \text{N}\cdot\text{m}}{R_1} = \frac{I_1}{R_1}\alpha_1 + \frac{I_2}{R_2}\alpha_2$$

Use equation (3) to eliminate α_2:

$$\frac{2.0\ \text{N}\cdot\text{m}}{R_1} = \frac{I_1}{R_1}\alpha_1 + \frac{I_2}{2R_2}\alpha_1$$

Solving for α_1 yields:

$$\alpha_1 = \frac{2.0\ \text{N}\cdot\text{m}}{I_1 + \frac{R_1}{2R_2}I_2}$$

Substitute numerical values and evaluate α_1:

$$\alpha_1 = \frac{2.0\ \text{N}\cdot\text{m}}{1.0\ \text{kg}\cdot\text{m}^2 + \frac{0.50\ \text{m}}{2(1.0\ \text{m})}\left(16\ \text{kg}\cdot\text{m}^2\right)}$$
$$= 0.400\ \text{rad/s}^2 = \boxed{0.40\ \text{rad/s}^2}$$

Use equation (3) to evaluate α_2:

$$\alpha_2 = \tfrac{1}{2}\left(0.400\ \text{rad/s}^2\right) = \boxed{0.20\ \text{rad/s}^2}$$

(*b*) To counterbalance the 2.0-N·m torque, a counter torque of 2.0 N·m must be applied to the first gear:

$$2.0\ \text{N}\cdot\text{m} - FR_1 = 0 \Rightarrow F = \frac{2.0\ \text{N}\cdot\text{m}}{R_1}$$

Substitute numerical values and evaluate F:

$$F = \frac{2.0\ \text{N}\cdot\text{m}}{0.50\ \text{m}} = \boxed{4.0\ \text{N, clockwise}}$$

100 ••• As the chief design engineer for a major toy company, you are in charge of designing a "loop-the-loop" toy for youngsters. The idea, as shown in Figure 9-67, is that a ball of mass m and radius r will roll down an inclined track and around the loop without slipping. The ball starts from rest at a height h above the tabletop that supports the whole track. The loop radius is R. Determine the minimum height h, in terms of R and r, for which the ball will remain in contact with the track during the whole of its loop-the-loop journey. (Do not neglect the size of the ball's radius when doing this calculation.)

Picture the Problem Choose $U_g = 0$ at the bottom of the loop and let the system include the ball, the track, and Earth. We can apply conservation of energy to this system to relate h to the speed of the ball v at the top of the "loop-to-loop", the radius r of the ball, and the radius R of the "loop-to-loop." We can then apply Newton's second law to the ball at the top of the loop to eliminate v.

Apply conservation of energy to the system to obtain:

$$W_{\text{ext}} = \Delta K + \Delta U_{\text{g}}$$

and, because $W_{\text{ext}} = 0$,

$$\Delta K_{\text{trans}} + \Delta K_{\text{rot}} + \Delta U_{\text{g}} = 0$$

Substituting for ΔK_{trans}, ΔK_{rot}, and ΔU_{g} yields:

$$K_{\text{trans,f}} - K_{\text{trans, i}} + K_{\text{rot,f}} - K_{\text{rot,i}} + U_{\text{g,f}} - U_{\text{g,i}} = 0$$

Because $K_{\text{trans,i}} = K_{\text{rot,i}} = 0$:

$$K_{\text{trans,f}} + K_{\text{rot,f}} + U_{\text{g,f}} - U_{\text{g,i}} = 0$$

Substitute for $K_{\text{trans,f}}$, $K_{\text{rot,f}}$, $U_{\text{g,f}}$, and $U_{\text{g,i}}$ to obtain:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 + mg(2R - r) - mgh = 0$$

Noting that, because the ball rolls without slipping, $v = r\omega$ and that $I_{\text{sphere}} = \tfrac{2}{5}mr^2$, substitute to obtain:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 + mg(2R - r) - mgh = 0 \Rightarrow h = \frac{7v^2}{10g} + 2R - r$$

Noting that, for the minimum speed for which the ball remains in contact with the track at the top of the loop the centripetal force is the gravitational force, apply Newton's second law to the ball to obtain:

$$mg = m\frac{v^2}{R - r} \Rightarrow v^2 = g(R - r)$$

Substituting for v^2 and simplifying yields:

$$h = \frac{7g(R - r)}{10g} + 2R - r$$
$$= \boxed{2.7R - 1.7r}$$

Rolling With Slipping

101 •• A bowling ball of mass M and radius R is released so that at the instant it touches the floor it is moving horizontally with a speed v_0 and is not rotating. It slides for a time t_1 a distance s_1 before it begins to roll without slipping. (*a*) If μ_{k} is the coefficient of kinetic friction between the ball and the floor, find s_1, t_1, and the final speed v_1 of the ball. (*b*) Find the ratio of the final kinetic energy to the initial kinetic energy of the ball. (*c*) Evaluate s_1, t_1, and v_1 for $v_0 = 8.0$ m/s and $\mu_{\text{k}} = 0.060$.

Picture the Problem Part (*a*) of this problem is identical to Example 9-16. In Part (*b*) we can use the definitions of translational and rotational kinetic energy to find the ratio of the final and initial kinetic energies.

(*a*) From Example 9-19:

$$s_1 = \boxed{\frac{12}{49}\frac{v_0^2}{\mu_k g}}, t_1 = \boxed{\frac{2}{7}\frac{v_0}{\mu_k g}}, \text{ and}$$

$$v_1 = \tfrac{5}{2}\mu_k g t_1 = \boxed{\frac{5}{7}v_0}$$

(*b*) When the ball rolls without slipping, $v_1 = r\omega$. The final kinetic energy of the ball is given by:

$$K_f = K_{trans} + K_{rot} = \tfrac{1}{2}Mv_1^2 + \tfrac{1}{2}I\omega^2$$

Substituting for I and simplifying yields:

$$K_f = \tfrac{1}{2}Mv_1^2 + \tfrac{1}{2}\left(\tfrac{2}{5}Mr^2\right)\frac{v_1^2}{r^2} = \tfrac{7}{10}Mv_1^2 = \tfrac{5}{14}Mv_0^2$$

Express the ratio of the final and initial kinetic energies:

$$\frac{K_f}{K_i} = \frac{\frac{5}{14}Mv_0^2}{\frac{1}{2}Mv_0^2} = \boxed{\frac{5}{7}}$$

(*c*) Substitute numerical values in the expression from (*a*) and evaluate s_1:

$$s_1 = \frac{12}{49}\frac{(8.0\,\text{m/s})^2}{(0.060)(9.81\,\text{m/s}^2)} = \boxed{27\,\text{m}}$$

Substitute numerical values in the expression from (*a*) and evaluate t_1:

$$t_1 = \frac{2}{7}\frac{8.0\,\text{m/s}}{(0.060)(9.81\,\text{m/s}^2)} = \boxed{3.9\,\text{s}}$$

Substitute numerical values in the expression from (*a*) and evaluate v_1:

$$v_1 = \frac{5}{7}(8.0\,\text{m/s}) = \boxed{5.7\,\text{m/s}}$$

102 •• During a game of pool, the cue ball (a uniform sphere of radius r) is at rest on the horizontal pool table (Figure 9-68). You strike the ball horizontally with your cue stick, which delivers a large horizontal force of magnitude F_0 for a short time. The stick strikes the ball at a point a vertical height h above the tabletop. Assume the striking location is above the ball's center. Show that the ball's angular speed ω is related to the initial linear speed of its center of mass v_{cm} by $\omega = (5/2)v_{cm}(h - r)/r^2$. Estimate the ball's rotation rate just after the hit using reasonable estimates for h, r and v_{cm}.

Picture the Problem We can apply Newton's second law in rotational form and the impulse-momentum theorem to obtain two equations that we can solve simultaneously for ω.

Apply Newton's second law in rotational form to the cue ball:

$$F_0(h-r)=I_{\text{cm}}\alpha=\tfrac{2}{5}mr^2\frac{\Delta\omega}{\Delta t}$$
$$=\tfrac{2}{5}mr^2\frac{\omega}{\Delta t}$$

Solving for ω gives:

$$\omega=\frac{5F_0(h-r)\Delta t}{2mr^2} \qquad (1)$$

From the impulse-momentum theorem:

$$F_0\Delta t=\Delta p=mv_{\text{cm}}\Rightarrow\Delta t=\frac{mv_{\text{cm}}}{F_0}$$

Substitute for Δt in equation (1) and simplify to obtain:

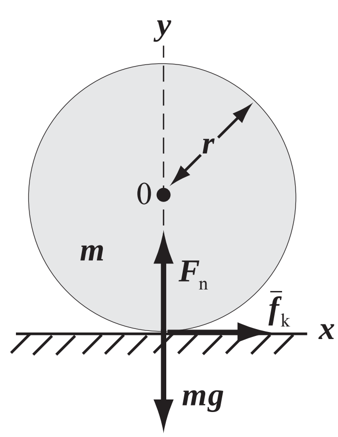
$$\omega=\frac{5F_0(h-r)\dfrac{mv_{\text{cm}}}{F_0}}{2mr^2}=\boxed{\frac{5v_{\text{cm}}(h-r)}{2r^2}}$$

Assume that $v_{\text{cm}}\approx 1.0$ m/s, $r\approx 4.0$ cm, and $h\approx 6.0$ cm, to estimate ω:

$$\omega=\frac{5(1.0\text{ m/s})(6.0\text{ cm}-4.0\text{ cm})}{2(4.0\text{ cm})^2}$$
$$\approx\boxed{30\text{ rad/s}}$$

103 •• A uniform solid sphere is set rotating about a horizontal axis at an angular speed ω_0 and then is placed on the floor with its center of mass at rest. If the coefficient of kinetic friction between the sphere and the floor is μ_k, find the speed of the center of mass of the sphere when the sphere begins to roll without slipping.

Picture the Problem The angular speed of the rotating sphere will decrease until the condition for rolling without slipping is satisfied and then it will begin to roll. The force diagram shows the forces acting on the sphere. We can apply Newton's second law to the sphere and use the condition for rolling without slipping to find the speed of the center of mass when the sphere begins to roll without slipping.



Relate the velocity of the sphere when it begins to roll to its acceleration at that moment and the elapsed time:

$$v = a\Delta t \qquad (1)$$

Apply Newton's second law to the sphere to obtain:

$$\sum F_x = f_k = ma\ , \qquad (2)$$

$$\sum F_y = F_n - mg = 0\,, \qquad (3)$$

and

$$\sum \tau_0 = f_k r = I_0\alpha \qquad (4)$$

Using the definition of f_k and F_n from equation (3), substitute in equation (2) and solve for a:

$$a = \mu_k g$$

Substitute in equation (1) to obtain:

$$v = a\Delta t = \mu_k g\Delta t \qquad (5)$$

Solving for α in equation (4) yields:

$$\alpha = \frac{f_k r}{I_0} = \frac{mar}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu_k g}{r}$$

Express the angular speed of the sphere when it has been moving for a time Δt:

$$\omega = \omega_0 - \alpha\,\Delta t = \omega_0 - \frac{5\mu_k g}{2r}\Delta t \quad (6)$$

Express the condition that the sphere rolls without slipping:

$$v = r\omega$$

Substitute from equations (5) and (6) and solve for the elapsed time until the sphere begins to roll:

$$\Delta t = \frac{2}{7}\frac{r\omega_0}{\mu_k g}$$

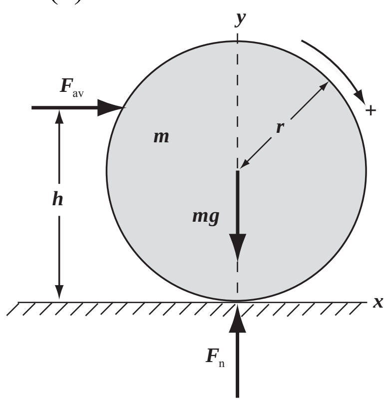
Use equation (5) to find v when the sphere begins to roll:

$$v = \mu_k g\Delta t = \frac{2}{7}\frac{r\omega_0\mu_k g}{\mu_k g} = \boxed{\frac{2r\omega_0}{7}}$$

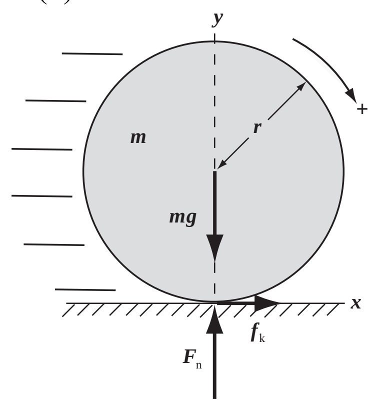
104 •• A uniform solid ball that has a mass of 20 g and a radius of 5.0 cm rests on a horizontal surface. A sharp force is applied to the ball in a horizontal direction 9.0 cm above the horizontal surface. During impact the force increases linearly from 0 N to 40 kN in 1.0×10^{-4} s, and then decreases linearly to 0 N in 1.0×10^{-4} s. (*a*) What is the speed of the ball just after impact? (*b*) What is the angular speed of the ball after impact? (*c*) What is the speed of the ball when it begins to roll without slipping? (*d*) How far does the ball travel along the surface before it begins to roll without slipping? Assume that $\mu_k = 0.50$.

Picture the Problem (*a*) The sharp force delivers a translational impulse to the ball that changes its linear momentum. We can use the impulse-momentum theorem to find the speed of the ball after impact. (*b*) We can find the angular speed of the ball after impact by applying Newton's second law in rotational form. Because the ball has a forward spin, the friction force is in the direction of motion and will cause the ball's translational speed to increase. In Parts (*c*) and (*d*) we can apply Newton's second law to the ball to obtain equations describing both the translational and rotational motion of the ball. We can then solve these equations to find the constant accelerations that allow us to apply constant-acceleration equations to find the velocity of the ball when it begins to roll and its sliding time.

(*a*) and (*b*)



(*c*) and (*d*)



(*a*) Apply the impulse-momentum theorem to the ball. The frictional force can be neglected during impact:

$$I = F_{av}\Delta t = \Delta p = mv_0$$

Solving for v_0 yields:

$$v_0 = \frac{F_{av}\Delta t}{m} \qquad (1)$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{(20\,\text{kN})(2.0\times10^{-4}\ \text{s})}{0.020\,\text{kg}} = \boxed{2.0\times10^{2}\ \text{m/s}}$$

(*b*) Referring to the force diagram for (*a*) and (*b*), apply Newton's second law in rotational form to the ball to obtain:

$$F_{av}(h-r) = I_{cm}\alpha = I_{cm}\frac{\omega_0}{\Delta t}$$

Solving for ω_0 yields:

$$\omega_0 = \frac{F_{\text{av}}(h-r)\Delta t}{I_{\text{cm}}} = \frac{F_{\text{av}}(h-r)\Delta t}{\frac{2}{5}mr^2}$$

Solve equation (1) for Δt:

$$\Delta t = \frac{mv_0}{F_{\text{av}}}$$

Substitute for Δt and simplify to obtain:

$$\omega_0 = \frac{F_{\text{av}}(h-r)\frac{mv_0}{F_{\text{av}}}}{\frac{2}{5}mr^2} = \frac{5v_0(h-r)}{2r^2}$$

Substitute numerical values and evaluate ω_0:

$$\omega_0 = \frac{5(2.0\times10^2\ \text{m/s})(0.090\,\text{m} - 0.050\,\text{m})}{2(.050\,\text{m})^2} = \boxed{8.0\times10^3\ \text{rad/s}}$$

(*c*) Use a constant-acceleration equation to relate the speed of the ball to the acceleration and the time:

$$v_{\text{cm}\,x} = v_0 + a_{\text{cm}\,x}t \qquad (1)$$

To determine the direction of the frictional force we first determine whether the angular speed after impact is less than, equal to, or greater than the angular speed v_0/r for rolling without slipping:

$$\frac{v_0}{r} = \frac{2.0\times10^2\ \text{m/s}}{0.050\ \text{m}} = 4.0\times10^3\ \text{rad/s}$$

If the angular speed ω_0 is greater than v_0/r, then the frictional force will be in the forward direction, as shown in the force diagram for (*c*) and (*d*):

$$\omega_0 > \frac{v_0}{r}$$

Referring to the force diagram for (*c*) and (*d*), apply Newton's second law to the ball to obtain:

$$\sum F_x = f_{\text{k}} = ma_{\text{cm}\,x}, \qquad (2)$$

$$\sum F_y = F_{\text{n}} - mg = 0, \qquad (3)$$

and

$$\sum \tau_{\text{cm}} = -f_{\text{k}}r = I_{\text{cm}}\alpha \qquad (4)$$

Substituting for f_{k} in equation (2) gives:

$$\mu_{\text{k}}mg = ma_{\text{cm}\,x} \Rightarrow a_{\text{cm}\,x} = \mu_{\text{k}}g$$

Substitute for $a_{\text{cm}\,x}$ in equation (1) to obtain:

$$v_{\text{cm}\,x} = v_0 + \mu_{\text{k}} gt \qquad (5)$$

Solving for α in equation (4) yields:

$$\alpha = -\frac{f_{\text{k}} r}{I_{\text{cm}}} = -\frac{\mu_{\text{k}} mgr}{\frac{2}{5}mr^2} = -\frac{5\mu_{\text{k}} g}{2r}$$

Use a constant-acceleration equation to relate the angular speed of the ball to the angular acceleration and the time:

$$\omega = \omega_0 + \alpha t = \omega_0 - \frac{5\mu_{\text{k}} g}{2r} t \qquad (6)$$

When the ball rolls without slipping:

$$v_{\text{cm}\,x} = r\omega = r\left(\omega_0 - \frac{5\mu_{\text{k}} g}{2r} t\right) = r\omega_0 - \frac{5\mu_{\text{k}} g}{2} t \qquad (7)$$

Equate equations (5) and (7) to obtain:

$$r\omega_0 - \frac{5\mu_{\text{k}} g}{2} t = v_0 + \mu_{\text{k}} gt$$

Solve for the time t at which $v_{\text{cm}\,x} = r\omega$:

$$t = \frac{2(r\omega_0 - v_0)}{7\mu_{\text{k}} g}$$

Substitute numerical values and evaluate t:

$$t = \frac{2\left[(0.050\text{ m})\left(8.0\times10^3\text{ rad/s}\right) - 2.0\times10^2\text{ m/s}\right]}{7(0.50)\left(9.81\text{ m/s}^2\right)} = 11.65\text{ s}$$

Substitute numerical values in equation (5) and evaluate $v_{\text{cm}\,x}(11.65\text{ s})$:

$$v_{\text{cm}\,x}(11.65\text{ s}) = 2.0\times10^2\text{ m/s} + (0.50)\left(9.81\text{ m/s}^2\right)(11.65\text{ s}) = 2.571\times10^2\text{ m/s} = \boxed{2.6\times10^2\text{ m/s}}$$

(*d*) The distance traveled in time t is:

$$\Delta x = v_0 t + \tfrac{1}{2} a_{\text{cm}} t^2 = v_0 t + \tfrac{1}{2}\mu_{\text{k}} g t^2$$

Substitute numerical values and evaluate $\Delta x(11.65\text{ s})$:

$$\Delta x(11.65\text{ s}) = \left(2.0\times10^2\text{ m/s}\right)(11.65\text{ s}) + \tfrac{1}{2}(0.50)\left(9.81\text{ m/s}^2\right)(11.65\text{ s})^2 = \boxed{3.0\text{ km}}$$

105 •• **[SSM]** A 0.16-kg billiard ball whose radius is 3.0 cm is given a sharp blow by a cue stick. The applied force is horizontal and the line of action of the force passes through the center of the ball. The speed of the ball just after the

blow is 4.0 m/s, and the coefficient of kinetic friction between the ball and the billiard table is 0.60. (*a*) How long does the ball slide before it begins to roll without slipping? (*b*) How far does it slide? (*c*) What is its speed once it begins rolling without slipping?

Picture the Problem Because the impulse is applied through the center of mass, $\omega_0 = 0$. We can use the results of Example 9-16 to find the rolling time without slipping, the distance traveled to rolling without slipping, and the velocity of the ball once it begins to roll without slipping.

(*a*) From Example 9-19 we have:

$$t_1 = \frac{2}{7}\frac{v_0}{\mu_k g}$$

Substitute numerical values and evaluate t_1:

$$t_1 = \frac{2}{7}\frac{4.0\,\text{m/s}}{(0.60)(9.81\,\text{m/s}^2)} = \boxed{0.19\,\text{s}}$$

(*b*) From Example 9-19 we have:

$$s_1 = \frac{12}{49}\frac{v_0^2}{\mu_k g}$$

Substitute numerical values and evaluate s_1:

$$s_1 = \frac{12}{49}\frac{(4.0\,\text{m/s})^2}{(0.60)(9.81\,\text{m/s}^2)} = \boxed{0.67\,\text{m}}$$

(*c*) From Example 9-16 we have:

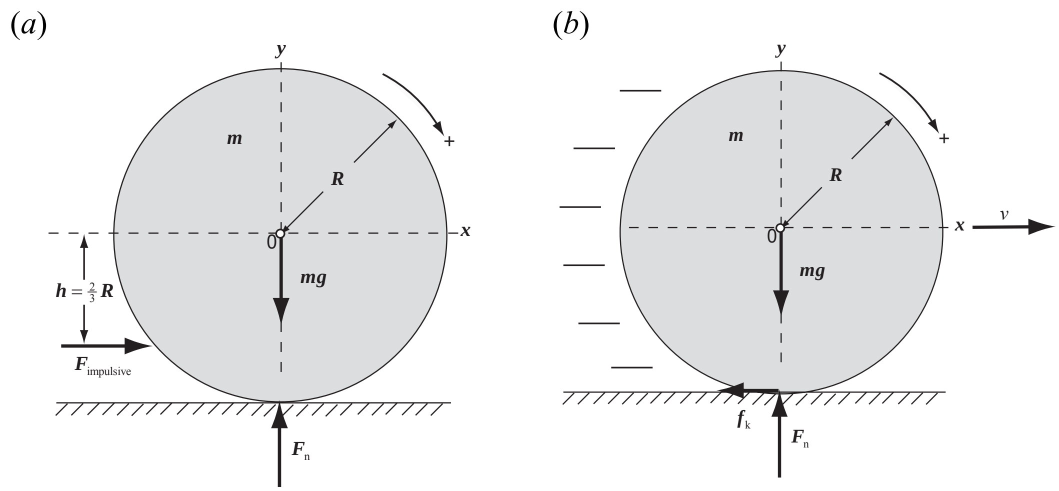
$$v_1 = \frac{5}{7}v_0$$

Substitute numerical values and evaluate v_1:

$$v_1 = \frac{5}{7}(4.0\,\text{m/s}) = \boxed{2.9\,\text{m/s}}$$

106 •• A billiard ball that is initially at rest is given a sharp blow by a cue stick. The force is horizontal and is applied at a distance $2R/3$ below the centerline, as shown in Figure 9-69. The speed of the ball just after the blow is v_0 and the coefficient of kinetic friction between the ball and the billiard table is μ_k. (*a*) What is the angular speed of the ball just after the blow? (*b*) What is the speed of the ball once it begins to roll without slipping? (*c*) What is the kinetic energy of the ball just after the hit?

Picture the Problem Because the impulsive force is applied below the center line, the ball will have a backward spin and the direction of the friction force is opposite the direction of motion. This will cause the ball's translational speed to decrease. We'll use the impulse-momentum theorem and Newton's second law in rotational form to find the linear and rotational speeds and accelerations of the ball and constant-acceleration equations to relate these quantities to each other and to the elapsed time-to-rolling without slipping.



(*a*) Apply Newton's second law in rotational form to the ball to obtain:

$$\sum \tau_0 = F_{av}(h-R) = I_{cm}\alpha = I_{cm}\frac{\omega_0}{\Delta t}$$

Solving for ω_0 yields:

$$\omega_0 = \frac{F_{av}(h-R)\Delta t}{I_{cm}} = \frac{F_{av}(h-R)\Delta t}{\frac{2}{5}mr^2}$$

Because $F_{av}\Delta t = mv_0$:

$$\Delta t = \frac{mv_0}{F_{av}}$$

Substitute for Δt and simplify to obtain:

$$\omega_0 = \frac{F_{av}(h-R)\frac{mv_0}{F_{av}}}{\frac{2}{5}mR^2} = \boxed{\frac{5v_0(h-R)}{2R^2}}$$

(*b*) Apply Newton's second law to the ball to obtain:

$$\sum \tau_0 = f_k R = I_{cm}\alpha\,, \qquad (1)$$

$$\sum F_y = F_n - mg = 0\,, \qquad (2)$$

and

$$\sum F_x = -f_k = ma \qquad (3)$$

Substitute for f_k in equation (1), and solve for α:

$$\alpha = \frac{\mu_k mgR}{I_{cm}} = \frac{\mu_k mgR}{\frac{2}{5}mR^2} = \frac{5\mu_k g}{2R}$$

Using a constant-acceleration equation, relate the angular speed of the ball to its acceleration:

$$\omega = \omega_0 + \alpha\Delta t = \omega_0 + \frac{5\mu_k g}{2R}\Delta t$$

Substituting for f_k in equation (3) and solving for a gives:

$$a = -\mu_k g$$

Using a constant-acceleration equation, relate the speed of the ball to its acceleration:

$$v = v_0 + a\Delta t = v_0 - \mu_k g\Delta t \qquad (4)$$

Impose the condition for rolling without slipping to obtain:

$$R\left(\omega_0 + \frac{5\mu_k g}{2R}\Delta t\right) = v_0 - \mu_k g\Delta t$$

Solving for Δt yields:

$$\Delta t = \frac{16}{21}\frac{v_0}{\mu_k g}$$

Substitute in equation (4) to obtain:

$$v = v_0 - \mu_k g\left(\frac{16}{21}\frac{v_0}{\mu_k g}\right) = \boxed{\frac{5}{21}v_0}$$

(*c*) Express the initial kinetic energy of the ball and simplify to obtain:

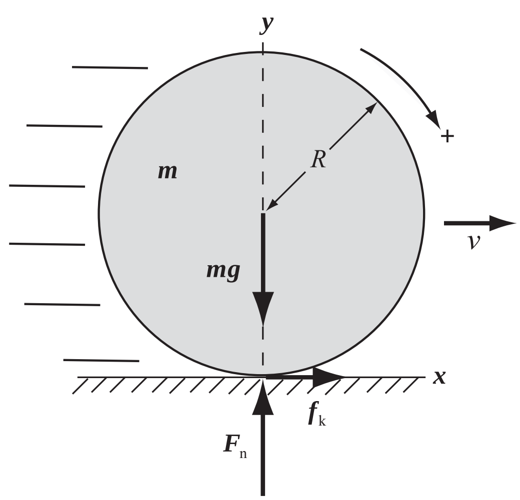
$$K_i = K_{trans} + K_{rot} = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I\omega_0^2$$

$$= \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}\left(\tfrac{2}{5}mR^2\right)\left(\frac{5v_0}{3R}\right)^2$$

$$= \boxed{\frac{19}{18}mv_0^2}$$

107 •• A bowling ball of radius R has an initial speed v_0 down the lane and a forward spin $\omega_0 = 3v_0/R$ just after its release. The coefficient of kinetic friction is μ_k. (*a*) What is the speed of the ball just as it begins rolling without slipping? (*b*) For how long a time does the ball slide before it begins rolling without slipping? (*c*) What distance does the ball slide down the lane before it begins rolling without slipping?

Picture the Problem The figure shows the forces acting on the bowling during the sliding phase of its motion. Because the ball has a forward spin, the friction force is in the direction of motion and will cause the ball's translational speed to increase. We'll apply Newton's second law to find the linear and rotational velocities and accelerations of the ball and constant-acceleration equations to relate these quantities to each other and to the elapsed time to rolling without slipping.



(*a*) and (*b*) Relate the velocity of the ball when it begins to roll to its acceleration and the elapsed time:

$$v = v_0 + a\Delta t \qquad (1)$$

Apply Newton's second law to the ball:

$$\sum F_x = f_k = ma\,, \qquad (2)$$

$$\sum F_y = F_n - mg = 0\,, \qquad (3)$$

and

$$\sum \tau_0 = -f_k R = I_0\alpha \qquad (4)$$

Substituting for f_k in equation (2) and solving for a gives:

$$a = \mu_k g$$

Substitute in equation (1) to obtain:

$$v = v_0 + a\Delta t = v_0 + \mu_k g\Delta t \qquad (5)$$

Solve for α in equation (4):

$$\alpha = -\frac{f_k R}{I_0} = -\frac{maR}{\frac{2}{5}mR^2} = -\frac{5}{2}\frac{\mu_k g}{R}$$

Relate the angular speed of the ball to its acceleration:

$$\omega = \omega_0 - \frac{5}{2}\frac{\mu_k g}{R}\Delta t$$

Apply the condition for rolling without slipping and simplify to obtain:

$$v = R\omega = R\left(\omega_0 - \frac{5}{2}\frac{\mu_k g}{R}\Delta t\right)$$

$$= R\left(\frac{3v_0}{R} - \frac{5}{2}\frac{\mu_k g}{R}\Delta t\right) \qquad (6)$$

$$= 3v_0 - \frac{5}{2}\mu_k g\Delta t$$

Equate equations (5) and (6) and solve for Δt:

$$\Delta t = \boxed{\frac{4}{7}\frac{v_0}{\mu_k g}}$$

Substitute for Δt in equation (6) to obtain:

$$v = \boxed{\frac{11}{7}v_0}$$

(*c*) Relate Δx to the average speed of the ball and the time it moves before beginning to roll without slipping:

$$\Delta x = v_{av}\Delta t = \tfrac{1}{2}(v_0 + v)\Delta t$$

Substitute for $v_0 + v$ and Δt and simplify to obtain:

$$\Delta x = \tfrac{1}{2}\left(v_0 + \frac{11}{7}v_0\right)\left(\frac{4v_0}{7\mu_k g}\right) = \boxed{\frac{36}{49}\frac{v_0^2}{\mu_k g}}$$

General Problems

108 •• The radius of a small playground merry-go-round is 2.2 m. To start it rotating, you wrap a rope around its perimeter and pull with a force of 260 N for 12 s. During this time, the merry-go-round makes one complete rotation. Neglect any effects of friction. (*a*) Find the angular acceleration of the merry-go-round. (*b*) What torque is exerted by the rope on the merry-go-round? (*c*) What is the moment of inertia of the merry-go-round?

Picture the Problem The force you exert on the rope results in a net torque that accelerates the merry-go-round. The moment of inertia of the merry-go-round, its angular acceleration, and the torque you apply are related through Newton's second law.

(*a*) Using a constant-acceleration equation, relate the angular displacement of the merry-go-round to its angular acceleration and acceleration time:

$$\Delta\theta = \omega_0 \Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

or, because $\omega_0 = 0$,

$$\Delta\theta = \tfrac{1}{2}\alpha(\Delta t)^2 \Rightarrow \alpha = \frac{2\Delta\theta}{(\Delta t)^2}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{2(2\pi\,\text{rad})}{(12\,\text{s})^2} = 0.0873\,\text{rad/s}^2 = \boxed{0.087\,\text{rad/s}^2}$$

(*b*) Use the definition of torque to obtain:

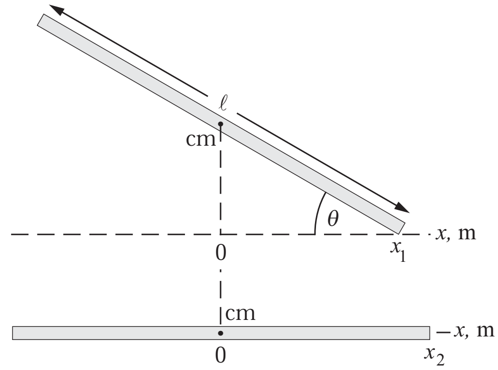
$$\tau = Fr = (260\,\text{N})(2.2\,\text{m}) = 572\,\text{N}\cdot\text{m} = \boxed{0.57\,\text{kN}\cdot\text{m}}$$

(*c*) Use Newton's second law to find the moment of inertia of the merry-go-round:

$$I = \frac{\tau_{\text{net}}}{\alpha} = \frac{572\,\text{N}\cdot\text{m}}{0.0873\,\text{rad/s}^2} = \boxed{6.6\times10^3\,\text{kg}\cdot\text{m}^2}$$

109 •• A uniform 2.00-m-long stick is raised at an angle of 30° to the horizontal above a sheet of ice. The bottom end of the stick rests on the ice. The stick is released from rest. The bottom of the stick remains in contact with the ice at all times. How far will the bottom end of the stick have traveled during the time the rest of the stick is falling to the ice? Assume that the ice is frictionless.

Picture the Problem Because there are no horizontal forces acting on the stick, the center of mass of the stick will not move in the horizontal direction. Choose a coordinate system in which the origin is at the horizontal position of the center of mass. The diagram shows the stick in its initial raised position and when it has fallen to the ice.



Express the displacement of the right end of the stick Δx as the difference between the position coordinates x_1 and x_2:

$$\Delta x = x_2 - x_1 \qquad (1)$$

Using trigonometry, find the initial coordinate of the right end of the stick:

$$x_1 = \tfrac{1}{2}\ell\cos\theta = \tfrac{1}{2}(2.00\,\text{m})\cos 30° = 0.866\,\text{m}$$

Because the center of mass has not moved horizontally:

$$x_2 = \tfrac{1}{2}\ell = 1.00\,\text{m}$$

Substitute for x_1 and x_2 in equation (1) to find the displacement of the right end of the stick:

$$\Delta \mathbf{x} = 1.00\,\text{m} - 0.866\,\text{m} = \boxed{13\,\text{cm}}$$

110 •• A uniform 5.0-kg disk that has a 0.12-m radius is pivoted so that it rotates freely about its axis (Figure 9-70). A string wrapped around the disk is pulled with a force equal to 20 N. (*a*) What is the torque being exerted by this force about the rotation axis? (*b*) What is the angular acceleration of the disk? (*c*) If the disk starts from rest, what is its angular speed after 5.0 s? (*d*) What is its kinetic energy after the 5.0 s? (*e*) What is the angular displacement of the disk during the 5.0 s? (*f*) Show that the work done by the torque, $\tau\,\Delta\theta$, equals the kinetic energy.

Picture the Problem The force applied to the string results in a torque about the center of mass of the disk that accelerates it. We can relate these quantities to the moment of inertia of the disk through Newton's second law and then use constant-acceleration equations to find the disks angular speed the angle through which it has rotated in a given period of time. The disk's rotational kinetic energy can be found from its definition.

(*a*) Use the definition of torque to obtain:

$$\tau = FR = (20\,\text{N})(0.12\,\text{m}) = 2.40\,\text{N}\cdot\text{m} = \boxed{2.4\,\text{N}\cdot\text{m}}$$

(*b*) Use Newton's second law to express the angular acceleration of the disk in terms of the net torque acting on it and its moment of inertia:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{\tau_{\text{net}}}{\frac{1}{2}MR^2} = \frac{2\tau_{\text{net}}}{MR^2}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{2(2.40\,\text{N}\cdot\text{m})}{(5.0\,\text{kg})(0.12\,\text{m})^2} = 66.7\,\text{rad/s}^2$$
$$= \boxed{67\,\text{rad/s}^2}$$

(*c*) Using a constant-acceleration equation, relate the angular speed of the disk to its angular acceleration and the elapsed time:

$$\omega = \omega_0 + \alpha\Delta t$$

or, because $\omega_0 = 0$,

$$\omega = \alpha\Delta t$$

Substitute numerical values and evaluate ω:

$$\omega = (66.7\,\text{rad/s}^2)(5.0\,\text{s}) = 333\,\text{rad/s}$$
$$= \boxed{3.3\times10^2\ \text{rad/s}}$$

(*d*) Use the definition of rotational kinetic energy to obtain:

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\omega^2$$
$$= \tfrac{1}{4}MR^2\omega^2$$

Substitute numerical values and evaluate K_{rot}:

$$K_{\text{rot}} = \tfrac{1}{4}(5.0\,\text{kg})(0.12\,\text{m})^2(333\,\text{rad/s})^2$$
$$= \boxed{2.0\,\text{kJ}}$$

(*e*) Using a constant-acceleration equation, relate the angle through which the disk turns to its angular acceleration and the elapsed time:

$$\Delta\theta = \omega_0\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

or, because $\omega_0 = 0$,

$$\Delta\theta = \tfrac{1}{2}\alpha(\Delta t)^2$$

Substitute numerical values and evaluate $\Delta\theta$:

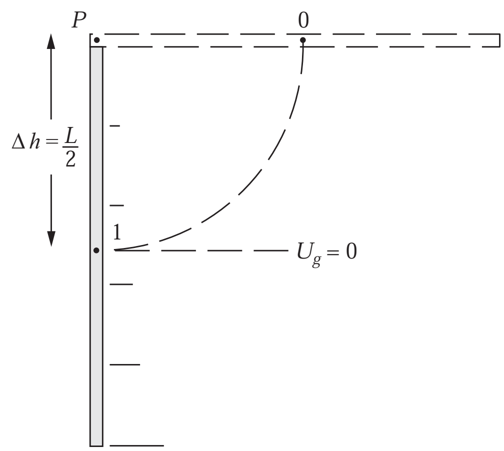
$$\Delta\theta = \tfrac{1}{2}(66.7\,\text{rad/s}^2)(5.0\,\text{s})^2$$
$$= \boxed{8.3\times10^2\ \text{rad}}$$

(*f*) Substitute for I and ω^2 in the expression for K and simplify to obtain:

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\frac{\tau}{\alpha}\right)(\alpha\Delta t)^2 = \tfrac{1}{2}\alpha\tau(\Delta t)^2$$
$$= \boxed{\tau\Delta\theta}$$

111 •• A uniform 0.25-kg thin rod that has a 80-cm length is free to rotate about a fixed horizontal axis perpendicular to, and through one end, of the rod. It is held horizontal and released. Immediately after it is released, what is (*a*) the acceleration of the center of the rod, and (*b*) the initial acceleration of the free end of the rod? (*c*) What is the speed of the center of mass of the rod when the rod is (momentarily) vertical.

Picture the Problem The diagram shows the rod in its initial horizontal position and then, later, as it swings through its vertical position. The center of mass is denoted by the numerals 0 and 1. Let the length of the rod be represented by L and its mass by m. We can use Newton's second law in rotational form to find, first, the angular acceleration of the rod and then, from α, the acceleration of any point on the rod. We can use conservation of energy to find the angular speed of the center of mass of the rod when it is vertical and then use this value to find its linear velocity.



(*a*) Relate the acceleration of the center of the rod to the angular acceleration of the rod:

$$a = \ell\alpha = \frac{L}{2}\alpha$$

Use Newton's second law to relate the torque about the suspension point of the rod (exerted by the weight of the rod) to the rod's angular acceleration:

$$\alpha = \frac{\tau}{I_{\mathrm{P}}} = \frac{Mg\frac{L}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{3(9.81\,\mathrm{m/s^2})}{2(0.80\,\mathrm{m})} = 18.4\,\mathrm{rad/s^2}$$

Substitute numerical values and evaluate a:

$$a = \tfrac{1}{2}(0.80\,\mathrm{m})(18.4\,\mathrm{rad/s^2}) = \boxed{7.4\,\mathrm{m/s^2}}$$

(*b*) Relate the acceleration of the end of the rod to α:

$$a_{\text{end}} = L\alpha$$

Substitute numerical values and evaluate a_{end}:

$$a_{\text{end}} = (0.80\,\text{m})(18.4\,\text{rad/s}^2) = \boxed{15\,\text{m/s}^2}$$

(*c*) Relate the linear velocity of the center of mass of the rod to its angular speed as it passes through the vertical:

$$v = \omega\Delta h = \tfrac{1}{2}\omega L \qquad (1)$$

Use conservation of energy to relate the changes in the kinetic and potential energies of the rod as it swings from its initial horizontal orientation through its vertical orientation:

$$\Delta K + \Delta U = K_1 - K_0 + U_1 - U_0 = 0$$

or, because $K_0 = U_1 = 0$,

$$K_1 - U_0 = 0$$

Substitute for K_1 and U_0 to obtain:

$$\tfrac{1}{2}I_{\text{P}}\omega^2 - mg\Delta h = 0$$

Substituting for I_{P} and Δh yields:

$$\tfrac{1}{2}\left(\tfrac{1}{3}mL^2\right)\omega^2 - mg\left(\tfrac{1}{2}L\right) = 0 \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

Substituting for ω in equation (1) yields:

$$v = \tfrac{1}{2}L\sqrt{\frac{3g}{L}} = \tfrac{1}{2}\sqrt{3gL}$$

Substitute numerical values and evaluate v:

$$v = \tfrac{1}{2}\sqrt{3(9.81\,\text{m/s}^2)(0.80\,\text{m})} = \boxed{2.4\,\text{m/s}}$$

112 •• A marble of mass M and radius R rolls without slipping down the track on the left from a height h_1, as shown in Figure 9-71. The marble then goes up the *frictionless* track on the right to a height h_2. Find h_2.

Picture the Problem Let the zero of gravitational potential energy be at the bottom of the track. The initial potential energy of the marble is transformed into translational and rotational kinetic energy and gravitational potential energy (note that its center of mass is a distance R above the bottom of the track when it reaches the lowest point on the track) as it rolls down the track to its lowest point and then, because the portion of the track to the right is frictionless, into translational kinetic energy and, eventually, into gravitational potential energy.

Using conservation of energy, relate h_2 to the mechanical energy of the marble at the bottom of the track:

$$\Delta K + \Delta U = 0$$

or, because $K_f = 0$,

$$-K_i + U_f - U_i = 0$$

Substitute for K_i, U_f, and U_i to obtain:

$$-\tfrac{1}{2}Mv^2 + Mgh_2 - MgR = 0$$

Solving for h_2 gives:

$$h_2 = \frac{2gR + v^2}{2g} \qquad (1)$$

Using conservation of energy, relate h_1 to the energy of the marble at the bottom of the track:

$$\Delta K + \Delta U = 0$$

or, because $K_i = 0$,

$$K_{\text{f, trans}} + K_{\text{f, rot}} + U_f - U_i = 0$$

Substitute for the energies to obtain:

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 + MgR - Mgh_1 = 0$$

The moment of inertia of a sphere of mass M and radius R, about an axis through its center, is $\tfrac{2}{5}MR^2$:

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}MR^2\right)\omega^2 + MgR - Mgh_1 = 0$$

Because the marble is rolling without slipping, $v = R\omega$ and:

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 + MgR - Mgh_1 = 0$$

Solve for v^2 to obtain:

$$v^2 = \tfrac{10}{7}g(h_1 - R)$$

Substitute for v^2 in equation (1) and simplify to obtain:

$$h_2 = \frac{2gR + \tfrac{10}{7}g(h_1 - R)}{2g} = \boxed{\tfrac{1}{7}(5h_1 + 2R)}$$

Remarks: If h_1, h_2 >> R, then our result becomes $h_2 = \tfrac{5}{7}h_1$.

113 •• **[SSM]** A uniform 120-kg disk with a radius equal to 1.4 m initially rotates with an angular speed of 1100 rev/min. A constant tangential force is applied at a radial distance of 0.60 m from the axis. (*a*) How much work must this force do to stop the wheel? (*b*) If the wheel is brought to rest in 2.5 min, what torque does the force produce? What is the magnitude of the force? (*c*) How many revolutions does the wheel make in these 2.5 min?

Picture the Problem To stop the wheel, the tangential force will have to do an amount of work equal to the initial rotational kinetic energy of the wheel. We can find the stopping torque and the force from the average power delivered by the

force during the slowing of the wheel. The number of revolutions made by the wheel as it stops can be found from a constant-acceleration equation.

(*a*) Relate the work that must be done to stop the wheel to its kinetic energy:

$$W=\tfrac{1}{2}I\omega^2=\tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\omega^2=\tfrac{1}{4}mr^2\omega^2$$

Substitute numerical values and evaluate W:

$$W=\tfrac{1}{4}(120\,\text{kg})(1.4\,\text{m})^2\left[1100\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right]^2=780\,\text{kJ}=\boxed{7.8\times10^2\,\text{kJ}}$$

(*b*) Express the stopping torque in terms of the average power required:

$$P_\text{av}=\tau\omega_\text{av}\Rightarrow\tau=\frac{P_\text{av}}{\omega_\text{av}}$$

Substitute numerical values and evaluate τ:

$$\tau=\frac{\dfrac{780\,\text{kJ}}{(2.5\,\text{min})(60\,\text{s/min})}}{\dfrac{(1100\,\text{rev/min})(2\pi\,\text{rad/rev})(1\,\text{min}/60\,\text{s})}{2}}=90.3\,\text{N}\cdot\text{m}=\boxed{90\,\text{N}\cdot\text{m}}$$

Relate the stopping torque to the magnitude of the required force and evaluate F:

$$F=\frac{\tau}{R}=\frac{90.3\,\text{N}\cdot\text{m}}{0.60\,\text{m}}=\boxed{0.15\text{ kN}}$$

(*c*) Using a constant-acceleration equation, relate the angular displacement of the wheel to its average angular speed and the stopping time:

$$\Delta\theta=\omega_\text{av}\Delta t$$

Substitute numerical values and evaluate $\Delta\theta$:

$$\Delta\theta=\left(\frac{1100\,\text{rev/min}}{2}\right)(2.5\,\text{min})$$
$$=\boxed{1.4\times10^3\text{ rev}}$$

114 •• A day-care center has a merry-go-round that consists of a uniform 240-kg circular wooden platform 4.00 m in diameter. Four children run alongside the merry-go-round and push tangentially along the platform's circumference until, starting from rest, the merry-go-round is spinning at 2.14 rev/min. During the spin up: (*a*) If each child exerts a sustained force equal to 26 N, how far does each child run? (*b*) What is the angular acceleration of the merry-go-round?

(*c*) How much work does each child do? (*d*) What is the increase in the kinetic energy of the merry-go-round?

Picture the Problem The work done by the four children on the merry-go-round will change its kinetic energy. We can use the work-energy theorem to relate the work done by the children to the distance they ran and Newton's second law to find the angular acceleration of the merry-go-round.

(*a*) Use the work-kinetic energy theorem to relate the work done by the children to the kinetic energy of the merry-go-round:

$$W_{\text{net force}} = \Delta K = K_{\text{f}}$$

or

$$4F\Delta s = \tfrac{1}{2}I\omega^2$$

Substitute for I to obtain:

$$4F\Delta s = \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\omega^2 = \tfrac{1}{4}mr^2\omega^2$$

Solving for Δs yields:

$$\Delta s = \frac{mr^2\omega^2}{16F}$$

Substitute numerical values and evaluate Δs:

$$\Delta s = \frac{(240\,\text{kg})(2.00\,\text{m})^2\left[2.14\,\frac{\text{rev}}{\text{min}}\times\frac{1\,\text{min}}{60\,\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right]^2}{16(26\,\text{N})}$$

$$= 0.116\,\text{m} = \boxed{0.12\,\text{m}}$$

(*b*) Apply Newton's second law to express the angular acceleration of the merry-go-round:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{4Fr}{\tfrac{1}{2}mr^2} = \frac{8F}{mr}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{8(26\,\text{N})}{(240\,\text{kg})(2.00\,\text{m})} = \boxed{0.43\,\text{rad/s}^2}$$

(*c*) Use the definition of work to relate the force exerted by each child to the distance over which that force is exerted:

$$W = F\Delta s = (26\,\text{N})(0.116\,\text{m}) = \boxed{3.0\,\text{J}}$$

(*d*) Relate the kinetic energy of the merry-go-round to the work that was done on it:

$$W_{\text{by net force}} = \Delta K = K_{\text{f}} - 0 = 4F\Delta s$$

Substitute numerical values and evaluate $W_{\text{by net force}}$:

$$W_{\text{by net force}} = 4(26\,\text{N})(0.116\,\text{m}) = \boxed{12\,\text{J}}$$

115 •• A uniform 1.5-kg hoop with a 65-cm radius has a string wrapped around its circumference and lies flat on a horizontal frictionless table. The free end of the string is pulled with a constant horizontal force equal to 5.0 N and the string does not slip on the hoop. (*a*) How far does the center of the hoop travel in 3.0 s? (*b*) What is the angular speed of the hoop after 3.0 s?

Picture the Problem Because the center of mass of the hoop is at its center, we can use Newton's second law to relate the acceleration of the hoop to the net force acting on it. The distance moved by the center of the hoop can be determined using a constant-acceleration equation, as can the angular speed of the hoop.

(*a*) Using a constant-acceleration equation, relate the distance the center of the hoop travels in 3.0 s to the acceleration of its center of mass:

$$\Delta s = \tfrac{1}{2} a_{\text{cm}} (\Delta t)^2$$

Relate the acceleration of the center of mass of the hoop to the net force acting on it:

$$a_{\text{cm}} = \frac{F_{\text{net}}}{m}$$

Substitute for a_{cm} to obtain:

$$\Delta s = \frac{F(\Delta t)^2}{2m}$$

Substitute numerical values and evaluate Δs:

$$\Delta s = \frac{(5.0\,\text{N})(3.0\,\text{s})^2}{2(1.5\,\text{kg})} = \boxed{15\,\text{m}}$$

(*b*) Relate the angular speed of the hoop to its angular acceleration and the elapsed time:

$$\omega = \alpha\,\Delta t$$

Use Newton's second law to relate the angular acceleration of the hoop to the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{FR}{mR^2} = \frac{F}{mR}$$

Substitute for α in the expression for ω to obtain:

$$\omega = \frac{F\Delta t}{mR}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{(5.0\,\text{N})(3.0\,\text{s})}{(1.5\,\text{kg})(0.65\,\text{m})} = \boxed{15\,\text{rad/s}}$$

116 •• A hand-driven grinding wheel is a uniform 60-kg disk with a 45-cm radius. It has a handle of negligible mass 65 cm from the rotation axis. A compact 25-kg load is attached to the handle when it is at the same height as the horizontal rotation axis. Ignoring the effects of friction, find (*a*) the initial angular acceleration of the wheel, and (*b*) the maximum angular speed of the wheel.

Picture the Problem Let R represent the radius of the grinding wheel, M its mass, r the radius of the handle, and m the mass of the load attached to the handle. In the absence of information to the contrary, we'll treat the 25-kg load as though it were concentrated at a point. Let the zero of gravitational potential energy be where the 25-kg load is at its lowest point. We'll apply Newton's second law and the conservation of mechanical energy to determine the initial angular acceleration and the maximum angular speed of the wheel.

(*a*) Use Newton's second law to relate the acceleration of the wheel to the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{mgr}{I}$$

The moment of inertia of the wheel-and-load system is the sum of their moments of inertia. Substituting for I yields:

$$\alpha = \frac{mgr}{\frac{1}{2}MR^2 + mr^2}$$

Substitute numerical values and evaluate α:

$$\alpha = \frac{(25\,\text{kg})(9.81\,\text{m/s}^2)(0.65\,\text{m})}{\frac{1}{2}(60\,\text{kg})(0.45\,\text{m})^2 + (25\,\text{kg})(0.65\,\text{m})^2} = \boxed{9.6\,\text{rad/s}^2}$$

(*b*) Use conservation of mechanical energy to relate the initial potential energy of the load to its kinetic energy and the rotational kinetic energy of the wheel when the load is directly below the center of mass of the wheel:

$$\Delta K + \Delta U = 0$$

or, because $K_{\text{i}} = U_{\text{f}} = 0$,

$$K_{\text{f,trans}} + K_{\text{f,rot}} - U_{\text{i}} = 0\,.$$

Substitute for $K_{\text{f,trans}}$, $K_{\text{f,rot}}$, and U_{i} to obtain:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\omega^2 - mgr = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{4mgr}{2mr^2 + MR^2}}$$

Substitute numerical values and evaluate ω:

$$\omega=\sqrt{\frac{4(25\,\text{kg})(9.81\,\text{m/s}^2)(0.65\,\text{m})}{2(25\,\text{kg})(0.65\,\text{m})^2+(60\,\text{kg})(0.45\,\text{m})^2}}=\boxed{4.4\,\text{rad/s}}$$

117 •• A uniform disk of radius R and mass M is pivoted about a horizontal axis parallel to its symmetry axis and passing through a point on its perimeter, so that it can swing freely in a vertical plane (Figure 9-72). It is released from rest with its center of mass at the same height as the pivot. (*a*) What is the angular speed of the disk when its center of mass is directly below the pivot? (*b*) What force is exerted by the pivot on the disk at this moment?

Picture the Problem Let the zero of gravitational potential energy be at the center of the disk when it is directly below the pivot. The initial gravitational potential energy of the disk is transformed into rotational kinetic energy when its center of mass is directly below the pivot. We can use Newton's second law to relate the force exerted by the pivot to the weight of the disk and the centripetal force acting on it at its lowest point.

(*a*) Use conservation of mechanical energy to relate the initial potential energy of the disk to its kinetic energy when its center of mass is directly below the pivot:

$$\Delta K+\Delta U=0$$

or, because $K_i = U_f = 0$,

$$K_{\text{f,rot}}-U_i=0$$

Substitute for $K_{\text{f,rot}}$ and U_i to obtain:

$$\tfrac{1}{2}I\omega^2-Mgr=0 \qquad (1)$$

Use the parallel-axis theorem to relate the moment of inertia of the disk about the pivot to its moment of inertia with respect to an axis through its center of mass:

$$I=I_{\text{cm}}+Mh^2$$

or

$$I=\tfrac{1}{2}Mr^2+Mr^2=\tfrac{3}{2}Mr^2$$

Substituting for I in equation (1) yields:

$$\tfrac{1}{2}\left(\tfrac{3}{2}Mr^2\right)\omega^2-Mgr=0\Rightarrow\omega=\boxed{\sqrt{\frac{4g}{3r}}}$$

(*b*) Letting F represent the force exerted by the pivot, use Newton's second law to express the net force acting on the swinging disk as it passes through its lowest point:

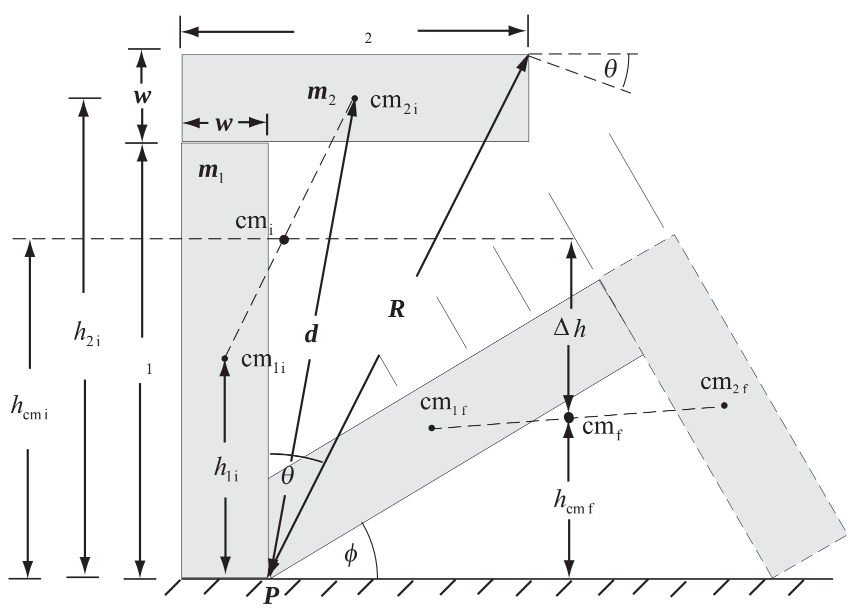
$$F_{\text{net}}=F-Mg=Mr\omega^2$$

Solve for F to obtain: $F = Mg + Mr\omega^2$

Substituting for ω^2 and simplifying yields: $F = Mg + Mr\dfrac{4g}{3r} = \boxed{\tfrac{7}{3}Mg}$

118 •• The roof of the student dining hall at your college will be supported by high cross-braced wooden beams attached in the shape of an upside-down *L* (Figure 9-73). Each vertical member is 10.0 ft high and 2.0 ft wide, and the horizontal cross-member is 8.0 ft long. The mass of the vertical beam is 350 kg, and the mass of the horizontal beam is 280 kg. As the workers were building the hall, one of these structures started to fall over before it was anchored into place. (Luckily, they stopped it before it fell.) (*a*) If it started falling from an upright position, what was the initial angular acceleration of the structure? Assume that the bottom did not slide across the floor and that it did not fall ″out of plane,″ that is, that during the fall the structure remained in the vertical plane defined by the initial position of the structure. (*b*) What would be the magnitude of the initial linear acceleration of the upper right corner of the horizontal beam? (*c*) What would the horizontal component of the initial linear acceleration be at this same location? (*d*) Estimate the structure's rotational speed at impact.

Picture the Problem The following pictorial representation shows one of the structures initially in its upright position and, later, as it is about to strike the floor. Because we need to take moments about the axis of rotation (a line through point *P*), we'll need to use the parallel-axis theorem to find the moments of inertia of the two parts of this composite structure. Let the numeral 1 denote the vertical member and the numeral 2 the horizontal member. We can apply Newton's second law in rotational form to the structure to express its angular acceleration in terms of the net torque causing it to fall and its moment of inertia with respect to point *P*.



(*a*) Taking clockwise rotation to be positive (this is the direction the structure is going to rotate), apply $\sum \tau = I_P \alpha$:

$$m_2 g\left(\tfrac{1}{2}\ell_2 - w\right) - m_1 g\left(\tfrac{1}{2}w\right) = I_P \alpha$$

Solving for α and substituting $I_{1P} + I_{2P}$ for I_P gives:

$$\alpha = \frac{\left[m_2\left(\tfrac{1}{2}\ell_2 - w\right) - \tfrac{1}{2}m_1 w\right]g}{I_{1P} + I_{2P}} \quad (1)$$

Using Table 9-1 and the parallel-axis theorem, express the moment of inertia of the vertical member about an axis through point *P*:

$$I_{1P} = \tfrac{1}{12}m_1\left(\ell_1^2 + w^2\right) + m_1\left[\left(\tfrac{1}{2}\ell_1\right)^2 + \left(\tfrac{1}{2}w\right)^2\right]$$
$$= \tfrac{1}{3}m_1\left(\ell_1^2 + w^2\right)$$

Substitute numerical values and evaluate I_{1P}:

$$I_{1P} = \tfrac{1}{3}(350\,\text{kg})\left[(10.0\,\text{ft})^2 + (2.0\,\text{ft})^2\right]\left(0.3048\frac{\text{m}}{\text{ft}}\right)^2 = 1.127\times10^3\,\text{kg}\cdot\text{m}^2$$

Using the parallel-axis theorem, express the moment of inertia of the horizontal member about an axis through point *P*:

$$I_{2P} = I_{2,\text{cm}} + m_2 d^2 \quad (2)$$

where

$$d^2 = \left(\ell_1 + \tfrac{1}{2}w\right)^2 + \left(\tfrac{1}{2}\ell_2 - w\right)^2$$

From Table 9-1 we have:

$$I_{2\,\text{cm}} = \tfrac{1}{12}m_2\left(\ell_2^2 + w^2\right)$$

Substitute for $I_{2\,\text{cm}}$ and d^2 in equation (2) and simplify to obtain:

$$I_{2P} = m_2\left[\tfrac{1}{3}\ell_2^2 + \tfrac{4}{3}w^2 + \ell_1^2 + \left(\ell_1 - \ell_2\right)w\right]$$

Substitute numerical values and evaluate I_{2P}:

$$I_{2P} = (280\,\text{kg})\left[\tfrac{1}{3}(8.0\,\text{ft})^2 + \tfrac{4}{3}(2.0\,\text{ft})^2 + (10.0\,\text{ft})^2 + (10.0\,\text{ft} - 8.0\,\text{ft})(2.0\,\text{ft})\right]$$
$$\times\left(0.3048\frac{\text{m}}{\text{ft}}\right)^2$$
$$= 3.399\times10^3\,\text{kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) and evaluate α:

$$\alpha = \frac{\left[(280\,\text{kg})\left[\tfrac{1}{2}(8.0\text{ ft}) - 2(2.0\text{ ft})\right] - \tfrac{1}{2}(350\,\text{kg})(2.0\text{ ft})\right]\left(9.81\,\text{m/s}^2\right)\left(0.3048\,\frac{\text{m}}{\text{ft}}\right)}{1.127\times10^3\ \text{kg}\cdot\text{m}^2 + 3.399\times10^3\ \text{kg}\cdot\text{m}^2}$$

$$= 0.1387\,\text{rad/s}^2 = \boxed{0.14\,\text{rad/s}^2}$$

(b) Express the magnitude of the acceleration of the top right-corner of the cross member:

$$a = \alpha R$$

Because $R = \sqrt{(\ell_1 + w)^2 + (\ell_2 - w)^2}$:

$$a = \alpha\sqrt{(\ell_1 + w)^2 + (\ell_2 - w)^2}$$

Substitute numerical values and evaluate a:

$$a = \left(0.1387\text{ rad/s}^2\right)\sqrt{(10.0\text{ ft} + 2.0\text{ ft})^2 + (8.0\text{ ft} - 2.0\text{ ft})^2}\left(0.3048\,\frac{\text{m}}{\text{ft}}\right)$$

$$= 0.5672\text{ m/s}^2 = \boxed{57\text{ cm/s}^2}$$

(c) a_x is given by:

$$a_x = a\cos\theta$$

Substitute for $\cos\theta$ (see the diagram) to obtain:

$$a_x = a\left(\frac{\ell_1 + w}{R}\right) = a\left(\frac{\ell_1 + w}{\sqrt{(\ell_1 + w)^2 + (\ell_2 - w)^2}}\right)$$

Substitute numerical values and evaluate a_x:

$$a_x = \left(0.5672\,\text{m/s}^2\right)\left(\frac{10.0\,\text{ft} + 2.0\text{ ft}}{\sqrt{(10.0\text{ ft} + 2.0\text{ ft})^2 + (8.0\text{ ft} - 2.0\text{ ft})^2}}\right) = 0.5073\,\text{m/s}^2$$

$$= \boxed{51\,\text{cm/s}^2}$$

(d) Letting the system include the beam and Earth, apply conservation of mechanical energy to the beam to obtain:

$$\Delta K + \Delta U_{\text{g}} = K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} = 0$$

where the subscript "f" refers to the instant just before it is caught and the subscript "i" refers to the instant just after it starts to move.

Let $U_g = 0$ at the location of the center of mass of the composite beam when it is about to hit the floor. Then:

$$K_f - U_i = 0$$

Substituting for K_f and U_i yields:

$$\tfrac{1}{2} I_P \omega_f^2 - (m_1 + m_2) g |\Delta h| = 0$$

where $\Delta h = h_{cm\,f} - h_{cm\,i}$ is the change in height of the center of mass of the composite beam.

Solving for ω_f gives:

$$\omega_f = \sqrt{\frac{2(m_1 + m_2) g |\Delta h|}{I_P}} = \sqrt{\frac{2(m_1 + m_2) g |\Delta h|}{I_{1P} + I_{2P}}} \qquad (3)$$

$h_{cm\,i}$ is given by:

$$h_{cm\,i} = \frac{\frac{1}{2}\ell_1 m_1 + \left(\ell_1 + \frac{1}{2}w\right) m_2}{m_1 + m_2}$$

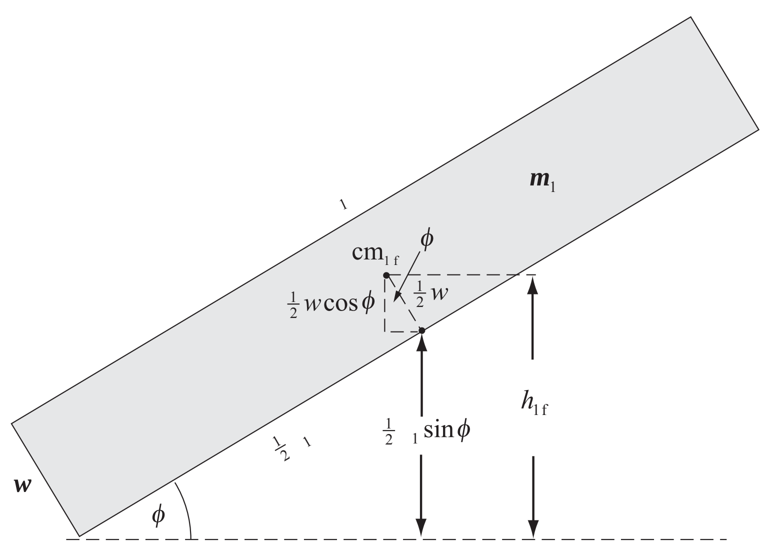
Substitute numerical values and evaluate $h_{cm\,i}$:

$$h_{cm\,i} = \frac{\left[\frac{1}{2}(10.0\text{ ft})(350\text{ kg}) + \left[10.0\text{ ft} + \frac{1}{2}(2.0\text{ ft})\right](280\text{ kg})\right]}{350\text{ kg} + 280\text{ kg}} = 7.667\text{ ft}$$

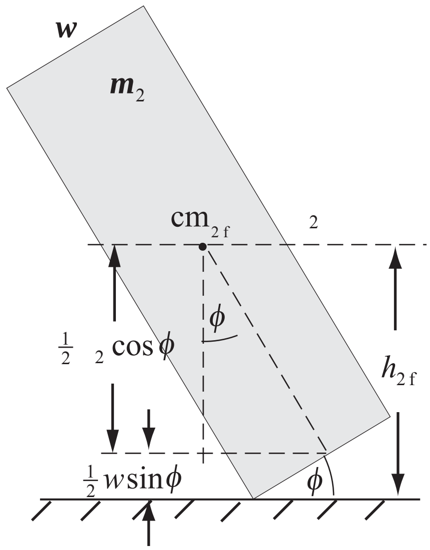
$h_{cm\,f}$ is given by:

$$h_{cm\,f} = \frac{h_{1f} m_1 + h_{2f} m_2}{m_1 + m_2}$$

Use the following diagram to show that $h_{1f} = \frac{1}{2}\ell_1 \sin\phi + \frac{1}{2} w \cos\phi$:



Use the following diagram to show that $h_{2\,\mathrm{f}} = \frac{1}{2}w\sin\phi + \frac{1}{2}\ell_2\cos\phi$.



Substituting for $h_{1\,\mathrm{f}}$ and $h_{2\,\mathrm{f}}$ gives:

$$h_{\mathrm{cm\,f}} = \frac{\left(\frac{1}{2}\ell_1\sin\phi + \frac{1}{2}w\cos\phi\right)m_1 + \left(\frac{1}{2}w\sin\phi + \frac{1}{2}\ell_2\cos\phi\right)m_2}{m_1 + m_2}$$

Noting that $\phi = \tan^{-1}\left(\dfrac{6.0\text{ ft}}{10.0\text{ ft}}\right) = 30.96°$, substitute numerical values and evaluate $h_{\mathrm{cm\,f}}$:

$$\begin{aligned} h_{\mathrm{cm\,f}} &= \frac{\left[\frac{1}{2}(10.0\text{ ft})\sin 30.96° + \frac{1}{2}(2.0\text{ ft})\cos 30.96°\right](350\text{ kg})}{350\text{ kg} + 280\text{ kg}} \\ &\quad + \frac{\left[\frac{1}{2}(2.0\text{ ft})\sin 30.96° + \frac{1}{2}(8.0\text{ ft})\cos 30.96°\right](280\text{ kg})}{350\text{ kg} + 280\text{ kg}} \\ &= 3.658\text{ ft} \end{aligned}$$

Substitute numerical values and evaluate $|\Delta h| = |h_{\mathrm{cm\,f}} - h_{\mathrm{cm\,i}}|$:

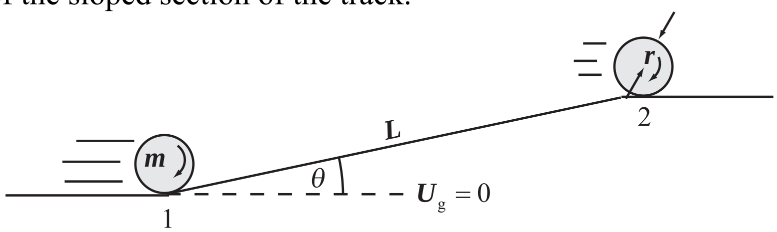
$$\begin{aligned} |\Delta h| &= |3.658\text{ ft} - 7.667\text{ ft}| = 4.009\text{ ft} \\ &= (4.009\text{ ft})\left(0.3048\frac{\text{m}}{\text{ft}}\right) = 1.222\text{ m} \end{aligned}$$

Substitute numerical values in equation (3) and evaluate ω_f:

$$\omega_\mathrm{f} = \sqrt{\frac{2(350\text{ kg} + 280\text{ kg})(9.81\text{ m/s}^2)(1.222\text{ m})}{1.127\times10^3\text{ kg}\cdot\text{m}^2 + 3.399\times10^3\text{ kg}\cdot\text{m}^2}} \approx \boxed{1.83\text{ rad/s}}$$

119 •• **[SSM]** You are participating in league bowling with your friends. Time after time, you notice that your bowling ball rolls back to you without slipping on the flat section of track. When the ball encounters the slope that brings it up to the ball return, it is moving at 3.70 m/s. The length of the sloped part of the track is 2.50 m. The radius of the bowling ball is 11.5 cm. (*a*) What is the angular speed of the ball before it encounters the slope? (*b*) If the speed with which the ball emerges at the top of the incline is 0.40 m/s, what is the angle (assumed constant), that the sloped section of the track makes with the horizontal? (*c*) What is the magnitude of the angular acceleration of the ball while it is on the slope?

Picture the Problem The pictorial representation shows the bowling ball slowing down as it rolls up the slope. Let the system include the ball, the incline, and Earth. Then $W_{ext} = 0$ and we can use conservation of mechanical energy to find the angle of the sloped section of the track.



(*a*) Because the bowling ball rolls without slipping, its angular speed is directly proportional to its linear speed:

$$\omega = \frac{v}{r}$$

where r is the radius of the bowling ball.

Substitute numerical values and evaluate ω:

$$\omega = \frac{3.70\ \text{m/s}}{0.115\ \text{m}} = 32.17\ \text{rad/s} = \boxed{32.2\ \text{rad/s}}$$

(*b*) Apply conservation of mechanical energy to the system as the bowling ball rolls up the incline:

$$W_{ext} = \Delta K + \Delta U$$

or, because $W_{ext} = 0$,

$$K_{t,2} - K_{t,1} + K_{r,2} - K_{r,1} + U_2 - U_1 = 0$$

Substituting for the kinetic and potential energies yields:

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}I_{ball}\omega_2^2 - \tfrac{1}{2}I_{ball}\omega_1^2 + mgL\sin\theta = 0$$

Solving for θ yields:

$$\theta = \sin^{-1}\left[\frac{m\left(v_1^2 - v_2^2\right) + I_{ball}\left(\omega_1^2 - \omega_2^2\right)}{2mgL}\right]$$

Because $I_{\text{ball}} = \frac{2}{5}mr^2$:

$$\theta = \sin^{-1}\left[\frac{m\left(v_1^2 - v_2^2\right) + \frac{2}{5}mr^2\left(\omega_1^2 - \omega_2^2\right)}{2mgL}\right]$$
$$= \sin^{-1}\left[\frac{7\left(v_1^2 - v_2^2\right)}{10gL}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{7\left((3.70\text{ m/s})^2 - (0.40\text{ m/s})^2\right)}{10\left(9.81\text{ m/s}^2\right)(2.50\text{ m})}\right]$$
$$= \boxed{23°}$$

(*c*) The angular acceleration of the bowling ball is directly proportional to its translational acceleration:

$$\alpha = \frac{a}{r} \qquad (1)$$

Use a constant-acceleration equation to relate the speeds of the ball at points 1 and 2 to its acceleration:

$$v_2^2 = v_1^2 + 2aL \Rightarrow a = \frac{v_2^2 - v_1^2}{2L}$$

Substitute in equation (1) to obtain:

$$\alpha = \frac{v_2^2 - v_1^2}{2rL}$$

Substitute numerical values and evaluate $|\alpha|$:

$$|\alpha| = \left|\frac{(0.40\text{ m/s})^2 - (3.70\text{ m/s})^2}{2(0.115\text{ m})(2.50\text{ m})}\right|$$
$$= \boxed{24\text{ rad/s}^2}$$

120 •• Figure 9-74 shows a hollow cylinder that has a length equal to 1.80 m, a mass equal to 0.80 kg, and radius equal to 0.20 m. The cylinder is free to rotate about a vertical axis that passes through its center and is perpendicular to the cylinder. Two objects are inside the cylinder. Each object has a mass equal to 0.20 kg, and is attached to a spring that has a force constant *k* and has an unstressed length equal to 0.40 m. The inside walls of the cylinder are frictionless. (*a*) Determine the value of the force constant if the objects are located 0.80 m from the center of the cylinder when the cylinder rotates at 24 rad/s. (*b*) How much work is required to bring the system from rest to an angular speed of 24 rad/s?

Picture the Problem Let *m* represent the mass of the 0.20-kg objects, *M* the mass of the 0.80-kg hollow cylinder, *L* the 1.8-m length, and $x + \Delta x$ the distance from the center of the objects whose mass is *m*. We can use Newton's second law to relate the radial forces on the masses to the spring's force constant and use the work-energy theorem to find the work done as the system accelerates to its final angular speed.

(*a*) Express the net inward force acting on each of the 0.2-kg masses:

$$\sum F_{\text{radial}} = k\Delta x = m(x + \Delta x)\omega^2$$

Solving for *k* yields:

$$k = \frac{m(x + \Delta x)\omega^2}{\Delta x}$$

Substitute numerical values and evaluate *k*:

$$k = \frac{(0.20\,\text{kg})(0.80\,\text{m})(24\,\text{rad/s})^2}{0.40\,\text{m}} = 230\,\text{N/m} = \boxed{0.23\,\text{kN/m}}$$

(*b*) Using the work-energy theorem, relate the work done to the change in energy of the system:

$$W = K_{\text{rot}} + \Delta U_{\text{spring}} = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}k(\Delta x)^2 \qquad (1)$$

From Table 9-1 we have, for a cylindrical shell about a diameter through its center:

$$I = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2$$

where L is the length of the cylinder.

Express I as the sum of the moments of inertia of the cylindrical shell and the masses:

$$I = I_M + I_{2m} = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2I_m$$

For a disk (thin cylinder), L is small and:

$$I = \tfrac{1}{4}mr^2$$

Apply the parallel-axis theorem to obtain:

$$I_m = \tfrac{1}{4}mr^2 + mx^2$$

Substitute to obtain:

$$I = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2\left(\tfrac{1}{4}mr^2 + mx^2\right) = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2m\left(\tfrac{1}{4}r^2 + x^2\right)$$

Substitute numerical values and evaluate I:

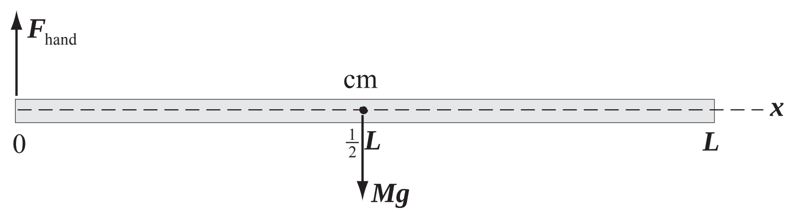
$$I = \tfrac{1}{2}(0.80\,\text{kg})(0.20\,\text{m})^2 + \tfrac{1}{12}(0.80\,\text{kg})(1.8\,\text{m})^2 + 2(0.20\,\text{kg})\left[\tfrac{1}{4}(0.20\,\text{m})^2 + (0.80\,\text{m})^2\right] = 0.492\,\text{N}\cdot\text{m}^2$$

Substitute in equation (1) to obtain:

$$W = \tfrac{1}{2}(0.492\,\text{N}\cdot\text{m}^2)(24\,\text{rad/s})^2 + \tfrac{1}{2}(230\,\text{N/m})(0.40\,\text{m})^2 = \boxed{0.16\,\text{kJ}}$$

121 •• **[SSM]** A popular classroom demonstration involves taking a meterstick and holding it horizontally at the 0.0-cm end with a number of pennies spaced evenly along its surface. If the hand is suddenly relaxed so that the meterstick pivots freely about the 0.0-cm mark under the influence of gravity, an interesting thing is seen during the first part of the stick's rotation: the pennies nearest the 0.0-cm mark remain on the meterstick, while those nearest the 100-cm mark are left behind by the falling meterstick. (This demonstration is often called the ″faster than gravity″ demonstration.) Suppose this demonstration is repeated without any pennies on the meterstick. (*a*) What would the initial acceleration of the 100.0-cm mark then be? (The initial acceleration is the acceleration just after the release.) (*b*) What point on the meterstick would then have an initial acceleration greater than *g*?

Picture the Problem The diagram shows the force the hand supporting the meterstick exerts at the pivot point and the force Earth exerts on the meterstick acting at the center of mass. We can relate the angular acceleration to the acceleration of the end of the meterstick using $a = L\alpha$ and use Newton's second law in rotational form to relate α to the moment of inertia of the meterstick.



(*a*) Relate the acceleration of the far end of the meterstick to the angular acceleration of the meterstick:

$$a = L\alpha \qquad (1)$$

Apply $\sum \tau_P = I_P\alpha$ to the meterstick:

$$Mg\left(\frac{L}{2}\right) = I_P\alpha \Rightarrow \alpha = \frac{MgL}{2I_P}$$

From Table 9-1, for a rod pivoted at one end, we have:

$$I_P = \frac{1}{3}ML^2$$

Substitute for I_P in the expression for α to obtain:

$$\alpha = \frac{3MgL}{2ML^2} = \frac{3g}{2L}$$

Substitute for α in equation (1) to obtain:

$$a = \frac{3g}{2}$$

Substitute numerical values and evaluate *a*:

$$a = \frac{3\left(9.81\,\text{m/s}^2\right)}{2} = \boxed{14.7\,\text{m/s}^2}$$

(*b*) Express the acceleration of a point on the meterstick a distance x from the pivot point:

$$a = \alpha x = \frac{3g}{2L}x$$

Express the condition that the meterstick have an initial acceleration greater than g:

$$\frac{3g}{2L}x > g \Rightarrow x > \frac{2L}{3}$$

Substitute the numerical value of L and evaluate x:

$$x > \frac{2(100.0\,\text{cm})}{3} = \boxed{66.7\,\text{cm}}$$

122 •• A solid metal rod, 1.5 m long, is free to pivot without friction about a fixed horizontal axis perpendicular to the rod and passing through one of its ends. The rod is held in a horizontal position. Small coins, each of mass *m,* are placed on the rod 25 cm, 50 cm, 75 cm, 1 m, 1.25 m, and 1.5 m from the pivot. If the free end is now released, calculate the initial force exerted on each coin by the rod. Assume that the masses of the coins can be neglected in comparison to the mass of the rod.

Picture the Problem While the angular acceleration of the rod is the same at each point along its length, the linear acceleration and, hence, the force exerted on each coin by the rod, varies along its length. We can relate this force the linear acceleration of the rod through Newton's second law and the angular acceleration of the rod.

Letting x be the distance from the pivot, use Newton's second law to express the force F acting on a coin:

$$F_{\text{net}} = mg - F(x) = ma(x)$$

or

$$F(x) = m(g - a(x)) \qquad (1)$$

Use Newton's second law to relate the angular acceleration of the system to the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{Mg\frac{L}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

Relate $a(x)$ and α:

$$a(x) = x\alpha = x\frac{3g}{2(1.5\,\text{m})} = gx$$

Substitute in equation (1) to obtain:

$$F(x) = m(g - gx) = mg(1 - x)$$

Evaluate $F(0.25\text{ m})$:

$$F(0.25\,\text{m}) = mg(1 - 0.25\,\text{m}) = \boxed{0.75mg}$$

Evaluate $F(0.50\text{ m})$:

$$F(0.50\,\text{m}) = mg(1 - 0.50\,\text{m}) = \boxed{0.50mg}$$

Evaluate $F(0.75\text{ m})$:

$$F(0.75\,\text{m}) = mg(1 - 0.75\,\text{m}) = \boxed{0.25mg}$$

Evaluate $F(1.0\text{ m})$:

$$F(1.0\,\text{m}) = F(1.25\,\text{m}) = F(1.5\,\text{m}) = \boxed{0}$$

123 •• Suppose that for the system described in Problem 120, the force constants are each 60 N/m. The system starts from rest and slowly accelerates until the masses are 0.80 m from the center of the cylinder. How much work was done in the process?

Picture the Problem Let m represent the mass of the 0.20-kg cylinder, M the mass of the 0.80-kg cylinder, L the 1.80-m length, and $x + \Delta x$ the distance from the center of the objects whose mass is m. We can use Newton's second law to relate the radial forces on the masses to the spring's force constant and use the work-energy theorem to find the work done as the system accelerates to its final angular speed.

Using the work-energy theorem, relate the work done to the change in energy of the system:

$$W = K_{\text{rot}} + \Delta U_{\text{spring}} = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}k(\Delta x)^2 \qquad (1)$$

From Table 9-1 we have, for a cylindrical shell about a diameter through its center:

$$I = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2$$

where L is the length of the cylinder.

Express I as the sum of the moments of inertia of the cylinder and the masses:

$$I = I_M + I_{2m} = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2I_m$$

For a disk (thin cylinder), L is small and:

$$I = \tfrac{1}{4}mr^2$$

Apply the parallel-axis theorem to obtain:

$$I_m = \tfrac{1}{4}mr^2 + mx^2$$

Substitute for the moments of inertia and simplify to obtain:

$$I = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2\left(\tfrac{1}{4}mr^2 + mx^2\right) = \tfrac{1}{2}Mr^2 + \tfrac{1}{12}ML^2 + 2m\left(\tfrac{1}{4}r^2 + x^2\right)$$

Substitute numerical values and evaluate I:

$$I = \tfrac{1}{2}(0.80\,\text{kg})(0.20\,\text{m})^2 + \tfrac{1}{12}(0.80\,\text{kg})(1.80\,\text{m})^2 + 2(0.20\,\text{kg})\left[\tfrac{1}{4}(0.20\,\text{m})^2 + (0.80\,\text{m})^2\right]$$
$$= 0.492\,\text{N}\cdot\text{m}^2$$

Express the net inward force acting on each of the 0.2-kg masses:

$$\sum F_{\text{radial}} = k\Delta x = m(x+\Delta x)\omega^2$$

Solving for ω yields:

$$\omega = \sqrt{\frac{k\Delta x}{m(x+\Delta x)}}$$

Substitute numerical values and evaluate ω:

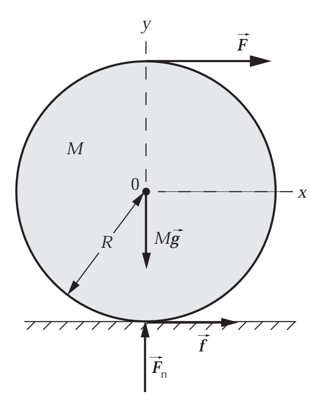
$$\omega = \sqrt{\frac{(60\,\text{N/m})(0.40\,\text{m})}{(0.20\,\text{kg})(0.80\,\text{m})}} = 12.25\,\text{rad/s}$$

Substitute numerical values in equation (1) and evaluate W:

$$W = \tfrac{1}{2}(0.492\,\text{N}\cdot\text{m}^2)(12.25\,\text{rad/s})^2 + \tfrac{1}{2}(60\,\text{N/m})(0.40\,\text{m})^2 = \boxed{42\,\text{J}}$$

124 ••• A string is wrapped around a uniform solid cylinder of radius R and mass M that rests on a horizontal frictionless surface (The string does not touch the surface because there is a groove cut in the surface to provide space for the string to clear). The string is pulled horizontally from the top with force F. (*a*) Show that the magnitude of the angular acceleration of the cylinder is twice the magnitude of the angular acceleration needed for rolling without slipping, so that the bottom point on the cylinder slides backward against the table. (*b*) Find the magnitude and direction of the frictional force between the table and cylinder that would be needed for the cylinder to roll without slipping. What would be the magnitude of the acceleration of the cylinder in this case?

Picture the Problem The force diagram shows the forces acting on the cylinder. Because F causes the cylinder to rotate clockwise, f, which opposes this motion, is to the right. We can use Newton's second law in both translational and rotational forms to relate the linear and angular accelerations to the forces acting on the cylinder.



(*a*) Use Newton's second law to relate the angular acceleration of the center of mass of the cylinder to *F*:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{FR}{\frac{1}{2}MR^2} = \frac{2F}{MR}$$

Use Newton's second law to relate the acceleration of the center of mass of the cylinder to *F*:

$$a_{\text{cm}} = \frac{F_{\text{net}}}{M} = \frac{F}{M}$$

Apply the rolling-without-slipping condition to the linear and angular accelerations:

$$\alpha' = \frac{a_{\text{cm}}}{R}$$

Substituting for a_{cm} yields:

$$\alpha' = \frac{F}{MR} = \boxed{\tfrac{1}{2}\alpha}$$

(*b*) Take the point of contact with the floor as the ″pivot″ point, express the net torque about that point:

$$\tau_{\text{net}} = 2FR = I\alpha \Rightarrow \alpha = \frac{2FR}{I} \qquad (1)$$

Express the moment of inertia of the cylinder with respect to the pivot point:

$$I = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

Substitute for *I* in equation (1) to obtain:

$$\alpha = \frac{2FR}{\frac{3}{2}MR^2} = \frac{4F}{3MR}$$

The linear acceleration of the cylinder is:

$$a_{\text{cm}} = R\alpha = \boxed{\frac{4F}{3M}}$$

Apply Newton's second law to the cylinder to obtain:

$$\sum F_x = F + f = Ma_{\text{cm}}$$

Solving for *f* and substituting for a_{cm} yields:

$$f = Ma_{\text{cm}} - F = \frac{4F}{3} - F$$

$$= \boxed{\tfrac{1}{3}F \text{ in the } +x \text{ direction.}}$$

125 ••• **[SSM]** Let's calculate the position *y* of the falling load attached to the winch in Example 9-8 as a function of time by numerical integration. Let the +*y* direction be straight downward. Then, $v(y) = dy/dt$, or

$$t=\int_0^y \frac{1}{v(y')}dy' \approx \sum_{i=0}^{N} \frac{1}{v(y_i')}\Delta y'$$

where t is the time taken for the bucket to fall a distance y, $\Delta y'$ is a small increment of y', and $y' = N\Delta y'$. Hence, we can calculate t as a function of d by numerical summation. Make a graph of y versus t between 0 s and 2.00 s. Assume m_w = 10.0 kg, R = 0.50 m, m_b = 5.0 kg, L = 10.0 m, and m_c = 3.50 kg. Use $\Delta y'$ = 0.10 m. Compare this position to the position of the falling load if it were in free-fall.

Picture the Problem As the load falls, mechanical energy is conserved. As in Example 9-7, choose the initial potential energy to be zero and let the system include the winch, the bucket, and Earth. Apply conservation of mechanical energy to obtain an expression for the speed of the bucket as a function of its position and use the given expression for t to determine the time required for the bucket to travel a distance y.

Apply conservation of mechanical energy to the system to obtain:

$$U_f + K_f = U_i + K_i = 0 + 0 = 0 \qquad (1)$$

Express the total potential energy when the bucket has fallen a distance y:

$$U_f = U_{bf} + U_{cf} + U_{wf} = -mgy - m_c'g\left(\frac{y}{2}\right)$$

where m_c' is the mass of the hanging part of the cable.

Assume the cable is uniform and express m_c' in terms of m_c, y, and L:

$$\frac{m_c'}{y} = \frac{m_c}{L} \text{ or } m_c' = \frac{m_c}{L}y$$

Substitute for m_c' to obtain:

$$U_f = -mgy - \frac{m_c g y^2}{2L}$$

Noting that bucket, cable, and rim of the winch have the same speed v, express the total kinetic energy when the bucket is falling with speed v:

$$\begin{aligned} K_f &= K_{bf} + K_{cf} + K_{wf} \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}m_c v^2 + \tfrac{1}{2}I\omega_f^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}m_c v^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\frac{v^2}{R^2} \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}m_c v^2 + \tfrac{1}{4}Mv^2 \end{aligned}$$

Substituting in equation (1) yields:

$$-mgy - \frac{m_c g y^2}{2L} + \tfrac{1}{2}mv^2 + \tfrac{1}{2}m_c v^2 + \tfrac{1}{4}Mv^2 = 0$$

Solving for v yields:

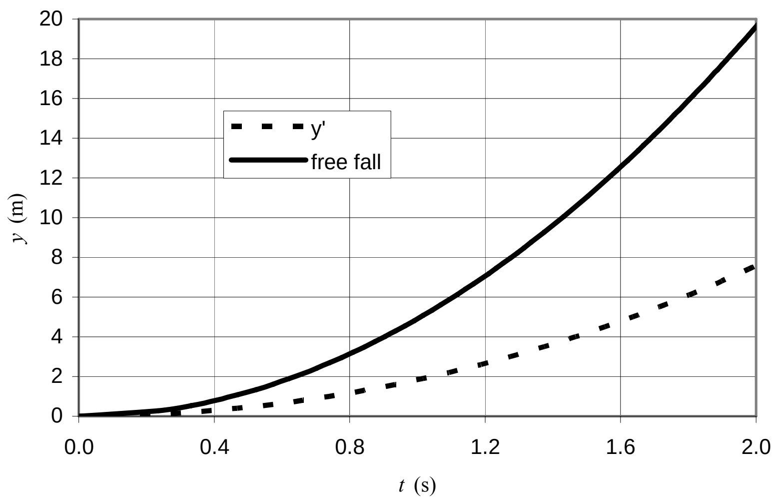
$$v = \sqrt{\frac{4mgy + \frac{2m_c gy^2}{L}}{M + 2m + 2m_c}}$$

A spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
D9	0	y_0
D10	D9+$B$8	$y + \Delta y$
E9	0	v_0
E10	((4*$B$3*$B$7*D10+2*$B$7*D10^2/(2*$B$5))/ ($B$1+2*$B$3+2*$B$4))^0.5	$\sqrt{\frac{4mgy + \frac{2m_c gy^2}{L}}{M + 2m + 2m_c}}$
F10	F9+$B$8/((E10+E9)/2)	$t_{n-1} + \left(\frac{v_{n-1} + v_n}{2}\right)\Delta y$
J9	0.5*$B$7*H9^2	$\frac{1}{2}gt^2$

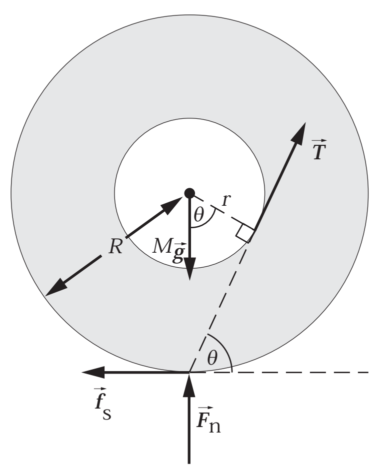
	A	B	C	D	E	F	G	H	I	J
1	M=	10	kg							
2	R=	0.5	m							
3	m=	5	kg							
4	m_c=	3.5	kg							
5	L=	10	m							
6										
7	g=	9.81	m/s^2							
8	dy=	0.1	m	y	$v(y)$	$t(y)$		$t(y)$	y	$0.5gt^2$
9				0.0	0.00	0.00		0.00	0.0	0.00
10				0.1	0.85	0.23		0.23	0.1	0.27
11				0.2	1.21	0.33		0.33	0.2	0.54
12				0.3	1.48	0.41		0.41	0.3	0.81
13				0.4	1.71	0.47		0.47	0.4	1.08
15				0.5	1.91	0.52		0.52	0.5	1.35
105				9.6	9.03	2.24		2.24	9.6	24.61
106				9.7	9.08	2.25		2.25	9.7	24.85
107				9.8	9.13	2.26		2.26	9.8	25.09
108				9.9	9.19	2.27		2.27	9.9	25.34
109				10.0	9.24	2.28		2.28	10.0	25.58

The solid line on the following graph shows the position of the bucket as a function of time when it is in free fall and the dashed line shows its position as a function of time under the conditions modeled in this problem.



126 ••• Figure 9-75 shows a solid cylinder that has mass M and radius R to which a second solid cylinder that has mass m and radius r is attached. A string is wound about the smaller cylinder. The larger cylinder rests on a horizontal surface. The coefficient of static friction between the larger cylinder and the surface is μ_s. If a light tension is applied to the string in the vertical direction, the cylinder will roll to the left; if the tension is applied with the string horizontally to the right, the cylinder rolls to the right. Find the angle between the string and the horizontal that will allow the cylinder to remain stationary when a light tension is applied to the string.

Picture the Problem The pictorial representation shows the forces acting on the cylinder when it is stationary. First, we note that if the tension is small, then there can be no slipping, and the system must roll. Now consider the point of contact of the cylinder with the surface as the "pivot" point. If τ about that point is zero, the system will not roll. This will occur if the line of action of the tension passes through the pivot point.

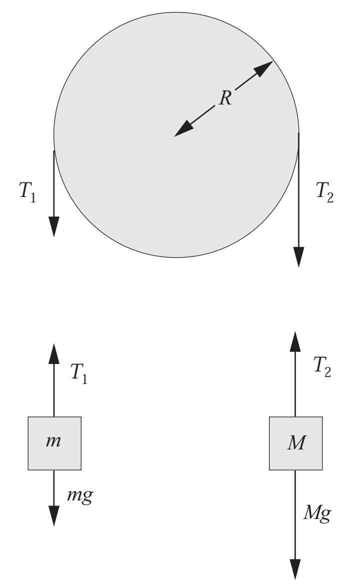


From the diagram we see that:

$$\theta = \boxed{\cos^{-1}\left(\frac{r}{R}\right)}$$

127 ••• [SSM] In problems dealing with a pulley with a nonzero moment of inertia, the magnitude of the tensions in the ropes hanging on either side of the pulley are not equal. The difference in the tension is due to the static frictional force between the rope and the pulley; however, the static frictional force cannot be made arbitrarily large. Consider a massless rope wrapped partly around a cylinder through an angle $\Delta\theta$ (measured in radians). It can be shown that if the tension on one side of the pulley is T, while the tension on the other side is T' ($T' > T$), the maximum value of T' that can be maintained without the rope slipping is $T'_{\text{max}} = Te^{\mu_s \Delta\theta}$, where μ_s is the coefficient of static friction. Consider the Atwood's machine in Figure 9-77: the pulley has a radius $R = 0.15$ m, the moment of inertia is $I = 0.35\ \text{kg}\cdot\text{m}^2$, and the coefficient of static friction between the wheel and the string is $\mu_s = 0.30$. (*a*) If the tension on one side of the pulley is 10 N, what is the maximum tension on the other side that will prevent the rope from slipping on the pulley? (*b*) What is the acceleration of the blocks in this case? (*c*) If the mass of one of the hanging blocks is 1.0 kg, what is the maximum mass of the other block if, after the blocks are released, the pulley is to rotate without slipping?

Picture the Problem Free-body diagrams for the pulley and the two blocks are shown to the right. Choose a coordinate system in which the direction of motion of the block whose mass is M (downward) is the $+y$ direction. We can use the given relationship $T'_{\text{max}} = Te^{\mu_s \Delta\theta}$ to relate the tensions in the rope on either side of the pulley and apply Newton's second law in both rotational form (to the pulley) and translational form (to the blocks) to obtain a system of equations that we can solve simultaneously for a, T_1, T_2, and M.



(*a*) Use $T'_{\text{max}} = Te^{\mu_s \Delta\theta}$ to evaluate the maximum tension required to prevent the rope from slipping on the pulley:

$$T'_{\text{max}} = (10\ \text{N})e^{(0.30)\pi} = 25.66\ \text{N} = \boxed{26\ \text{N}}$$

(*b*) Given that the angle of wrap is π radians, express T_2 in terms of T_1:

$$T_2 = T_1 e^{(0.30)\pi} \qquad (1)$$

Because the rope doesn't slip, we can relate the angular acceleration, α, of the pulley to the acceleration, a, of the hanging masses by:

$$\alpha = \frac{a}{R}$$

Apply $\sum \tau = I\alpha$ to the pulley to obtain:

$$(T_2 - T_1)R = I\frac{a}{R} \qquad (2)$$

Substitute for T_2 from equation (1) in equation (2) to obtain:

$$\left(T_1 e^{(0.30)\pi} - T_1\right)R = I\frac{a}{R}$$

Solving for T_1 yields:

$$T_1 = \frac{I}{\left(e^{(0.30)\pi} - 1\right)R^2}a$$

Apply $\sum F_y = ma_y$ to the block whose mass is m to obtain:

$$T_1 - mg = ma$$

and

$$T_1 = ma + mg \qquad (3)$$

Equating these two expressions for T_1 and solving for a yields:

$$a = \frac{g}{\dfrac{I}{\left(e^{(0.30)\pi} - 1\right)mR^2} - 1}$$

Substitute numerical values and evaluate a:

$$a = \frac{9.81\,\text{m/s}^2}{\dfrac{0.35\,\text{kg}\cdot\text{m}^2}{\left(e^{(0.30)\pi} - 1\right)(1.0\,\text{kg})(0.15\,\text{m})^2} - 1}$$

$$= 1.098\,\text{m/s}^2 = \boxed{1.1\,\text{m/s}^2}$$

(*c*) Apply $\sum F_y = ma_y$ to the block whose mass is M to obtain:

$$Mg - T_2 = Ma \Rightarrow M = \frac{T_2}{g - a}$$

Substitute for T_2 (from equation (1)) and T1 (from equation (3)) yields:

$$M = \frac{m(a + g)e^{(0.30)\pi}}{g - a}$$

Substitute numerical values and evaluate M:

$$M = \frac{(1.0\,\text{kg})\left(1.098\,\text{m/s}^2 + 9.81\,\text{m/s}^2\right)e^{(0.30)\pi}}{9.81\,\text{m/s}^2 - 1.098\,\text{m/s}^2} = \boxed{3.2\,\text{kg}}$$

128 ••• A massive, uniform cylinder has a mass m and a radius R (Figure 9-77). It is accelerated by a tension force $\vec{T}$ that is applied through a rope wound around a light drum of radius r that is attached to the cylinder. The coefficient of static friction is sufficient for the cylinder to roll without slipping. (*a*) Find the frictional force. (*b*) Find the acceleration a of the center of the cylinder. (*c*) Show that it is possible to choose r so that a is greater than T/m. (*d*) What is the direction of the frictional force in the circumstances of Part (*c*)?

Picture the Problem When the tension is horizontal, the cylinder will roll forward and the friction force will be in the direction of $\vec{T}$. We can use Newton's second law to obtain equations that we can solve simultaneously for a and f.

(*a*) Apply Newton's second law to the cylinder:

$$\sum F_x = T + f = ma \qquad (1)$$

and

$$\sum \tau = Tr - fR = I\alpha \qquad (2)$$

Substitute for I and α in equation (2) to obtain:

$$Tr - fR = \tfrac{1}{2}mR^2\frac{a}{R} = \tfrac{1}{2}mRa \qquad (3)$$

Solve equation (3) for f:

$$f = \frac{Tr}{R} - \tfrac{1}{2}ma \qquad (4)$$

Substitute equation (4) in equation (1) to obtain:

$$T + \frac{Tr}{R} - \tfrac{1}{2}ma = ma$$

Solving for a gives:

$$a = \frac{2T}{3m}\left(1 + \frac{r}{R}\right) \qquad (5)$$

Substitute equation (5) in equation (4) to obtain:

$$f = \frac{Tr}{R} - \frac{1}{2}m\left[\frac{2T}{3m}\left(1 + \frac{r}{R}\right)\right]$$

Simplifying yields:

$$f = \boxed{\frac{T}{3}\left(\frac{2r}{R} - 1\right)}$$

(*b*) Solve equation (4) for a:

$$a = \frac{2\left(f - \frac{Tr}{R}\right)}{m}$$

Substituting for f yields:

$$a = \frac{2\left(\frac{T}{3}\left(\frac{2r}{R} - 1\right) - \frac{Tr}{R}\right)}{m}$$

$$= \boxed{-\frac{2T}{3m}\left(1 + \frac{r}{R}\right)}$$

(c) Express the condition that $a > \frac{T}{m}$:

$$\frac{2T}{3m}\left(1+\frac{r}{R}\right) > \frac{T}{m} \Rightarrow \frac{2}{3}\left(1+\frac{r}{R}\right) > 1$$

or

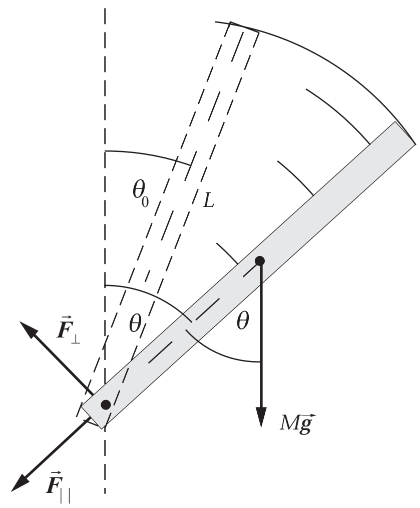
$$r > \boxed{\tfrac{1}{2}R}$$

(d) If $r > \frac{1}{2}R$:

$$\boxed{f > 0, \text{i.e., in the direction of } \vec{\boldsymbol{T}}.}$$

129 ••• A uniform rod that has a length L and a mass M is free to rotate about a horizontal axis through one end, as shown in Figure 9-78. The rod is released from rest at $\theta = \theta_0$. Show that the parallel and perpendicular components of the force exerted by the axis on the rod are given by $F_{\|} = \frac{1}{2}Mg(5\cos\theta - 3\cos\theta_0)$ and $F_{\perp} = \frac{1}{4}Mg\sin\theta$, where $F_{\|}$ is the component parallel with the rod and $F_{\perp}$ is the component perpendicular to the rod.

Picture the Problem The system is shown in the drawing in two positions, with angles θ_0 and θ with the vertical. The drawing also shows all the forces that act on the rod. These forces result in a rotation of the rod—and its center of mass—about the pivot, and a tangential acceleration of the center of mass. We'll apply the conservation of mechanical energy and Newton's second law to relate the radial and tangential forces acting on the stick.



Apply conservation of mechanical energy to relate the kinetic energy of the stick when it makes an angle θ with the vertical to its initial potential energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substituting for K_f, U_f and U_i gives:

$$\tfrac{1}{2}I\omega^2 + Mg\frac{L}{2}\cos\theta - Mg\frac{L}{2}\cos\theta_0 = 0$$

Substitute for I to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML^2\right)\omega^2 + Mg\frac{L}{2}\cos\theta - Mg\frac{L}{2}\cos\theta_0 = 0$$

Solving for ω^2 yields:

$$\omega^2 = \frac{3g}{L}\left(\cos\theta_0 - \cos\theta\right)$$

Applying Newton's second law in the centripetal direction gives:

$$F_{\parallel} + F_{\text{g c}} = M\frac{L}{2}\omega^2$$

where toward the pivot is the positive centripetal direction.

Substituting for the gravitational force component yields:

$$F_{\parallel} + Mg\cos\theta = M\frac{L}{2}\omega^2$$

or, solving for $F_{\parallel}$,

$$F_{\parallel} = M\frac{L}{2}\omega^2 - Mg\cos\theta$$

Substitute for ω^2 and simplify to obtain:

$$\begin{aligned} F_{\parallel} &= M\frac{L}{2}\left[\frac{3g}{L}\left(\cos\theta_0 - \cos\theta\right)\right] - Mg\cos\theta \\ &= \tfrac{1}{2}Mg\left[\left(3\cos\theta_0 - 3\cos\theta\right)\right] - Mg\cos\theta \\ &= \tfrac{1}{2}Mg\left[\left(3\cos\theta_0 - 3\cos\theta - 2\cos\theta\right)\right] \\ &= \boxed{-\tfrac{1}{2}Mg\left[\left(5\cos\theta - 3\cos\theta_0\right)\right]} \end{aligned}$$

where the minus sign is a consequence of our choice of the positive direction for the centripetal force.

Relate the tangential acceleration of the center of mass to its angular acceleration:

$$a_{\perp} = \tfrac{1}{2}L\alpha$$

Use Newton's second law to relate the angular acceleration of the stick to the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{Mg\frac{L}{2}\sin\theta}{\frac{1}{3}ML^2} = \frac{3g\sin\theta}{2L}$$

Express $a_{\perp}$ in terms of α:⊥

$$a_{\perp} = \tfrac{1}{2}L\alpha = \tfrac{3}{4}g\sin\theta = g\sin\theta + \frac{F_{\perp}}{M}$$

Solving for $F_{\perp}$ yields:

$F_{\perp} = \boxed{-\tfrac{1}{4}Mg\sin\theta}$ where the minus sign indicates that the force is directed oppositely to the tangential component of $M\vec{g}$.

Chapter 10
Angular Momentum

Conceptual Problems

1 • True or false:

(*a*) If two vectors are exactly opposite in direction, their vector product must be zero.
(*b*) The magnitude of the vector product of two vectors is at a minimum when the two vectors are perpendicular.
(*c*) Knowing the magnitude of the vector product of two nonzero vectors and the vectors′ individual magnitudes uniquely determines the angle between them.

Determine the Concept The vector product of $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$ where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$ and ϕ is the angle between $\vec{A}$ and $\vec{B}$.

(*a*) True. If $\vec{A}$ and $\vec{B}$ are in opposite direction, then $\sin\phi = \sin 180° = 0$.

(*b*) False. If $\vec{A}$ and $\vec{B}$ are perpendicular, then $\sin\phi = \sin 90° = 1$ and the vector product of $\vec{A}$ and $\vec{B}$ is a maximum.

(*c*) False. Knowing the magnitude of the vector product and the vectors′ individual magnitudes only gives the magnitude of the sine of the angle between the vectors. It does not determine the angle uniquely, nor does this knowledge tell us if the sine of the angle is positive or negative.

2 • Consider two nonzero vectors $\vec{A}$ and $\vec{B}$. Their vector product has the greatest magnitude if $\vec{A}$ and $\vec{B}$ are (*a*) parallel, (*b*) perpendicular, (*c*) antiparallel, (*d*) at an angle of 45° to each other.

Determine the Concept The vector product of the vectors $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$ where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$ and ϕ is the angle between $\vec{A}$ and $\vec{B}$. Hence, the vector product of $\vec{A}$ and $\vec{B}$ is a maximum when $\sin\phi = 1$. This condition is satisfied provided $\vec{A}$ and $\vec{B}$ are *perpendicular*. $\boxed{(b)}$ is correct.

3 • What is the angle between a force vector $\vec{F}$ and a torque vector $\vec{\tau}$ produced by $\vec{F}$?

Determine the Concept Because $\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\phi\,\hat{n}$, where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{r}$ and $\vec{F}$, the angle between $\vec{F}$ and $\vec{\tau}$ is $\boxed{90°.}$

4 • A point particle of mass *m* is moving with a constant speed *v* along a straight line that passes through point *P*. What can you say about the angular momentum of the particle relative to point *P*? (*a*) Its magnitude is *mv*. (*b*) Its magnitude is zero. (*c*) Its magnitude changes sign as the particle passes through point *P*. (*d*) It varies in magnitude as the particle approaches point *P*.

Determine the Concept $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r} \times \vec{p}$ and the magnitude of $\vec{L}$ is $L = rp\sin\phi$ where ϕ is the angle between $\vec{r}$ and $\vec{p}$. Because the motion is along a line that passes through point *P*, $r = 0$ and so is *L*. $\boxed{(b)}$ is correct.

5 • **[SSM]** A particle travels in a circular path and point *P* is at the center of the circle. (*a*) If the particle's linear momentum $\vec{p}$ is doubled without changing the radius of the circle, how is the magnitude of its angular momentum about *P* affected? (*b*) If the radius of the circle is doubled but the speed of the particle is unchanged, how is the magnitude of its angular momentum about *P* affected?

Determine the Concept $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r} \times \vec{p}$ and the magnitude of $\vec{L}$ is $L = rp\sin\phi$ where ϕ is the angle between $\vec{r}$ and $\vec{p}$.

(*a*) Because $\vec{L}$ is directly proportional to $\vec{p}$, *L* is doubled.

(*b*) Because $\vec{L}$ is directly proportional to $\vec{r}$, *L* is doubled.

6 • A particle moves along a straight line at constant speed. How does its angular momentum about any fixed point vary with time?

Determine the Concept We can determine how the angular momentum of the particle about any fixed point varies with time by examining the derivative of the cross product of $\vec{r}$ and $\vec{p}$.

The angular momentum of the particle is given by: $\vec{L} = \vec{r} \times \vec{p}$

Differentiate $\vec{L}$ with respect to time to obtain:

$$\frac{d\vec{L}}{dt} = \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) + \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) \qquad (1)$$

Because $\vec{p} = m\vec{v}$, $\frac{d\vec{p}}{dt} = \vec{F}_{\text{net}}$, and $\frac{d\vec{r}}{dt} = \vec{v}$:

$$\frac{d\vec{L}}{dt} = \left(\vec{r} \times \vec{F}_{\text{net}}\right) + \left(\vec{v} \times \vec{p}\right)$$

Because the particle moves along a straight line at constant speed:

$$F_{\text{net}} = 0 \Rightarrow \vec{r} \times \vec{F}_{\text{net}} = 0$$

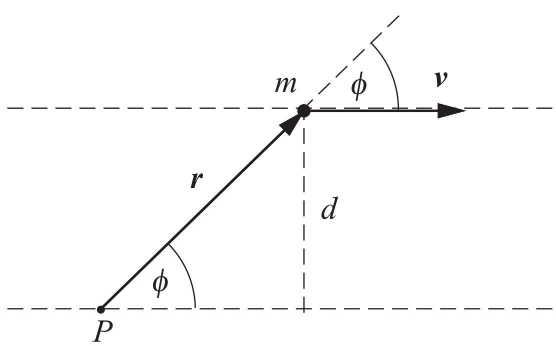
Because $\vec{v}$ and $\vec{p}(= m\vec{v})$ are parallel:

$$\vec{v} \times \vec{p} = 0$$

Substitute in equation (1) to obtain:

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \boxed{\text{constant}}$$

<u>**An alternate solution**</u> The following diagram shows a particle whose mass is *m* moving along a straight line at constant speed. The point identified as *P* is any fixed point and the vector $\vec{r}$ is the position vector, relative to this fixed point, of the particle moving with constant velocity.



From the definition of angular momentum, the magnitude of the angular momentum of the particle, relative to the fixed point at *P,* is given by:

$$\left|\vec{L}\right| = rmv\sin\phi = mv(r\sin\phi)$$

Because $d = r\sin\phi$:

$$\left|\vec{L}\right| = mvd$$

or, because *m, v* and *d* are constants,

$$\left|\vec{L}\right| = \boxed{\text{constant}}$$

7 •• True or false: If the net torque on a rotating system is zero, the angular velocity of the system cannot change. If your answer is false, give an example of such a situation.

False. The net torque acting on a rotating system equals the change in the system's angular momentum; that is, $\tau_{\text{net}} = dL/dt$ where $L = I\omega$. Hence, if τ_{net} is zero, all we can say for sure is that the angular momentum (the product of I and ω) is constant. If I changes, so must ω. An example is a high diver going from a tucked to a layout position.

8 •• You are standing on the edge of a turntable with frictionless bearings that is initially rotating when you catch a ball that is moving in the same direction but faster than you are moving and on a line tangent to the edge of the turntable. Assume you do not move relative to the turntable. (*a*) Does the angular speed of the turntable increase, decrease, or remain the same during the catch? (*b*) Does the magnitude of your angular momentum (about the rotation axis of the table) increase, decrease, or remain the same after the catch? (*c*) How does the ball's angular momentum (about the rotation axis of the table) change after the catch? (*d*) How does the total angular momentum of the system, you-table-ball (about the rotation axis of the table) change after the catch?

Determine the Concept You can apply conservation of angular momentum to the you-table-ball system to answer each of these questions.

(*a*) Because the ball is moving in the same direction that you are moving, your angular speed will $\boxed{\text{increase}}$ when you catch it. Your Newton's 3rd law interaction with the ball causes a torque that acts on the you-table-ball system to increase ω.

(*b*) The ball has angular momentum relative to the rotation axis of the table before you catch it and so catching it $\boxed{\text{increases}}$ your angular momentum relative to the rotation axis of the table.

(*c*) The ball will slow down as a result of your catch and so its angular momentum relative to the center of the table will $\boxed{\text{decrease.}}$

(*d*) Because there is zero net torque on the you-table-ball system, its angular momentum $\boxed{\text{remains the same.}}$

9 •• If the angular momentum of a system about a fixed point P is constant, which one of the following statements must be true?

(*a*) No torque about P acts on any part of the system.
(*b*) A constant torque about P acts on each part of the system.
(*c*) Zero net torque about P acts on each part of the system.
(*d*) A constant external torque about P acts on the system.
(*e*) Zero net external torque about P acts on the system.

Determine the Concept If L is constant, we know that the *net* torque acting on the system is zero. There may be multiple constant or time-dependent torques acting on the system as long as the net torque is zero. $\boxed{(e)}$ is correct.

10 •• A block sliding on a frictionless table is attached to a string that passes through a narrow hole through the tabletop. Initially, the block is sliding with speed v_0 in a circle of radius r_0. A student under the table pulls slowly on the string. What happens as the block spirals inward? Give supporting arguments for your choice. (The term angular momentum refers to the angular momentum about a vertical axis through the hole.) (*a*) Its energy and angular momentum are conserved. (*b*) Its angular momentum is conserved and its energy increases. (*c*) Its angular momentum is conserved and its energy decreases. (*d*) Its energy is conserved and its angular momentum increases. (*e*) Its energy is conserved and its angular momentum decreases.

Determine the Concept The pull that the student exerts on the block is at right angles to its motion and exerts no torque (recall that $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$ and $\tau = rF\sin\phi$). Therefore, we can conclude that the angular momentum of the block is conserved. The student does, however, do work in displacing the block in the direction of the radial force and so the block's energy increases. $\boxed{(b)}$ is correct.

11 •• **[SSM]** One way to tell if an egg is hardboiled or uncooked without breaking the egg is to lay the egg flat on a hard surface and try to spin it. A hardboiled egg will spin easily, while an uncooked egg will not. However, once spinning, the uncooked egg will do something unusual; if you stop it with your finger, it may start spinning again. Explain the difference in the behavior of the two types of eggs.

Determine the Concept The hardboiled egg is solid inside, so everything rotates with a uniform angular speed. By contrast, when you start an uncooked egg spinning, the yolk will not immediately spin with the shell, and when you stop it from spinning the yolk will continue to spin for a while.

12 •• Explain why a helicopter with just one main rotor has a second smaller rotor mounted on a horizontal axis at the rear as in Figure 10-40. Describe the resultant motion of the helicopter if this rear rotor fails during flight.

Determine the Concept The purpose of the second smaller rotor is to prevent the body of the helicopter from rotating. If the rear rotor fails, the body of the helicopter will tend to rotate on the main axis due to angular momentum being conserved.

13 •• The spin angular-momentum vector for a spinning wheel is parallel with its axle and is pointed east. To cause this vector to rotate toward the south, in which direction must a force be exerted on the east end of the axle? (*a*) up, (*b*) down, (*c*) north, (*d*) south, (*e*) east.

Determine the Concept The vector $\Delta\vec{L} = \vec{L}_f - \vec{L}_i$ (and the torque that is responsible for this change in the direction of the angular momentum vector) initially points to the south. One can use a right-hand rule to determine the direction of this torque, and hence the force exerted on the east end of the axle, required to cause the angular-momentum vector to rotate toward the south. Letting the fingers of your right hand point east, rotate your wrist until your thumb points south. Note that your curled fingers, which point in the direction of the force that must be exerted on the east end of the axle, point upward. $\boxed{(a)}$ is correct.

14 •• You are walking toward the north and in your left hand you are carrying a suitcase that contains a massive spinning wheel mounted on an axle attached to the front and back of the case. The angular velocity of the gyroscope points north. You now begin to turn to walk toward the east. As a result, the front end of the suitcase will (*a*) resist your attempt to turn it and will try to maintain its original orientation, (*b*) resist your attempt to turn and will pull to the west, (*c*) rise upward, (*d*) dip downward, (*e*) show no effect whatsoever.

Determine the Concept In turning toward the east, you redirect the angular momentum vector from north to east by exerting a torque on the spinning wheel. The force that you must exert to produce this torque (use a right-hand rule with your thumb pointing either east of north and note that your fingers point upward) is upward. That is, the force you exert on the front end of the suitcase is upward and the force the suitcase exerts on you is downward. Consequently, the front end of the suitcase will dip downward. $\boxed{(d)}$ is correct.

15 •• [SSM] The angular momentum of the propeller of a small single-engine airplane points forward. The propeller rotates clockwise if viewed from behind. (*a*) Just after liftoff, as the nose of the plane tilts upward, the airplane tends to veer to one side. To which side does it tend to veer and why? (*b*) If the plane is flying horizontally and suddenly turns to the right, does the nose of the plane tend to veer upward or downward? Why?

(*a*) The plane tends to veer to the right. The change in angular momentum $\Delta\vec{L}_{\text{prop}}$ for the propeller is upward, so the net torque $\vec{\tau}$ on the propeller is upward as well. The propeller must exert an equal but opposite torque on the plane. This downward torque exerted on the plane by the propeller tends to cause a downward

change in the angular momentum of the plane. This means the plane tends to rotate clockwise as viewed from above.

(*b*) The nose of the plane tends to veer downward. The change in angular momentum $\Delta\vec{\boldsymbol{L}}_{\text{prop}}$ for the propeller is to the right, so the net torque $\vec{\boldsymbol{\tau}}$ on the propeller is toward the right as well. The propeller must exert an equal but opposite torque on the plane. This leftward directed torque exerted by the propeller on the plane tends to cause a leftward-directed change in angular momentum for the plane. This means the plane tends to rotate clockwise as viewed from the right.

16 •• You have designed a car that is powered by the energy stored in a single flywheel with a spin angular momentum $\vec{L}$. In the morning, you plug the car into an electrical outlet and a motor spins the flywheel up to speed, adding a huge amount of rotational kinetic energy to it—energy that will be changed into translational kinetic energy of the car during the day. Having taken a physics course involving angular momentum and torques, you realize that problems would arise during various maneuvers of the car. Discuss some of these problems. For example, suppose the flywheel is mounted so $\vec{\boldsymbol{L}}$ points vertically upward when the car is on a horizontal road. What would happen as the car travels over a hilltop? Through a valley? Suppose the flywheel is mounted so $\vec{\boldsymbol{L}}$ points forward, or to one side, when the car is on a horizontal road. Then what would happen as the car attempts to turn to the left or right? In each case that you examine, consider the direction of the torque exerted on the car by the road.

Determine the Concept If $\vec{\boldsymbol{L}}$ points upward and the car travels over a hill or through a valley, the force the road exerts on the wheels on one side (or the other) will increase and car will tend to tip. If $\vec{\boldsymbol{L}}$ points forward and the car turns left or right, the front (or rear) of the car will tend to lift. These problems can be averted by having two identical flywheels that rotate on the same shaft in opposite directions.

17 •• [SSM] You are sitting on a spinning piano stool with your arms folded. (*a*) When you extend your arms out to the side, what happens to your kinetic energy? What is the cause of this change? (*b*) Explain what happens to your moment of inertia, angular speed and angular momentum as you extend your arms.

Determine the Concept The rotational kinetic energy of the you-stool system is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2 = \dfrac{L^2}{2I}$. Because the net torque acting on the you-stool system is zero, its angular momentum $\vec{\boldsymbol{L}}$ is conserved.

(*a*) Your kinetic energy decreases. Increasing your moment of inertia I while conserving your angular momentum L decreases you kinetic energy $K = \frac{L^2}{2I}$.

(*b*) Extending your arms out to the side increases your moment of inertia, decreases your angular speed. The angular momentum of the system is unchanged.

18 •• A uniform rod of mass M and length L rests on a horizontal frictionless table. A blob of putty of mass $m = M/4$ moves along a line perpendicular to the rod, strikes the rod near its end, and sticks to the rod. Describe qualitatively the subsequent motion of the rod and putty.

Determine the Concept The center of mass of the rod-and-putty system moves in a straight line, and the system rotates about its center of mass.

Estimation and Approximation

19 •• **[SSM]** An ice-skater starts her pirouette with arms outstretched, rotating at 1.5 rev/s. Estimate her rotational speed (in revolutions per second) when she brings her arms flat against her body.

Picture the Problem Because we have no information regarding the mass of the skater, we'll assume that her body mass (not including her arms) is 50 kg and that each arm has a mass of 4.0 kg. Let's also assume that her arms are 1.0 m long and that her body is cylindrical with a radius of 20 cm. Because the net external torque acting on her is zero, her angular momentum will remain constant during her pirouette.

Because the net external torque acting on her is zero:

$$\Delta L = L_\mathrm{f} - L_\mathrm{i} = 0$$

or

$$I_{\text{arms in}}\omega_{\text{arms in}} - I_{\text{arms out}}\omega_{\text{arms out}} = 0 \quad (1)$$

Express her total moment of inertia with her arms outstretched:

$$I_{\text{arms out}} = I_{\text{body}} + I_{\text{arms}}$$

Treating her body as though it is cylindrical, calculate the moment of inertia of her body, minus her arms:

$$I_{\text{body}} = \tfrac{1}{2}mr^2 = \tfrac{1}{2}(50\,\mathrm{kg})(0.20\,\mathrm{m})^2 = 1.00\,\mathrm{kg\cdot m^2}$$

Modeling her arms as though they are rods, calculate their moment of inertia when they are outstretched:

$$I_{\text{arms}} = 2\left[\tfrac{1}{3}(4\,\text{kg})(1.0\,\text{m})^2\right] = 2.67\,\text{kg}\cdot\text{m}^2$$

Substitute to determine her total moment of inertia with her arms outstretched:

$$I_{\text{arms out}} = 1.00\,\text{kg}\cdot\text{m}^2 + 2.67\,\text{kg}\cdot\text{m}^2 = 3.67\,\text{kg}\cdot\text{m}^2$$

Express her total moment of inertia with her arms flat against her body:

$$\begin{aligned} I_{\text{arms in}} &= I_{\text{body}} + I_{\text{arms}} \\ &= 1.00\,\text{kg}\cdot\text{m}^2 + 2\left[(4.0\,\text{kg})(0.20\,\text{m})^2\right] \\ &= 1.32\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Solve equation (1) for $\omega_{\text{arms in}}$ to obtain:

$$\omega_{\text{arms in}} = \frac{I_{\text{arms out}}}{I_{\text{arms in}}}\omega_{\text{arms out}}$$

Substitute numerical values and evaluate $\omega_{\text{arms in}}$:

$$\omega_{\text{arms in}} = \frac{3.67\,\text{kg}\cdot\text{m}^2}{1.32\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) \approx \boxed{4\,\text{rev/s}}$$

20 •• Estimate the ratio of angular velocities for the rotation of a diver between the full tuck position and the full-layout position.

Picture the Problem Because the net external torque acting on the diver is zero, the diver's angular momentum will remain constant as she rotates from the full tuck to the full layout position. Assume that, in layout position, the diver is a thin rod of length 2.5 m and that, in the full tuck position, the diver is a sphere of radius 0.50 m.

Because the net external torque acting on the diver is zero:

$$\Delta L = L_{\text{layout}} - L_{\text{tuck}} = 0$$

or

$$I_{\text{layout}}\omega_{\text{layout}} - I_{\text{tuck}}\omega_{\text{tuck}} = 0$$

Solving for the ratio of the angular velocities gives:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}} = \frac{I_{\text{layout}}}{I_{\text{tuck}}}$$

Substituting for the moment of inertia of a thin rod relative to an axis through its center of mass and the moment of inertia of a sphere relative to its center of mass and simplifying yields:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}}=\frac{\frac{1}{12}m\ell^2}{\frac{2}{5}mr^2}=\frac{5\ell^2}{24r^2}$$

Substitute numerical values and evaluate $\omega_{\text{tuck}}/\omega_{\text{layout}}$:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}}=\frac{5(2.5\text{ m})^2}{24(0.50\text{ m})^2}\approx\boxed{5}$$

21 •• The days on Mars and Earth are of nearly identical length. Earth's mass is 9.35 times Mars's mass, Earth's radius is 1.88 times Mars's radius, and Mars is on average 1.52 times farther away from the Sun than Earth is. The Martian year is 1.88 times longer than Earth's year. Assume that they are both uniform spheres and that their orbits about the Sun are circles. Estimate the ratio (Earth to Mars) of (*a*) their spin angular momenta, (*b*) their spin kinetic energies, (*c*) their orbital angular momenta, and (*d*) their orbital kinetic energies.

Picture the Problem We can use the definitions of spin angular momentum, spin kinetic energy, orbital angular momentum, and orbital kinetic energy to evaluate these ratios.

(*a*) The ratio of the spin angular momenta of Earth and Mars is:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}=\frac{I_{\text{E}}\omega_{\text{E}}}{I_{\text{M}}\omega_{\text{M}}}$$

Because Mars and Earth have nearly identical lengths of days, $\omega_{\text{E}}\approx\omega_{\text{M}}$:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx\frac{I_{\text{E}}}{I_{\text{M}}}$$

Substituting for the moments of inertia and simplifying yields:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx\frac{\frac{2}{5}M_{\text{E}}R_{\text{E}}^2}{\frac{2}{5}M_{\text{M}}R_{\text{M}}^2}=\frac{M_{\text{E}}}{M_{\text{M}}}\left(\frac{R_{\text{E}}}{R_{\text{M}}}\right)^2$$

Substitute numerical values for the ratios and evaluate $\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}$:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx 9.35(1.88)^2\approx\boxed{33}$$

(*b*) The ratio of the spin kinetic energies of Earth and Mars is:

$$\left(\frac{K_{\text{E}}}{K_{\text{M}}}\right)_{\text{spin}}=\frac{\dfrac{L_{\text{E}}^2}{2I_{\text{E}}}}{\dfrac{L_{\text{M}}^2}{2I_{\text{M}}}}=\frac{L_{\text{E}}^2}{L_{\text{M}}^2}\frac{I_{\text{M}}}{I_{\text{E}}}$$

Because $\left(\frac{L_E}{L_M}\right)_{spin} \approx \frac{I_E}{I_M}$:

$$\left(\frac{K_E}{K_M}\right)_{spin} = \frac{L_E^2}{L_M^2}\frac{L_M}{L_E} = \left(\frac{L_E}{L_M}\right)_{spin}$$

From (*a*) $\left(\frac{L_E}{L_M}\right)_{spin} \approx 33$. Hence:

$$\left(\frac{K_E}{K_M}\right)_{spin} \approx \boxed{33}$$

(*c*) Treating Earth and Mars as point objects, the ratio of their orbital angular momenta is:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \frac{I_E \omega_{orb,\,E}}{I_M \omega_{orb,\,M}}$$

Substituting for the moments of inertia and orbital angular speeds yields:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \frac{M_E r_E^2 \left(\frac{2\pi}{T_E}\right)}{M_M r_M^2 \left(\frac{2\pi}{T_M}\right)}$$

where r_E and r_M are the radii of the orbits of Earth and Mars, respectively.

Simplify to obtain:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \left(\frac{M_E}{M_M}\right)\left(\frac{r_E}{r_M}\right)^2\left(\frac{T_M}{T_E}\right)$$

Substitute numerical values for the three ratios and evaluate $\left(\frac{L_E}{L_M}\right)_{orb}$:

$$\left(\frac{L_E}{L_M}\right)_{orb} = (9.35)\left(\frac{1}{1.52}\right)^2(1.88) \approx \boxed{8}$$

(*d*) The ratio of the orbital kinetic energies of Earth and Mars is:

$$\left(\frac{K_E}{K_M}\right)_{orb} = \frac{\frac{1}{2}I_E \omega_{orb,\,E}^2}{\frac{1}{2}I_M \omega_{orb,\,M}^2}$$

Substituting for the moments of inertia and angular speeds and simplifying gives:

$$\left(\frac{K_E}{K_M}\right)_{orb} = \frac{M_E r_E^2 \left(\frac{2\pi}{T_E}\right)^2}{M_M r_M^2 \left(\frac{2\pi}{T_M}\right)^2} = \left(\frac{M_E}{M_M}\right)\left(\frac{r_E}{r_M}\right)^2\left(\frac{T_M}{T_E}\right)^2$$

Substitute numerical values for the ratios and evaluate $\left(\frac{K_E}{K_M}\right)_{orb}$:

$$\left(\frac{K_E}{K_M}\right)_{orb} = (9.35)\left(\frac{1}{1.52}\right)^2(1.88)^2 \approx \boxed{14}$$

22 •• The polar ice caps contain about 2.3×10^{19} kg of ice. This mass contributes negligibly to the moment of inertia of Earth because it is located at the poles, close to the axis of rotation. Estimate the change in the length of the day that would be expected if the polar ice caps were to melt and the water were distributed uniformly over the surface of Earth.

Picture the Problem The change in the length of the day is the difference between its length when the ice caps have melted and the water has been distributed over the surface of Earth and the length of the day before the ice caps melt. Because the net torque acting on Earth during this process is zero, angular momentum is conserved and we can relate the angular speed (which are related to the length of the day) of Earth before and after the ice caps melt to the moments of inertia of the Earth-plus-spherical shell the ice caps melt.

Express the change in the length of a day as:

$$\Delta T = T_{\text{after}} - T_{\text{before}} \qquad (1)$$

Because the net torque acting on Earth during this process is zero, angular momentum is conserved:

$$\Delta L = L_{\text{after}} - L_{\text{before}} = 0$$

Substituting for L_{after} and L_{before} yields:

$$\left(I_{\text{sphere}} + I_{\text{shell}}\right)\omega_{\text{after}} - I_{\text{sphere}}\omega_{\text{before}} = 0$$

Because $\omega = 2\pi/T$:

$$\left(I_{\text{sphere}} + I_{\text{shell}}\right)\frac{2\pi}{T_{\text{after}}} - I_{\text{sphere}}\frac{2\pi}{T_{\text{before}}} = 0$$

or, simplifying,

$$\frac{I_{\text{sphere}} + I_{\text{shell}}}{T_{\text{after}}} - \frac{I_{\text{sphere}}}{T_{\text{before}}} = 0$$

Solve for T_{after} to obtain:

$$T_{\text{after}} = \left(1 + \frac{I_{\text{shell}}}{I_{\text{sphere}}}\right)T_{\text{before}}$$

Substituting for T_{after} in equation (1) and simplifying yields:

$$\Delta T = \left(1 + \frac{I_{\text{shell}}}{I_{\text{sphere}}}\right)T_{\text{before}} - T_{\text{before}}$$

$$= \frac{I_{\text{shell}}}{I_{\text{sphere}}}T_{\text{before}}$$

Substitute for I_{shell} and I_{sphere} and simplify to obtain:

$$\Delta T = \frac{\frac{2}{3}mr^2}{\frac{2}{5}M_E R_E^2} T_{\text{before}} = \frac{5m}{3M_E} T_{\text{before}}$$

Substitute numerical values and evaluate ΔT:

$$\Delta T = \frac{5(2.3\times10^{19}\ \text{kg})}{3(5.98\times10^{24}\ \text{kg})}\left(1\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}\right) = \boxed{0.55\,\text{s}}$$

23 •• [SSM] A 2.0-g particle moves at a constant speed of 3.0 mm/s around a circle of radius 4.0 mm. (*a*) Find the magnitude of the angular momentum of the particle. (*b*) If $L = \sqrt{\ell(\ell+1)}\hbar$, where ℓ is an integer, find the value of $\ell(\ell+1)$ and the approximate value of ℓ. (*c*) By how much does ℓ change if the particle's speed increases by one-millionth of a percent, and nothing else changing? Use your result to explain why the quantization of angular momentum is not noticed in macroscopic physics.

Picture the Problem We can use $L = mvr$ to find the angular momentum of the particle. In (*b*) we can solve $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$ and the approximate value of ℓ.

(*a*) Use the definition of angular momentum to obtain:

$$L = mvr = (2.0\times10^{-3}\ \text{kg})(3.0\times10^{-3}\ \text{m/s})(4.0\times10^{-3}\ \text{m}) = 2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}$$
$$= \boxed{2.4\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Solve the equation $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$:

$$\ell(\ell+1) = \frac{L^2}{\hbar^2} \qquad (1)$$

Substitute numerical values and evaluate $\ell(\ell+1)$:

$$\ell(\ell+1) = \left(\frac{2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.05\times10^{-34}\ \text{J}\cdot\text{s}}\right)^2 = \boxed{5.2\times10^{52}}$$

Because $\ell >> 1$, approximate its value with the square root of $\ell(\ell+1)$:

$$\ell \approx \boxed{2.3\times10^{26}}$$

(*c*) The change in ℓ is:

$$\Delta\ell = \ell_{\text{new}} - \ell \qquad (2)$$

If the particle's speed increases by one-millionth of a percent while nothing else changes:

$$v \to v + 10^{-8} v = \left(1 + 10^{-8}\right)v$$
and
$$L \to L + 10^{-8} L = \left(1 + 10^{-8}\right)L$$

Equation (1) becomes:

$$\ell_{\text{new}}\left(\ell_{\text{new}} + 1\right) = \frac{\left[\left(1 + 10^{-8}\right)L\right]^2}{\hbar^2}$$
and
$$\ell_{\text{new}} \approx \frac{\left(1 + 10^{-8}\right)L}{\hbar}$$

Substituting in equation (2) yields:

$$\Delta\ell = \ell_{\text{new}} - \ell \approx \frac{\left(1 + 10^{-8}\right)L}{\hbar} - \frac{L}{\hbar} = 10^{-8}\frac{L}{\hbar}$$

Substitute numerical values and evaluate $\Delta\ell$:

$$\Delta\ell = 10^{-8}\left(\frac{2.40 \times 10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.05 \times 01^{-34}\ \text{J}\cdot\text{s}}\right)$$
$$= \boxed{2.3 \times 10^{18}}$$
and
$$\frac{\Delta\ell}{\ell} = \frac{2.3 \times 10^{18}}{2.3 \times 10^{26}} \approx 10^{-6}\%$$

The quantization of angular momentum is not noticed in macroscopic physics because no experiment can detect a fractional change in ℓ of $10^{-6}\%$.

24 ••• Astrophysicists in the 1960s tried to explain the existence and structure of *pulsars*—extremely regular astronomical sources of radio pulses whose periods ranged from seconds to milliseconds. At one point, these radio sources were given the acronym LGM (Little Green Men), a reference to the idea that they might be signals of extraterrestrial civilizations. The explanation given today is no less interesting. Consider the following. Our Sun, which is a fairly typical star, has a mass of 1.99×10^{30} kg and a radius of 6.96×10^{8} m. Although it does not rotate uniformly, because it is not a solid body, its average rate of rotation is about 1 rev/25 d. Stars larger than the Sun can end their life in spectacular explosions called supernovae, leaving behind a collapsed remnant of the star called a *neutron star*. These neutron-stars have masses comparable to the original masses of the stars but radii of only a few kilometers! The high rotation rates are due to the conservation of angular momentum during the collapses. These stars emit beams of radio waves. Because of the rapid angular speed of the stars, the beam sweeps past Earth at regular, very short, intervals. To produce the observed radio-wave pulses, the star has to rotate at rates that range from about 1 rev/s to 1000 rev/s. (*a*) Using data from the textbook, estimate the rotation rate of the Sun if it were to collapse into a neutron star of radius 10 km. The Sun is not a uniform sphere of

gas, and its moment of inertia is given by $I = 0.059MR^2$. Assume that the neutron star is spherical and has a uniform mass distribution. (*b*) Is the rotational kinetic energy of the Sun greater or smaller after the collapse? By what factor does it change, and where does the energy go to or come from?

Picture the Problem We can use conservation of angular momentum in Part (*a*) to relate the before-and-after collapse rotation rates of the Sun. In Part (*b*), we can express the fractional change in the rotational kinetic energy of the Sun as it collapses into a neutron star to decide whether its rotational kinetic energy is greater initially or after the collapse.

(*a*) Use conservation of angular momentum to relate the angular momenta of the Sun before and after its collapse:

$$I_b\omega_b = I_a\omega_a \Rightarrow \omega_a = \frac{I_b}{I_a}\omega_b \qquad (1)$$

Using the given formula, approximate the moment of inertia I_b of the Sun before collapse:

$$I_b = 0.059MR_{sun}^2 = 0.059(1.99\times10^{30}\text{ kg})(6.96\times10^5\text{ km})^2 = 5.69\times10^{46}\text{ kg}\cdot\text{m}^2$$

Find the moment of inertia I_a of the Sun when it has collapsed into a spherical neutron star of radius 10 km and uniform mass distribution:

$$\begin{aligned} I_a &= \tfrac{2}{5}MR^2 \\ &= \tfrac{2}{5}(1.99\times10^{30}\text{ kg})(10\text{ km})^2 \\ &= 7.96\times10^{37}\text{ kg}\cdot\text{m}^2 \end{aligned}$$

Substitute numerical values in equation (1) and simplify to obtain:

$$\begin{aligned} \omega_a &= \frac{5.69\times10^{46}\text{ kg}\cdot\text{m}^2}{7.96\times10^{37}\text{ kg}\cdot\text{m}^2}\omega_b \\ &= 7.15\times10^8\,\omega_b \end{aligned}$$

Given that ω_b = 1 rev/25 d, evaluate ω_a:

$$\begin{aligned} \omega_a &= 7.15\times10^8\left(\frac{1\text{rev}}{25\text{d}}\right) = 2.86\text{ rev/d} \\ &= \boxed{2.9\times10^7\text{ rev/d}} \end{aligned}$$

Note that the rotational period decreases by the same factor of I_b/I_a and becomes:

$$T_a = \frac{2\pi}{\omega_a} = \frac{2\pi}{2.86\times10^7\,\frac{\text{rev}}{\text{d}}\times\frac{2\pi\text{ rad}}{\text{rev}}\times\frac{1\text{d}}{24\text{h}}\times\frac{1\text{h}}{3600\text{s}}} = 3.0\times10^{-3}\text{ s}$$

(*b*) Express the fractional change in the Sun's rotational kinetic energy as a consequence of its collapse:

$$\frac{\Delta K}{K_b}=\frac{K_a-K_b}{K_b}=\frac{K_a}{K_b}-1$$

Substituting for the kinetic energies and simplifying yields:

$$\frac{\Delta K}{K_b}=\frac{\frac{1}{2}I_a\omega_a^2}{\frac{1}{2}I_b\omega_b^2}-1=\frac{I_a\omega_a^2}{I_b\omega_b^2}-1$$

Substitute numerical values and evaluate $\Delta K/K_b$:

$$\frac{\Delta K}{K_b}=\left(\frac{1}{7.15\times10^8}\right)\left(\frac{2.86\times10^7\text{ rev/d}}{1\text{rev/25d}}\right)^2-1=\boxed{7.1\times10^8}$$

That is, the rotational kinetic energy *increases* by a factor of approximately 7×10^8. The additional rotational kinetic energy comes at the expense of gravitational potential energy, which decreases as the Sun gets smaller.

25 •• The moment of inertia of Earth about its spin axis is approximately 8.03×10^{37} kg·m^2. (*a*) Because Earth is nearly spherical, assume that the moment of inertia can be written as $I = CMR^2$, where C is a dimensionless constant, $M = 5.98\times10^{24}$ kg is the mass of Earth, and $R = 6370$ km is its radius. Determine C. (*b*) If Earth's mass were distributed uniformly, C would equal 2/5. From the value of C calculated in Part (*a*), is Earth's density greater near its center or near its surface? Explain your reasoning.

Picture the Problem We can solve $I=CMR^2$ for C and substitute numerical values in order to determine an experimental value of C for Earth. We can then compare this value to those for a spherical shell and a sphere in which the mass is uniformly distributed to decide whether Earth's mass density is greatest near its core or near its crust.

(*a*) Express the moment of inertia of Earth in terms of the constant C:

$$I=CMR^2\Rightarrow C=\frac{I}{MR^2}$$

Substitute numerical values and evaluate C:

$$C=\frac{8.03\times10^{37}\text{ kg}\cdot\text{m}^2}{(5.98\times10^{24}\text{ kg})(6370\text{ km})^2}=\boxed{0.331}$$

(*b*) Because experimentally $C < 0.4$, the mass density must be greater near the center of Earth.

26 ••• Estimate Timothy Goebel's initial takeoff speed, rotational velocity, and angular momentum when he performs a quadruple Lutz (Figure 10-41). Make any assumptions you think reasonable, but justify them. Goebel's mass is about 60 kg and the height of the jump is about 0.60 m. Note that his angular speed will

change quite a bit during the jump, as he begins with arms outstretched and pulls them in. Your answer should be accurate to within a factor of 2, if you're careful.

Picture the Problem We'll assume that he launches himself at an angle of 45° with the horizontal with his arms spread wide, and then pulls them in to increase his rotational speed during the jump. We'll also assume that we can model him as a 2.0-m long cylinder with an average radius of 0.15 m and a mass of 60 kg. We can then find his take-off speed and ″air time″ using constant-acceleration equations, and use the latter, together with the definition of rotational velocity, to find his initial rotational velocity. Finally, we can apply conservation of angular momentum to find his initial angular momentum.

Using a constant-acceleration equation, relate his takeoff speed v_0 to his maximum elevation Δy:

$$v^2 = v_{0y}^2 + 2a_y\Delta y$$

or, because $v_{0y} = v_0\sin(45°)$, $v = 0$, and $a_y = -g$,

$$0 = v_0^2\sin^2 45° - 2g\Delta y$$

Solving for v_0 and simplifying yields:

$$v_0 = \sqrt{\frac{2g\Delta y}{\sin^2 45°}} = \frac{\sqrt{2g\Delta y}}{\sin 45°}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{\sqrt{2\left(9.81\,\text{m/s}^2\right)\left(0.60\,\text{m}\right)}}{\sin 45°} = \boxed{4.9\,\text{m/s}}$$

Use its definition to express Goebel's angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Use a constant-acceleration equation to express Goebel's ″air time″ Δt:

$$\Delta t = 2\Delta t_{\text{rise }0.6\text{ m}} = 2\sqrt{\frac{2\Delta y}{g}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = 2\sqrt{\frac{2\left(0.60\,\text{m}\right)}{9.81\,\text{m/s}^2}} = 0.699\,\text{s}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{4\,\text{rev}}{0.699\,\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}} = \boxed{36\,\text{rad/s}}$$

Use conservation of angular momentum to relate his take-off angular velocity ω_0 to his average angular velocity ω as he performs a quadruple Lutz:

$$I_0\omega_0 = I\omega$$

Assuming that he can change his moment of inertia by a factor of 2 by pulling his arms in, solve for and evaluate ω_0:

$$\omega_0 = \frac{I}{I_0}\omega = \frac{1}{2}(36\,\text{rad/s}) = \boxed{18\,\text{rad/s}}$$

Express his take-off angular momentum:

$$L_0 = I_0\omega_0$$

Assuming that we can model him as a solid cylinder of length ℓ with an average radius r and mass m, express his moment of inertia with arms outstretched (his take-off configuration):

$$I_0 = 2\left(\tfrac{1}{2}mr^2\right) = mr^2$$

where the factor of 2 represents our assumption that he can double his moment of inertia by extending his arms.

Substituting for I_0 gives:

$$L_0 = mr^2\omega_0$$

Substitute numerical values and evaluate L_0:

$$L_0 = (60\,\text{kg})(0.15\,\text{m})^2(18\,\text{rad/s}) = \boxed{24\,\text{kg}\cdot\text{m}^2/\text{s}}$$

The Vector Product and the Vector Nature of Torque and Rotation

27 • [SSM] A force of magnitude F is applied horizontally in the negative x direction to the rim of a disk of radius R as shown in Figure 10-42. Write $\vec{F}$ and $\vec{r}$ in terms of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, and compute the torque produced by this force about the origin at the center of the disk.

Picture the Problem We can express $\vec{F}$ and $\vec{r}$ in terms of the unit vectors $\hat{i}$ and $\hat{j}$ and then use the definition of the vector product to find $\vec{\tau}$.

Express $\vec{F}$ in terms of F and the unit vector $\hat{i}$:

$$\vec{F} = -F\hat{i}$$

Express $\vec{r}$ in terms of R and the unit vector $\hat{j}$:

$$\vec{r} = R\hat{j}$$

Calculate the vector product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r}\times\vec{F} = FR\left(\hat{j}\times -\hat{i}\right) = FR\left(\hat{i}\times\hat{j}\right) = \boxed{FR\,\hat{k}}$$

28 • Compute the torque about the origin of the gravitational force $\vec{F} = -mg\hat{j}$ acting on a particle of mass m located at $\vec{r} = x\hat{i} + y\hat{j}$ and show that this torque is independent of the y coordinate.

Picture the Problem We can find the torque from the vector product of $\vec{r}$ and $\vec{F}$.

Compute the vector product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(x\hat{i} + y\hat{j}\right)\left(-mg\hat{j}\right)$$
$$= -mgx\left(\hat{i} \times \hat{j}\right) - mgy\left(\hat{j} \times \hat{j}\right)$$
$$= \boxed{-mgx\hat{k}}$$

29 • Find $\vec{A} \times \vec{B}$ for the following choices: (a) $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$, ($b$) $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$, and ($c$) $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$.

Picture the Problem We can use the definitions of the vector products of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ to evaluate $\vec{A} \times \vec{B}$ in each case.

(a) Evaluate $\vec{A} \times \vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$:

$$\vec{A} \times \vec{B} = 4\hat{i} \times \left(6\hat{i} + 6\hat{j}\right)$$
$$= 24\left(\hat{i} \times \hat{i}\right) + 24\left(\hat{i} \times \hat{j}\right)$$
$$= 24(0) + 24\hat{k}$$
$$= \boxed{24\hat{k}}$$

(b) Evaluate $\vec{A} \times \vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$:

$$\vec{A} \times \vec{B} = 4\hat{i} \times \left(6\hat{i} + 6\hat{k}\right)$$
$$= 24\left(\hat{i} \times \hat{i}\right) + 24\left(\hat{i} \times \hat{k}\right)$$
$$= 24(0) + 24\left(-\hat{j}\right)$$
$$= \boxed{-24\hat{j}}$$

(c) Evaluate $\vec{A} \times \vec{B}$ for $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$:

$$\vec{A} \times \vec{B} = \left(2\hat{i} + 3\hat{j}\right) \times \left(3\hat{i} + 2\hat{j}\right)$$
$$= 6\left(\hat{i} \times \hat{i}\right) + 4\left(\hat{i} \times \hat{j}\right) + 9\left(\hat{j} \times \hat{i}\right)$$
$$+ 6\left(\hat{j} \times \hat{j}\right)$$
$$= 6(0) + 4\left(\hat{k}\right) + 9\left(-\hat{k}\right) + 6(0)$$
$$= \boxed{-5\hat{k}}$$

30 •• For each case in Problem 31, compute $\left|\vec{A} \times \vec{B}\right|$. Compare it to $\left|\vec{A}\right|\left|\vec{B}\right|$ to estimate which of the pairs of vectors are closest to being perpendicular. Verify your answers by calculating the angle using the dot product.

Picture the Problem Because $\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\phi$, if vectors $\vec{A}$ and $\vec{B}$ are perpendicular, then $\left|\vec{A}\times\vec{B}\right| = \vec{A}\left|\vec{B}\right|$ or $\dfrac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = 1$. The scalar product of vectors $\vec{A}$ and $\vec{B}$ is $\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\phi$. We can verify our estimations using this definition to calculate ϕ for each pair of vectors.

(*a*) For $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|4\hat{i}\times\left(6\hat{i}+6\hat{j}\right)\right|}{(4)\left(6\sqrt{2}\right)} = \frac{\left|24\hat{k}\right|}{24\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\approx 0.707$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right) = \cos^{-1}\left(\frac{4\hat{i}\cdot\left(6\hat{i}+6\hat{j}\right)}{24\sqrt{2}}\right)$$
$$= \cos^{-1}\frac{24}{24\sqrt{2}} = 45°,$$

a result confirming that obtained above.

(*b*) For $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|4\hat{i}\times\left(6\hat{i}+6\hat{k}\right)\right|}{(4)\left(6\sqrt{2}\right)} = \frac{\left|-24\hat{j}\right|}{24\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\approx 0.707$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right) = \cos^{-1}\left(\frac{4\hat{i}\cdot\left(6\hat{i}+6\hat{k}\right)}{24\sqrt{2}}\right)$$
$$= \cos^{-1}\frac{24}{24\sqrt{2}} = 45°,$$

a result confirming that obtained above.

(*c*) For $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|\left(2\hat{i}+3\hat{j}\right)\times\left(3\hat{i}+2\hat{j}\right)\right|}{\sqrt{13}\sqrt{13}} = \frac{\left|-5\hat{k}\right|}{13}$$

$$= \frac{5}{13} \approx 0.385$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right)$$

$$= \cos^{-1}\left(\frac{\left(2\hat{i}+3\hat{j}\right)\cdot\left(3\hat{i}+2\hat{j}\right)}{\sqrt{13}\sqrt{13}}\right)$$

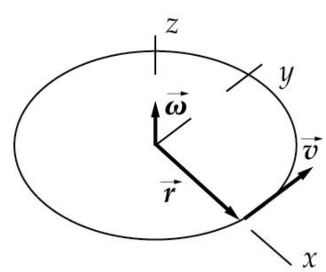
$$= \cos^{-1}\left(\frac{12}{13}\right) = 23°,$$

a result confirming that obtained above.

While none of these sets of vectors are perpendicular, those in (*a*) and (*b*) are the closest, with $\phi = 45°$, to being perpendicular.

31 •• A particle moves in a circle that is centered at the origin. The particle has position $\vec{r}$ and angular velocity $\vec{\omega}$. (*a*) Show that its velocity is given by $\vec{v} = \vec{\omega}\times\vec{r}$. (*b*) Show that its centripetal acceleration is given by $\vec{a}_c = \vec{\omega}\times\vec{v} = \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$.

Picture the Problem Let $\vec{r}$ be in the *xy* plane and point in the +*x* direction. Then $\vec{\omega}$ points in the +*z* direction. We can establish the results called for in this problem by forming the appropriate vector products and by differentiating $\vec{v}$.



(*a*) Express $\vec{\omega}$ using unit vector notation:

$$\vec{\omega} = \omega\hat{k}$$

Express $\vec{r}$ using unit vector notation:

$$\vec{r} = r\hat{i}$$

Form the vector product of $\vec{\omega}$ and $\vec{r}$:

$$\vec{\omega}\times\vec{r} = \omega\hat{k}\times r\hat{i} = r\omega\left(\hat{k}\times\hat{i}\right) = r\omega\,\hat{j}$$

$$= v\hat{j}$$

and $\vec{v} = \boxed{\vec{\omega}\times\vec{r}}$

(*b*) Differentiate $\vec{v}$ with respect to *t* to express $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{a}_\mathrm{t} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$= \vec{a}_\mathrm{t} + \vec{a}_\mathrm{c}$$

where $\vec{a}_\mathrm{c} = \boxed{\vec{\omega}\times(\vec{\omega}\times\vec{r})}$ and $\vec{a}_\mathrm{t}$ and $\vec{a}_\mathrm{c}$ are the tangential and centripetal accelerations, respectively.

32 •• You are given three vectors and their components in the form: $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$, $\vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, and $\vec{C} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$. Show that the following equalities hold: $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$

Picture the Problem We can establish these equalities by carrying out the details of the vector- and scalar-products and comparing the results of these operations.

Evaluate the vector product of $\vec{B}$ and $\vec{C}$ to obtain:

$$\vec{B}\times\vec{C} = (b_yc_z - b_zc_y)\hat{i} + (b_zc_x - b_xc_z)\hat{j} + (b_xc_y - b_yc_x)\hat{k}$$

Form the scalar product of $\vec{A}$ with $\vec{B}\times\vec{C}$ to obtain:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = a_xb_yc_z - a_xb_zc_y + a_yb_zc_x - a_yb_xc_z + a_zb_xc_y - a_zb_yc_x \qquad (1)$$

Evaluate the vector product of $\vec{A}$ and $\vec{B}$ to obtain:

$$\vec{A}\times\vec{B} = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} + (a_xb_y - a_yb_x)\hat{k}$$

Form the scalar product of $\vec{C}$ with $\vec{A}\times\vec{B}$ to obtain:

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = c_xa_yb_z - c_xa_zb_y + c_ya_zb_x - c_ya_xb_z + c_za_xb_y - c_za_yb_x \qquad (2)$$

Evaluate the vector product of $\vec{C}$ and $\vec{A}$ to obtain:

$$\vec{C}\times\vec{A} = (c_ya_z - a_za_y)\hat{i} + (c_za_x - a_xa_z)\hat{j} + (c_xa_y - a_ya_x)\hat{k}$$

Form the scalar product of $\vec{B}$ with $\vec{C}\times\vec{A}$ to obtain:

$$\vec{B}\cdot(\vec{C}\times\vec{A}) = b_xc_ya_z - b_xc_za_y + b_yc_za_x - b_yc_xa_z + b_zc_xa_y - b_zc_ya_x \qquad (3)$$

The equality of equations (1), (2), and (3) establishes the equalities.

33 •• If $\vec{A} = 3\hat{j}$, $\vec{A}\times\vec{B} = 9\hat{i}$, and $\vec{A}\cdot\vec{B} = 12$, find $\vec{B}$.

Picture the Problem We can write $\vec{B}$ in the form $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and use the scalar product of $\vec{A}$ and $\vec{B}$ to find B_y and their vector product to find B_x and B_z.

Express $\vec{B}$ in terms of its components:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \qquad (1)$$

Evaluate $\vec{A}\cdot\vec{B}$:

$$\vec{A}\cdot\vec{B} = 3B_y = 12 \Rightarrow B_y = 4$$

Evaluate $\vec{A}\times\vec{B}$:

$$\vec{A}\times\vec{B} = 3\hat{j}\times\left(B_x\hat{i} + 4\hat{j} + B_z\hat{k}\right) = -3B_x\hat{k} + 3B_z\hat{i}$$

Because $\vec{A}\times\vec{B} = 9\hat{i}$:

$B_x = 0$ and $B_z = 3$.

Substitute for B_y and B_z in equation (1) to obtain:

$$\vec{B} = \boxed{4\hat{j} + 3\hat{k}}$$

34 •• If $\vec{A} = 4\hat{i}$, $B_z = 0$, $\left|\vec{B}\right| = 5$, and $\vec{A}\times\vec{B} = 12\hat{k}$, determine $\vec{B}$.

Picture the Problem Because $B_z = 0$, we can express $\vec{B}$ as $\vec{B} = B_x\hat{i} + B_y\hat{j}$ and form its vector product with $\vec{A}$ to determine B_x and B_y.

Express $\vec{B}$ in terms of its components:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} \qquad (1)$$

Express $\vec{A}\times\vec{B}$:

$$\vec{A}\times\vec{B} = 4\hat{i}\times\left(B_x\hat{i} + B_y\hat{j}\right) = 4B_y\hat{k} = 12\hat{k}$$

Solving for B_y yields:

$$B_y = 3$$

Relate B to B_x and B_y:

$$B^2 = B_x^2 + B_y^2$$

Solve for and evaluate B_x:

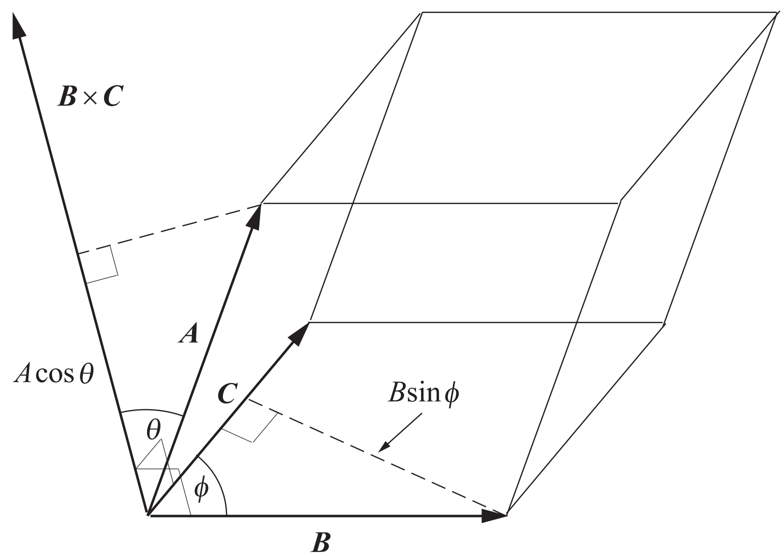
$$B_x = \sqrt{B^2 - B_y^2} = \sqrt{5^2 - 3^2} = 4$$

Substitute for B_x and B_y in equation (1) to obtain:

$$\vec{B} = \boxed{4\hat{i} + 3\hat{j}}$$

35 ••• Given three noncoplanar vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, show that $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ is the volume of the parallelepiped formed by the three vectors.

Picture the Problem Let, without loss of generality, the vector $\vec{C}$ lie along the x axis and the vector $\vec{B}$ lie in the xy plane as shown below to the left. The diagram to the right shows the parallelepiped spanned by the three vectors. We can apply the definitions of the vector- and scalar-products to show that $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ is the volume of the parallelepiped.



The magnitude of the vector product of $\vec{B}$ and $\vec{C}$ is:

$$\left|\vec{B}\times\vec{C}\right| = BC\sin\phi$$

and

$$\begin{aligned}\left|\vec{B}\times\vec{C}\right| &= (B\sin\phi)C \\ &= \text{area of the base parallelogram}\end{aligned}$$

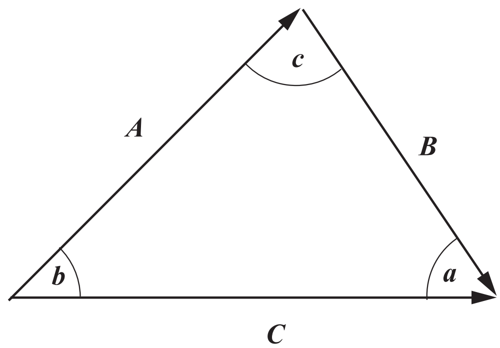
Forming the scalar-product of $\vec{A}$ with the vector-product of $\vec{B}$ and $\vec{C}$ gives:

$$\begin{aligned}\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) &= A(B\sin\phi)C\cos\theta \\ &= (BC\sin\phi)(A\cos\theta) \\ &= (\text{area of base})(\text{height}) \\ &= \boxed{V_{\text{parallelepiped}}}\end{aligned}$$

36 ••• Using the cross product, prove the *law of sines* for the triangle shown in Figure 10-43. That is, if A, B, and C are the lengths of each side of the triangle, show that $A/\sin a = B/\sin b = C/\sin c$.

Picture the Problem Draw the triangle using the three vectors as shown below. Note that $\vec{A}+\vec{B}=\vec{C}$. We can find the magnitude of the vector product of $\vec{A}$ and $\vec{B}$ and of $\vec{A}$ and $\vec{C}$ and then use the vector product of $\vec{A}$ and $\vec{C}$, using

$\vec{A}+\vec{B}=\vec{C},$ to show that $AC\sin b = AB\sin c$ or $\dfrac{B}{\sin b}=\dfrac{C}{\sin c}$. Proceeding similarly, we can extend the law of sines to the third side of the triangle and the angle opposite it.



Express the magnitude of the vector product of $\vec{A}$ and $\vec{B}$:

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\left(180° - c\right) = AB\sin c$$

Express the magnitude of the vector product of $\vec{A}$ and $\vec{C}$:

$$\left|\vec{A}\times\vec{C}\right| = AC\sin b$$

Form the vector product of $\vec{A}$ with $\vec{C}$ to obtain:

$$\begin{aligned}\vec{A}\times\vec{C} &= \vec{A}\times\left(\vec{A}+\vec{B}\right)\\ &= \vec{A}\times\vec{A}+\vec{A}\times\vec{B}\\ &= \vec{A}\times\vec{B}\end{aligned}$$

because $\vec{A}\times\vec{A}=0$.

Because $\vec{A}\times\vec{C}=\vec{A}\times\vec{B}$:

$$\left|\vec{A}\times\vec{C}\right| = \left|\vec{A}\times\vec{B}\right|$$

and

$$AC\sin b = AB\sin c$$

Simplify and rewrite this expression to obtain:

$$\boxed{\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Proceed similarly to extend this result to the law of sines:

$$\boxed{\frac{A}{\sin a}=\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Torque and Angular Momentum

37 • [SSM] A 2.0-kg particle moves directly eastward at a constant speed of 4.5 m/s along an east-west line. (*a*) What is its angular momentum (including direction) about a point that lies 6.0 m north of the line? (*b*) What is its angular momentum (including direction) about a point that lies 6.0 m south of the line?

(*c*) What is its angular momentum (including direction) about a point that lies 6.0 m directly east of the particle?

Picture the Problem The angular momentum of the particle is $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the vector locating the particle relative to the reference point and $\vec{p}$ is the particle's linear momentum.

(*a*) The magnitude of the particle's angular momentum is given by:

$$L = rp\sin\phi = rmv\sin\phi = mv(r\sin\phi)$$

Substitute numerical values and evaluate *L*:

$$L = (2.0\,\text{kg})(4.5\,\text{m/s})(6.0\,\text{m}) = 54\,\text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}$:

$$L = \boxed{54\,\text{kg}\cdot\text{m}^2/\text{s, upward}}$$

(*b*) Because the distance to the line along which the particle is moving is the same, only the direction of $\vec{L}$ differs:

$$L = \boxed{54\,\text{kg}\cdot\text{m}^2/\text{s, downward}}$$

(*c*) Because $\vec{r} \times \vec{p} = 0$ for a point on the line along which the particle is moving:

$$\vec{L} = \boxed{0}$$

38 • You observe a 2.0-kg particle moving at a constant speed of 3.5 m/s in a clockwise direction around a circle of radius 4.0 m. (*a*) What is its angular momentum (including direction) about the center of the circle? (*b*) What is its moment of inertia about an axis through the center of the circle and perpendicular to the plane of the motion? (*c*) What is the angular velocity of the particle?

Picture the Problem The angular momentum of the particle is $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the vector locating the particle relative to the reference point and $\vec{p}$ is the particle's linear momentum.

(*a*) The magnitude of the particle's angular momentum is given by:

$$L = rp\sin\phi = rmv\sin\phi = mv(r\sin\phi)$$

Substitute numerical values and evaluate the magnitude of *L*:

$$L = (2.0\,\text{kg})(3.5\,\text{m/s})(4.0\,\text{m}) = 28\,\text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}$: $L = \boxed{28\,\text{kg}\cdot\text{m}^2/\text{s, away from you}}$

(*b*) Treat the 2.0-kg particle as a point particle to obtain: $I = mr^2$

Substitute numerical values and evaluate *I*: $I = (2.0\,\text{kg})(4.0\,\text{m})^2 = \boxed{32\,\text{kg}\cdot\text{m}^2}$

(*c*) Because $L = I\omega$, the angular speed of the particle is the ratio of its angular momentum and its moment of inertia: $\omega = \frac{L}{I}$

Substitute numerical values and evaluate ω: $\omega = \frac{28\,\text{kg}\cdot\text{m}^2/\text{s}}{32\,\text{kg}\cdot\text{m}^2} = \boxed{0.88\,\text{rad/s}^2}$

39 •• (*a*) A particle moving at constant velocity has zero angular momentum about a particular point. Use the definition of angular momentum to show that under this condition the particle is moving either directly toward or directly away from the point. (*b*) You are a right-handed batter and let a waist-high fastball go past you without swinging. What is the direction of its angular momentum relative to your navel? (Assume the ball travels in a straight horizontal line as it passes you.)

Picture the Problem $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r}\times\vec{p}$. If $\vec{L} = 0$, then examination of the magnitude of $\vec{r}\times\vec{p}$ will allow us to conclude that $\sin\phi = 0$ and that the particle is moving either directly toward the point, directly away from the point, or through the point.

(*a*) Because $\vec{L} = 0$: $\vec{r}\times\vec{p} = \vec{r}\times m\vec{v} = m\vec{r}\times\vec{v} = 0$

or

$\vec{r}\times\vec{v} = 0$

Express the magnitude of $\vec{r}\times\vec{v}$: $|\vec{r}\times\vec{v}| = rv\sin\phi = 0$

Because neither *r* nor *v* is zero: $\sin\phi = 0$

where ϕ is the angle between $\vec{r}$ and $\vec{v}$.

Solving for ϕ yields: $\phi = \sin^{-1}(0) = \boxed{0°\text{ or }180°}$

(*b*) Use the right-hand rule to establish that the ball's angular momentum is downward.

40 •• A particle that has a mass m is traveling with a constant velocity $\vec{v}$ along a straight line that is a distance b from the origin O (Figure 10-44). Let dA be the area swept out by the position vector from O to the particle during a time interval dt. Show that dA/dt is constant and is equal to $L/2m$, where L is the magnitude of the angular momentum of the particle about the origin.

Picture the Problem The area swept out by the position vector (the shaded area in Figure 10-44) is the difference between the area of a trapezoid and the area of a triangle. Let x_1 be the x component of $\vec{r}_1$ and $x_1 + \Delta x$ be the x component of $\vec{r}_2$. Use the formulas for the areas of a trapezoid and a triangle to express ΔA and then take the limit as $\Delta t \to 0$ to express dA/dt. Letting θ be the angle between $\vec{r}_1$ and the horizontal axis, we can express b as a function of $|\vec{r}_1|$ and θ.

The area swept out by the position vector (the shaded area in Figure 10-44) is given by:

$$\begin{aligned}\Delta A &= A_{\text{trap}} - A_{\text{triangle}}\\ &= \tfrac{1}{2}b[\Delta x + (x_1 + \Delta x)] - \tfrac{1}{2}b(x_1 + \Delta x)\\ &= \tfrac{1}{2}b\Delta x\end{aligned}$$

In the limit as $\Delta t \to 0$:

$$\frac{dA}{dt} = \tfrac{1}{2}b\frac{dx}{dt} = \tfrac{1}{2}bv = \text{constant}$$

Because $b = |\vec{r}_1|\sin\theta$:

$$\begin{aligned}\frac{dA}{dt} &= \tfrac{1}{2}bv = \tfrac{1}{2}(|\vec{r}_1|\sin\theta)v = \frac{(|\vec{r}_1|\sin\theta)mv}{2m}\\ &= \frac{1}{2m}(|\vec{r}_1|p\sin\theta) = \boxed{\frac{L}{2m}}\end{aligned}$$

41 •• A 15-g coin that has a diameter of 1.5 cm is spinning at 10 rev/s about a fixed vertical axis. The coin is spinning on edge with its center directly above the point of contact with the tabletop. As you look down on the tabletop, the coin spins clockwise. (*a*) What is the angular momentum (including direction) of the coin about its center of mass? (To find the moment of inertia about the axis, see Table 9-1.) Model the coin as a cylinder of length L and take the limit as L approaches zero. (*b*) What is the coin's angular momentum (including direction) about a point on the tabletop 10 cm from the axis? (*c*) Now the coin's center of mass travels at 5.0 cm/s in a straight line east across the tabletop, while spinning the same way as in Part (*a*). What is the angular momentum (including direction) of the coin about a point on the line of motion of the center of mass? (*d*) When it is both spinning and sliding, what is the angular momentum of the coin (including direction) about a point 10 cm north of the line of motion of the center of mass?

Picture the Problem We can find the total angular momentum of the coin from the sum of its spin and orbital angular momenta.

(*a*) The spin angular momentum of the coin is:

$$L_{\text{spin}} = I\omega_{\text{spin}}$$

From Table 9-1, for *L* negligible compared to *R*:

$$I = \tfrac{1}{4}MR^2$$

Substitute for *I* to obtain:

$$L_{\text{spin}} = \tfrac{1}{4}MR^2\omega_{\text{spin}}$$

Substitute numerical values and evaluate L_{spin}:

$$L_{\text{spin}} = \tfrac{1}{4}(0.015\,\text{kg})(0.0075\,\text{m})^2\left(10\frac{\text{rev}}{\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 1.33\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}_{\text{spin}}$:

$$L_{\text{spin}} = \boxed{1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, away from you.}}$$

(*b*)The total angular momentum of the coin is the sum of its orbital and spin angular momenta:

$$L_{\text{total}} = L_{\text{orbital}} + L_{\text{spin}}$$

Substitute numerical values and evaluate L_{total}:

$$L_{\text{total}} = 0 + L_{\text{spin}} = 1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}_{\text{total}}$:

$$L_{\text{total}} = \boxed{1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*c*) Because $L_{\text{orbital}} = 0$:

$$L_{\text{total}} = \boxed{1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*d*) When it is both spinning and sliding, the total angular momentum of the coin is:

$$L_{\text{total}} = L_{\text{orbital}} + L_{\text{spin}}$$

The orbital angular momentum of the coin is: $L_{\text{orbital}} = MvR$

The spin angular momentum of the coin is: $L_{\text{spin}} = I_{\text{spin}}\omega_{\text{spin}} = \frac{1}{4}MR^2\omega_{\text{spin}}$

Substituting for L_{orbital} and L_{spin} yields: $L_{\text{total}} = MvR + \frac{1}{4}MR^2\omega_{\text{spin}}$

Substitute numerical values and evaluate L_{total}:

$$L_{\text{total}} = (0.015\,\text{kg})(0.050\,\text{m/s})(0.10\,\text{m}) + \frac{1}{4}(0.015\,\text{kg})(0.0075\,\text{m})^2\left(10\,\frac{\text{rev}}{\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right)$$
$$= \boxed{8.8\times10^{-5}\,\text{kg}\cdot\text{m}^2\text{/s, pointing toward you}}$$

42 •• (*a*) Two stars of masses m_1 and m_2 are located at $\vec{r}_1$ and $\vec{r}_2$ relative to some origin *O*, as shown in Figure 10-45. They exert equal and opposite attractive gravitational forces on each other. For this two-star system, calculate the net torque exerted by these internal forces about the origin *O* and show that it is zero only if both forces lie along the line joining the particles. (*b*)The fact that the Newton's third-law pair of forces are not only equal and oppositely directed but also lie along the line connecting the two objects is sometimes called the strong form of Newton's third law. Why is it important to add that last phrase? *Hint: Consider what would happen to these two objects if the forces were offset from each other.*

Picture the Problem Both the forces acting on the particles exert torques with respect to an axis perpendicular to the page and through point O and the net torque about this axis is their vector sum.

(*a*) The net torque about an axis perpendicular to the page and through point O is given by:

$$\vec{\tau}_{\text{net}} = \sum_{\text{i}} \vec{\tau}_{\text{i}} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

or, because $\vec{F}_2 = -\vec{F}_1$,

$$\vec{\tau}_{\text{net}} = (\vec{r}_1 - \vec{r}_2)\times\vec{F}_1$$

Because $\vec{r}_1 - \vec{r}_2$ points along $-\vec{F}_1$: $\vec{\tau}_{\text{net}} = (\vec{r}_1 - \vec{r}_2)\times\vec{F}_1 = \boxed{0}$

(*b*) If the forces are not along the same line, there will be a net torque (but still no net force) acting on the system. This net torque would cause the system to accelerate angularly, contrary to observation, and hence makes no sense physically.

43 •• A 1.8-kg particle moves in a circle of radius 3.4 m. As you look down on the plane of its orbit, it is initially moving clockwise. If we call the clockwise direction positive, its angular momentum relative to the center of the circle varies with time according to $L(t) = 10\ \mathrm{N\cdot m\cdot s} - (4.0\ \mathrm{N\cdot m})t$. (*a*) Find the magnitude and direction of the torque acting on the particle. (*b*) Find the angular velocity of the particle as a function of time.

Picture the Problem The angular momentum of the particle changes because a *net* torque acts on it. Because we know how the angular momentum depends on time, we can find the net torque acting on the particle by differentiating its angular momentum. We can use a constant-acceleration equation and Newton's second law to relate the angular speed of the particle to its angular acceleration.

(*a*) The magnitude of the torque acting on the particle is the rate at which its angular momentum changes:

$$\tau_{\text{net}} = \frac{dL}{dt}$$

Evaluate dL/dt to obtain:

$$\tau_{\text{net}} = \frac{d}{dt}\left[10\ \mathrm{N\cdot m\cdot s} - (4.0\ \mathrm{N\cdot m})t\right] = \boxed{-4.0\ \mathrm{N\cdot m}}$$

Note that, because L decreases as the particle rotates clockwise, the angular acceleration and the net torque are both upward.

(*b*) The angular speed of the particle is given by:

$$\omega_{\text{orbital}} = \frac{L_{\text{orbital}}}{I_{\text{orbital}}}$$

Treating the 1.8-kg particle as a point particle, express its moment of inertia relative to an axis through the center of the circle and normal to it:

$$I_{\text{orbital}} = MR^2$$

Substitute for I_{orbital} and L_{orbital} to obtain:

$$\omega_{\text{orbital}} = \frac{10\ \mathrm{N\cdot m\cdot s} - (4.0\ \mathrm{N\cdot m})t}{MR^2}$$

Substitute numerical values and evaluate ω_{orbital}:

$$\omega_{\text{orbital}} = \frac{10\,\text{N}\cdot\text{m}\cdot\text{s} - (4.0\,\text{N}\cdot\text{m})t}{(1.8\,\text{kg})(3.4\,\text{m})^2} = \boxed{0.48\,\text{rad/s} - (0.19\,\text{rad/s}^2)t}$$

Note that the direction of the angular velocity is downward.

44 •• You are designing a lathe motor and part of it consists of a uniform cylinder whose mass is 90 kg and radius is 0.40 m that is mounted so that it turns without friction on its axis, which is fixed. The cylinder is driven by a belt that wraps around its perimeter and exerts a constant torque. At $t = 0$, the cylinder's angular velocity is zero. At $t = 25$ s, its angular speed is 500 rev/min. (*a*) What is the magnitude of its angular momentum at $t = 25$ s? (*b*) At what rate is the angular momentum increasing? (*c*) What is the magnitude of the torque acting on the cylinder? (*d*) What is the magnitude of the frictional force acting on the rim of the cylinder?

Picture the Problem The angular momentum of the cylinder changes because a *net* torque acts on it. We can find the angular momentum at $t = 25$ s from its definition and the magnitude of the *net* torque acting on the cylinder from the rate at which the angular momentum is changing. The magnitude of the frictional force acting on the rim can be found using the definition of torque.

(*a*) The angular momentum of the cylinder is given by:

$$L = I\omega = \tfrac{1}{2}mr^2\omega$$

Substitute numerical values and evaluate L:

$$L = \tfrac{1}{2}(90\,\text{kg})(0.40\,\text{m})^2\left(500\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)$$
$$= 377\,\text{kg}\cdot\text{m}^2/\text{s} = \boxed{3.8\times10^2\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) The rate at which the angular momentum of the cylinder is increasing is given by:

$$\frac{dL}{dt} = \frac{(377\,\text{kg}\cdot\text{m}^2/\text{s})}{25\,\text{s}} = 15\,\text{kg}\cdot\text{m}^2/\text{s}^2$$
$$= \boxed{15\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

(*c*) Because the torque acting on the uniform cylinder is constant, the rate of change of the angular momentum is constant and hence the instantaneous rate of change of the angular momentum at any instant is equal to the average rate of change over the time during which the torque acts:

$$\tau = \frac{dL}{dt} = \boxed{15\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

(*d*) The magnitude of the frictional force *f* acting on the rim is:

$$f = \frac{\tau}{\ell} = \frac{15.1\,\text{kg}\cdot\text{m}^2/\text{s}^2}{0.40\,\text{m}} = \boxed{38\,\text{N}}$$

45 •• [SSM] In Figure 10-46, the incline is frictionless and the string passes through the center of mass of each block. The pulley has a moment of inertia *I* and radius *R*. (*a*) Find the net torque acting on the system (the two masses, string, and pulley) about the center of the pulley. (*b*)Write an expression for the total angular momentum of the system about the center of the pulley. Assume the masses are moving with a speed *v*. (*c*) Find the acceleration of the masses by using your results for Parts (*a*) and (*b*) and by setting the net torque equal to the rate of change of the system's angular momentum.

Picture the Problem Let the system include the pulley, string, and the blocks and assume that the mass of the string is negligible. The angular momentum of this system changes because a *net* torque acts on it. We'll take clockwise to be positive to be consistent with a positive upward velocity of the block whose mass is m_1 as indicated in the figure.

(*a*) Express the net torque about the center of mass of the pulley:

$$\tau_{\text{net}} = -m_1 gR + T_1 R - T_1 R + T_2 R + (m_2 g\sin\theta)R - Rm_1 g - T_2 R = \boxed{Rg(m_2\sin\theta - m_1)}$$

(*b*) Express the total angular momentum of the system about an axis through the center of the pulley:

$$L = I\omega + m_1 vR + m_2 vR$$

$$= \boxed{vR\left(\frac{I}{R^2} + m_1 + m_2\right)}$$

(*c*) Express τ as the time derivative of the angular momentum:

$$\tau = \frac{dL}{dt} = \frac{d}{dt}\left[vR\left(\frac{I}{R^2} + m_1 + m_2\right)\right]$$
$$= aR\left(\frac{I}{R^2} + m_1 + m_2\right)$$

Equate this result to that of Part (*a*) and solve for *a* to obtain:

$$a = \boxed{\frac{g(m_2 \sin\theta - m_1)}{\frac{I}{R^2} + m_1 + m_2}}$$

46 •• Figure 10-47 shows the rear view of a space capsule that was left rotating rapidly about its longitudinal axis at 30 rev/min after a collision with another capsule. You are the flight controller and have just moments to tell the crew how to stop this rotation before they become ill from the rotation and the situation becomes dangerous. You know that they have access to two small jets mounted tangentially at a distance of 3.0 m from the axis, as indicated in the figure. These jets can each eject 10 g/s of gas with a nozzle speed of 800 m/s. Determine the length of time these jets must run to stop the rotation. In flight, the moment of inertia of the ship about its axis (assumed constant) is known to be 4000 kg·m^2.

Picture the Problem The forces resulting from the release of gas from the jets will exert a torque on the spaceship that will slow and eventually stop its rotation. We can relate this net torque to the angular momentum of the spaceship and to the time the jets must fire.

Relate the firing time of the jets to the desired change in angular momentum:

$$\Delta t = \frac{\Delta L}{\tau_{\text{net}}} = \frac{I\Delta\omega}{\tau_{\text{net}}} \qquad (1)$$

Express the magnitude of the net torque exerted by the jets:

$$\tau_{\text{net}} = 2FR$$

Letting $\Delta m/\Delta t'$ represent the mass of gas per unit time exhausted from the jets, relate the force exerted by the gas on the spaceship to the rate at which the gas escapes:

$$F = \frac{\Delta m}{\Delta t'}v$$

Substituting for *F* yields:

$$\tau_{\text{net}} = 2vR\frac{\Delta m}{\Delta t'}$$

Substitute for τ_{net} in equation (1) to obtain:

$$\Delta t = \frac{I\Delta\omega}{2vR\dfrac{\Delta m}{\Delta t'}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{\left(4000\,\text{kg}\cdot\text{m}^2\right)\left(30\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\pi\,\text{rad}}{\text{rev}}\times\dfrac{1\,\text{min}}{60\,\text{s}}\right)}{2\left(10^{-2}\,\text{kg/s}\right)\left(800\,\text{m/s}\right)\left(3.0\,\text{m}\right)} = \boxed{2.6\times10^{2}\,\text{s}}$$

47 •• A projectile (mass M) is launched at an angle θ with an initial speed v_0. Considering the torque and angular momentum about the launch point, explicitly show that $dL/dt = \tau$. Ignore the effects of air resistance. (The equations for projectile motion are found in Chapter 3.)

Picture the Problem We can use constant-acceleration equations to express the projectile's position and velocity coordinates as functions of time. We can use these coordinates to express the particle's position and velocity vectors $\vec{r}$ and $\vec{v}$. Using its definition, we can express the projectile's angular momentum $\vec{L}$ as a function of time and then differentiate this expression to obtain $d\vec{L}/dt$. Finally, we can use the definition of the torque, relative to an origin located at the launch position, the gravitational force exerts on the projectile to express $\vec{\tau}$ and complete the demonstration that $d\vec{L}/dt = \vec{\tau}$.

Using its definition, express the angular momentum vector $\vec{L}$ of the projectile:

$$\vec{L} = \vec{r}\times m\vec{v} \qquad (1)$$

Using constant-acceleration equations, express the position coordinates of the projectile as a function of time:

$$x = v_{0x}t = \left(v_0\cos\theta\right)t$$

and

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_yt^2 = \left(v_0\sin\theta\right)t - \tfrac{1}{2}gt^2$$

Express the projectile's position vector $\vec{r}$:

$$\vec{r} = \left[\left(v_0\cos\theta\right)t\right]\hat{i} + \left[\left(v_0\sin\theta\right)t - \tfrac{1}{2}gt^2\right]\hat{j}$$

Using constant-acceleration equations, express the velocity of the projectile as a function of time:

$$v_x = v_{0x} = v_0\cos\theta$$

and

$$v_y = v_{0y} + a_yt = v_0\sin\theta - gt$$

Express the projectile's velocity vector $\vec{v}$:

$$\vec{v} = \left[v_0\cos\theta\right]\hat{i} + \left[v_0\sin\theta - gt\right]\hat{j}$$

Substituting in equation (1) and simplifying yields:

$$\vec{L} = \left\{\left[(V\cos\theta)t\right]\hat{i} + \left[(V\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j}\right\} \times m\left\{\left[V\cos\theta\right]\hat{i} + \left[V\sin\theta - gt\right]\hat{j}\right\}$$
$$= \left(-\tfrac{1}{2}mgt^2V\cos\theta\right)\hat{k}$$

Differentiate $\vec{L}$ with respect to t to obtain:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(-\tfrac{1}{2}mgt^2V\cos\theta\right)\hat{k} = \left(-mgtV\cos\theta\right)\hat{k} \quad (2)$$

Using its definition, express the torque acting on the projectile:

$$\vec{\tau} = \vec{r} \times (-mg)\hat{j} = \left[(v_0\cos\theta)t\right]\hat{i} + \left[(v_0\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j} \times (-mg)\hat{j} = \left(-mgtV\cos\theta\right)\hat{k} \quad (3)$$

Comparing equations (2) and (3) we see that:

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Conservation of Angular Momentum

48 • A planet moves in an elliptical orbit about the Sun, with the Sun at one focus of the ellipse, as in Figure 10-48. (*a*) What is the torque about the center of the Sun due to the gravitational force of attraction of the Sun on the planet? (*b*) At position A, the planet has an orbital radius r_1 and is moving with a speed v_1 perpendicular to the line from the sun to the planet. At position B, the planet has an orbital radius r_2 and is moving with speed v_2, again perpendicular to the line from the sun to the planet. What is the ratio of v_1 to v_2 in terms of r_1 and r_2?

Picture the Problem Let m represent the mass of the planet and apply the definition of torque to find the torque produced by the gravitational force of attraction. We can use Newton's second law of motion in the form $\vec{\tau} = d\vec{L}/dt$ to show that $\vec{L}$ is constant and apply conservation of angular momentum to the motion of the planet at points A and B.

(*a*) Express the torque produced by the gravitational force of attraction of the sun for the planet:

$\vec{\tau} = \vec{r} \times \vec{F} = \boxed{0}$ because $\vec{F}$ acts along the direction of $\vec{r}$.

(*b*) Because $\vec{\tau} = 0$:

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \vec{r} \times m\vec{v} = \text{constant}$$

Noting that at points A and B $|\vec{r}\times\vec{v}| = rv$, express the relationship between the distances from the sun and the speeds of the planets:

$$r_1v_1 = r_2v_2 \Rightarrow \frac{v_1}{v_2} = \boxed{\frac{r_2}{r_1}}$$

49 •• **[SSM]** You stand on a frictionless platform that is rotating at an angular speed of 1.5 rev/s. Your arms are outstretched, and you hold a heavy weight in each hand. The moment of inertia of you, the extended weights, and the platform is 6.0 kg·m^2. When you pull the weights in toward your body, the moment of inertia decreases to 1.8 kg·m^2. (*a*) What is the resulting angular speed of the platform? (*b*) What is the change in kinetic energy of the system? (*c*) Where did this increase in energy come from?

Picture the Problem Let the system consist of you, the extended weights, and the platform. Because the net external torque acting on this system is zero, its angular momentum remains constant during the pulling in of the weights.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the system to its initial and final moments of inertia:

$$I_i\omega_i - I_f\omega_f = 0 \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \frac{6.0\,\text{kg}\cdot\text{m}^2}{1.8\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) = \boxed{5.0\,\text{rev/s}}$$

(*b*) Express the change in the kinetic energy of the system:

$$\Delta K = K_f - K_i = \tfrac{1}{2}I_f\omega_f^2 - \tfrac{1}{2}I_i\omega_i^2$$

Substitute numerical values and evaluate ΔK:

$$\Delta K = \tfrac{1}{2}(1.8\,\text{kg}\cdot\text{m}^2)\left(5.0\frac{\text{rev}}{\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right)^2 - \tfrac{1}{2}(6.0\,\text{kg}\cdot\text{m}^2)\left(1.5\frac{\text{rev}}{\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right)^2$$
$$= \boxed{0.62\,\text{kJ}}$$

(*c*) Because no external agent does work on the system, the energy comes from your internal energy.

50 •• A small blob of putty of mass m falls from the ceiling and lands on the outer rim of a turntable of radius R and moment of inertia I_0 that is rotating freely with angular speed ω_0 about its vertical fixed-symmetry axis. (*a*) What is the post-collision angular speed of the turntable-putty system? (*b*) After several turns, the

blob flies off the edge of the turntable. What is the angular speed of the turntable after the blob's departure?

Picture the Problem Let the system consist of the blob of putty and the turntable. Because the net external torque acting on this system is zero, its angular momentum remains constant when the blob of putty falls onto the turntable.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the turntable to its initial and final moments of inertia and solve for ω_f:

$$I_0\omega_0 - I_f\omega_f = 0 \Rightarrow \omega_f = \frac{I_0}{I_f}\omega_0 \quad (1)$$

Express the final rotational inertia of the turntable-plus-blob:

$$I_f = I_0 + I_{blob} = I_0 + mR^2$$

Substitute for I_f in equation (1) and simplify to obtain:

$$\omega_f = \frac{I_0}{I_0 + mR^2}\omega_0 = \boxed{\frac{1}{1+\dfrac{mR^2}{I_0}}\omega_0}$$

(*b*) If the blob flies off tangentially to the turntable, its angular momentum doesn't change (with respect to an axis through the center of turntable). Because there is no external torque acting on the blob-turntable system, the total angular momentum of the system will remain constant and the angular momentum of the turntable will not change. The turntable will continue to spin at $\boxed{\omega' = \omega_f}$.

51 •• [SSM] A lazy Susan consists of a heavy plastic disk mounted on a frictionless bearing resting on a vertical shaft through its center. The cylinder has a radius R = 15 cm and mass M = 0.25 kg. A cockroach (mass m = 0.015 kg) is on the lazy Susan, at a distance of 8.0 cm from the center. Both the cockroach and the lazy Susan are initially at rest. The cockroach then walks along a circular path concentric with the center of the Lazy Susan at a constant distance of 8.0 cm from the axis of the shaft. If the speed of the cockroach with respect to the lazy Susan is 0.010 m/s, what is the speed of the cockroach with respect to the room?

Picture the Problem Because the net external torque acting on the lazy Susan-cockroach system is zero, the net angular momentum of the system is constant (equal to zero because the lazy Susan is initially at rest) and we can use conservation of angular momentum to find the angular velocity ω of the lazy Susan. The speed of the cockroach relative to the floor v_f is the difference between its speed with respect to the lazy Susan and the speed of the lazy Susan at the location of the cockroach with respect to the floor.

Relate the speed of the cockroach with respect to the floor v_f to the speed of the lazy Susan at the location of the cockroach:

$$v_f = v - \omega r \quad (1)$$

Use conservation of angular momentum to obtain:

$$L_{LS} - L_C = 0 \quad (2)$$

Express the angular momentum of the lazy Susan:

$$L_{LS} = I_{LS}\omega = \tfrac{1}{2}MR^2\omega$$

Express the angular momentum of the cockroach:

$$L_C = I_C\omega_C = mr^2\left(\frac{v}{r} - \omega\right)$$

Substitute for L_{LS} and L_C in equation (2) to obtain:

$$\tfrac{1}{2}MR^2\omega - mr^2\left(\frac{v}{r} - \omega\right) = 0$$

Solving for ω yields:

$$\omega = \frac{2mrv}{MR^2 + 2mr^2}$$

Substitute for ω in equation (1) to obtain:

$$v_f = v - \frac{2mr^2v}{MR^2 + 2mr^2}$$

Substitute numerical values and evaluate v_f:

$$v_f = 0.010\,\text{m/s} - \frac{2(0.015\,\text{kg})(0.080\,\text{m})^2(0.010\,\text{m/s})}{(0.25\,\text{m})(0.15\,\text{m})^2 + 2(0.015\,\text{kg})(0.080\,\text{m})^2} = \boxed{10\,\text{mm/s}}$$

Remarks: Because the moment of inertia of the lazy Susan is so much larger than the moment of inertia of the cockroach, after the cockroach begins moving, the angular speed of the lazy Susan is very small. Therefore, the speed of the cockroach relative to the floor is almost the same as the speed relative to the lazy Susan.

52 •• Two disks of identical mass but different radii (r and $2r$) are spinning on frictionless bearings at the same angular speed ω_0 but in opposite directions (Figure 10-49). The two disks are brought slowly together. The resulting frictional force between the surfaces eventually brings them to a common angular velocity. (a) What is the magnitude of that final angular velocity in terms of ω_0? (b) What is the change in rotational kinetic energy of the system? Explain.

Picture the Problem The net external torque acting on this system is zero and so we know that angular momentum is conserved as these disks are brought together. Let the numeral 1 refer to the disk to the left and the numeral 2 to the disk to the right. Let the angular momentum of the disk with the larger radius be positive.

(*a*) Using conservation of angular momentum, relate the initial angular speeds of the disks to their common final speed and to their moments of inertia:

$$I_i\omega_i = I_f\omega_f$$

or

$$I_1\omega_0 - I_2\omega_0 = (I_1 + I_2)\omega_f$$

Solving for ω_f yields:

$$\omega_f = \frac{I_1 - I_2}{I_1 + I_2}\omega_0 \qquad (1)$$

Express I_1 and I_2:

$$I_1 = \tfrac{1}{2}m(2r)^2 = 2mr^2$$

and

$$I_2 = \tfrac{1}{2}mr^2$$

Substitute for I_1 and I_2 in equation (1) and simplify to obtain:

$$\omega_f = \frac{2mr^2 - \frac{1}{2}mr^2}{2mr^2 + \frac{1}{2}mr^2}\omega_0 = \boxed{\tfrac{3}{5}\omega_0}$$

(*b*) The change in kinetic energy of the system is given by:

$$\Delta K = K_f - K_i \qquad (2)$$

The initial kinetic energy of the system is the sum of the kinetic energies of the two disks:

$$K_i = K_1 + K_2 = \tfrac{1}{2}I_1\omega_0^2 + \tfrac{1}{2}I_2\omega_0^2 = \tfrac{1}{2}(I_1 + I_2)\omega_0^2$$

Substituting for K_f and K_i in equation (2) yields:

$$\Delta K = \tfrac{1}{2}(I_1 + I_2)\omega_f^2 - \tfrac{1}{2}(I_1 + I_2)\omega_0^2$$

Substitute for ω_f from part (*a*) and simplify to obtain:

$$\Delta K = \tfrac{1}{2}(I_1 + I_2)\left(\tfrac{3}{5}\omega_0\right)^2 - \tfrac{1}{2}(I_1 + I_2)\omega_0^2 = -\tfrac{16}{25}\left[\tfrac{1}{2}(I_1 + I_2)\omega_0^2\right]$$

Noting that the quantity in brackets is K_i, substitute to obtain:

$$\Delta K = \boxed{-\tfrac{16}{25}K_i}$$

The frictional force between the surfaces is responsible for some of the initial kinetic energy being converted to thermal energy as the two disks come together.

53 •• A block of mass m sliding on a frictionless table is attached to a string that passes through a narrow hole through the center of the table. The block is sliding with speed v_0 in a circle of radius r_0. Find (*a*) the angular momentum of the block, (*b*) the kinetic energy of the block, and (*c*) the tension in the string. (*d*) A student under the table now slowly pulls the string downward. How much work is required to reduce the radius of the circle from r_0 to $r_0/2$?

Picture the Problem (*a*) and (*b*) We can express the angular momentum and kinetic energy of the block directly from their definitions. (*c*) The tension in the string provides the centripetal force required for the uniform circular motion and can be expressed using Newton's second law. (*d*) Finally, we can use the work-kinetic energy theorem to express the work required to reduce the radius of the circle by a factor of two.

(*a*) Express the initial angular momentum of the block:

$$L_0 = \boxed{r_0 m v_0}$$

(*b*) Express the initial kinetic energy of the block:

$$K_0 = \boxed{\tfrac{1}{2} m v_0^2}$$

(*c*) Using Newton's second law, relate the tension in the string to the centripetal force required for the circular motion:

$$T = F_c = \boxed{m\frac{v_0^2}{r_0}}$$

(*d*) Use the work-kinetic energy theorem to relate the required work to the change in the kinetic energy of the block:

$$W = \Delta K = K_f - K_0 = \frac{L_f^2}{2I_f} - \frac{L_0^2}{2I_0}$$

$$= \frac{L_0^2}{2I_f} - \frac{L_0^2}{2I_0} = \frac{L_0^2}{2}\left(\frac{1}{I_f} - \frac{1}{I_0}\right)$$

$$= \frac{L_0^2}{2}\left(\frac{1}{m\left(\frac{1}{2}r_0\right)^2} - \frac{1}{mr_0^2}\right)$$

$$= \frac{L_0^2}{2}\left(\frac{4}{mr_0^2} - \frac{1}{mr_0^2}\right) = \frac{L_0^2}{2}\left(\frac{3}{mr_0^2}\right)$$

Substituting for L_0 from Part (*a*) and simplifying gives:

$$W = \frac{(r_0 m v_0)^2}{2}\left(\frac{3}{mr_0^2}\right) = \boxed{\tfrac{3}{2} m v_0^2}$$

54 ••• A 0.20-kg point mass moving on a frictionless horizontal surface is attached to a rubber band whose other end is fixed at point P. The rubber band exerts a force whose magnitude is $F = bx$, where x is the length of the rubber band

and b is an unknown constant. The rubber band force points inward towards P. The mass moves along the dotted line in Figure 10-50. When it passes point A, its velocity is 4.0 m/s, directed as shown. The distance AP is 0.60 m and BP is 1.0 m. (*a*) Find the speed of the mass at points B and C. (*b*) Find b.

Picture the Problem Because the force exerted by the rubber band is parallel to the position vector of the point mass, the net external torque acting on it is zero and we can use the conservation of angular momentum to determine the speeds of the ball at points B and C. We'll use mechanical energy conservation to find b by relating the kinetic and elastic potential energies at A and B.

(*a*) Use conservation of momentum to relate the angular momenta at points A, B and C:

$$L_A = L_B = L_C$$

or

$$mv_A r_A = mv_B r_B = mv_C r_C \qquad (1)$$

Solve for v_B in terms of v_A:

$$v_B = v_A \frac{r_A}{r_B}$$

Substitute numerical values and evaluate v_B:

$$v_B = (4.0\,\text{m/s})\frac{0.60\,\text{m}}{1.0\,\text{m}} = \boxed{2.4\,\text{m/s}}$$

Solve equation (1) for v_C in terms of v_A:

$$v_C = v_A \frac{r_A}{r_C}$$

Substitute numerical values and evaluate v_C:

$$v_C = (4.0\,\text{m/s})\frac{0.60\,\text{m}}{0.60\,\text{m}} = \boxed{4.0\,\text{m/s}}$$

(*b*) Use conservation of mechanical energy between points A and B to relate the kinetic energy of the point mass and the energy stored in the stretched rubber band:

$$\Delta E = E_A - E_B = 0$$

or

$$\tfrac{1}{2}mv_A^2 + \tfrac{1}{2}br_A^2 - \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}br_B^2 = 0$$

Solving for b yields:

$$b = \frac{m\left(v_B^2 - v_A^2\right)}{r_A^2 - r_B^2}$$

Substitute numerical values and evaluate b:

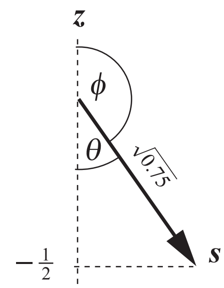
$$b = \frac{(0.20\,\text{kg})\left[(2.4\,\text{m/s})^2 - (4.0\,\text{m/s})^2\right]}{(0.60\,\text{m})^2 - (1.0\,\text{m})^2}$$

$$= \boxed{3\,\text{N/m}}$$

*Quantization of Angular Momentum

55 •• **[SSM]** The z component of the spin of an electron is $-\frac{1}{2}\hbar$, but the magnitude of the spin vector is $\sqrt{0.75}\hbar$. What is the angle between the electron's spin angular momentum vector and the positive z-axis?

Picture the Problem The electron's spin angular momentum vector is related to its z component as shown in the diagram. The angle between $\vec{s}$ and the $+z$-axis is ϕ.



Express ϕ in terms of θ to obtain: $\phi = 180° - \theta$

Using trigonometry, relate the magnitude of $\vec{s}$ to its $-z$ component:

$$\theta = \cos^{-1}\left(\frac{\frac{1}{2}\hbar}{\sqrt{0.75}\hbar}\right)$$

Substitute for θ in the expression for ϕ to obtain:

$$\theta = 180° - \cos^{-1}\left(\frac{\frac{1}{2}\hbar}{\sqrt{0.75}\hbar}\right) = \boxed{125°}$$

56 •• Show that the energy difference between one rotational state of a molecule and the next higher state is proportional to $\ell + 1$.

Picture the Problem Equation 10-29*a* describes the quantization of rotational energy. We can show that the energy difference between a given state and the next higher state is proportional to $\ell + 1$ by using Equation 10-27*a* to express the energy difference.

From Equation 10-29*a* we have: $K_\ell = \ell(\ell + 1)E_{0r}$

Using this equation, express the difference between one rotational state and the next higher state:

$$\Delta E = (\ell + 1)(\ell + 2)E_{0r} - \ell(\ell + 1)E_{0r}$$
$$= \boxed{2(\ell + 1)E_{0r}}$$

57 •• **[SSM]** You work in a bio-chemical research lab, where you are investigating the rotational energy levels of the HBr molecule. After consulting the periodic chart, you know that the mass of the bromine atom is 80 times that of the hydrogen atom. Consequently, in calculating the rotational motion of the molecule, you assume, to a good approximation, that the Br nucleus remains stationary as the H atom (mass 1.67×10^{-27} kg) revolves around it. You also know

that the separation between the H atom and bromine nucleus is 0.144 nm. Calculate (*a*) the moment of inertia of the HBr molecule about the bromine nucleus, and (*b*) the rotational energies for the bromine nucleus's *ground state* (lowest energy) $\ell = 0$, and the next two states of higher energy (called the first and second *excited states*) described by $\ell = 1$, and $\ell = 2$.

Picture the Problem The rotational energies of HBr molecule are related to ℓ and E_{0r} according to $K_\ell = \ell(\ell+1)E_{0r}$ where $E_{0r} = \hbar^2/2I$.

(*a*) Neglecting the motion of the bromine molecule:

$$I_{HBr} \approx m_p r^2 = m_H r^2$$

Substitute numerical values and evaluate I_{HBr}:

$$I_{HBr} \approx (1.67\times10^{-27}\ \text{kg})(0.144\times10^{-9}\ \text{m})^2 = 3.463\times10^{-47}\ \text{kg}\cdot\text{m}^2 = \boxed{3.46\times10^{-47}\ \text{kg}\cdot\text{m}^2}$$

(*b*) Relate the rotational energies to ℓ and E_{0r}:

$$K_\ell = \ell(\ell+1)E_{0r} \text{ where } E_{0r} = \frac{\hbar^2}{2I_{HBr}}$$

Substitute numerical values and evaluate E_{0r}:

$$E_{0r} = \frac{\hbar^2}{2I} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(3.463\times10^{-47}\ \text{kg}\cdot\text{m}^2)} = 1.607\times10^{-22}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = 1.003\,\text{meV}$$

Evaluate E_0 to obtain:

$$E_0 = K_0 = \boxed{1.00\,\text{meV}}$$

Evaluate E_1 to obtain:

$$E_1 = K_1 = (1+1)(1.003\,\text{meV}) = \boxed{2.01\,\text{meV}}$$

Evaluate E_2 to obtain:

$$E_2 = K_2 = 2(2+1)(1.003\,\text{meV}) = \boxed{6.02\,\text{meV}}$$

58 ••• The equilibrium separation between the nuclei of the nitrogen molecule (N_2) is 0.110 nm and the mass of each nitrogen nucleus is 14.0 u, where u = 1.66×10^{-27} kg. For rotational energies, the total energy is due to rotational kinetic energy. (*a*) Approximate the nitrogen molecule as a rigid dumbbell of two equal point masses and calculate the moment of inertia about its center of mass.

(*b*) Find the energy E_ℓ of the lowest three energy levels using $E_\ell = K_\ell = \ell(\ell+1)\hbar^2/(2I)$. (*c*) Molecules emit a particle (or *quantum*) of light called a *photon* when they make a *transition* from a higher energy state to a lower one. Determine the energy of a photon emitted when a nitrogen molecule drops from the $\ell = 2$ to the $\ell = 1$ state. Visible light photons each have between 2 and 3 eV of energy. Are these photons in the visible region?

Picture the Problem We can use the definition of the moment of inertia of point particles to calculate the rotational inertia of the nitrogen molecule. The rotational energies of nitrogen molecule are related to ℓ and E_{0r} according to $E_\ell = K_\ell = \ell(\ell+1)E_{0r}$ where $E_{0r} = \hbar^2/2I$.

(*a*) Using a rigid dumbbell model, express and evaluate the moment of inertia of the nitrogen molecule about its center of mass:

$$I_{N_2} = \sum_i m_i r_i^2 = m_N r^2 + m_N r^2 = 2m_N r^2$$

Substitute numerical values and evaluate I_{N_2}:

$$I_{N_2} = 2(14)(1.66\times10^{-27}\text{ kg})\left(\frac{0.110\text{ nm}}{2}\right)^2 = 1.406\times10^{-46}\text{ kg}\cdot\text{m}^2 = \boxed{1.41\times10^{-46}\text{ kg}\cdot\text{m}^2}$$

(*b*) Relate the rotational energies to ℓ and E_{0r}:

$$E_\ell = K_\ell = \ell(\ell+1)E_{0r}$$

where

$$E_{0r} = \frac{\hbar^2}{2I_{N_2}}$$

Substitute numerical values and evaluate E_{0r}:

$$E_{0r} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(1.406\times10^{-46}\text{ kg}\cdot\text{m}^2)} = 3.958\times10^{-23}\text{ J}\times\frac{1\text{eV}}{1.60\times10^{-19}\text{ J}} = 0.2474\text{ meV}$$

Evaluate E_0 to obtain:

$$E_0 = \boxed{0.247\text{ meV}}$$

Evaluate E_1 to obtain:

$$E_1 = (1+1)(0.2474\text{ meV}) = \boxed{0.495\text{ meV}}$$

Evaluate E_2 to obtain:

$$E_2 = 2(2+1)(0.2474\,\text{meV}) = \boxed{1.48\,\text{meV}}$$

(c) The energy of a photon emitted when a nitrogen molecule drops from the $\ell = 2$ to the $\ell = 1$ state is:

$$\Delta E_{\ell=2\to\ell=1} = E_2 - E_1 = 1.48\,\text{meV} - 0.495\,\text{meV} = \boxed{0.99\,\text{meV}}$$

No. This energy is too low to produce radiation in the visible portion of the spectrum.

59 ••• Consider a *transition* from a lower energy state to a higher one. That is, the absorption of a quantum of energy resulting in an increase in the rotational energy of an N_2 molecule (see Problem 64). Suppose such a molecule, initially in its ground rotational state, was exposed to photons each with energy equal to the three times the energy of its first excited state. (a) Would the molecule be able to absorb this photon energy? Explain why or why not and if it can, determine the energy level to which it goes. (b) To make a transition from its ground state to its second excited state requires how many times the energy of the first excited state?

Picture the Problem The rotational energies of a nitrogen molecule depend on the quantum number ℓ according to $E_\ell = L^2/2I = \ell(\ell+1)\hbar^2/2I$.

(a) No. None of the allowed values of E_ℓ are equal to $E_{0\text{r}}$.

(b) The upward transition from the ground state to the second excited state requires energy given by:

$$\Delta E_{\ell=0\to\ell=2} = E_2 - E_0$$

Set this energy difference equal to a constant n times the energy of the 1st excited state:

$$E_2 - E_0 = nE_1 \Rightarrow n = \frac{E_2 - E_0}{E_1}$$

Substitute numerical values and evaluate n:

$$n = \frac{2(2+1)E_{0\text{r}} - E_{0\text{r}}}{(1+1)E_{0\text{r}}} = \boxed{2.5}$$

Collisions with Rotations

60 •• A 16.0-kg, 2.40-m-long rod is supported at its midpoint on a knife edge. A 3.20-kg ball of clay drops from rest from a height of 1.20 m and makes a perfectly inelastic collision with the rod 0.90 m from the point of support (Figure 10-51). Find the angular momentum of the rod and clay system about the point of support immediately after the inelastic collision.

Picture the Problem Let the zero of gravitational potential energy be at the elevation of the rod. Because the net external torque acting on this system is zero, we know that angular momentum is conserved in the collision. We'll use the definition of angular momentum to express the angular momentum just after the collision and conservation of mechanical energy to determine the speed of the ball just before it makes its perfectly inelastic collision with the rod.

Use conservation of angular momentum to relate the angular momentum before the collision to the angular momentum just after the perfectly inelastic collision:

$$L_f = L_i = mvr \qquad (1)$$

Use conservation of mechanical energy to relate the kinetic energy of the ball just before impact to its initial potential energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0$$

Letting h represent the distance the ball falls, substitute for K_f and U_i to obtain:

$$\tfrac{1}{2}mv^2 - mgh = 0 \Rightarrow v = \sqrt{2gh}$$

Substituting for v in equation (1) yields:

$$L_f = mr\sqrt{2gh}$$

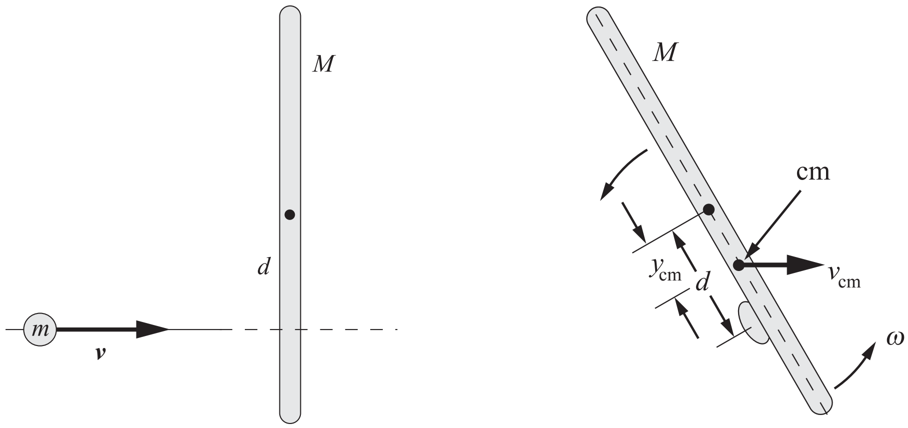
Substitute numerical values and evaluate L_f:

$$L_f = (3.20\,\text{kg})(0.90\,\text{m})\sqrt{2(9.81\,\text{m/s}^2)(1.20\,\text{m})} = \boxed{14\,\text{J}\cdot\text{s}}$$

61 •• [SSM] Figure 10-52 shows a thin uniform bar of length L and mass M and a small blob of putty of mass m. The system is supported by a frictionless horizontal surface. The putty moves to the right with velocity $\vec{v}$, strikes the bar at a distance d from the center of the bar, and sticks to the bar at the point of contact. Obtain expressions for the velocity of the system's center of mass and for the angular speed following the collision.

Picture the Problem The velocity of the center of mass of the bar-blob system does not change during the collision and so we can calculate it before the collision using its definition. Because there are no external forces or torques acting on the bar-blob system, both linear and angular momentum are conserved in the collision. Let the direction the blob of putty is moving initially be the $+x$ direction. Let lower-case letters refer to the blob of putty and upper-case letters refer to the bar.

The diagram to the left shows the blob of putty approaching the bar and the diagram to the right shows the bar-blob system rotating about its center of mass and translating after the perfectly inelastic collision.



The velocity of the center of mass before the collision is given by:

$$(M+m)\vec{v}_{\text{cm}} = m\vec{v} + M\vec{V}$$

or, because $\vec{V} = 0$,

$$\vec{v}_{\text{cm}} = \boxed{\frac{m}{M+m}\vec{v}}$$

Using its definition, express the location of the center of mass relative to the center of the bar:

$$(M+m)y_{\text{cm}} = md \Rightarrow y_{\text{cm}} = \frac{md}{M+m}$$

below the center of the bar.

Express the angular momentum, relative to the center of mass, of the bar-blob system:

$$L_{\text{cm}} = I_{\text{cm}}\omega \Rightarrow \omega = \frac{L_{\text{cm}}}{I_{\text{cm}}} \qquad (1)$$

Express the angular momentum about the center of mass:

$$L_{\text{cm}} = mv(d - y_{\text{cm}}) = mv\left(d - \frac{md}{M+m}\right) = \frac{mMvd}{M+m}$$

Using the parallel axis theorem, express the moment of inertia of the system relative to its center of mass:

$$I_{\text{cm}} = \tfrac{1}{12}ML^2 + My_{\text{cm}}^2 + m(d - y_{\text{cm}})^2$$

Substitute for y_{cm} and simplify to obtain:

$$I_{\text{cm}} = \tfrac{1}{12}ML^2 + M\left(\frac{md}{M+m}\right)^2 + m\left(d - \frac{md}{M+m}\right)^2 = \tfrac{1}{12}ML^2 + \frac{mMd^2}{M+m}$$

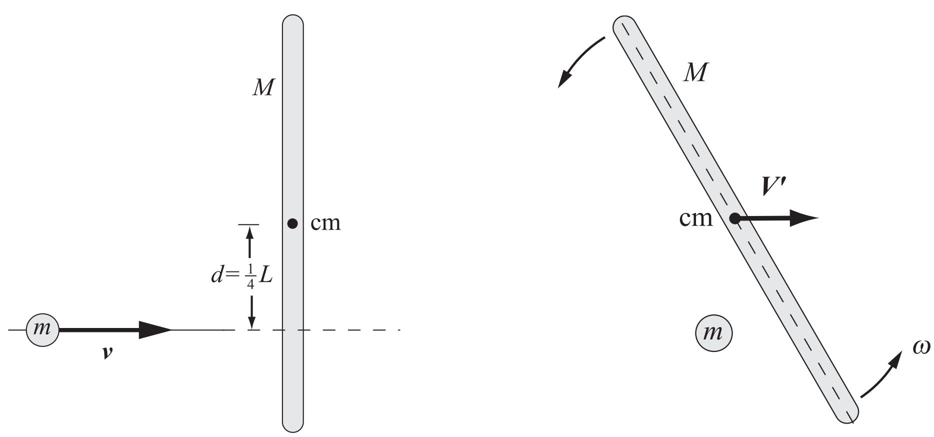
Substitute for I_{cm} and L_{cm} in equation (1) and simplify to obtain:

$$\omega = \boxed{\frac{mMvd}{\frac{1}{12}ML^2(M+m)+Mmd^2}}$$

Remarks: You can verify the expression for I_{cm} by letting $m \to 0$ to obtain $I_{cm} = \frac{1}{12}ML^2$ and letting $M \to 0$ to obtain $I_{cm} = 0$.

62 •• Figure 10-52 shows a thin uniform bar whose length is L and mass is M and a compact hard sphere whose mass is m. The system is supported by a frictionless horizontal surface. The sphere moves to the right with velocity $\vec{v}$, strikes the bar at a distance $\frac{1}{4}L$ from the center of the bar. The collision is elastic, and following the collision the sphere is at rest. Find the value of the ratio m/M.

Picture the Problem Because there are no external forces or torques acting on the bar-sphere system, both linear and angular momentum are conserved in the collision. Kinetic energy is also conserved in the elastic collision of the hard sphere with the bar. Let the direction the sphere is moving initially be the $+x$ direction Let lower-case letters refer to the compact hard sphere and upper-case characters refer to the bar. Let unprimed characters refer to before the collision and primed characters to after the collision. The diagram to the left shows the path of the sphere before its collision with the bar and the diagram to the right shows the sphere at rest after the collision and the bar rotating about its center of mass and translating to the right.



Apply conservation of linear momentum to the collision to obtain:

$$mv = 0 + MV' \Rightarrow V' = \frac{m}{M}v \qquad (1)$$

Apply conservation of angular momentum to the collision to obtain:

$$mvd = 0 + I_{cm}\omega \qquad (2)$$

Apply conservation of mechanical energy to the elastic collision to obtain:

$$\tfrac{1}{2}mv^2 = 0 + \tfrac{1}{2}MV'^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 \quad (3)$$

Use Table 9-1 to find the moment of inertia of a thin bar about an axis through its center:

$$I_{\text{cm}} = \tfrac{1}{12}ML^2$$

Substitute for I_{cm} in equation (2) and simplify to obtain:

$$mvd = \tfrac{1}{12}ML^2\omega \Rightarrow \omega = \left(\frac{12vd}{L^2}\right)\frac{m}{M}$$

Substitute for I_{cm} and V' in equation (3) and simplify to obtain:

$$mv^2 = M\left(\frac{m}{M}\right)^2 v^2 + \tfrac{1}{12}ML^2\omega^2$$

Substituting for ω yields:

$$mv^2 = M\left(\frac{m}{M}\right)^2 v^2 + \tfrac{1}{12}ML^2\left[\left(\frac{12vd}{L^2}\right)\frac{m}{M}\right]^2$$

Solve this equation for $\frac{m}{M}$ to obtain:

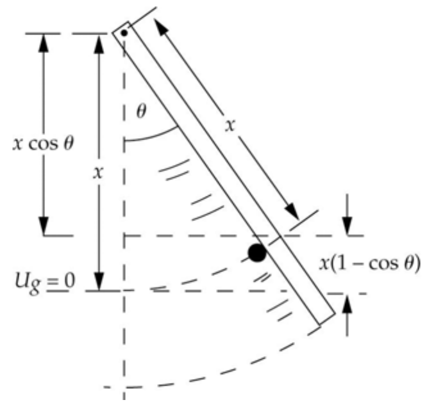
$$\frac{m}{M} = \frac{1}{1 + 12\left(\frac{d}{L}\right)^2}$$

Because $d = L/4$:

$$\frac{m}{M} = \frac{1}{1 + 12\left(\frac{1}{4}\right)^2} = \boxed{\frac{4}{7}}$$

63 •• Figure 10-53 shows a uniform rod whose length is L and whose mass is M pivoted at the top. The rod, which is initially at rest, is struck by a particle whose mass is m at a point $x = 0.8L$ below the pivot. Assume that the particle sticks to the rod. What must be the speed v of the particle so that following the collision the maximum angle between the rod and the vertical is 90°?

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot and ignore friction between the rod and the pivot. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:

$\Delta K + \Delta U = 0$

or, because $K_f = U_i = 0$,

$-K_i + U_f = 0$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}I\omega^2 + \left(Mg\frac{L}{2} + mgx\right)(1-\cos\theta) = 0 \quad (1)$$

Applying conservation of angular momentum to the collision gives:

$$\Delta L = L_f - L_i = 0 \quad (2)$$

The moment of inertia of the system about the pivot is given by:

$$I_f = m(0.8L)^2 + \tfrac{1}{3}ML^2 \quad (3)$$

Substituting for I_f and I_i in equation (2) yields:

$$\left[\tfrac{1}{3}ML^2 + (0.8L)^2 m\right]\omega - 0.8Lmv = 0$$

Solving for ω yields:

$$\omega = \frac{0.8Lmv}{\tfrac{1}{3}ML^2 + 0.64mL^2} \quad (4)$$

Substitute equations (3) and (4) in equation (1) and simplify to obtain:

$$\frac{0.32(Lmv)^2}{\tfrac{1}{3}ML^2 + 0.64mL^2} - \left(Mg\frac{L}{2} + mg(0.8L)\right)(1-\cos\theta) = 0$$

Solving for v gives:

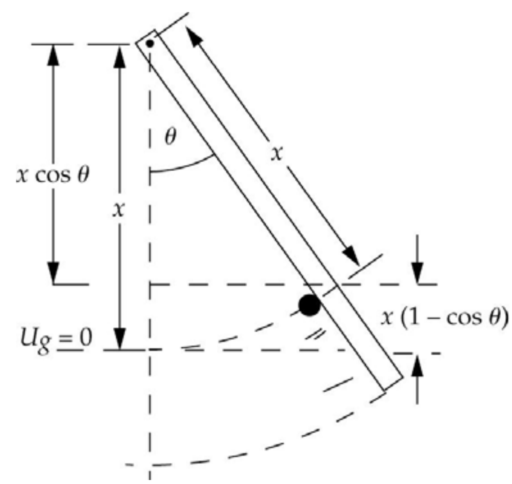
$$v = \sqrt{\frac{(0.5M + 0.8m)\left(\frac{1}{3}ML^2 + 0.64mL^2\right)g(1 - \cos\theta)}{0.32Lm^2}}$$

Evaluate v for $\theta = 90°$ to obtain:

$$\boxed{v = \sqrt{\frac{(0.5M + 0.8m)\left(\frac{1}{3}ML^2 + 0.64mL^2\right)g}{0.32Lm^2}}}$$

64 •• If, for the system of Problem 63, L = 1.2 m, M = 0.80 kg, m = 0.30 kg, and the maximum angle between the rod and the vertical following the collision is 60°, find the speed of the particle before impact.

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot and ignore friction between the rod and pivot. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}I\omega^2 + \left(Mg\frac{L}{2} + mgx\right)(1 - \cos\theta) = 0 \qquad (1)$$

Apply conservation of momentum to the collision:

$$\Delta L = L_f - L_i = 0 \qquad (2)$$

The moment of inertia of the system about the pivot is:

$$I = m(0.80L)^2 + \tfrac{1}{3}ML^2 = \left(0.64m + \tfrac{1}{3}M\right)L^2$$

Substituting for L_f and L_i in equation (2) gives:

$$\left(\tfrac{1}{3}M + 0.64m\right)L^2\omega - 0.80Lmv = 0$$

Solving for ω yields:

$$\omega = \frac{0.80Lmv}{\tfrac{1}{2}ML^2 + 0.64mL^2}$$

Substitute for ω in equation (1) and simplify to obtain:

$$-\frac{0.32(Lmv)^2}{I} + \left(Mg\frac{L}{2} + 0.80Lmg\right)(1-\cos\theta) = 0$$

Solving for v yields:

$$v = \sqrt{\frac{g(0.50M + 0.80m)(1-\cos\theta)I}{0.32Lm^2}}$$

Substitute numerical values and evaluate I:

$$I = \left[0.64(0.30\,\text{kg}) + \tfrac{1}{3}(0.80\,\text{kg})\right](1.2\,\text{m})^2 = 0.660\,\text{kg}\cdot\text{m}^2$$

Substitute numerical values and evaluate v for $\theta = 60°$ to obtain:

$$v = \sqrt{\frac{(9.81\,\text{m/s}^2)[0.50(0.80\,\text{kg}) + (0.80)(0.30\,\text{kg})](0.50)(0.660\,\text{kg}\cdot\text{m}^2)}{0.32(1.2\,\text{m})(0.30\,\text{kg})^2}} = \boxed{7.7\,\text{m/s}}$$

65 •• A uniform rod is resting on a frictionless table when it is suddenly struck at one end by a sharp horizontal blow in a direction perpendicular to the rod. The mass of the rod is M and the magnitude of the impulse applied by the blow is J. Immediately after the rod is struck, (*a*) what is the velocity of the center of mass of the rod, (*b*) what is the velocity of the end that is struck, (*c*) and what is the velocity of the other end of the rod? (*d*) Is there a point on the rod that remains motionless?

Picture the Problem Let the length of the uniform stick be ℓ. We can use the impulse-change in momentum theorem to express the velocity of the center of mass of the stick. By expressing the velocity V of the end of the stick in terms of the velocity of the center of mass and applying the angular impulse-change in angular momentum theorem we can find the angular velocity of the stick and, hence, the velocity of the end of the stick.

(*a*) Apply the impulse-change in momentum theorem to obtain:

$$J = \Delta p = p - p_0 = p$$

or, because $p_0 = 0$ and $p = Mv_{cm}$,

$$J = Mv_{cm} \Rightarrow v_{cm} = \boxed{\frac{J}{M}}$$

(*b*) Relate the velocity V of the end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} + v_{\text{rel to cm}} = v_{cm} + \omega\left(\tfrac{1}{2}\ell\right) \quad (1)$$

Relate the angular impulse to the change in the angular momentum of the stick:

$$J\left(\tfrac{1}{2}\ell\right) = \Delta L = L - L_0 = I_{cm}\omega$$

or, because $L_0 = 0$,

$$J\left(\tfrac{1}{2}\ell\right) = I_{cm}\omega \quad (2)$$

Refer to Table 9-1 to find the moment of inertia of the stick with respect to its center of mass:

$$I_{cm} = \tfrac{1}{12}M\ell^2$$

Substitute for I_{cm} in equation (2) to obtain:

$$J\left(\tfrac{1}{2}\ell\right) = \tfrac{1}{12}M\ell^2\omega \Rightarrow \omega = \frac{6J}{M\ell}$$

Substituting for ω in equation (1) yields:

$$V = \frac{J}{M} + \left(\frac{6J}{M\ell}\right)\frac{\ell}{2} = \boxed{\frac{4J}{M}}$$

(*c*) Relate the velocity V' of the other end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} - v_{\text{rel to cm}} = v_{cm} - \omega\left(\tfrac{1}{2}\ell\right)$$
$$= \frac{J}{M} - \left(\frac{6J}{M\ell}\right)\frac{\ell}{2} = \boxed{-\frac{2J}{M}}$$

(*d*) Yes, one point remains motionless, but only for a very brief time.

66 •• A projectile of mass m_p is traveling at a constant velocity $\vec{v}_0$ toward a stationary disk of mass M and radius R that is free to rotate about its axis O (Figure 10-54). Before impact, the projectile is traveling along a line displaced a distance b below the axis. The projectile strikes the disk and sticks to point B. Model the projectile as a point mass. (*a*) Before impact, what is the total angular momentum L_0 of the disk-projectile system about the axis? Answer the following questions in terms of the symbols given at the start of this problem. (*b*) What is

the angular speed ω of the disk-projectile system just after the impact? (*c*) What is the kinetic energy of the disk-projectile system after impact? (*d*) How much mechanical energy is lost in this collision?

Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision.

(*a*) Use its definition to express the total angular momentum of the disk and projectile just before impact:

$$L_0 = \boxed{m_\mathrm{p} v_0 b}$$

(*b*) Use conservation of angular momentum to relate the angular momenta just before and just after the collision:

$$L_0 = L = I\omega \Rightarrow \omega = \frac{L_0}{I}$$

The moment of inertia of the disk-projectile after the impact is:

$$I = \tfrac{1}{2}MR^2 + m_\mathrm{p}R^2 = \tfrac{1}{2}\left(M + 2m_\mathrm{p}\right)R^2$$

Substitute for I in the expression for ω to obtain:

$$\omega = \boxed{\frac{2m_\mathrm{p} v_0 b}{\left(M + 2m_\mathrm{p}\right)R^2}}$$

(*c*) Express the kinetic energy of the system after impact in terms of its angular momentum:

$$K_\mathrm{f} = \frac{L^2}{2I} = \frac{\left(m_\mathrm{p} v_0 b\right)^2}{2\left[\tfrac{1}{2}\left(M + 2m_\mathrm{p}\right)R^2\right]}$$

$$= \boxed{\frac{\left(m_\mathrm{p} v_0 b\right)^2}{\left(M + 2m_\mathrm{p}\right)R^2}}$$

(*d*) Express the difference between the initial and final kinetic energies, substitute, and simplify to obtain:

$$\Delta E = K_\mathrm{i} - K_\mathrm{f}$$

$$= \tfrac{1}{2}m_\mathrm{p} v_0^2 - \frac{\left(m_\mathrm{p} v_0 b\right)^2}{\left(M + 2m_\mathrm{p}\right)R^2}$$

$$= \boxed{\tfrac{1}{2}m_\mathrm{p} v_0^2 \left[1 - \frac{2m_\mathrm{p} b^2}{\left(M + 2m_\mathrm{p}\right)R^2}\right]}$$

67 •• [SSM] A uniform rod of length L_1 and mass M equal to 0.75 kg is attached to a hinge of negligible mass at one end and is free to rotate in the vertical plane (Figure 10-55). The rod is released from rest in the position shown. A particle of mass m = 0.50 kg is supported by a thin string of length L_2 from the hinge. The particle sticks to the rod on contact. What should the ratio L_2/L_1 be so that θ_{max} = 60° after the collision?

Picture the Problem Assume that there is no friction between the rod and the hinge. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational kinetic of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the masses M and m.

Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the rod-and-particle system just after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$\Delta L = L_f - L_i = 0$$

or

$$\left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega' - \left(\tfrac{1}{3}ML_1^2\right)\omega = 0$$

Solve for ω' to obtain:

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\omega$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_f - K_i + U_f - U_i = 0$$

Because $K_f = 0$:

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)\left(1-\cos\theta_{max}\right) + mgL_2\left(1-\cos\theta_{max}\right) = 0 \qquad (1)$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1):

$$\frac{3\dfrac{g}{L_1}\left(\tfrac{1}{3}ML_1^2\right)^2}{\tfrac{1}{3}ML_1^2 + mL_2^2} = Mg\left(\tfrac{1}{2}L_1\right) + mgL_2$$

Simplify to obtain:

$$L_1^3 = 2\frac{m}{M}L_1^2L_2 + 3\frac{m}{M}L_2^2L_1 + 6\frac{m}{M}L_2^3$$

Dividing both sides of the equation by L_1^3 yields:

$$1 = 2\left(\frac{m}{M}\right)\left(\frac{L_2}{L_1}\right) + 3\left(\frac{m}{M}\right)\left(\frac{L_2}{L_1}\right)^2 + 6\left(\frac{m}{M}\right)^2\left(\frac{L_2}{L_1}\right)^3$$

Let $\alpha = m/M$ and $\beta = L_2/L_1$ to obtain:

$$6\alpha^2\beta^3 + 3\alpha\beta^2 + 2\alpha\beta - 1 = 0$$

Substitute for α and simplify to obtain the cubic equation in β:

$$8\beta^3 + 6\beta^2 + 4\beta - 3 = 0$$

Use the solver function of your calculator to find the only real value of β:

$$\beta = \boxed{0.39}$$

68 •• A uniform rod that has a length L_1 equal to 1.2 m and a mass M equal to 2.0 kg is supported by a hinge at one end and is free to rotate in the vertical plane (Figure 10-55). The rod is released from rest in the position shown. A particle whose mass is m is supported by a thin string that has a length L_2 equal to 0.80 m from the hinge. The particle sticks to the rod on contact, and after the collision the rod continues to rotate until $\theta_{max} = 37°$. (*a*) Find m. (*b*) How much energy is dissipated during the collision?

Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational

kinetic energy of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the mass M to m.

(*a*) Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the system after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$\Delta L = L_f - L_i = 0$$

or

$$\left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega' - \left(\tfrac{1}{3}ML_1^2\right)\omega = 0 \quad (1)$$

Solving for ω' and simplifying yields:

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\omega = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\sqrt{\frac{3g}{L_1}}$$

Substitute numerical values and simplify to obtain:

$$\omega' = \frac{\tfrac{1}{3}(2.0\,\text{kg})(1.2\,\text{m})^2}{\tfrac{1}{3}(2.0\,\text{kg})(1.2\,\text{m})^2 + m(0.80\,\text{m})^2}\sqrt{\frac{3(9.81\,\text{m/s}^2)}{1.2\,\text{m}}} = \frac{4.75\,\text{kg/s}}{0.960\,\text{kg} + 0.64m}$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-K_i + U_f - U_i = 0$$

Substitute for K_i, U_f, and U_i to obtain:

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)\left(1-\cos\theta_{max}\right) + mgL_2\left(1-\cos\theta_{max}\right) = 0$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1) and simplify to obtain:

$$\frac{\tfrac{1}{2}(4.75\,\text{kg/s})^2}{0.960\,\text{kg} + 0.64m} = 0.2g\left(ML_1 + mL_2\right)$$

Substitute for M, L_1 and L_2 and simplify to obtain:

$$\frac{\tfrac{1}{2}(4.75\,\text{kg/s})^2}{0.960\,\text{kg} + 0.64m} = 0.2g\left(2.4\,\text{kg}\cdot\text{m} + (0.80\,\text{m})m\right)$$

Solve for m to obtain:

$$m = 1.18\,\text{kg} = \boxed{1.2\,\text{kg}}$$

(*b*) The energy dissipated in the inelastic collision is:

$$\Delta E = U_i - U_f \qquad (2)$$

Express U_i:

$$U_i = Mg\frac{L_1}{2}$$

Express U_f:

$$U_f = \left(1-\cos\theta_{max}\right)g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute for U_i and U_f in equation (2) to obtain:

$$\Delta E = Mg\frac{L_1}{2} - \left(1-\cos\theta_{max}\right)g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \frac{(2.0\,\text{kg})(9.81\,\text{m/s}^2)(1.2\,\text{m})}{2} - (1-\cos 37°)(9.81\,\text{m/s}^2)\left(\frac{(2.0\,\text{kg})(1.2\,\text{m})}{2} + (1.18\,\text{kg})(0.80\,\text{m})\right)$$

$$= \boxed{7.5\,\text{J}}$$

Precession

69 •• [SSM] A bicycle wheel that has a radius equal to 28 cm is mounted at the middle of an axle 50 cm long. The tire and rim weigh 30 N. The wheel is spun at 12 rev/s, and the axle is then placed in a horizontal position with one end resting on a pivot. (*a*) What is the angular momentum due to the spinning of the wheel? (Treat the wheel as a hoop.) (*b*) What is the angular velocity of precession? (*c*) How long does it take for the axle to swing through 360° around the pivot? (*d*) What is the angular momentum associated with the motion of the center of mass, that is, due to the precession? In what direction is this angular momentum?

Picture the Problem We can determine the angular momentum of the wheel and the angular velocity of its precession from their definitions. The period of the precessional motion can be found from its angular velocity and the angular momentum associated with the motion of the center of mass from its definition.

(*a*) Using the definition of angular momentum, express the angular momentum of the spinning wheel:

$$L = I\omega = MR^2\omega = \frac{w}{g}R^2\omega$$

Substitute numerical values and evaluate L:

$$L = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.28\,\text{m})^2 \times \left(12\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = 18.1\,\text{J}\cdot\text{s} = \boxed{18\,\text{J}\cdot\text{s}}$$

(*b*) Using its definition, express the angular velocity of precession:

$$\omega_\text{p} = \frac{d\phi}{dt} = \frac{MgD}{L}$$

Substitute numerical values and evaluate ω_p:

$$\omega_p = \frac{(30\,\text{N})(0.25\,\text{m})}{18.1\,\text{J}\cdot\text{s}} = 0.414\,\text{rad/s}$$
$$= \boxed{0.41\,\text{rad/s}}$$

(*c*) Express the period of the precessional motion as a function of the angular velocity of precession:

$$T = \frac{2\pi}{\omega_p} = \frac{2\pi}{0.414\,\text{rad/s}} = \boxed{15\,\text{s}}$$

(*d*) Express the angular momentum of the center of mass due to the precession:

$$L_p = I_{cm}\omega_p = MD^2\omega_p$$

Substitute numerical values and evaluate L_p:

$$L_p = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.25\,\text{m})^2(0.414\,\text{rad/s})$$
$$= \boxed{0.079\,\text{J}\cdot\text{s}}$$

The direction of L_p is either up or down, depending on the direction of L.

70 •• A uniform disk, whose mass is 2.50 kg and radius is 6.40 cm is mounted at the center of a 10.0-cm-long axle and spun at 700 rev/min. The axle is then placed in a horizontal position with one end resting on a pivot. The other end is given an initial horizontal velocity such that the precession is smooth with no nutation. (*a*) What is the angular velocity of precession? (*b*) What is the speed of the center of mass during the precession? (*c*) What is the acceleration (magnitude and direction) of the center of mass? (*d*) What are the vertical and horizontal components of the force exerted by the pivot on the axle?

Picture the Problem The angular velocity of precession can be found from its definition. Both the speed and the magnitude of the acceleration of the center of mass during precession are related to the angular velocity of precession. We can use Newton's second law to find the vertical and horizontal components of the force exerted by the pivot on the axle.

(*a*) The angular velocity of precession is given by:

$$\omega_p = \frac{d\phi}{dt} = \frac{MgD}{I_s\omega_s}$$

Substituting for I_s and simplifying yields:

$$\omega_p = \frac{MgD}{\frac{1}{2}MR^2\omega_s} = \frac{2gD}{R^2\omega_s}$$

Substitute numerical values and evaluate ω_p:

$$\omega_p = \frac{2(9.81\,\text{m/s}^2)(0.050\,\text{m})}{(0.064\,\text{m})^2\left(700\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)} = 3.27\,\text{rad/s} = \boxed{3.3\,\text{rad/s}}$$

(*b*) Express the speed of the center of mass in terms of its angular speed of precession:

$$v_{cm} = D\omega_p = (0.050\,\text{m})(3.27\,\text{rad/s}) = \boxed{16\,\text{cm/s}}$$

(*c*) Relate the acceleration of the center of mass to its angular speed of precession:

$$a_{cm} = D\omega_p^2 = (0.050\,\text{m})(3.27\,\text{rad/s})^2 = 0.535\,\text{m/s}^2 = \boxed{54\,\text{cm/s}^2}$$

(*d*) Use Newton's second law to relate the vertical component of the force exerted by the pivot to the weight of the disk:

$$F_v = Mg = (2.5\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{25\,\text{N}}$$

Relate the horizontal component of the force exerted by the pivot on the axle to the acceleration of the center of mass:

$$F_H = Ma_{cm} = (2.5\,\text{kg})(0.535\,\text{m/s}^2) = \boxed{1.3\,\text{N}}$$

General Problems

71 • [SSM] A particle whose mass is 3.0 kg moves in the *xy* plane with velocity $\vec{v} = (3.0\text{ m/s})\hat{i}$ along the line $y = 5.3$ m. (*a*) Find the angular momentum $\vec{L}$ about the origin when the particle is at (12 m, 5.3 m). (*b*) A force $\vec{F} = (-3.9\text{ N})\hat{i}$ is applied to the particle. Find the torque about the origin due to this force as the particle passes through the point (12 m, 5.3 m).

Picture the Problem While the 3-kg particle is moving in a straight line, it has angular momentum given by $\vec{L} = \vec{r}\times\vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque due to the applied force is given by $\vec{\tau} = \vec{r}\times\vec{F}$.

(*a*) The angular momentum of the particle is given by:

$$\vec{L} = \vec{r}\times\vec{p}$$

Express the vectors $\vec{r}$ and $\vec{p}$:

$$\vec{r} = (12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}$$

and

$$\vec{p} = mv\hat{\mathbf{i}} = (3.0\,\text{kg})(3.0\,\text{m/s})\hat{i} = (9.0\,\text{kg}\cdot\text{m/s})\hat{i}$$

Substitute for $\vec{r}$ and $\vec{p}$:and simplify to find $\vec{L}$:

$$\vec{L} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right]\times(9.0\,\text{kg}\cdot\text{m/s})\hat{i} = (47.7\,\text{kg}\cdot\text{m}^2/\text{s})(\hat{j}\times\hat{i}) = \boxed{-(48\,\text{kg}\cdot\text{m}^2/\text{s})\hat{k}}$$

(*b*) Using its definition, express the torque due to the force:

$$\vec{\tau} = \vec{r}\times\vec{F}$$

Substitute for $\vec{r}$ and $\vec{F}$ and simplify to find $\vec{\tau}$:

$$\vec{\tau} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right]\times(-3.9\,\text{N})\hat{i} = -(15.9\,\text{N}\cdot\text{m})(\hat{j}\times\hat{i}) = \boxed{(21\,\text{N}\cdot\text{m})\hat{k}}$$

72 • The position vector of a particle whose mass is 3.0 kg is given by $\vec{r} = 4.0\,\hat{i} + 3.0t^2\,\hat{j}$, where $\vec{r}$ is in meters and *t* is in seconds. Determine the angular momentum and net torque, about the origin, acting on the particle.

Picture the Problem The angular momentum of the particle is given by $\vec{L} = \vec{r}\times\vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque acting on the particle is given by $\vec{\tau} = d\vec{L}/dt$.

The angular momentum of the particle is given by:

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v} = m\vec{r}\times\vec{v} = m\vec{r}\times\frac{d\vec{r}}{dt}$$

Evaluating $\frac{d\vec{r}}{dt}$ yields:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[4.0\hat{i} + 3.0t^2\hat{j}\right] = (6.0t)\hat{j}$$

Substitute for $m\vec{r}$ and $\frac{d\vec{r}}{dt}$ and simplify to find $\vec{L}$:

$$\vec{L} = \left[(3.0\,\text{kg})\left\{(4.0\,\text{m})\hat{i} + (3.0t^2\,\text{m/s}^2)\hat{j}\right\}\right]\times(6.0t\,\text{m/s})\hat{j} = \boxed{(72t\,\text{J}\cdot\text{s})\hat{k}}$$

Find the net torque due to the force:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left[(72t\,\text{J}\cdot\text{s})\hat{k}\right] = \boxed{(72\,\text{N}\cdot\text{m})\hat{k}}$$

73 •• Two ice skaters, whose masses are 55 kg and 85 kg, hold hands and rotate about a vertical axis that passes between them, making one revolution in 2.5 s. Their centers of mass are separated by 1.7 m and their center of mass is stationary. Model each skater as a point particle and find (*a*) the angular momentum of the system about their center of mass and (*b*) the total kinetic energy of the system.

Picture the Problem The ice skaters rotate about their center of mass; a point we can locate using its definition. Knowing the location of the center of mass we can determine their moment of inertia with respect to an axis through this point. The angular momentum of the system is then given by $L = I_{\text{cm}}\omega$ and its kinetic energy can be found from $K = L^2/(2I_{\text{cm}})$.

(*a*) Express the angular momentum of the system about the center of mass of the skaters:

$$L = I_{\text{cm}}\omega$$

Using its definition, locate the center of mass, relative to the 85-kg skater, of the system:

$$x_{\text{cm}} = \frac{(55\,\text{kg})(1.7\,\text{m}) + (85\,\text{kg})(0)}{55\,\text{kg} + 85\,\text{kg}} = 0.668\,\text{m}$$

Calculate I_{cm}:

$$I_{\text{cm}} = (55\,\text{kg})(1.7\,\text{m} - 0.668\,\text{m})^2 + (85\,\text{kg})(0.668\,\text{m})^2 = 96.5\,\text{kg}\cdot\text{m}^2$$

Substitute to determine L:

$$L = \left(96.5\,\text{kg}\cdot\text{m}^2\right)\left(\frac{1\,\text{rev}}{2.5\,\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = 243\,\text{J}\cdot\text{s} = \boxed{0.24\,\text{kJ}\cdot\text{s}}$$

(*b*) Relate the total kinetic energy of the system to its angular momentum and evaluate K:

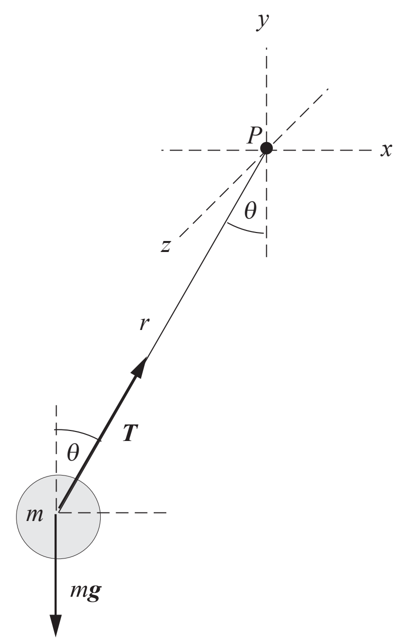
$$K = \frac{L^2}{2I_{\text{cm}}}$$

Substitute numerical values and evaluate K:

$$K = \frac{(243\,\text{J}\cdot\text{s})^2}{2\left(96.5\,\text{kg}\cdot\text{m}^2\right)} = \boxed{0.31\,\text{kJ}}$$

74 •• A 2.0-kg ball attached to a string whose length is 1.5 m moves counterclockwise (as viewed from above) in a horizontal circle (Figure 10-56). The string makes an angle $\theta = 30°$ with the vertical. (*a*) Determine both the horizontal and vertical components of the angular momentum $\vec{L}$ of the ball about the point of support *P*. (*b*) Find the magnitude of $d\vec{L}/dt$ and verify that it equals the magnitude of the torque exerted by gravity about the point of support.

Picture the Problem Let the origin of the coordinate system be at the pivot. The diagram shows the forces acting on the ball. We'll apply Newton's second law to the ball to determine its speed. We'll then use the derivative of its position vector to express its velocity and the definition of angular momentum to show that $\vec{L}$ has both horizontal and vertical components. We can use the derivative of $\vec{L}$ with respect to time to show that the rate at which the angular momentum of the ball changes is equal to the torque, relative to the pivot point, acting on it.



(*a*) Express the angular momentum of the ball about the point of support:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \qquad (1)$$

Apply Newton's second law to the ball:

$$\sum F_x = T\sin\theta = m\frac{v^2}{r\sin\theta}$$

and

$$\sum F_z = T\cos\theta - mg = 0$$

Eliminate *T* between these equations and solve for *v* to obtain:

$$v = \sqrt{rg\sin\theta\tan\theta}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{(1.5\,\text{m})(9.81\,\text{m/s}^2)\sin 30°\tan 30°} = 2.06\,\text{m/s}$$

Express the position vector of the ball:

$$\vec{r} = (1.5\,\text{m})\sin 30°\left(\cos\omega t\,\hat{\boldsymbol{i}} + \sin\omega t\hat{\boldsymbol{j}}\right) - (1.5\,\text{m})\cos 30°\hat{\boldsymbol{k}}$$

The velocity of the ball is:

$$\vec{v} = \frac{d\vec{r}}{dt} = (0.75\omega\,\text{m/s})\left(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right)$$

Evaluating ω yields:

$$\omega = \frac{2.06\,\text{m/s}}{(1.5\,\text{m})\sin 30°} = 2.75\,\text{rad/s}$$

Substitute for ω to obtain:

$$\vec{v} = (2.06\,\text{m/s})\left(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j}\right)$$

Substitute in equation (1) and evaluate $\vec{L}$:

$$\begin{aligned}\vec{L} &= (2.0\,\text{kg})\left[(1.5\,\text{m})\sin 30°\left(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right) - (1.5\,\text{m})\cos 30°\hat{k}\right] \\ &\qquad \times\left[(2.06\,\text{m/s})\left(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right)\right] \\ &= (5.35\,\text{J}\cdot\text{s})\cos\omega t\,\hat{i} + (5.35\,\text{J}\cdot\text{s})\sin\omega t\,\hat{j} + (3.09\,\text{J}\cdot\text{s})\hat{k}\end{aligned}$$

The horizontal component of $\vec{L}$ is the component in the xy plane:

$$\vec{L}_{\text{hor}} = \boxed{(5.4\,\text{J}\cdot\text{s})\cos\omega t\,\hat{i} + (5.4\,\text{J}\cdot\text{s})\sin\omega t\,\hat{j}}$$

The vertical component of $\vec{L}$ is its z component:

$$\vec{L}_{\text{vertical}} = \boxed{(3.1\,\text{J}\cdot\text{s})\hat{k}}$$

(*b*) Evaluate $\dfrac{d\vec{L}}{dt}$:

$$\frac{d\vec{L}}{dt} = \left[5.36\omega\left(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j}\right)\right]\text{J}$$

Evaluate the magnitude of $\dfrac{d\vec{L}}{dt}$:

$$\left|\frac{d\vec{L}}{dt}\right| = (5.36\,\text{N}\cdot\text{m}\cdot\text{s})(2.75\,\text{rad/s}) = \boxed{15\,\text{N}\cdot\text{m}}$$

Express the magnitude of the torque exerted by gravity about the point of support:

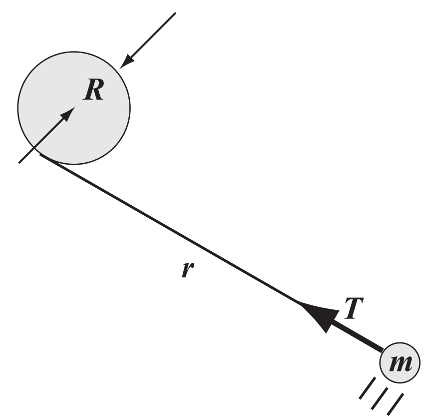
$$\tau = mgr\sin\theta$$

Substitute numerical values and evaluate τ:

$$\tau = (2.0\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m})\sin 30° = \boxed{15\,\text{N}\cdot\text{m}}$$

75 •• A compact object whose mass is m resting on a horizontal, frictionless surface is attached to a string that wraps around a vertical cylindrical post attached to the surface. Thus, when the object is set into motion, it follows a path that spirals inward. (*a*) Is the angular momentum of the object about the axis of the post conserved? Explain your answer. (*b*) Is the energy of the object conserved? Explain your answer. (*c*) If the speed of the object is v_0 when the unwrapped length of the string is r, what is its speed when the unwrapped length has shortened to $r/2$?

Picture the Problem The pictorial representation depicts the object rotating counterclockwise around the cylindrical post. Let the system be the object. In Part (*a*) we need to decide whether a net torque acts on the object and in Part (*b*) the issue is whether any external forces act on the object. In Part (*c*) we can apply the definition of kinetic energy to find the speed of the object when the unwrapped length has shortened to $r/2$.



(*a*) The net torque acting on the object is given by:

$$\tau_{\text{net}} = \frac{dL}{dt} = RT$$

Because $\tau_{\text{net}} \neq 0$, angular momentum is not conserved.

(*b*) Because, in this frictionless environment, the net external force acting on the object is the tension force and it acts at right angles to the object's velocity, the energy of the object is conserved.

(*c*) Apply conservation of mechanical energy to the object to obtain:

$$\Delta E = \Delta K + \Delta U = 0$$

or, because $\Delta U = 0$,

$$\Delta K_{\text{rot}} = 0$$

Substituting for the kinetic energies yields:

$$\tfrac{1}{2}I'\omega'^2 - \tfrac{1}{2}I\omega_0^2 = 0$$

or

$$I'\omega'^2 - I\omega_0^2 = 0$$

Substitute for I, I', ω', and ω_0 to obtain:

$$\tfrac{1}{2}m\left(\frac{r}{2}\right)^2\left(\frac{v'}{\frac{r}{2}}\right)^2 - \tfrac{1}{2}mr^2\left(\frac{v_0}{r}\right)^2 = 0$$

Solving for v' yields:

$$v' = \boxed{v_0}$$

76 •• Figure 10-57 shows a hollow cylindrical tube that has a mass M, a length L, and a moment of inertia $ML^2/10$. Inside the cylinder are two disks each of mass m, separated by a distance ℓ, and tied to a central post by a thin string. The system can rotate about a vertical axis through the center of the cylinder. You are designing this cylinder-disk apparatus to shut down the rotations when the strings break by triggering an electronic "shutoff" signal (sent to the rotating motor) when the disks hit the ends of the cylinder. During development, you notice that with the system rotating at some critical angular speed ω, the string suddenly breaks. When the disks reach the ends of the cylinder, they stick. Obtain expressions for the final angular speed and the initial and final kinetic energies of the system. Assume that the inside walls of the cylinder are frictionless.

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular velocities of the system. See Table 9-1 for the moment of inertia of a disk.

Using conservation of angular momentum, relate the initial and final angular speeds to the initial and final moments of inertia:

$$\Delta L = L_f - L_i = 0$$

or

$$I_f\omega_f - I_i\omega_i = 0$$

Solving for ω_f yields:

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{I_i}{I_f}\omega \qquad (1)$$

Use the parallel-axis theorem to express the moment of inertia of each of the disks with respect to the axis of rotation:

$$\begin{aligned} I_{\text{i, each disk}} &= I_{\text{cm}} + m\left(\tfrac{1}{2}\ell\right)^2 \\ &= \tfrac{1}{4}mr^2 + \tfrac{1}{4}m\ell^2 \\ &= \tfrac{1}{4}m\left(r^2 + \ell^2\right) \end{aligned}$$

Express the initial moment of inertia I_i of the cylindrical tube plus disks system:

$$\begin{aligned} I_i &= I_{\substack{\text{cylindrical}\\ \text{tube}}} + 2I_{\text{i, each disk}} \\ &= \tfrac{1}{10}ML^2 + 2\left[\tfrac{1}{4}m\left(r^2+\ell^2\right)\right] \\ &= \tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2+\ell^2\right) \end{aligned}$$

When the disks have moved out to the end of the cylindrical tube:

$$I_f = \tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + L^2\right)$$

Substitute for I_i and I_f in equation (1) and simplify to obtain:

$$\omega_f = \frac{\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + \ell^2\right)}{\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + L^2\right)}\omega = \boxed{\frac{ML^2 + 5m\left(r^2 + \ell^2\right)}{ML^2 + 5m\left(r^2 + L^2\right)}\omega}$$

The initial kinetic energy of the system is:

$$K_i = \tfrac{1}{2}I_i\omega^2$$

Substituting for I_i and simplifying yields:

$$K_i = \tfrac{1}{2}\left[\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + \ell^2\right)\right]\omega^2 = \boxed{\left[\tfrac{1}{20}ML^2 + \tfrac{1}{4}m\left(r^2 + \ell^2\right)\right]\omega^2}$$

The final kinetic energy of the system is:

$$K_f = \tfrac{1}{2}I_f\omega_f^2$$

Substitute for I_f and ω_f and simplify to obtain:

$$K_f = \tfrac{1}{2}\left[\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + L^2\right)\right]\left(\frac{ML^2 + 5m\left(r^2 + \ell^2\right)}{ML^2 + 5m\left(r^2 + L^2\right)}\omega\right)^2 = \boxed{\tfrac{1}{20}\left[\frac{\left[ML^2 + 5m\left(r^2 + \ell^2\right)\right]^2}{ML^2 + 5m\left(r^2 + L^2\right)}\right]\omega^2}$$

77 •• [SSM] Repeat Problem 76, this time friction between the disks and the walls of the cylinder is not negligible. However, the coefficient of friction is not great enough to prevent the disks from reaching the ends of the cylinder. Can the final kinetic energy of the system be determined without knowing the coefficient of kinetic friction?

Determine the Concept Yes. The solution depends only upon conservation of angular momentum of the system, so it depends only upon the initial and final moments of inertia.

78 •• Suppose that in Figure 10-57 $\ell = 0.60$ m, $L = 2.0$ m, $M = 0.80$ kg, and $m = 0.40$ kg. The string breaks when the system's angular speed approaches the critical angular speed ω_i, at which time the tension in the string is 108 N. The masses then move radially outward until they undergo perfectly inelastic collisions with the ends of the cylinder. Determine the critical angular speed and the angular speed of the system after the inelastic collisions. Find the total kinetic

energy of the system at the critical angular speed, and again after the inelastic collisions. Assume that the inside walls of the cylinder are frictionless.

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular speeds of the system.

Using conservation of angular momentum, relate the initial and final angular speeds to the initial and final moments of inertia:

$$\Delta L = L_f - L_i = 0$$

or

$$I_f\omega_f - I_i\omega_i = 0 \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i \quad (1)$$

Express the tension in the string as a function of the critical angular speed of the system:

$$T = mr\omega_i^2 = m\frac{\ell}{2}\omega_i^2 \Rightarrow \omega_i = \sqrt{\frac{2T}{m\ell}}$$

Substitute numerical values and evaluate ω_i:

$$\omega_i = \sqrt{\frac{2(108\,\text{N})}{(0.40\,\text{kg})(0.60\,\text{m})}} = 30.0\,\text{rad/s}$$
$$= \boxed{30\,\text{rad/s}}$$

Express I_i:

$$I_i = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)$$

Substitute numerical values and evaluate I_i:

$$I_i = \tfrac{1}{10}(0.80\,\text{kg})(2.0\,\text{m})^2 + \tfrac{1}{2}(0.40\,\text{kg})(0.60\,\text{m})^2$$
$$= 0.392\,\text{kg}\cdot\text{m}^2$$

Express I_f:

$$I_f = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)$$

Substitute numerical values and evaluate I_f:

$$I_f = \tfrac{1}{10}(0.80\,\text{kg})(2.0\,\text{m})^2 + \tfrac{1}{2}(0.40\,\text{kg})(2.0\,\text{m})^2$$
$$= 1.12\,\text{kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) and evaluate ω_f:

$$\omega_f = \frac{0.392\,\text{kg}\cdot\text{m}^2}{1.12\,\text{kg}\cdot\text{m}^2}(30.0\,\text{rad/s})$$
$$= 10.5\,\text{rad/s} = \boxed{11\,\text{rad/s}}$$

The total kinetic energy of the system at the critical angular speed is:

$$K_i = \tfrac{1}{2} I_i \omega_i^2$$

Substitute numerical values and evaluate K_i:

$$K_i = \tfrac{1}{2}\left(0.392\,\text{kg}\cdot\text{m}^2\right)\left(30.0\,\text{rad/s}\right)^2 = 176\,\text{J} = \boxed{0.18\,\text{kJ}}$$

The total kinetic energy of the system after the inelastic collisions is:

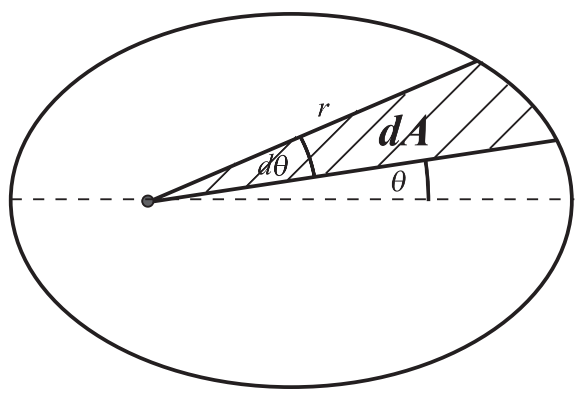
$$K_f = \tfrac{1}{2} I_f \omega_f^2$$

Substitute numerical values and evaluate K_f:

$$K_f = \tfrac{1}{2}\left(1.12\,\text{kg}\cdot\text{m}^2\right)\left(10.5\,\text{rad/s}\right)^2 = \boxed{62\,\text{J}}$$

79 •• [SSM] Kepler's second law states: *The line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.* Show that this law follows directly from the law of conservation of angular momentum and the fact that the force of gravitational attraction between a planet and the Sun acts along the line joining the centers of the two celestial objects.

Picture the Problem The pictorial representation shows an elliptical orbit. The triangular element of the area is $dA = \frac{1}{2}r(rd\theta) = \frac{1}{2}r^2 d\theta$.



Differentiate dA with respect to t to obtain:

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}r^2\omega \qquad (1)$$

Because the gravitational force acts along the line joining the two objects, $\tau = 0$. Hence:

$$L = mr^2\omega = \text{constant} \qquad (2)$$

Eliminate $r^2\omega$ between equations (1) and (2) to obtain:

$$\frac{dA}{dt}=\boxed{\frac{L}{2m}=\text{constant}}$$

80 •• Consider a cylindrical turntable whose mass is *M* and radius is *R*, turning with an initial angular speed ω_1. (*a*) A parakeet of mass *m*, hovering in flight above the outer edge of the turntable, gently lands on it and stays in one place on it as shown in Figure 10-58. What is the angular speed of the turntable after the parakeet lands? (*b*) Becoming dizzy, the parakeet jumps off (not flies off) with a velocity $\vec{v}$ relative to the turntable. The direction of $\vec{v}$ is tangent to the edge of the turntable, and in the direction of its rotation. What will be the angular speed of the turntable afterwards? Express your answer in terms of the two masses *m* and *M*, the radius *R*, the parakeet speed *v* and the initial angular speed ω_1.

Picture the Problem The angular momentum of the turntable-parakeet is conserved in both parts of this problem.

(*a*) Apply conservation of angular momentum to the turntable-parakeet system as the parakeet lands to obtain:

$$\Delta L=L_f-L_i=0 \qquad (1)$$

The final angular momentum of the system is given by:

$$\begin{aligned}L_f&=L_{\text{turntable}}+L_{\text{parakeet}}\\&=I_{\text{turntable}}\omega_f+\left|\vec{r}\times\vec{p}_{\text{parakeet}}\right|\end{aligned}$$

Because $I_{\text{turntable}}=\frac{1}{2}MR^2$ and $\left|\vec{r}\times\vec{p}_{\text{parakeet}}\right|=Rmv_{\text{parakeet}}$:

$$\begin{aligned}L_f&=\tfrac{1}{2}MR^2\omega_f+Rmv_{\text{parakeet}}\\&=\tfrac{1}{2}MR^2\omega_f+Rm(R\omega_f)\\&=\tfrac{1}{2}MR^2\omega_f+mR^2\omega_f\end{aligned}$$

The initial angular momentum of the system is given by:

$$L_i=I_{\text{turntable}}\omega_i=\tfrac{1}{2}MR^2\omega_i$$

Substituting for L_f and L_i in equation (1) yields:

$$\tfrac{1}{2}MR^2\omega_f+mR^2\omega_f-\tfrac{1}{2}MR^2\omega_i=0$$

Solve for ω_f to obtain:

$$\omega_f=\boxed{\frac{M}{M+2m}\omega_i}$$

(*b*) Apply conservation of angular momentum to the turntable-parakeet system as the parakeet jumps off to obtain:

$$\Delta L = L_f - L_i = 0 \qquad (2)$$

The final angular momentum of the system is given by:

$$L_f = L_{\text{turntable}} + L_{\text{parakeet}} = I_{\text{turntable}}\omega_f + \left|\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}_{\text{parakeet}}\right|$$

Because $I_{\text{turntable}} = \frac{1}{2}MR^2$ and $\left|\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}_{\text{parakeet}}\right| = Rmv_{\text{parakeet}}$:

$$L_f = \tfrac{1}{2}MR^2\omega_f + Rmv_{\text{parakeet}} \qquad (3)$$

Express the speed of the parakeet relative to the ground:

$$v_{\text{parakeet}} = v_{\text{turntable}} + v = R\omega_f + v$$

Using the expression derived in (*a*), substitute for ω_f to obtain:

$$v_{\text{parakeet}} = \frac{M}{M+2m}R\omega_i + v$$

Substituting for v_{parakeet} in equation (3) and simplifying yields:

$$L_f = \tfrac{1}{2}MR^2\omega_f + mR\left(\frac{M}{M+2m}R\omega_i + v\right)$$

The initial angular momentum of the system is the same as the final angular momentum in (*a*):

$$L_i = \tfrac{1}{2}MR^2\omega_i$$

Substituting for L_f and L_i in equation (2) yields:

$$\tfrac{1}{2}MR^2\omega_f + mR\left[\left(\frac{M}{M+2m}\right)R\omega_i + v\right] - \tfrac{1}{2}MR^2\omega_i = 0$$

Solving for ω_f gives:

$$\omega_f = \boxed{\frac{M}{M+2m}\omega_i - 2\left(\frac{m}{M}\right)\frac{v}{R}}$$

81 •• You are given a heavy but thin metal disk (like a coin, but larger; Figure 10-59). (Objects like this are called *Euler disks.*) Placing the disk on a turntable, you spin the disk, on edge, about a vertical axis through a diameter of the disk and the center of the turntable. As you do this, you hold the turntable still with your other hand, letting it go immediately after you spin the disk. The turntable is a uniform solid cylinder with a radius equal to 0.250 m and a mass equal to 0.735 kg and rotates on a frictionless bearing. The disk has an initial angular speed of 30.0 rev/min. (*a*) The disk spins down and falls over, finally

coming to rest on the turntable with its symmetry axis coinciding with the turntable's. What is the final angular speed of the turntable? (*b*) What will be the final angular speed if the disk's symmetry axis ends up 0.100 m from the axis of the turntable?

Picture the Problem Let the letters *d*, *m*, and *r* denote the disk and the letters *t*, *M*, and *R* the turntable. We can use conservation of angular momentum to relate the final angular speed of the turntable to the initial angular speed of the Euler disk and the moments of inertia of the turntable and the disk. In part (*b*) we'll need to use the parallel-axis theorem to express the moment of inertia of the disk with respect to the rotational axis of the turntable. You can find the moments of inertia of the disk in its two orientations and that of the turntable in Table 9-1.

(*a*) Use conservation of angular momentum to relate the initial and final angular momenta of the system:

$$I_{\text{df}}\omega_{\text{df}} + I_{\text{tf}}\omega_{\text{tf}} - I_{\text{di}}\omega_{\text{di}} = 0$$

Because $\omega_{\text{tf}} = \omega_{\text{df}}$:

$$I_{\text{df}}\omega_{\text{tf}} + I_{\text{tf}}\omega_{\text{tf}} - I_{\text{di}}\omega_{\text{di}} = 0$$

Solving for ω_{tf} yields:

$$\omega_{\text{tf}} = \frac{I_{\text{di}}}{I_{\text{df}} + I_{\text{tf}}}\omega_{\text{di}} \qquad (1)$$

Ignoring the negligible thickness of the disk, express its initial moment of inertia:

$$I_{\text{di}} = \tfrac{1}{4}mr^2$$

Express the final moment of inertia of the disk:

$$I_{\text{df}} = \tfrac{1}{2}mr^2$$

Express the final moment of inertia of the turntable:

$$I_{\text{tf}} = \tfrac{1}{2}MR^2$$

Substitute in equation (1) and simplify to obtain:

$$\omega_{\text{tf}} = \frac{\tfrac{1}{4}mr^2}{\tfrac{1}{2}mr^2 + \tfrac{1}{2}MR^2}\omega_{\text{di}} = \frac{1}{2 + 2\dfrac{MR^2}{mr^2}}\omega_{\text{di}} \qquad (2)$$

Express ω_{di} in rad/s:

$$\omega_{\text{di}} = 30.0\frac{\text{rev}}{\text{min}} \times \frac{2\pi\,\text{rad}}{\text{rev}} \times \frac{1\,\text{min}}{60\,\text{s}} = \pi\ \text{rad/s}$$

Substitute numerical values in equation (2) and evaluate ω_{tf}:

$$\omega_{\text{tf}} = \frac{\pi\ \text{rad/s}}{2 + 2\dfrac{(0.735\,\text{kg})(0.250\,\text{m})^2}{(0.500\,\text{kg})(0.125\,\text{m})^2}} = \boxed{0.228\,\text{rad/s}}$$

(*b*) Use the parallel-axis theorem to express the final moment of inertia of the disk when it is a distance L from the center of the turntable:

$$I_{\text{df}} = \tfrac{1}{2}mr^2 + mL^2 = m\left(\tfrac{1}{2}r^2 + L^2\right)$$

Substitute in equation (1) to obtain:

$$\omega_{\text{tf}} = \frac{\frac{1}{4}mr^2}{m\left(\frac{1}{2}r^2 + L^2\right) + \frac{1}{2}MR^2}\omega_{\text{di}} = \frac{1}{2 + 4\dfrac{L^2}{r^2} + 2\dfrac{MR^2}{mr^2}}\omega_{\text{di}}$$

Substitute numerical values and evaluate ω_{tf}:

$$\omega_{\text{tf}} = \frac{\pi\ \text{rad/s}}{2 + 4\dfrac{(0.100\,\text{m})^2}{(0.125\,\text{m})^2} + 2\dfrac{(0.735\,\text{kg})(0.250\,\text{m})^2}{(0.500\,\text{kg})(0.125\,\text{m})^2}} = \boxed{0.192\,\text{rad/s}}$$

82 •• (*a*) Assuming Earth to be a homogeneous sphere that has a radius r and a mass m, show that the period T (time for one daily rotation) of Earth's rotation about its axis is related to its radius by $T = br^2$, where $b = (4/5)\pi m/L$. Here L is the magnitude of the spin angular momentum of Earth. (*b*) Suppose that the radius r changes by a very small amount Δr due to some internal cause such as thermal expansion. Show that the fractional change in the period ΔT is given approximately by $\Delta T/T = 2\Delta r/r$. (*c*) By how many kilometers would r need to increase for the period to change by 0.25 d/y (so that leap years would no longer be necessary)?

Picture the Problem We can express the period of Earth's rotation in terms of its angular velocity of rotation and relate its angular velocity to its angular momentum and moment of inertia with respect to an axis through its center. We can differentiate this expression with respect to T and then use differentials to approximate the changes in r and T.

(*a*) Express the period of Earth's rotation in terms of its angular velocity of rotation:

$$T = \frac{2\pi}{\omega}$$

Relate Earth's angular speed to its angular momentum and moment of inertia:

$$\omega = \frac{L}{I} = \frac{L}{\frac{2}{5}mr^2}$$

Substitute for ω and simplify to obtain:

$$T = \frac{2\pi\left(\frac{2}{5}mr^2\right)}{L} = \boxed{\frac{4\pi m}{5L}r^2}$$

(*b*) Find dT/dr:

$$\frac{dT}{dr} = \frac{d}{dr}\left(\frac{4\pi m}{5L}r^2\right) = 2\left(\frac{4\pi m}{5L}\right)r$$
$$= 2\left(\frac{T}{r^2}\right)r = \frac{2T}{r}$$

Solving for dT/T yields:

$$\frac{dT}{T} = 2\frac{dr}{r} \Rightarrow \frac{\Delta T}{T} \approx \boxed{2\frac{\Delta r}{r}}$$

(*c*) Using the equation we just derived, substitute for the change in the period of Earth:

$$\frac{\Delta T}{T} = \frac{\frac{1}{4}\,\text{d}}{\text{y}} \times \frac{1\,\text{y}}{365.24\,\text{d}} = \frac{1}{1460} = 2\frac{\Delta r}{r}$$

Solving for Δr yields:

$$\Delta r = \frac{r}{2(1460)}$$

Substitute numerical values and evaluate Δr:

$$\Delta r = \frac{6.37\times10^3\ \text{km}}{2(1460)} = \boxed{2.18\,\text{km}}$$

83 •• [SSM] The term *precession of the equinoxes* refers to the fact that Earth's spin axis does not stay fixed but sweeps out a cone once every 26,000 y. (This explains why our pole star, Polaris, will not remain the pole star forever.) The reason for this instability is that Earth is a giant gyroscope. The spin axis of Earth precesses because of the torques exerted on it by the gravitational forces of the Sun and moon. The angle between the direction of Earth's spin axis and the normal to the ecliptic plane (the plane of Earth's orbit) is 22.5 degrees. Calculate an approximate value for this torque, given that the period of rotation of Earth is 1.00 d and its moment of inertia is 8.03×10^{37} kg·m^2.

Picture the Problem Let ω_P be the angular velocity of precession of Earth-as-gyroscope, ω_s its angular velocity about its spin axis, and I its moment of inertia with respect to an axis through its poles, and relate ω_P to ω_s and I using its definition.

Use its definition to express the precession rate of Earth as a giant gyroscope:

$$\omega_P = \frac{\tau}{L}$$

Substitute for I and solve for τ to obtain:

$$\tau = L\omega_P = I\omega\omega_P$$

The angular velocity ω_s of Earth about its spin axis is given by:

$\omega = \dfrac{2\pi}{T}$ where T is the period of rotation of Earth.

Substitute for ω to obtain:

$$\tau = \frac{2\pi\, I\omega_P}{T}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{2\pi\left(8.03\times10^{37}\ \text{kg}\cdot\text{m}^2\right)\left(7.66\times10^{-12}\ \text{s}^{-1}\right)}{1\,\text{d}\times\dfrac{24\,\text{h}}{\text{d}}\times\dfrac{3600\,\text{s}}{\text{h}}} = \boxed{4.47\times10^{22}\ \text{N}\cdot\text{m}}$$

84 •• As indicated in the text, according to the Standard Model of Particle Physics, electrons are point-like particles having no spatial extent. (This assumption has been confirmed experimentally, and the radius of the electron has been shown to be less than 10^{-18} m.) The intrinsic spin of an electron could *in principle* be due to its rotation. Let's check to see if this conclusion is feasible. (*a*) Assuming that the electron is a uniform sphere whose radius is 1.00×10^{-18} m, what angular speed would be necessary to produce the observed intrinsic angular momentum of $\hbar/2$? (*b*) Using this value of angular speed, show that the speed of a point on the ″equator″ of a ″spinning″ electron would be moving faster than the speed of light. What is your conclusion about the spin angular momentum being analogous to a spinning sphere with spatial extent?

Picture the Problem We can use the definition of the angular momentum of a spinning sphere, together with the expression for its moment of inertia, to find the angular speed of a point on the surface of a spinning electron. The speed of such a point is directly proportional to the angular speed of the sphere.

(*a*) Express the angular momentum of the spinning electron:

$$L = I\omega = \tfrac{1}{2}\hbar$$

Assuming a spherical electron of radius R, its moment of inertia, relative to its spin axis, is:

$$I = \tfrac{2}{5}MR^2$$

Substituting for I yields:

$$\tfrac{2}{5}MR^2\omega = \tfrac{1}{2}\hbar \Rightarrow \omega = \frac{5\hbar}{4MR^2}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{5\left(1.05\times10^{-34}\ \text{J}\cdot\text{s}\right)}{4\left(9.11\times10^{-31}\ \text{kg}\right)\left(10^{-18}\ \text{m}\right)^2} = \boxed{1.44\times10^{32}\ \text{rad/s}}$$

(b) The speed of a point on the "equator" of a spinning electron of radius R is given by:

$$v = R\omega$$

Substitute numerical values and evaluate v:

$$v = \left(10^{-18}\ \text{m}\right)\left(1.44\times10^{32}\ \text{rad/s}\right) = \boxed{1.44\times10^{14}\ \text{m/s}} > c$$

Given that our model predicts a value for the speed of a point on the "equator" of a spinning electron that is greater than the speed of light, the idea that the spin angular momentum of an electron is analogous to that of a spinning sphere with spatial extent lacks credibility.

85 •• An interesting phenomenon occurring in certain *pulsars* (see Problem 24) is an event known as a "spin glitch," that is, a quick change in the spin rate of the pulsar due to a shift in mass location and a resulting rotational inertia change. Imagine a pulsar whose radius is 10.0 km and whose period of rotation is 25.032 ms. The rotation period is observed to suddenly decrease from 25.032 ms to 25.028 ms. If that decrease was related to a contraction of the star, by what amount would the pulsar radius have had to change?

Picture the Problem We can apply the conservation of angular momentum to the shrinking pulsar to relate its radii to the observed periods.

The change in the radius of the pulsar is:

$$\Delta R = R_\text{f} - R_\text{i} \qquad (1)$$

Apply conservation of angular momentum to the shrinking pulsar to obtain:

$$\Delta L = L_\text{f} - L_\text{i} = 0$$

or

$$I_\text{f}\omega_\text{f} - I_\text{i}\omega_\text{i} = 0$$

Substituting for I_f and I_i yields:

$$\tfrac{2}{5}MR_\text{f}^2\omega_\text{f} - \tfrac{2}{5}MR_\text{i}^2\omega_\text{i} = 0$$

Solve for ω_f to obtain:

$$\omega_f = \frac{R_i^2}{R_f^2}\omega_i$$

Because $\omega = 2\pi/T$, where T is the rotation period:

$$\frac{2\pi}{T_f} = \frac{R_i^2}{R_f^2}\frac{2\pi}{T_i} \Rightarrow R_f = \sqrt{\frac{T_f}{T_i}}R_i$$

Substitute for R_f in equation (1) and simplify to obtain:

$$\Delta R = \sqrt{\frac{T_f}{T_i}}R_i - R_i = \left(\sqrt{\frac{T_f}{T_i}} - 1\right)R_i$$

Substitute numerical values and evaluate ΔR:

$$\Delta R = \left(\sqrt{\frac{25.028\ \text{ms}}{25.032\ \text{ms}}} - 1\right)(10.0\ \text{km})$$
$$= \boxed{-79.9\ \text{cm}}$$

a 0.008% decrease in radius.

86 ••• Figure 10.60 shows a pulley in the form of a uniform disk with a rope hanging over it. The circumference of the pulley is 1.2 m and its mass is 2.2 kg. The rope is 8.0 m long and its mass is 4.8 kg. At the instant shown in the figure, the system is at rest and the difference in height of the two ends of the rope is 0.60 m. (*a*) What is the angular speed of the pulley when the difference in height between the two ends of the rope is 7.2 m? (*b*) Obtain an expression for the angular momentum of the system as a function of time while neither end of the rope is above the center of the pulley. There is no slippage between rope and pulley wheel.

Picture the Problem Let the origin of the coordinate system be at the center of the pulley with the upward direction positive. Let λ be the linear density (mass per unit length) of the rope and L_1 and L_2 the lengths of the hanging parts of the rope. We can use conservation of mechanical energy to find the angular speed of the pulley when the difference in height between the two ends of the rope is 7.2 m.

(*a*) Apply conservation of energy to relate the final kinetic energy of the system to the change in potential energy:

$$\Delta K + \Delta U = 0$$

or, because $K_i = 0$,

$$K + \Delta U = 0 \qquad (1)$$

Express the change in potential energy of the system:

$$\Delta U = U_f - U_i = -\tfrac{1}{2}L_{1f}(L_{1f}\lambda)g - \tfrac{1}{2}L_{2f}(L_{2f}\lambda)g - \left[-\tfrac{1}{2}L_{1i}(L_{1i}\lambda)g - \tfrac{1}{2}L_{2i}(L_{2i}\lambda)g\right]$$
$$= -\tfrac{1}{2}\left(L_{1f}^2 + L_{2f}^2\right)\lambda g + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g = -\tfrac{1}{2}\lambda g\left[\left(L_{1f}^2 + L_{2f}^2\right) - \left(L_{1i}^2 + L_{2i}^2\right)\right]$$

Because $L_1 + L_2 = 7.4$ m, $L_{2i} - L_{1i} = 0.6$ m, and $L_{2f} - L_{1f} = 7.2$ m, we obtain:

$L_{1i} = 3.4$ m, $L_{2i} = 4.0$ m, $L_{1f} = 0.1$ m, and $L_{2f} = 7.3$ m.

Substitute numerical values and evaluate ΔU:

$$\begin{aligned}\Delta U &= -\tfrac{1}{2}(0.60\,\text{kg/m})(9.81\,\text{m/s}^2)\left[(0.10\,\text{m})^2 + (7.3\,\text{m})^2 - (3.4\,\text{m})^2 - (4.0\,\text{m})^2\right] \\ &= -75.75\,\text{J}\end{aligned}$$

Express the kinetic energy of the system when the difference in height between the two ends of the rope is 7.2 m:

$$\begin{aligned}K &= \tfrac{1}{2}I_\text{p}\omega^2 + \tfrac{1}{2}Mv^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}M_\text{p}R^2\right)\omega^2 + \tfrac{1}{2}MR^2\omega^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}M_\text{p} + M\right)R^2\omega^2\end{aligned}$$

Substitute numerical values and simplify:

$$\begin{aligned}K &= \tfrac{1}{2}\left[\tfrac{1}{2}(2.2\,\text{kg}) + 4.8\,\text{kg}\right]\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\omega^2 \\ &= \left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2\end{aligned}$$

Substitute in equation (1) and solve for ω:

$$\left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2 - 75.75\,\text{J} = 0$$

and

$$\omega = \sqrt{\frac{75.75\,\text{J}}{0.1076\,\text{kg}\cdot\text{m}^2}} = \boxed{27\,\text{rad/s}}$$

(*b*) Noting that the moment arm of each portion of the rope is the same, express the total angular momentum of the system:

$$\begin{aligned}L &= L_\text{p} + L_\text{r} = I_\text{p}\omega + M_\text{r}R^2\omega \\ &= \left(\tfrac{1}{2}M_\text{p}R^2 + M_\text{r}R^2\right)\omega \qquad (2) \\ &= \left(\tfrac{1}{2}M_\text{p} + M_\text{r}\right)R^2\omega\end{aligned}$$

Letting θ be the angle through which the pulley has turned, express $U(\theta)$:

$$U(\theta) = -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g$$

Express ΔU and simplify to obtain:

$$\begin{aligned}\Delta U &= U_\text{f} - U_\text{i} = U(\theta) - U(0) \\ &= -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g \\ &\quad + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g \\ &= -R^2\theta^2\lambda g + (L_{1i} - L_{2i})R\theta\lambda g\end{aligned}$$

Assuming that, at $t = 0$, $L_{1i} \approx L_{2i}$:

$$\Delta U \approx -R^2\theta^2\lambda g$$

Substitute for K and ΔU in equation (1) to obtain:

$$\left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2 - R^2\theta^2\lambda g = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{R^2\theta^2\lambda g}{0.1076\,\text{kg}\cdot\text{m}^2}}$$

Substitute numerical values to obtain:

$$\omega = \sqrt{\frac{\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\left(0.6\,\text{kg/m}\right)\left(9.81\,\text{m/s}^2\right)}{0.1076\,\text{kg}\cdot\text{m}^2}}\,\theta = \left(1.41\,\text{s}^{-1}\right)\theta$$

Express ω as the rate of change of θ:

$$\frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)\theta \Rightarrow \frac{d\theta}{\theta} = \left(1.41\,\text{s}^{-1}\right)dt$$

Integrate θ from 0 to θ to obtain:

$$\ln\theta = \left(1.41\,\text{s}^{-1}\right)t$$

Transform from logarithmic to exponential form to obtain:

$$\theta(t) = e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Differentiate to express ω as a function of time:

$$\omega(t) = \frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute for ω in equation (2) to obtain:

$$L = \left(\tfrac{1}{2}M_\text{p} + M_\text{r}\right)R^2\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute numerical values and evaluate L:

$$L = \left[\tfrac{1}{2}\left(2.2\,\text{kg}\right) + \left(4.8\,\text{kg}\right)\right]\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\left[\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}\right] = \boxed{\left(0.30\,\text{kg}\cdot\text{m}^2/\text{s}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}}$$

Chapter R
Relativity

Conceptual Problems

1 • You are standing on a corner and a friend is driving past in an automobile. Each of you is wearing a wristwatch. Both of you note the times when the car passes two different intersections and determine from your watch readings the time that elapses between the two events. Have either of you determined the proper time interval? Explain your answer.

Determine the Concept In the reference frame of the car both events occur at the same location (the location of the car). Thus, your friend's watch measures the proper time between the two events.

2 • In Problem 1, suppose your friend in the car measures the width of the car door to be 90 cm. You also measure the width as he goes by you.
(*a*) Does either of you measures the proper width of the door? Explain your answer. (*b*) How will your value for the door width compare to his? (1) Yours will be smaller, (2) yours will be larger, (3) yours will be the same, (4) you can't compare the widths, as the answer depends on the car's speed.

Determine the Concept The proper length of an object is the length of the object in the rest frame of the object. The proper length of a meter stick is one meter.

(*a*) Because the door is at rest in the reference frame of the car, its width in that frame is its proper width. If your friend measures this width, say by placing a meter stick against the door, then he will measure the proper width of the door.

(*b*) In the reference frame in which you are at rest, the door is moving, so its width is less than its proper width. To measure this width would be challenging. (You could measure the width by measuring the time for the door to go by. The width of the door is the product of the speed of the car and the time.)

3 • **[SSM]** If event A occurs at a different location than event B in some reference frame, might it be possible for there to be a second reference frame in which they occur at the same location? If so, give an example. If not, explain why not.

Determine the Concept Yes. Let the initial frame of reference be frame 1. In frame 1 let L be the distance between the events, let T be the time between the events, and let the $+x$ direction be the direction of event B relative to event A. Next, calculate the value of L/T. If L/T is less than c, then consider the two events in a reference frame 2, a frame moving at speed $v = L/T$ in the $+x$ direction. In frame 2 both events occur at the same location.

4 • [SSM] If event A occurs prior to event B in some frame, might it be possible for there to be a second reference frame in which event B occurs prior to event A? If so, give an example. If not, explain why not.

Determine the Concept Yes. Let L and T be the distance and time between the two events in reference frame 1. If $L \le cT$, then something moving at a speed less than or equal to c could travel from the location of event A to the location of event B in a time less than T. Thus, it is possible that event A could cause event B. For events like these, causality demands that event A must precede event B in all reference frames. However, if $L > cT$ then event A cannot be the cause of event B. For events like these, event B does precede event A in certain reference frames.

5 • [SSM] Two events are simultaneous in a frame in which they also occur at the same location. Are they simultaneous in all other reference frames?

Determine the Concept Yes. If two events occur at the same time *and* place in one reference frame they occur at the same time *and* place in all reference frames. (Any pair of events that occur at the same time *and* at the same place in one reference frame are called a spacetime coincidence.)

6 • Two inertial observers are in relative motion. Under what circumstances can they agree on the simultaneity of two different events?

Determine the Concept We will refer to the two events as event A and event B. Assume that in the reference frame of the first observer there is a stationary clock at the location of each event, with clock A at the location of event A and clock B at the location of event B, and that the two clocks are synchronized. Because the two events are simultaneous in this frame, the readings of the two clocks at the time the events occur are the same. Also, event A and the reading of clock A at the time of event A are a spacetime coincidence, so all observers must agree with that clock reading. In like manner, event B and the reading of clock B at the time of event B are a spacetime coincidence. If observer B is moving parallel with the line joining the two clocks then the clocks readings will differ by Lv/c^2 in the reference frame of B, where L is the distance between the clocks in the reference frame of observer A. This means that observer B will agree that the two clock readings at the times of the events are the same, but will not agree that the events occurred at the same time unless $L = 0$.

7 • The approximate total energy of a particle of mass m moving at speed $v << c$ is (*a*) $mc^2 + \frac{1}{2}mv^2$, (*b*) mv^2, (*c*) cmv, (*d*) $\frac{1}{2}mc^2$, (*e*) cmv.

Picture the Problem We can use the expression for the total relativistic energy of the particle to express its energy in terms of its speed and expand the radical factor binomially to find the correct approximate expression for the total energy when $v << c$.

Express the total relativistic energy of the particle as the sum of its kinetic energy and the rest energy:

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Expand the radical factor binomially to obtain:

$$E = mc^2\left(1+\tfrac{1}{2}\frac{v^2}{c^2}+...\right) = mc^2 + \tfrac{1}{2}mv^2 + ...$$

$$\approx mc^2 + \tfrac{1}{2}mv^2 \text{ for } v << c.$$

and $\boxed{(a)}$ is correct.

8 • True or false:

(*a*) The speed of light is the same in all reference frames.
(*b*) The proper time interval is the shortest time interval between two events.
(*c*) Absolute motion can be determined by means of length contraction.
(*d*) The light-year is a unit of distance.
(*e*) For two events to form a space-time coincidence they must occur at the same place.
(*f*) If two events are not simultaneous in one frame, they cannot be simultaneous in any other frame.

(*a*) True. This is Einstein's second postulate.

(*b*) True. The time between events that happen at the *same place* in a reference frame is called the proper time and the time interval Δt measured in any other reference frame is always longer than the proper time.

(*c*) False. Absolute motion cannot be detected.

(*d*) True. A light-year is the distance light travels (in a vacuum) in one year.

(*e*) True. For two events to form a space-time coincidence, they must not only occur at the same place but must also occur at the same location.

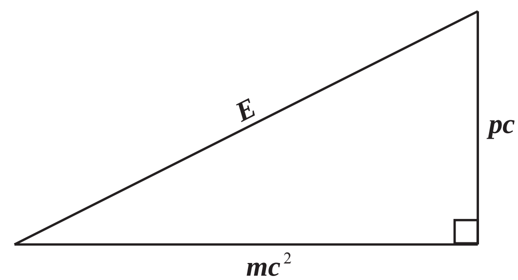
(*f*) False. The fact that two events are not simultaneous in one frame tells us nothing about their simultaneity in any other frame.

9 •• (*a*) Show that *pc* has dimensions of energy. (*b*) There is a geometrical interpretation of Equation R-17 based on the Pythagorean Theorem. Draw a picture of a triangle illustrating this interpretation.

(*a*) Express the dimensions of *pc* and simplify them to obtain:

$$[p][c]=\left(M\cdot\frac{L}{T}\right)\left(\frac{L}{T}\right)=M\cdot\left(\frac{L}{T}\right)^2$$
$$=\frac{ML^2}{T^2}=[E]$$

(*b*) The geometrical interpretation, based on the Pythagorean Theorem, of Equation R-17 is shown to the right:



10 •• A wad of putty of mass m_1 strikes and sticks to a second wad of putty of mass m_2 which is initially at rest. Do you expect that after the collision the combined putty mass will be (*a*) greater than, (*b*) less than, (*c*) the same as $m_1 + m_2$. Explain your answer.

Determine the Concept Because the collision is inelastic, kinetic energy is converted into mass energy. $\boxed{(a)}$ is correct.

11 •• **[SSM]** Many nuclei of atoms are unstable; for example, ^{14}C, an isotope of carbon, has a half-life of 5700 years. (By definition, the *half-life* is the time it takes for any given number of unstable particles to decay to half that number of particles.) This fact is used extensively for archeological and biological dating of old artifacts. Such unstable nuclei decay into several decay products, each with significant kinetic energy. Which of the following is true? (*a*) The mass of the unstable nucleus is larger than the sum of the masses of the decay products. (*b*) The mass of the unstable nucleus is smaller than the sum of the masses of the decay products. (*c*) The mass of the unstable nucleus is the same as the sum of the masses of the decay products. Explain your choice.

Determine the Concept Because mass is converted into the kinetic energy of the fragments, the mass of the unstable nucleus is larger than the sum of the masses of the decay products. $\boxed{(a)}$ is correct.

12 •• Positron emission tomography **(**PET) scans are common in modern medicine. In this procedure, positrons (a positron has the same mass, but the opposite charge as an electron) are emitted by radioactive nuclei that have been introduced into the body. Assume that an emitted positron, traveling slowly (with negligible kinetic energy) collides with an electron traveling at the same slow speed in the opposite direction. They undergo annihilation and two quanta of light (photons) are formed. You are in charge of designing detectors to receive these photons and measure their energy. (*a*) Explain why you would expect these two photons to come off in exactly opposite directions. (*b*) In terms of the electron mass m_e , how much energy would each photon have? (1) less than mc^2, (2) greater than mc^2, (3) exactly mc^2. Explain your choice.

Determine the Concept
(*a*) The linear momentum of the electron-positron pair was zero just before their annihilation and the emission of the two photons and therefore the linear momentum of the two photons must be zero.

(*b*) $\boxed{(3)}$ is correct. Each photon has the rest mass energy, m_ec^2, equivalent of an electron. Because there are two photons and they have the same momentum, they share equally, each getting half the available initial rest energy of $2m_ec^2$.

Estimation and Approximation

13 •• **[SSM]** In 1975, an airplane carrying an atomic clock flew back and forth at low altitude for 15 hours at an average speed of 140 m/s as part of a time-dilation experiment. The time on the clock was compared to the time on an atomic clock kept on the ground. What is the time difference between the atomic clock on the airplane and the atomic clock on the ground? (Ignore any effects that accelerations of the airplane have on the atomic clock which is on the airplane. Also assume that the airplane travels at constant speed.)

Picture the Problem We can use the time dilation equation to relate the elapsed time in the frame of reference of the airborne clock to the elapsed time in the frame of reference of the clock kept on the ground.

Use the time dilation equation to relate the elapsed time Δt according to the clock on the ground to the elapsed time Δt_0 according to the airborne atomic clock:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \qquad (1)$$

Because $v << c$, we can use the approximation $\frac{1}{\sqrt{1-x}} \approx 1+\frac{1}{2}x$ to obtain:

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \approx 1+\frac{1}{2}\left(\frac{v}{c}\right)^2$$

Substitute in equation (1) to obtain:

$$\Delta t = \left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right]\Delta t_0 = \Delta t_0 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \Delta t_0 \qquad (2)$$

where the second term represents the additional time measured by the clock on the ground.

Evaluate the proper elapsed time according to the clock on the airplane:

$$\Delta t_0 = (15\,\text{h})\left(3600\,\frac{\text{s}}{\text{h}}\right) = 5.40\times 10^4\ \text{s}$$

Substitute numerical values and evaluate the second term in equation (2):

$$\Delta t' = \frac{1}{2}\left(\frac{140\,\text{m/s}}{2.998\times 10^8\ \text{m/s}}\right)^2 (5.40\times 10^4\ \text{s}) = 5.89\times 10^{-9}\ \text{s} \approx \boxed{5.9\,\text{ns}}$$

14 •• (*a*) By making any necessary assumptions and finding certain stellar distances, estimate the speed at which a spaceship would have to travel for its passengers to make a trip to the nearest star (not the Sun!) and back to Earth in 1.0 Earth years, as measured by an observer on the ship. Assume the passengers make the outgoing and return trips at constant speed and ignore any effects due to the spaceship stopping and starting. (*b*) How much time would elapse on Earth during their roundtrip? Include 2.0 Earth years for a low-speed exploration of the planets in the vicinity of this star.

Picture the Problem We can use the length-contraction equation $L = L_0\sqrt{1-(v/c)^2}$ and $\Delta t = L/v$ to estimate the speed of the spaceship. Note that there is no time dilation effect during the two-year exploration period. Take the distance to the nearest sun to be 4.0 c·y.

(*a*) According to the travelers, the elapsed time for the trip is:

$$\Delta t = \frac{L}{v} = \frac{L_0\sqrt{1-(v/c)^2}}{v}$$

Multiplying both sides of the equation by c yields:

$$c\Delta t = \frac{cL_0\sqrt{1-(v/c)^2}}{v} = \frac{L_0\sqrt{1-(v/c)^2}}{v/c}$$

Because $c\Delta t$ = 1.0 c·y and L_0 = 4.0 c·y:

$$1.0 = \frac{4.0\sqrt{1-(v/c)^2}}{v/c}$$

Solve for v to obtain:

$$v=\sqrt{\frac{16}{17}}c=\boxed{0.97c}$$

(*b*) Express the elapsed time as the sum of the travel and exploration times:

$$\Delta t=\Delta t_{\text{travel}}+\Delta t_{\text{exploration}}=2\Delta t_{1\text{ wayl}}+\Delta t_{\text{exploration}}\qquad(1)$$

In the Earth frame-of reference:

$$\Delta t_{1\text{ way}}=\frac{L_0}{v}=\frac{4.0\,c\cdot\text{y}}{0.97c}=4.12\text{ y}$$

Substitute numerical values in equation (1) and evaluate Δt:

$$\Delta t=2(4.12\text{ y})+2.0\text{ y}=\boxed{10.2\text{ y}}$$

15 •• (*a*) Compare the kinetic energy of a moving car to its rest energy. (*b*) Compare the total energy of a moving car to its rest energy. (*c*) Estimate the error made in computing the kinetic energy of a moving car using non-relativistic expressions compared to the relativistically correct expressions. *Hint: Use of the binomial expansion may help.*

Picture the Problem We can compare these energies by expressing their ratios. Assume that the speed of the car is 30 m/s (≈ 67 mi/h).

(*a*) Express the ratio of the kinetic energy of a moving car to its rest energy:

$$\frac{K}{mc^2}=\frac{\frac{1}{2}mv^2}{mc^2}=\frac{1}{2}\left(\frac{v}{c}\right)^2$$

For a car moving at 30 m/s:

$$\frac{K}{mc^2}=\frac{1}{2}\left(\frac{30\text{ m/s}}{2.998\times10^8\text{ m/s}}\right)^2\approx\boxed{5\times10^{-13}\%}$$

(*b*) Express the ratio of the total energy of a moving car to its rest energy:

$$\frac{E}{mc^2}=\frac{K+mc^2}{mc^2}=\frac{K}{mc^2}+1\approx\boxed{1.000\,000\,000\,000\,005}$$

(*c*) The error made in computing the kinetic energy of a moving car using the non-relativistic expression compared to the relativistically correct expression is given by:

$$\frac{\Delta K}{K_{\text{rel}}}=\left|\frac{K_{\text{non-rel}}-K_{\text{rel}}}{K_{\text{rel}}}\right|=\left|\frac{K_{\text{non-rel}}}{K_{\text{rel}}}-1\right|\qquad(1)$$

From Equation R-14, the relativistic kinetic energy is given by:

$$K_{\text{rel}}=\left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}-1\right]mc^2\qquad(2)$$

Expand $\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ binomially to obtain:

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\ldots \approx 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}$$

Substituting in equation (2) and simplifying yields:

$$\begin{aligned} K_{\text{rel}} &= \left[\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\right]mc^2 \\ &= \frac{1}{2}mv^2+\frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) \\ &= K_{\text{non-rel}}\left(1+\frac{3}{4}\frac{v^2}{c^2}\right) \end{aligned}$$

Substitute for K_{rel} in equation (1) and simplify to obtain:

$$\begin{aligned} \frac{\Delta K}{K_{\text{rel}}} &= \left|\frac{K_{\text{non-rel}}}{K_{\text{non-rel}}\left(1+\frac{3}{4}\frac{v^2}{c^2}\right)}-1\right| \\ &= \left|\left(1+\frac{3}{4}\frac{v^2}{c^2}\right)^{-1}-1\right| \end{aligned}$$

Expanding $\left(1+\frac{3}{4}\frac{v^2}{c^2}\right)^{-1}$ binomially yields:

$$\left(1+\frac{3}{4}\frac{v^2}{c^2}\right)^{-1} = 1+(-1)\left(\frac{3}{4}\frac{v^2}{c^2}\right)+\frac{(-1)(-2)}{2}\left(\frac{3}{4}\frac{v^2}{c^2}\right)^2+\ldots \approx 1-\left(\frac{3}{4}\frac{v^2}{c^2}\right)$$

Substitute for $\left(1+\frac{3}{4}\frac{v^2}{c^2}\right)^{-1}$ in the expression for $\frac{\Delta K}{K_{\text{rel}}}$ to obtain:

$$\frac{\Delta K}{K_{\text{rel}}} = \left|\left(1-\frac{3}{4}\frac{v^2}{c^2}\right)-1\right| = \frac{3}{4}\frac{v^2}{c^2}$$

Substitute numerical values and evaluate $\frac{\Delta K}{K_{\text{rel}}}$:

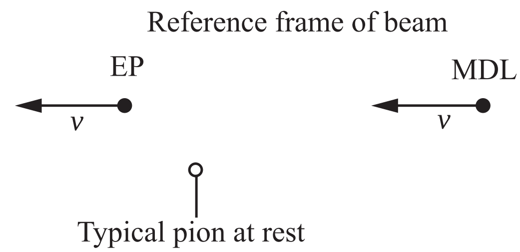
$$\frac{\Delta K}{K_{\text{rel}}} = \frac{3}{4}\frac{(30\text{ m/s})^2}{(2.998\times10^8\text{ m/s})^2} = \boxed{7.5\times10^{-15}}$$

Length Contraction and Time Dilation

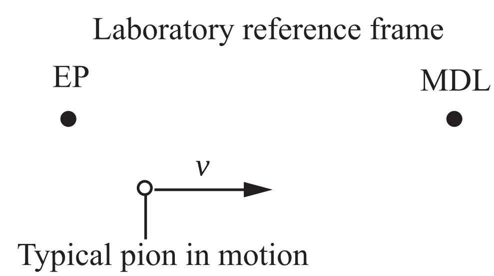
16 • The proper average (or mean) lifetime of a pion (a subatomic particle) is 2.6×10^{-8} s. (A neutral pion has a much shorter lifetime. See Chapter 41.) A beam of pions has a speed of $0.85c$ relative to a laboratory. (*a*) What would be their mean lifetime as measured in that laboratory? (*b*) On average, how far would they travel in that laboratory before they decay? (*c*) What would be your answer to Part (*b*) if you had neglected time dilation?

Picture the Problem The pion beam enters the laboratory at a point called the entrance port (EP). A typical pion in the beam disintegrates at a second point called the mean disintegration location (MDL). Event A occurs when a typical pion enters the room at EP, and event B occurs when the pion disintegrates at MDL. (*a*) In the reference frame of the beam these two events occur at the same location, so the time Δt_0 between these two events in this frame is the proper time between the events. We can use the time dilation formula to calculate the time Δt between these events in the reference frame of the laboratory, a reference frame that moves with speed $v = 0.85c$ relative to the reference frame of the beam.
(*b*) In the reference frame of the room a typical pion travels at speed v for time Δt.

(*a*) Sketch a typical pion in the reference frame of the pion beam. In this frame the typical pion is at rest and both EP and MDL are moving at speed v. Event A occurs when EP is next to a typical pion, and event B occurs when MDL reaches the pion at the instant the pion disintegrates:



Sketch a typical pion in the reference frame of the laboratory. In this frame the typical pion moves at speed v. Event A occurs when a typical pion enters the room at EP, and event B occurs when the pion reaches MDL at the instant the pion disintegrates:



Use the time-dilation equation to relate the mean lifetime Δt of a typical pion in the laboratory frame to the proper mean lifetime Δt_0 in the reference frame of the beam:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2.6\times10^{-8}\ \text{s}}{\sqrt{1-\left(\frac{0.85c}{c}\right)^2}} = 4.94\times10^{-8}\ \text{s}$$
$$= \boxed{4.9\times10^{-8}\ \text{s}}$$

(*b*) Express the distance a typical pion will travel in the lab frame before it decays in terms of its speed and its mean lifetime in the lab frame:

$$\Delta x = v\Delta t$$

Substitute numerical values and evaluate Δx:

$$\Delta \boldsymbol{x} = (0.85\boldsymbol{c})(4.94\times10^{-8}\ \text{s}) = \boxed{13\ \text{m}}$$

(*c*) Neglecting time dilation, $\Delta t = 2.6 \times 10^{-8}$ s, and:

$$\Delta \boldsymbol{x} = (0.85\boldsymbol{c})(2.6\times10^{-8}\ \text{s}) = \boxed{6.6\ \text{m}}$$

17 • **[SSM]** In the reference frame of a pion in Problem 16, how far does the laboratory travel in 2.6×10^{-8} s?

Picture the Problem We can use $\Delta x = v\Delta t_\pi$, where Δt_π is the proper mean lifetime of the pions, to find the distance traveled by the laboratory frame in a typical pion lifetime.

The average distance the laboratory will travel before the pions decay is the product of the speed of the pions and their proper mean lifetime:

$$\Delta x = v\Delta t_\pi$$

Substitute numerical values and evaluate Δx:

$$\Delta x = (0.85c)(2.6\times10^{-8}\ \text{s}) = 6.63\ \text{m}$$
$$= \boxed{6.6\ \text{m}}$$

18 • The proper average (or mean) lifetime of a muon (a subnuclear particle) particle is 2.20 μs. Muons in a beam are traveling at 0.999c relative to a laboratory. (*a*) What is their lifetime as measured in that laboratory? (*b*) On average, how far do they travel in that laboratory before they decay?

Picture the Problem We can express the mean lifetimes of the muons in the laboratory in terms of their proper lifetimes using $\Delta t = \Delta t_0 \big/ \sqrt{1-(v/c)^2}$. The average distance the muons will travel before they decay is related to their speed and mean lifetime in the laboratory frame according to $\Delta x = v\Delta t$.

(*a*) Use the time-dilation equation to relate the mean lifetime of a muon in the laboratory Δt to their proper mean lifetime Δt_0:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2.20\times10^{-6}\ \text{s}}{\sqrt{1-\left(\frac{0.999c}{c}\right)^2}} = 49.2\ \mu\text{s}$$

$$= \boxed{49\ \mu\text{s}}$$

(*b*) The average distance muons travel before they decay in terms of their speed and mean lifetime in the laboratory frame of reference is:

$$\Delta x = v\Delta t$$

Substitute numerical values and evaluate Δx:

$$\Delta x = (0.999c)(49.2\ \mu\text{s}) = \boxed{15\,\text{km}}$$

19 • In the reference frame of the muon in Problem 20, how far does the laboratory travel in 2.20 μs?

Picture the Problem We can use $\Delta x = v\Delta t_\pi$, where Δt_π is the proper mean lifetime of the pions, to find the distance traveled by the laboratory frame in a typical pion lifetime and $\Delta x_0 = \Delta x \big/ \sqrt{1-(v/c)^2}$ to find this distance in the laboratory's frame.

Express the average distance the laboratory will travel before the pions decay in terms of the speed of the pions and their proper mean lifetime:

$$\Delta x = v\Delta t_\pi$$

Substitute numerical values and evaluate Δx:

$$\Delta x = (0.999c)(2.20\ \mu\text{s}) = \boxed{659\,\text{m}}$$

20 • You have been posted in a remote region of space to monitor traffic. Toward the end of a quiet shift, a spacecraft goes by and you measure its length using a laser device. This device reports a length of 85.0 m. You flip open your handy reference catalogue and identify the craft as a CCCNX-22, which has a proper length of 100 m. When you phone in your report, what speed should you give for this spacecraft?

Picture the Problem The measured length L of the spacecraft is related to its proper length L_0 and its speed according to $L = L_0\sqrt{1-\left(v^2/c^2\right)}$. We can solve this equation for v as a function of c, L, and L_0.

Express the length L of the spacecraft in terms of its proper length L_0:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} \Rightarrow v = c\sqrt{1-\left(\frac{L}{L_0}\right)^2}$$

Substitute numerical values and evaluate v:

$$v = \left(2.998\times10^8\ \text{m/s}\right)\sqrt{1-\left(\frac{85.0\,\text{m}}{100\,\text{m}}\right)^2}$$

$$= \boxed{1.58\times10^8\ \text{m/s}}$$

21 • **[SSM]** A spaceship travels from Earth to a star 95 light-years away at a speed of 2.2×10^8 m/s. How long does the spaceship take to get to the star (*a*) as measured on Earth and (*b*) as measured by a passenger on the spaceship?

Picture the Problem We can use $\Delta x = v\Delta t$ to find the time for the trip as measured on Earth and $\Delta t_0 = \Delta t\sqrt{1-\left(v/c\right)^2}$ to find the time measured by a passenger on the spaceship.

(*a*) Express the elapsed time, as measured on Earth, in terms of the distance traveled and the speed of the spaceship:

$$\Delta t = \frac{\Delta x}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{95\,c\cdot\text{y}}{2.2\times10^8\ \text{m/s}}\times\frac{9.461\times10^{15}\ \text{m}}{c\cdot\text{y}}$$

$$= 4.09\times10^9\ \text{s}\times\frac{1\,\text{y}}{31.56\,\text{Ms}} = 129\,\text{y}$$

$$= \boxed{1.3\times10^2\ \text{y}}$$

(*b*) A passenger on the spaceship will measure the proper time:

$$\Delta t_0 = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate the proper time:

$$\Delta t_0 = (129\,\text{y})\sqrt{1-\frac{(2.2\times10^8\ \text{m/s})^2}{(2.998\times10^8\ \text{m/s})^2}} = \boxed{88\,\text{y}}$$

22 • The average lifetime of a beam of subatomic particles called pions (see Problem 16 for details on these particles) traveling at high speed is measured to be 7.5 × 10^{-8} s. Their average lifetime at rest is known to be 2.6 × 10^{-8} s. How fast is this pion beam traveling?

Picture the Problem We can express the mean lifetime of the pion in the laboratory in terms of its proper lifetime using $\Delta t = \Delta t_0 \big/ \sqrt{1-(v/c)^2}$ and solve for v to find its speed.

Use the time-dilation equation to relate the mean lifetime of a muon in the laboratory Δt to their proper mean lifetime Δt_0:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = c\sqrt{1-\left(\frac{\Delta t_0}{\Delta t}\right)^2}$$

Substitute numerical values and evaluate v:

$$v = (2.998\times10^8\ \text{m/s})\sqrt{1-\left(\frac{2.6\times10^{-8}\ \text{s}}{7.5\times10^{-8}\ \text{s}}\right)^2} = \boxed{2.8\times10^8\ \text{m/s}}$$

23 • A meterstick moves with speed 0.80*c* relative to you in the direction parallel to the stick. (*a*) Find the length of the stick as measured by you. (*b*) How long does it take for the stick to pass you?

Picture the Problem We can find the measured length L of the meterstick using $L = L_0\sqrt{1-(v^2/c^2)}$ and the time it takes to pass you using $L = v\Delta t$.

(*a*) Express the length L of the meterstick in terms of its proper length L_0:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

Substitute numerical values and evaluate L:

$$L = (1.0\,\text{m})\sqrt{1-\frac{(0.8c)^2}{c^2}} = \boxed{60\,\text{cm}}$$

(*b*) Express the time it takes for the meterstick to pass you in terms of its apparent length and speed:

$$\Delta t = \frac{L}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{0.60\,\text{m}}{0.8c} = \boxed{2.5\,\text{ns}}$$

24 • Recall that the half-life is the time it takes for any given amount of unstable particles to decay to half that amount of particles. The proper half-life of a species of charged subatomic particles called pions is 1.80×10^{-8} s (See Problem 16 for details on pions.) . Suppose a group of these pions are produced in an accelerator and emerge with a speed of 0.998*c*. How far do these particles travel in the accelerator's laboratory before half of them have decayed?

Picture the Problem We can express the distance the pions will travel in the laboratory using $\Delta x = v\Delta t$ and find their half-life in the accelerator laboratory using $\Delta t = \Delta t_0 \big/ \sqrt{1-(v/c)^2}$.

Express the average distance the pions will travel before decaying in terms of their speed proper mean lifetime:

$$\Delta x = v\Delta t$$

Use the time-dilation equation to relate the mean lifetimes of the pions in the accelerator laboratory Δt to their proper mean lifetime Δt_0:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute for Δt to obtain:

$$\Delta x = \frac{v\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

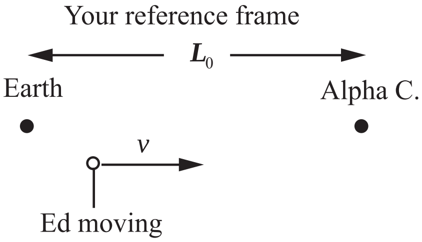
Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{(0.998c)(1.80\times10^{-8}\,\text{s})}{\sqrt{1-\left(\frac{0.998c}{c}\right)^2}} = \boxed{85.2\,\text{m}}$$

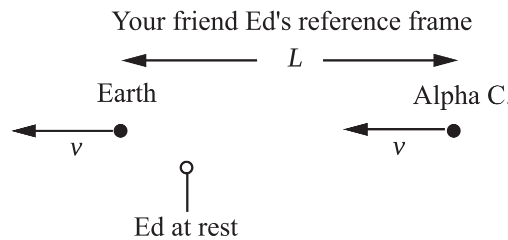
25 •• [SSM] Your friend, who is the same age as you, travels to the star Alpha Centauri, which is 4.0 light-years away, and returns immediately. He claims that the entire trip took just 6.0 y. What was his speed? Ignore any accelerations of your friend's spaceship and assume the spaceship traveled at the same speed during the entire trip.

Picture the Problem To calculate the speed in the reference frame of the friend, who is named Ed, we consider each leg of the trip separately. Consider an imaginary stick extending from Earth to Alpha Centauri that is at rest relative to Earth. In Ed's frame the length of the stick, and thus the distance between Earth and Alpha Centauri, is shortened in accord with the length contraction formula. As three years pass on Ed's watch Alpha Centauri travels at speed v from its initial location to him.

Sketch the situation as it is in your reference frame. The distance between Earth and Alpha C. is the rest length L_0 of the stick discussed in Picture the Problem:



Sketch the situation as it is in Ed's reference frame. The distance between Earth and Alpha C. is the length L of the moving stick discussed in Picture the Problem:



The two events are Ed leaves Earth and Ed arrive at Alpha Centauri. In Ed's frame these two events occur at the same place (next to Ed). Thus, the time between those two events Δt_0 is the proper time between the two events.

$$\Delta t_0 = 3\text{ y}$$

The distance L traveled by Alpha Centauri in Ed's frame during the first three years equals the speed multiplied by the time in Ed's frame:

$$L = v\Delta t_0$$

The distance between Earth and Alpha Centauri in Ed's frame is the contracted length of the imaginary stick:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

Equate these expressions for L to obtain:

$$L_0\sqrt{1-\frac{v^2}{c^2}} = v\Delta t_0$$

Substituting numerical values yields:

$$(4.0\,c\cdot\text{y})\sqrt{1-\frac{v^2}{c^2}}=v(3.0\text{ y})$$

or

$$1-\frac{v^2}{c^2}=\frac{9}{16}\frac{v^2}{c^2}\Rightarrow\frac{25}{16}\left(\frac{v^2}{c^2}\right)=1$$

Solving for v gives:

$$v=\boxed{0.80c}$$

26 •• Two spaceships pass each other traveling in opposite directions. A passenger in ship A knows that her ship is 100 m long. She notes that ship B is moving with a speed of 0.92c relative to A and that the length of B is 36 m. What are the lengths of the two spaceships as measured by a passenger in ship B?

Picture the Problem We can use the relationship between the measured length L of the spaceships and their proper lengths L_0 to find the lengths of the two spaceships as measured by a passenger in ship B.

Relate the measured length L_A of ship A to its proper length:

$$L_A=L_{0,A}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate L_A:

$$L_A=(100\text{ m})\sqrt{1-\left(\frac{0.92c}{c}\right)^2}=\boxed{39\text{ m}}$$

Relate the proper length $L_{0,B}$ of ship B to its measured length L_B:

$$L_{0,B}=\frac{L_B}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate $L_{0,B}$:

$$L_{0,B}=\frac{36\text{ m}}{\sqrt{1-\left(\frac{0.92c}{c}\right)^2}}=\boxed{92\text{ m}}$$

27 •• Supersonic jets achieve maximum speeds of about $3.00\times10^{-6}c$. (*a*) By what percentage would a jet traveling at this speed contract in length? (*b*) During a time of exactly one year or 3.15×10^{7} s on your clock, how much time would elapse on the pilot's clock? How much time is lost by the pilot's clock in one year of your time? Assume you are on the ground and the pilot is flying at the specified speed for the entire year.

Picture the Problem We can use the relationship between the measured length L of the jet and its proper length to express the fractional change in the length of the jet traveling at its maximum speed. In Part (*b*) we can express the elapsed time on the pilot's clock Δt_0 in terms of the elapsed time Δt on your clock.

(*a*) Express the fractional change in length of the jet:

$$\frac{L_0 - L}{L_0} = 1 - \frac{L}{L_0}$$

Relate L to L_0:

$$L = L_0\sqrt{1-\left(\frac{v}{c}\right)^2} \Rightarrow \frac{L}{L_0} = \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute for L/L_0 to obtain:

$$\frac{L_0 - L}{L_0} = 1 - \sqrt{1-\frac{v^2}{c^2}} \qquad (1)$$

Expand $\sqrt{1-\frac{v^2}{c^2}}$ binomially to obtain:

$$\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$$

$$\approx 1 - \frac{1}{2}\frac{v^2}{c^2}$$

Substituting in equation (1) yields:

$$\frac{L_0 - L}{L_0} = 1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}\frac{v^2}{c^2}$$

Substitute numerical values and evaluate the jet's fractional change in length:

$$\frac{L_0 - L}{L_0} \approx \frac{1}{2}\frac{\left(3.00\times10^{-6}c\right)^2}{c^2}$$

$$= \boxed{4.50\times10^{-10}\%}$$

(*b*) Express the elapsed time on the pilot's clock Δt_0 in terms of the elapsed time Δt on your clock:

$$\Delta t_0 = \left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\Delta t \approx \Delta t\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$= \Delta t - \frac{1}{2}\frac{v^2}{c^2}\Delta t$$

where the second term represents the time lost on the pilot's clock.

Substitute numerical values and evaluate the elapsed time on the pilot's clock in 1 y = 3.15 × 10^7 s:

$$\Delta t_0 = \left(1-\frac{1}{2}\frac{v^2}{c^2}\right)\Delta t = \left(1-\frac{1}{2}\frac{\left(3.00\times10^{-6}\,c\right)^2}{c^2}\right)\left(3.15\times10^{7}\text{ s}\right) = \boxed{1.00\text{ y}}$$

The time lost by the pilot's clock in time Δt of your time is given by:

$$\Delta t - \Delta t_0 = \frac{1}{2}\frac{v^2}{c^2}\Delta t$$

Substitute numerical values and evaluate $\Delta t - \Delta t_0$ for 1 y = 3.15 × 10^7 s:

$$\Delta t - \Delta t_0 = \frac{1}{2}\frac{\left(3.00\times10^{-6}\,c\right)^2}{c^2}\left(3.15\times10^{7}\text{ s}\right) = \left(142\ \mu\text{s}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = \boxed{2.37\ \mu\text{min}}$$

28 •• The proper mean lifetime of a muon (see Problems 18 and 19 for details) is 2.20 μs. Consider a muon, created in Earth's atmosphere, speeding toward the surface 8.00 km below, at a speed of 0.980c. (*a*) What is the likelihood that it will survive its trip to ground before decaying? The probability of a muon decaying is given by $P = 1 - e^{-\Delta t/\tau}$, where Δt is the time interval as measured in the reference frame in question. (*b*) Calculate this from the point of view of an observer moving with the muon. Show that, from the point of view of an observer on Earth, the answer is the same.

Picture the Problem We can use the given probability function to find the probability of the muon's survival from both the point of view of an Earth observer and the muon. To do so, we'll need to use the time-dilation relationship in (*a*) and the length-contraction relationship in (*b*).

(*a*) The probability of a muon decaying is $P = 1 - e^{-\Delta t/\tau}$, so the probability of survival is:

$$P_{\text{survival}} = e^{-\Delta t/\tau} \qquad (1)$$

From the point of view of an observer on Earth:

$$\Delta t = \frac{L}{v} = \frac{8000\text{ m}}{(0.980)\left(2.998\times10^{8}\text{ m/s}\right)} = 27.229\ \mu\text{s}$$

The mean lifetime of the muon is given by:

$$\tau = \frac{\tau_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{2.20\ \mu\text{s}}{\sqrt{1-(0.980)^2}} = 11.055\ \mu\text{s}$$

Substitute numerical values in equation (1) and evaluate P_{survival}:

$$P = e^{-\frac{27.229\ \mu\text{s}}{11.055\ \mu\text{s}}} = \boxed{8.5\%}$$

(*b*) From the point of view of the muon:

$$\Delta t = \frac{L}{v} \qquad (2)$$

The muon's reference frame is the one in which it is at rest and Earth is rushing toward it at 0.980*c*. Express the thickness of Earth's atmosphere as measured in the muon's frame of reference:

$$L = L_0\sqrt{1 - v^2/c^2}$$

Substituting for L in equation (2) yields:

$$\Delta t = \frac{L_0\sqrt{1 - v^2/c^2}}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{8000\ \text{m}\sqrt{1 - (0.980)^2}}{(0.980)(2.998\times10^8\ \text{m/s})} = 5.419\ \mu\text{s}$$

Substituting numerical values in equation (1) yields:

$$P_{\text{survival}} = e^{-\frac{5.419\ \mu\text{s}}{2.20\ \mu\text{s}}} = \boxed{8.5\%}$$

a result that agrees, to two significant figures, with the result obtained in (*a*).

29 •• A spaceship commander traveling to the Magellanic Clouds, travels at a uniform speed of 0.800c. When leaving the Kuiper belt, whose outer edge is 50.0 AU from Earth (*Note*: 1 AU = 150,000,000 km, and represents the average distance between Earth and the Sun; AU = astronomical unit), he sends a message to ground control, Houston, Texas, saying that he is fine. Fifteen minutes later (according to him) he realizes he has made a typo, so he sends a correction. How much time passes at Houston between the receipt of his initial message and the second message?

Picture the Problem Let ΔT be the proper time interval between event A, the sending of the first message, and event B, the sending of the correction. These two events take place on the ship, so the time interval between the two events it the time interval in the reference frame of the ship. Thus, ΔT = 15 min. Let $\Delta T'$ be the time interval between these same two events in the reference frame of Houston. Then the time between the arrival of the two messages in Houston is $\Delta T_{\substack{\text{between}\\\text{messages}}} = \Delta T' + \Delta T'_{\text{travel}}$.

The time between the arrival of the two messages in Houston is:

$$\Delta T_{\substack{\text{between}\\\text{messages}}} = \Delta T' + \Delta T'_{\text{travel}}$$

Because Houston is moving at speed $v = 0.800c$ relative to the ship:

$$\Delta T' = \frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}}$$

Substituting for $\Delta T'$ gives:

$$\Delta T_{\substack{\text{between}\\ \text{messages}}} = \frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}} + \Delta T'_{\text{travel}} \quad (1)$$

During the time interval $\Delta T'$ the ship travels a distance:

$$\Delta x = v\Delta T'$$

The reply has to travel the distance Δx farther than the first message and this takes time $\Delta T'_{\text{travel}}$. The reply travels at speed c, so:

$$\Delta x = c\Delta T'_{\text{travel}}$$

Equating these expressions for Δx gives:

$$v\Delta T' = c\Delta T'_{\text{travel}} \Rightarrow \Delta T'_{\text{travel}} = \frac{v}{c}\Delta T'$$

Substituting for $\Delta T'$ yields:

$$\Delta T'_{\text{travel}} = \frac{v}{c}\frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}}$$

Substitute for $\Delta T'_{\text{travel}}$ and simplify to obtain:

$$\begin{aligned}\Delta T_{\substack{\text{between}\\ \text{messages}}} &= \frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}} + \frac{v}{c}\frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}}\\ &= \frac{\Delta T_{\text{spaceship}}}{\sqrt{1-\left(v^2/c^2\right)}}\left(1+\frac{v}{c}\right)\end{aligned}$$

Substitute numerical values and evaluate $\Delta T_{\substack{\text{between}\\ \text{messages}}}$:

$$\begin{aligned}\Delta T_{\substack{\text{between}\\ \text{messages}}} &= \frac{15\text{ min}}{\sqrt{1-(0.800)^2}}(1+0.800)\\ &= \boxed{45\text{ min}}\end{aligned}$$

The Relativity of Simultaneity

30 • According to Jamal: (*a*) what is the distance between the flashbulb and clock A? (*b*) How far does the flash travel to reach clock A? (*c*) How far does clock A travel while the flash is traveling from the flashbulb to it?

Picture the Problem The proper distance L_0 between the flashbulb and clock A is 50.0 c·min and is related to the distance L measured by Jamal according to $L = L_0\sqrt{1-(v/c)^2}$. Because clock A and the flashbulb are at rest relative to each other, we can find the distance between them using the same relationship with $L_0 = 50.0\ c$·min.

(*a*) Express L in terms of L_0:

$$L = L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate L:

$$L = (50.0\,c\cdot\text{min})\sqrt{1-\left(\frac{0.600c}{c}\right)^2}$$
$$= \boxed{40.0\,c\cdot\text{min}}$$

(*b*) Jamal sees the flash traveling away from him at 1.0*c* and clock A approaching at 0.6*c*. So:

$$\Delta t = \frac{40.0\,c\cdot\text{min}}{1.600c} = 25.0\,\text{min}$$

and so

$$L = \boxed{25.0\,c\cdot\text{min}}$$

(*c*) Express the distance d clock A travels while the flash is traveling to it in terms of the speed of clock A and the time Δt it takes the flash to reach it:

$$d = v\Delta t = v\left(\frac{L}{c}\right)$$

Substitute numerical values and evaluate L:

$$d = (0.600c)\left(\frac{25.0\,c\cdot\text{min}}{c}\right)$$
$$= \boxed{15.0\,c\cdot\text{min}}$$

31 •• According to Jamal, how long does it take the flash to travel to clock A, and what does clock C read as the flash reaches clock A?

Picture the Problem As the light pulse from the flashbulb travels toward clock A with speed *c*, clock A travels toward clock C with speed 0.600*c*. The sum of the distances traveled by the flash and clock *A* must equal the distance separating them as seen in Jamal's frame of reference.

Express the sum of the distances between clock A and the flashbulb as seen in Jamal's frame of reference:

$$0.600ct + ct = \tfrac{1}{2}\ell \Rightarrow t = \frac{\ell}{3.200c}$$

where ℓ is the distance separating clocks A and B.

Express ℓ in terms of ℓ_0:

$$\ell = \ell_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute for ℓ to obtain:

$$t=\frac{\ell_0}{3.200c}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate t:

$$t=\frac{100c\cdot\text{min}}{3.200\text{c}}\sqrt{1-\left(\frac{0.600c}{c}\right)^2}=\boxed{25\,\text{min}}$$

Because clock C starts from zero when the flashbulb passes next to it, its reading when the flash reaches clock A will equal the time required for the flash to travel to clock A. Hence clock C will read:

$$t_C=\boxed{25\,\text{min}}$$

32 •• Show that clock C reads 100 min as the light flash reaches clock B, which is traveling away from clock C with speed 0.600*c*.

Picture the Problem As the light flash from the flashbulb travels toward clock B with speed *c*, clock B travels away from clock C with speed 0.600*c*. The difference between these distances must equal the distance between clock B and the flashbulb as seen in Jamal's frame of reference.

Express the difference between the distances traveled by the flash of light and clock B as seen in Jamal's frame of reference:

$$ct-0.600ct=\tfrac{1}{2}L\Rightarrow t=\frac{L}{0.800c}$$

where L is the separation of clocks A and B.

Use the length contraction equation to express L in terms of L_0:

$$L=L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute for L to obtain:

$$t=\frac{L_0}{0.800c}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate t:

$$t=\frac{100c\cdot\text{min}}{0.800c}\sqrt{1-\left(\frac{0.600c}{c}\right)^2}=\boxed{100\,\text{min}}$$

33 •• According to Jamal, the reading on clock C advances from 25 min to 100 min between the reception of the flashes by clocks A and B in Problems 31 and 32. According to Jamal, how much will the reading on clock A advance during this 75-min interval?

Picture the Problem We can find the elapsed time on the clock at A (the proper time) from the elapsed time on the clock at B using $\Delta t_0 = \Delta t\sqrt{1-(v/c)^2}$.

Express the elapsed proper time Δt_0 on clock A in terms of the elapsed time Δt measured by Jamal:

$$\Delta t_0 = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate Δt_0:

$$\Delta t_0 = (75\,\text{min})\sqrt{1-\left(\frac{0.600c}{c}\right)^2} = \boxed{60\,\text{min}}$$

34 •• The advance of clock A calculated in Problem 33 is the amount that clock A leads clock B according to Jamal. Compare this result with vL_0/c^2, where $v = 0.600c$.

Picture the Problem We can compare the time calculated in Problem 33 with L_0v/c^2 by evaluating their ratio.

Express the ratio r of L_0v/c^2 to Δt_0:

$$r = \frac{\frac{L_0 v}{c^2}}{\Delta t_0} = \frac{\frac{L_0 v}{c^2}}{\Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{L_0 v}{c^2 \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate r:

$$r = \frac{(100\,c\cdot\text{min})(0.600c)}{c^2(75\,\text{min})\sqrt{1-\left(\frac{0.600c}{c}\right)^2}} = \boxed{1}$$

35 •• **[SSM]** In an inertial reference frame S, event B occurs 2.00 μs after event A and 1.50 km distant from event A. How fast must an observer be moving along the line joining the two events so that the two events occur simultaneously?

For an observer traveling fast enough is it possible for event B to precede event A?

Picture the Problem Because event A is ahead of event B by $L_p v/c^2$ where v is the speed of the observer moving along the line joining the two events, we can use this expression and the given time between the events to find v.

Express the time Δt between events A and B in terms of L_p, v, and c:

$$\Delta t = \frac{L_p v}{c^2} \Rightarrow v = \frac{c^2 \Delta t}{L_p}$$

Substitute numerical values and evaluate v:

$$v = \frac{\left(2.998\times10^8\ \text{m/s}\right)^2 (2.00\ \mu\text{s})}{1.50\ \text{km}}$$

$$= \left(1.20\times10^8\ \text{m/s}\right)\left(\frac{c}{2.998\times10^8\ \text{m/s}}\right)$$

$$= \boxed{0.400c}$$

Rewrite the expression for v in terms of t_A and t_B yields:

$$v = \frac{c^2 (t_B - t_A)}{L_p}$$

Express the condition on $t_B - t_A$ if event B is to precede event A:

$$t_B - t_A < 0 \text{ and } v > \frac{c^2 (t_B - t_A)}{L_p}$$

Substitute numerical values and evaluate v:

$$v > \frac{\left(2.998\times10^8\ \text{m/s}\right)^2 (2.00\ \mu\text{s})}{1.50\ \text{km}}$$

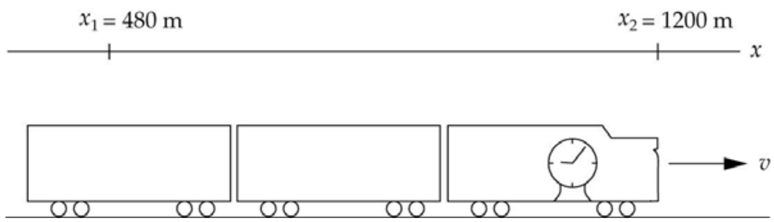
$$= 1.20\times10^8\ \text{m/s} = 0.400c$$

Event B can precede event A provided $v > 0.400c$.

36 •• A large flat space platform has an x axis painted on it. A firecracker explodes on the x axis at $x_1 = 480$ m, and a second firecracker explodes on the x axis 5.00 μs later at $x_2 = 1200$ m. In the reference frame of a train traveling alongside the x axis at speed v relative to the platform, these two explosions occur at the same place on that axis. What is the separation in time between the two explosions in the reference frame of the train?

Picture the Problem Let both explosions occur at the front end of the train—where there is a clock fastened to the train. In the reference frame of the train, both explosions occur at the same place. The clock is stationary in this frame, so it does not run slow. Thus, in this frame the amount the clock reading advances is also the time T_0 between the two events. In the reference frame of the platform the clock (and the train) moves with speed $v = L/T$, where $L = x_2 - x_1$ and

$T = 5.00\ \mu\text{s}$. In this frame the clock is moving so it runs slow, advancing by only $T\sqrt{1-\left(v^2/c^2\right)}$ during time T. All observers must agree with both the clock reading when the first explosion occurs (a spacetime coincidence), and with the clock reading when the second firecracker goes off (also a spacetime coincidence). Thus, all observers agree with the amount the clock reading changes by in the interval between the two events.



Equate the expressions for the amount the clock advances to obtain:

$$T_0 = T\sqrt{1-\left(v^2/c^2\right)}$$

Substitute L/T for v to obtain:

$$T_0 = T\sqrt{1-\left(\frac{L}{cT}\right)^2}$$

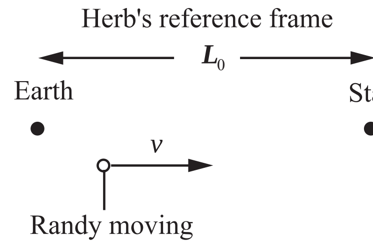
Substitute numerical values and evaluate T_0:

$$T_0 = (5.00\ \mu\text{s})\sqrt{1-\left(\frac{1200\,\text{m}-480\,\text{m}}{\left(2.998\times10^8\ \text{m/s}\right)\left(5.00\times10^{-6}\ \text{s}\right)}\right)^2} = \boxed{4.39\ \mu\text{s}}$$

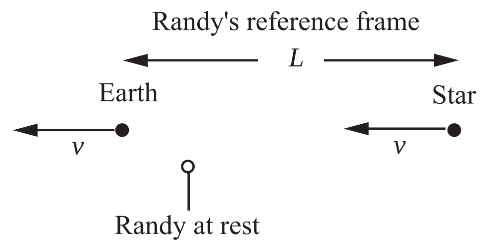
37 •• Herb and Randy are twin jazz musicians who perform as a trombone–saxophone duo. At the age of twenty, however, Randy got an irresistible offer to perform on a star 15 light-years away. To celebrate his good fortune, he bought a new vehicle for the trip—a deluxe space-coupe that travels at 0.99*c*. Each of the twins promises to practice diligently, so they can reunite afterward. However, Randy's gig goes so well that he stays for a full 10 years before returning to Herb. After their reunion, (*a*) how many years of practice will Randy have had; (*b*) how many years of practice will Herb have had?

Picture the Problem Randy's clocks, including his biological clocks, are always at rest in Randy's reference frame, so they advance at the same rate that time advances in Randy's frame. The same is true for Herb and the clocks in his reference frame. Think of a long imaginary stick, at rest relative to Earth, with one end at Earth and the other at the star. The rest length of this stick is 15 c·y. The time Randy has to practice is the time for the trip in Randy's reference frame, and the time Herb has to practice is the time for Randy's trip in Herb's reference frame.

Sketch the situation as it is in Herb's reference frame. The distance between Earth and the star is the rest length L_0 of the stick (discussed in **Picture the Problem**):



Sketch the situation as it is in Randy's reference frame. The distance between Earth and the star is the length L of the moving stick (discussed in Picture the Problem):



To find the time T_0 in Randy's reference frame for the star to travel the distance L we apply the formula distance equals speed multiplied by time. However, to do this we must find L, which is related to L_0 by the length contraction formula:

$$L = vT_0 \Rightarrow T_0 = \frac{L}{v} = \frac{L_0\sqrt{1-\dfrac{v^2}{c^2}}}{v}$$

Substitute numerical values and evaluate T_0:

$$T_0 = \frac{15c\cdot\text{y}\sqrt{1-(0.99)^2}}{0.99c} = 2.137\ \text{y}$$

To find the time T in Herb's reference frame for Randy to travel to the star we apply the formula distance equals speed multiplied by the time:

$$L_0 = vT \Rightarrow T = \frac{L_0}{v}$$

Substitute numerical values and evaluate T:

$$T = \frac{15c\cdot\text{y}}{0.99c} = 15.152\ \text{y}$$

While Randy is at the Star the two reference frames are at rest relative to each other. Thus, the time that Randy is at the star is the same in both frames. The calculations for the return trip give the same result as for the outgoing trip.

(*a*) The time for the trip in Randy's frame is $10\ \text{y} + 2(2.137\ \text{y}) = \boxed{14\ \text{y}}$.

(*b*) The time for the trip in Herb's frame is $10\ \text{y} + 2(15.152\ \text{y}) = \boxed{40\ \text{y}}$.

38 •• Al and Bert are twins. Al travels at $0.600c$ to Alpha Centauri (which is 4.00 c·y from Earth as measured in the reference frame of Earth) and returns immediately. Each twin sends the other a light signal every 0.0100 y as measured in his own reference frame. (*a*) At what rate does Bert receive signals as Al is moving away from him? (*b*) How many signals does Bert receive at this rate? (*c*) How many total signals are received by Bert before Al returns to Earth? (*d*) At what rate does Al receive signals as Bert is moving away from him? (*e*) How many signals does Al receive at this rate? (*f*) How many total signals are received by Al before Al returns to Earth? (*g*) Which twin is younger at the end of the trip and by how many years?

Picture the Problem (*a*) In Bert's reference frame each successive signal from Al travels an additional distance equal to vT, where v is Al's speed and T is the interval between successive signals in Al's reference frame. We also need to take into account the dilation of the time intervals measured on Earth due to Al's motion. In Parts (*b*) and (*c*) we can use $N = f_0\Delta t_{\text{Al}}$ and $\Delta t_{\text{Al}} = \Delta t_{\text{Bert}}\sqrt{1-v^2/c^2}$ to find the number of signals received by Bert. In Parts (*d*) and (*e*) we can proceed similarly to find the number of signals received by Al and the total number he receives before he returns to Bert. Finally, we can use the number of signals each received to find the difference in their ages resulting from Al's trip.

(*a*) The rate at which Bert receives signals f_{Bert} is the reciprocal of the time between the arrival of successive signals on Earth:

$$f_{\text{Bert}} = \frac{1}{T_{\text{between signals}}}$$

The time between the arrival of successive signals on Earth is the sum of the time T between signals in Al's frame of reference and the delay introduced by the fact that Al travels a distance vT between signals:

$$T_{\text{between signals}} = T + T_{\text{delay}}$$

or, because $T_{\text{delay}} = \dfrac{vT}{c} = 0.600T$,

$$T_{\text{between signals}} = T + 0.600T = 1.600T$$

Substitute for $T_{\text{between signals}}$ to obtain:

$$f_{\text{Bert}} = \frac{1}{1.600T}$$

T, the time between the arrival of signals on Earth is related to the proper time interval T_0 (the time between signals in Al's frame of reference) through the time-dilation equation:

$$T = \frac{T_0}{\sqrt{1-\left(v^2/c^2\right)}}$$

Substituting for T and simplifying yields:

$$f_{\text{Bert}} = \frac{\sqrt{1-\left(v^2/c^2\right)}}{1.600T_0}$$

Substitute numerical values and evaluate f_{Bert}:

$$f_{\text{Bert}} = \frac{\sqrt{1-(0.600)^2}}{(1.600)(0.0100\text{ y})} = \boxed{50\text{ y}^{-1}}$$

(b) Express the number of signals N received by Bert in terms of the number of signals sent by Al:

$$N = f_0 \Delta t_{\text{Al}}$$

Use the time dilation equation to express the elapsed time in Al's frame of reference in terms of the elapsed time in Bert's frame (the proper elapsed time):

$$\Delta t_{\text{Al}} = \Delta t_{\text{Bert}} \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute for Δt_{Al} to obtain:

$$N = f_0 \Delta t_{\text{Bert}} \sqrt{1-\left(\frac{v}{c}\right)^2} \qquad (1)$$

Find Δt_{Bert}:

$$\Delta t_{\text{Bert}} = \frac{4.00\,c\cdot\text{y}}{0.600c} = 6.667\text{ y}$$

Substitute numerical values in equation (1) and evaluate N:

$$N = \left(100\text{ y}^{-1}\right)(6.667\text{ y})\sqrt{1-\left(\frac{0.600c}{c}\right)^2}$$

$$= \boxed{533}$$

(c) Express the number of signals N received by Bert in terms of the number of signals sent by Al before he returns:

$$N = f_0 \Delta T_{\text{Al}}$$

Because $\Delta t_{\text{Al}} = \Delta t_{\text{Bert}} \sqrt{1-\left(\frac{v}{c}\right)^2}$, the time in Al's frame for the round trip $\Delta \boldsymbol{T}_{\text{Al}}$ is given by:

$$\Delta T_{\text{Al}} = 2\Delta t_{\text{Al}} = 2\Delta t_{\text{Bert}} \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substituting for ΔT_{Al} in the expression for N yields:

$$N = 2 f_0 \Delta t_{\text{Bert}} \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate N:

$$N = 2\left(100\ \text{y}^{-1}\right)\left(6.667\ \text{y}\right)\sqrt{1-\left(\frac{0.600c}{c}\right)^2} = \boxed{1.07\times10^3}$$

(d) Proceed as in Part (a) to find the rate at which Al receives signals as Bert is moving away from him:

$$f_{\text{Al}} = \boxed{50.0\ \text{y}^{-1}}$$

(e) Express the number of signals N received by Al:

$$N = f_{\text{Al}}\Delta t_{\text{Al}}$$

Substitute for Δt_{Al} to obtain:

$$N = f_{\text{Al}}\Delta t_{\text{Bert}}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate N:

$$N = \left(50.0\ \text{y}^{-1}\right)\left(6.667\ \text{y}\right)\sqrt{1-\left(\frac{0.600c}{c}\right)^2} = \boxed{267}$$

(f) Express the total number of signals received by Al:

$$N_{\text{tot}} = N_{\text{outbound}} + N_{\text{return}} = 267 + N_{\text{return}}$$

The number, N_{return}, of signals received by Al on his return trip is given by:

$$N_{\text{return}} = f_{\text{Al, return}}\Delta t_{\text{Al}} = f_{\text{Al, return}}\Delta t_{\text{Bert}}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute for N_{return} to obtain:

$$N_{\text{tot}} = 267 + f_{\text{Al, return}}\Delta t_{\text{Bert}}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Proceeding as in Part (a), find the rate at which signals are received by Al on the return trip:

$$f_{\text{Al, return}} = 200\ \text{y}^{-1}$$

Substitute numerical values and evaluate N_{tot}:

$$N_{\text{tot}} = 267 + \left(200\ \text{y}^{-1}\right)\left(6.667\ \text{y}\right)\sqrt{1-\left(\frac{0.600c}{c}\right)^2} = 1334 = \boxed{1.33\times10^3}$$

(*g*) Their age difference is:

$$\Delta t = \Delta t_{\text{Bert}} - \Delta t_{\text{Al}}$$

Substitute numerical values to obtain:

$$\Delta t = \frac{1334}{100\ \text{y}^{-1}} - \frac{1067}{100\ \text{y}^{-1}} = 2.67\ \text{y}$$

and

$$\boxed{\text{Al is } 2.67\ \text{y younger than Bert.}}$$

Relativistic Energy and Momentum

39 • Find the ratio of the total energy to the rest energy of a particle of mass *m* moving with speed (*a*) 0.100*c*, (*b*) 0.500*c*, (*c*) 0.800*c*, and (*d*) 0.990*c*.

Picture the Problem We can use Equation R-15 to find the ratio of the total energy to the rest energy for the given particle.

From Equation R-15:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Solve for the ratio of E to E_0 to obtain:

$$\frac{E}{E_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

(*a*) Evaluate E/E_0 for $v = 0.100c$:

$$\left.\frac{E}{E_0}\right]_{v=0.100c} = \frac{1}{\sqrt{1-\frac{(0.100c)^2}{c^2}}} = \boxed{1.01}$$

(*b*) Evaluate E/E_0 for $v = 0.500c$:

$$\left.\frac{E}{E_0}\right]_{v=0.500c} = \frac{1}{\sqrt{1-\frac{(0.500c)^2}{c^2}}} = \boxed{1.15}$$

(*c*) Evaluate E/E_0 for $v = 0.800c$:

$$\left.\frac{E}{E_0}\right]_{v=0.800c} = \frac{1}{\sqrt{1-\frac{(0.800c)^2}{c^2}}} = \boxed{1.67}$$

(*d*) Evaluate E/E_0 for $v = 0.990c$:

$$\left.\frac{E}{E_0}\right]_{v=0.999c} = \frac{1}{\sqrt{1-\frac{(0.990c)^2}{c^2}}} = \boxed{7.09}$$

40 • A proton (rest energy 938 MeV) has a total energy of 1400 MeV. (*a*) What is its speed? (*b*) What is its momentum?

Picture the Problem The rest energy E_0 is equal to mc^2. We are given E_0 and E, where E_0 = 938 MeV and the total energy E = 1400 MeV. (The total energy is the rest energy plus the kinetic energy). We can find the momentum p of the proton using $E^2 = p^2c^2 + m^2c^4$ (Equation R-17), and once we have p we can solve for the speed v using $v/c = pc/E$ (Equation R-16).

(*b*) Use Equation R-17 to relate the momentum to the total energy and the rest energy:

$$E^2 = p^2c^2 + m^2c^4 \text{ and } E_0 = mc^2$$

so

$$E^2 = p^2c^2 + E_0^2 \Rightarrow p = \frac{\sqrt{E^2 - E_0^2}}{c}$$

Substitute numerical values and evaluate p:

$$p = \frac{\sqrt{(1400\text{ Mev})^2 - (938\text{ Mev})^2}}{c}$$
$$= 1039\text{ MeV}/c$$
$$= \boxed{1040\text{ MeV}/c}$$

(*a*) Use Equation R-16 to express the speed of the proton:

$$\frac{v}{c} = \frac{pc}{E} \Rightarrow v = \left(\frac{pc}{E}\right)c$$

Substitute numerical values and evaluate v:

$$v = \left(\frac{(1039\text{ MeV}/c)}{1400\text{ MeV}}\right)c = \boxed{0.742c}$$

41 • **[SSM]** How much energy would be required to accelerate a particle of mass m from rest to (*a*) 0.500c, (*b*) 0.900c, and (*c*) 0.990c? Express your answers as multiples of the rest energy, mc^2.

Picture the Problem We can use Equation R-14 to find the energy required to accelerate this particle from rest to the given speeds.

From Equation R-14 we have:

$$K = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2$$
$$= \left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right)mc^2$$

(*a*) Substitute numerical values and evaluate $K(0.500c)$:

$$K(0.500c)=\left(\frac{1}{\sqrt{1-(0.500c/c)^2}}-1\right)mc^2=\boxed{0.155mc^2}$$

(*b*) Substitute numerical values and evaluate $K(0.900c)$:

$$K(0.900c)=\left(\frac{1}{\sqrt{1-(0.900c/c)^2}}-1\right)mc^2=\boxed{1.29mc^2}$$

(*c*) Substitute numerical values and evaluate $K(0.990c)$:

$$K(0.990c)=\left(\frac{1}{\sqrt{1-(0.990c/c)^2}}-1\right)mc^2=\boxed{6.09mc^2}$$

42 • If the kinetic energy of a particle equals its rest energy, what percentage error is made by using $p = mv$ for its momentum? Is the non-relativistic expression always low or high compared to the relativistically correct expression for momentum?

Picture the Problem We can use Equations R-10 and R-14 to express the error made in using $p = mv$ for the momentum of the particle when $K = E_0$.

The error in using $p = mv$ for the momentum of the particle is given by:

$$\frac{p_{\text{rel}}-p}{p_{\text{rel}}}=1-\frac{p}{p_{\text{rel}}} \qquad (1)$$

From Equation R-14 we have:

$$K=\frac{mc^2}{\sqrt{1-(v/c)^2}}-mc^2=\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)mc^2$$

For $K = E_0$:

$$K=\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)K$$

or

$$\frac{1}{\sqrt{1-(v/c)^2}}=2$$

From Equation R-10, the relativistic momentum of the particle is:

$$p_{\text{rel}} = \frac{mv}{\sqrt{1-(v/c)^2}} = 2mv$$

Substitute in equation (1) and simplify to obtain:

$$1-\frac{p}{p_{\text{rel}}} = 1-\frac{mv}{2mv} = \boxed{50\%}$$

The ratio of the non-relativistic momentum of a particle to its relativistic momentum is given by:

$$\frac{p}{p_{\text{rel}}} = \frac{p}{\dfrac{p}{\sqrt{1-v^2/c^2}}} = \sqrt{1-v^2/c^2}$$

Because $\sqrt{1-v^2/c^2} < 1$, the non-relativistic expression is always low compared to the relativistically correct expression for momentum.

43 • What is the total energy of a proton whose momentum is $3mc$?

Picture the Problem We can use Equation R-17 to find the total energy of any proton whose momentum is given. See Problem 42 for the rest energy of a proton.

The total energy, momentum, and rest energy of the proton are related by Equation R-17:

$$E^2 = p^2c^2 + (mc^2)^2$$

Substitute for the momentum of the proton:

$$E^2 = (3mc)^2c^2 + (mc^2)^2 = 9m^2c^4 + m^2c^4 = 10m^2c^4$$

Solving for E yields:

$$E = \sqrt{10}mc^2$$

Substitute for m and evaluate E:

$$E = \sqrt{10}(938\,\text{MeV}/c^2)c^2 = \boxed{2.97\,\text{GeV}}$$

44 •• Using a **spreadsheet** program or graphing calculator, make a graph of the kinetic energy of a particle with rest energy of 100 MeV for speeds between 0 and c. On the same graph, plot $\frac{1}{2}mv^2$ by way of comparison. Using the graph, estimate at about what speed the non-relativistic expression is no longer a good approximation to the kinetic energy. As a suggestion, plot the energy in units of MeV and the speed in the dimensionless form v/c.

Picture the Problem We can create a spreadsheet program to plot both the classical and relativistic kinetic energy of the particle.

The relativistic kinetic energy of the particle is given by:

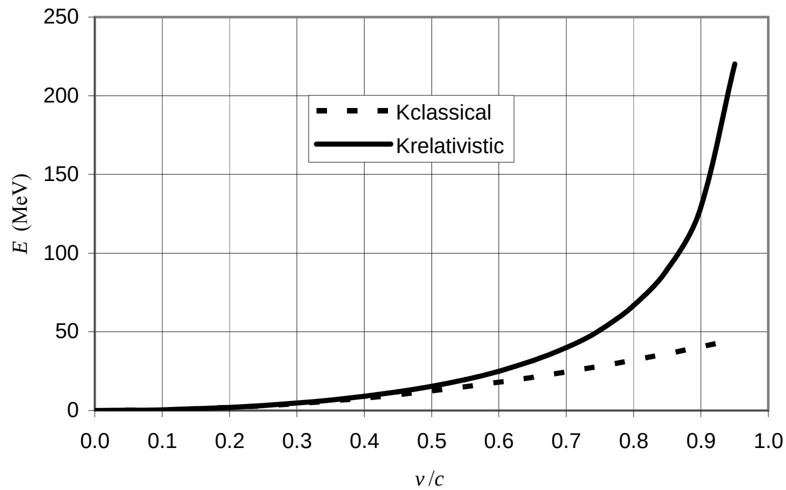
$$K_{\text{relativistic}} = mc^2\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)$$

A spreadsheet program to graph $K_{\text{relativistic}}$ and $K_{\text{classical}}$ is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A8	A7+0.05	$v/c + 0.05$
B7	0.5*$B$3*A7^2	$\frac{1}{2}mv^2$
C7	$B$3*(1/((1–A7^2)^0.5)–1)	$mc^2\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)$

	A	B	C
1			
2			
3	mc^2=	100	MeV
4			
5	v/c	$\frac{1}{2}mv^2$	$\left(\frac{1}{\sqrt{1-(v/c)^2}}\right)mc^2$
6		$K_{\text{classical}}$	$K_{\text{relativistic}}$
7	0.00	0.00	0.00
8	0.05	0.13	0.13
9	0.10	0.50	0.50
10	0.15	1.13	1.14
23	0.80	32.00	66.67
24	0.85	36.13	89.83
25	0.90	40.50	129.42
26	0.95	45.13	220.26

The solid curve is the graph of the relativistic kinetic energy. The relativistic formula, represented by the continuous curve, begins to deviate from the classical curve around $v/c \approx 0.4$.



45 •• **[SSM]** (*a*) Show that the speed *v* of a particle of mass *m* and total energy *E* is given by $\frac{v}{c}=\left[1-\frac{\left(mc^2\right)^2}{E^2}\right]^{1/2}$ and that when *E* is much greater than mc^2, this can be approximated by $\frac{v}{c}\approx 1-\frac{\left(mc^2\right)^2}{2E^2}$. Find the speed of an electron with kinetic energy of (*b*) 0.510 MeV and (*c*) 10.0 MeV.

Picture the Problem We can solve the equation for the relativistic energy of a particle to obtain the first result and then use the binomial expansion subject to $E >> mc^2$ to obtain the second result. In Parts (*b*) and (*c*) we can use the first expression obtained in (*a*), with $E = E_0 + K$, to find the speeds of electrons with the given kinetic energies. See Table 39-1 for the rest energy of an electron.

(*a*) The relativistic energy of a particle is given by Equation R-15:

$$E=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Solving for *v*/*c* yields:

$$\frac{v}{c}=\boxed{\left[1-\frac{\left(mc^2\right)^2}{E^2}\right]^{1/2}} \quad (1)$$

Expand the radical expression binomially to obtain:

$$\frac{v}{c}=\sqrt{1-\frac{\left(mc^2\right)^2}{E^2}}=1-\frac{1}{2}\frac{\left(mc^2\right)^2}{E^2}+\text{higher-order terms}$$

Because the higher-order terms are much smaller than the 2nd-degree term when $E >> mc^2$:

$$\frac{v}{c} \approx \boxed{1 - \frac{\left(mc^2\right)^2}{2E^2}}$$

(*b*) Solve equation (1) for *v*:

$$v = c\sqrt{1 - \frac{\left(mc^2\right)^2}{E^2}}$$

Because $E = E_0 + K$:

$$v = c\sqrt{1 - \frac{E_0^2}{\left(E_0 + K\right)^2}} = c\sqrt{1 - \frac{1}{\left(1 + \frac{K}{E_0}\right)^2}}$$

For an electron whose kinetic energy is 0.510 MeV:

$$v(0.510\,\text{MeV}) = c\sqrt{1 - \frac{1}{\left(1 + \frac{0.510\,\text{MeV}}{0.511\,\text{MeV}}\right)^2}} = \boxed{0.866c}$$

(*c*) For an electron whose kinetic energy is 10.0 MeV:

$$v(10.0\,\text{MeV}) = c\sqrt{1 - \frac{1}{\left(1 + \frac{10.0\,\text{MeV}}{0.511\,\text{MeV}}\right)^2}} = \boxed{0.999c}$$

46 •• Use the binomial expansion and Equation R-17 to show that when $pc << mc^2$, the total energy is given approximately by $E \approx mc^2 + p^2/(2m)$.

Picture the Problem We can solve Equation R-17 for *E* and factor $\left(mc^2\right)^2$ from under the resulting radical to obtain an expression to which we can apply the binomial expansion to write the radical expression as a power series. Finally, we can invoke the condition that $pc << mc^2$ to complete the argument that the total energy is given approximately by $E \approx mc^2 + p^2/(2m)$.

From Equation R-17 we have:

$$E = \sqrt{p^2c^2 + \left(mc^2\right)^2}$$

Factor $\left(mc^2\right)^2$ under the radical and simplify to obtain:

$$E = \sqrt{\left(mc^2\right)^2\left[\frac{p^2c^2}{\left(mc^2\right)^2} + 1\right]} = mc^2\sqrt{\frac{p^2c^2}{\left(mc^2\right)^2} + 1} = mc^2\sqrt{1 + \frac{p^2}{mc^2}}$$

Expand the radical factor binomially to obtain:

$$\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}=1+\frac{1}{2}\frac{p^2}{m^2c^2}+\text{higher - order terms}$$

For $pc << mc^2$:

$$\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}\approx 1+\frac{1}{2}\frac{p^2}{m^2c^2}$$

Substituting for $\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}$ and simplifying yields:

$$E=mc^2\left(1+\frac{1}{2}\frac{p^2}{m^2c^2}\right)=\boxed{mc^2+\frac{p^2}{2m}}$$

$$=\text{rest energy}+\text{classical kinetic energy}$$

47 •• Derive the equation $E^2 = p^2c^2 + m^2c^4$ (Equation R-17) by eliminating v from Equations R-10 and R-16.

Picture the Problem We can solve Equation R-16 for v and substitute in Equation R-10 to eliminate v. Simplification of the resulting expression leads to $E^2 = p^2c^2 + \left(mc^2\right)^2$.

Express the relativistic momentum of a particle using Equation R-10:

$$p=\frac{mv}{\sqrt{1-v^2/c^2}}$$

From Equation R-16 we have:

$$\frac{v}{c}=\frac{pc}{E}\Rightarrow v=\frac{pc^2}{E}$$

Substitute for v and simplify to obtain:

$$p=\frac{m\dfrac{pc^2}{E}}{\sqrt{1-\dfrac{\left(pc^2/E\right)^2}{c^2}}}$$

or

$$1=\frac{mc^2}{E\sqrt{1-\dfrac{p^2c^2}{E^2}}}$$

Multiply both sides of the equation by $E\sqrt{1-\frac{p^2c^2}{E^2}}$:

$$E\sqrt{1-\frac{p^2c^2}{E^2}}=mc^2$$

Square both sides of the equation:

$$E^2\left(1-\frac{p^2c^2}{E^2}\right)=\left(mc^2\right)^2$$

Solve for E^2 to obtain Equation R-17:

$$E^2=\boxed{p^2c^2+\left(mc^2\right)^2}$$

48 •• The rest energy of a proton is about 938 MeV. If its kinetic energy is also 938 MeV, find (*a*) its momentum and (*b*) its speed.

Picture the Problem (*a*) We can solve the relation for total energy, momentum, and rest energy of the proton for its momentum and evaluate this expression for $K = E_0$. In Part (*b*) we can equate the expression for the momentum of the proton under the conditions that exist in this problem with the general expression for the relativistic momentum of the proton and solve the resulting equation for the speed of the proton.

(*a*) Relate the total energy, momentum, and rest energy of a proton:

$$E^2=p^2c^2+\left(mc^2\right)^2$$

Solving for p yields:

$$p=\frac{\sqrt{E^2-E_0^2}}{c}=\frac{\sqrt{\left(E_0+K\right)^2-E_0^2}}{c}$$
$$=\frac{\sqrt{K^2+2E_0K}}{c}$$

If $K = E_0$:

$$p_{K=E_0}=\frac{\sqrt{E_0^2+2E_0^2}}{c}=\frac{\sqrt{3}E_0}{c}$$

Substitute numerical values and evaluate p:

$$p_{K=E_0}=\frac{\sqrt{3}\left(938\text{ MeV}\right)}{c}$$
$$=\boxed{1.62\text{ GeV}/c}$$

(*b*) In Part (*a*) we established that the momentum of the proton is given by:

$$p_{K=E_0}=\frac{\sqrt{3}E_0}{c}$$

Substituting for E_0 and simplifying yields:

$$p_{K=E_0}=\frac{\sqrt{3}mc^2}{c}=\sqrt{3}mc$$

The relativistic momentum of the proton is also given by:

$$p = \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Equating these expressions gives:

$$\sqrt{3}mc = \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Solve for v to obtain:

$$v = \frac{\sqrt{3}}{2}c = \boxed{0.866c}$$

49 •• What percentage error is made in using $\frac{1}{2}mv^2$ for the kinetic energy of a particle if its speed is (*a*) 0.10*c* and (*b*) 0.90*c*?

Picture the Problem We can use the expressions for the classical and relativistic kinetic energies of a particle to obtain a general expression for the fractional error made in using $\frac{1}{2}mv^2$ for the kinetic energy of a particle as a function of its speed.

The percentage error made in using $\frac{1}{2}mv^2$ for the kinetic energy of a particle is given by:

$$\frac{\Delta K}{K_{\text{rel}}} = \left|\frac{K_{\text{rel}} - K_{\text{classical}}}{K_{\text{rel}}}\right| = \left|1 - \frac{K_{\text{classical}}}{K_{\text{rel}}}\right|$$

The relativistic kinetic energy of the particle is given by Equation R-14:

$$K_{\text{rel}} = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2$$

$$= \left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right)mc^2$$

Substitute for K_{rel} in the expression for $\dfrac{\Delta K}{K_{\text{rel}}}$ and simplify to obtain:

$$\frac{\Delta K}{K_{\text{rel}}} = \left|1 - \frac{\frac{1}{2}mv^2}{\left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right)mc^2}\right| = \left|1 - \frac{v^2}{2\left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right)c^2}\right|$$

(a) Evaluate f for $v = 0.10c$:

$$f(0.10c) = \left| 1 - \frac{(0.10c)^2}{2\left(\frac{1}{\sqrt{1-(0.10)^2}} - 1\right)c^2} \right| = \boxed{0.75\%}$$

(b) Evaluate f for $v = 0.90c$:

$$f(0.90c) = \left| 1 - \frac{(0.90c)^2}{2\left(\frac{1}{\sqrt{1-(0.90)^2}} - 1\right)c^2} \right| = \boxed{69\%}$$

General Problems

50 • A spaceship departs from Earth for the star Alpha Centauri, which is 4.0 $c{\cdot}$y away in the reference frame of Earth. The spaceship travels at 0.75c. How long does it take to get there (a) as measured on Earth and (b) as measured by a passenger on the spaceship?

Picture the Problem We can find the duration of the trip, as measured on Earth, using the definition of average speed; that is, $\Delta t = L/v$. The elapsed time measured by the passenger is the proper time and is related to Δt through the time dilation equation (Equation R-3).

(a) Relate the time Δt for the trip as measured on Earth to its length L and the speed u of the spaceship:

$$\Delta t = \frac{L}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{4.0\,c\cdot\text{y}}{0.75c} = 5.33\,\text{y} = \boxed{5.3\,\text{y}}$$

(b) Use Equation R-3, the time dilation equation, to express the proper time measured by a passenger:

$$\Delta t_0 = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate Δt_0:

$$\Delta t_\text{p} = (5.33\,\text{y})\sqrt{1-\frac{(0.75c)^2}{c^2}} = \boxed{3.5\,\text{y}}$$

51 • The total energy of a particle is three times its rest energy. (*a*) Find v/c for the particle. (*b*) Show that its momentum is given by $p = \sqrt{8}mc$.

Picture the Problem (*a*) We can solve the expression for the relativistic energy of the particle for v/c and evaluate this expression for $E = 3E_0$. In Part (*b*) we can solve the expression relating the total energy, momentum, and rest energy of the particle for p and evaluate it for $E = 3E_0$ to show that its momentum is given by $p = \sqrt{8}\,mc$.

(*a*) The relativistic energy of the particle is given by Equation R-15:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{E_0}{\sqrt{1 - v^2/c^2}}$$

Solving for v/c yields:

$$\frac{v}{c} = \sqrt{1 - \frac{E_0^2}{E^2}}$$

For $E = 3E_0$:

$$\frac{v}{c} = \sqrt{1 - \frac{E_0^2}{9E_0^2}} = \sqrt{0.889} = \boxed{0.943}$$

(*b*) Equation R-17 relates the total energy, momentum, and rest energy of the particle:

$$E^2 = p^2c^2 + \left(mc^2\right)^2 = p^2c^2 + E_0^2$$

Solving for p yields:

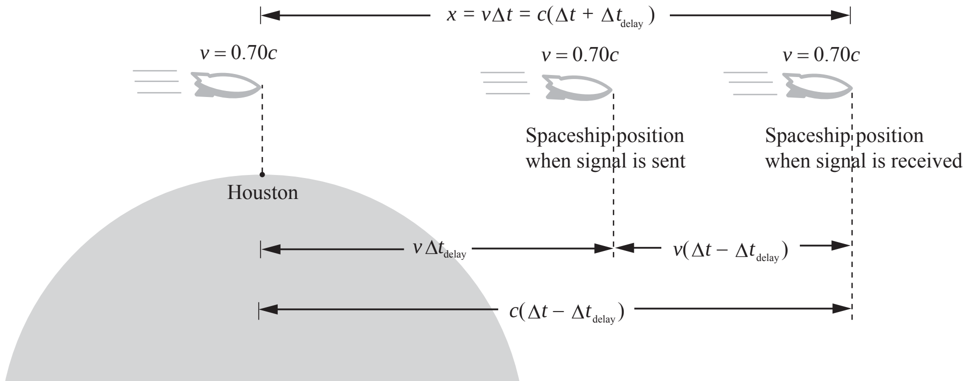
$$p = \frac{\sqrt{E^2 - E_0^2}}{c}$$

Evaluating p for $E = 3E_0$ yields:

$$p = \frac{\sqrt{(3E_0)^2 - E_0^2}}{c} = \frac{\sqrt{8}E_0}{c} = \frac{\sqrt{8}mc^2}{c}$$
$$= \boxed{\sqrt{8}mc}$$

52 •• A spaceship travels past Earth moving at $0.70c$ relative to Earth. 5.0 minutes after the spaceship passes closest to Earth, a message is sent from Houston, Texas, to the craft. (Neglect all effects of the rotational motion of Earth.) (*a*) How long does it take for the signal to arrive? (*b*) The spaceship and the control center agree on the time when the ship passes closest to Earth. Five minutes after the message is received aboard the ship, a return message is sent by the ship back to Houston. What is the time interval in Houston between the time their message was sent, and the time the return message is received?

Picture the Problem The pictorial representation summarizes the information given in the problem statement. Note that the spaceship, moving at $0.70c$, has a 5.0 min head start. This means that, in the Earth frame of reference, the message is received by the spaceship after a time Δt has passed since the spaceship passed closest to Earth. In Part (*b*) we'll have to determine the time-dilated interval of 5.0 min aboard the spaceship.



(*a*) The distance Δx from Earth to the ship is given by:

$$\Delta x = c\left(\Delta t - \Delta t_{\text{delay}}\right)$$

Δx is also given by:

$$\Delta x = v\Delta t$$

Equate these expressions to obtain:

$$c\left(\Delta t - \Delta t_{\text{delay}}\right) = v\Delta t \Rightarrow \Delta t = \frac{\Delta t_{\text{delay}}}{1 - \frac{v}{c}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{5.0\text{ min}}{1 - \frac{0.70c}{c}} = 16.67\text{ min}$$

$$= \boxed{17\text{ min}}$$

(*b*) Express the total time from the sending of the original message (5.0 min after the closest passage) to the receipt of the return message:

$$\Delta t_{\text{tot}} = \Delta t_{\text{to ship}} + \Delta t_{\text{return}} - 5.0\text{ min} \qquad (2)$$

According to observers on Earth, the time between receipt of the message and its return is:

$$\Delta t_{\text{return}} = \frac{5.0\text{ min}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{5.0\text{ min}}{\sqrt{1-(0.70)^2}}$$
$$= 7.001\text{ min}$$

and

$$\Delta t_{\text{to ship}} = 16.67\text{ min} + 7.001\text{ min}$$
$$= 23.67\text{ min}$$

In Earth's frame of reference, the spaceship is at a distance from Earth given by:

$$\Delta x_{\text{to ship}} = v\Delta t_{\text{to ship}} = 0.70c\Delta t_{\text{to ship}}$$

Substitute numerical values and evaluate $\Delta x_{\text{to ship}}$:

$$\Delta x_{\text{to ship}} = (0.70)(2.998\times10^8\text{ m/s})\left(23.67\text{ min}\times\frac{60\text{ s}}{\text{min}}\right) = 2.980\times10^{11}\text{ m}$$

The time for the signal to return to Earth from this distance is:

$$\Delta t_{\text{return message}} = \frac{\Delta x}{c} = \frac{2.980\times10^{11}\text{ m}}{2.998\times10^8\text{ m/s}}$$
$$= 994\text{ s} = 17\text{ min}$$

Substitute numerical values in equation (2) and evaluate Δt_{tot}:

$$\Delta t_{\text{tot}} = 19\text{ min} + 17\text{ min} - 5.0\text{ min}$$
$$= \boxed{31\text{ min}}$$

53 •• [SSM] Particles called muons traveling at 0.99995c are detected at the surface of Earth. One of your fellow students claims that the muons might have originated from the Sun. Prove him wrong. (The proper mean lifetime of the muon is 2.20 μs.)

Picture the Problem Your fellow student is thinking that the time dilation factor might allow muons to travel the 150,000,000,000 m from the Sun to Earth. You can discredit your classmate's assertion by considering the mean lifetime of the muon from Earth's reference frame. Doing so will demonstrate that the distance traveled during as many as 5 proper mean lifetimes is consistent with the origination of muons within Earth's atmosphere.

The distance, in the Earth frame of reference, a muon can travel in n mean lifetimes τ is given by:

$$d = n\frac{d_0}{\sqrt{1-(v/c)^2}} = \frac{nv\tau}{\sqrt{1-(v/c)^2}} = \frac{n(v/c)c\tau}{\sqrt{1-(v/c)^2}}$$

Substitute numerical values for v, c, and τ and simplify to obtain:

$$d = n\frac{(0.99995)(2.998\times10^8\ \text{m/s})(2.20\ \mu\text{s})}{\sqrt{1-(0.99995)^2}} = (66.0\ \text{km})n$$

In 5 lifetimes a muon would travel a distance:

$d = (66.0\ \text{km})(5) = 330\ \text{km}$, a distance approximating a low-Earth orbit.

In 100 lifetimes, $d \approx 6600$ km, or approximately one Earth radius. This relatively short distance should convince your classmate that the origin of the muons that are observed on Earth is within our atmosphere and that they certainly are not from the Sun.

54 •• (*a*) How tall is Mount Everest in a reference frame traveling with a cosmic ray muon that is traveling straight down, relative to Earth, at 0.99c? Take the height of Mount Everest according to an Earth-based observer to be 8846 m. (*b*) How long does it take the muon to travel the height of the mountain from the reference frame traveling with the muon? (*c*) How long does it take the muon to travel the height of the mountain from the Earth-based reference frame?

Picture the Problem Apply the length contraction equation to find the height of Mount Everest in the frame of reference traveling with the muon and use the relationship between the distance an object travels, its speed, and the elapsed time to solve Parts (*b*) and (*c*).

(*a*) The height h of Mount Everest in the frame of reference traveling with the muon is given by:

$$h = \sqrt{1-(v/c)^2}\,h_0$$

where h_0 is the height of Mount Everest in Earth's frame of reference.

Substitute numerical values and evaluate h:

$$h = \sqrt{1-(0.99)^2}\,(8.846\ \text{km}) = 1.248\ \text{km}$$
$$= \boxed{1.2\ \text{km}}$$

(*b*) In the frame of reference traveling with the muon, the mountain is traveling toward it at a speed of 0.99*c*. Hence the time to travel the height of the mountain is given by:

$$\Delta t_{\text{muon frame}} = \frac{h}{v_{\text{mountain}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t_{\text{muon frame}} = \frac{1.248\ \text{km}}{0.99\left(2.998\times10^{8}\ \text{m/s}\right)} = \boxed{4.2\ \mu\text{s}}$$

(*c*) From the Earth-based reference frame, the time it takes the muon to travel the height of the mountain is the ratio of the height of the mountain to the speed of the muon:

$$\Delta t_{\text{earth frame}} = \frac{h_0}{v_{\text{muon}}}$$

Substitute numerical values and evaluate $\Delta t_{\text{earth frame}}$:

$$\Delta t_{\text{earth frame}} = \frac{8.846\ \text{km}}{(0.99)\left(2.998\times10^{8}\ \text{m/s}\right)} = \boxed{30\ \mu\text{s}}$$

55 ••• A gold nucleus has a radius of 3.00×10^{-14} m, and a mass of 197 amu. (1 amu has a rest energy of 932 MeV.) During experiments at Brookhaven National Laboratory, these nuclei are routinely accelerated to a kinetic energy of 3.35×10^{4} GeV. (*a*) How much less than the speed of light are they traveling? (*b*) At these energies, how long does it take them to travel 100 m in the laboratory frame?

Picture the Problem We can apply Equation R-15, the expression for the total relativistic energy of a particle, to find how much less than the speed of light the gold nuclei are traveling. The time it takes the nuclei to travel 100 m in the laboratory frame can be found using the distance, rate, and time equation.

(*a*) Express the difference between the speed of light and the speed of the gold nucleus:

$$\Delta v = c - v \qquad (1)$$

The total relativistic energy of a gold nucleus is given by Equation R-15:

$$K + mc^2 = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

Solving for $\sqrt{1-(v/c)^2}$ yields:

$$\sqrt{1-(v/c)^2} = \frac{\frac{mc^2}{K}}{1+\frac{mc^2}{K}} \qquad (2)$$

Evaluate $\frac{mc^2}{K}$ to obtain:

$$\frac{mc^2}{K} = \frac{(197\text{ amu})(932\text{ MeV/amu})}{3.35\times10^4\text{ GeV}} = 5.48\times10^{-3} << 1$$

Because $mc^2 << K$, equation (2) can be approximated by:

$$\sqrt{1-(v/c)^2} \approx \frac{mc^2}{K} \qquad (3)$$

Solve for v/c to obtain:

$$\frac{v}{c} = \sqrt{1-\left(\frac{mc^2}{K}\right)^2}$$

Expanding the radical binomially yields:

$$\left[1-\left(\frac{mc^2}{K}\right)^2\right]^{\frac{1}{2}} = 1-\frac{1}{2}\left(\frac{mc^2}{K}\right)^2 + \text{higher order terms}$$

Because $mc^2 << K$:

$$\left[1-\left(\frac{mc^2}{K}\right)^2\right]^{\frac{1}{2}} \approx 1-\frac{1}{2}\left(\frac{mc^2}{K}\right)^2$$

and

$$v = c\left[1-\frac{1}{2}\left(\frac{mc^2}{K}\right)^2\right]$$

Substituting for v in equation (1) yields:

$$\Delta v = c - c\left[1-\frac{1}{2}\left(\frac{mc^2}{K}\right)^2\right] = \frac{1}{2}\left(\frac{mc^2}{K}\right)^2 c$$

Substitute numerical values and evaluate Δv:

$$\Delta v = \frac{1}{2}\left(\frac{197\text{ amu}\times932\text{ MeV/amu}}{3.35\times10^4\text{ GeV}}\right)^2\left(2.998\times10^8\text{ m/s}\right) = \boxed{4.50\text{ km/s}}$$

(*b*) Because the nuclei are, to three significant figures, traveling at the speed of light:

$$\Delta t \approx \frac{\Delta x}{c} = \frac{100\ \text{m}}{2.998\times 10^{8}\ \text{m/s}} = \boxed{0.334\ \mu\text{s}}$$

56 ••• Consider the flight of a beam of neutrons produced in a nuclear reactor. These neutrons have kinetic energies of up to 1.00 MeV. The rest energy of a neutron is 939 MeV. (*a*) What is the speed of 1.00 MeV neutrons? Express your answer in terms of *v*/*c*. (*b*) If the average lifetime of such a neutron is 15.0 min, what is the maximum length of a beam of such neutrons (in a vacuum, in the absence of any interactions between the neutrons and other material)? Estimate this maximum range by calculating the length corresponding to five mean lifetimes. After five mean lifetimes only e^{-5} or 0.007 (0.7%) of the neutrons remain. (*c*) Compare this to the range of so-called ″thermally moderated″ neutrons, whose kinetic energies are around 0.025 eV. Express your answer as a percentage. That is, what percent of the 1.00 MeV-neutron range is the thermally moderated-neutron range? (Note that our assumption of a vacuum continues, however in reality neutrons of this energy interact readily with matter, such as air or water, and ″real″ ranges are much shorter.)

Picture the Problem We can apply $K + mc^2 = mc^2 \Big/ \sqrt{1-(v/c)^2}$, the expression for the total relativistic energy of a particle, to find the speed of 1.00 MeV neutrons. We can use the equation relating distance, rate, and time to estimate their maximum beam length. Finally, we can use the non-relativistic expression for their kinetic energy to find the range of ″thermally moderated″ neutrons.

(*a*) The total relativistic energy of a neutron is given by Equation R-15:

$$K + mc^2 = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

Solving for *v*/*c* yields:

$$\frac{v}{c} = \sqrt{1-\left(\frac{1}{1+\dfrac{K}{mc^2}}\right)^2}$$

Substitute numerical values and evaluate *v*/*c*:

$$\frac{v}{c} = \sqrt{1-\left(\frac{1}{1+\dfrac{1.00\ \text{MeV}}{939\ \text{MeV}}}\right)^2} = \boxed{4.61\%}$$

(*b*) Express the maximum expected beam length of the neutrons as a function of the number *n* of lifetimes:

$$\Delta x = n\left(\frac{v}{c}\right)c\Delta t \qquad (1)$$

Evaluate this expression for $n = 5$:

$$\Delta x = (5)(0.0461)\left(2.998\times10^{8}\ \text{m/s}\right)\left(15.0\ \text{min}\times\frac{60\ \text{s}}{\text{min}}\right) = \boxed{6.22\times10^{10}\ \text{m}}$$

(*c*) Express the ratio *r* of the range of "thermally moderated" neutrons to the range of 1.00 MeV neutrons:

$$r = \frac{\Delta x_{\text{thermally moderated}}}{\Delta x_{1.00\ \text{MeV}}} \qquad (2)$$

Because the energy of the "thermally moderated" neutrons is 0.025 eV, we can use the non-relativistic equation for their kinetic energy:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{v}{c}\right)^2 c^2 \Rightarrow \frac{v}{c} = \sqrt{\frac{2K}{mc^2}}$$

Substitute for v/c in equation (1) to obtain:

$$\Delta x_{\text{thermally moderated}} = n\sqrt{\frac{2K}{mc^2}}c\Delta t$$

Substitute numerical values and evaluate $\Delta x_{\text{thermally moderated}}$:

$$\Delta x_{\text{thermally moderated}} = 5\sqrt{\frac{2(0.025\ \text{eV})}{939\ \text{MeV}}}\left(2.998\times10^{8}\ \text{m/s}\right)(15.0\ \text{min})\left(\frac{60\ \text{s}}{\text{min}}\right) = 9.84\times10^{6}\ \text{m}$$

Substitute numerical values in equation (2) and evaluate *r*:

$$r = \frac{9.84\times10^{6}\ \text{m}}{6.22\times10^{10}\ \text{m}} = \boxed{0.016\%}$$

57 ••• You and Ernie are trying to fit a 15-ft-long ladder into a 10-ft-long shed with doors at each end. You suggest to Ernie that you open the front door to the shed and have him run toward it with the ladder at a speed such that the length contraction of the ladder shortens it enough so that it fits in the shed. As soon as the back end of the ladder passes through the door, you will slam it shut. (*a*) What is the minimum speed at which Ernie must run to fit the ladder into the shed? Express it as a fraction of the speed of light. (*b*) As Ernie runs toward the shed at a speed of 0.866*c*, he realizes that in the reference frame of the ladder, it is the *shed* which is shorter, not the ladder. How long is the shed in the rest frame of the ladder? (*c*) In the reference frame of the ladder is there any instant that

both ends of the ladder are simultaneously inside the shed? Examine this from the point of view of relativistic simultaneity.

Picture the Problem Let the letter ″L″ denote the ladder and the letter ″S″ the shed. We can apply the length contraction equation to the determination of the minimum speed at which Ernie must run to fit the ladder into the shed as well as the length of the shed in rest frame of the ladder.

(*a*) Express the length L_L of the shed in Ernie's frame of reference in terms of its proper length $L_{L,0}$:

$$L_L = L_{L,0}\sqrt{1-\left(\frac{v}{c}\right)^2} \Rightarrow v = c\sqrt{1-\frac{L_L^2}{L_{L,0}^2}}$$

Substitute numerical values and evaluate v:

$$v = c\sqrt{1-\left(\frac{10\,\text{ft}}{15\,\text{ft}}\right)^2} = \boxed{0.75c}$$

(*b*) Express the length L_S of the shed in the rest frame of the ladder in terms of its proper length $L_{L,0}$:

$$L_S = L_{S,0}\sqrt{1-\frac{v^2}{c^2}}$$

Substitute numerical values and evaluate L_S:

$$L_S = (10\,\text{ft})\sqrt{1-\left(\frac{0.866c}{c}\right)^2} = \boxed{5.0\,\text{ft}}$$

(*c*) No. In your rest frame, the back end of the ladder will clear the door before the front end hits the wall of the shed, while in Ernie's rest frame, the front end will hit the wall of the shed while the back end has yet to clear the other door.

Let the ladder be traveling from left to right. To ″explain″ the simultaneity issue we first describe the situation in the reference frame of the shed. In this frame the ladder has length L_L = 7.5 m and the shed has length $L_{S,0}$ = 10 m. We (mentally) put a clock at each end of the shed. Let both clocks read zero at the instant the left end of the ladder enters the shed. At this instant the right end of the ladder is a distance $L_{S,0} - L_L$ = 2.5 m from the right end of the shed. At the instant the right end of the ladder exits the shed both clocks read Δt, where $\Delta t = (L_{S,0} - L_L)/v = 9.63\,\text{ns}$. There are two space-time coincidences to consider: the left end of the ladder enters the shed and the clock at the left end of the shed reads zero, and the right end of the ladder exits the shed and the clock on the right end of the shed reads $(L_{S,0} - L_L)/v = 9.63\,\text{ns}$. In the reference frame of the ladder the two clocks are moving to the left at speed $v = 0.866c$. In this frame the clock on the right (the trailing clock) is ahead of the clock on the left by $vL_{S,0}/c^2 = 28.9$ ns, so when the clock on the right reads 9.63 ns the one on the left reads –19.3 ns. This means the left end of the ladder is yet to enter the shed when

the right end of the ladder is exiting the shed. This is consistent with the assertion that in the rest frame of the ladder, the ladder is longer than the shed, so the entire ladder is never entirely inside the shed.

Chapter 11
Gravity

Conceptual Problems

1 • [SSM] True or false:

(*a*) For Kepler's law of equal areas to be valid, the force of gravity must vary inversely with the square of the distance between a given planet and the Sun.
(*b*) The planet closest to the Sun has the shortest orbital period.
(*c*) Venus's orbital speed is larger than the orbital speed of Earth.
(*d*) The orbital period of a planet allows accurate determination of that planet's mass.

(*a*) False. Kepler's law of equal areas is a consequence of the fact that the gravitational force acts along the line joining two bodies but is independent of the manner in which the force varies with distance.

(*b*) True. The periods of the planets vary with the three-halves power of their distances from the Sun. So the shorter the distance from the Sun, the shorter the period of the planet's motion.

(*c*) True. Setting up a proportion involving the orbital speeds of the two planets in terms of their orbital periods and mean distances from the Sun (see Table 11-1) shows that $v_{\text{Venus}} = 1.17 v_{\text{Earth}}$.

(*d*) False. The orbital period of a planet is independent of the planet's mass.

2 • If the mass of a small Earth-orbiting satellite is doubled, the radius of its orbit can remain constant if the speed of the satellite (*a*) increases by a factor of 8, (*b*) increases by a factor of 2, (*c*) does not change, (*d*) is reduced by a factor of 8, (*e*) is reduced by a factor of 2.

Determine the Concept We can apply Newton's second law and the law of gravity to the satellite to obtain an expression for its speed as a function of the radius of its orbit.

Apply Newton's second law to the satellite to obtain:

$$\sum F_{\text{radial}} = \frac{GMm}{r^2} = m\frac{v^2}{r}$$

where M is the mass of the object the satellite is orbiting and m is the mass of the satellite.

Solving for v yields:

$$v = \sqrt{\frac{GM}{r}}$$

Thus the speed of the satellite is independent of its mass and $\boxed{(c)}$ is correct.

3 • **[SSM]** During what season in the northern hemisphere does Earth attain its maximum orbital speed about the Sun? What season is related to its minimum orbital speed? *Hint: Earth is at perihelion in early January.*

Determine the Concept Earth is closest to the Sun during winter in the northern hemisphere. This is the time of fastest orbital speed. Summer would be the time for minimum orbital speed.

4 • Haley's comet is in a highly elliptical orbit about the Sun with a period of about 76 y. Its last closest approach to the Sun occurred in 1987. In what years of the twentieth century was it traveling at its fastest or slowest orbital speed about the Sun?

Determine the Concept Haley's comet was traveling at its fastest orbital speed in 1987, and at its slowest orbital speed 38 years previously in 1949.

5 • Venus has no natural satellites. However artificial satellites have been placed in orbit around it. To use one of their orbits to determine the mass of Venus, what orbital parameters would you have to measure? How would you then use them to do the mass calculation?

Determine the Concept To obtain the mass M of Venus you need to measure the period T and semi-major axis a of the orbit of one of the satellites, substitute the measured values into $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$ (Kepler's third law), and solve for M.

6 • A majority of the asteroids are in approximately circular orbits in a ″belt″ between Mars and Jupiter. Do they all have the same orbital period about the Sun? Explain.

Determine the Concept No. As described by Kepler's third law, the asteroids closer to the Sun have a shorter ″year″ and are orbiting faster.

7 • **[SSM]** At the surface of the moon, the acceleration due to the gravity of the moon is a. At a distance from the center of the moon equal to four times the radius of the moon, the acceleration due to the gravity of the moon is (*a*) $16a$, (*b*) $a/4$, (*c*) $a/3$, (*d*) $a/16$, (*e*) None of the above.

Picture the Problem The acceleration due to gravity varies inversely with the square of the distance from the center of the moon.

Express the dependence of the acceleration due to the gravity of the moon on the distance from its center:

$$a' \propto \frac{1}{r^2}$$

Express the dependence of the acceleration due to the gravity of the moon at its surface on its radius:

$$a \propto \frac{1}{R_M^2}$$

Divide the first of these expressions by the second to obtain:

$$\frac{a'}{a} = \frac{R_M^2}{r^2}$$

Solving for a' and simplifying yields:

$$a' = \frac{R_M^2}{r^2}a = \frac{R_M^2}{(4R_M)^2}a = \tfrac{1}{16}a$$

and $\boxed{(d)}$ is correct.

8 • At a depth equal to half the radius of Earth, the acceleration due to gravity is about (*a*) *g* (*b*) 2*g* (*c*) *g*/2, (*d*) *g*/4, (*e*) *g*/8, (*f*) You cannot determine the answer based on the data given.

Picture the Problem We can use Newton's law of gravity and the assumption of uniform density to express the ratio of the acceleration due to gravity at a depth equal to half the radius of Earth to the acceleration due to gravity at the surface of Earth.

The acceleration due to gravity at a depth equal to half the radius of Earth is given by:

$$g_{\frac{1}{2}r} = \frac{GM'}{\left(\frac{1}{2}r\right)^2} = \frac{4GM'}{r^2}$$

where M' is the mass of Earth between the location of interest and the center of Earth.

The acceleration due to gravity at the surface of Earth is given by:

$$g = \frac{GM}{r^2}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{g_{\frac{1}{2}r}}{g} = \frac{\frac{4GM'}{r^2}}{\frac{GM}{r^2}} = \frac{4M'}{M} \qquad (1)$$

Express M' in terms of the average density of Earth ρ and the volume V' of Earth between the location of interest and the center of Earth:

$$M' = \rho V' = \rho\left[\tfrac{4}{3}\pi\left(\tfrac{1}{2}r\right)^3\right] = \tfrac{1}{6}\pi\rho r^3$$

Express M in terms of the average density of Earth ρ and the volume V of Earth:

$$M = \rho V = \rho\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{4}{3}\pi\rho r^3$$

Substitute for M and M' in equation (1) and simplify to obtain:

$$\frac{g_{\frac{1}{2}r}}{g} = \frac{4\left(\frac{1}{6}\pi\rho r^3\right)}{\frac{4}{3}\pi\rho r^3} = \frac{1}{2}$$

and $\boxed{(c)}$ is correct.

9 •• Two stars orbit their common center of mass as a *binary* star system. If each of their masses were doubled, what would have to happen to the distance between them in order to maintain the same gravitational force? The distance would have to (*a*) remain the same (*b*) double (*c*) quadruple (*d*) be reduced by a factor of 2 (*e*) You cannot determine the answer based on the data given.

Picture the Problem We can use Newton's law of gravity to express the ratio of the forces and then solve this proportion for the separation of the stars that would maintain the same gravitational force.

The gravitational force acting on the stars before their masses are doubled is given by:

$$F_g = \frac{GmM}{r^2}$$

The gravitational force acting on the stars after their masses are doubled is given by:

$$F_g' = \frac{G(2m)(2M)}{r'^2} = \frac{4GmM}{r'^2}$$

Dividing the second of these equations by the first yields:

$$\frac{F_g'}{F_g} = 1 = \frac{\dfrac{4GmM}{r'^2}}{\dfrac{GmM}{r^2}} = \frac{4r^2}{r'^2}$$

Solving for r' yields:

$r' = 2r$ and $\boxed{(b)}$ is correct.

10 •• If you had been working for NASA in the 1960's and planning the trip to the moon, you would have determined that there exists a unique location somewhere between Earth and the moon, where a spaceship is, for an instant, truly weightless. [Consider only the moon, Earth and the Apollo spaceship, and

neglect other gravitational forces.] Explain this phenomenon and explain whether this location is closer to the moon, midway on the trip, or closer to Earth.

Determine the Concept Between Earth and the moon, the gravitational pulls on the spaceship are oppositely directed. Because of the moon's relatively small mass compared to the mass of Earth, the location where the gravitational forces cancel (thus producing no net gravitational force, a weightless condition) is considerably closer to the moon.

11 •• [SSM] Suppose the escape speed from a planet was only slightly larger than the escape speed from Earth, yet the planet is considerably larger than Earth. How would the planet's (average) density compare to Earth's (average) density? (*a*) It must be denser. (*b*) It must be less dense. (*c*) It must be the same density. (*d*) You cannot determine the answer based on the data given.

Picture the Problem The densities of the planets are related to the escape speeds from their surfaces through $v_e = \sqrt{2GM/R}$.

The escape speed from the planet is given by:

$$v_{\text{planet}} = \sqrt{\frac{2GM_{\text{planet}}}{R_{\text{planet}}}}$$

The escape speed from Earth is given by:

$$v_{\text{Earth}} = \sqrt{\frac{2GM_{\text{Earth}}}{R_{\text{Earth}}}}$$

Expressing the ratio of the escape speed from the planet to the escape speed from Earth and simplifying yields:

$$\frac{v_{\text{planet}}}{v_{\text{Earth}}} = \frac{\sqrt{\dfrac{2GM_{\text{planet}}}{R_{\text{planet}}}}}{\sqrt{\dfrac{2GM_{\text{Earth}}}{R_{\text{Earth}}}}} = \sqrt{\frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}}$$

Because $v_{\text{planet}} \approx v_{\text{Earth}}$:

$$1 \approx \sqrt{\frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}}$$

Squaring both sides of the equation yields:

$$1 \approx \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}$$

Express M_{planet} and M_{Earth} in terms of their densities and simplify to obtain:

$$1 \approx \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}V_{\text{planet}}}{\rho_{\text{Earth}}V_{\text{Earth}}} = \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}V_{\text{planet}}}{\rho_{\text{Earth}}V_{\text{Earth}}} = \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}\,\frac{4}{3}\pi R_{\text{planet}}^3}{\rho_{\text{Earth}}\,\frac{4}{3}\pi R_{\text{Earth}}^3} = \frac{\rho_{\text{planet}}R_{\text{planet}}^2}{\rho_{\text{Earth}}R_{\text{Earth}}^2}$$

Solving for the ratio of the densities yields:

$$\frac{\rho_{\text{planet}}}{\rho_{\text{Earth}}} \approx \frac{R_{\text{Earth}}^2}{R_{\text{planet}}^2}$$

Because the planet is considerably larger than Earth:

$$\frac{\rho_{\text{planet}}}{\rho_{\text{Earth}}} << 1$$

and $\boxed{(b)}$ is correct.

12 •• Suppose that, using a telescope in your backyard, you discovered a distant object approaching the Sun, and were able to determine both its distance from the Sun and its speed. How would you be able to predict whether the object will remain "bound" to the Solar System, or if it is an interstellar interloper and would come in, turn around and escape, never to return?

Determine the Concept You could take careful measurements of its position as a function of time in order to determine whether its trajectory is an ellipse, a hyperbola, or a parabola. If the path is an ellipse, it will return; if its path is hyperbolic or parabolic, it will not return. Alternatively, by measuring its distance from the Sun, you can estimate the gravitational potential energy (per kg of its mass, and neglecting the planets) of the object, and by determining its position on several successive nights, the speed of the object can be determined. From this, its kinetic energy (per kg) can be determined. The sum of these two gives the comet's total energy (per kg) and if it is positive, it will likely swing once around the Sun and then leave the Solar System forever.

13 •• **[SSM]** Near the end of their useful lives, several large Earth-orbiting satellites have been maneuvered so they burn up as they enter Earth's atmosphere. These maneuvers have to be done carefully so large fragments do not impact populated land areas. You are in charge of such a project. Assuming the satellite of interest has on-board propulsion, in what direction would you fire the rockets for a short burn time to start this downward spiral? What would happen to the kinetic energy, gravitational potential energy and total mechanical energy following the burn as the satellite came closer and closer to Earth?

Determine the Concept You should fire the rocket in a direction to oppose the orbital motion of the satellite. As the satellite gets closer to Earth after the burn, the potential energy will decrease. However, the total mechanical energy will decrease due to the frictional drag forces transforming mechanical energy into thermal energy. The kinetic energy will increase until the satellite enters the atmosphere where the drag forces slow its motion.

14 •• During a trip back from the moon, the Apollo spacecraft fires its rockets to leave its lunar orbit. Then it coasts back to Earth where it enters the atmosphere at high speed, survives a blazing re-entry and parachutes safely into the ocean. In what direction do you fire the rockets to initiate this return trip? Explain the changes in kinetic energy, gravitational potential and total mechanical energy that occur to the spacecraft from the beginning to the end of this journey.

Determine the Concept Near the moon you would fire the rockets to accelerate the spacecraft with the thrust acting in the direction of your ship's velocity at the time. When the rockets have shut down, as you leave the moon, your kinetic energy will initially decrease (the moon's gravitational pull exceeds that of Earth), and your potential energy will increase. After a certain point, Earth's gravitational attraction would begin accelerating the ship and the kinetic energy would increase at the expense of the gravitational potential energy of the spacecraft-Earth-moon system. The spacecraft will enter Earth's atmosphere with its maximum kinetic energy. Eventually, landing in the ocean, the kinetic energy would be zero, the gravitational potential energy a minimum, and the total mechanical energy of the ship will have been dramatically reduced due to air drag forces producing heat and light during re-entry.

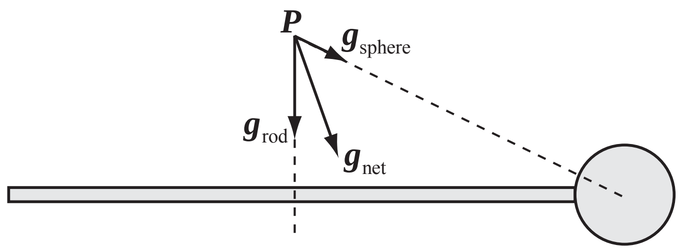
15 •• Explain why the gravitational field inside a solid sphere of uniform mass is directly proportional to r rather than inversely proportional to r.

Determine the Concept At a point inside the sphere a distance r from its center, the gravitational field strength is directly proportional to the amount of mass within a distance r from the center, and inversely proportional to the square of the distance r from the center. The mass within a distance r from the center is proportional to the cube of r. Thus, the gravitational field strength is directly proportional to r.

16 •• In the movie *2001: A Space Odyssey*, a spaceship containing two astronauts is on a long-term mission to Jupiter. A model of their ship could be a uniform pencil-like rod (containing the propulsion systems) with a uniform sphere (the crew habitat and flight deck) attached to one end (Figure 11-23). The design is such that the radius of the sphere is much smaller than the length of the rod. At a location a few meters away from the ship, at point P on the perpendicular bisector of the rod-like section, what would be the direction of the gravitational field due to the ship alone (that is, assuming all other gravitational fields are

negligible)? Explain your answer. At a large distance from the ship, what would be the dependence of its gravitational field on the distance from the ship?

Determine the Concept The pictorial representation shows the point of interest P and the gravitational fields $\vec{g}_{\text{rod}}$ and $\vec{g}_{\text{sphere}}$ due to the rod and the sphere as well as the resultant field $\vec{g}_{\text{net}}$. Note that the net field (the sum of $\vec{g}_{\text{rod}}$ and $\vec{g}_{\text{sphere}}$) points slightly toward the habitat end of the ship. At very large distances, the rod+sphere mass distribution looks like a point mass, so the field's distance dependence is an inverse square dependence.



Estimation and Approximation

17 • **[SSM]** Estimate the mass of our galaxy (the Milky Way) if the Sun orbits the center of the galaxy with a period of 250 million years at a mean distance of 30,000 $c\cdot$y. Express the mass in terms of multiples of the solar mass M_S. (Neglect the mass farther from the center than the Sun, and assume that the mass closer to the center than the Sun exerts the same force on the Sun as would a point particle of the same mass located at the center of the galaxy.)

Picture the Problem To approximate the mass of the galaxy we'll assume the galactic center to be a point mass with the Sun in orbit about it and apply Kepler's third law. Let M_G represent the mass of the galaxy.

Using Kepler's third law, relate the period of the Sun T to its mean distance R_0 from the center of the galaxy:

$$T^2 = \frac{4\pi^2}{GM_G}R_0^3 \Rightarrow \frac{M_G}{M_S} = \frac{4\pi^2 R_0^3}{GM_S T^2}$$

Substitute numerical values and evaluate M_G/M_s:

$$\frac{M_G}{M_s} = \frac{4\pi^2\left(3.00\times10^4\ c\cdot\text{y}\times\dfrac{9.461\times10^{15}\ \text{m}}{c\cdot\text{y}}\right)^3}{\left(6.6742\times10^{-11}\ \dfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\left(1.99\times10^{30}\ \text{kg}\right)\left(250\times10^6\ \text{y}\times3.156\times10^7\ \dfrac{\text{s}}{\text{y}}\right)^2}$$

$$\approx 1.1\times10^{11} \Rightarrow M_{\text{galaxy}} \approx \boxed{1.1\times10^{11} M_s}$$

18 •• Besides studying samples of the lunar surface, the Apollo astronauts had several ways of determining that the moon is *not* made of green cheese. Among these ways are measurements of the gravitational acceleration at the lunar surface. Estimate the gravitational acceleration at the lunar surface if the moon were, in fact, a solid block of green cheese and compare it to the known value of the gravitational acceleration at the lunar surface.

Picture the Problem The density of a planet or other object determines the strength of the gravitational force it exerts on other objects. We can use Newton's law of gravity to express the acceleration due to gravity at the surface of the moon as a function of the density of the moon. Estimating the density of cheese will then allow us to calculate what the acceleration due to gravity at the surface of the moon would be if the moon were made of cheese. Finally, we can compare this value to the measured value of 1.62 m/s^2.

Apply Newton's law of gravity to an object of mass m at the surface of the moon to obtain:

$$F_g = ma_g = \frac{GmM}{r_{moon}^2} \Rightarrow a_g = \frac{GM}{r_{moon}^2}$$

Assuming the moon to be made of cheese, substitute for its mass to obtain:

$$a_{g,cheese} = \frac{G\rho_{cheese}V_{moon}}{r_{moon}^2}$$

Substituting for the volume of the moon and simplifying yields:

$$a_{g,cheese} = \frac{G\rho_{cheese}\frac{4}{3}\pi r_{moon}^3}{r_{moon}^2} = \frac{4\pi}{3}G\rho_{cheese}r_{moon}$$

Substitute numerical values and evaluate a_g:

$$a_{g,cheese} = \frac{4\pi}{3}\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(0.80\frac{\text{g}}{\text{cm}^3}\times\frac{1\ \text{kg}}{10^3\ \text{g}}\times\frac{10^6\ \text{cm}^3}{\text{m}^3}\right)\left(1.738\times10^6\ \text{m}\right) = \boxed{0.39\ \text{m/s}^2}$$

Express the ratio of $a_{g,cheese}$ to the measured value of $a_{g,moon}$:

$$\frac{a_{g,cheese}}{a_{g,moon}} = \frac{0.388\ \text{m/s}^2}{1.62\ \text{m/s}^2} = 0.24$$

or

$$a_{g,cheese} \approx \boxed{0.24a_{g,moon}}$$

19 •• You are in charge of the first manned exploration of an asteroid. You are concerned that, due to the weak gravitational field and resulting low escape speed, tethers might be required to bind the explorers to the surface of the

asteroid. Therefore, if you do not wish to use tethers, you have to be careful about which asteroids to choose to explore. Estimate the largest radius the asteroid can have that would still allow you to escape its surface by jumping. Assume spherical geometry and reasonable rock density.

Picture the Problem The density of an asteroid determines the strength of the gravitational force it exerts on other objects. We can use the equation for the escape speed from an asteroid of mass M_{asteroid} and radius R_{asteroid} to derive an expression for the radius of an asteroid as a function of its escape speed and density. We can approximate the escape speed from the asteroid by determining one's push-off speed for a jump at the surface of Earth.

The escape speed from an asteroid is given by:

$$v_{\text{e,asteroid}} = \sqrt{\frac{2GM_{\text{asteroid}}}{R_{\text{asteroid}}}}$$

In terms of the density of the asteroid, $v_{\text{e,asteroid}}$ becomes:

$$v_{\text{e,asteroid}} = \sqrt{\frac{2G\rho_{\text{asteroid}}\frac{4}{3}\pi R_{\text{asteroid}}^3}{R_{\text{asteroid}}}} = \sqrt{\tfrac{8}{3}\pi G\rho_{\text{asteroid}}}\,R_{\text{asteroid}}$$

Solving for R_{asteroid} yields:

$$R_{\text{asteroid}} = \frac{v_{\text{e,asteroid}}}{\sqrt{\frac{8}{3}\pi G\rho_{\text{asteroid}}}} \qquad (1)$$

Using a constant-acceleration equation, relate the height h to which you can jump on the surface of Earth to your push-off speed:

$$v^2 = v_0^2 - 2gh$$

or, because $v = 0$,

$$0 = v_0^2 - 2gh \Rightarrow v_0 = \sqrt{2gh}$$

Letting $v_0 = v_{\text{e,asteroid}}$, substitute in equation (1) and simplify to obtain:

$$R_{\text{asteroid}} = \frac{\sqrt{2gh}}{\sqrt{\frac{8}{3}\pi G\rho_{\text{asteroid}}}} = \sqrt{\frac{3gh}{4\pi G\rho_{\text{asteroid}}}}$$

Assuming that you can jump 0.75 m and that the average density of an asteroid is 3.0 g/cm^3, substitute numerical values and evaluate R_{asteroid}:

$$R_{\text{asteroid}} = \sqrt{\frac{3\left(9.81\ \text{m/s}^2\right)(0.75\ \text{m})}{4\pi\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(3.0\frac{\text{g}}{\text{cm}^3}\times\frac{1\ \text{kg}}{10^3\ \text{g}}\times\frac{10^6\ \text{cm}^3}{\text{m}^3}\right)}} \approx \boxed{3.0\ \text{km}}$$

20 ••• One of the great discoveries in astronomy in the past decade is the detection of planets outside the Solar System. Since 1996, more than 100 planets have been detected orbiting stars other than the Sun. While the planets themselves cannot be seen directly, telescopes can detect the small periodic motion of the star

as the star and planet orbit around their common center of mass. (This is measured using the *Doppler effect*, which is discussed in Chapter 15.) Both the period of this motion and the variation in the speed of the star over the course of time can be determined observationally. The mass of the star is found from its observed luminance and from the theory of stellar structure. *Iota Draconis* is the 8th brightest star in the constellation Draco. Observations show that a planet, with an orbital period of 1.50 y, is orbiting this star. The mass of Iota Draconis is $1.05M_{\text{Sun}}$. (*a*) Estimate the size (in AU) of the semimajor axis of this planet's orbit. (*b*) The radial speed of the star is observed to vary by 592 m/s. Use conservation of momentum to find the mass of the planet. Assume the orbit is circular, we are observing the orbit edge-on, and no other planets orbit Iota Draconis. Express the mass as a multiple of the mass of Jupiter.

Picture the Problem We can use Kepler's third law to find the size of the semi-major axis of the planet's orbit and the conservation of momentum to find its mass.

(*a*) Using Kepler's third law, relate the period of this planet T to the length r of its semi-major axis and simplify to obtain:

$$T^2 = \frac{4\pi^2}{GM_{\text{Iota Draconis}}}r^3 = \frac{\dfrac{4\pi^2}{M_{\text{s}}}}{G\dfrac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3 = \frac{\dfrac{4\pi^2}{GM_{\text{s}}}}{\dfrac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3$$

If we measure time in years, distances in AU, and masses in terms of the mass of the sun:

$$\frac{4\pi^2}{MG_{\text{s}}} = 1 \text{ and } T^2 = \frac{1}{\dfrac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3$$

Solving for r yields:

$$r = \sqrt[3]{\frac{M_{\text{Iota Draconis}}}{M_{\text{s}}}T^2}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\left(\frac{1.05\,M_{\text{s}}}{M_{\text{s}}}\right)(1.50\,\text{y})^2} = \boxed{1.33\,\text{AU}}$$

(*b*) Apply conservation of momentum to the planet (mass m and speed v) and the star (mass $M_{\text{Iota Draconis}}$ and speed V) to obtain:

$$mv = M_{\text{Iota Draconis}}V$$

Solve for m to obtain:

$$m = M_{\text{Iota Draconis}}\frac{V}{v} \qquad (1)$$

The speed v of the orbiting planet is given by:

$$v = \frac{\Delta d}{\Delta t} = \frac{2\pi r}{T}$$

Substitute numerical values and evaluate v:

$$v = \frac{2\pi\left(1.33\,\text{AU}\times\frac{1.50\times10^{11}\,\text{m}}{\text{AU}}\right)}{1.50\,\text{y}\times\frac{365.24\,\text{d}}{\text{y}}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}} = 2.648\times10^{4}\,\text{m/s}$$

Substitute numerical values in equation (1) and evaluate m:

$$m = (1.05M_{\text{sun}})\left(\frac{296\,\text{m/s}}{2.648\times10^{4}\,\text{m/s}}\right) = (1.05)(1.99\times10^{30}\,\text{kg})(0.0112) = 2.336\times10^{28}\,\text{kg}$$

Express m as a fraction of the mass M_{J} of Jupiter:

$$\frac{m}{M_{\text{J}}} = \frac{2.336\times10^{28}\,\text{kg}}{1.90\times10^{27}\,\text{kg}} = 12.3$$

or

$$m = \boxed{12.3M_{\text{J}}}$$

Remarks: A more sophisticated analysis, using the eccentricity of the orbit, leads to a lower bound of 8.7 Jovian masses. (Only a lower bound can be established, as the plane of the orbit is not known.)

21 ••• One of the biggest unresolved problems in the theory of the formation of the solar system is that, while the mass of the Sun is 99.9 percent of the total mass of the Solar System, it carries only about 2 percent of the total angular momentum. The most widely accepted theory of solar system formation has as its central hypothesis the collapse of a cloud of dust and gas under the force of gravity, with most of the mass forming the Sun. However, because the net angular momentum of this cloud is conserved, a simple theory would indicate that the Sun should be rotating much more rapidly than it currently is. In this problem, you will show why it is important that most of the angular momentum was somehow transferred to the planets. (*a*) The Sun is a cloud of gas held together by the force of gravity. If the Sun were rotating too rapidly, gravity couldn't hold it together. Using the known mass of the Sun (1.99×10^{30} kg) and its radius (6.96×10^{8} m), estimate the maximum angular speed that the Sun can have if it is to stay intact. What is the period of rotation corresponding to this rotation rate? (*b*) Calculate the orbital angular momentum of Jupiter and of Saturn from their masses (318 and 95.1 Earth masses, respectively), mean distances from the Sun (778 and 1430 million km, respectively), and orbital periods (11.9 and 29.5 y, respectively). Compare them to the experimentally measured value of the Sun's angular momentum of 1.91×10^{41} kg·m^2/s. (*c*) If we were to somehow transfer all of

Jupiter's and Saturn's angular momentum to the Sun, what would be the Sun's new rotational period? The Sun is not a uniform sphere of gas, and its moment of inertia is given by the formula $I = 0.059MR^2$. Compare this to the maximum rotational period of Part (*a*).

Picture the Problem We can apply Newton's law of gravity to estimate the maximum angular speed that the Sun can have if it is to stay together and use the definition of angular momentum to find the orbital angular momenta of Jupiter and Saturn. In Part (*c*) we can relate the final angular speed of the Sun to its initial angular speed, its moment of inertia, and the orbital angular momenta of Jupiter and Saturn.

(*a*) Gravity must supply the centripetal force which keeps an element of the sun's mass *m* rotating around it. Letting R_S represent the radius of the Sun and M_S the mass of the Sun, apply Newton's law of gravity to an object of mass *m* on the surface of the Sun to obtain:

$$m\omega^2 R_S < \frac{GM_S m}{R_S^2}$$

or

$$\omega^2 R_S < \frac{GM_S}{R_S^2} \Rightarrow \omega < \sqrt{\frac{GM_S}{R_S^3}}$$

where we've used the inequality because we're estimating the *maximum* angular speed which the sun can have if it is to stay together.

Substitute numerical values and evaluate ω:

$$\omega < \sqrt{\frac{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{(6.96\times10^{8}\ \text{m})^3}} = \boxed{6.28\times10^{-4}\ \text{rad/s}}$$

Calculate the maximum period of this motion from its angular speed:

$$T_{\max} = \frac{2\pi}{\omega} = \frac{2\pi}{6.28\times10^{-4}\ \text{rad/s}}$$
$$= 1.00\times10^{4}\ \text{s}\times\frac{1\ \text{h}}{3600\ \text{s}} = \boxed{2.78\ \text{h}}$$

(*b*) Express the orbital angular momenta of Jupiter and Saturn:

$$L_J = m_J r_J v_J \text{ and } L_S = m_S r_S v_S$$

Express the orbital speeds of Jupiter and Saturn in terms of their periods and distances from the sun:

$$v_J = \frac{2\pi r_J}{T_J} \text{ and } v_S = \frac{2\pi r_S}{T_S}$$

Substitute to obtain:

$$L_J = \frac{2\pi m_J r_J^2}{T_J} \text{ and } L_S = \frac{2\pi m_S r_S^2}{T_S}$$

Substitute numerical values and evaluate L_J and L_S:

$$L_J = \frac{2\pi(318M_E)r_J^2}{T_J} = \frac{2\pi(318)(5.98\times10^{24}\text{ kg})(778\times10^9\text{ m})^2}{11.9\text{ y}\times\frac{365.24\text{ d}}{\text{y}}\times\frac{24\text{ h}}{\text{d}}\times\frac{3600\text{ s}}{\text{h}}}$$
$$= \boxed{1.93\times10^{43}\text{ kg}\cdot\text{m}^2/\text{s}}$$

and

$$L_S = \frac{2\pi(95.1M_E)r_S^2}{T_S} = \frac{2\pi(95.1)(5.98\times10^{24}\text{ kg})(1430\times10^9\text{ m})^2}{29.5\text{ y}\times\frac{365.24\text{ d}}{\text{y}}\times\frac{24\text{ h}}{\text{d}}\times\frac{3600\text{ s}}{\text{h}}}$$
$$= \boxed{7.85\times10^{42}\text{ kg}\cdot\text{m}^2/\text{s}}$$

Express the angular momentum of the Sun as a fraction of the sum of the angular momenta of Jupiter and Saturn:

$$\frac{L_{\text{sun}}}{L_J + L_S} = \frac{1.91\times10^{41}\text{ kg}\cdot\text{m}^2/\text{s}}{(19.3+7.85)\times10^{42}\text{ kg}\cdot\text{m}^2/\text{s}} = \boxed{0.70\%}$$

(*c*) The Sun's rotational period depends on its rotational speed:

$$T_{\text{Sun}} = \frac{2\pi}{\omega_f} \qquad (1)$$

Relate the final angular momentum of the sun to its initial angular momentum and the angular momenta of Jupiter and Saturn:

$$L_f = L_i + L_J + L_S$$
or
$$I_{\text{Sun}}\omega_f = I_{\text{Sun}}\omega_i + L_J + L_S$$

Solve for ω_f to obtain:

$$\omega_f = \omega_i + \frac{L_J + L_S}{I_{\text{sun}}}$$

Substitute for ω_i and I_{Sun}:

$$\omega_f = \frac{2\pi}{T_{\text{Sun}}} + \frac{L_J + L_S}{0.059M_{\text{sun}}R_{\text{sun}}^2}$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \frac{2\pi}{30\text{ d}\times\frac{24\text{ h}}{\text{d}}\times\frac{3600\text{ s}}{\text{h}}} + \frac{(19.3+7.85)\times10^{42}\text{ kg}\cdot\text{m}^2/\text{s}}{0.059(1.99\times10^{30}\text{ kg})(6.96\times10^8\text{ m})^2}$$
$$= 4.798\times10^{-4}\text{ rad/s}$$

Substitute numerical values in equation (1) and evaluate T_{Sun}:

$$T_{Sun} = \frac{2\pi}{4.798\times10^{-4}\,\frac{\text{rad}}{\text{s}}\times\frac{3600\,\text{s}}{\text{h}}} = \boxed{3.64\,\text{h}}$$

Compare this to the maximum rotational period of Part (*a*).

$$\frac{T_{Sun}}{T_{max}} = \frac{3.64\,\text{h}}{2.78\,\text{h}} = 1.31$$

or

$$T_{Sun} = \boxed{1.31T_{max}}$$

Kepler's Laws

22 • The new comet Alex-Casey has a very elliptical orbit with a period of 127.4 y. If the closest approach of Alex-Casey to the Sun is 0.1 AU, what is its greatest distance from the Sun?

Picture the Problem We can use the relationship between the semi-major axis and the distances of closest approach and greatest separation, together with Kepler's third law, to find the greatest separation of Alex-Casey from the Sun.

Letting x represent the greatest distance from the Sun, express the relationship between x, the distance of closest approach, and its semi-major axis R:

$$R = \frac{x + 0.1\,\text{AU}}{2} \Rightarrow x = 2R - 0.1\,\text{AU} \quad (1)$$

Apply Kepler's third law, with the period T measured in years and R in AU to obtain:

$$T^2 = R^3 \Rightarrow R = \sqrt[3]{T^2}$$

Substituting for R in equation (1) yields:

$$x = 2\sqrt[3]{T^2} - 0.1\,\text{AU}$$

Substitute numerical values and evaluate x:

$$x = 2\sqrt[3]{(127.4\,\text{y})^2} - 0.1\,\text{AU} = \boxed{50.5\,\text{AU}}$$

23 • The radius of Earth's orbit is 1.496×10^{11} m and that of Uranus is 2.87×10^{12} m. What is the orbital period of Uranus?

Picture the Problem We can use Kepler's third law to relate the orbital period of Uranus to the orbital period of Earth.

Using Kepler's third law, relate the orbital period of Uranus to its mean distance from the Sun:

$$T_{\text{Uranus}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Uranus}}^3$$

.

Using Kepler's third law, relate the orbital period of Earth to its mean distance from the Sun:

$$T_{\text{Earth}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{T_{\text{Uranus}}^2}{T_{\text{Earth}}^2} = \frac{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Uranus}}^3}{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3} = \frac{r_{\text{Uranus}}^3}{r_{\text{Earth}}^3}$$

Solve for T_{Uranus} to obtain:

$$T_{\text{Uranus}} = T_{\text{Earth}} \left(\frac{r_{\text{Uranus}}}{r_{\text{Earth}}} \right)^{3/2}$$

Substitute numerical values and evaluate T_{Uranus}:

$$T_{\text{Uranus}} = (1.00\ \text{y}) \left(\frac{2.87 \times 10^{12}\ \text{m}}{1.496 \times 10^{11}\ \text{m}} \right)^{3/2} = \boxed{84.0\ \text{y}}$$

24 • The asteroid Hektor, discovered in 1907, is in a nearly circular orbit of radius 5.16 *AU* about the Sun. Determine the period of this asteroid.

Picture the Problem We can use Kepler's third law to relate the orbital period of Hector to the orbital period of Earth.

Using Kepler's third law, relate the orbital period of Hektor to its mean distance from the Sun:

$$T_{\text{Hector}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Hector}}^3$$

Using Kepler's third law, relate the orbital period of Earth to its mean distance from the Sun:

$$T_{\text{Earth}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{T_{\text{Hector}}^2}{T_{\text{Earth}}^2} = \frac{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Hector}}^3}{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3} = \frac{r_{\text{Hector}}^3}{r_{\text{Earth}}^3}$$

Solve for T_{Hector} to obtain:

$$T_{\text{Hector}} = T_{\text{Earth}}\left(\frac{r_{\text{Hector}}}{r_{\text{Earth}}}\right)^{3/2}$$

Substitute numerical values and evaluate T_{Hector}:

$$T_{\text{Hector}} = (1.00\ \text{y})\left(\frac{5.16\ \text{AU}}{1.00\ \text{AU}}\right)^{3/2} = \boxed{11.7\ \text{y}}$$

25 •• **[SSM]** One of the so-called "Kirkwood gaps" in the asteroid belt occurs at an orbital radius at which the period of the orbit is half that of Jupiter's. The reason there is a gap for orbits of this radius is because of the periodic pulling (by Jupiter) that an asteroid experiences at the same place in its orbit every *other* orbit around the sun. Repeated tugs from Jupiter of this kind would eventually change the orbit of such an asteroid. Therefore, all asteroids that would otherwise have orbited at this radius have presumably been cleared away from the area due to this resonance phenomenon. How far from the Sun is this particular 2:1 resonance "Kirkwood" gap?

Picture the Problem The period of an orbit is related to its semi-major axis (for circular orbits this distance is the orbital radius). Because we know the orbital periods of Jupiter and a hypothetical asteroid in the Kirkwood gap, we can use Kepler's third law to set up a proportion relating the orbital periods and average distances of Jupiter and the asteroid from the Sun from which we can obtain an expression for the orbital radius of an asteroid in the Kirkwood gap.

Use Kepler's third law to relate Jupiter's orbital period to its mean distance from the Sun:

$$T_{\text{Jupiter}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Jupiter}}^3$$

Use Kepler's third law to relate the orbital period of an asteroid in the Kirkwood gap to its mean distance from the Sun:

$$T_{\text{Kirkwood}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Kirkwood}}^3$$

Dividing the second of these equations by the first and simplifying yields:

$$\frac{T_{\text{Kirkwood}}^2}{T_{\text{Jupiter}}^2} = \frac{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Kirkwood}}^3}{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Jupiter}}^3} = \frac{r_{\text{Kirkwood}}^3}{r_{\text{Jupiter}}^3}$$

Solving for r_{Kirkwood} yields:

$$r_{\text{Kirkwood}} = r_{\text{Jupiter}}\left(\frac{T_{\text{Kirkwood}}}{T_{\text{Jupiter}}}\right)^{2/3}$$

Because the period of the orbit of an asteroid in the Kirkwood gap is half that of Jupiter's:

$$r_{\text{Kirkwood}} = \left(77.8\times10^{10}\ \text{m}\right)\left(\frac{\frac{1}{2}T_{\text{Jupiter}}}{T_{\text{Jupiter}}}\right)^{2/3} = \boxed{4.90\times10^{11}\ \text{m}}$$

$$= 4.90\times10^{11}\ \text{m}\times\frac{1\ \text{AU}}{1.50\times10^{11}\text{m}} = \boxed{3.27\ \text{AU}}$$

Remarks: There are also significant Kirkwood gaps at 3:1, 5:2, and 7:3 and resonances at 2.5 AU, 2.82 AU, and 2.95 AU.

26 •• The tiny Saturnian moon, Atlas, is locked into what is known as an orbital resonance with another moon, Mimas, whose orbit lies outside of Atlas's. The ratio between periods of these orbits is 3:2 – that means, for every 3 orbits of Atlas, Mimas completes 2 orbits. Thus, Atlas, Mimas and Saturn are aligned at intervals equal to two orbital periods of Atlas. If Mimas orbits Saturn at a radius of 186,000 km, what is the radius of Atlas's orbit?

Picture the Problem The period of an orbit is related to its semi-major axis (for circular orbits this distance is the orbital radius). Because we know the orbital periods of Atlas and Mimas, we can use Kepler's third law to set up a proportion relating the orbital periods and average distances from Saturn of Atlas and Mimas from which we can obtain an expression for the radius of Atlas's orbit.

Use Kepler's third law to relate Atlas's orbital period to its mean distance from Saturn:

$$T_{\text{Atlas}}^2 = \frac{4\pi^2}{GM_{\text{Saturn}}}r_{\text{Atlas}}^3$$

Use Kepler's third law to relate the orbital period of Mimas to its mean distance from Saturn:

$$T_{\text{Mimas}}^2 = \frac{4\pi^2}{GM_{\text{Saturn}}}r_{\text{Mimas}}^3$$

Dividing the second of these equations by the first and simplifying yields:

$$\frac{T_{\text{Mimas}}^2}{T_{\text{Atlas}}^2} = \frac{\dfrac{4\pi^2}{GM_{\text{Saturn}}}r_{\text{Mimas}}^3}{\dfrac{4\pi^2}{GM_{\text{Saturn}}}r_{\text{Atlas}}^3} = \frac{r_{\text{Mimas}}^3}{r_{\text{Atlas}}^3}$$

Solving for r_{Atlas} yields:

$$r_{\text{Atlas}} = r_{\text{Mimas}}\left(\frac{T_{\text{Atlas}}}{T_{\text{Mimas}}}\right)^{2/3}$$

Because for every 3 orbits of Atlas, Mimas has completed 2:

$$r_{\text{Atlas}} = \left(1.86\times10^{5}\ \text{km}\right)\left(\frac{2}{3}\right)^{2/3} = \boxed{1.42\times10^{5}\ \text{km}}$$

27 •• The asteroid Icarus, discovered in 1949, was so named because its highly eccentric elliptical orbit brings it close to the Sun at perihelion. The eccentricity e of an ellipse is defined by the relation $r_{\text{p}} = a(1 - e)$, where r_{p} is the perihelion distance and a is the semimajor axis. Icarus has an eccentricity of 0.83 and a period of 1.1 y. (*a*) Determine the semimajor axis of the orbit of Icarus. (*b*) Find the perihelion and aphelion distances of the orbit of Icarus.

Picture the Problem Kepler's third law relates the period of Icarus to the length of its semimajor axis. The aphelion distance r_{a} is related to the perihelion distance r_{p} and the semimajor axis by $r_{\text{a}} + r_{\text{p}} = 2a$.

(*a*) Using Kepler's third law, relate the period T of Icarus to the length a of its semimajor axis:

$$T^2 = \frac{4\pi^2}{GM_{\text{Sun}}}a^3 \Rightarrow a = \sqrt[3]{\frac{T^2}{C}}$$

where $C = \dfrac{4\pi^2}{GM_{\text{Sun}}} = 2.973\times10^{-19}\ \text{s}^2/\text{m}^3$.

Substitute numerical values and evaluate a:

$$a = \sqrt[3]{\frac{\left(1.1\text{y}\times\dfrac{365.24\,\text{d}}{\text{y}}\times\dfrac{24\,\text{h}}{\text{d}}\times\dfrac{3600\,\text{s}}{\text{h}}\right)^2}{2.973\times10^{-19}\ \text{s}^2/\text{m}^3}} = \boxed{1.6\times10^{11}\ \text{m}}$$

(*b*) Use the definition of the eccentricity of an ellipse to determine the perihelion distance of Icarus:

$$r_{\text{p}} = a(1-e) = \left(1.59\times10^{11}\ \text{m}\right)(1-0.83) = 2.71\times10^{10}\ \text{m} = \boxed{2.7\times10^{10}\ \text{m}}$$

Express the relationship between r_{p} and r_{a} for an ellipse:

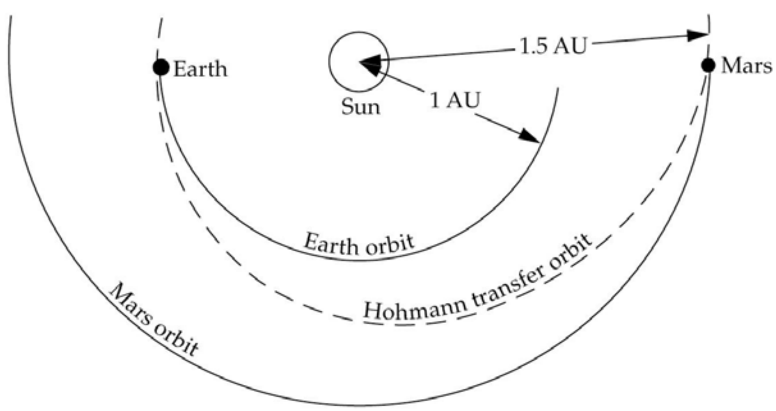
$$r_{\text{a}} + r_{\text{p}} = 2a \Rightarrow r_{\text{a}} = 2a - r_{\text{p}}$$

Substitute numerical values and evaluate r_{a}:

$$r_{\text{a}} = 2\left(1.59\times10^{11}\ \text{m}\right) - 2.71\times10^{10}\ \text{m} = \boxed{2.9\times10^{11}\ \text{m}}$$

28 •• A manned mission to Mars and its attendant problems due to the extremely long time the astronauts would spend weightless and without supplies in space have been extensively discussed. To examine this issue in a simple way, consider one possible trajectory for the spacecraft: the "Hohmann transfer orbit." This orbit consists of an elliptical orbit tangent to the orbit of Earth at its perihelion and tangent to the orbit of Mars at its aphelion. Given that Mars has a mean distance from the Sun of 1.52 times the mean Sun–Earth distance, calculate the time spent by the astronauts during the out-bound part of the trip to Mars. Many adverse biological effects (such as muscle atrophy, decreased bone density, etc.) have been observed in astronauts returning from near-Earth orbit after only a few months in space. As the flight doctor, are there any health issues you should be aware of?

Picture the Problem The Hohmann transfer orbit is shown in the diagram. We can apply Kepler's third law to relate the time-in-orbit to the period of the spacecraft in its Hohmann Earth-to-Mars orbit. The period of this orbit is, in turn, a function of its semi-major axis, which we can find from the average of the lengths of the semi-major axes of Earth and Mars orbits.



Using Kepler's third law, relate the period T of the spacecraft to the semi-major axis of its orbit:

$$T^2 = R^3 \Rightarrow T = \sqrt{R^3}$$
where T is in years and R is in AU.

Relate the out-bound transit time to the period of this orbit:

$$t_{\text{out-bound}} = \tfrac{1}{2}T = \tfrac{1}{2}\sqrt{R^3}$$

Express the semi-major axis of the Hohmann transfer orbit in terms of the mean Sun-Mars and Sun-Earth distances:

$$R = \frac{1.52\,\text{AU} + 1.00\,\text{AU}}{2} = 1.26\,\text{AU}$$

Substitute numerical values and evaluate $t_{\text{out-bound}}$:

$$t_{\text{out-bound}} = \tfrac{1}{2}\sqrt{(1.26\,\text{AU})^3} = 0.707\,\text{y} \times \frac{365.24\,\text{d}}{1\,\text{y}} = \boxed{258\ \text{d}}$$

In order for bones and muscles to maintain their health, they need to be under compression as they are on Earth. Due to the long duration (well over a year) of the round trip, you would want to design an exercise program that would maintain the strength of their bones and muscles.

29 •• [SSM] Kepler determined distances in the Solar System from his data. For example, he found the relative distance from the Sun to Venus (as compared to the distance from the Sun to Earth) as follows. Because Venus's orbit is closer to the Sun than is Earth's orbit, Venus is a morning or evening star—its position in the sky is never very far from the Sun (Figure 11-24). If we suppose the orbit of Venus is a perfect circle, then consider the relative orientation of Venus, Earth, and the Sun at maximum extension, that is when Venus is farthest from the Sun in the sky. (*a*) Under this condition, show that angle *b* in Figure 11-24 is 90º. (*b*) If the maximum elongation angle *a* between Venus and the Sun is 47º, what is the distance between Venus and the Sun in AU? (*c*) Use this result to estimate the length of a Venusian ″year.″

Picture the Problem We can use a property of lines tangent to a circle and radii drawn to the point of contact to show that $b = 90°$. Once we've established that *b* is a right angle we can use the definition of the sine function to relate the distance from the Sun to Venus to the distance from the Sun to Earth.

(*a*) The line from Earth to Venus′ orbit is tangent to the orbit of Venus at the point of maximum extension. Venus will appear farther from the Sun in Earth's sky when it passes the line drawn from Earth that is tangent to its orbit. Hence $b = \boxed{90°}$

(*b*) Using trigonometry, relate the distance from the Sun to Venus d_{SV} to the angle *a*:

$$\sin a = \frac{d_{\text{SV}}}{d_{\text{SE}}} \Rightarrow d_{\text{SV}} = d_{\text{SE}} \sin a$$

Substitute numerical values and evaluate d_{SV}:

$$d_{\text{SV}} = (1.00\,\text{AU})\sin 47° = 0.731\,\text{AU} = \boxed{0.73\,\text{AU}}$$

(*c*) Use Kepler's third law to relate Venus's orbital period to its mean distance from the Sun:

$$T_{\text{Venus}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Venus}}^3$$

Use Kepler's third law to relate Earth's orbital period to its mean distance from the Sun:

$$T_{\text{Earth}}^2 = \frac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{T_{\text{Venus}}^2}{T_{\text{Earth}}^2} = \frac{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Venus}}^3}{\dfrac{4\pi^2}{GM_{\text{Sun}}} r_{\text{Earth}}^3} = \frac{r_{\text{Venus}}^3}{r_{\text{Earth}}^3}$$

Solving for T_{Venus} yields:

$$T_{\text{Venus}} = T_{\text{Earth}} \left(\frac{r_{\text{Venus}}}{r_{\text{Earth}}} \right)^{3/2}$$

Using the result from Part (*b*) yields:

$$T_{\text{Venus}} = (1.00\ \text{y}) \left(\frac{0.731\ \text{AU}}{1.00\ \text{AU}} \right)^{3/2} = \boxed{0.63\ \text{y}}$$

Remarks: The correct distance from the sun to Venus is closer to 0.723 AU.

30 •• At apogee the center of the moon is 406,395 km from the center of Earth and at perigee the moon is 357,643 km from the center of Earth. What is the orbital speed of the moon at perigee and at apogee? The mass of Earth is 5.98×10^{24} kg.

Picture the Problem Because the gravitational force Earth exerts on the moon is along the line joining their centers, the net torque acting on the moon is zero and its angular momentum is conserved in its orbit about Earth. Because energy is also conserved, we can combine these two expressions to solve for either v_p or v_a initially and then use conservation of angular momentum to find the other.

Letting *m* be the mass of the moon, apply conservation of angular momentum to the moon at apogee and perigee to obtain:

$$mv_p r_p = mv_a r_a \Rightarrow v_a = \frac{r_p}{r_a} v_p \qquad (1)$$

Apply conservation of energy to the moon-Earth system to obtain:

$$\tfrac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \tfrac{1}{2} m v_a^2 - \frac{GMm}{r_a}$$

or

$$\tfrac{1}{2} v_p^2 - \frac{GM}{r_p} = \tfrac{1}{2} v_a^2 - \frac{GM}{r_a}$$

Substitute for v_a to obtain:

$$\tfrac{1}{2}v_p^2 - \frac{GM}{r_p} = \tfrac{1}{2}\left(\frac{r_p}{r_a}v_p\right)^2 - \frac{GM}{r_a}$$
$$= \tfrac{1}{2}\left(\frac{r_p}{r_a}\right)^2 v_p^2 - \frac{GM}{r_a}$$

Solving for v_p yields:

$$v_p = \sqrt{\frac{2GM}{r_p}\left(\frac{1}{1+r_p/r_a}\right)}$$

Substitute numerical values and evaluate v_p:

$$v_p = \sqrt{\frac{2\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.98\times10^{24}\ \text{kg}\right)}{3.576\times10^{8}\ \text{m}}\left(\frac{1}{1+\dfrac{3.576\times10^{8}\ \text{m}}{4.064\times10^{8}\ \text{m}}}\right)} = \boxed{1.09\ \text{km/s}}$$

Substitute numerical values in equation (1) and evaluate v_a:

$$v_a = \frac{3.576\times10^{8}\ \text{m}}{4.064\times10^{8}\ \text{m}}\left(1.09\,\text{km/s}\right)$$
$$= \boxed{959\ \text{m/s}}$$

Newton's Law of Gravity

31 • **[SSM]** Jupiter's satellite Europa orbits Jupiter with a period of 3.55 d at an average orbital radius of 6.71 × 10^8 m. (*a*) Assuming that the orbit is circular, determine the mass of Jupiter from the data given. (*b*) Another satellite of Jupiter, Callisto, orbits at an average radius of 18.8 × 10^8 m with an orbital period of 16.7 d. Show that these data are consistent with an inverse square force law for gravity (*Note*: DO NOT use the value of *G* anywhere in Part (*b*)).

Picture the Problem While we could apply Newton's law of gravitation and second law of motion to solve this problem from first principles, we'll use Kepler's third law (derived from these laws) to find the mass of Jupiter in Part (*a*). In Part (*b*) we can compare the ratio of the centripetal accelerations of Europa and Callisto to show that these data are consistent with an inverse square law for gravity.

(*a*) Assuming a circular orbit, apply Kepler's third law to the motion of Europa to obtain:

$$T_{\text{E}}^2 = \frac{4\pi^2}{GM_{\text{J}}}R_{\text{E}}^3 \Rightarrow M_{\text{J}} = \frac{4\pi^2}{GT_{\text{E}}^2}R_{\text{E}}^3$$

Substitute numerical values and evaluate M_J:

$$M_J = \frac{4\pi^2\left(6.71\times10^8\ \text{m}\right)^3}{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(3.55\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}\right)^2} = \boxed{1.90\times10^{27}\ \text{kg}}$$

Note that this result is in excellent agreement with the accepted value of 1.902×10^{27} kg.

(*b*) Express the centripetal acceleration of both of the moons to obtain:

$$a_{\text{centripetal}} = \frac{v^2}{R} = \frac{\left(\frac{2\pi R}{T}\right)^2}{R} = \frac{4\pi^2 R}{T^2}$$

where R and T are the radii and periods of their motion.

Using this result, express the centripetal accelerations of Europa and Callisto:

$$a_E = \frac{4\pi^2 R_E}{T_E^2} \text{ and } a_C = \frac{4\pi^2 R_C}{T_C^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{a_E}{a_C} = \frac{\frac{4\pi^2 R_E}{T_E^2}}{\frac{4\pi^2 R_C}{T_C^2}} = \frac{T_C^2}{T_E^2}\frac{R_E}{R_C}$$

Substitute for the periods of Callisto and Europa using Kepler's third law to obtain:

$$\frac{a_E}{a_C} = \frac{CR_C^3}{CR_E^3}\frac{R_E}{R_C} = \frac{R_C^2}{R_E^2}$$

This result, together with the fact that the gravitational force is directly proportional to the acceleration of the moons, demonstrates that the gravitational force varies inversely with the square of the distance.

32 • Some people think that shuttle astronauts are ″weightless″ because they are ″beyond the pull of Earth's gravity. ″ In fact, this is completely untrue. (*a*) What is the magnitude of the gravitational field in the vicinity of a shuttle orbit? A shuttle orbit is about 400 km above the ground. (*b*) Given the answer in Part (*a*), explain why shuttle astronauts do suffer from adverse biological affects such as muscle atrophy even though they are actually not ″weightless″?

Determine the Concept The weight of anything, including astronauts, is the reading of a scale from which the object is suspended or on which it rests. That is, it is the magnitude of the normal force acting on the object. If the scale reads zero, then we say the object is ″weightless.″ The pull of Earth's gravity, on the other hand, depends on the local value of the acceleration of gravity and we can use Newton's law of gravity to find this acceleration at the elevation of the shuttle.

(*a*) Apply Newton's law of gravitation to an astronaut of mass *m* in a shuttle at a distance *h* above the surface of Earth:

$$mg_{\text{shuttle}} = \frac{GmM_{\text{E}}}{(h+R_{\text{E}})^2}$$

Solving for g_{shuttle} yields:

$$g_{\text{shuttle}} = \frac{GM_{\text{E}}}{(h+R_{\text{E}})^2}$$

Substitute numerical values and evaluate g_{shuttle}:

$$g_{\text{shuttle}} = \frac{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}{(400\,\text{km}+6370\,\text{km})^2} = \boxed{8.71\,\text{m/s}^2}$$

(*b*) In orbit, the astronauts experience only one (the gravitational force) of the two forces (the second being the normal force – a compressive force – exerted by Earth) that normally acts on them. Lacking this compressive force, their bones and muscles, in the absence of an exercise program, will weaken. In orbit the astronauts are not weightless, they are normal–forceless.

33 • **[SSM]** The mass of Saturn is 5.69×10^{26} kg. (*a*) Find the period of its moon Mimas, whose mean orbital radius is 1.86×10^8 m. (*b*) Find the mean orbital radius of its moon Titan, whose period is 1.38×10^6 s.

Picture the Problem While we could apply Newton's law of gravitation and second law of motion to solve this problem from first principles, we'll use Kepler's third law (derived from these laws) to find the period of Mimas and to relate the periods of the moons of Saturn to their mean distances from its center.

(*a*) Using Kepler's third law, relate the period of Mimas to its mean distance from the center of Saturn:

$$T_{\text{M}}^2 = \frac{4\pi^2}{GM_{\text{S}}}r_{\text{M}}^3 \Rightarrow T_{\text{M}} = \sqrt{\frac{4\pi^2}{GM_{\text{S}}}r_{\text{M}}^3}$$

Substitute numerical values and evaluate T_{M}:

$$T_{\text{M}} = \sqrt{\frac{4\pi^2(1.86\times10^8\ \text{m})^3}{(6.6726\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.69\times10^{26}\ \text{kg})}} = 8.18\times10^4\ \text{s} \approx \boxed{22.7\ \text{h}}$$

(*b*) Using Kepler's third law, relate the period of Titan to its mean distance from the center of Saturn:

$$T_T^2 = \frac{4\pi^2}{GM_S} r_T^3 \Rightarrow r_T = \sqrt[3]{\frac{T_T^2 GM_S}{4\pi^2}}$$

Substitute numerical values and evaluate r_T:

$$r_T = \sqrt[3]{\frac{\left(1.38\times10^6\ \text{s}\right)^2\left(6.6726\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.69\times10^{26}\ \text{kg}\right)}{4\pi^2}} = \boxed{1.22\times10^9\ \text{m}}$$

34 • Calculate the mass of Earth from the period of the moon, $T = 27.3$ d; its mean orbital radius, $r_m = 3.84 \times 10^8$ m; and the known value of G.

Picture the Problem While we could apply Newton's law of gravitation and second law of motion to solve this problem from first principles, we'll use Kepler's third law (derived from these laws) to relate the period of the moon to the mass of Earth and the mean Earth-moon distance.

Using Kepler's third law, relate the period of the moon to its mean orbital radius:

$$T_m^2 = \frac{4\pi^2}{GM_E} r_m^3 \Rightarrow M_E = \frac{4\pi^2}{GT_m^2} r_m^3$$

Substitute numerical values and evaluate M_E:

$$M_E = \frac{4\pi^2\left(3.84\times10^8\ \text{m}\right)^3}{\left(6.6726\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(27.3\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\ \text{s}}{\text{h}}\right)^2} = \boxed{6.02\times10^{24}\ \text{kg}}$$

Remarks: This analysis neglects the mass of the moon; consequently the mass calculated here is slightly too large.

35 • Suppose you leave the Solar System and arrive at a planet that has the same mass-to-volume ratio as Earth but has 10 times Earth's radius. What would you weigh on this planet compared with what you weigh on Earth?

Picture the Problem Your weight is the local gravitational force exerted on you. We can use the definition of density to relate the mass of the planet to the mass of Earth and the law of gravity to relate your weight on the planet to your weight on Earth.

Using the definition of density, relate the mass of Earth to its radius:

$$M_E = \rho V_E = \tfrac{4}{3}\rho\pi R_E^3$$

Relate the mass of the planet to its radius:

$$M_P = \rho V_P = \tfrac{4}{3}\rho\pi R_P^3 = \tfrac{4}{3}\rho\pi(10R_E)^3$$

Divide the second of these equations by the first to express M_P in terms of M_E:

$$\frac{M_P}{M_E} = \rho\frac{\tfrac{4}{3}\rho\pi(10R_E)^3}{\tfrac{4}{3}\rho\pi R_E^3} \Rightarrow M_P = 10^3 M_E$$

Letting w' represent your weight on the planet, use the law of gravity to relate w' to your weight on Earth:

$$w' = \frac{GmM_P}{R_P^2} = \frac{Gm(10^3 M_E)}{(10R_E)^2} = 10\frac{GmM_E}{R_E^2} = 10w$$

Your weight would be ten times your weight on Earth.

36 • Suppose that Earth retained its present mass but was somehow compressed to half its present radius. What would be the value of g at the surface of this new, compact planet?

Picture the Problem We can relate the acceleration due to gravity of a test object at the surface of the new planet to the acceleration due to gravity at the surface of Earth through use of the law of gravity and Newton's second law of motion.

Letting a represent the acceleration due to gravity at the surface of this new planet and m the mass of a test object, apply Newton's second law and the law of gravity to obtain:

$$\sum F_{\text{radial}} = \frac{GmM_E}{\left(\tfrac{1}{2}R_E\right)^2} = ma \Rightarrow a = \frac{GM_E}{\left(\tfrac{1}{2}R_E\right)^2}$$

Simplify this expression to obtain:

$$a = 4\left(\frac{GM_E}{R_E^2}\right) = 4g = \boxed{39.2\,\text{m/s}^2}$$

37 • A planet orbits a massive star. When the planet is at perihelion, it has a speed of 5.0×10^4 m/s and is 1.0×10^{15} m from the star. The orbital radius increases to 2.2×10^{15} m at aphelion. What is the planet's speed at aphelion?

Picture the Problem We can use conservation of angular momentum to relate the planet's speeds at aphelion and perihelion.

Using conservation of angular momentum, relate the angular momenta of the planet at aphelion and perihelion:

$$L_a = L_p$$

or

$$mv_p r_p = mv_a r_a \Rightarrow v_a = \frac{v_p r_p}{r_a}$$

Substitute numerical values and evaluate v_a:

$$v_a = \frac{(5.0\times10^4\text{ m/s})(1.0\times10^{15}\text{ m})}{2.2\times10^{15}\text{ m}} = \boxed{2.3\times10^4\text{ m/s}}$$

38 • What is the magnitude of the gravitational field at the surface of a neutron star whose mass is 1.60 times the mass of the Sun and whose radius is 10.5 km?

Picture the Problem We can use Newton's law of gravity to express the gravitational force acting on an object at the surface of the neutron star in terms of the weight of the object. We can then simplify this expression by dividing out the mass of the object … leaving an expression for the magnitude of the gravitational field at the surface of the neutron star.

Apply Newton's law of gravity to an object of mass m at the surface of the neutron star to obtain:

$$\frac{GM_{\text{Neutron Star}}m}{R^2_{\text{Neutron Star}}} = mg$$

where g represents the magnitude of the gravitational field at the surface of the neutron star.

Solve for g and substitute for the mass of the neutron star:

$$g = \frac{GM_{\text{Neutron Star}}}{R^2_{\text{Neutron Star}}} = \frac{G(1.60M_{\text{sun}})}{R^2_{\text{Neutron Star}}}$$

Substitute numerical values and evaluate g:

$$g = \frac{1.60(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\text{ kg})}{(10.5\text{ km})^2} = \boxed{1.93\times10^{12}\text{ m/s}^2}$$

39 •• The speed of an asteroid is 20 km/s at perihelion and 14 km/s at aphelion. (*a*) Determine the ratio of the aphelion to perihelion distances. (*b*) Is this asteroid farther from the Sun or closer to the Sun than Earth, on average? Explain.

Picture the Problem We can use conservation of angular momentum to relate the asteroid's aphelion and perihelion distances.

(*a*) Using conservation of angular momentum, relate the angular momenta of the asteroid at aphelion and perihelion:

$$L_a - L_p = 0$$

or

$$mv_a r_a - mv_p r_p = 0 \Rightarrow \frac{r_a}{r_p} = \frac{v_p}{v_a}$$

Substitute numerical values and evaluate the ratio of the asteroid's aphelion and perihelion distances:

$$\frac{r_a}{r_p} = \frac{20\,\text{km/s}}{14\,\text{km/s}} = \boxed{1.4}$$

(*b*) It is farther from the Sun than Earth. Kepler's third law ($T^2 = Cr_{av}^3$) tells us that longer orbital periods together with larger orbital radii means slower orbital speeds, so the speed of objects orbiting the Sun decreases with distance from the Sun. The average orbital speed of Earth, given by $v = 2\pi r_{ES}/T_{ES}$, is approximately 30 km/s. Because the given maximum speed of the asteroid is only 20 km/s, the asteroid is farther from the Sun.

40 •• A satellite that has a mass of 300 kg moves in a circular orbit 5.00×10^7 m above Earth's surface. (*a*) What is the gravitational force on the satellite? (*b*) What is the speed of the satellite? (*c*) What is the period of the satellite?

Picture the Problem We'll use the law of gravity to find the gravitational force acting on the satellite. The application of Newton's second law will lead us to the speed of the satellite and its period can be found from its definition.

(*a*) Letting *m* represent the mass of the satellite and *h* its elevation, use the law of gravity to express the gravitational force acting on it:

$$F_g = \frac{GmM_E}{(R_E + h)^2}$$

Because $GM_E = gR_E^2$:

$$F_g = \frac{mR_E^2 g}{(R_E + h)^2}$$

Divide the numerator and denominator of this expression by R_E^2 to obtain:

$$F_g = \frac{mg}{\left(1 + \frac{h}{R_E}\right)^2}$$

Substitute numerical values and evaluate F_g:

$$F_g = \frac{(300\,\text{kg})(9.81\,\text{N/kg})}{\left(1+\dfrac{5.00\times10^7\text{ m}}{6.37\times10^6\text{ m}}\right)^2} = 37.58\,\text{N}$$

$$= \boxed{37.6\,\text{N}}$$

(*b*) Using Newton's second law, relate the gravitational force acting on the satellite to its centripetal acceleration:

$$F_g = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{F_g r}{m}}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{(37.58\,\text{N})(6.37\times10^6\text{ m} + 5.00\times10^7\text{ m})}{300\,\text{kg}}} = 2.657\text{ km/s} = \boxed{2.66\text{ km/s}}$$

(*c*) The period of the satellite is given by:

$$T = \frac{2\pi r}{v}$$

Substitute numerical values and evaluate T:

$$T = \frac{2\pi(6.37\times10^6\text{ m} + 5.00\times10^7\text{ m})}{2.657\times10^3\text{ m/s}}$$

$$= 1.333\times10^5\text{ s}\times\frac{1\,\text{h}}{3600\,\text{s}} = \boxed{37.0\,\text{h}}$$

41 •• [SSM] A superconducting gravity meter can measure changes in gravity of the order $\Delta g/g = 1.00 \times 10^{-11}$. (*a*) You are hiding behind a tree holding the meter, and your 80-kg friend approaches the tree from the other side. How close to you can your friend get before the meter detects a change in g due to his presence? (*b*) You are in a hot air balloon and are using the meter to determine the rate of ascent (assume the balloon has constant acceleration). What is the smallest change in altitude that results in a detectable change in the gravitational field of Earth?

Picture the Problem We can determine the maximum range at which an object with a given mass can be detected by substituting the equation for the gravitational field in the expression for the resolution of the meter and solving for the distance. Differentiating $g(r)$ with respect to r, separating variables to obtain dg/g, and approximating Δr with dr will allow us to determine the vertical change in the position of the gravity meter in Earth's gravitational field is detectable.

(*a*) Earth's gravitational field is given by:

$$g_E = \frac{GM_E}{R_E^2}$$

Express the gravitational field due to the mass *m* (assumed to be a point mass) of your friend and relate it to the resolution of the meter:

$$g(r) = \frac{Gm}{r^2} = 1.00\times10^{-11}\, g_E$$
$$= 1.00\times10^{-11}\frac{GM_E}{R_E^2}$$

Solving for *r* yields:

$$r = R_E\sqrt{\frac{1.00\times10^{11}\, m}{M_E}}$$

Substitute numerical values and evaluate *r*:

$$r = (6.37\times10^6\ \text{m})\sqrt{\frac{1.00\times10^{11}(80\,\text{kg})}{5.98\times10^{24}\ \text{kg}}}$$
$$= \boxed{7.37\,\text{m}}$$

(*b*) Differentiate *g*(*r*) and simplify to obtain:

$$\frac{dg}{dr} = \frac{-2Gm}{r^3} = -\frac{2}{r}\left(\frac{Gm}{r^2}\right) = -\frac{2}{r}g$$

Separate variables to obtain:

$$\frac{dg}{g} = -2\frac{dr}{r} = 10^{-11}$$

Approximating *dr* with Δr, evaluate Δr with $r = R_E$:

$$\Delta r = \left|-\tfrac{1}{2}(1.00\times10^{-11})(6.37\times10^6\ \text{m})\right|$$
$$= \boxed{31.9\,\mu\text{m}}$$

42 •• Suppose that the attractive interaction between a star of mass *M* and a planet of mass $m \ll M$ is of the form $F = KMm/r$, where *K* is the gravitational constant. What would be the relation between the radius of the planet's circular orbit and its period?

Picture the Problem We can use the law of gravity and Newton's second law to relate the force exerted on the planet by the star to its orbital speed and the definition of the period to relate it to the radius of the orbit.

The period of the planet is related to its orbital speed:

$$T = \frac{2\pi r}{v} \qquad (1)$$

Using the law of gravity and Newton's second law, relate the force exerted on the planet by the star to its centripetal acceleration:

$$F_{net} = \frac{KMm}{r} = m\frac{v^2}{r} \Rightarrow v = \sqrt{KM}$$

Substitute for v in equation (1) to obtain:

$$T = \boxed{\frac{2\pi}{\sqrt{KM}}r}$$

43 •• **[SSM]** Earth's radius is 6370 km and the moon's radius is 1738 km. The acceleration of gravity at the surface of the moon is 1.62 m/s^2. What is the ratio of the average density of the moon to that of Earth?

Picture the Problem We can use the definitions of the gravitational fields at the surfaces of Earth and the moon to express the accelerations due to gravity at these locations in terms of the average densities of Earth and the moon. Expressing the ratio of these accelerations will lead us to the ratio of the densities.

Express the acceleration due to gravity at the surface of Earth in terms of Earth's average density:

$$g_E = \frac{GM_E}{R_E^2} = \frac{G\rho_E V_E}{R_E^2} = \frac{G\rho_E \frac{4}{3}\pi R_E^3}{R_E^2}$$
$$= \tfrac{4}{3}G\rho_E\pi R_E$$

The acceleration due to gravity at the surface of the moon in terms of the moon's average density is:

$$g_M = \tfrac{4}{3}G\rho_M\pi R_M$$

Divide the second of these equations by the first to obtain:

$$\frac{g_M}{g_E} = \frac{\rho_M R_M}{\rho_E R_E} \Rightarrow \frac{\rho_M}{\rho_E} = \frac{g_M R_E}{g_E R_M}$$

Substitute numerical values and evaluate $\frac{\rho_M}{\rho_E}$:

$$\frac{\rho_M}{\rho_E} = \frac{(1.62\,\text{m/s}^2)(6.37\times10^6\,\text{m})}{(9.81\,\text{m/s}^2)(1.738\times10^6\,\text{m})}$$
$$= \boxed{0.605}$$

Gravitational and Inertial Mass

44 • The weight of a standard object defined as having a mass of exactly 1.00… kg is measured to be 9.81 N. In the same laboratory, a second object weighs 56.6 N. (*a*) What is the mass of the second object? (*b*) Is the mass you determined in Part (*a*) gravitational or inertial mass?

Picture the Problem Newton's second law of motion relates the weights of these two objects to their masses and the acceleration due to gravity.

(*a*) Apply Newton's second law to the standard object:

$$F_{net} = w_1 = m_1 g$$

Apply Newton's second law to the object of unknown mass:

$$F_{net} = w_2 = m_2 g$$

Eliminate g between these two equations and solve for m_2:

$$m_2 = \frac{w_2}{w_1} m_1$$

Substitute numerical values and evaluate m_2:

$$m_2 = \frac{56.6\,\text{N}}{9.81\,\text{N}}(1.00\,\text{kg}) = \boxed{5.77\,\text{kg}}$$

(*b*) Because this result is determined by the effect on m_2 of Earth's gravitational field, it is the *gravitational* mass of the second object.

45 • The *Principle of Equivalence* states that the free-fall acceleration of any object in a gravitational field is independent of the mass of the object. This can be deduced from the law of universal gravitation, but how well does it hold up experimentally? The Roll-Krotkov-Dicke experiment performed in the 1960s indicates that the free-fall acceleration is independent of mass to at least 1 part in 10^{12}. Suppose two objects are simultaneously released from rest in a uniform gravitational field. Also, suppose one of the objects falls with a constant acceleration of exactly 9.81 m/s^2 while the other falls with a constant acceleration that is greater than 9.81 m/s^2 by one part in 10^{12}. How far will the first object have fallen when the second object has fallen 1.00 mm farther than it has? Note that this estimate provides only an upper bound on the difference in the accelerations; most physicists believe that there is no difference in the accelerations.

Picture the Problem Noting that $g_1 \sim g_2 \sim g$, let the acceleration of gravity on the first object be g_1, and on the second be g_2. We can use a constant-acceleration equation to express the difference in the distances fallen by each object and then relate the average distance fallen by the two objects to obtain an expression from which we can approximate the distance they would have to fall before we might measure a difference in their fall distances greater than 1 mm.

Express the difference Δd in the distances fallen by the two objects in time t:

$$\Delta d = d_1 - d_2$$

Express the distances fallen by each of the objects in time t:

$$d_1 = \tfrac{1}{2} g_1 t^2 \text{ and } d_2 = \tfrac{1}{2} g_2 t^2$$

Substitute for d_1 and d_2 to obtain:

$$\Delta d = \tfrac{1}{2} g_1 t^2 - \tfrac{1}{2} g_2 t^2 = \tfrac{1}{2}\left(g_1 - g_2\right) t^2$$

Relate the average distance d fallen by the two objects to their time of fall:

$$d = \tfrac{1}{2} g t^2 \Rightarrow t^2 = \frac{2d}{g}$$

Substitute for t^2 to obtain:

$$\Delta d \approx \tfrac{1}{2} \Delta g \frac{2d}{g} = d \frac{\Delta g}{g} \Rightarrow d = \Delta d \frac{g}{\Delta g}$$

Substitute numerical values and evaluate d:

$$d = \left(10^{-3}\text{ m}\right)\left(10^{12}\right) = \boxed{10^9\text{ m}}$$

Gravitational Potential Energy

46 • (*a*) If we take the potential energy of a 100-kg object and Earth to be zero when the two are separated by an infinite distance, what is the potential energy when the object is at the surface of Earth? (*b*) Find the potential energy of the same object at a height above Earth's surface equal to Earth's radius. (*c*) Find the escape speed for a body projected from this height.

Picture the Problem Choosing the zero of gravitational potential energy to be at infinite separation yields, as the potential energy of a two-body system in which the objects are separated by a distance r, $U(r) = -GMm/r$, where M and m are the masses of the two bodies. In order for an object to just escape a gravitational field from a particular location, it must have enough kinetic energy so that its total energy is zero.

(*a*) Letting $U(\infty) = 0$, express the gravitational potential energy of the Earth-object system:

$$U(r) = -\frac{GM_\text{E} m}{r} \qquad (1)$$

Substitute for GM_E and simplify to obtain:

$$U\left(R_\text{E}\right) = -\frac{GM_\text{E} m}{R_\text{E}} = -\frac{gR_\text{E}^2 m}{R_\text{E}} = -mgR_\text{E}$$

Substitute numerical values and evaluate $U(R_\text{E})$:

$$U\left(R_\text{E}\right) = -(100\text{ kg})(9.81\text{ N/kg})\left(6.37\times10^6\text{ m}\right) = \boxed{-6.25\times10^9\text{ J}}$$

(*b*) Evaluate equation (1) with $r = 2R_\text{E}$:

$$U\left(2R_\text{E}\right) = -\frac{GM_\text{E} m}{2R_\text{E}} = -\frac{gR_\text{E}^2 m}{2R_\text{E}} = -\tfrac{1}{2} mgR_\text{E}$$

Substitute numerical values and evaluate $U(2R_E)$:

$$U(2R_E) = -\tfrac{1}{2}(100\,\text{kg})(9.81\,\text{N/kg})(6.37\times10^6\,\text{m}) = -3.124\times10^9\,\text{J} = \boxed{-3.12\times10^9\,\text{J}}$$

(*c*) Express the condition that an object must satisfy in order to escape from Earth's gravitational field from a height R_E above its surface:

$$K_e(2R_E) + U(2R_E) = 0$$

or

$$\tfrac{1}{2}mv_e^2 + U(2R_E) = 0$$

Solving for v_e yields:

$$v_e = \sqrt{\frac{-2U(2R_E)}{m}}$$

Substitute numerical values and evaluate v_e:

$$v_e = \sqrt{\frac{-2(-3.124\times10^9\,\text{J})}{100\,\text{kg}}} = \boxed{7.90\,\text{km/s}}$$

47 • [SSM] Knowing that the acceleration of gravity on the moon is 0.166 times that on Earth and that the moon's radius is $0.273R_E$, find the escape speed for a projectile leaving the surface of the moon.

Picture the Problem The escape speed from the moon is given by $v_{e,m} = \sqrt{2GM_m/R_m}$, where M_m and R_m represent the mass and radius of the moon, respectively.

Express the escape speed from the moon:

$$v_{e.m} = \sqrt{\frac{2GM_m}{R_m}} = \sqrt{2g_mR_m}$$

Because $g_m = 0.166g_E$ and $R_m = 0.273R_E$:

$$v_{e.m} = \sqrt{2(0.166g_E)(0.273R_E)}$$

Substitute numerical values and evaluate $v_{e,m}$:

$$v_{e.m} = \sqrt{2(0.166)(9.81\,\text{m/s}^2)(0.273)(6.371\times10^6\,\text{m})} = \boxed{2.38\,\text{km/s}}$$

48 •• What initial speed would a particle need to be given at the surface of Earth if it is to have a final speed that is equal to its escape speed when it is very far from Earth? Neglect any effects due to air resistance.

Picture the Problem Let the zero of gravitational potential energy be at infinity, *m* represent the mass of the particle, and the subscript E refer to Earth. When the particle is very far from Earth, the gravitational potential energy of the

Earth-particle system is zero. We'll use conservation of energy to relate the initial potential and kinetic energies of the particle-Earth system to the final kinetic energy of the particle.

Use conservation of energy to relate the initial energy of the system to its energy when the particle is very far away:

$$K_f - K_i + U_f - U_i = 0$$

or, because $U_f = 0$,

$$K(\infty) - K(R_E) - U(R_E) = 0 \qquad (1)$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv_\infty^2 - \tfrac{1}{2}mv_i^2 + \frac{GM_E m}{R_E} = 0$$

or, because $GM_E = gR_E^2$,

$$\tfrac{1}{2}mv_\infty^2 - \tfrac{1}{2}mv_i^2 + mgR_E = 0$$

Solving for v_i yields:

$$v_i = \sqrt{v_\infty^2 + 2gR_E}$$

Substitute numerical values and evaluate v_i:

$$v_i = \sqrt{(11.2\times10^3\ \text{m/s})^2 + 2(9.81\,\text{m/s}^2)(6.37\times10^6\ \text{m})} = \boxed{15.8\,\text{km/s}}$$

49 •• While trying to work out its budget for the next fiscal year, NASA wants to report to the nation a rough estimate of the cost (per kilogram) of launching a modern satellite into near-Earth orbit. You are chosen for this task, because you know physics and accounting. (*a*) Determine the energy, in kW·h, necessary to place 1.0-kg object in low-Earth orbit. In low-Earth orbit, the height of the object above the surface of Earth is much smaller than Earth's radius. Take the orbital height to be 300 km. (*b*) If this energy can be obtained at a typical electrical energy rate of \$0.15/kW·h, what is the minimum cost of launching a 400-kg satellite into low-Earth orbit? Neglect any effects due to air resistance.

Picture the Problem We can use the expression for the total energy of a satellite to find the energy required to place it in a low-Earth orbit.

(*a*) The total energy of a satellite in a low-Earth orbit is given by:

$$E = K + U_g = \tfrac{1}{2}U_g$$

Substituting for U_g yields:

$$E = -\frac{GM_{\text{Earth}}m_{\text{satellite}}}{2r}$$

where r is the orbital radius and the minus sign indicates the satellite is bound to Earth.

For a near-Earth orbit, $r \approx R_{\text{Earth}}$ and the amount of energy required to place the satellite in orbit becomes:

$$E = \frac{GM_{\text{Earth}}m_{\text{satellite}}}{2R_{\text{Earth}}}$$

Substitute numerical values and evaluate E:

$$E = -\frac{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.98\times10^{24}\ \text{kg}\right)\left(1.0\ \text{kg}\right)}{2\left(6.37\times10^{6}\ \text{m}\right)}$$

$$= 31.31\ \text{MJ}\times\frac{1\,\text{kW}\cdot\text{h}}{3.6\,\text{MJ}} = \boxed{8.7\,\text{kW}\cdot\text{h}}$$

(*b*) Express the cost of this project in terms of the mass of the satellite:

$$\text{Cost} = \text{rate}\times\frac{\text{required energy}}{\text{kg}}\times m_{\text{satellite}}$$

Substitute numerical values and find the cost:

$$\text{Cost} = \frac{\$0.15}{\text{kW}\cdot\text{h}}\times\frac{8.7\,\text{kW}\cdot\text{h}}{\text{kg}}\left(400\,\text{kg}\right) \approx \boxed{\$500}$$

50 •• The science fiction writer Robert Heinlein once said, "If you can get into orbit, then you're halfway to anywhere." Justify this statement by comparing the minimum energy needed to place a satellite into low Earth orbit (h = 400 km) to that needed to set it completely free from the bonds of Earth's gravity. Neglect any effects of air resistance.

Picture the Problem We'll consider a rocket of mass m which is initially on the surface of Earth (mass M and radius R) and compare the kinetic energy needed to get the rocket to its escape speed with its kinetic energy in a low circular orbit around Earth. We can use conservation of energy to find the escape kinetic energy and Newton's law of gravity to derive an expression for the low-Earth orbit kinetic energy.

Apply conservation of energy to relate the initial energy of the rocket to its escape kinetic energy:

$$K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} = 0$$

Letting the zero of gravitational potential energy be at infinity we have $U_{\text{f}} = K_{\text{f}} = 0$ and:

$$-K_{\text{i}} - U_{\text{i}} = 0$$

or

$$K_{\text{e}} = -U_{\text{i}} = \frac{GMm}{R}$$

Apply Newton's law of gravity to the rocket in orbit at the surface of Earth to obtain:

$$\frac{GMm}{R^2} = m\frac{v^2}{R}$$

Rewrite this equation to express the low-Earth orbit kinetic energy K_o of the rocket:

$$K_o = \tfrac{1}{2}mv^2 = \frac{GMm}{2R}$$

Express the ratio of K_o to K_e and simplify to obtain:

$$\frac{K_o}{K_e} = \frac{\frac{GMm}{2R}}{\frac{GMm}{R}} = \frac{1}{2}$$

Solving for K_e yields:

$K_e = \boxed{2K_o}$ as asserted by Heinlein.

51 •• **[SSM]** An object is dropped from rest from a height of 4.0×10^6 m above the surface of Earth. If there is no air resistance, what is its speed when it strikes Earth?

Picture the Problem Let the zero of gravitational potential energy be at infinity and let m represent the mass of the object. We'll use conservation of energy to relate the initial potential energy of the object-Earth system to the final potential and kinetic energies.

Use conservation of energy to relate the initial potential energy of the system to its energy as the object is about to strike Earth:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K(R_E) + U(R_E) - U(R_E + h) = 0 \qquad (1)$$

where h is the initial height above Earth's surface.

Express the potential energy of the object-Earth system when the object is at a distance r from the surface of Earth:

$$U(r) = -\frac{GM_E m}{r}$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv^2 - \frac{GM_E m}{R_E} + \frac{GM_E m}{R_E + h} = 0$$

Solving for v yields:

$$v=\sqrt{2\left(\frac{GM_E}{R_E}-\frac{GM_E}{R_E+h}\right)}$$
$$=\sqrt{2gR_E\left(\frac{h}{R_E+h}\right)}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{\frac{2(9.81\,\text{m/s}^2)(6.37\times10^6\,\text{m})(4.0\times10^6\,\text{m})}{6.37\times10^6\,\text{m}+4.0\times10^6\,\text{m}}}=\boxed{6.9\,\text{km/s}}$$

52 •• An object is projected straight upward from the surface of Earth with an initial speed of 4.0 km/s. What is the maximum height it reaches?

Picture the Problem Let the zero of gravitational potential energy be at infinity, m represent the mass of the object, and h the maximum height reached by the object. We'll use conservation of energy to relate the initial potential and kinetic energies of the object-Earth system to the final potential energy.

Use conservation of energy to relate the initial potential energy of the system to its energy as the object is at its maximum height:

$$K_f-K_i+U_f-U_i=0$$

or, because $K_f = 0$,

$$K(R_E)+U(R_E)-U(R_E+h)=0 \quad (1)$$

where h is the maximum height above Earth's surface.

Express the potential energy of the object-Earth system when the object is at a distance r from the surface of Earth:

$$U(r)=-\frac{GM_Em}{r}$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv^2-\frac{GM_Em}{R_E}+\frac{GM_Em}{R_E+h}=0$$

Solving for h yields:

$$h=\frac{R_E}{\dfrac{2gR_E}{v^2}-1}$$

Substitute numerical values and evaluate h:

$$h = \frac{6.37\times10^{6}\text{ m}}{\dfrac{2\left(9.81\text{ m/s}^2\right)\left(6.37\times10^{6}\text{ m}\right)}{\left(4.0\times10^{3}\text{ m/s}\right)^2}-1} = \boxed{9.4\times10^{5}\text{ m}}$$

53 •• A particle is projected from the surface of Earth with a speed twice the escape speed. When it is very far from Earth, what is its speed?

Picture the Problem Let the zero of gravitational potential energy be at infinity, m represent the mass of the particle, and the subscript E refer to Earth. When the particle is very far from Earth, the gravitational potential energy of Earth-particle system will be zero. We'll use conservation of energy to relate the initial potential and kinetic energies of the particle-Earth system to the final kinetic energy of the particle.

Use conservation of energy to relate the initial energy of the system to its energy when the particle is very far from Earth:

$$K_f - K_i + U_f - U_i = 0$$

or, because $U_f = 0$,

$$K(\infty) - K(R_E) - U(R_E) = 0 \qquad (1)$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv_\infty^2 - \tfrac{1}{2}m(2v_e)^2 + \frac{GM_E m}{R_E} = 0$$

or, because $GM_E = gR_E^2$,

$$\tfrac{1}{2}mv_\infty^2 - 2mv_e^2 + mgR_E = 0$$

Solving for v_∞ yields:

$$v_\infty = \sqrt{2\left(2v_e^2 - gR_E\right)}$$

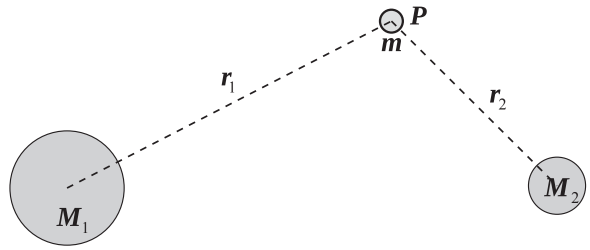
Substitute numerical values and evaluate v_∞:

$$v_\infty = \sqrt{2\left[2\left(11.2\times10^{3}\text{ m/s}\right)^2 - \left(9.81\text{ m/s}^2\right)\left(6.37\times10^{6}\text{ m}\right)\right]} = \boxed{19.4\text{ km/s}}$$

54 ••• When we calculate escape speeds, we usually do so with the assumption that the body from which we are calculating escape speed is isolated. This is, of course, generally not true in the Solar system. Show that the escape speed at a point near a system that consists of two massive spherical bodies is equal to the square root of the sum of the squares of the escape speeds from each of the two bodies considered individually.

Picture the Problem The pictorial representation shows the two massive objects from which the object (whose mass is m), located at point P, is to escape. This

object will have escaped the gravitational fields of the two massive objects provided, when its gravitational potential energy has become zero, its kinetic energy will also be zero.



Express the total energy of the system consisting of the two massive objects and the object whose mass is m:

$$E = \tfrac{1}{2}mv^2 - \frac{GM_1 m}{r_1} - \frac{GM_2 m}{r_2}$$

When the object whose mass is m has escaped, $E = 0$ and:

$$0 = \tfrac{1}{2}mv_e^2 - \frac{GM_1 m}{r_1} - \frac{GM_2 m}{r_2}$$

Solving for v_e yields:

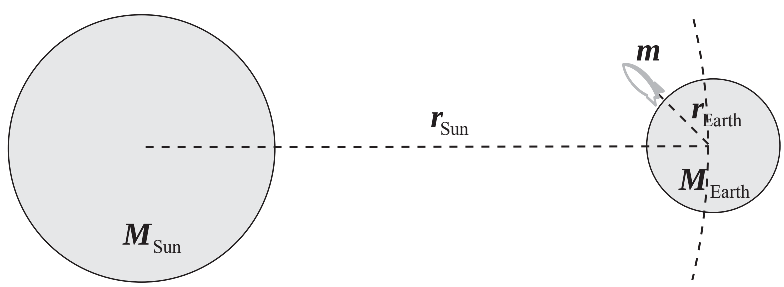
$$v_e^2 = \frac{2GM_1}{r_1} + \frac{2GM_2}{r_2}$$

The terms on the right-hand side of the equation are the squares of the escape speeds from the objects whose masses are M_1 and M_2. Hence;

$$v_e^2 = \boxed{v_{e,1}^2 + v_{e,2}^2}$$

55 ••• Calculate the minimum necessary speed, relative to Earth, for a projectile launched from the surface of Earth to escape the solar system. The answer will depend on the direction of the launch. Explain the choice of direction you'd make for the direction of the launch in order to minimize the necessary launch speed relative to Earth. Neglect Earth's rotational motion and any effects due to air resistance.

Picture the Problem The pictorial representation summarizes the initial positions of the Sun, Earth, and rocket.



From Problem 54, the escape speed from the Earth-Sun system is given by:

$$v_e^2 = \frac{2GM_{Earth}}{r_{Earth}} + \frac{2GM_{Sun}}{r_{Sun}}$$

Solving for v_e yields:

$$v_e = \sqrt{2G\left(\frac{M_{Earth}}{r_{Earth}} + \frac{M_{Sun}}{r_{Sun}}\right)}$$

Substitute numerical values and evaluate v_e:

$$v_e = \sqrt{2\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(\frac{5.98\times10^{24}\ \text{kg}}{6.37\times10^{6}\ \text{m}} + \frac{1.99\times10^{30}\ \text{kg}}{1.496\times10^{11}\ \text{m}}\right)} = 43.6\ \text{km/s}$$

What we have just calculated is the escape speed from Earth's surface, at Earth's orbit. Because the launch will take place from a moving Earth, we need to consider Earth's motion and use it to our advantage. If we launch at sunrise from the Equator, zenith will be pointed directly along the direction of motion of Earth and the required launch speed will be the *minimum* launch speed.

Express the minimum launch speed in terms of the escape speed calculated above and Earth's orbital speed:

$$v_{min} = v_e - v_{orbital}$$

The orbital speed of Earth is given by:

$$v_{orbital} = \frac{2\pi\, r_{orbital}}{T_{orbital}}$$

Substituting for $v_{orbital}$ yields:

$$v_{min} = v_e - \frac{2\pi\, r_{orbital}}{T_{orbital}}$$

Substitute numerical values and evaluate v_{min}:

$$v_{min} = 43.6\ \text{km/s} - \frac{2\pi\left(1.496\times10^{11}\ \text{m}\right)}{1\ \text{y}\times\frac{365.24\ \text{d}}{\text{y}}\times\frac{8.64\times10^{4}\ \text{s}}{\text{d}}} = \boxed{13.8\ \text{km/s}}$$

56 ••• An object is projected vertically from the surface of Earth at less than the escape speed. Show that the maximum height reached by the object is $H = R_E H'/(R_E - H')$, where H' is the height that it would reach if the gravitational field were constant. Neglect any effects due to air resistance.

Picture the Problem Let m represent the mass of the body that is projected vertically from the surface of Earth. We'll begin by using conservation of energy under the assumption that the gravitational field is constant to determine H'. We'll apply conservation of energy a second time, with the zero of gravitational potential energy at infinity, to express H. Finally, we'll solve these two equations simultaneously to express H in terms of H'.

Assuming the gravitational field to be constant and letting the zero of potential energy be at the surface of Earth, apply conservation of mechanical energy to relate the initial kinetic energy and the final potential energy of the object-Earth system:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}mv^2 + mgH' = 0 \Rightarrow H' = \frac{v^2}{2g} \quad (1)$$

Letting the zero of gravitational potential energy be at infinity, use conservation of mechanical energy to relate the initial kinetic energy and the final potential energy of the object-Earth system:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-K_i + U_f - U_i = 0$$

Substitute for K_i, U_f, and U_i and simplify to obtain:

$$-\tfrac{1}{2}mv^2 - \frac{GMm}{R_E + H} + \frac{GMm}{R_E} = 0$$

or

$$-\tfrac{1}{2}v^2 - \frac{gR_E^2}{R_E + H} + \frac{gR_E^2}{R_E} = 0$$

Solving for v^2 yields:

$$v^2 = 2gR_E^2\left(\frac{1}{R_E} - \frac{1}{R_E + H}\right) = 2gR_E\left(\frac{H}{R_E + H}\right)$$

Substitute in equation (1) to obtain:

$$H' = R_E\left(\frac{H}{R_E + H}\right) \Rightarrow H = \boxed{\frac{H'R_E}{R_E - H'}}$$

Gravitational Orbits

57 •• A 100-kg spacecraft is in a circular orbit about Earth at a height $h = 2R_E$. (*a*) What is the orbital period of the spacecraft? (*b*) What is the spacecraft's kinetic energy? (*c*) Express the angular momentum L of the spacecraft about the center of Earth in terms of its kinetic energy K and find the numerical value of L.

Picture the Problem We can use its definition to express the period of the spacecraft's motion and apply Newton's second law to the spacecraft to determine its orbital speed. We can then use this orbital speed to calculate the kinetic energy of the spacecraft. We can relate the spacecraft's angular momentum to its kinetic energy and moment of inertia.

(*a*) Express the period of the spacecraft's orbit about Earth:

$$T = \frac{2\pi R}{v} = \frac{2\pi(3R_E)}{v} = \frac{6\pi R_E}{v}$$

where v is the orbital speed of the spacecraft.

Use Newton's second law to relate the gravitational force acting on the spacecraft to its orbital speed:

$$F_{\text{radial}} = \frac{GM_E m}{(3R_E)^2} = m\frac{v^2}{3R_E} \Rightarrow v = \sqrt{\frac{gR_E}{3}}$$

Substitute for v in our expression for T to obtain:

$$T = 6\sqrt{3}\pi\sqrt{\frac{R_E}{g}}$$

Substitute numerical values and evaluate T:

$$T = 6\sqrt{3}\pi\sqrt{\frac{6.37\times10^6\ \text{m}}{9.81\ \text{m/s}^2}}$$

$$= 2.631\times10^4\ \text{s}\times\frac{1\ \text{h}}{3600\ \text{s}} = \boxed{7.31\ \text{h}}$$

(*b*) Using its definition, express the spacecraft's kinetic energy:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\tfrac{1}{3}gR_E\right)$$

Substitute numerical values and evaluate K:

$$K = \tfrac{1}{6}(100\ \text{kg})(9.81\ \text{m/s}^2)(6.37\times10^6\ \text{m})$$

$$= 1.041\ \text{GJ} = \boxed{1.04\ \text{GJ}}$$

(*c*) Express the kinetic energy of the spacecraft in terms of its angular momentum:

$$K = \frac{L^2}{2I} \Rightarrow L = \sqrt{2IK}$$

Express the moment of inertia of the spacecraft with respect to an axis through the center of Earth:

$$I = m(3R_E)^2 = 9mR_E^2$$

Substitute for I in the expression for L and simplify to obtain:

$$L = \sqrt{2(9mR_E^2)K} = 3R_E\sqrt{2mK}$$

Substitute numerical values and evaluate L:

$$L = 3(6.37\times10^6\ \text{m})\sqrt{2(100\ \text{kg})(1.041\times10^9\ \text{J})} = \boxed{8.72\times10^{12}\ \text{J}\cdot\text{s}}$$

58 •• The orbital period of the moon is 27.3 d, the average center-to-center distance between the moon and Earth is 3.82×10^8 m, the length of a year is 365.24 d, and the average center-to-center distance between Earth and the Sun is 1.50×10^{11} m. Use this data to estimate the ratio of the mass of the Sun to the mass of Earth. Compare this to the measured ratio of 3.33×10^5. List some neglected factors that might account for any discrepancy.

Picture the Problem We can use Kepler's third law to relate the periods of the moon and Earth, in their orbits about Earth and the Sun, to their mean center-to-center distances from the objects about which they orbit. We can solve these equations for the masses of the Sun and Earth and then divide one by the other to establish a value for the ratio of the mass of the Sun to the mass of Earth.

Using Kepler's third law, relate the period of the moon to its mean distance from Earth:

$$T_m^2 = \frac{4\pi^2}{GM_E}r_m^3 \qquad (1)$$

where r_m is the distance between the centers of Earth and the moon.

Using Kepler's third law, relate the period of Earth to its mean distance from the Sun:

$$T_E^2 = \frac{4\pi^2}{GM_s}r_E^3 \qquad (2)$$

where r_E is the distance between the centers of Earth and the Sun.

Solve equation (1) for M_E:

$$M_E = \frac{4\pi^2}{GT_m^2}r_m^3 \qquad (3)$$

Solve equation (2) for M_s:

$$M_s = \frac{4\pi^2}{GT_E^2}r_E^3 \qquad (4)$$

Divide equation (4) by equation (3) and simplify to obtain:

$$\frac{M_s}{M_E} = \left(\frac{r_E}{r_m}\right)^3\left(\frac{T_m}{T_E}\right)^2$$

Substitute numerical values and evaluate M_s/M_E:

$$\frac{M_s}{M_E} = \left(\frac{1.50\times10^{11}\text{ m}}{3.82\times10^{8}\text{ m}}\right)^3\left(\frac{27.3\text{d}}{365.24\text{d}}\right)^2 = \boxed{3.38\times10^5}$$

Express the difference between this value and the measured value of 3.33×10^5:

$$\%\,\text{diff} = \frac{3.38\times10^5 - 3.33\times10^5}{3.33\times10^5} = \boxed{1.50\,\%}$$

In this analysis we've neglected gravitational forces exerted by other planets and the Sun.

59 •• [SSM] Many satellites orbit Earth with maximum altitudes of 1000 km or less. *Geosynchronous* satellites, however, orbit at an altitude of 35 790 km above Earth's surface. How much more energy is required to launch a 500-kg satellite into a geosynchronous orbit than into an orbit 1000 km above the surface of Earth?

Picture the Problem We can express the energy difference between these two orbits in terms of the total energy of a satellite at each elevation. The application of Newton's second law to the force acting on a satellite will allow us to express the total energy of each satellite as a function of its mass, the radius of Earth, and its orbital radius.

Express the energy difference:

$$\Delta E = E_{\text{geo}} - E_{1000} \qquad (1)$$

Express the total energy of an orbiting satellite:

$$E_{\text{tot}} = K + U = \tfrac{1}{2}mv^2 - \frac{GM_E m}{R} \qquad (2)$$

where R is the orbital radius.

Apply Newton's second law to a satellite to relate the gravitational force to the orbital speed:

$$F_{\text{radial}} = \frac{GM_E m}{R^2} = m\frac{v^2}{R}$$

Solving for v^2 yields:

$$v^2 = \frac{gR_E^2}{R}$$

Substitute in equation (2) to obtain:

$$E_{\text{tot}} = \tfrac{1}{2}m\frac{gR_E^2}{R} - \frac{gR_E^2 m}{R} = -\frac{mgR_E^2}{2R}$$

Substituting in equation (1) and simplifying yields:

$$\Delta E = -\frac{mgR_E^2}{2R_{geo}} + \frac{mgR_E^2}{2R_{1000}} = \frac{mgR_E^2}{2}\left(\frac{1}{R_{1000}} - \frac{1}{R_{geo}}\right)$$
$$= \frac{mgR_E^2}{2}\left(\frac{1}{R_E + 1000\,\text{km}} - \frac{1}{R_E + 35\ 790\,\text{km}}\right)$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \frac{(500\,\text{kg})(9.81\,\text{N/kg})(6.37\times10^6\,\text{m})^2}{2}\left(\frac{1}{7.37\times10^6\,\text{m}} - \frac{1}{4.22\times107\,\text{m}}\right) = \boxed{11.1\,\text{GJ}}$$

60 ••• The idea of a spaceport orbiting Earth is an attractive proposition for launching probes and/or manned missions to the outer planets of the Solar System. Suppose such a ″platform″ has been constructed, and orbits Earth at a distance of 450 km above Earth's surface. Your research team is launching a lunar probe into an orbit that has its perigee at the spaceport's orbital radius, and its apogee at the moon's orbital radius. (*a*) To launch the probe successfully, first determine the orbital speed for the platform. (*b*) Next, determine the necessary speed relative to the platform that is necessary to launch the probe so that it attains the desired orbit. Assume that any effects due to the gravitational pull of the moon on the probe are negligible. In addition, assume that the launch takes place in a negligible amount of time. (*c*) You have the probe designed to radio back when it has reached apogee. How long after launch should you expect to receive this signal from the probe (neglect the second or so delay for the transit time of the signal back to the platform)?

Picture the Problem We can use the fact that the kinetic energy of the orbiting platform equals half its gravitational potential energy to find the orbital speed of the platform. In Part (*b*), the required launch speed is the difference between the speed of the probe at perigee and the orbital speed of the platform. To find the speed of the probe at perigee, we can use conservation of mechanical energy and Kepler's law of equal areas. Finally, in Part (*c*), we can use Kepler's third law to find the time after launch that you would expect to receive a signal from the probe announcing that it had reached apogee.

(*a*) The kinetic energy of the orbiting platform equals half its gravitational potential energy:

$$K = \frac{1}{2}U_g$$

Substitute for K and U to obtain:

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GM_{\text{Earth}}m}{r}$$

where m is the mass of the platform and r is the distance from the center of Earth to the platform.

Solving for v yields:

$$v = \sqrt{\frac{GM_{\text{Earth}}}{r}}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}{6.37\times10^6\ \text{m} + 0.450\times10^6\ \text{m}}} = \boxed{7.65\ \text{km/s}}$$

(*b*) The required launch speed is the difference between the speed of the probe at perigee and the orbital speed of the platform:

$$v_{\text{rel to platform}} = v_{\text{p}} - v \quad (1)$$

Here the platform's orbital radius will be the probe's orbit's perigee; the speed of launch will point in the direction the platform is moving at launch time. Neglecting everything in the universe but Earth and probe, apply conservation of energy from perigee to apogee to obtain:

$$\tfrac{1}{2}mv_{\text{p}}^2 - \tfrac{1}{2}mv_{\text{a}}^2 - \frac{GmM_{\text{Earth}}}{r_{\text{p}}} - \left(-\frac{GmM_{\text{Earth}}}{r_{\text{a}}}\right) = 0$$

Simplify this expression to obtain:

$$v_{\text{p}}^2\left[1-\left(\frac{v_{\text{a}}}{v_{\text{p}}}\right)\right] = 2GM_{\text{Earth}}\left(\frac{1}{r_{\text{p}}} - \frac{1}{r_{\text{a}}}\right) \quad (2)$$

Applying Kepler's law of equal areas between perigee and apogee yields:

$$v_{\text{p}}r_{\text{p}} = v_{\text{a}}r_{\text{a}} \Rightarrow v_{\text{a}} = \frac{r_{\text{p}}}{r_{\text{a}}}v_{\text{p}}$$

Substituting for v_{a} in equation (2) yields:

$$v_{\text{p}}^2\left[1-\left(\frac{r_{\text{p}}}{r_{\text{a}}}\right)\right] = 2GM_{\text{Earth}}\left(\frac{1}{r_{\text{p}}} - \frac{1}{r_{\text{a}}}\right) \quad (3)$$

where

$$r_{\text{p}} = R_{\text{Earth}} + h = 6370\ \text{km} + 450\ \text{km}$$
$$= 6820\ \text{km}$$

and

$$r_{\text{a}} = 3.84\times10^5\ \text{km}$$

Because $r_p << r_a$, equation (3) becomes:

$$v_p^2 \approx 2GM_{\text{Earth}}\left(\frac{1}{r_p}\right) = \frac{2GM_{\text{Earth}}}{r_p}$$

Solving for v_p yields:

$$v_p \approx \sqrt{\frac{2GM_{\text{Earth}}}{r_p}}$$

Substitute numerical values and evaluate v_p:

$$v_p \approx \sqrt{\frac{2(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}{6820\ \text{km}}} = 10.8\ \text{km/s}$$

Substitute numerical values in equation (1) and evaluate $v_{\text{rel to platform}}$:

$$v_{\text{rel to platform}} = 10.8\ \text{km/s} - 7.65\ \text{km/s} = \boxed{3.2\ \text{km/s}}$$

(*c*) The time after launch that you should expect to receive this signal from the probe is half the period of the probe's motion:

$$\Delta t = \tfrac{1}{2}T \qquad (4)$$

Apply Kepler's third law to the probe-Earth system to obtain:

$$T^2 = \frac{4\pi^2}{GM_{\text{Earth}}}a^3 \Rightarrow T = \sqrt{\frac{4\pi^2 a^3}{GM_{\text{Earth}}}}$$

where a is the semi-major axis.

Substitute for T in equation (4) to obtain:

$$\Delta t = \sqrt{\frac{\pi^2 a^3}{GM_{\text{Earth}}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{\pi^2\left[\frac{1}{2}\left(3.844\times10^8\ \text{m} + (6.37+0.45)\times10^6\ \text{m}\right)\right]^3}{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}} = 4.303\times10^5\ \text{s} \approx \boxed{5.0\ \text{d}}$$

The Gravitational Field ($\vec{g}$)

61 • A 3.0-kg space probe experiences a gravitational force of $12\ \text{N}\,\hat{i}$ as it passes through point P. What is the gravitational field at point P?

Picture the Problem The gravitational field at any point is defined by $\vec{g} = \vec{F}/m$.

Using its definition, express the gravitational field at a point in space:

$$\vec{g}=\frac{\vec{F}}{m}=\frac{(12\,\text{N})\hat{i}}{3.0\,\text{kg}}=\boxed{(4.0\,\text{N/kg})\hat{i}}$$

62 • The gravitational field at some point is given by $\vec{g}=2.5\times10^{-6}$ N/kg $\hat{j}$. What is the gravitational force on a 0.0040-kg object located at that point?

Picture the Problem The gravitational force acting on an object of mass m where the gravitational field is $\vec{g}$ is given by $\vec{F}=m\vec{g}$.

The gravitational force acting on the object is the product of the mass of the object and the gravitational field:

$$\vec{F}=m\vec{g}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{F}=(0.0040\,\text{kg})(2.5\times10^{-6}\,\text{N/kg})\hat{j}$$
$$=\boxed{(1.0\times10^{-8}\,N)\hat{j}}$$

63 •• **[SSM]** A point particle of mass m is on the x axis at $x = L$ and an identical point particle is on the y axis at $y = L$. (a) What is the direction of the gravitational field at the origin? (b) What is the magnitude of this field?

Picture the Problem We can use the definition of the gravitational field due to a point mass to find the x and y components of the field at the origin and then add these components to find the resultant field. We can find the magnitude of the field from its components using the Pythagorean theorem.

(a) The gravitational field at the origin is the sum of its x and y components:

$$\vec{g}=\vec{g}_x+\vec{g}_y \qquad (1)$$

Express the gravitational field due to the point mass at $x = L$:

$$\vec{g}_x=\frac{Gm}{L^2}\hat{i}$$

Express the gravitational field due to the point mass at $y = L$:

$$\vec{g}_y=\frac{Gm}{L^2}\hat{j}$$

Substitute in equation (1) to obtain:

$$\vec{g}=\vec{g}_x+\vec{g}_y=\frac{Gm}{L^2}\hat{i}+\frac{Gm}{L^2}\hat{j}\Rightarrow$$ the direction of the gravitational field is along a line at 45° above the $+x$ axis.

(b) The magnitude of $\vec{g}$ is given by:

$$|\vec{g}| = \sqrt{g_x^2 + g_y^2}$$

Substitute for g_x and g_y and simplify to obtain:

$$|\vec{g}| = \sqrt{\left(\frac{Gm}{L^2}\right)^2 + \left(\frac{Gm}{L^2}\right)^2} = \boxed{\sqrt{2}\,\frac{Gm}{L^2}}$$

64 •• Five objects, each of mass M, are equally spaced on the arc of a semicircle of radius R as in Figure 11-25. An object of mass m is located at the center of curvature of the arc. (a) If M is 3.0 kg, m is 2.0 kg, and R is 10 cm, what is the gravitational force on the particle of mass m due to the five objects? (b) If the object whose mass is m is removed, what is the gravitational field at the center of curvature of the arc?

Picture the Problem We can find the net force acting on m by superposition of the forces due to each of the objects arrayed on the circular arc. Once we have expressed the net force, we can find the gravitational field at the center of curvature from its definition. Choose a coordinate system in which the $+x$ direction is to the right and the $+y$ direction is upward.

(a) Express the net force acting on the object whose mass is m:

$$\vec{F} = F_x\hat{i} + F_y\hat{j} \qquad (1)$$

F_x is given by:

$$F_x = \frac{GMm}{R^2} - \frac{GMm}{R^2} + \frac{GMm}{R^2}\cos 45° - \frac{GMm}{R^2}\cos 45° = 0$$

F_y is given by:

$$F_y = \frac{GMm}{R^2} + \frac{GMm}{R^2}\sin 45° + \frac{GMm}{R^2}\sin 45° = \frac{GMm}{R^2}\left(2\sin 45° + 1\right)$$

Substitute numerical values and evaluate F_y:

$$F_y = \frac{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)(3.0\,\text{kg})}{(0.10\,\text{m})^2}(2.0\,\text{kg})\left(2\sin 45° + 1\right) = 9.67\times10^{-8}\ \text{N}$$

Substitute in equation (1) to obtain:

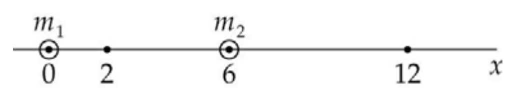
$$\vec{F} = \boxed{0\hat{i} + \left(9.7\times10^{-8}\ \text{N}\right)\hat{j}}$$

(*b*) Using its definition, express $\vec{g}$ at the center of curvature of the arc:

$$\vec{g} = \frac{\vec{F}}{m} = \frac{0\hat{i} + (9.67\times10^{-8}\ \text{N})\hat{j}}{2.0\,\text{kg}}$$
$$= \boxed{(4.8\times10^{-8}\ \text{N/kg})\hat{j}}$$

65 •• A point particle of mass $m_1 = 2.0$ kg is at the origin and a second point particle of mass $m_2 = 4.0$ kg is on the x axis at $x = 6.0$ m. Find the gravitational field $\vec{g}$ at (*a*) $x = 2.0$ m and (*b*) $x = 12$ m. (*c*) Find the point on the x axis for which $g = 0$.

Picture the Problem The configuration of point masses is shown to the right. The gravitational field at any point can be found by superimposing the fields due to each of the point masses.



(*a*) Express the gravitational field at $x = 2.0$ m as the sum of the fields due to the point masses m_1 and m_2:

$$\vec{g} = \vec{g}_1 + \vec{g}_2 \qquad (1)$$

Express $\vec{g}_1$ and $\vec{g}_2$:

$$\vec{g}_1 = -\frac{Gm_1}{x_1^2}\hat{i} \text{ and } \vec{g}_2 = \frac{Gm_2}{x_2^2}\hat{i}$$

Substitute in equation (1) to obtain:

$$\vec{g} = -\frac{Gm_1}{x_1^2}\hat{i} + \frac{Gm_2}{x_2^2}\hat{i} = -\frac{Gm_1}{x_1^2}\hat{i} + \frac{Gm_2}{(2x_1)^2}\hat{i} = -\frac{G}{x_1^2}\left(m_1 - \tfrac{1}{4}m_2\right)\hat{i}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{g} = -\frac{6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2}{(2.0\,\text{m})^2}\left[2.0\,\text{kg} - \tfrac{1}{4}(4.0\,\text{kg})\right]\hat{i} = \boxed{(-1.7\times10^{-11}\ \text{N/kg})\hat{i}}$$

(*b*) Express $\vec{g}_1$ and $\vec{g}_2$:

$$\vec{g}_1 = -\frac{Gm_1}{x_1^2}\hat{i} \text{ and } \vec{g}_2 = -\frac{Gm_2}{x_2^2}\hat{i}$$

Substitute in equation (1) and simplify to obtain:

$$\vec{g} = -\frac{Gm_1}{x_1^2}\hat{i} - \frac{Gm_2}{x_2^2}\hat{i} = -\frac{Gm_1}{(2x_2)^2}\hat{i} - \frac{Gm_2}{x_2^2}\hat{i} = -\frac{G}{x_2^2}\left(\tfrac{1}{4}m_1 + m_2\right)\hat{i}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{g} = -\frac{6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2}{(6.0\,\text{m})^2}\left[\tfrac{1}{4}(2.0\,\text{kg})+4.0\,\text{kg}\right]\hat{i} = \boxed{\left(-8.3\times10^{-12}\ \text{N/kg}\right)\hat{i}}$$

(c) Express the condition that $\vec{g} = 0$:

$$\frac{Gm_1}{x^2}-\frac{Gm_2}{(6.0-x)^2}=0$$

or

$$\frac{2.0}{\boldsymbol{x}^2}-\frac{4.0}{(6.0-\boldsymbol{x})^2}=0$$

Solve this quadratic equation to obtain:

$$x = 2.48\,\text{m} \text{ and } x = -14.5\,\text{m}$$

From the diagram it is clear that the physically meaningful root is the positive one at:

$$x = \boxed{2.5\,\text{m}}$$

66 •• Show that on the x axis, the maximum value of g for the field of Example 11-7 occurs at points $x = \pm a/\sqrt{2}$.

Picture the Problem To show that, on the x axis, the maximum value of g for the field of Example 11-7 occurs at the points $x = \pm a/\sqrt{2}$, we can differentiate g_x with respect to x and set the derivative equal to zero.

From Example 11-7:

$$g_x = -\frac{2GMx}{\left(x^2+a^2\right)^{3/2}}$$

Differentiate g_x with respect to x and set the derivative equal to zero to find extreme values:

$$\frac{dg_x}{dx} = -2GM\left[\left(x^2+a^2\right)^{-3/2} - 3x^2\left(x^2+a^2\right)^{-5/2}\right] = 0 \text{ for extrema.}$$

Solve for x to obtain:

$$x = \boxed{\pm\frac{a}{\sqrt{2}}}$$

Remarks: To establish that this value for *x* corresponds to a relative maximum, we need to either evaluate the second derivative of g_x at

$x = \pm a/\sqrt{2}$ or examine the graph of $|g_x|$ at $x = \pm a/\sqrt{2}$ for concavity downward.

67 ••• [SSM] A nonuniform thin rod of length L lies on the x axis. One end of the rod is at the origin, and the other end is at $x = L$. The rod's mass per unit length λ varies as $\lambda = Cx$, where C is a constant. (Thus, an element of the rod has mass $dm = \lambda\, dx$.) (*a*) What is the total mass of the rod? (*b*) Find the gravitational field due to the rod on the x axis at $x = x_0$, where $x_0 > L$.

Picture the Problem We can find the mass of the rod by integrating dm over its length. The gravitational field at $x_0 > L$ can be found by integrating $d\vec{g}$ at x_0 over the length of the rod.

(*a*) The total mass of the stick is given by:

$$M = \int_0^L \lambda\, dx$$

Substitute for λ and evaluate the integral to obtain"

$$M = C\int_0^L x dx = \boxed{\tfrac{1}{2}CL^2}$$

(*b*) Express the gravitational field due to an element of the stick of mass dm:

$$d\vec{g} = -\frac{Gdm}{(x_0 - x)^2}\hat{i} = -\frac{G\lambda\, dx}{(x_0 - x)^2}\hat{i} = -\frac{GCxdx}{(x_0 - x)^2}\hat{i}$$

Integrate this expression over the length of the stick to obtain:

$$\vec{g} = -GC\int_0^L \frac{xdx}{(x_0 - x)^2}\hat{i} = \boxed{\frac{2GM}{L^2}\left[\ln\left(\frac{x_0}{x_0 - L}\right) - \left(\frac{L}{x_0 - L}\right)\right]\hat{i}}$$

68 •• A uniform thin rod of mass M and length L lies on the positive x axis with one end at the origin. Consider an element of the rod of length dx, and mass dm, at point x, where $0 < x < L$. (*a*) Show that this element produces a gravitational field at a point x_0 on the x axis in the region $x_0 > L$ is given by $dg_x = -\dfrac{GM}{L(x_0 - x)^2}dx$. (*b*) Integrate this result over the length of the rod to find the total gravitational field at the point x_0 due to the rod. (*c*) Find the gravitational force on a point particle of mass m_0 at x_0. (*d*) Show that for $x_0 >> L$, the field of the rod approximates the field of a point particle of mass M at $x = 0$.

Picture the Problem The elements of the rod of mass dm and length dx produce a gravitational field at any point P located a distance $x_0 > L$ from the origin. We can calculate the total field by integrating the magnitude of the field due to dm from $x = 0$ to $x = L$.

(*a*) Express the gravitational field at P due to the element dm:

$$d\vec{g}_x = -\frac{Gdm}{r^2}\hat{i}$$

Relate dm to dx:

$$dm = \frac{M}{L}dx$$

Express the distance r between dm and point P in terms of x and x_0:

$$r = x_0 - x$$

Substitute these results to express $d\vec{g}_x$ in terms of x and x_0:

$$d\vec{g}_x = \boxed{\left\{-\frac{GM}{L(x_0 - x)^2}dx\right\}\hat{i}}$$

(*b*) Integrate from $x = 0$ to $x = L$ to find the total gravitational field at point P:

$$\vec{g}_x = -\frac{GM}{L}\int_0^L \frac{dx}{(x_0 - x)^2}\hat{i}$$

$$= \left\{-\frac{GM}{L}\left[\frac{1}{x_0 - x}\right]_0^L\right\}\hat{i}$$

$$= \boxed{-\frac{GM}{x_0(x_0 - L)}\hat{i}}$$

(*c*) Use the definition of gravitational field and the result from Part (*b*) to express $\vec{F}_g$ at $x = x_0$:

$$\vec{F}_g = m_0\vec{g} = \boxed{-\frac{GMm_0}{x_0(x_0 - L)}\hat{i}}$$

(*d*) Factor x_0 from the denominator of the expression for $\vec{g}_x$ to obtain:

$$\vec{g}_x = -\frac{GM}{x_0^2\left(1 - \frac{L}{x_0}\right)}\hat{i}$$

For $x_0 >> L$ the second term in parentheses is very small and:

$$\vec{g}_x \approx \boxed{-\frac{GM}{x_0^2}\hat{i}}$$

which is the gravitational field of a point mass M located at the origin.

The Gravitational Field ($\vec{g}$) due to Spherical Objects

69 • A uniform thin spherical shell has a radius of 2.0 m and a mass of 300 kg. What is the gravitational field at the following distances from the center of the shell: (*a*) 0.50 m, (*b*) 1.9 m, (*c*) 2.5 m?

Picture the Problem The gravitational field inside a spherical shell is zero and the field at the surface of and outside the shell is given by $g = GM/r^2$.

(*a*) Because 0.50 m < *R*: $g(0.50\text{ m}) = \boxed{0}$

(*b*) Because 1.9 m < *R*: $g(1.9\text{ m}) = \boxed{0}$

(*c*) Because 2.5 m > *R*:

$$g(2.5\text{ m}) = \frac{GM}{r^2} = \frac{\left(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2\right)(300\text{ kg})}{(2.5\text{ m})^2} = \boxed{3.2\times10^{-9}\text{ N/kg}}$$

70 • A uniform thin spherical shell has a radius of 2.00 m and a mass of 300 kg, and its center is located at the origin of a coordinate system. Another uniform thin spherical shell with a radius of 1.00 m and a mass of 150 kg is inside the larger shell with its center at 0.600 m on the *x* axis. What is the gravitational force of attraction between the two shells?

Determine the Concept The gravitational force is zero. The gravitational field inside the 2.00 m shell due to that shell is zero; therefore, it exerts no force on the 1.00 m-shell, and, by Newton's third law, that shell exerts no force on the larger shell.

71 •• **[SSM]** Two widely separated solid spheres, S_1 and S_2, each have radius *R* and mass *M*. Sphere S_1 is uniform, whereas the density of sphere S_2 is given by $\rho(r) = C/r$, where *r* is the distance from its center. If the gravitational field strength at the surface of S_1 is g_1, what is the gravitational field strength at the surface of S_2?

Picture the Problem The gravitational field strength at the surface of a sphere is given by $g = GM/R^2$, where *R* is the radius of the sphere and *M* is its mass.

Express the gravitational field strength on the surface of S_1: $g_1 = \dfrac{GM}{R^2}$

Express the gravitational field strength on the surface of S_2:

$$g_2 = \frac{GM}{R^2}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{g_2}{g_1} = \frac{\frac{GM}{R^2}}{\frac{GM}{R^2}} = 1 \Rightarrow \boxed{g_1 = g_2}$$

72 •• Two widely separated uniform solid spheres, S_1 and S_2, have equal masses but different radii, R_1 and R_2. If the gravitational field strength on the surface of S_1 is g_1, what is the gravitational field strength on the surface of S_2?

Picture the Problem The gravitational field strength at the surface of a sphere is given by $g = GM/R^2$, where R is the radius of the sphere and M is its mass.

Express the gravitational field strength on the surface of S_1:

$$g_1 = \frac{GM}{R_1^2}$$

Express the gravitational field strength on the surface of S_2:

$$g_2 = \frac{GM}{R_2^2}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{g_2}{g_1} = \frac{\frac{GM}{R_2^2}}{\frac{GM}{R_1^2}} = \frac{R_1^2}{R_2^2} \Rightarrow g_2 = \boxed{\frac{R_1^2}{R_2^2} g_1}$$

Remarks: The gravitational field strengths depend only on the masses and radii because the points of interest are outside spherically symmetric distributions of mass.

73 •• Two concentric uniform thin spherical shells have masses M_1 and M_2 and radii a and $2a$, as in Figure 11-26. What is the magnitude of the gravitational force on a point particle of mass m (not shown) located (*a*) a distance $3a$ from the center of the shells? (*b*) a distance $1.9a$ from the center of the shells? (*c*) a distance $0.9a$ from the center of the shells?

Picture the Problem The magnitude of the gravitational force is $F_g = mg$ where g inside a spherical shell is zero and outside is given by $g = GM/r^2$.

(*a*) The gravitational force on a particle of mass m is given by:

$$F_g = mg$$

At $r = 3a$, the masses of both spheres contribute to g:

$$F_g(3a) = m\frac{G(M_1 + M_2)}{(3a)^2} = \boxed{\frac{Gm(M_1 + M_2)}{9a^2}}$$

(*b*) At $r = 1.9a$, g due to M_2 is zero and:

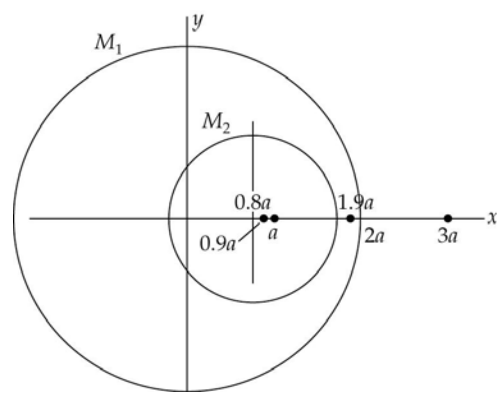
$$F_g(1.9a) = m\frac{GM_1}{(1.9a)^2} = \boxed{\frac{GmM_1}{3.61a^2}}$$

(*c*) At $r = 0.9a$, $g = 0$ and:

$$F_g(0.9a) = \boxed{0}$$

74 •• The inner spherical shell in Problem 73 is shifted so that its center is now on the x axis at $x = 0.8a$. What is the magnitude of the gravitational force on a particle of point mass m located on the x axis at (*a*) $x = 3a$, (*b*) $x = 1.9a$, (*c*) $x = 0.9a$?

Picture the Problem The configuration is shown on the right. The centers of the spheres are indicated by the center-lines. The x coordinates of the mass m for Parts (*a*), (*b*), and (*c*) are indicated along the x axis. The magnitude of the gravitational force is $F_g = mg$ where g inside a spherical shell is zero and outside is given by $g = \dfrac{GM}{r^2}$.



(*a*) Express the gravitational force acting on the object whose mass is m:

$$F_g = m(g_{1x} + g_{2x}) \qquad (1)$$

Find g_{1x} at $x = 3a$:

$$g_{1x}(3a) = \frac{GM_1}{(3a)^2} = \frac{GM_1}{9a^2}$$

Find g_{2x} at $x = 3a$:

$$g_{2x}(3a) = \frac{GM_2}{(3a - 0.8a)^2} = \frac{GM_2}{4.84a^2}$$

Substitute for $g_{1x}(3a)$ and $g_{2x}(3a)$ in equation (1) and simplify to obtain:

$$F(3a) = m\left(\frac{GM_1}{9a^2} + \frac{GM_2}{4.84a^2}\right)$$
$$= \boxed{\frac{Gm}{a^2}\left(\frac{M_1}{9} + \frac{M_2}{4.84}\right)}$$

(*b*) Find g_{2x} at $x = 1.9a$:

$$g_{2x}(1.9a) = \frac{GM_2}{(1.9a - 0.8a)^2}$$
$$= \frac{GM_2}{1.21a^2}$$

Find g_{1x} at $x = 1.9a$:

$$g_{1x}(1.9a) = 0$$

Substitute for $g_{1x}(1.9a)$ and $g_{2x}(1.9a)$ and simplify to obtain:

$$F(1.9a) = mg = \boxed{\frac{GmM_2}{1.21a^2}}$$

(*c*) At $x = 0.9a$, $g_{1x} = g_{2x} = 0$ and:

$$F(0.9a) = \boxed{0}$$

75 •• [SSM] Suppose you are standing on a spring scale in an elevator that is descending at constant speed in a mine shaft located on the equator. Model Earth as a homogeneous sphere. (*a*) Show that the force on you due to Earth's gravity alone is proportional to your distance from the center of the planet. (*b*) Assume that the mine shaft located on the equator and is vertical. Do not neglect Earth's rotational motion. Show that the reading on the spring scale is proportional to your distance from the center of the planet.

Picture the Problem There are two forces acting on you as you descend in the elevator and are at a distance *r* from the center of Earth; an upward normal force (F_N) exerted by the scale, and a downward gravitational force (mg) exerted by Earth. Because you are in equilibrium (you are descending at constant speed) under the influence of these forces, the normal force exerted by the scale is equal in magnitude to the gravitational force acting on you. We can use Newton's law of gravity to express this gravitational force.

(*a*) Express the force of gravity acting on you when you are a distance *r* from the center of Earth:

$$F_g = \frac{GM(r)m}{r^2} \qquad (1)$$

where $M(r)$ is the mass of Earth enclosed within the radius *r*: that is, loser to the center of Earth than your position.

Using the definition of density, express the density of Earth between you and the center of Earth and the density of Earth as a whole:

$$\rho = \frac{M(r)}{V(r)} = \frac{M(r)}{\frac{4}{3}\pi r^3}$$

The density of Earth is also given by:

$$\rho = \frac{M_\mathrm{E}}{V_\mathrm{E}} = \frac{M_\mathrm{E}}{\frac{4}{3}\pi R^3}$$

Equating these two expressions for ρ and solving for $M(r)$ yields:

$$M(r) = M_\mathrm{E}\left(\frac{r}{R}\right)^3$$

Substitute for $M(r)$ in equation (1) and simplify to obtain:

$$F_g = \frac{GM_\mathrm{E}\left(\frac{r}{R}\right)^3 m}{r^2} = \frac{GM_\mathrm{E}m}{R^2}\frac{r}{R} \qquad (2)$$

Apply Newton's law of gravity to yourself at the surface of Earth to obtain:

$$mg = \frac{GM_\mathrm{E}m}{R^2} \Rightarrow g = \frac{GM_\mathrm{E}}{R^2}$$

where g is the magnitude of the gravitational field at the surface of Earth.

Substitute for g in equation (2) to obtain:

$$F_g = \boxed{\left(\frac{mg}{R}\right)r}$$

That is, the force of gravity on you is proportional to your distance from the center of Earth.

(*b*) Apply Newton's second law to your body to obtain:

$$F_\mathrm{N} - mg\frac{r}{R} = -mr\omega^2$$

where the net force $\left(-mr\omega^2\right)$, directed toward the center of Earth, is the centripetal force acting on your body.

Solving for F_N yields:

$$F_\mathrm{N} = \left(\frac{mg}{R}\right)r - mr\omega^2$$

Note that this equation tells us that your effective weight increases linearly with distance from the center of Earth. However, due just to the effect of rotation, as you approach the center the centripetal force decreases linearly and, doing so, increases your effective weight.

76 •• Suppose Earth were a nonrotating uniform sphere. As a reward for earning the highest lab grade, your physics professor chooses your laboratory team to participate in a gravitational experiment at a deep mine on the equator. This mine has an elevator shaft going 15.0 km into Earth. Before making the measurement, you are asked to predict the decrease in the weight of a team member, who weighs 800 N at the surface of Earth, when she is at the bottom of the shaft. The density of Earth's crust actually increases with depth. Is your answer higher or lower than the actual experimental result?

Picture the Problem We can find the loss in weight of your team member at this depth by taking the difference between her weight at the surface of Earth and her weight at a depth d = 15.0 km. To find the gravitational field at depth d, we'll use its definition and the mass of Earth that is between the bottom of the shaft and the center of Earth. We'll assume (incorrectly) that the density of Earth is constant.

The loss in weight of a team member is given by:

$$\Delta w = w(R_E) - w(R) \quad (1)$$

The mass M inside $R = R_E - d$ is given by:

$$M = \rho V = \tfrac{4}{3}\rho\pi(R_E - d)^3$$

Relate the mass of Earth to its density and volume:

$$M_E = \rho V_E = \tfrac{4}{3}\rho\pi R_E^3$$

Divide the first of these equations by the second to obtain:

$$\frac{M}{M_E} = \frac{\tfrac{4}{3}\rho\pi(R_E - d)^3}{\tfrac{4}{3}\rho\pi R_E^3} = \frac{(R_E - d)^3}{R_E^3}$$

Solving for M yields:

$$M = M_E\frac{(R_E - d)^3}{R_E^3}$$

Express the gravitational field at $R = R_E - d$:

$$g = \frac{GM}{R^2} = \frac{GM_E(R_E - d)^3}{(R_E - d)^2 R_E^3} = \frac{GM_E(R_E - d)}{R_E^3} \quad (2)$$

Express the gravitational field at $R = R_E$:

$$g_E = \frac{GM_E}{R_E^2} \quad (3)$$

Divide equation (2) by equation (3) to obtain:

$$\frac{g}{g_E}=\frac{\dfrac{GM_E(R_E-d)}{R_E^3}}{\dfrac{GM_E}{R_E^2}}=\frac{R_E-d}{R_E}$$

Solving for g yields:

$$g=\frac{R_E-d}{R_E}g_E$$

The weight of the student at $R=R_E-d$ is given by:

$$w(R)=mg(R)=\frac{R_E-d}{R_E}mg_E$$
$$=\left(1-\frac{d}{\mathrm{R}_E}\right)mg_E$$

Substitute for $w(R)$ in equation (1) and simplify to obtain:

$$\Delta w=mg_E-\left(1-\frac{d}{\mathrm{R}_E}\right)mg_E=\frac{mg_E d}{R_E}$$

Substitute numerical values and evaluate Δw:

$$\Delta w=\frac{(800\,\mathrm{N})(15.0\,\mathrm{km})}{6370\,\mathrm{km}}=\boxed{1.88\,\mathrm{N}}$$

If Earth's crustal density actually increased with depth, this increase with depth would partially compensate for the decrease in the fraction of Earth's mass between a descending team member and the center of Earth; with the result that the loss in weight would be lower than the actual experimental result.

77 •• [SSM] A solid sphere of radius R has its center at the origin. It has a uniform mass density ρ_0, except that there is a spherical cavity in it of radius $r=\frac{1}{2}R$ centered at $x=\frac{1}{2}R$ as in Figure 11-27. Find the gravitational field at points on the x axis for $|x|>R$. *Hint: The cavity may be thought of as a sphere of mass $m=(4/3)\pi r^3\rho_0$ plus a sphere of "negative" mass –m.*

Picture the Problem We can use the hint to find the gravitational field along the x axis.

Using the hint, express $g(x)$:

$$g(x)=g_{\text{solid sphere}}+g_{\text{hollow sphere}}$$

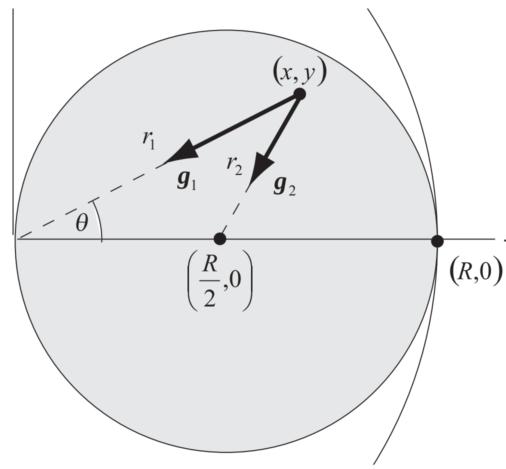
Substitute for $g_{\text{solid sphere}}$ and $g_{\text{hollow sphere}}$ and simplify to obtain:

$$g(x) = \frac{GM_{\text{solid sphere}}}{x^2} + \frac{GM_{\text{hollow sphere}}}{\left(x - \frac{1}{2}R\right)^2} = \frac{G\rho_0\left(\frac{4}{3}\pi R^3\right)}{x^2} + \frac{G\rho_0\left[-\frac{4}{3}\pi\left(\frac{1}{2}R\right)^3\right]}{\left(x - \frac{1}{2}R\right)^2}$$

$$= \boxed{G\left(\frac{4\pi\rho_0 R^3}{3}\right)\left[\frac{1}{x^2} - \frac{1}{8\left(x - \frac{1}{2}R\right)^2}\right]}$$

78 ••• For the sphere with the cavity in Problem 77, show that the gravitational field is uniform throughout the cavity, and find its magnitude and direction there.

Picture the Problem The diagram shows the portion of the solid sphere in which the hollow sphere is embedded. $\vec{g}_1$ is the field due to the solid sphere of radius R and density ρ_0 and $\vec{g}_2$ is the field due to the sphere of radius $\frac{1}{2}\boldsymbol{R}$ and negative density ρ_0 centered at $\frac{1}{2}\boldsymbol{R}$. We can find the resultant field by adding the x and y components of $\vec{g}_1$ and $\vec{g}_2$.



Use its definition to express $|\vec{g}_1|$:

$$|\vec{g}_1| = \frac{F_g}{m}$$

Substitute for the gravitational force to obtain:

$$|\vec{g}_1| = \frac{\frac{GMm}{r_1^2}}{m} = \frac{GM}{r_1^2}$$

Substituting the product of its density and volume for the mass of the sphere and simplifying yields:

$$|\vec{g}_1| = \frac{G\rho_0 V}{r_1^2} = \frac{4\pi\rho_0 r_1^3 G}{3r_1^2} = \frac{4\pi\rho_0 r_1 G}{3}$$

Find the x and y components of $\vec{g}_1$:

$$g_{1x} = -g_1 \cos\theta = -g_1\left(\frac{x}{r_1}\right) = -\frac{4\pi\rho_0 Gx}{3}$$

and

$$g_{1y} = -g_1 \sin\theta = -g_1\left(\frac{y}{r_1}\right) = -\frac{4\pi\rho_0 Gy}{3}$$

where the negative signs indicate that the field points inward.

Proceed similarly to express $\left|\vec{g}_2\right|$:

$$\left|\vec{\mathbf{g}}_2\right| = \frac{4\pi\rho_0 r_2 G}{3}$$

Express the x and y components of $\vec{g}_2$:

$$g_{2x} = g_2\left(\frac{x - \frac{1}{2}R}{r_2}\right) = \frac{4\pi\rho_0 G\left(x - \frac{1}{2}R\right)}{3}$$

and

$$g_{2y} = g_2\left(\frac{y}{r_2}\right) = \frac{4\pi\rho_0 Gy}{3}$$

Add the x components and simplify to obtain the x component of the resultant field:

$$\begin{aligned} g_x &= g_{1x} + g_{2x} \\ &= -\frac{4\pi\rho_0 Gx}{3} + \frac{4\pi\rho_0 G\left(x - \frac{1}{2}R\right)}{3} \\ &= -\frac{2\pi\rho_0 GR}{3} \end{aligned}$$

where the negative sign indicates that the field points inward.

Add the y components and simplify to obtain the y component of the resultant field:

$$\begin{aligned} g_y &= g_{1y} + g_{2y} \\ &= -\frac{4\pi\rho_0 Gy}{3} + \frac{4\pi\rho_0 Gy}{3} = 0 \end{aligned}$$

Express $\vec{g}$ in vector form:

$$\vec{g} = g_x\hat{i} + g_y\hat{j} = \boxed{\left(-\frac{2\pi\rho_0 GR}{3}\right)\hat{i} + 0\hat{j}}$$

The magnitude of $\vec{g}$ is:

$$\left|\vec{g}\right| = \sqrt{g_x^2 + g_y^2} = \boxed{\frac{2\pi\rho_0 GR}{3}}$$

79 ••• A straight, smooth tunnel is dug through a uniform spherical planet of mass density ρ_0. The tunnel passes through the center of the planet and is perpendicular to the planet's axis of rotation, which is fixed in space. The planet

rotates with a constant angular speed ω so objects in the tunnel have no apparent weight. Find the required angular speed of the planet ω.

Picture the Problem The gravitational field will exert an inward radial force on the objects in the tunnel. We can relate this force to the angular speed of the planet by using Newton's second law of motion.

Letting r be the distance from the objects to the center of the planet, use Newton's second law to relate the gravitational force acting on the objects to their angular speed:

$$F_{\text{net}} = F_{\text{g}} = mr\omega^2$$

or

$$mg = mr\omega^2 \Rightarrow \omega = \sqrt{\frac{g}{r}} \qquad (1)$$

Use its definition to express g:

$$g = \frac{F_{\text{g}}}{m}$$

Substitute for F_{g} to obtain:

$$g = \frac{F_{\text{g}}}{m} = \frac{\frac{GMm}{r^2}}{m} = \frac{GM}{r^2}$$

Substituting the product of its density and volume for the mass of planet and simplifying yields:

$$g = \frac{G\rho_0 V}{r^2} = \frac{4\pi\rho_0 r^3 G}{3r^2} = \frac{4\pi\rho_0 rG}{3}$$

Substituting for g in equation (1) and simplifying yields:

$$\omega = \sqrt{\frac{\frac{4\pi\rho_0 rG}{3}}{r}} = \boxed{\sqrt{\frac{4\pi\rho_0 G}{3}}}$$

80 ••• The density of a sphere is given by $\rho(r) = C/r$. The sphere has a radius of 5.0 m and a mass of 1.0×10^{11} kg. (*a*) Determine the constant C. (*b*) Obtain expressions for the gravitational field for the regions (1) $r > 5.0$ m and (2) $r < 5.0$ m.

Picture the Problem Because we're given the mass of the sphere, we can find C by expressing the mass of the sphere in terms of C. We can use its definition to find the gravitational field of the sphere both inside and outside its surface.

(*a*) Express the mass of a differential element of the sphere:

$$dm = \rho\, dV = \rho\left(4\pi r^2 dr\right)$$

Integrate to express the mass of the sphere in terms of C:

$$M = 4\pi C \int_0^{5.0\,\text{m}} r dr = (50\,\text{m}^2)\pi C$$

Solving for C yields:

$$C = \frac{M}{(50\,\text{m}^2)\pi}$$

Substitute numerical values and evaluate C:

$$C = \frac{1.0\times10^{11}\,\text{kg}}{(50\,\text{m}^2)\pi} = 6.37\times10^{8}\,\text{kg/m}^2$$

$$= \boxed{6.4\times10^{8}\,\text{kg/m}^2}$$

(b) Use its definition to express the gravitational field of the sphere at a distance from its center greater than its radius:

$$g = \frac{GM}{r^2}$$

(1) For $r > 5.0$ m:

$$g(r > 5.0\,\text{m}) = \frac{(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(1.0\times10^{11}\,\text{kg})}{r^2} = \boxed{\frac{6.7\,\text{N}\cdot\text{m}^2/\text{kg}}{r^2}}$$

Use its definition to express the gravitational field of the sphere at a distance from its center less than its radius:

$$g = G\frac{\int_0^r 4\pi r^2 \rho\, dr}{r^2} = G\frac{\int_0^r 4\pi r^2 \frac{C}{r}\, dr}{r^2}$$

$$= G\frac{4\pi C\int_0^r r\, dr}{r^2} = 2\pi GC$$

(2) For $r < 5.0$ m:

$$g(r < 5.0\,\text{m}) = 2\pi(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(6.37\times10^{8}\,\text{kg/m}^2) = \boxed{0.27\,\text{N/kg}}$$

81 ••• [SSM] A small diameter hole is drilled into the sphere of Problem 80 toward the center of the sphere to a depth of 2.0 m below the sphere's surface. A small mass is dropped from the surface into the hole. Determine the speed of the small mass as it strikes the bottom of the hole.

Picture the Problem We can use conservation of energy to relate the work done by the gravitational field to the speed of the small object as it strikes the bottom of the hole. Because we're given the mass of the sphere, we can find C by expressing the mass of the sphere in terms of C. We can then use the definition of

the gravitational field to find the gravitational field of the sphere inside its surface. The work done by the field equals the negative of the change in the potential energy of the system as the small object falls in the hole.

Use conservation of energy to relate the work done by the gravitational field to the speed of the small object as it strikes the bottom of the hole:

$$K_f - K_i + \Delta U = 0$$

or, because $K_i = 0$ and $W = -\Delta U$,

$$W = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2W}{m}} \qquad (1)$$

where v is the speed with which the object strikes the bottom of the hole and W is the work done by the gravitational field.

Express the mass of a differential element of the sphere:

$$dm = \rho\, dV = \rho\left(4\pi r^2 dr\right)$$

Integrate to express the mass of the sphere in terms of C:

$$M = 4\pi C \int_0^{5.0\,\text{m}} r dr = \left(50\,\text{m}^2\right)\pi C$$

Solving for C yields:

$$C = \frac{M}{\left(50\,\text{m}^2\right)\pi}$$

Substitute numerical values and evaluate C:

$$C = \frac{1.0\times10^{11}\ \text{kg}}{\left(50\,\text{m}^2\right)\pi} = 6.37\times10^{8}\ \text{kg/m}^2$$

Use its definition to express the gravitational field of the sphere at a distance from its center less than its radius:

$$g = \frac{F_g}{m} = \frac{GM}{r^2} = G\frac{\int_0^r 4\pi r^2 \rho\, dr}{r^2} = G\frac{\int_0^r 4\pi r^2 \frac{C}{r} dr}{r^2} = G\frac{4\pi C\int_0^r r\, dr}{r^2} = 2\pi GC$$

Express the work done on the small object by the gravitational force acting on it:

$$W = -\int_{5.0\,\text{m}}^{3.0\,\text{m}} mgdr = (2\,\text{m})mg$$

Substitute in equation (1) and simplify to obtain:

$$v = \sqrt{\frac{2(2.0\,\text{m})m(2\pi GC)}{m}} = \sqrt{(8.0\,\text{m})\pi GC}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{(8.0\,\text{m})\pi\left(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(6.37\times10^{8}\,\text{kg/m}^2\right)}=\boxed{1.0\,\text{m/s}}$$

82 ••• As a geologist for a mining company, you are working on a method for determining possible locations of underground ore deposits. Assume that where the company owns land the crust of Earth is 40.0 km thick and has a density of about 3000 kg/m^3. Suppose a spherical deposit of heavy metals with a density of 8000 kg/m^3 and radius of 1000 m was centered 2000 m below the surface. You propose to detect it by determining its affect on the local surface value of g. Find $\Delta g/g$ at the surface directly above this deposit, where Δg is the increase in the gravitational field due to the deposit.

Picture the Problem The spherical deposit of heavy metals will increase the gravitational field at the surface of Earth. We can express this increase in terms of the difference in densities of the deposit and Earth and then form the quotient $\Delta g/g$.

Express Δg due to the spherical deposit:

$$\Delta g=\frac{G\Delta M}{r^2}\qquad(1)$$

Express the mass of the spherical deposit:

$$M=\Delta\rho V=\Delta\rho\left(\tfrac{4}{3}\pi R^3\right)=\tfrac{4}{3}\pi\,\Delta\rho\,R^3$$

Substitute in equation (1):

$$\Delta g=\frac{\frac{4}{3}G\pi\,\Delta\rho\,R^3}{r^2}$$

Express $\Delta g/g$:

$$\frac{\Delta g}{g}=\frac{\dfrac{\frac{4}{3}G\pi\,\Delta\rho\,R^3}{r^2}}{g}=\frac{\frac{4}{3}G\pi\,\Delta\rho\,R^3}{gr^2}$$

Substitute numerical values and evaluate $\Delta g/g$:

$$\frac{\Delta g}{g}=\frac{\frac{4}{3}\pi\left(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5000\,\text{kg/m}^3\right)(1000\,\text{m})^3}{(9.81\,\text{N/kg})(2000\,\text{m})^2}=\boxed{3.56\times10^{-5}}$$

83 ••• **[SSM]** Two identical spherical cavities are made in a lead sphere of radius R. The cavities each have a radius $R/2$. They touch the outside surface of the sphere and its center as in Figure 11-28. The mass of a solid uniform lead sphere of radius R is M. Find the force of attraction on a point particle of mass m located at a distance d from the center of the lead sphere.

Picture the Problem The force of attraction of the small sphere of mass m to the lead sphere of mass M is the sum of the forces due to the solid sphere ($\vec{F}_S$) and the cavities ($\vec{F}_C$) of negative mass.

Express the force of attraction:

$$\vec{F} = \vec{F}_S + \vec{F}_C \quad (1)$$

Use the law of gravity to express the force due to the solid sphere:

$$\vec{F}_S = -\frac{GMm}{d^2}\hat{i}$$

Express the magnitude of the force acting on the small sphere due to one cavity:

$$F_C = \frac{GM'm}{d^2 + \left(\frac{R}{2}\right)^2}$$

where M' is the negative mass of a cavity.

Relate the negative mass of a cavity to the mass of the sphere before hollowing:

$$M' = -\rho V = -\rho\left[\tfrac{4}{3}\pi\left(\frac{R}{2}\right)^3\right] = -\tfrac{1}{8}\left(\tfrac{4}{3}\pi\rho R^3\right) = -\tfrac{1}{8}M$$

Letting θ be the angle between the x axis and the line joining the center of the small sphere to the center of either cavity, use the law of gravity to express the force due to the two cavities:

$$\vec{F}_C = 2\frac{GMm}{8\left(d^2 + \frac{R^2}{4}\right)}\cos\theta\,\hat{i}$$

because, by symmetry, the y components add to zero.

Use the figure to express $\cos\theta$:

$$\cos\theta = \frac{d}{\sqrt{d^2 + \frac{R^2}{4}}}$$

Substitute for $\cos\theta$ and simplify to obtain:

$$\vec{F}_C = \frac{GMm}{4\left(d^2 + \frac{R^2}{4}\right)}\frac{d}{\sqrt{d^2 + \frac{R^2}{4}}}\hat{i} = \frac{GMmd}{4\left(d^2 + \frac{R^2}{4}\right)^{3/2}}\hat{i}$$

Substitute in equation (1) and simplify:

$$\vec{F} = -\frac{GMm}{d^2}\hat{i} + \frac{GMmd}{4\left(d^2 + \frac{R^2}{4}\right)^{3/2}}\hat{i}$$

$$= \boxed{-\frac{GMm}{d^2}\left[1 - \frac{\frac{d^3}{4}}{\left\{d^2 + \frac{R^2}{4}\right\}^{3/2}}\right]\hat{i}}$$

84 ••• A globular cluster is a roughly spherical collection of up to millions of stars bound together by the force of gravity. Astronomers can measure the velocities of stars in the cluster to study its composition and to get an idea of the mass distribution within the cluster. Assuming that all of the stars have about the same mass and are distributed uniformly within the cluster, show that the mean speed of a star in a circular orbit around the center of the cluster should increase linearly with its distance from the center.

Picture the Problem Let R be the size of the cluster, and N the total number of stars in it. We can apply Newton's law of gravity and the second law of motion to relate the net force (which depends on the number of stars $N(r)$ in a sphere whose radius is equal to the distance between the star of interest and the center of the cluster) acting on a star at a distance r from the center of the cluster to its speed. We can use the definition of density, in conjunction with the assumption of uniform distribution of the starts within the cluster, to find $N(r)$ and, ultimately, express the orbital speed v of a star in terms of the total mass of the cluster.

Using Newton's law of gravity and second law, express the force acting on a star at a distance r from the center of the cluster:

$$F(r) = \frac{GN(r)M^2}{r^2} = M\frac{v^2}{r} \qquad (1)$$

where $N(r)$ is the number of stars within a distance r of the center of the cluster and M is the mass of an individual star.

Using the uniform distribution assumption and the definition of density, relate the number of stars $N(r)$ within a distance r of the center of the cluster to the total number N of stars in the cluster:

$$\rho = \frac{N(r)M}{\frac{4}{3}\pi r^3} = \frac{NM}{\frac{4}{3}\pi R^3} \Rightarrow N(r) = N\frac{r^3}{R^3}$$

Substitute for $N(r)$ in equation (1) to obtain:

$$\frac{GNM^2}{r^2}\frac{r^3}{R^3} = M\frac{v^2}{r}$$

Solving for v yields:

$$v = r\sqrt{\frac{GNM}{R^3}} \Rightarrow v \propto r$$

The mean speed v of a star in a circular orbit about the center of the cluster increases linearly with distance r from the center.

General Problems

85 • The mean distance of Pluto from the Sun is 39.5 AU. Calculate the period of Pluto′s orbital motion.

Picture the Problem We can use Kepler's third law to relate Pluto's period to its mean distance from the Sun.

Using Kepler's third law, relate the period of Pluto to its mean distance from the Sun:

$$T_{\text{Pluto}}^2 = Cr_{\text{Pluto}}^3 \Rightarrow T_{\text{Pluto}} = \sqrt{Cr_{\text{Pluto}}^3}$$

where $C = \frac{4\pi^2}{GM_{\text{s}}} = 2.973\times10^{-19}\ \text{s}^2/\text{m}^3$.

Substitute numerical values and evaluate T_{Pluto}:

$$T_{\text{Pluto}} = \sqrt{\left(2.973\times10^{-19}\ \text{s}^2/\text{m}^3\right)\left(39.5\,\text{AU}\times\frac{1.50\times10^{11}\ \text{m}}{\text{AU}}\right)^3}$$
$$= 7.864\times10^9\ \text{s}\times\frac{1\,\text{h}}{3600\,\text{s}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{y}}{365.25\,\text{d}} = \boxed{249\,\text{y}}$$

86 • Calculate the mass of Earth using the known values of G, g, and R_{E}.

Picture the Problem Consider an object of mass m at the surface of Earth. We can relate the weight of this object to the gravitational field of Earth and to the mass of Earth.

Using Newton's second law, relate the weight of an object at the surface of Earth to the gravitational force acting on it:

$$w = mg = \frac{GM_{\text{E}}m}{R_{\text{E}}^2} \Rightarrow M_{\text{E}} = \frac{gR_{\text{E}}^2}{G}$$

Substitute numerical values and evaluate M_{E}:

$$M_{\text{E}} = \frac{(9.81\,\text{N/kg})(6.37\times10^6\,\text{m})^2}{6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2} = \boxed{5.97\times10^{24}\,\text{kg}}$$

87 •• The force exerted by Earth on a particle of mass m a distance r ($r > R_{\text{E}}$) from the center of Earth has the magnitude mgR_{E}^2/r^2, where $g = GM_{\text{E}}/R_{\text{E}}^2$. (*a*) Calculate the work you must do to move the particle from distance r_1 to distance r_2. (*b*) Show that when $r_1 = R_{\text{E}}$ and $r_2 = R_{\text{E}} + h$, the result can be written as $W = mgR_{\text{E}}^2\left[(1/R_{\text{E}}) - 1/(R_{\text{E}}+h)\right]$. (*c*) Show that when $h \ll R_{\text{E}}$, the work is given approximately by $W = mgh$.

Picture the Problem The work you must do against gravity to move the particle from a distance r_1 to r_2 is the negative of the change in the particle's gravitational potential energy.

(*a*) Relate the work you must do to the change in the gravitational potential energy of Earth-particle system:

$$W = -\Delta U = -\int_{r_1}^{r_2}\vec{F}_{\text{g}}\cdot d\vec{r} = -\int_{r_1}^{r_2}F_{\text{g}}dr(\cos180°) = \int_{r_1}^{r_2}F_{\text{g}}dr$$

Substitute for F_{g} and evaluate the integral to obtain:

$$W = GM_{\text{E}}m\int_{r_1}^{r_2}\frac{dr}{r^2} = -GM_{\text{E}}m\left(\frac{1}{r_2}-\frac{1}{r_1}\right) = \boxed{GM_{\text{E}}m\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}$$

(*b*) Substitute gR_{E}^2 for GM_{E}, R_{E} for r_1, and $R_{\text{E}} + h$ for r_2 to obtain:

$$W = \boxed{mgR_{\text{E}}^2\left(\frac{1}{R_{\text{E}}}-\frac{1}{R_{\text{E}}+h}\right)} \quad (1)$$

(c) Rewrite equation (1) with a common denominator and simplify to obtain:

$$W = mgR_E^2\left(\frac{R_E + h - R_E}{R_E(R_E + h)}\right) = mgh\left(\frac{R_E}{R_E + h}\right) = mgh\left(\frac{1}{1+\dfrac{h}{R_E}}\right) \approx \boxed{mgh}$$

provided $h << R_E$.

88 •• The average density of the moon is $\rho = 3340\ \mathrm{kg/m^3}$. Find the minimum possible period T of a spacecraft orbiting the moon.

Picture the Problem Let m represent the mass of the spacecraft. From Kepler's third law we know that its period will be a minimum when it is in orbit just above the surface of the moon. We'll use Newton's second law to relate the angular speed of the spacecraft to the gravitational force acting on it.

Relate the period of the spacecraft to its angular speed:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Using Newton's second law, relate the gravitational force acting on the spacecraft when it is in orbit at the surface of the moon to the angular speed of the spacecraft:

$$\sum F_{\text{radial}} = \frac{GM_M m}{R_M^2} = mR_M\omega^2$$

Solving for ω and simplifying yields:

$$\omega = \sqrt{\frac{GM_M}{R_M^3}} = \sqrt{\frac{G\left(\frac{4}{3}\pi\rho R_M^3\right)}{R_M^3}} = \sqrt{\tfrac{4}{3}G\pi\rho}$$

Substitute for ω in equation (1) and simplify to obtain:

$$T_{\min} = \frac{2\pi}{\sqrt{\frac{4}{3}G\pi\rho}} = \sqrt{\frac{3\pi}{\rho G}}$$

Substitute numerical values and evaluate $T_{\min}$:

$$T_{\min} = \sqrt{\frac{3\pi}{\left(6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(3340\ \mathrm{kg/m^3}\right)}} = 6503\ \mathrm{s} = \boxed{1\ \mathrm{h}\ 48\ \mathrm{min}}$$

89 •• **[SSM]** A *neutron star* is a highly condensed remnant of a massive star in the last phase of its evolution. It is composed of neutrons (hence the name) because the star's gravitational force causes electrons and protons to "coalesce" into the neutrons. Suppose at the end of its current phase, the Sun collapsed into a neutron star (it can't in actuality because it does not have enough mass) of radius 12.0 km, without losing any mass in the process. (*a*) Calculate the ratio of the gravitational acceleration at the surface of the Sun following its collapse compared to its value at the surface of the Sun today. (*b*) Calculate the ratio of the escape speed from the surface of the neutron-Sun to its value today.

Picture the Problem We can apply Newton's second law and the law of gravity to an object of mass *m* at the surface of the Sun and the neutron-Sun to find the ratio of the gravitational accelerations at their surfaces. Similarly, we can express the ratio of the corresponding expressions for the escape speeds from the two suns to determine their ratio.

(*a*) Express the gravitational force acting on an object of mass *m* at the surface of the Sun:

$$F_g = ma_g = \frac{GM_{Sun}m}{R_{Sun}^2}$$

Solving for a_g yields:

$$a_g = \frac{GM_{Sun}}{R_{Sun}^2} \qquad (1)$$

The gravitational force acting on an object of mass *m* at the surface of a neutron-Sun is:

$$F_g = ma'_g = \frac{GM_{neutron\text{-}Sun}m}{R_{neutron\text{-}Sun}^2}$$

Solving for a'_g yields:

$$a'_g = \frac{GM_{neutron\text{-}Sun}}{R_{neutron\text{-}Sun}^2} \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{a'_g}{a_g} = \frac{\dfrac{GM_{neutron\text{-}Sun}}{R_{neutron\text{-}Sun}^2}}{\dfrac{GM_{Sun}}{R_{Sun}^2}} = \frac{\dfrac{M_{neutron\text{-}Sun}}{R_{neutron\text{-}Sun}^2}}{\dfrac{M_{Sun}}{R_{Sun}^2}}$$

Because $M_{neutron\text{-}Sun} = M_{Sun}$:

$$\frac{a'_g}{a_g} = \frac{R_{Sun}^2}{R_{neutron\text{-}Sun}^2}$$

Substitute numerical values and evaluate the ratio a'_g/a_g:

$$\frac{a'_g}{a_g} = \left(\frac{6.96\times10^8\ \text{m}}{12.0\times10^3\ \text{m}}\right)^2 = \boxed{3.36\times10^9}$$

(*b*) The escape speed from the neutron-Sun is given by:

$$v'_e = \sqrt{\frac{GM_{\text{neutron-Sun}}}{R_{\text{neutron-Sun}}}}$$

The escape speed from the Sun is given by:

$$v_e = \sqrt{\frac{GM_{\text{Sun}}}{R_{\text{Sun}}}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{v'_e}{v_e} = \sqrt{\frac{R_{\text{Sun}}}{R_{\text{neutron-Sun}}}}$$

Substitute numerical values and evaluate v'_e/v_e:

$$\frac{v'_e}{v_e} = \sqrt{\frac{6.96\times10^8\ \text{m}}{12.0\times10^3\ \text{m}}} = \boxed{241}$$

90 •• Suppose the Sun could collapse into a neutron star of radius 12.0 km, as in Problem 89. Your research team is in charge of sending a probe from Earth to study the transformed Sun, and the probe needs to end up in a circular orbit 4500 km from the neutron-Sun's center. (*a*) Calculate the orbital speed of the probe. (*b*) Later on, plans call for construction of a permanent spaceport in that same orbit. To transport equipment and supplies, scientists on Earth need you to determine the escape speed for rockets launched from the spaceport (relative to the spaceport) in the direction of the spaceport's orbital velocity at takeoff time. What is that speed, and how does it compare to the escape speed at the surface of Earth?

Picture the Problem We can apply Newton's law law to the probe orbiting the Sun to determine its orbital speed. Using the escape-speed equation will allow us to find the escape speed for rockets launched from the spaceport.

(*a*) Apply Newton's second law to the probe of mass *m* in orbit about the Sun:

$$\sum F_{\text{radial}} = \frac{GmM_{\text{Sun}}}{r^2} = m\frac{v_{\text{orbital}}^2}{r}$$

where *r* is the orbital radius.

Solving for v_{orbital} yields:

$$v_{\text{orbital}} = \sqrt{\frac{GM_{\text{Sun}}}{r}}$$

Substitute numerical values and evaluate v_{orbital}:

$$v_{\text{orbital}} = \sqrt{\frac{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(1.99\times10^{30}\ \text{kg}\right)}{4.50\times10^6\ \text{m}}} = 5.432\times10^6\ \text{m/s}$$

$$= \boxed{5.43\times10^6\ \text{m/s}}$$

(*b*) The escape speed (relative to the spaceport) for rockets launched from the spaceport is given by:

$$v_{\substack{\text{rel to}\\\text{spaceport}}} = v_e - v_{\text{orbital}} \quad (1)$$

The escape speed at a distance r from the center of the neutron-Sun is given by:

$$v_e = \sqrt{\frac{2GM_{\text{neutron-Sun}}}{r}} = \sqrt{2}\sqrt{\frac{GM_{\text{neutron-Sun}}}{r}} = \sqrt{2}v_{\text{orbital}}$$

Substituting for v_e in equation (1) yields:

$$v_{\substack{\text{rel to}\\\text{spaceport}}} = \sqrt{2}v_{\text{orbital}} - v_{\text{orbital}} = \left(\sqrt{2}-1\right)v_{\text{orbital}}$$

Substitute numerical values and evaluate $v_{\substack{\text{rel to}\\\text{spaceport}}}$:

$$v_{\substack{\text{rel to}\\\text{spaceport}}} = \left(\sqrt{2}-1\right)\left(5.431\times10^{6}\ \text{m/s}\right) = \boxed{2.25\times10^{6}\ \text{m/s}}$$

Express the ratio of $v_{\substack{\text{rel to}\\\text{spaceport}}}$ to $v_{\text{e,Earth}}$:

$$\frac{v_{\substack{\text{rel to}\\\text{spaceport}}}}{v_{\text{e,Earth}}} = \frac{2.25\times10^{6}\ \text{m/s}}{11.2\ \text{km/s}} \approx \boxed{201}$$

91 •• A satellite is circling the moon (radius 1700 km) close to the surface at a speed *v*. A projectile is launched from the moon vertically up at the same initial speed *v*. How high will it rise?

Picture the Problem We can use conservation of energy to establish a relationship between the height h to which the projectile will rise and its initial speed. The application of Newton's second law will relate the orbital speed, which is equal to the initial speed of the projectile, to the mass and radius of the moon.

Use conservation of energy to relate the initial energies of the projectile to its final energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-\tfrac{1}{2}mv^2 - \frac{GM_{\text{moon}}m}{R_{\text{moon}}+h} + \frac{GM_{\text{moon}}m}{R_{\text{moon}}} = 0$$

Solving for h yields:

$$h = R_M\left(\frac{1}{1-\dfrac{v^2R_{\text{moon}}}{2GM_{\text{moon}}}} - 1\right) \quad (1)$$

Use Newton's second law to relate the orbital speed of the satellite to the gravitational force acting on it:

$$\sum F_{\text{radial}} = \frac{GM_{\text{M}}m}{R_{\text{M}}^2} = m\frac{v^2}{R_{\text{M}}}$$

Solve for v^2 to obtain:

$$v^2 = \frac{GM_{\text{M}}}{R_{\text{M}}}$$

Substitute for v^2 in equation (1) and simplify to obtain:

$$h = R_{\text{M}}\left(\frac{1}{1-\frac{1}{2}} - 1\right) = R_{\text{M}} = \boxed{1.70\,\text{Mm}}$$

92 •• *Black holes* are objects whose gravitational field is so strong that not even light can escape. One way of thinking about this is to consider a spherical object whose density is so large that the escape speed at its surface is greater than the speed of light c. If a star's radius is smaller than a value called the *Schwarzchild radius* R_{S}, then the star will be a black hole, that is, light originating from its surface cannot escape. (a) For a non-rotating black hole, the Schwarzchild radius depends only upon the mass of the black hole. Show that it is related to that mass M by $R_{\text{S}} = 2GM/c^2$. (b) Calculate the value of the Schwarzchild radius for a black hole whose mass is ten solar masses.

Picture the Problem We can use the escape-speed equation, with $v_{\text{e}} = c$, to derive the expression for the Schwarzchild radius of a non-rotating black hole.

(a) The escape speed at a distance r from the center of a spherical object of mass M is given by:

$$v_{\text{e}} = \sqrt{\frac{2GM}{r}}$$

Setting $v_{\text{e}} = c$ yields:

$$c = \sqrt{\frac{2GM}{R_{\text{S}}}} \Rightarrow R_{\text{S}} = \boxed{\frac{2GM}{c^2}}$$

(b) For a black hole whose mass is ten solar masses:

$$R_{\text{S}} = \frac{2G(10M_{\text{Sun}})}{c^2} = \frac{20GM_{\text{Sun}}}{c^2}$$

Substitute numerical values and evaluate R_{S}:

$$R_{\text{S}} = \frac{20(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{(2.998\times10^{8}\ \text{m/s})^2} = \boxed{29.5\,\text{km}}$$

93 •• In a binary star system, two stars follow circular orbits about their common center of mass. If the stars have masses m_1 and m_2 and are separated by a distance *r,* show that the period of rotation is related to *r* by $T^2 = \dfrac{4\pi^2 r^3}{G(m_1 + m_2)}$.

Picture the Problem Let the origin of our coordinate system be at the center of mass of the binary star system and let the distances of the stars from their center of mass be r_1 and r_2. The period of rotation is related to the angular speed of the star system and we can use Newton's second law of motion to relate this speed to the separation of the stars.

Relate the square of the period of the motion of the stars to their angular speed:

$$T^2 = \frac{4\pi^2}{\omega^2} \qquad (1)$$

Using Newton's second law, relate the gravitational force acting on the star whose mass is m_2 to the angular speed of the system:

$$\sum F_{\text{radial}} = \frac{Gm_1 m_2}{(r_1 + r_2)^2} = m_2 r_2 \omega^2$$

Solving for ω^2 yields:

$$\omega^2 = \frac{Gm_1}{r_2 (r_1 + r_2)^2} \qquad (2)$$

From the definition of the center of mass we have:

$$m_1 r_1 = m_2 r_2 \qquad (3)$$

where $r = r_1 + r_2 \qquad (4)$

Eliminate r_1 from equations (3) and (4) and solve for r_2 to obtain:

$$r_2 = \frac{r m_1}{m_1 + m_2}$$

Eliminate r_2 from equations (3) and (4) and solve for r_1 to obtain:

$$r_1 = \frac{r m_2}{m_1 + m_2}$$

Substituting for r_1 and r_2 in equation (2) yields:

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

Finally, substitute for ω^2 in equation (1) and simplify:

$$T^2 = \frac{4\pi^2}{\dfrac{G(m_1 + m_2)}{r^3}} = \boxed{\frac{4\pi^2 r^3}{G(m_1 + m_2)}}$$

94 •• Two particles of masses m_1 and m_2 are released from rest at a large separation distance. Find their speeds v_1 and v_2 when their separation distance is r. The initial separation distance is given as large, but large is a relative term. Relative to what distance is it large?

Picture the Problem Because the two-particle system has zero initial energy and zero initial linear momentum; we can use energy and momentum conservation to obtain simultaneous equations in the variables r, v_1 and v_2. We'll assume that initial separation distance of the particles and their final separation *r is large compared to the size of the particles* so that we can treat them as though they are point particles.

Use conservation of energy to relate the speeds of the particles when their separation distance is r:

$$E_\mathrm{i} = E_\mathrm{f}$$

or

$$0 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{r} \quad (1)$$

Use conservation of linear momentum to obtain a second relationship between the speeds of the particles and their masses:

$$p_\mathrm{i} = p_\mathrm{f}$$

or

$$0 = m_1v_1 + m_2v_2 \quad (2)$$

Solve equation (2) for v_1 and substitute in equation (1) to obtain:

$$v_2^2\left(m_2 + \frac{m_2^2}{m_1}\right) = \frac{2Gm_1m_2}{r} \quad (3)$$

Solve equation (3) for v_2:

$$v_2 = \boxed{\sqrt{\frac{2Gm_1^2}{r(m_1 + m_2)}}}$$

Solve equation (2) for v_1 and substitute for v_2 to obtain:

$$v_1 = \boxed{\sqrt{\frac{2Gm_2^2}{r(m_1 + m_2)}}}$$

95 •• [SSM] Uranus, the seventh planet in the Solar System, was first observed in 1781 by William Herschel. Its orbit was then analyzed in terms of Kepler's Laws. By the 1840s, observations of Uranus clearly indicated that its true orbit was different from the Keplerian calculation by an amount that could not be accounted for by observational uncertainty. The conclusion was that there must be another influence other than the Sun and the known planets lying inside Uranus's orbit. This influence was hypothesized to be due to an eighth planet, whose predicted orbit was described independently in 1845 by two astronomers: John Adams (no relation to the former president of the United States) and Urbain LeVerrier. In September of 1846, John Galle, searching in the sky at the place

predicted by Adams and LeVerrier, made the first observation of Neptune. Uranus and Neptune are in orbit about the Sun with periods of 84.0 and 164.8 years, respectively. To see the effect that Neptune had on Uranus, determine the ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another (i.e. when aligned with the Sun). The masses of the Sun, Uranus, and Neptune are 333,000, 14.5 and 17.1 times that of Earth, respectively.

Picture the Problem We can use the law of gravity and Kepler's third law to express the ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another.

The ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another is given by:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{\dfrac{GM_{\text{N}}M_{\text{U}}}{(r_{\text{N}}-r_{\text{U}})^2}}{\dfrac{GM_{\text{U}}M_{\text{S}}}{r_{\text{U}}^2}}=\frac{M_{\text{N}}r_{\text{U}}^2}{M_{\text{S}}(r_{\text{N}}-r_{\text{U}})^2} \quad (1)$$

Applying Kepler's third law to Uranus yields:

$$T_{\text{U}}^2=Cr_{\text{U}}^3 \quad (2)$$

Applying Kepler's third law to Neptune yields:

$$T_{\text{N}}^2=Cr_{\text{N}}^3 \quad (3)$$

Divide equation (3) by equation (2) to obtain:

$$\frac{T_{\text{N}}^2}{T_{\text{U}}^2}=\frac{Cr_{\text{N}}^3}{Cr_{\text{U}}^3}=\frac{r_{\text{N}}^3}{r_{\text{U}}^3}\Rightarrow r_{\text{N}}=r_{\text{U}}\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}$$

Substitute for r_{N} in equation (1) to obtain:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{M_{\text{N}}r_{\text{U}}^2}{M_{\text{S}}\left(r_{\text{U}}\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-r_{\text{U}}\right)^2}$$

Simplifying this expression yields:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{M_{\text{N}}}{M_{\text{S}}\left(\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-1\right)^2}$$

Because $M_N = 17.1M_E$ and $M_S = 333{,}000M_E$:

$$\frac{F_{g,N\text{-}U}}{F_{g,U\text{-}S}} = \frac{17.1M_E}{3.33\times10^5 M_E\left(\left(\dfrac{T_N}{T_U}\right)^{2/3}-1\right)^2} = \frac{17.1}{3.33\times10^5\left(\left(\dfrac{T_N}{T_U}\right)^{2/3}-1\right)^2}$$

Substitute numerical values and evaluate $\dfrac{F_{g,N\text{-}U}}{F_{g,U\text{-}S}}$:

$$\frac{F_{g,N\text{-}U}}{F_{g,U\text{-}S}} = \frac{17.1}{3.33\times10^5\left(\left(\dfrac{164.8\text{ y}}{84.0\text{ y}}\right)^{2/3}-1\right)^2} \approx 2\times10^{-4}$$

Because this ratio is so small, during the time at which Neptune is closest to Uranus, the force exerted on Uranus by Neptune is much less than the force exerted on Uranus by the Sun.

96 •• It is believed that there is a ″super-massive″ black hole at the center of our galaxy. One datum that leads to this conclusion is the important recent observation of stellar motion in the vicinity of the galactic center. If one such star moves in an elliptical orbit with a period of 15.2 years and has a semi-major axis of 5.5 light-days (the distance light travels in 5.5 days), what is the mass around which the star moves in its Keplerian orbit?

Picture the Problem We can apply Kepler’s third law to the orbital motion of the star to find the effective mass around which it is moving.

Using Kepler’s third law, relate the orbital period of the star to the semi-major axis of its orbit:

$$T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow M = \frac{4\pi^2 a^3}{GT^2}$$

where M is the mass around which the star moves in its Keplerian orbit.

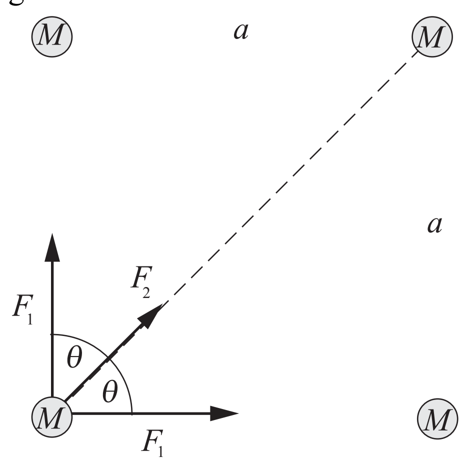
Substitute numerical values and evaluate M:

$$M = \frac{4\pi^2\left(5.5\ \text{d}\times\frac{86400\ \text{s}}{\text{d}}\times 2.998\times10^8\ \text{m/s}\right)^3}{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(15.2\ \text{y}\times\frac{3.156\times10^7\ \text{s}}{\text{y}}\right)^2} = 7.434\times10^{36}\ \text{kg}$$

$$= 7.434\times10^{36}\ \text{kg}\times\frac{1\,M_{\text{Sun}}}{1.99\times10^{30}\ \text{kg}} = \boxed{3.7\times10^6\ M_{\text{Sun}}}$$

97 •• [SSM] Four identical planets are arranged in a square as shown in Figure 11-29. If the mass of each planet is M and the edge length of the square is a, what must be their speed if they are to orbit their common center under the influence of their mutual attraction?

Picture the Problem Note that, due to the symmetrical arrangement of the planets, each experiences the same centripetal force. We can apply Newton's second law to any of the four planets to relate its orbital speed to this net (centripetal) force acting on it.



Applying $\sum \vec{F}_{\text{radial}} = m\vec{a}_{\text{radial}}$ to one of the planets gives:

$$F_c = 2F_1\cos\theta + F_2$$

Substituting for F_1, F_2, and θ and simplifying yields:

$$F_c = \frac{2GM^2}{a^2}\cos 45° + \frac{GM^2}{\left(\sqrt{2}a\right)^2} = \frac{2GM^2}{a^2}\frac{1}{\sqrt{2}} + \frac{GM^2}{2a^2} = \frac{GM^2}{a^2}\left(\sqrt{2}+\frac{1}{2}\right)$$

Because $F_c = \dfrac{Mv^2}{a/\sqrt{2}} = \dfrac{\sqrt{2}Mv^2}{a}$

$$\frac{\sqrt{2}Mv^2}{a} = \frac{GM^2}{a^2}\left(\sqrt{2}+\frac{1}{2}\right)$$

Solve for v to obtain:

$$v = \sqrt{\frac{GM}{a}\left(1+\frac{1}{2\sqrt{2}}\right)} = \boxed{1.16\sqrt{\frac{GM}{a}}}$$

98 •• A hole is drilled from the surface of Earth to its center as in Figure 11-30. Ignore Earth's rotation and any effects due to air resistance, and model Earth as a uniform sphere. (*a*) How much work is required to lift a particle of mass *m* from the center of Earth to Earth's surface? (*b*) If the particle is dropped from rest at the surface of Earth, what is its speed when it reaches the center of Earth? (*c*) What is the escape speed for a particle projected from the center of Earth? Express your answers in terms of *m, g,* and R_E.

Picture the Problem Let *r* represent the separation of the particle from the center of Earth and assume a uniform density for Earth. The work required to lift the particle from the center of Earth to its surface is the line integral of the gravitational force function. This function can be found from the law of gravity and by relating the mass of Earth between the particle and the center of Earth to Earth's mass. We can use the work-kinetic energy theorem to find the speed with which the particle, when released from the surface of Earth, will strike the center of Earth. Finally, the energy required for the particle to escape Earth from the center of Earth is the sum of the energy required to get it to the surface of Earth and the kinetic energy it must have to escape from the surface of Earth.

(*a*) Express the work required to lift the particle from the center of Earth to Earth's surface:

$$W = \int_0^R \vec{F}\cdot d\vec{r} = -\int_0^R \vec{F}_g \cdot d\vec{r} = \int_0^{R_E} F_g dr \quad (1)$$

where F_g is the gravitational force acting on the particle.

Using the law of gravity, express the force acting on the particle as a function of its distance from the center of Earth:

$$F_g = \frac{GmM}{r^2} \quad (2)$$

where M is the mass of a sphere whose radius is r.

Express the ratio of M to M_E and simplify to obtain:

$$\frac{M}{M_E} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\rho\left(\frac{4}{3}\pi R_E^3\right)} = \frac{r^3}{R_E^3} \Rightarrow M = M_E\frac{r^3}{R_E^3}$$

Substitute for M in equation (2) to obtain:

$$F_g = \frac{GmM_E}{R_E^3}r = \frac{mgR_E^2}{R_E^3}r = \frac{mg}{R_E}r$$

Substitute for F_g in equation (1) and evaluate the integral:

$$W = \frac{mg}{R_E}\int_0^{R_E} r\,dr = \boxed{\tfrac{1}{2}gmR_E}$$

(*b*) Use the work-kinetic energy theorem to relate the kinetic energy of the particle as it reaches the center of Earth to the work done on it in moving it to the surface of Earth:

$$W = \Delta K = \tfrac{1}{2}mv^2$$

Substituting for W yields:

$$\tfrac{1}{2}gmR_E = \tfrac{1}{2}mv^2 \Rightarrow v = \boxed{\sqrt{gR_E}}$$

(*c*) Express the total energy required for the particle to escape when projected from the center of Earth:

$$E_{esc} = W + \tfrac{1}{2}mv_e^2 = \tfrac{1}{2}mv_{esc}^2$$

where v_e is the escape speed from the surface of Earth.

Substituting for W yields:

$$\tfrac{1}{2}gmR_E + \tfrac{1}{2}mv_e^2 = \tfrac{1}{2}mv_{esc}^2$$

or, simplifying,

$$gR_E + v_e^2 = v_{esc}^2$$

Because $v_e^2 = \dfrac{2GM}{R_E}$:

$$gR_E + \frac{2GM}{R_E} = v_{esc}^2 \qquad (3)$$

Apply Newton's second law to an object of mass m at the surface of Earth to obtain:

$$mg = \frac{GMm}{R_E^2} \Rightarrow \frac{GM}{R_E} = gR_E$$

Substitute for GM/R_E in equation (3) to obtain:

$$gR_E + 2gR_E = v_{esc}^2 \Rightarrow v_{esc} = \boxed{\sqrt{3gR_E}}$$

Remarks: This escape speed is approximately 122% of the escape speed from the surface of Earth.

99 •• A thick spherical shell of mass M and uniform density has an inner radius R_1 and an outer radius R_2. Find the gravitational field g_r as a function of r for $0 < r < \infty$. Sketch a graph of g_r versus r.

Picture the Problem We need to find the gravitational field in three regions: $r < R_1$, $R_1 < r < R_2$, and $r > R_2$.

For $r < R_1$: $$g(r < R_1) = \boxed{0}$$

For $r > R_2$, $g(r)$ is the field due to the thick spherical shell of mass M centered at the origin: $$g(r > R_2) = \boxed{\frac{GM}{r^2}}$$

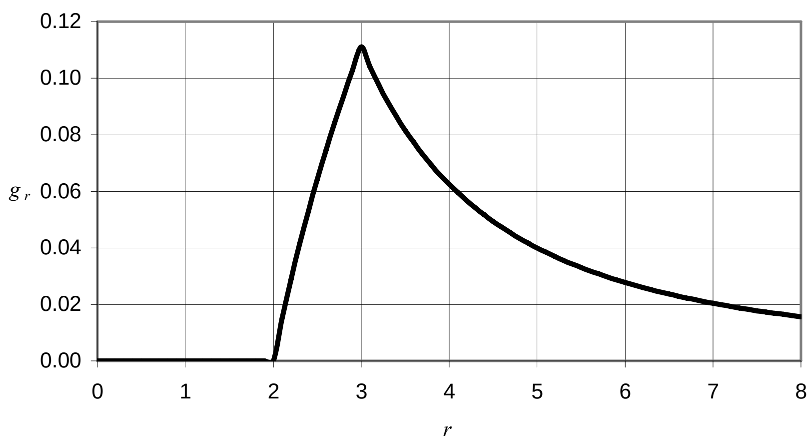
For $R_1 < r < R_2$, $g(r)$ is determined by the mass within the shell of radius r: $$g(R_1 < r < R_2) = \frac{Gm}{r^2} \qquad (1)$$
where $m = \frac{4}{3}\pi\rho\left(r^3 - R_1^3\right) \qquad (2)$

Express the density of the spherical shell: $$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi\left(R_2^3 - R_1^3\right)}$$

Substitute for ρ in equation (2) and simplify to obtain: $$m = \frac{M\left(r^3 - R_1^3\right)}{R_2^3 - R_1^3}$$

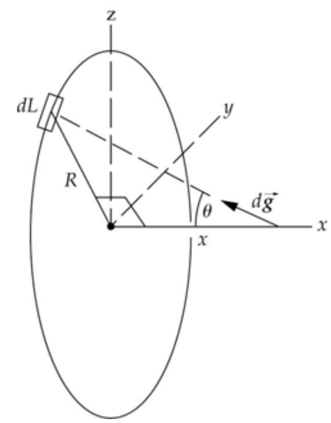
Substitute for m in equation (1) to obtain: $$g_r = \boxed{\frac{GM\left(r^3 - R_1^3\right)}{r^2\left(R_2^3 - R_1^3\right)}}$$

A graph of g_r with $R_1 = 2$, $R_2 = 3$, and $GM = 1$ follows:



100 •• (*a*) A thin uniform ring of mass M and radius R lies in the $x = 0$ plane and is centered at the origin. Sketch a plot of the gravitational field g_x versus x for all points on the x axis. (*b*) At what point, or points, on the axis is the magnitude of g_x a maximum?

Picture the Problem A ring of radius R is shown to the right. Choose a coordinate system in which the origin is at the center of the ring and x axis is as shown. An element of length dL and mass dm is responsible for the field dg at a distance x from the center of the ring. We can express the x component of dg and then integrate over the circumference of the ring to find the total field as a function of x.



(a) Express the differential gravitational field at a distance x from the center of the ring in terms of the mass of an elemental segment of length dL:

$$dg = \frac{Gdm}{R^2 + x^2}$$

Relate the mass of the element to its length:

$$dm = \lambda\, dL$$

where λ is the linear density of the ring.

Substitute for dm to obtain:

$$dg = \frac{G\lambda\, dL}{R^2 + x^2}$$

By symmetry, the y and z components of g vanish. The x component of dg is:

$$dg_x = dg\cos\theta = \frac{G\lambda\, dL}{R^2 + x^2}\cos\theta$$

Refer to the figure to obtain:

$$\cos\theta = \frac{x}{\sqrt{R^2 + x^2}}$$

Substituting for $\cos\theta$ yields:

$$dg_x = \frac{G\lambda\, dL}{R^2 + x^2}\frac{x}{\sqrt{R^2 + x^2}} = \frac{G\lambda\, xdL}{\left(R^2 + x^2\right)^{3/2}}$$

Because $\lambda = \dfrac{M}{2\pi R}$:

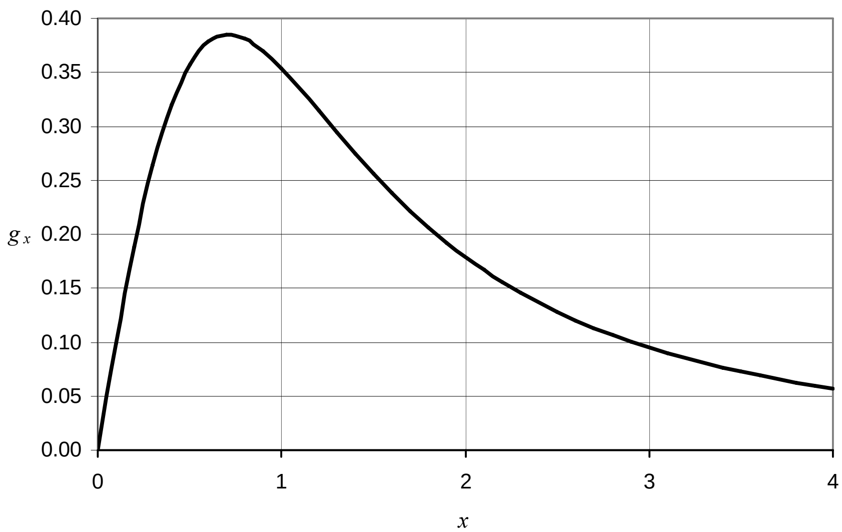
$$dg_x = \frac{GM\, xdL}{2\pi R\left(R^2 + x^2\right)^{3/2}}$$

Integrate to find g_x:

$$g_x = \frac{GM\,x}{2\pi R\left(R^2 + x^2\right)^{3/2}} \int_0^{2\pi R} dL$$

$$= \boxed{\frac{GM}{\left(R^2 + x^2\right)^{3/2}}\,x}$$

A graph of g_x follows. The curve is normalized with $R = 1$ and $GM = 1$.



(b) Differentiate $g(x)$ with respect to x and set the derivative equal to zero to identify extreme values:

$$\frac{dg}{dx} = GM\left[\frac{\left(x^2 + R^2\right)^{3/2} - x\left(\frac{3}{2}\right)\left(x^2 + R^2\right)^{1/2}(2x)}{\left(R^2 + x^2\right)^3}\right] = 0 \text{ for extrema}$$

Simplify to obtain:

$$\left(x^2 + R^2\right)^{3/2} - 3x^2\left(x^2 + R^2\right)^{1/2} = 0$$

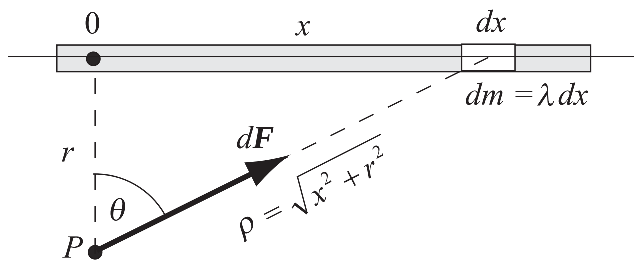
Solving for x yields:

$$x = \boxed{\pm\frac{R}{\sqrt{2}}}$$

Because the curve is concave downward, we can conclude that this result corresponds to a maximum. Note that this result agrees with our graphical maximum.

101 ••• Find the magnitude of the gravitational field a distance r from an infinitely long uniform thin rod whose mass per unit length is λ.

Picture the Problem The diagram shows a segment of the wire of length dx and mass $dm = \lambda dx$ at a distance x from the origin of our coordinate system. We can find the magnitude of the gravitational field at a distance r from the wire from the resultant gravitational force acting on a particle of mass m located at point P and then integrating over the length of the wire.



The magnitude of the gravitational field at P is given by:

$$g = \frac{F}{m}$$

Express the gravitational force acting on a particle of mass m at a distance r from the wire due to the segment of the wire of length dx:

$$F = 2\int_0^{90^\circ} \frac{Gm\,dm\cos\theta}{\rho^2}$$

Substituting for F and dm gives:

$$g = 2\int_0^{90^\circ} \frac{G\lambda dx\cos\theta}{\rho^2}$$

Because $\tan\theta = \frac{x}{r}$:

$$dx = r\sec^2\theta d\theta = \frac{r d\theta}{\cos^2\theta}$$

Note that $\rho = r/\cos\theta$ and substitute for dx and ρ in the expression for x and simplify to obtain:

$$g = 2\int_0^{90^\circ} \frac{G\lambda r d\theta\cos\theta}{\cos^2\theta\left(\frac{r^2}{\cos^2\theta}\right)}$$

$$= \frac{2G\lambda}{r}\int_0^{90^\circ}\cos\theta d\theta = \boxed{\frac{2G\lambda}{r}}$$

102 ••• One question in early planetary science was whether each of the rings of Saturn were solid or were, instead, composed of individual chunks, each in its own orbit. The issue could be resolved by an observation in which astronomers would measure the speed of the inner and outer portions of the ring. If the inner portion of the ring moved more slowly than the outer portion, then the ring was solid; if the opposite was true, then it was actually composed of separate chunks. Let us see how this results from a theoretical viewpoint. Let the radial

width of a given ring (there are many) be Δr, the average distance of that ring from the center of Saturn be represented by R, and the average speed of that ring be v_{avg}. (*a*) If the ring is solid, show that the *difference* in speed between its outermost and innermost portions, Δv, is given by the approximate expression $\Delta v = v_{\text{out}} - v_{\text{in}} \approx v_{\text{avg}} \dfrac{\Delta r}{R}$. Here, v_{out} is the speed of the outermost portion of the ring, and v_{in} is the speed of the innermost portion. (*b*) If, however, the ring is composed of many small chunks, show that $\Delta v \approx -\frac{1}{2}\left(v_{\text{avg}}\,\Delta r/R\right)$. (Assume that $\Delta r << R$.)

Picture the Problem We can use the relationship between the angular speed of an orbiting object and its tangential velocity to express the speeds v_{in} and v_{out} of the innermost and outermost portions of the ring. In Part (*b*) we can use Newton's law of gravity, in conjunction with Newton's second law, to relate the tangential speed of a chunk of the ring to the gravitational force acting on it. As in Part (*a*), once we know v_{in} and v_{out}, we can express the difference between them to obtain the desired results.

(*a*) The difference between v_{out} and v_{in} is:

$$\Delta v = v_{\text{out}} - v_{\text{in}} \qquad (1)$$

The speed of a point in the ring at the average distance R from the center of Saturn under the assumption that the ring is solid and rotates with an angular speed ω is given by:

$$v(R) = \omega R$$

Express the speeds v_{in} and v_{out} of the innermost and outermost portions of the ring:

$$v_{\text{in}} = \left(R - \tfrac{1}{2}\Delta r\right)\omega$$
and
$$v_{\text{out}} = \left(R + \tfrac{1}{2}\Delta r\right)\omega$$

Substituting for v_{in} and v_{out} in equation (1) and simplifying yields:

$$\Delta v = \left(R + \tfrac{1}{2}\Delta r\right)\omega - \left(R - \tfrac{1}{2}\Delta r\right)\omega$$
$$= \omega\Delta r = \frac{v_{\text{avg}}}{R}\Delta r = \boxed{v_{\text{avg}}\frac{\Delta r}{R}}$$

(*b*) Assume that a chunk of the ring is moving in a circular orbit around the center of Saturn under the force of gravity and apply Newton's second law to obtain:

$$\sum F_{\text{radial}} = \frac{GMm}{R'^2} = m\frac{v^2}{R'} \Rightarrow v = \sqrt{\frac{GM}{R'}}$$
where M is the mass of Saturn and R' the distance from its center.

Express v_{out} by substituting for $R+\frac{1}{2}\Delta r$ for R' and simplifying:

$$v_{out}=\sqrt{\frac{GM}{R+\frac{1}{2}\Delta r}}=\sqrt{\frac{GM}{R\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)}}$$

$$=\sqrt{\frac{GM}{R}}\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)^{-1/2}$$

Expanding $\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)^{-1/2}$ binomially, discarding higher-order terms, and simplifying yields:

$$v_{out}=\sqrt{\frac{GM}{R}}\left(1-\frac{1}{2}\left(\frac{1}{2}\frac{\Delta r}{R}\right)\right.$$
$$\left.+\text{higher order terms}\right)$$

$$\approx\sqrt{\frac{GM}{R}}\left(1-\frac{1}{4}\frac{\Delta r}{R}\right)$$

Proceed similarly to obtain, for v_{in}:

$$v_{in}\approx\sqrt{\frac{GM}{R}}\left(1+\frac{1}{4}\frac{\Delta r}{R}\right)$$

Express the difference between v_{out} and v_{in} and simplify to obtain:

$$\Delta v=v_{out}-v_{in}\approx\sqrt{\frac{GM}{R}}\left(1-\frac{1}{4}\frac{\Delta r}{R}\right)-\sqrt{\frac{GM}{R}}\left(1+\frac{1}{4}\frac{\Delta r}{R}\right)=\sqrt{\frac{GM}{R}}\left(-\frac{1}{2}\frac{\Delta r}{R}\right)$$

Because $v_{avg}=\sqrt{\frac{GM}{R}}$:

$$\Delta v\approx\boxed{-\frac{1}{2}\left(v_{avg}\frac{\Delta r}{R}\right)}$$

103 ••• [SSM] In this problem you are to find the gravitational potential energy of the thin rod in Example 11-8 and a point particle of mass m_0 that is on the x axis at $x = x_0$. (*a*) Show that the potential energy shared by an element of the rod of mass dm (shown in Figure 11-14) and the point particle of mass m_0 located at $x_0\geq\frac{1}{2}L$ is given by

$$dU=-\frac{Gm_0dm}{x_0-x_s}=-\frac{GMm_0}{L(x_0-x_s)}dx_s$$

where $U = 0$ at $x_0 = \infty$. (*b*) Integrate your result for Part (*a*) over the length of the rod to find the total potential energy for the system. Generalize your function $U(x_0)$ to any place on the x axis in the region $x > L/2$ by replacing x_0 by a general coordinate x and write it as $U(x)$. (*c*) Compute the force on m_0 at a general point x using $F_x = –dU/dx$ and compare your result with m_0g, where g is the field at x_0 calculated in Example 11-8.

Picture the Problem Let $U = 0$ at $x = \infty$. The potential energy of an element of the stick dm and the point mass m_0 is given by the definition of gravitational potential energy: $dU=-Gm_0dm/r$ where r is the separation of dm and m_0.

(*a*) Express the potential energy of the masses m_0 and dm:

$$dU = -\frac{Gm_0 dm}{x_0 - x_s}$$

The mass dm is proportional to the size of the element dx_s:

$$dm = \lambda\, dx_s$$

where $\lambda = \frac{M}{L}$.

Substitute for dm and λ to express dU in terms of x_s:

$$dU = -\frac{Gm_0 \lambda\, dx_s}{x_0 - x_s} = \boxed{-\frac{GMm_0\, dx_s}{L(x_0 - x_s)}}$$

(*b*) Integrate dU to find the total potential energy of the system:

$$U = -\frac{GMm_0}{L}\int_{-L/2}^{L/2}\frac{dx_s}{x_0 - x_s} = \frac{GMm_0}{L}\left[\ln\left(x_0 - \frac{L}{2}\right) - \ln\left(x_0 + \frac{L}{2}\right)\right]$$

$$= \boxed{-\frac{GMm_0}{L}\ln\left(\frac{x_0 + L/2}{x_0 - L/2}\right)}$$

(*c*) Because x_0 is a general point along the x axis:

$$F(x_0) = -\frac{dU}{dx_0} = \frac{GMm_0}{L}\left[\frac{1}{x_0 + \frac{L}{2}} - \frac{1}{x_0 - \frac{L}{2}}\right]$$

Further simplification yields:

$$F(x_0) = -\frac{Gmm_0}{x^2 - L^2/4}$$

This answer and the answer given in Example 11-8 are the same.

104 ••• A uniform sphere of mass M is located near a thin, uniform rod of mass m and length L as in Figure 11-31. Find the gravitational force of attraction exerted by the sphere on the rod.

Picture the Problem Choose a mass element dm of the rod of thickness dx at a distance x from the origin. All such elements of the rod experience a gravitational force dF due to presence of the sphere centered at the origin. We can find the total gravitational force of attraction experienced by the rod by integrating dF from $x = a$ to $x = a + L$.

Express the gravitational force dF acting on the element of the rod of mass dm:

$$dF = \frac{GMdm}{x^2}$$

Express dm in terms of the mass m and length L of the rod:

$$dm = \frac{m}{L}dx$$

Substitute for dm to obtain:

$$dF = \frac{GMm}{L}\frac{dx}{x^2}$$

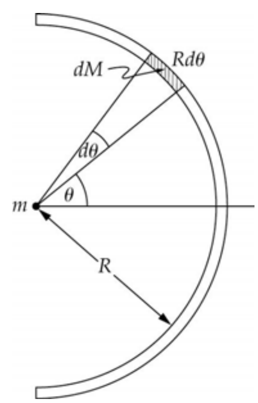
Integrate dF from $x = a$ to $x = a + L$ to find the total gravitational force acting on the rod:

$$F = \frac{GMm}{L}\int_a^{a+L} x^{-2}dx = -\frac{GMm}{L}\left[\frac{1}{x}\right]_a^{a+L}$$

$$= \boxed{\frac{GMm}{a(a+L)}}$$

105 ••• A thin uniform 20-kg rod with a length equal to 5.0 m is bent into a semicircle. What is the gravitational force exerted by the rod on a 0.10-kg point mass located at the center of curvature of the circular arc?

Picture the Problem The semicircular rod is shown in the figure. We'll use an element of length $Rd\theta = (L/\pi)d\theta$ whose mass dM is $(M/\pi)d\theta$. By symmetry, $F_y = 0$. We'll first find dF_x and then integrate over θ from $-\pi/2$ to $\pi/2$.



Express dF_x:

$$dF_x = \frac{GmdM}{R^2} = \frac{GMm}{\pi\left(\frac{L}{\pi}\right)^2}d\theta\cos\theta$$

Integrate dF_x over θ from $-\pi/2$ to $\pi/2$:

$$F_x = \frac{\pi\, GMm}{L^2}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{2\pi\, GMm}{L^2}$$

Substitute numerical values and evaluate F_x:

$$F_x = \frac{2\pi\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)(20\,\text{kg})(0.10\,\text{kg})}{(5.0\,\text{m})^2} = \boxed{34\,\text{pN}}$$

106 ••• Both the Sun and the moon exert gravitational forces on the oceans of Earth, causing tides. (*a*) Show that the ratio of the force exerted on a point particle on the surface of Earth by the Sun to that exerted by the moon is $M_s r_m^2 / M_m r_s^2$. Here M_s and M_m represent the masses of the Sun and moon and r_s and r_m are the distances of the particle from Earth to the Sun and Earth to the moon, respectively. Evaluate this ratio numerically. (*b*) Even though the Sun exerts a much greater force on the oceans than does the moon, the moon has a greater effect on the tides because it is the *difference* in the force from one side of Earth to the other that is important. Differentiate the expression $F = G\, m_1 m_2/r^2$ to calculate the change in F due to a small change in r. Show that $dF/F = (-2\ dr)/r$. (*c*) The oceanic *tidal bulge* (that is, the elongation of the liquid water of the oceans causing two opposite high and two opposite low spots) is caused by the difference in gravitational force on the oceans from one side of Earth to the other. Show that for a small difference in distance compared to the average distance, the ratio of the differential gravitational force exerted by the Sun to the differential gravitational force exerted by the moon on Earth's oceans is given by $\Delta F_s / \Delta F_m \approx \left(M_s r_m^3\right) / \left(M_m r_s^3\right)$. Calculate this ratio. What is your conclusion? Which object, the moon or the Sun, is the main cause of the tidal stretching of the oceans on Earth?

Picture the Problem We can begin by expressing the forces exerted by the Sun and the moon on a body of water of mass *m* and taking the ratio of these forces. In (*b*) we'll simply follow the given directions and in (*c*) we can approximate differential quantities with finite quantities to establish the given ratio.

(*a*) Express the force exerted by the sun on a body of water of mass *m*:

$$F_S = \frac{GM_S m}{r_S^2}$$

Express the force exerted by the moon on a body of water of mass *m*:

$$F_m = \frac{GM_m m}{r_m^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{F_S}{F_m} = \boxed{\frac{M_S r_m^2}{M_m r_S^2}}$$

Substitute numerical values and evaluate this ratio:

$$\frac{F_S}{F_m} = \frac{\left(1.99\times10^{30}\ \text{kg}\right)\left(3.84\times10^{8}\ \text{m}\right)^2}{\left(7.36\times10^{22}\ \text{kg}\right)\left(1.50\times10^{11}\ \text{m}\right)^2} = \boxed{177}$$

(*b*) Find $\dfrac{dF}{dr}$:

$$\frac{dF}{dr} = -\frac{2Gm_1 m_2}{r^3} = -2\frac{F}{r}$$

Solve for the ratio $\frac{dF}{F}$:

$$\frac{dF}{F} = \boxed{-2\frac{dr}{r}}$$

(*c*) Express the change in force ΔF for a small change in distance Δr:

$$\Delta F = -2\frac{F}{r}\Delta r$$

Express ΔF_S:

$$\Delta F_S = -2\frac{\frac{GmM_S}{r_S^2}}{r_S}\Delta r_S = -2\frac{GmM_S}{r_S^3}\Delta r_S$$

Express ΔF_m:

$$\Delta F_m = -2\frac{GmM_m}{r_m^3}\Delta r_m$$

Divide the first of these equations by the second and simplify:

$$\frac{\Delta F_S}{\Delta F_m} = \frac{\frac{M_S}{r_S^3}\Delta r_S}{\frac{M_m}{r_m^3}\Delta r_m} = \frac{M_S r_m^3}{M_m r_S^3}\frac{\Delta r_S}{\Delta r_m}$$

Because $\frac{\Delta r_S}{\Delta r_m} = 1$:

$$\frac{\Delta F_S}{\Delta F_m} = \boxed{\frac{M_S r_m^3}{M_m r_S^3}}$$

Substitute numerical values and evaluate this ratio:

$$\frac{\Delta F_S}{\Delta F_m} = \frac{(1.99\times10^{30}\text{ kg})(3.84\times10^{8}\text{ m})^3}{(7.36\times10^{22}\text{ kg})(1.50\times10^{11}\text{ m})^3} = \boxed{0.454}$$

Because the ratio of the forces is less than one, the moon is the main cause of the tidal stretching of the oceans on Earth.

107 ••• United Federation Starship *Excelsior* is dropping two small robotic probes towards the surface of a neutron star for exploration. The mass of the star is the same as that of the Sun, but the star's diameter is only 10 km. The robotic probes are linked together by a 1.0-m-long steel cord (which includes communication lines between the two probes), and are dropped vertically (that is, one always above the other). The ship hovers at rest above the star's surface. As the Chief of Materials Engineering on the ship, you are concerned that the communication between the two probes, a crucial aspect of the mission, will not survive. (*a*) Outline your briefing session to the mission commander and explain the existence of a "stretching force" that will try to pull the robots apart as they fall toward the planet. (See Problem 106 for hints.) (*b*) Assume that the cord in use has a breaking tension of 25 kN and that the robots each have a mass of

1.0 kg. How close will the robots be to the surface of the star before the cord breaks?

Picture the Problem Let M_{NS} be the mass of the neutron star and m the mass of each robot. We can use Newton's law of gravity to express the difference in the tidal-like forces acting on the coupled robots. Expanding the expression for the force on the robot further from the neutron star binomially will lead us to an expression for the distance at which the breaking tension in the connecting cord will be exceeded.

(*a*) The gravitational force is greater on the lower robot, so if it were not for the cable its acceleration would be greater than that of the upper robot and they would separate. In opposing this separation the cable is stressed.

(*b*) Letting the separation of the two robots be Δr, and the distance from the center of the star to the lower robot be r, use Newton's law of gravity to express the difference in the forces acting on the robots:

$$F_{\text{tide}} = \frac{GM_{NS}m}{r^2} - \frac{GM_{NS}m}{(r+\Delta r)^2}$$

$$= GM_{NS}m\left[\frac{1}{r^2} - \frac{1}{r^2\left(1+\frac{\Delta r}{r}\right)^2}\right]$$

$$= \frac{GM_{NS}m}{r^2}\left[1-\left(1+\frac{\Delta r}{r}\right)^{-2}\right]$$

Expand the expression in the square brackets binomially and simplify to obtain:

$$1-\left(1+\frac{\Delta r}{r}\right)^{-2} \approx 1-\left(1-2\frac{\Delta r}{r}\right)$$

$$= 2\frac{\Delta r}{r}$$

Substituting yields:

$$F_{\text{tide}} \approx \frac{2GM_{NS}m}{r^3}\Delta r$$

Letting F_B be the breaking tension of the cord, substitute for F_{tide} and solve for the value of r corresponding to the breaking strain being exceeded:

$$r = \sqrt[3]{\frac{2GM_{NS}m}{F_B}\Delta r}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\frac{2(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\,\text{kg})(1.0\,\text{kg})}{25\,\text{kN}}(1.0\,\text{m})} = \boxed{2.2\times10^5\ \text{m}}$$

Chapter 12
Static Equilibrium and Elasticity

Conceptual Problems

1 • **[SSM]** True or false:

(*a*) $\sum_i \vec{F}_i = 0$ is sufficient for static equilibrium to exist.
(*b*) $\sum_i \vec{F}_i = 0$ is necessary for static equilibrium to exist.
(*c*) In static equilibrium, the net torque about any point is zero.
(*d*) An object in equilibrium cannot be moving.

(*a*) False. The conditions $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ must be satisfied.

(*b*) True. The necessary and sufficient conditions for static equilibrium are $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$.

(*c*) True. The conditions $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ must be satisfied.

(*d*) False. An object can be moving with constant speed (translational or rotational) when the conditions $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ are satisfied.

2 • True or false:

(*a*) The center of gravity is always at the geometric center of a body.
(*b*) The center of gravity must be located inside an object.
(*c*) The center of gravity of a baton is located between the two ends.
(*d*) The torque produced by the force of gravity about the center of gravity is always zero.

(*a*) False. The location of the center of gravity depends on how an object's mass is distributed.

(*b*) False. An example of an object for which the center of gravity is outside the object is a donut.

(*c*) True. The structure of a baton and the definition of the center of gravity guarantee that the center of gravity of a baton is located between the two ends.

(*d*) True. Because the force of gravity acting on an object acts through the center of gravity of the object, its lever (or moment) arm is always zero.

3 • The horizontal bar in Figure 12-27 will remain horizontal if (a) $L_1 = L_2$ and $R_1 = R_2$, (b) $M_1R_1 = M_2R_2$, (c) $M_2R_1 = R_2M_1$, (d) $L_1M_1 = L_2M_2$, (e) $R_1L_1 = R_2L_2$.

Determine the Concept The condition that the bar is in rotational equilibrium is that the net torque acting on it equal zero; i.e., $R_1M_1 - R_2M_2 = 0$. $\boxed{(b)}$ is correct.

4 • Sit in a chair with your back straight. Now try to stand up without leaning forward. Explain why you cannot do it.

Determine the Concept You cannot stand up because your body's center of gravity must be above your feet.

5 • You have a job digging holes for posts to support signs for a Louisiana restaurant (called Mosca's). Explain why the higher above the ground a sign is mounted, the farther the posts should extend into the ground.

Determine the Concept The higher the sign, the greater the torque about the horizontal axis through the lowest extremes of the posts for a given wind speed. In addition, the deeper the post holes the greater the maximum opposing torque about the same axis for a given consistency of the dirt. Thus, higher signs require deeper holes for the signposts.

6 • A father (mass M) and his son, (mass m) begin walking out towards opposite ends of a balanced see-saw. As they walk, the see-saw stays exactly horizontal. What can be said about the relationship between the father's speed V and the son's speed v?

Determine the Concept The question is about a situation in which an object is in static equilibrium. Both the father and son are walking outward from the center of the see-saw, which always remains in equilibrium. In order for this to happen, at any time, the net torque about any point (let's say, the pivot point at the center of the see-saw) must be zero. We can denote the father's position as X, and the son's position as x, and choose the origin of coordinates to be at the pivot point. At each moment, the see-saw exerts normal forces on the son and his father equal to their respective weights, mg and Mg. By Newton's third law, the father exerts a downward force equal in magnitude to the normal force, and the son exerts a downward force equal in magnitude to the normal force acting on him.

Apply $\sum\tau_{\text{pivot point}} = 0$ to the see-saw (assume that the father walks to the left and that counterclockwise torques are positive):

$$MgX - mgx = 0 \quad (1)$$

Express the distance both the father and his son walk as a function of time:

$X = V\Delta t$ and $x = v\Delta t$

Substitute for X and x in equation (1) to obtain:

$$MgV\Delta t - mgv\Delta t = 0 \Rightarrow V = \boxed{\frac{m}{M}v}$$

Remarks: The father's speed is less than the son's speed by a factor of *m*/*M*.

7 • Travel mugs that people might set on the dashboards of their cars are often made with broad bases and relatively narrow mouths. Why would travel mugs be designed with this shape, rather than have the roughly cylindrical shape that mugs normally have?

Determine the Concept The main reason this is done is to lower the center of gravity of the mug. The lower the center of gravity the more stable the mug is.

8 •• The sailors in the photo are using a technique called ″hiking out″. What purpose does positioning themselves in this way serve? If the wind were stronger, what would they need to do in order to keep their craft stable?

Determine the Concept Dynamically the boats are in equilibrium along their line of motion, but in the plane of their sail and the sailors, they are in static equilibrium. The torque on the boat, applied by the wind acting on the sail, has a tendency to tip the boat. The rudder counteracts that tendency to some degree, but in particularly strong winds, when the boat is sailing at particular angles with respect to the wind, the sailors need to ″hike out″ to apply some torque (due to the gravitational force of Earth on the sailors) by leaning outward on the beam of the boat. If the wind strengthens, they need to extend their bodies further over the side and may need to get into a contraption called a ″trapeze″ that enables the sailor to have his or her entire body outside the boat.

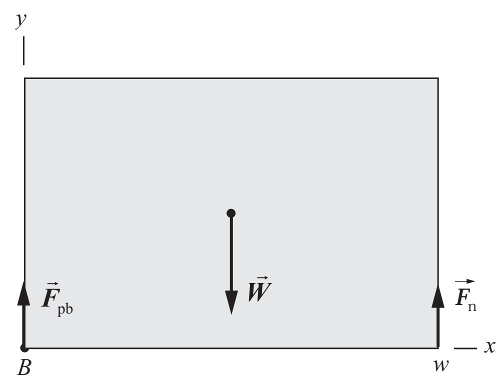
9 •• **[SSM]** An aluminum wire and a steel wire of the same length L and diameter D are joined end-to-end to form a wire of length $2L$. One end of the wire is then fastened to the ceiling and an object of mass M is attached to the other end. Neglecting the mass of the wires, which of the following statements is true? (*a*) The aluminum portion will stretch by the same amount as the steel portion. (*b*) The tensions in the aluminum portion and the steel portion are equal. (*c*) The tension in the aluminum portion is greater than that in the steel portion. (*d*) None of the above

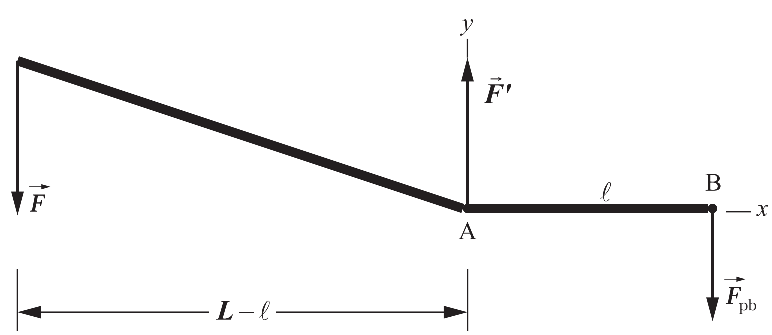
Determine the Concept We know that equal lengths of aluminum and steel wire of the same diameter will stretch different amounts when subjected to the same tension. Also, because we are neglecting the mass of the wires, the tension in them is independent of which one is closer to the roof and depends only on Mg. $\boxed{(b)}$ is correct.

Estimation and Approximation

10 •• A large crate weighing 4500 N rests on four 12-cm-high blocks on a horizontal surface (Figure 12-28). The crate is 2.0 m long, 1.2 m high and 1.2 m deep. You are asked to lift one end of the crate using a long steel pry bar. The fulcrum on the pry bar is 10 cm from the end that lifts the crate. Estimate the length of the bar you will need to lift the end of the crate.

Picture the Problem The diagram to the right shows the forces acting on the crate as it is being lifted at its left end. Note that when the crowbar lifts the crate, only half the weight of the crate is supported by the bar. Choose the coordinate system shown and let the subscript ″pb″ refer to the pry bar. The diagram below shows the forces acting on the pry bar as it is being used to lift the end of the crate.





Assume that the maximum force F you can apply is 500 N (about 110 lb). Let ℓ be the distance between the points of contact of the steel bar with the floor and the crate, and let L be the total length of the bar. Lacking information regarding the bend in pry bar at the fulcrum, we'll assume that it is small enough to be negligible. We can apply the condition for rotational equilibrium to the pry bar and a condition for translational equilibrium to the crate when its left end is on the verge of lifting.

Apply $\sum F_y = 0$ to the crate: $F_{\text{pb}} - W + F_{\text{n}} = 0$ (1)

Apply $\sum \vec{\tau} = 0$ to the crate about an axis through point B and perpendicular to the plane of the page to obtain: $wF_{\text{n}} - \frac{1}{2}wW = 0 \Rightarrow F_{\text{n}} = \frac{1}{2}W$ as noted in **Picture the Problem**.

Solve equation (1) for F_{pb} and substitute for F_{n} to obtain: $F_{\text{pb}} = W - \frac{1}{2}W = \frac{1}{2}W$

Apply $\sum \vec{\tau} = 0$ to the pry bar about an axis through point A and perpendicular to the plane of the page to obtain:

$$F(L-\ell) - \ell F_{\text{pb}} = 0 \Rightarrow L = \ell\left(1 + \frac{F_{\text{pb}}}{F}\right)$$

Substitute for F_{pb} to obtain:

$$L = \ell\left(1 + \frac{W}{2F}\right)$$

Substitute numerical values and evaluate L:

$$L = (0.10\,\text{m})\left(1 + \frac{4500\,\text{N}}{2(500\,\text{N})}\right) = \boxed{55\,\text{cm}}$$

11 •• **[SSM]** Consider an atomic model for Young's modulus. Assume that a large number of atoms are arranged in a cubic array, with each atom at a corner of a cube and each atom at a distance *a* from its six nearest neighbors. Imagine that each atom is attached to its 6 nearest neighbors by little springs each with force constant *k*. (*a*) Show that this material, if stretched, will have a Young's modulus $Y = k/a$. (*b*) Using Table 12-1 and assuming that $a \approx 1.0$ nm, estimate a typical value for the "atomic force constant" *k* in a metal.

Picture the Problem We can derive this expression by imagining that we pull on an area *A* of the given material, expressing the force each spring will experience, finding the fractional change in length of the springs, and substituting in the definition of Young's modulus.

(*a*) The definition of Young's modulus is:

$$Y = \frac{F/A}{\Delta L/L} \quad (1)$$

Express the elongation ΔL of each spring:

$$\Delta L = \frac{F_{\text{s}}}{k} \quad (2)$$

The force F_s each spring will experience as a result of a force F acting on the area A is:

$$F_s = \frac{F}{N}$$

Express the number of springs N in the area A:

$$N = \frac{A}{a^2}$$

Substituting for N yields:

$$F_s = \frac{Fa^2}{A}$$

Substitute F_s in equation (2) to obtain, for the extension of one spring:

$$\Delta L = \frac{Fa^2}{kA}$$

Assuming that the springs extend/compress linearly, the fractional extension of the springs is:

$$\frac{\Delta L_{\text{tot}}}{L} = \frac{\Delta L}{a} = \frac{1}{a}\frac{Fa^2}{kA} = \frac{Fa}{kA}$$

Substitute in equation (1) and simplify to obtain:

$$Y = \frac{\frac{F}{A}}{\frac{Fa}{kA}} = \boxed{\frac{k}{a}}$$

(*b*) From our result in Part (*a*):

$$k = Ya$$

From Table 12-1:

$$Y = 200\,\text{GN/m}^2 = 2.00\times10^{11}\ \text{N/m}^2$$

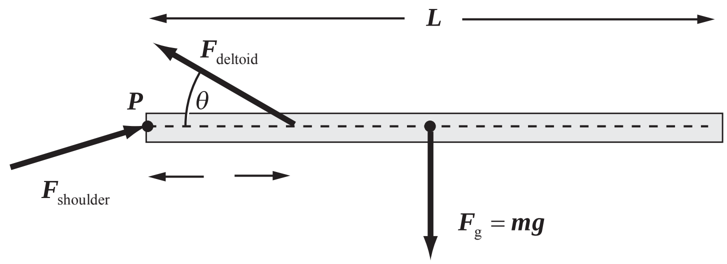
Substitute numerical values and evaluate k:

$$k = \left(2.00\times10^{11}\ \text{N/m}^2\right)\left(1.0\times10^{-9}\ \text{m}\right) = \boxed{2.0\,\text{N/cm}}$$

12 •• By considering the torques about the centers of the ball joints in your shoulders, estimate the force your deltoid muscles (those muscles on top of your shoulder) must exert on your upper arm, in order to keep your arm held out and extended at shoulder level. Then, estimate the force they must exert when you hold a 10-lb weight out to the side at arm's length.

Picture the Problem A model of your arm is shown in the pictorial representation. Your shoulder joint is at point P and the force the deltoid muscles exert on your extended arm $\vec{F}_{\text{deltoid}}$ is shown acting at an angle θ with the horizontal. The weight of your arm is the gravitational force $\vec{F}_g = m\vec{g}$ exerted by Earth through the center of gravity of your arm. We can use the condition for rotational equilibrium to estimate the forces exerted by your deltoid muscles. Note that, because its moment

arm is zero, the torque due to $\vec{F}_{\text{shoulder}}$ about an axis through point P and perpendicular to the page is zero.



Apply $\sum \tau_P = 0$ to your extended arm:

$$F_{\text{deltoid}}\ell\sin\theta - \tfrac{1}{2}mgL = 0 \qquad (1)$$

Solving for F_{deltoid} yields:

$$F_{\text{deltoid}} = \frac{mgL}{2\ell\sin\theta}$$

Assuming that $\ell \approx 20$ cm, $L \approx 60$ cm, $mg \approx 10$ lb, and $\theta \approx 10°$, substitute numerical values and evaluate F_{deltoid}:

$$F_{\text{deltoid}} = \frac{(10\text{ lb})(60\text{ cm})}{2(20\text{ cm})\sin 10°} \approx \boxed{86\text{ lb}}$$

If you hold a 10-lb weight at the end of your arm, equation (1) becomes:

$$F'_{\text{deltoid}}\ell\sin\theta - \tfrac{1}{2}mgL - m'gL = 0$$

where m' is the mass of the 10-lb weight.

Solving for F_{deltoid} yields:

$$F'_{\text{deltoid}} = \frac{mgL + 2m'gL}{2\ell\sin\theta}$$

Substitute numerical values and evaluate F'_{deltoid}:

$$F_{\text{deltoid}} = \frac{(10\text{ lb})(60\text{ cm}) + 2(10\text{ lb})(60\text{ cm})}{2(20\text{ cm})\sin 10°} \approx \boxed{260\text{ lb}}$$

Conditions for Equilibrium

13 • Your crutch is pressed against the sidewalk with a force $\vec{F}_{\text{c}}$ along its own direction, as shown in Figure 12-29. This force is balanced by the normal force $\vec{F}_{\text{n}}$ and a frictional force $\vec{f}_{\text{s}}$. (*a*) Show that when the force of friction is at its maximum value, the coefficient of friction is related to the angle θ by $\mu_{\text{s}} = \tan\theta$. (*b*) Explain how this result applies to the forces on your foot when you are not using a crutch. (*c*) Why is it advantageous to take short steps when walking on slippery surfaces?

Picture the Problem Choose a coordinate system in which upward is the $+y$ direction and to the right is the $+x$ direction and use the conditions for translational equilibrium.

(*a*) Apply $\sum \vec{F} = 0$ to the forces acting on the tip of the crutch:

$$\sum F_x = -f_s + F_c \sin\theta = 0 \qquad (1)$$

and

$$\sum F_y = F_n - F_c \cos\theta = 0 \qquad (2)$$

Solve equation (2) for F_n and assuming that $f_s = f_{s,max}$, obtain:

$$f_s = f_{s,max} = \mu_s F_n = \mu_s F_c \cos\theta$$

Substitute in equation (1) and solve for μ_s:

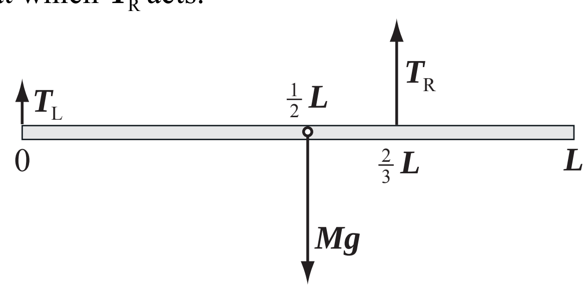
$$\mu_s = \boxed{\tan\theta}$$

(*b*) Taking long strides requires a large coefficient of static friction because θ is large for long strides.

(*c*) If μ_s is small, that is, there is ice on the surface, θ must be small to avoid slipping.

14 •• A thin uniform rod of mass M is suspended horizontally by two vertical wires. One wire is at the left end of the rod, and the other wire is 2/3 of the length of the rod from the left end. (*a*) Determine the tension in each wire. (*b*) An object is now hung by a string attached to the right end of the rod. When this happens, it is noticed that the rod remains horizontal but the tension in the wire on the left vanishes. Determine the mass of the object.

Picture the Problem The pictorial representation shows the thin rod with the forces described in Part (*a*) acting on it. We can apply $\sum \vec{\tau}_0 = 0$ to the rod to find the forces T_L and T_R. The simplest way to determine the mass m of the object suspended from the rod in (*b*) is to apply the condition for rotational equilibrium a second time, but this time with respect to an axis perpendicular to the page and through the point at which $\vec{T}_R$ acts.



(a) Apply $\sum \vec{\tau}_0 = 0$ to the rod: $\frac{2}{3}LT_R - \frac{1}{2}LMg = 0 \Rightarrow T_R = \boxed{\frac{3}{4}Mg}$

Apply $\sum \vec{F}_{\text{vertical}} = 0$ to the rod: $T_L - Mg + T_R = 0$

Substitute for T_R to obtain: $T_L - Mg + \frac{3}{4}Mg = 0 \Rightarrow T_L = \boxed{\frac{1}{4}Mg}$

(b) With an object of mass m suspended from the right end of the rod and $T_L = 0$, applying $\sum \vec{\tau} = 0$ about an axis perpendicular to the page and through the point at which $\vec{T}_R$ acts yields:

$$\left(\tfrac{2}{3} - \tfrac{1}{2}\right)LMg - \left(1 - \tfrac{2}{3}\right)Lmg = 0$$

Solving for m yields: $m = \boxed{\frac{1}{2}M}$

The Center of Gravity

15 • An automobile has 58 percent of its weight on the front wheels. The front and back wheels on each side are separated by 2.0 m. Where is the center of gravity located?

Picture the Problem Let the weight of the automobile be w. Choose a coordinate system in which the origin is at the point of contact of the front wheels with the ground and the positive x axis includes the point of contact of the rear wheels with the ground. Apply the definition of the center of gravity to find its location.

Use the definition of the center of gravity to obtain:

$$\begin{aligned} x_{cg}W &= \sum_i w_i x_i \\ &= 0.58w(0) + 0.42w(2.0\,\text{m}) \\ &= (0.84\,\text{m})w \end{aligned}$$

Because $W = w$: $x_{cg}w = (0.84\,\text{m})w \Rightarrow \boldsymbol{x}_{cg} = \boxed{84\,\text{cm}}$

Static Equilibrium

16 • Figure 12-30 shows a lever of negligible mass with a vertical force F_{app} being applied to lift a load F. The *mechanical advantage* of the lever is defined as $M = F/F_{\text{app, min}}$, where $F_{\text{app, min}}$ is the smallest force necessary to lift the load F. Show that for this simple lever system, $M = x/X$, where x is the moment arm (distance to the pivot) for the applied force and X is the moment arm for the load.

Picture the Problem We can use the given definition of the mechanical advantage of a lever and the condition for rotational equilibrium to show that $M = x/X$.

Express the definition of mechanical advantage for a lever:

$$M = \frac{F}{F_{\text{app, min}}} \tag{1}$$

Apply the condition for rotational equilibrium to the lever:

$$\sum \tau_{\text{fulcrum}} = xF_{\text{app, min}} - XF = 0$$

Solve for the ratio of F to $F_{\text{app, min}}$ to obtain:

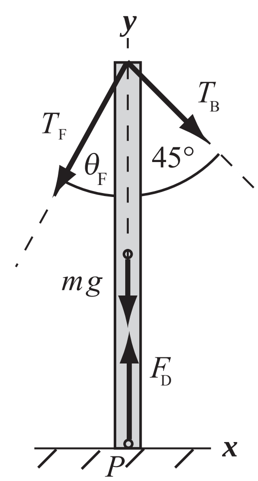
$$\frac{F}{F_{\text{app, min}}} = \frac{x}{X}$$

Substitute for $F/F_{\text{app, min}}$ in equation (1) to obtain:

$$M = \boxed{\frac{x}{X}}$$

17 • **[SSM]** Figure 12-31 shows a 25-foot sailboat. The mast is a uniform 120-kg pole that is supported on the deck and held fore and aft by wires as shown. The tension in the *forestay* (wire leading to the bow) is 1000 N. Determine the tension in the *backstay* (wire leading aft) and the normal force that the deck exerts on the mast. (Assume that the frictional force the deck exerts on the mast to be negligible.)

Picture the Problem The force diagram shows the forces acting on the mast. Let the origin of the coordinate system be at the foot of the mast with the $+x$ direction to the right and the $+y$ direction upward. Because the mast is in equilibrium, we can apply the conditions for translational and rotational equilibrium to find the tension in the backstay, T_B, and the normal force, F_D, that the deck exerts on the mast.



Apply $\sum \vec{\tau} = 0$ to the mast about an axis through point P:

$$(4.88\,\text{m})(1000\,\text{N})\sin\theta_F - (4.88\,\text{m})T_B \sin 45.0° = 0$$

Solve for T_B to obtain:

$$T_B = \frac{(1000\,\text{N})\sin\theta_F}{\sin 45.0°} \tag{1}$$

Find θ_F, the angle of the forestay with the vertical:

$$\theta_F = \tan^{-1}\left(\frac{2.74\,\text{m}}{4.88\,\text{m}}\right) = 29.3°$$

Substitute numerical values in equation (1) and evaluate T_B:

$$T_B = \frac{(1000\,\text{N})\sin 29.3°}{\sin 45.0°} = \boxed{692\,\text{N}}$$

Apply the condition for translational equilibrium in the y direction to the mast:

$$\sum F_y = F_D - T_F \cos\theta_F - T_B \cos 45° - mg = 0$$

Solving for F_D yields:

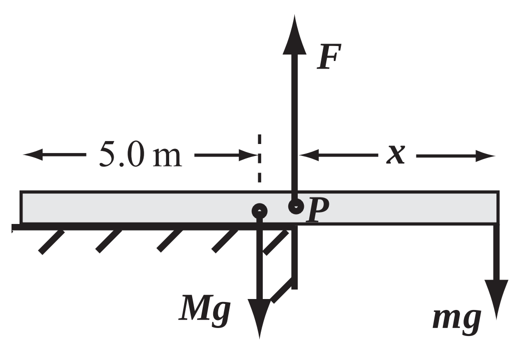
$$F_D = T_F \cos\theta_F + T_B \cos 45° + mg$$

Substitute numerical values and evaluate F_D:

$$F_D = (1000\text{ N})\cos 29.3° + (692\text{ N})\cos 45° + (120\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{2.54\,\text{kN}}$$

18 •• A uniform 10.0-m beam of mass 300 kg extends over a ledge as in Figure 12-32. The beam is not attached, but simply rests on the surface. A 60.0-kg student intends to position the beam so that he can walk to the end of it. What is the maximum distance the beam can extend past the end of the ledge and still allow him to perform this feat?

Picture the Problem The diagram shows $M\vec{g}$, the weight of the beam, $m\vec{g}$, the weight of the student, and the force the ledge exerts $\vec{F}$, acting on the beam. Because the beam is in equilibrium, we can apply the condition for rotational equilibrium to the beam to find the location of the pivot point P that will allow the student to walk to the end of the beam.



Apply $\sum \vec{\tau} = 0$ about an axis through the pivot point P:

$$Mg(5.0\,\text{m} - x) - mgx = 0$$

Solving for x yields:

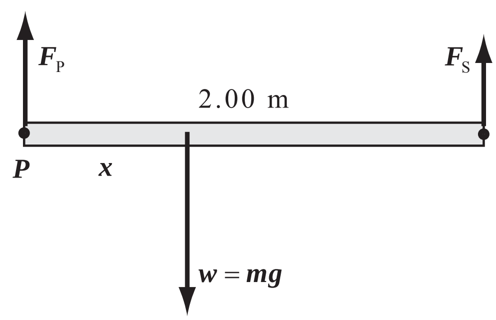
$$x = \frac{5.0M}{M+m}$$

Substitute numerical values and evaluate x:

$$x = \frac{(5.0\text{ m})(300\,\text{kg})}{300\,\text{kg} + 60.0\,\text{kg}} = \boxed{4.2\,\text{m}}$$

19 •• **[SSM]** A *gravity board* is a convenient and quick way to determine the location of the center of gravity of a person. It consists of a horizontal board supported by a fulcrum at one end and a scale at the other end. To demonstrate this in class, your physics professor calls on you to lie horizontally on the board with the top of your head directly above the fulcrum point as shown in Figure 12-33. The scale is 2.00 m from the fulcrum. In preparation for this experiment, you had accurately weighed yourself and determined your mass to be 70.0 kg. When you are at rest on the gravity board, the scale advances 250 N beyond its reading when the board is there by itself. Use this data to determine the location of your center of gravity relative to your feet.

Picture the Problem The diagram shows $\vec{w}$, the weight of the student, $\vec{F}_P$, the force exerted by the board at the pivot, and $\vec{F}_s$, the force exerted by the scale, acting on the student. Because the student is in equilibrium, we can apply the condition for rotational equilibrium to the student to find the location of his center of gravity.



Apply $\sum \vec{\tau} = 0$ about an axis through the pivot point *P*:

$$F_s(2.00\,\text{m}) - wx = 0$$

Solving for *x* yields:

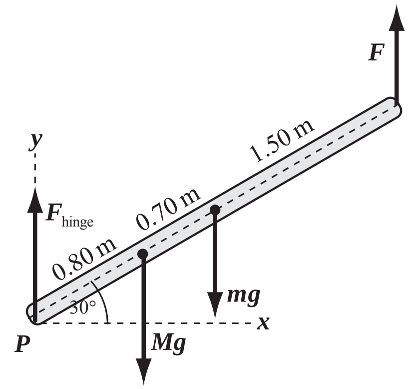
$$x = \frac{(2.00\,\text{m})F_s}{w}$$

Substitute numerical values and evaluate *x*:

$$x = \frac{(2.00\,\text{m})(250\,\text{N})}{(70.0\,\text{kg})(9.81\,\text{m/s}^2)} = \boxed{0.728\,\text{m}}$$

20 •• A stationary 3.0-m board of mass 5.0 kg is hinged at one end. A force $\vec{F}$ is applied vertically at the other end, and the board makes at 30° angle with the horizontal. A 60-kg block rests on the board 80 cm from the hinge as shown in Figure 12-34. (*a*) Find the magnitude of the force $\vec{F}$. (*b*) Find the force exerted by the hinge. (*c*) Find the magnitude of the force $\vec{F}$, as well as the force exerted by the hinge, if $\vec{F}$ is exerted, instead, at right angles to the board.

Picture the Problem The diagram shows $m\vec{g}$, the weight of the board, $\vec{F}_{\text{hinge}}$, the force exerted by the hinge, $M\vec{g}$, the weight of the block, and $\vec{F}$, the force acting vertically at the right end of the board. Because the board is in equilibrium, we can apply the condition for rotational equilibrium to it to find the magnitude of $\vec{F}$.



(a) Apply $\sum \vec{\tau} = 0$ about an axis through the hinge to obtain:

$$F[(3.0\,\text{m})\cos 30°] - mg[(1.50\,\text{m})\cos 30°] - Mg[(0.80\,\text{m})\cos 30°] = 0$$

Solving for F yields:

$$F = \frac{m(1.50\,\text{m}) + M(0.80\,\text{m})}{3.0\,\text{m}} g$$

Substitute numerical values and evaluate F:

$$F = \frac{(5.0\,\text{kg})(1.50\,\text{m}) + (60\,\text{kg})(0.80\,\text{m})}{3.0\,\text{m}}(9.81\,\text{m/s}^2) = 181\,\text{N} = \boxed{0.181\,\text{kN}}$$

Apply $\sum F_y = 0$ to the board to obtain:

$$F_{\text{hinge}} - Mg - mg + F = 0$$

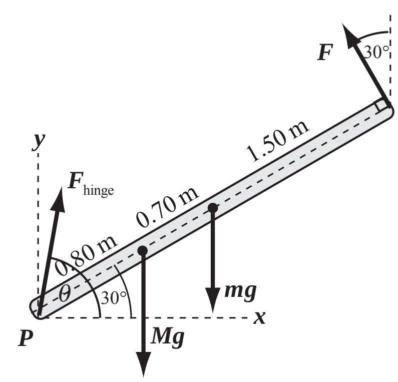
Solving for F_{hinge} yields:

$$F_{\text{hinge}} = Mg + mg - F = (M + m)g - F$$

Substitute numerical values and evaluate F_{hinge}:

$$F_{\text{hinge}} = (60\,\text{kg} + 5.0\,\text{kg})(9.81\,\text{m/s}^2) - 181\,\text{N} = \boxed{0.46\,\text{kN}}$$

(c) The force diagram showing the force $\vec{F}$ acting at right angles to the board is shown to the right:



Apply $\sum \vec{\tau} = 0$ about the hinge:

$$F(3.0\,\text{m}) - mg[(1.5\,\text{m})\cos 30°] - Mg[(0.80\,\text{m})\cos 30°] = 0$$

Solving for F yields:

$$F = \frac{m(1.5\,\text{m}) + M(0.80\,\text{m})}{3.0\,\text{m}} g\cos 30°$$

Substitute numerical values and evaluate F:

$$F = \frac{(5.0\,\text{kg})(1.5\,\text{m}) + (60\,\text{kg})(0.80\,\text{m})}{3.0\,\text{m}}(9.81\,\text{m/s}^2)\cos 30° = 157\,\text{N} = \boxed{0.16\,\text{kN}}$$

Apply $\sum F_y = 0$ to the board:

$$F_{\text{hinge}} \sin\theta - Mg - mg + F\cos 30° = 0$$

or

$$F_{\text{hinge}} \sin\theta = (M + m)g - F\cos 30° \quad (1)$$

Apply $\sum F_x = 0$ to the board:

$$F_{\text{hinge}} \cos\theta - F\sin 30° = 0$$

or

$$F_{\text{hinge}} \cos\theta = F\sin 30° \quad (2)$$

Divide the first of these equations by the second to obtain:

$$\frac{F_{\text{hinge}} \sin\theta}{F_{\text{hinge}} \cos\theta} = \frac{(M + m)g - F\cos 30°}{F\sin 30°}$$

Solving for θ yields:

$$\theta = \tan^{-1}\left[\frac{(M + m)g - F\cos 30°}{F\sin 30°}\right]$$

Substitute numerical values and evaluate θ:

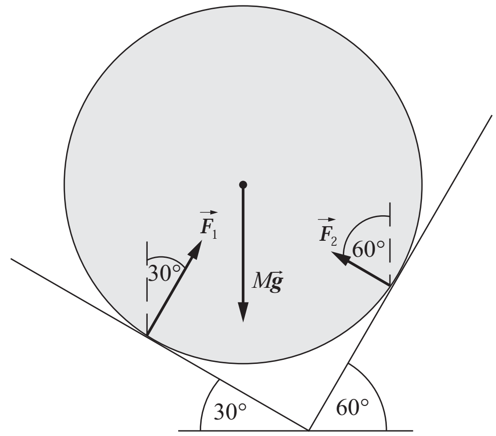
$$\theta = \tan^{-1}\left[\frac{(65\,\text{kg})(9.81\,\text{m/s}^2) - (157\,\text{N})\cos 30°}{(157\,\text{N})\sin 30°}\right] = 81.1°$$

Substitute numerical values in equation (2) and evaluate F_{hinge}:

$$F_{\text{hinge}} = \frac{(157\,\text{N})\sin 30°}{\cos 81.1°} = \boxed{0.51\,\text{kN}}$$

21 •• A cylinder of mass M is supported by a frictionless trough formed by a plane inclined at 30º to the horizontal on the left and one inclined at 60º on the right as shown in Figure 12-35. Find the force exerted by each plane on the cylinder.

Picture the Problem The planes are frictionless; therefore, the force exerted by each plane must be perpendicular to that plane. Let $\vec{F}_1$ be the force exerted by the 30° plane, and let $\vec{F}_2$ be the force exerted by the 60° plane. Choose a coordinate system in which the +*x* direction is to the right and the +*y* direction is upward. Because the cylinder is in equilibrium, we can use the conditions for translational equilibrium to find the magnitudes of $\vec{F}_1$ and $\vec{F}_2$.



Apply $\sum F_x = 0$ to the cylinder: $$F_1 \sin 30° - F_2 \sin 60° = 0 \quad (1)$$

Apply $\sum F_y = 0$ to the cylinder: $$F_1 \cos 30° + F_2 \cos 60° - W = 0 \quad (2)$$

Solve equation (1) for F_1: $$F_1 = \sqrt{3} F_2 \quad (3)$$

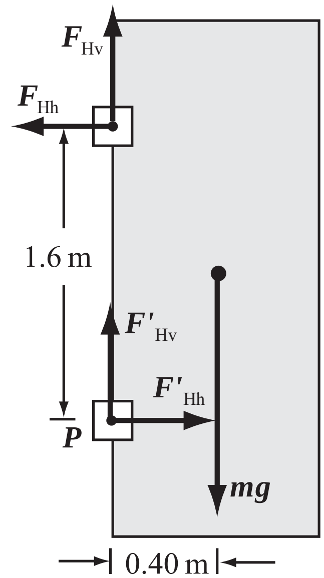
Substituting for F_1 in equation (2) yields: $$\sqrt{3} F_2 \cos 30° + F_2 \cos 60° - W = 0$$

Solving for F_2 gives: $$F_2 = \frac{W}{\sqrt{3}\cos 30° + \cos 60°} = \boxed{\tfrac{1}{2}W}$$

Substitute for F_2 in equation (3) to obtain: $$F_1 = \sqrt{3}\left(\tfrac{1}{2}W\right) = \boxed{\tfrac{\sqrt{3}}{2}W}$$

22 •• A uniform 18-kg door that is 2.0 m high by 0.80 m wide is hung from two hinges that are 20 cm from the top and 20 cm from the bottom. If each hinge supports half the weight of the door, find the magnitude and direction of the horizontal components of the forces exerted by the two hinges on the door.

Picture the Problem The drawing shows the door and its two supports. The center of gravity of the door is 0.80 m above (and below) the hinge, and 0.40 m from the hinges horizontally. Choose a coordinate system in which the +*x* direction is to the right and the +*y* direction is upward. Denote the horizontal and vertical components of the hinge force by F_{Hh} and F_{Hv}. Because the door is in equilibrium, we can use the conditions for translational and rotational equilibrium to determine the horizontal forces exerted by the hinges.



Apply $\sum\vec{\tau} = 0$ about an axis through the lower hinge:

$$F_{\text{Hh}}(1.6\,\text{m}) - mg(0.40\,\text{m}) = 0$$

Solve for F_{Hh}:

$$F_{\text{Hh}} = \frac{mg(0.40\,\text{m})}{1.6\,\text{m}}$$

Substitute numerical values and evaluate F_{Hh}:

$$F_{\text{Hh}} = \frac{(18\,\text{kg})(9.81\,\text{m/s}^2)(0.40\,\text{m})}{1.6\,\text{m}} = \boxed{44\,\text{N}}$$

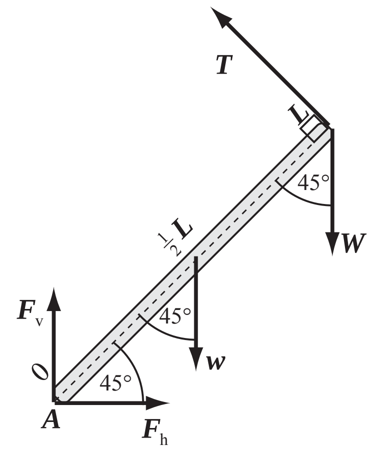
Apply $\sum F_x = 0$ to the door and solve for F'_{Hh}:

$$F'_{\text{Hh}} - F_{\text{Hh}} = 0$$
and
$$F'_{\text{Hh}} = \boxed{44\,\text{N}}$$

Remarks: Note that the upper hinge pulls on the door and the lower hinge pushes on it.

23 •• Find the force exerted on the strut by the hinge at A for the arrangement in Figure 12-36 if (*a*) the strut is weightless, and (*b*) the strut weighs 20 N.

Picture the Problem Let T be the tension in the line attached to the wall and L be the length of the strut. The figure includes w, the weight of the strut, for Part (b). Because the strut is in equilibrium, we can use the conditions for both rotational and translational equilibrium to find the force exerted on the strut by the hinge.



(a) Express the force exerted on the strut at the hinge:

$$\vec{F} = F_h\hat{i} + F_v\hat{j} \qquad (1)$$

Ignoring the weight of the strut, apply $\sum\vec{\tau} = 0$ at the hinge:

$$LT - (L\cos 45°)W = 0$$

Solve for the tension in the line:

$$T = W\cos 45° = (60\,\text{N})\cos 45° = 42.4\,\text{N}$$

Apply $\sum\vec{F} = 0$ to the strut:

$$\sum F_x = F_h - T\cos 45° = 0$$

and

$$\sum F_y = F_v + T\cos 45° - Mg = 0$$

Solve for and evaluate F_h:

$$F_h = T\cos 45° = (42.4\,\text{N})\cos 45° = 30\,\text{N}$$

Solve for and evaluate F_v:

$$F_v = Mg - T\cos 45° = 60\,\text{N} - (42.4\,\text{N})\cos 45° = 30\,\text{N}$$

Substitute in equation (1) to obtain:

$$\vec{F} = \boxed{(30\,\text{N})\hat{i} + (30\,\text{N})\hat{j}}$$

(b) Including the weight of the strut, apply $\sum\vec{\tau} = 0$ at the hinge:

$$LT - (L\cos 45°)W - \left(\frac{L}{2}\cos 45\right)w = 0$$

Solve for the tension in the line:

$$T = (\cos 45°)W + \left(\frac{1}{2}\cos 45°\right)w$$

Substitute numerical values and evaluate T:

$$T = (\cos 45°)(60\,\text{N}) + \left(\frac{1}{2}\cos 45°\right)(20\,\text{N}) = 49.5\,\text{N}$$

Apply $\sum \vec{F} = 0$ to the strut:

$$\sum F_x = F_\text{h} - T\cos 45° = 0$$

and

$$\sum F_y = F_\text{v} + T\cos 45° - W - w = 0$$

Solve for and evaluate F_h:

$$F_\text{h} = T\cos 45° = (49.5\,\text{N})\cos 45° = 35\,\text{N}$$

Solve for and evaluate F_v:

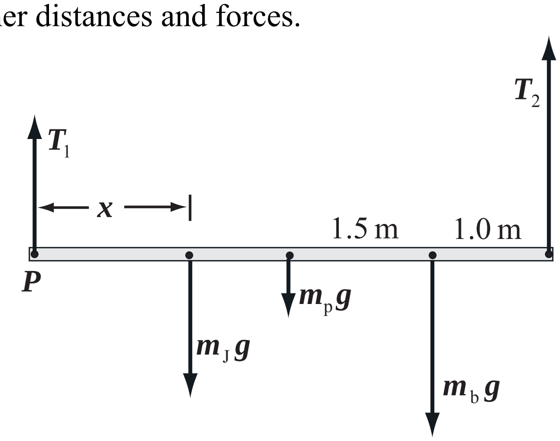
$$F_\text{v} = W + w - T\cos 45° = 60\,\text{N} + 20\,\text{N} - (49.5\,\text{N})\cos 45° = 45\,\text{N}$$

Substitute for F_h and F_v in equation (1) to obtain:

$$\vec{F} = \boxed{(35\,\text{N})\hat{\boldsymbol{i}} + (45\,\text{N})\hat{\boldsymbol{j}}}$$

24 •• Julie has been hired to help paint the trim of a building, but she is not convinced of the safety of the apparatus. A 5.0-m plank is suspended horizontally from the top of the building by ropes attached at each end. Julie knows from previous experience that the ropes being used will break if the tension exceeds 1.0 kN. Her 80-kg boss dismisses Julie's worries and begins painting while standing 1.0 m from the end of the plank. If Julie's mass is 60 kg and the plank has a mass of 20 kg, then over what range of positions can Julie stand to join her boss without causing the ropes to break?

Picture the Problem Note that if Julie is at the far left end of the plank, T_1 and T_2 are less than 1.0 kN. Let x be the distance of Julie from T_1. Because the plank is in equilibrium, we can apply the condition for rotational equilibrium to relate the distance x to the other distances and forces.



Apply $\sum\vec{\tau} = 0$ about an axis through the left end of the plank:

$$(5.0\,\text{m})T_2 - (4.0\,\text{m})m_\text{b}g - (2.5\,\text{m})m_\text{p}g - m_\text{J}gx = 0$$

Solving for x yields:

$$x = \frac{(5.0\,\text{m})T_2}{m_\text{J}g} - \frac{(4.0\,\text{m})m_\text{b}}{m_\text{J}} - \frac{(2.5\,\text{m})m_\text{p}}{m_\text{J}}$$

Substitute numerical values and simplify to obtain:

$$\boldsymbol{x} = \left(8.495\times10^{-3}\,\frac{\text{m}}{\text{N}}\right)\boldsymbol{T}_2 - 6.17\text{ m}$$

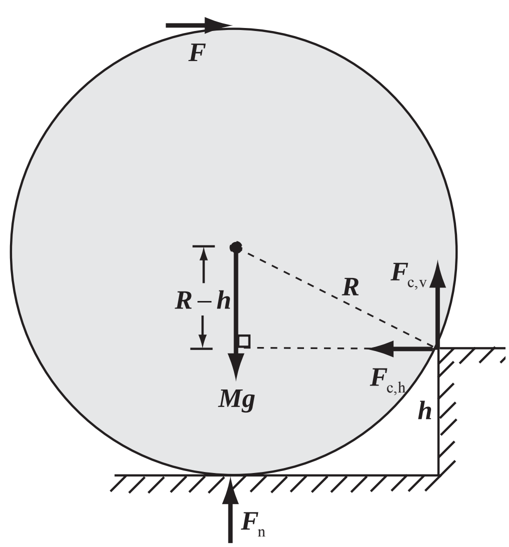
Set $T_2 = 1.0$ kN and evaluate x:

$$\boldsymbol{x} = \left(8.495\times10^{-3}\,\frac{\text{m}}{\text{N}}\right)(1.0\,\text{kN}) - 6.17\text{ m} = 2.3\,\text{m}$$

and Julie is safe provided $\boxed{\boldsymbol{x} < 2.3\,\text{m.}}$

25 •• **[SSM]** A cylinder of mass M and radius R rolls against a step of height h as shown in Figure 12-37. When a horizontal force of magnitude F is applied to the top of the cylinder, the cylinder remains at rest. (a) Find an expression for the normal force exerted by the floor on the cylinder. (b) Find an expression for the horizontal force exerted by the edge of the step on the cylinder. (c) Find an expression for the vertical component of the force exerted by the edge of the step on the cylinder.

Picture the Problem The figure to the right shows the forces acting on the cylinder. Choose a coordinate system in which the $+x$ direction is to the right and the $+y$ direction is upward. Because the cylinder is in equilibrium, we can apply the conditions for translational and rotational equilibrium to find F_n and the horizontal and vertical components of the force the corner of the step exerts on the cylinder.



(a) Apply $\sum\vec{\tau} = 0$ to the cylinder about the step's corner:

$$Mg\ell - F_\text{n}\ell - F(2R - h) = 0$$

Solving for F_n yields:

$$F_n = Mg - \frac{F(2R-h)}{\ell}$$

Express ℓ as a function of R and h:

$$\ell = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

Substitute for ℓ in the expression for F_n and simplify to obtain:

$$F_n = Mg - \frac{F(2R-h)}{\sqrt{2Rh-h^2}} = \boxed{Mg - F\sqrt{\frac{2R-h}{h}}}$$

(*b*) Apply $\sum F_x = 0$ to the cylinder:

$$-F_{c,h} + F = 0$$

Solve for $F_{c,h}$:

$$F_{c,h} = \boxed{F}$$

(*c*) Apply $\sum F_y = 0$ to the cylinder:

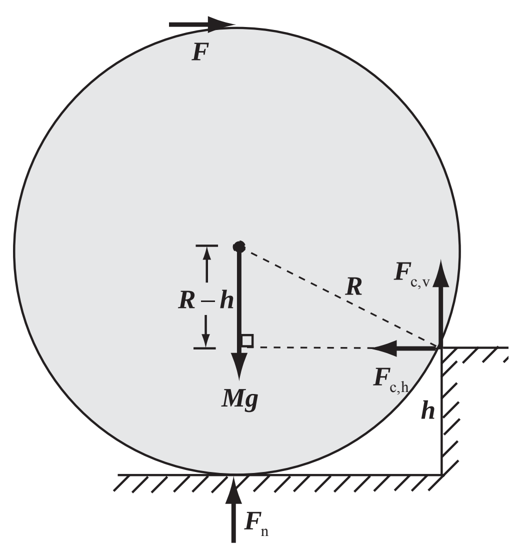
$$F_n - Mg + F_{c,v} = 0 \Rightarrow F_{c,v} = Mg - F_n$$

Substitute the result from Part (*a*) and simplify to obtain:

$$F_{c,v} = Mg - \left\{Mg - F\sqrt{\frac{2R-h}{h}}\right\} = \boxed{F\sqrt{\frac{2R-h}{h}}}$$

26 •• For the cylinder in Problem 25, find an expression for the minimum magnitude of the horizontal force $\vec{F}$ that will roll the cylinder over the step if the cylinder does not slide on the edge.

Picture the Problem The figure to the right shows the forces acting on the cylinder. Because the cylinder is in equilibrium, we can use the condition for rotational equilibrium to express F_n in terms of F. Because, to roll over the step, the cylinder must lift off the floor, we can set $F_n = 0$ in our expression relating F_n and F and solve for F.



Apply $\sum \vec{\tau} = 0$ about the step's corner:

$$Mg\ell - F_\text{n}\ell - F(2R - h) = 0$$

Solve for F_n:

$$F_\text{n} = Mg - \frac{F(2R-h)}{\ell}$$

Express ℓ as a function of R and h:

$$\ell = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

Substitute for ℓ in the expression for F_n and simplify to obtain:

$$F_\text{n} = Mg - \frac{F(2R-h)}{\sqrt{2Rh - h^2}} = Mg - F\sqrt{\frac{2R-h}{h}}$$

To roll over the step, the cylinder must lift off the floor. That is, $F_\text{n} = 0$:

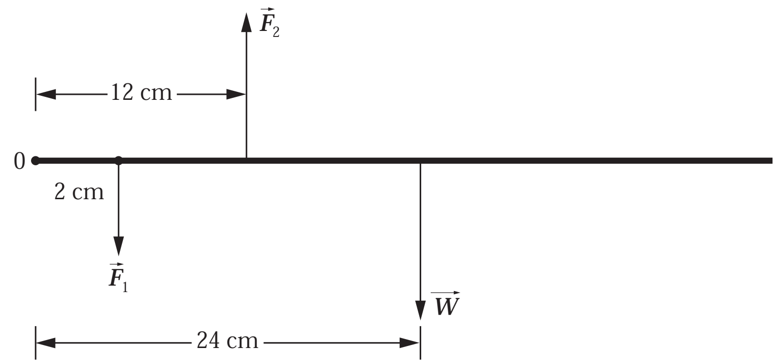
$$0 = Mg - F\sqrt{\frac{2R-h}{h}}$$

Solving for F yields:

$$F = \boxed{Mg\sqrt{\frac{h}{2R-h}}}$$

27 •• Figure 12-38 shows a hand holding an epee, a weapon used in the sport of fencing which you are taking as a physical education elective. The center of mass of your epee is 24 cm from the pommel (the end of the epee at the grip). You have weighed it so you know that the epee's mass is 0.700 kg and its full length is 110 cm. (*a*) At the beginning of a match you hold it straight out in static equilibrium. Find the total force exerted by your hand on the epee. (*b*) Find the torque exerted by your hand on the epee. (*c*) Your hand, being an extended object, actually exerts its force along the length of the epee grip. Model the total force exerted by your hand as two oppositely directed forces whose lines of action are separated by the width of your hand (taken to be 10.0 cm). Find the magnitudes and directions of these two forces.

Picture the Problem The diagram shows the forces F_1 and F_2 that the fencer's hand exerts on the epee. We can use a condition for translational equilibrium to find the upward force the fencer must exert on the epee when it is in equilibrium and the definition of torque to determine the total torque exerted. In Part (*c*) we can use the conditions for translational and rotational equilibrium to obtain two equations in F_1 and F_2 that we can solve simultaneously. In Part (*d*) we can apply Newton's second law in rotational form and the condition for translational equilibrium to obtain two equations in F_1 and F_2 that, again, we can solve simultaneously.



(a) Letting the upward force exerted by the fencer's hand be F, apply $\sum F_y = 0$ to the epee to obtain:

$F - W = 0$
and
$F = W = mg$

Substitute numerical values and evaluate F:

$$F = (0.700\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{6.87\,\text{N}}$$

(b) The torque due to the weight about the left end of the epee is equal in magnitude but opposite in direction to the torque exerted by your hand on the epee:

$$\tau = \ell w$$

Substitute numerical values and evaluate τ:

$$\tau = (0.24\,\text{m})(6.87\,\text{N}) = 1.65\,\text{N}\cdot\text{m} = \boxed{1.7\,\text{N}\cdot\text{m}}$$

(c) Apply $\sum F_y = 0$ to the epee to obtain:

$$-F_1 + F_2 - 6.87\,\text{N} = 0 \qquad (1)$$

Apply $\sum \vec{\tau}_0 = 0$ to obtain:

$$-(0.020\,\text{m})F_1 + (0.12\,\text{m})F_2 - 1.65\,\text{N}\cdot\text{m} = 0 \qquad (2)$$

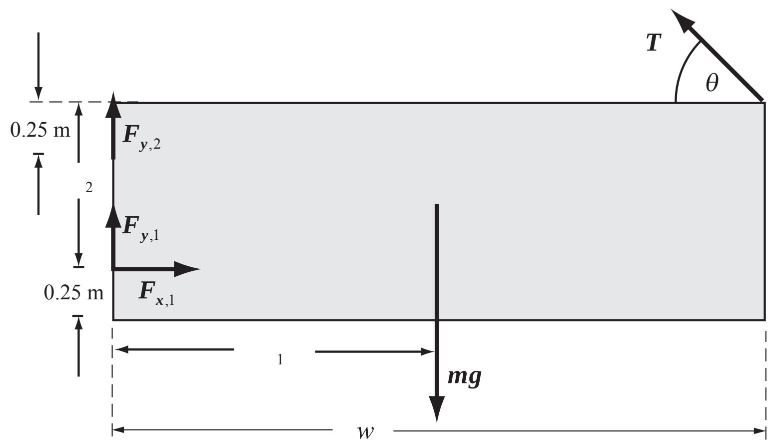
Solve equations (1) and (2) simultaneously to obtain:

$$F_1 = \boxed{8.3\,\text{N}} \text{ and } F_2 = \boxed{15\,\text{N}}.$$

Remarks: Note that the force nearest the butt of the epee is directed downward and the force nearest the hand guard is directed upward.

28 •• A large gate weighing 200 N is supported by hinges at the top and bottom and is further supported by a wire as shown in Figure 12-39. (a) What must be the tension in the wire for the force on the upper hinge to have no horizontal component? (b) What is the horizontal force on the lower hinge? (c) What are the vertical forces on the hinges?

Picture the Problem In the force diagram, the forces exerted by the hinges are $\vec{F}_{y,2}$, $\vec{F}_{y,1}$, and $\vec{F}_{x,1}$ where the subscript 1 refers to the lower hinge. Because the gate is in equilibrium, we can apply the conditions for translational and rotational equilibrium to find the tension in the wire and the forces at the hinges.



(*a*) Apply $\sum\vec{\tau} = 0$ about an axis through the lower hinge and perpendicular to the plane of the page:

$$wT\sin\theta + \ell_2 T\cos\theta - \ell_1 mg = 0$$

Solving for T yields:

$$T = \frac{\ell_1 mg}{w\sin\theta + \ell_2\cos\theta}$$

From Figure 12-39:

$$\theta = \tan^{-1}\left(\frac{1.50\ \text{m}}{3.00\ \text{m}}\right) = 26.57°$$

Substitute numerical values and evaluate T:

$$T = \frac{(1.50\,\text{m})(200\,\text{N})}{(3.00\,\text{m})\sin 26.57° + (0.75\,\text{m})\cos 26.57°} = 149.1\,\text{N} = \boxed{149\ \text{N}}$$

(*b*) Apply $\sum F_x = 0$ to the gate:

$$F_{x,1} - T\cos\theta = 0 \Rightarrow F_{x,1} = T\cos\theta$$

Substitute numerical values and evaluate $F_{x,1}$:

$$F_{x,1} = (149.1\,\text{N})\cos 26.57°$$
$$= 133.4\ \text{N} = \boxed{133\,\text{N}}$$

(*c*) Apply $\sum F_y = 0$ to the gate:

$$F_{y,1} + F_{y,2} + T\sin\theta - mg = 0$$

Because $F_{y,1}$ and $F_{y,2}$ cannot be determined independently, solve for their sum:

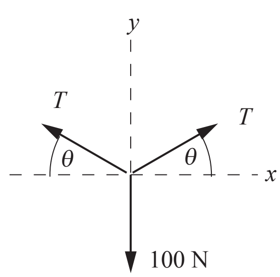
$$F_{y,1} + F_{y,2} = mg - T\sin\theta$$

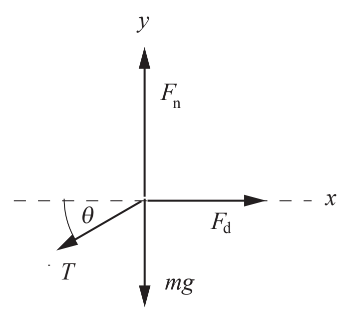
Substitute numerical values and evaluate $F_{y,1} + F_{y,2}$:

$$F_{y,1} + F_{y,2} = 200\text{ N} - (149.1\text{ N})\sin 26.57° = \boxed{133\text{ N}}$$

29 •• On a camping trip, you moor your boat at the end of a dock in a river that is rapidly flowing to the right. The boat is anchored to the dock by a chain 5.0 m long, as shown in Figure 12-40. A 100-N weight is suspended from the center in the chain. This will allow the tension in the chain to change as the force of the current which pulls the boat away from the dock and to the right varies. The drag force by the water on the boat depends on the speed of the water. You decide to apply the principles of statics you learned in physics class. (Ignore the weight of the chain.) The drag force on the boat is 50 N. (*a*) What is the tension in the chain? (*b*) How far is the boat from the dock? (*c*) The maximum tension the chain can sustain is 500 N. What is the minimum water drag force on the boat would snap the chain?

Picture the Problem The free-body diagram shown to the left below is for the weight and the diagram to the right is for the boat. Because both are in equilibrium under the influences of the forces acting on them, we can apply a condition for translational equilibrium to find the tension in the chain.





(*a*) Apply $\sum F_x = 0$ to the boat:

$$F_d - T\cos\theta = 0 \Rightarrow T = \frac{F_d}{\cos\theta}$$

Apply $\sum F_y = 0$ to the weight:

$$2T\sin\theta - 100\text{ N} = 0 \qquad (1)$$

Substitute for T to obtain:

$$2F_d \tan\theta - 100\text{ N} = 0$$

Solve for θ to obtain:

$$\theta = \tan^{-1}\left[\frac{100\text{ N}}{2F_d}\right]$$

Substitute for F_d and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{100\,\text{N}}{2(50\,\text{N})}\right] = 45°$$

Solve equation (1) for T:

$$T = \frac{100\,\text{N}}{2\sin\theta}$$

Substitute for θ and evaluate T:

$$T = \frac{100\,\text{N}}{2\sin 45°} = 70.7\,\text{N} = \boxed{71\,\text{N}}$$

(b) Relate the distance d of the boat from the dock to the angle θ the chain makes with the horizontal:

$$\cos\theta = \frac{\frac{1}{2}d}{\frac{1}{2}L} = \frac{d}{L} \Rightarrow d = L\cos\theta$$

Substitute numerical values and evaluate d:

$$d = (5.0\,\text{m})\cos 45° = \boxed{3.5\,\text{m}}$$

(c) Relate the resultant tension in the chain to the vertical component of the tension F_v and the maximum drag force exerted on the boat by the water $F_{d,max}$:

$$F_v^2 + F_{d,max}^2 = (500\,\text{N})^2$$

Solve for $F_{d,max}$:

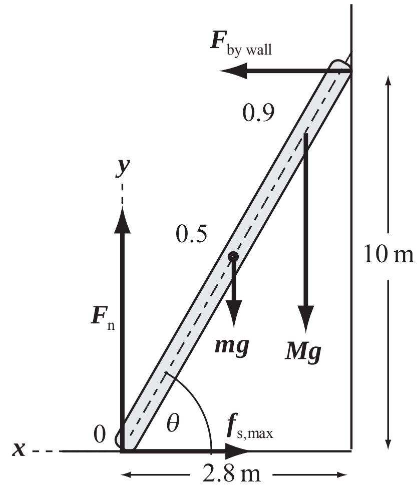
$$F_{d,max} = \sqrt{(500\,\text{N})^2 - F_v^2}$$

Because the vertical component of the tension is 50 N:

$$F_{d,max} = \sqrt{(500\,\text{N})^2 - (50\,\text{N})^2} = \boxed{0.50\,\text{kN}}$$

30 •• Romeo takes a uniform 10-m ladder and leans it against the smooth (frictionless) wall of the Capulet residence. The ladder's mass is 22 kg and the bottom rests on the ground 2.8 m from the wall. When Romeo, whose mass is 70 kg, gets 90 percent of the way to the top, the ladder begins to slip. What is the coefficient of static friction between the ground and the ladder?

Picture the Problem The ladder and the forces acting on it at the critical moment of slipping are shown in the diagram. Use the coordinate system shown. Because the ladder is in equilibrium, we can apply the conditions for translational and rotational equilibrium.



Using its definition, express μ_s:

$$\mu_s = \frac{f_{s,max}}{F_n} \qquad (1)$$

Apply $\sum \vec{\tau} = 0$ about the bottom of the ladder:

$$[0.9\ell\cos\theta]Mg + [0.5\ell\cos\theta]mg - [\ell\sin\theta]F_W = 0$$

Solving for F_W yields:

$$F_W = \frac{(0.9M + 0.5m)g\cos\theta}{\sin\theta}$$

Find the angle θ:

$$\theta = \cos^{-1}\left(\frac{2.8\,\text{m}}{10\,\text{m}}\right) = 73.74°$$

Substitute numerical values and evaluate F_W:

$$F_W = \frac{[0.9(70\,\text{kg}) + 0.5(22\,\text{kg})](9.81\,\text{m/s}^2)\cos 73.74°}{\sin 73.74°} = 211.7\,\text{N}$$

Apply $\sum F_x = 0$ to the ladder and solve for $f_{s,max}$:

$$F_W - f_{s,max} = 0$$

and

$$f_{s,max} = F_W = 211.7\,\text{N}$$

Apply $\sum F_y = 0$ to the ladder:

$$F_n - Mg - mg = 0 \Rightarrow F_n = (M + m)g$$

Substitute numerical values and evaluate F_n:

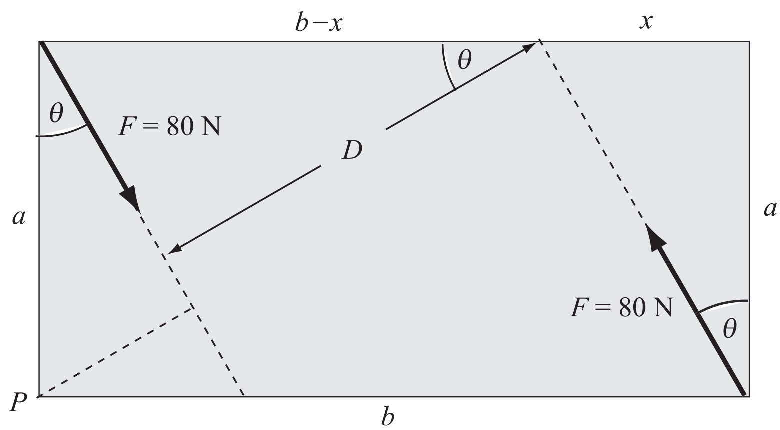
$$F_n = (70\,\text{kg} + 22\,\text{kg})(9.81\,\text{m/s}^2) = 902.5\,\text{N}$$

Substitute numerical values in equation (1) and evaluate μ_s:

$$\mu_s = \frac{211.7\,\text{N}}{902.5\,\text{N}} = \boxed{0.23}$$

31 •• [SSM] Two 80-N forces are applied to opposite corners of a rectangular plate as shown in Figure 12-41. (*a*) Find the torque produced by this couple using Equation 12-6. (*b*) Show that the result is the same as if you determine the torque about the lower left-hand corner.

Picture the Problem The forces shown in the figure constitute a couple and will cause the plate to experience a counterclockwise angular acceleration. The couple equation is $\tau = FD$. The following diagram shows the geometric relationships between the variables in terms of a generalized angle θ.



(*a*) The couple equation is: $\tau = FD$ (1)

From the diagram, D is given by: $D = (b - x)\cos\theta$ (2)

Again, referring to the diagram: $x = a\tan\theta$

Substituting for x in equation (2) and simplifying yields:

$$D = (b - a\tan\theta)\cos\theta$$
$$= b\cos\theta - a\sin\theta$$

Substituting for D in equation (1) yields: $\tau = F(b\cos\theta - a\sin\theta)$ (3)

Substitute numerical values and evaluate τ:

$$\tau = (80\,\text{N})(b\cos 30° - a\sin 30°)$$
$$= \boxed{(69\,\text{N})b - (40\,\text{N})a}$$

(*b*) Letting the counterclockwise direction be the positive direction, sum the torques about an axis normal to the plane of the rectangle and passing through point *P*:

$$\tau = F(\ell + D) - F\ell$$

Substituting for *D* yields:

$$\tau = F(\ell + b\cos\theta - a\sin\theta) - F\ell$$
$$= \boxed{F(b\cos\theta - a\sin\theta)}$$

in agreement with equation (3).

32 •• A uniform cube of side *a* and mass *M* rests on a horizontal surface. A horizontal force $\vec{F}$ is applied to the top of the cube as in Figure 12-42. This force is not sufficient to move or tip the cube. (*a*) Show that the force of static friction exerted by the surface and the applied force constitute a couple, and find the torque exerted by the couple. (*b*) The torque exerted by the couple is balanced by the torque exerted by the couple consisting of the normal force on the cube and the gravitational force on the cube. Use this fact to find the effective point of application of the normal force when $F = Mg/3$. (*c*) Find the greatest magnitude of $\vec{F}$ for which the cube will not tip (Assuming the cube does not slip.).

Picture the Problem We can use the condition for translational equilibrium and the definition of a couple to show that the force of static friction exerted by the surface and the applied force constitute a couple. We can use the definition of torque to find the torque exerted by the couple. We can use our result from (*b*) to find the effective point of application of the normal force when $F = Mg/3$ and the condition for rotational equilibrium to find the greatest magnitude of $\vec{F}$ for which the cube will not tip.

(*a*) Apply $\sum \vec{F}_x = 0$ to the stationary cube:

$$\vec{F} + \vec{f}_s = 0 \Rightarrow \vec{F} = -\vec{f}_s$$

Because $\vec{F} = -\vec{f}_s$, this pair of equal, parallel, and oppositely directed forces constitute a couple.

The torque of the couple is:

$$\tau_{\text{couple}} = \boxed{Fa}$$

(*b*) Let x equal the distance from the point of application of F_n to the center of the cube. Now, $F_n = Mg$, so applying $\sum \vec{\tau} = 0$ to the cube yields:

$$Mgx - Fa = 0 \Rightarrow x = \frac{Fa}{Mg} \qquad (1)$$

Substituting for F and simplifying yields:

$$x = \frac{\frac{Mg}{3}a}{Mg} = \boxed{\frac{a}{3}}$$

(*c*) Solve equation (1) for F:

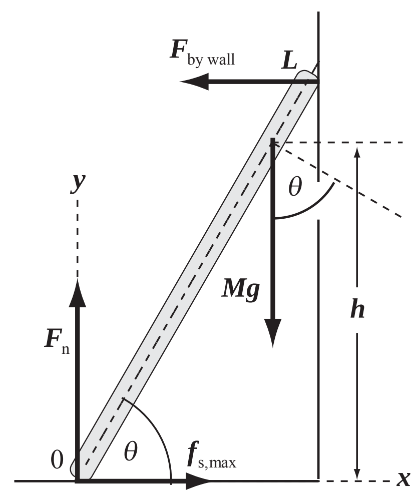
$$F = \frac{Mgx}{a}$$

Noting that $x_{max} = a/2$, express the condition that the cube will tip:

$$F > \frac{Mgx_{max}}{a} = \frac{Mg\frac{a}{2}}{a} = \boxed{\frac{Mg}{2}}$$

33 •• **[SSM]** A ladder of negligible mass and of length L leans against a slick wall making an angle of θ with the horizontal floor. The coefficient of friction between the ladder and the floor is μ_s. A man climbs the ladder. What height h can he reach before the ladder slips?

Picture the Problem Let the mass of the man be M. The ladder and the forces acting on it are shown in the diagram. Because the wall is slick, the force the wall exerts on the ladder must be horizontal. Because the ladder is in equilibrium, we can apply the conditions for translational and rotational equilibrium to it.



Apply $\sum F_y = 0$ to the ladder and solve for F_n:

$$F_n - Mg = 0 \Rightarrow F_n = Mg$$

Apply $\sum F_x = 0$ to the ladder and solve for $f_{s,max}$:

$$f_{s,max} - F_W = 0 \Rightarrow f_{s,max} = F_W$$

Apply $\sum \vec{\tau} = 0$ about the bottom of the ladder to obtain:

$$Mg\ell\cos\theta - F_W L\sin\theta = 0$$

Solving for ℓ and simplifying yields:

$$\ell = \frac{F_W L\sin\theta}{Mg\cos\theta} = \frac{f_{s,max}L}{Mg}\tan\theta$$
$$= \frac{\mu_s F_n L}{Mg}\tan\theta = \mu_s L\tan\theta$$

Referring to the figure, relate ℓ to h:

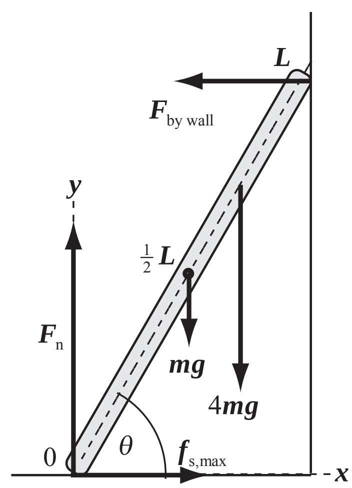
$$h = \ell\sin\theta$$

Substituting for ℓ yields:

$$h = \boxed{\mu_s L\tan\theta\sin\theta}$$

34 •• A uniform ladder of length L and mass m leans against a frictionless vertical wall, making an angle of 60º with the horizontal. The coefficient of static friction between the ladder and the ground is 0.45. If your mass is four times that of the ladder, how high can you climb before the ladder begins to slip?

Picture the Problem The ladder and the forces acting on it are shown in the drawing. Choose a coordinate system in which the $+x$ direction is to the right and the $+y$ direction is upward. Because the wall is smooth, the force the wall exerts on the ladder must be horizontal. Because the ladder is in equilibrium, we can apply the conditions for translational and rotational equilibrium.



Apply $\sum F_y = 0$ to the ladder and solve for F_n:

$$F_n - mg - 4mg = 0 \Rightarrow F_n = 5mg$$

Apply $\sum F_x = 0$ to the ladder and solve for $f_{s,max}$:

$$F_W - f_{s,max} = 0 \Rightarrow f_{s,max} = F_W$$

Apply $\sum \vec{\tau} = 0$ about an axis through the bottom of the ladder:

$$mg\frac{L}{2}\cos\theta + 4mg\ell\cos\theta - F_W L\sin\theta = 0$$

Substitute for F_W and $f_{s,max}$ and solve for ℓ:

$$\ell = \frac{5\mu_s mgL\sin\theta - \frac{1}{2}mgL\cos\theta}{4mg\cos\theta}$$

Simplify to obtain:

$$\ell = \left(\frac{5\mu_s}{4}\tan\theta - \frac{1}{8}\right)L$$

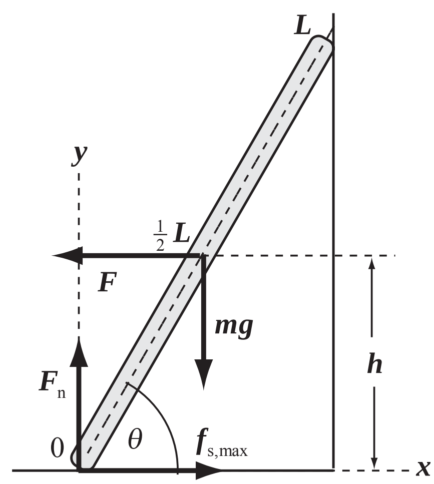
Substitute numerical values to obtain:

$$\ell = \left(\frac{5(0.45)}{4}\tan 60° - \frac{1}{8}\right)L = \boxed{0.85L}$$

That is, you can climb about 85% of the way to the top of the ladder before it begins to slip.

35 •• A ladder of mass m and length L leans against a frictionless vertical wall, so that it makes an angle θ with the horizontal. The center of mass of the ladder is a height h above the floor. A force F directed directly away from the wall pulls on the ladder at its midpoint. Find the minimum coefficient of static friction μ_s for which the top end of the ladder will separate from the wall before the lower end begins to slip.

Picture the Problem The ladder and the forces acting on it are shown in the figure. Because the ladder is separating from the wall, the force the wall exerts on the ladder is zero. Because the ladder is in equilibrium, we can apply the conditions for translational and rotational equilibrium.



To find the force required to pull the ladder away from the wall, apply $\sum\vec{\tau} = 0$ about an axis through the bottom of the ladder:

$$F\left(\tfrac{1}{2}L\sin\theta\right) - mg\left(\tfrac{1}{2}L\cos\theta\right) = 0$$

or, because $\frac{1}{2}L\cos\theta = \dfrac{h}{\tan\theta}$,

$$\tfrac{1}{2}LF\sin\theta - \frac{mgh}{\tan\theta} = 0$$

Solving for F yields:

$$F = \frac{2mgh}{L\tan\theta\sin\theta} \quad (1)$$

Apply $\sum F_x = 0$ to the ladder:

$$f_{s,max} - F = 0 \Rightarrow F = f_{s,max} = \mu_s F_n \quad (2)$$

Apply $\sum F_y = 0$ to the ladder: $F_n - mg = 0 \Rightarrow F_n = mg$

Equate equations (1) and (2) and substitute for F_n to obtain:

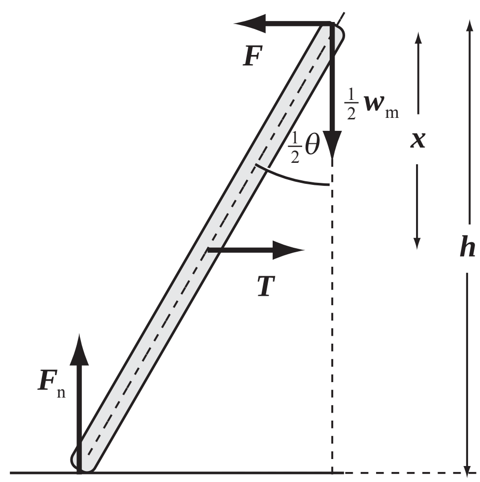
$$\mu_s mg = \frac{2mgh}{L\tan\theta\sin\theta}$$

Solving for μ_s yields:

$$\mu_s = \boxed{\frac{2h}{L\tan\theta\sin\theta}}$$

36 •• A 900-N man sits on top of a stepladder of negligible mass that rests on a frictionless floor as in Figure 12-43. There is a cross brace halfway up the ladder. The angle at the apex is θ = 30º. (*a*) What is the force exerted by the floor on each leg of the ladder? (*b*) Find the tension in the cross brace. (*c*) If the cross brace is moved down toward the bottom of the ladder (maintaining the same angle θ), will its tension be the same, greater, or less than when it was in its higher position? Explain your answer.

Picture the Problem Assume that half the man's weight acts on each side of the ladder. The force exerted by the frictionless floor must be vertical. D is the separation between the legs at the bottom and x is the distance of the cross brace from the apex. Because each leg of the ladder is in equilibrium, we can apply the condition for rotational equilibrium to the right leg to relate the tension in the cross brace to its distance from the apex.



(*a*) By symmetry, each leg carries half the total weight, and the force on each leg is $F_n = \frac{1}{2}w = \boxed{450\ \text{N}}$.

(*b*) Consider one of the ladder's legs and apply $\sum\vec{\tau} = 0$ about the apex:

$$F_n\frac{D}{2} - Tx = 0 \Rightarrow T = \frac{F_n D}{2x}$$

Using trigonometry, relate h and θ through the tangent function:

$$\tan\tfrac{1}{2}\theta = \frac{\frac{1}{2}D}{h} \Rightarrow D = 2h\tan\tfrac{1}{2}\theta$$

Substitute for D in the expression for T and simplify to obtain:

$$T = \frac{2F_n h\tan\frac{1}{2}\theta}{2x} = \frac{F_n h\tan\frac{1}{2}\theta}{x}$$

Substitute for F_n and simplify to obtain:

$$T = \frac{wh\tan\frac{1}{2}\theta}{2x} \quad (1)$$

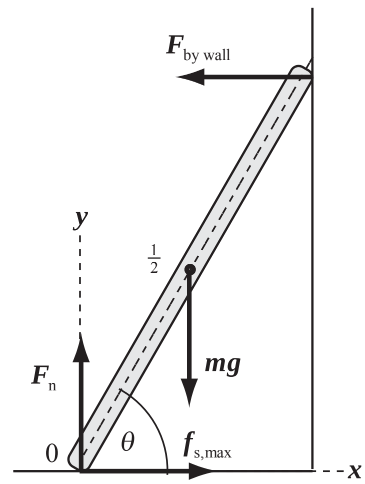
Substitute numerical values and evaluate T:

$$T = \frac{(900\,\text{N})(4.0\,\text{m})\tan 15°}{2(2.0\,\text{m})} = \boxed{0.24\,\text{kN}}$$

(c) From equation (1) we can see that T is inversely proportional to x. Hence, if the brace is moved lower, T will decrease.

37 •• A uniform ladder rests against a frictionless vertical wall. The coefficient of static friction between the ladder and the floor is 0.30. What is the smallest angle between the ladder and the horizontal such that the ladder will not slip?

Picture the Problem The figure shows the forces acting on the ladder. Because the wall is frictionless, the force the wall exerts on the ladder is perpendicular to the wall. Because the ladder is on the verge of slipping, the static friction force is $f_{s,max}$. Because the ladder is in equilibrium, we can apply the conditions for translational and rotational equilibrium.



Apply $\sum F_x = 0$ to the ladder:

$$f_{s,max} - F_W = 0 \Rightarrow F_W = f_{s,max} = \mu_s F_n$$

Apply $\sum F_y = 0$ to the ladder:

$$F_n - mg = 0 \Rightarrow F_n = mg$$

Apply $\sum \vec{\tau} = 0$ about an axis through the bottom of the ladder:

$$mg\left(\tfrac{1}{2}\ell\cos\theta\right) - F_W(\ell\sin\theta) = 0$$

Substitute for F_W and F_n and simplify to obtain:

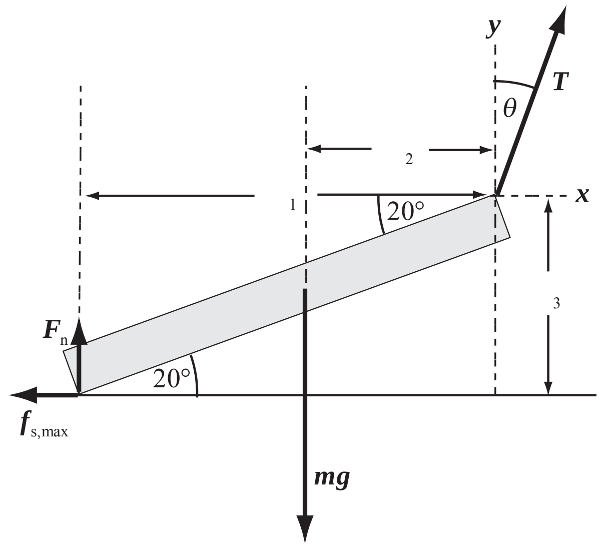
$$\tfrac{1}{2}\cos\theta - \mu_s\sin\theta = 0 \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2\mu_s}\right)$$

Substitute the numerical value of μ_s and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{1}{2(0.30)}\right] = \boxed{59°}$$

38 ••• A uniform log with a mass of 100 kg, a length of 4.0 m, and a radius of 12 cm is held in an inclined position, as shown in Figure 12-44. The coefficient of static friction between the log and the horizontal surface is 0.60. The log is on the verge of slipping to the right. Find the tension in the support wire and the angle the wire makes with the vertical wall.

Picture the Problem Let T = the tension in the wire; F_n = the normal force of the surface; and $f_{s,max} = \mu_s F_n$ the maximum force of static friction. Because the log is in equilibrium, we can apply the conditions for translational and rotational equilibrium to find the tension in the support wire and the angle the wire makes with the vertical wall.



Apply $\sum F_x = 0$ to the log:

$$T\sin\theta - f_{s,max} = 0$$

or

$$T\sin\theta = f_{s,max} = \mu_s F_n \qquad (1)$$

Apply $\sum F_y = 0$ to the log:

$$T\cos\theta + F_n - mg = 0$$

or

$$T\cos\theta = mg - F_n \qquad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{\mu_s F_n}{mg - F_n}$$

Solving for θ yields:

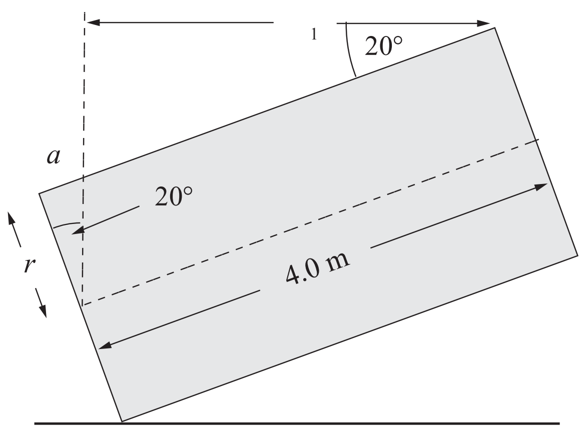
$$\theta = \tan^{-1}\left(\frac{\mu_s}{\frac{mg}{F_n} - 1}\right) \qquad (3)$$

Apply $\sum \vec{\tau} = 0$ about an axis through the origin: $\ell_2 mg - \ell_1 F_n - \ell_3 \mu_s F_n = 0$

Solve for F_n to obtain:

$$F_n = \frac{\ell_2 mg}{\ell_1 + \ell_3 \mu_s} \qquad (4)$$

Use the following diagram, in which the scale of the log has been modified to show important details, to show that $\ell_1 = 4.0\,\text{m} - a = [4.0\,\text{m} - r\tan 20°]\cos 20°$:



Substitute numerical values and evaluate ℓ_1:

$$\ell_1 = [4.0\,\text{m} - (0.12\,\text{m})\tan 20°]\cos 20° = 3.718\,\text{m}$$

Proceed similarly to show that ℓ_2 is given by:

$$\ell_2 = \left(4.0\,\text{m} - \tfrac{1}{2} r\tan 20°\right)\cos 20°$$

Substitute numerical values and evaluate ℓ_2:

$$\ell_2 = [2.0\,\text{m} - (0.06\,\text{m})\tan 20°]\cos 20° = 1.859\,\text{m}$$

Proceed as in the determination of the expression for ℓ_2 to show that ℓ_3 is given by:

$$\ell_3 = r\cos 20° + (4.0\,\text{m} - r\tan 20°)\sin 20°$$

Substitute numerical values and evaluate ℓ_3:

$$\ell_3 = (0.12\,\text{m})\cos 20° + [4.0\,\text{m} - (0.12\,\text{m})\tan 20°]\sin 20° = 1.466\,\text{m}$$

Substitute numerical values in equation (4) and evaluate F_n:

$$F_n = \frac{(1.859)(100\,\text{kg})(9.81\,\text{m/s}^2)}{3.718+(1.466)(0.60)} = 396.7\,\text{N}$$

Substitute numerical values in equation (3) and evaluate θ:

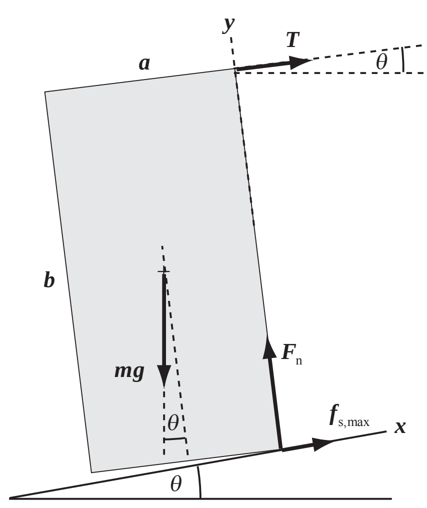
$$\theta = \tan^{-1}\left[\frac{0.60}{\frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{396.7\,\text{N}}-1}\right] = 22.16° = \boxed{22°}$$

Substitute numerical values in equation (1) and evaluate T:

$$T = \frac{(0.60)(396.7\,\text{N})}{\sin 22.16°} = \boxed{0.63\,\text{kN}}$$

39 ••• [SSM] A tall, uniform, rectangular block sits on an inclined plane as shown in Figure 12-45. A cord is attached to the top of the block to prevent it from falling down the incline. What is the maximum angle θ for which the block will not slide on the incline? Assume the block has a height-to-width ratio, b/a, of 4.0 and the coefficient of static friction between it and the incline is $\mu_s = 0.80$.

Picture the Problem Consider what happens just as θ increases beyond θ_{max}. Because the top of the block is fixed by the cord, the block will in fact rotate with only the lower right edge of the block remaining in contact with the plane. It follows that just prior to this slipping, F_n and $f_s = \mu_s F_n$ act at the lower right edge of the block. Choose a coordinate system in which up the incline is the $+x$ direction and the direction of $\vec{F}_n$ is the $+y$ direction. Because the block is in equilibrium, we can apply the conditions for translational and rotational equilibrium.



Apply $\sum F_x = 0$ to the block: $T + \mu_s F_n - mg\sin\theta = 0$ (1)

Apply $\sum F_y = 0$ to the block: $F_n - mg\cos\theta = 0$ (2)

Apply $\sum\vec{\tau} = 0$ about an axis through the lower right edge of the block: $\frac{1}{2}a(mg\cos\theta)+\frac{1}{2}b(mg\sin\theta)-bT=0$ (3)

Eliminate F_n between equations (1) and (2) and solve for T: $T = mg(\sin\theta - \mu_s\cos\theta)$

Substitute for T in equation (3):

$$\frac{1}{2}a(mg\cos\theta)+\frac{1}{2}b(mg\sin\theta) - b[mg(\sin\theta-\mu_s\cos\theta)]=0$$

Substitute $4a$ for b:

$$\frac{1}{2}a(mg\cos\theta)+\frac{1}{2}(4.0a)(mg\sin\theta) - (4.0a)[mg(\sin\theta-\mu_s\cos\theta)]=0$$

Simplify to obtain: $(1+8\mu_s)\cos\theta - 4\sin\theta = 0$

Solving for θ yields:

$$\theta = \tan^{-1}\left[\frac{1+8\mu_s}{4}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{1+(8.0)(0.80)}{4.0}\right] = \boxed{62°}$$

Stress and Strain

40 • A 50-kg ball is suspended from a steel wire of length 5.0 m and radius 2.0 mm. By how much does the wire stretch?

Picture the Problem L is the unstretched length of the wire, F is the force acting on it, and A is its cross-sectional area. The stretch in the wire ΔL is related to Young's modulus by $Y = (F/A)/(\Delta L/L)$. We can use Table 12-1 to find the numerical value of Young's modulus for steel.

Find the amount the wire is stretched from Young's modulus:

$$Y = \frac{F/A}{\Delta L/L} \Rightarrow \Delta L = \frac{FL}{YA}$$

Substitute for F and A to obtain:

$$\Delta L = \frac{mgL}{Y\pi r^2}$$

Substitute numerical values and evaluate ΔL:

$$\Delta L = \frac{(50\,\text{kg})(9.81\,\text{m/s}^2)(5.0\,\text{m})}{(200\,\text{GN/m}^2)(\pi)(2.0\times10^{-3}\,\text{m})^2} = \boxed{0.98\,\text{mm}}$$

41 • **[SSM]** Copper has a tensile strength of about 3.0×10^8 N/m^2. (*a*) What is the maximum load that can be hung from a copper wire of diameter 0.42 mm? (*b*) If half this maximum load is hung from the copper wire, by what percentage of its length will it stretch?

Picture the Problem L is the unstretched length of the wire, F is the force acting on it, and A is its cross-sectional area. The stretch in the wire ΔL is related to Young's modulus by $Y = \text{stress/strain} = (F/A)/(\Delta L/L)$.

(*a*) Express the maximum load in terms of the wire's tensile strength:

$$F_{\text{max}} = \text{tensile strength} \times A = \text{tensile strength} \times \pi r^2$$

Substitute numerical values and evaluate F_{max}:

$$F_{\text{max}} = (3.0\times10^8\,\text{N/m}^2)\pi(0.21\times10^{-3}\,\text{m})^2 = 41.6\,\text{N} = \boxed{42\,\text{N}}$$

(*b*) Using the definition of Young's modulus, express the fractional change in length of the copper wire:

$$\frac{\Delta L}{L} = \frac{F}{AY} = \frac{\frac{1}{2}F_{\text{max}}}{AY}$$

Substitute numerical values and evaluate $\frac{\Delta L}{L}$:

$$\frac{\Delta L}{L} = \frac{\frac{1}{2}(41.6\,\text{N})}{\pi(0.21\,\text{mm})^2(1.10\times10^{11}\,\text{N/m}^2)} = \boxed{0.14\%}$$

42 • A 4.0-kg mass is supported by a steel wire of diameter 0.60 mm and length 1.2 m. How much will the wire stretch under this load?

Picture the Problem L is the unstretched length of the wire, F is the force acting on it, and A is its cross-sectional area. The stretch in the wire ΔL is related to Young's modulus by $Y = (F/A)/(\Delta L/L)$. We can use Table 12-1 to find the numerical value of Young's modulus for steel.

Relate the amount the wire is stretched to Young's modulus:

$$Y = \frac{F/A}{\Delta L/L} \Rightarrow \Delta L = \frac{FL}{YA}$$

Substitute for F and A to obtain:

$$\Delta L = \frac{mgL}{Y\pi r^2}$$

Substitute numerical values and evaluate ΔL:

$$\Delta L = \frac{(4.0\,\text{kg})(9.81\,\text{m/s}^2)(1.2\,\text{m})}{2\pi\times10^{11}\,\text{N/m}^2\,(0.30\times10^{-3}\,\text{m})^2} = \boxed{0.83\,\text{mm}}$$

43 • **[SSM]** As a runner's foot pushes off on the ground, the shearing force acting on an 8.0-mm-thick sole is shown in Figure 12-46. If the force of 25 N is distributed over an area of 15 cm^2, find the angle of shear θ, given that the shear modulus of the sole is 1.9×10^5 N/m^2.

Picture the Problem The shear stress, defined as the ratio of the shearing force to the area over which it is applied, is related to the shear strain through the definition of the shear modulus; $M_\text{s} = \dfrac{\text{shear stress}}{\text{shear strain}} = \dfrac{F_\text{s}/A}{\tan\theta}$.

Using the definition of shear modulus, relate the angle of shear, θ to the shear force and shear modulus:

$$\tan\theta = \frac{F_\text{s}}{M_\text{s}A} \Rightarrow \theta = \tan^{-1}\left(\frac{F_\text{s}}{M_\text{s}A}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{25\,\text{N}}{(1.9\times10^5\,\text{N/m}^2)(15\times10^{-4}\,\text{m}^2)}\right] = \boxed{5.0^\circ}$$

44 •• A steel wire of length 1.50 m and diameter 1.00 mm is joined to an aluminum wire of identical dimensions to make a composite wire of length 3.00 m. Find the resulting change in the length of this composite wire if an object with a mass of 5.00 kg is hung vertically from one of its ends. (Neglect any effects the masses of the two wires have on the changes in their lengths.)

Picture the Problem The stretch in the wire ΔL is related to Young's modulus by $Y = (F/A)/(\Delta L/L)$, where L is the unstretched length of the wire, F is the force acting on it, and A is the cross-sectional area of the wire. The change in length of the composite wire is the sum of the changes in length of the steel and aluminum wires.

The change in length of the composite wire ΔL is the sum of the changes in length of the two wires:

$$\Delta L = \Delta L_\text{steel} + \Delta L_\text{Al}$$

Using the defining equation for Young's modulus, substitute for ΔL_{steel} and ΔL_{Al} in equation (1) and simplify to obtain:

$$\Delta L = \frac{F}{A}\frac{L_{\text{steel}}}{Y_{\text{steel}}} + \frac{F}{A}\frac{L_{\text{Al}}}{Y_{\text{Al}}} = \frac{F}{A}\left(\frac{L_{\text{steel}}}{Y_{\text{steel}}} + \frac{L_{\text{Al}}}{Y_{\text{Al}}}\right)$$

Substitute numerical values and evaluate ΔL:

$$\Delta L = \frac{(5.00\,\text{kg})(9.81\,\text{m/s}^2)}{\pi(0.500\times10^{-3}\,\text{m})^2}\left(\frac{1.50\,\text{m}}{2.00\times10^{11}\ \text{N/m}^2} + \frac{1.50\,\text{m}}{0.700\times10^{11}\ \text{N/m}^2}\right)$$
$$= 1.81\times10^{-3}\,\text{m} = \boxed{1.81\,\text{mm}}$$

45 •• [SSM] Equal but opposite forces of magnitude F are applied to both ends of a thin wire of length L and cross-sectional area A. Show that if the wire is modeled as a spring, the force constant k is given by $k = AY/L$ and the potential energy stored in the wire is $U = \frac{1}{2}F\Delta L$, where Y is Young's modulus and ΔL is the amount the wire has stretched.

Picture the Problem We can use Hooke's law and Young's modulus to show that, if the wire is considered to be a spring, the force constant k is given by $k = AY/L$. By treating the wire as a spring we can show the energy stored in the wire is $U = \frac{1}{2}F\Delta L$.

Express the relationship between the stretching force, the force constant, and the elongation of a spring:

$$F = k\Delta L \Rightarrow k = \frac{F}{\Delta L}$$

Using the definition of Young's modulus, express the ratio of the stretching force to the elongation of the wire:

$$\frac{F}{\Delta L} = \frac{AY}{L} \qquad (1)$$

Equate these two expressions for $F/\Delta L$ to obtain:

$$k = \boxed{\frac{AY}{L}}$$

Treating the wire as a spring, express its stored energy:

$$U = \tfrac{1}{2}k(\Delta L)^2 = \tfrac{1}{2}\frac{AY}{L}(\Delta L)^2 = \tfrac{1}{2}\left(\frac{AY\Delta L}{L}\right)\Delta L$$

Solving equation (1) for F yields:

$$F = \frac{AY\Delta L}{L}$$

Substitute for F in the expression for U to obtain:

$$U = \boxed{\tfrac{1}{2}F\Delta L}$$

46 •• The steel E string of a violin is under a tension of 53.0 N. The diameter of the string is 0.200 mm and the length under tension is 35.0 cm. Find (*a*) the unstretched length of this string and (*b*) the work needed to stretch the string.

Picture the Problem Let L' represent the stretched and L the unstretched length of the wire. The stretch in the wire ΔL is related to Young's modulus by $Y = (F/A)/(\Delta L/L)$, where F is the force acting on it, and A is its cross-sectional area. In Problem 45 we showed that the energy stored in the wire is $U = \frac{1}{2}F\Delta L$, where Y is Young's modulus and ΔL is the amount the wire has stretched.

(*a*) Express the stretched length L' of the wire:

$$L' = L + \Delta L$$

Using the definition of Young's modulus, express ΔL:

$$\Delta L = \frac{LF}{AY}$$

Substitute and simplify:

$$L' = L + \frac{LF}{AY} = L\left(1 + \frac{F}{AY}\right)$$

Solving for L yields:

$$L = \frac{L'}{1 + \dfrac{F}{AY}}$$

Substitute numerical values and evaluate L:

$$L = \frac{0.350\,\text{m}}{1 + \dfrac{53.0\,\text{N}}{\pi\left(0.100\times10^{-3}\,\text{m}\right)^2\left(2.00\times10^{11}\,\text{N/m}^2\right)}} = \boxed{34.7\,\text{cm}}$$

(*b*) From Problem 45, the work done in stretching the wire is:

$$W = \Delta U = \tfrac{1}{2}F\Delta L$$

Substitute numerical values and evaluate W:

$$W = \tfrac{1}{2}(53.0\,\text{N})(0.350\,\text{m} - 0.347\,\text{m}) \approx \boxed{0.08\,\text{J}}$$

47 •• During a materials science experiment on the Young's modulus of rubber, your teaching assistant supplies you and your team with a rubber strip that is rectangular in cross section. She tells you to first measure the cross section dimensions and their values are 3.0 mm × 1.5 mm. The lab write-up calls for the rubber strip to be suspended vertically and various (known) masses attached to it. Your team obtains the following data for the length of the strip as a function of the load (mass) on the end of the strip:

Load, kg	0.0	0.10	0.20	0.30	0.40	0.50
Length, cm	5.0	5.6	6.2	6.9	7.8	8.8

(*a*) Use a **spreadsheet** or **graphing calculator** to find Young's modulus for the rubber strip over this range of loads. *Hint: It is probably best to plot F/A versus ΔL/L. Why?*
(*b*) Find the energy stored in the strip when the load is 0.15 kg. (See Problem 45.)
(*c*) Find the energy stored in the strip when the load is 0.30 kg. Is it twice as much as your answer to Part (*b*)? Explain.

Picture the Problem We can use the definition of Young's modulus and your team's data to plot a graph whose slope is Young's modulus for the rubber strip over the given range of loads. Because the rubber strip stretches linearly for loads less than or equal to 0.20 kg, we can use linear interpolation in Part (*b*) to find the length of the rubber strip for a load of 0.15 kg. We can then use the result of Problem 45 to find the energy stored in the when the load is 0.15 kg. In Part (*c*) we can use the result of Problem 45 and the given length of the strip when its load is 0.30 kg to find the energy stored in the rubber strip.

(*a*) The equation for Young's modulus can be written as:

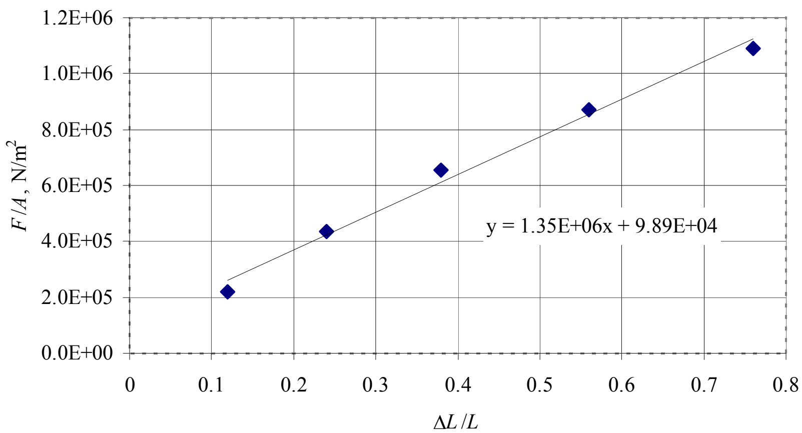
$$\frac{F}{A}=Y\frac{\Delta L}{L}$$

where Y is the slope of a graph of F/A as a function of $\Delta L/L$.

The following table summarizes the quantities, calculated using your team's data, used to plot the graph suggested in the problem statement.

Load	F	F/A	ΔL	$\Delta L/L$	U
(kg)	(N)	(N/m^2)	(m)		(J)
0.10	0.981	21.8×10^5	0.006	0.12	2.9×10^{-3}
0.20	1.962	4.36×10^5	0.012	0.24	12×10^{-3}
0.30	2.943	6.54×10^5	0.019	0.38	28×10^{-3}
0.40	3.924	8.72×10^5	0.028	0.56	55×10^{-3}
0.50	4.905	10.9×10^5	0.038	0.76	93×10^{-3}

A spreadsheet-generated graph of F/A as a function of $\Delta L/L$ follows. The spreadsheet program also plotted the regression line on the graph and added its equation to the graph.



From the regression function shown on the graph:

$$Y = \boxed{1.4\times10^{6}\ \text{N/m}^2}$$

(*b*) From Problem 45:

$$U = \tfrac{1}{2}\Delta LF$$

or, because $F = mg$,

$$U(m) = \tfrac{1}{2}\Delta Lmg$$

Interpolating from the data table we see that the length of the strip when the load on it is 0.15 kg is 5.9 cm. Substitute numerical values and evaluate U(0.15 kg):

$$U(0.15\ \text{kg}) = \tfrac{1}{2}(5.9\ \text{cm} - 5.0\ \text{cm})(0.15\ \text{kg})(9.81\ \text{m/s}^2) = \boxed{7\ \text{mJ}}$$

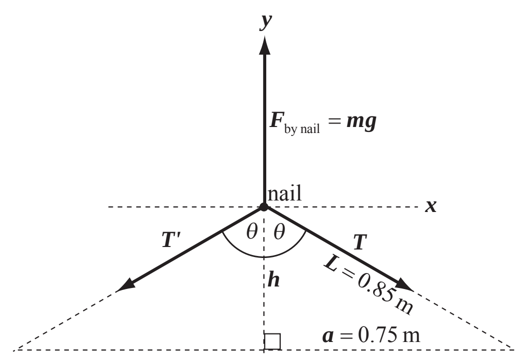
(*c*) Evaluate U(0.30 kg) to obtain:

$$U(0.30\ \text{kg}) = \tfrac{1}{2}(6.9\ \text{cm} - 5.0\ \text{cm})(0.30\ \text{kg})(9.81\ \text{m/s}^2) = \boxed{28\ \text{mJ}}$$

The energy stored in the strip when the load is 0.30 kg is four times as much as the energy stored when the load is 0.15 kg. Although the rubber strip does not stretch linearly (a conclusion you can confirm either graphically or by examining the data table), its stretch is sufficiently linear that, to a good approximation, the energy stored is quadrupled when the load is doubled.

48 •• A large mirror is hung from a nail as shown in Figure 12-47. The supporting steel wire has a diameter of 0.20 mm and an unstretched length of 1.7 m. The distance between the points of support at the top of the mirror's frame is 1.5 m. The mass of the mirror is 2.4 kg. How much will the distance between the nail and the mirror increase due to the stretching of the wire as the mirror is hung?

Picture the Problem The figure shows the forces acting on the wire where it passes over the nail. *m* represents the mass of the mirror and *T* is the tension in the supporting wires. The figure also shows the geometry of the right triangle defined by the support wires and the top of the mirror frame. The distance *a* is fixed by the geometry while *h* and *L* will change as the mirror is suspended from the nail.



Using the Pythagorean theorem, express the relationship between the sides of the right triangle in the diagram:

$$a^2 + h^2 = L^2$$

Express the differential of this equation and approximate differential changes with small changes:

$$2a\Delta a + 2h\Delta h = 2L\Delta L$$

or, because $\Delta a = 0$,

$$h\Delta h = L\Delta L \Rightarrow \Delta h = \frac{L\Delta L}{h}$$

Multiplying the numerator and denominator by L yields:

$$\Delta h = \frac{L^2}{h}\frac{\Delta L}{L} \qquad (1)$$

Solve the equation defining Young's modulus for $\Delta L/L$ to obtain:

$$\frac{\Delta L}{L} = \frac{T}{AY}$$

Substitute for $\Delta L/L$ in equation (1) to obtain:

$$\Delta h = \frac{L^2}{h}\frac{T}{AY} = \frac{L^2}{\sqrt{L^2 - a^2}}\frac{T}{\pi r^2 Y} \qquad (2)$$

where r is the radius of the wire.

Noting that $T = T'$, apply $\sum F_y = 0$ to the wire where it passes over the supporting nail:

$$mg - 2T\cos\theta = 0 \Rightarrow T = \frac{mg}{2\cos\theta}$$

Substituting for T in equation (2) yields:

$$\Delta h = \frac{L^2}{\sqrt{L^2 - a^2}} \frac{mg}{2\pi r^2 Y \cos\theta}$$

Because $\cos\theta = \dfrac{h}{L} = \dfrac{\sqrt{L^2 - a^2}}{L}$:

$$\Delta h = \frac{L^2}{\sqrt{L^2 - a^2}} \frac{mgL}{2\pi r^2 Y \sqrt{L^2 - a^2}} = \frac{mgL^3}{2\pi r^2 Y \left(L^2 - a^2\right)}$$

Substitute numerical values and evaluate Δh:

$$\Delta h = \frac{(2.4\ \text{kg})\left(9.81\ \text{m/s}^2\right)(0.85\ \text{m})^3}{2\pi\left(0.10\times10^{-3}\ \text{m}\right)^2\left(2.00\times10^{11}\ \text{N/m}^2\right)\left[(0.85\ \text{m})^2 - (0.75\ \text{m})^2\right]} = \boxed{7.2\ \text{mm}}$$

49 •• Two masses, M_1 and M_2, are supported by wires that have equal lengths when unstretched. The wire supporting M_1 is an aluminum wire 0.70 mm in diameter, and the one supporting M_2 is a steel wire 0.50 mm in diameter. What is the ratio M_1/M_2 if the two wires stretch by the same amount?

Picture the Problem Let the numeral 1 denote the aluminum wire and the numeral 2 the steel wire. Because their initial lengths and amount they stretch are the same, we can use the definition of Young's modulus to express the change in the lengths of each wire and then equate these expressions to obtain an equation solvable for the ratio M_1/M_2.

Using the definition of Young's modulus, express the change in length of the aluminum wire:

$$\Delta L_1 = \frac{M_1 g L_1}{A_1 Y_{\text{Al}}}$$

Using the definition of Young's modulus, express the change in length of the steel wire:

$$\Delta L_2 = \frac{M_2 g L_2}{A_2 Y_{\text{steel}}}$$

Because the two wires stretch by the same amount, equate ΔL_1 and ΔL_2 and simplify:

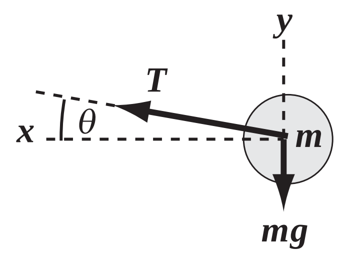
$$\frac{M_1}{A_1 Y_{\text{Al}}} = \frac{M_2}{A_2 Y_{\text{steel}}} \Rightarrow \frac{M_1}{M_2} = \frac{A_1 Y_{\text{Al}}}{A_2 Y_{\text{steel}}}$$

Substitute numerical values and evaluate M_1/M_2:

$$\frac{M_1}{M_2}=\frac{\frac{\pi}{4}(0.70\,\text{mm})^2\left(0.70\times10^{11}\ \text{N/m}^2\right)}{\frac{\pi}{4}(0.50\,\text{mm})^2\left(2.00\times10^{11}\ \text{N/m}^2\right)}=\boxed{0.69}$$

50 •• A 0.50-kg ball is attached to one end of an aluminum wire having a diameter of 1.6 mm and an unstretched length of 0.70 m. The other end of the wire is fixed to the top of a post. The ball rotates about the post in a horizontal plane at a rotational speed such that the angle between the wire and the horizontal is 5.0º. Find the tension in the wire and the increase in its length due to the tension in the wire.

Picture the Problem The free-body diagram shows the forces acting on the ball as it rotates around the post in a horizontal plane. We can apply Newton's second law to find the tension in the wire and use the definition of Young's modulus to find the amount by which the aluminum wire stretches.



Apply $\sum F_y=0$ to the ball:

$$T\sin\theta-mg=0\Rightarrow T=\frac{mg}{\sin\theta}$$

Substitute numerical values and evaluate T:

$$T=\frac{(0.50\,\text{kg})\left(9.81\,\text{m/s}^2\right)}{\sin5.0°}=56.3\ \text{N}=\boxed{56\,\text{N}}$$

Using the definition of Young's modulus, express ΔL:

$$\Delta L=\frac{FL}{AY}$$

Substitute numerical values and evaluate ΔL:

$$\Delta L=\frac{(56.3\,\text{N})(0.70\,\text{m})}{\frac{\pi}{4}\left(1.6\times10^{-3}\ \text{m}\right)^2\left(0.70\times10^{11}\ \text{N/m}^2\right)}=\boxed{0.28\,\text{mm}}$$

51 •• [SSM] An elevator cable is to be made of a new type of composite developed by Acme Laboratories. In the lab, a sample of the cable that is 2.00 m long and has a cross-sectional area of 0.200 mm^2 fails under a load of 1000 N. The actual cable used to support the elevator will be 20.0 m long and have a cross-sectional area of 1.20 mm^2. It will need to support a load of 20,000 N safely. Will it?

Picture the Problem We can use the definition of stress to calculate the failing stress of the cable and the stress on the elevator cable. Note that the failing stress of the composite cable is the same as the failing stress of the test sample.

The stress on the elevator cable is:

$$\text{Stress}_{\text{cable}} = \frac{F}{A} = \frac{20.0\,\text{kN}}{1.20\times10^{-6}\,\text{m}^2} = 1.67\times10^{10}\ \text{N/m}^2$$

The failing stress of the sample is:

$$\text{Stress}_{\text{failing}} = \frac{F}{A} = \frac{1000\,\text{N}}{0.2\times10^{-6}\,\text{m}^2} = 0.500\times10^{10}\ \text{N/m}^2$$

Because $\text{Stress}_{\text{failing}} < \text{Stress}_{\text{cable}}$, the cable will not support the elevator.

52 •• If a material's density remains constant when it is stretched in one direction, then (because its total volume remains constant), its length must decrease in one or both of the other directions. Take a rectangular block of length *x*, width *y*, and depth z, and pull on it so that its new length $x' = x + \Delta x$. If $\Delta x << x$ and $\Delta y/y = \Delta z/z$, show that $\Delta y/y = -\frac{1}{2}\Delta x/x$.

Picture the Problem Let the length of the sides of the rectangle be *x*, *y* and *z*. Then the volume of the rectangle will be $V = xyz$ and we can express the new volume V' resulting from the pulling in the *x* direction and the change in volume ΔV in terms of Δx, Δy, and Δz. Discarding the higher order terms in ΔV and dividing our equation by V and using the given condition that $\Delta y/y = \Delta z/z$ will lead us to the given expression for $\Delta y/y$.

Express the new volume of the rectangular box when its sides change in length by Δx, Δy, and Δz:

$$V' = (x+\Delta x)(y+\Delta y)(z+\Delta z) = xyz + \Delta x(yz) + \Delta y(xz) + \Delta z(xy) + \{z\Delta x\Delta y + y\Delta x\Delta z + x\Delta y\Delta z + \Delta x\Delta y\Delta z\}$$

where the terms in brackets are very small (i.e., second order or higher).

Discard the second order and higher terms to obtain:

$$V' = V + \Delta x(yz) + \Delta y(xz) + \Delta z(xy)$$

or

$$\Delta V = V' - V = \Delta x(yz) + \Delta y(xz) + \Delta z(xy)$$

Because $\Delta V = 0$:

$$\Delta x(yz) = -[\Delta y(xz) + \Delta z(xy)]$$

Divide both sides of this equation by $V = xyz$ to obtain:

$$\frac{\Delta x}{x} = -\left[\frac{\Delta y}{y} + \frac{\Delta z}{z}\right]$$

Because $\Delta y/y = \Delta z/z$, our equation becomes:

$$\frac{\Delta x}{x} = -2\frac{\Delta y}{y} \Rightarrow \frac{\Delta y}{y} = \boxed{-\frac{1}{2}\frac{\Delta x}{x}}$$

53 •• **[SSM]** You are given a wire with a circular cross-section of radius r and a length L. If the wire is made from a material whose density remains constant when it is stretched in one direction, then show that $\Delta r/r = -\frac{1}{2}\Delta L/L$, assuming that $\Delta L << L$. (See Problem 52.)

Picture the Problem We can evaluate the differential of the volume of the wire and, using the assumptions that the volume of the wire does not change under stretching and that the change in its length is small compared to its length, show that $\Delta r/r = -\frac{1}{2}\Delta L/L\,L$.

The volume of the wire is given by:

$$V = \pi r^2 L$$

Evaluate the differential of V to obtain:

$$dV = \pi r^2 dL + 2\pi rLdr$$

Because $dV = 0$:

$$0 = rdL + 2Ldr \Rightarrow \frac{dr}{r} = -\frac{1}{2}\frac{dL}{L}$$

Because $\Delta L << L$, we can approximate the differential changes dr and dL with small changes Δr and ΔL to obtain:

$$\frac{\Delta r}{r} = \boxed{-\frac{1}{2}\frac{\Delta L}{L}}$$

54 ••• For most materials listed in Table 12-1, the tensile strength is two to three orders of magnitude lower than Young's modulus. Consequently, most of these materials will break before their strain exceeds 1 percent. Of man-made materials, nylon has about the greatest extensibility—it can take strains of about 0.2 before breaking. But spider silk beats anything man-made. Certain forms of spider silk can take strains on the order of 10 before breaking! (*a*) If such a thread has a circular cross-section of radius r_0 and unstretched length L_0, find its new radius r when stretched to a length $L = 10L_0$. (Assume that the density of the thread remains

constant as it stretches.) (*b*) If the Young's modulus of the spider thread is Y, calculate the tension needed to break the thread in terms of Y and r_0.

Picture the Problem Because the density of the thread remains constant during the stretching process, we can equate the initial and final volumes to express r_0 in terms of r. We can also use Young's modulus to express the tension needed to break the thread in terms of Y and r_0.

(*a*) Because the volume of the thread is constant during the stretching of the spider's silk:

$$\pi r^2 L = \pi r_0^2 L_0 \Rightarrow r = r_0\sqrt{\frac{L_0}{L}}$$

Substitute for L and simplify to obtain:

$$r = r_0\sqrt{\frac{L_0}{10L_0}} = \boxed{0.316r_0}$$

(*b*) Express Young's modulus in terms of the breaking tension T:

$$Y = \frac{T/A}{\Delta L/L} = \frac{T/\pi r^2}{\Delta L/L} = \frac{10T/\pi r_0^2}{\Delta L/L}$$

Solving for T yields:

$$T = \frac{1}{10}\pi r_0^2 Y\frac{\Delta L}{L}$$

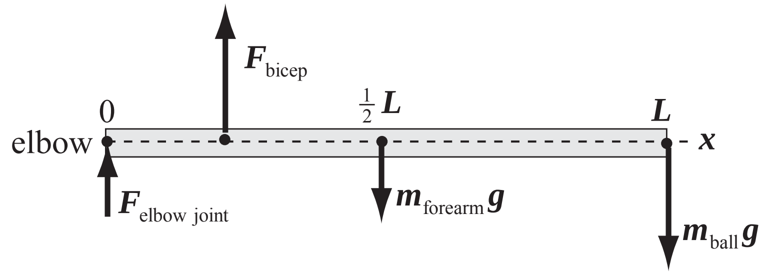
Because $\Delta L/L = 9$:

$$T = \boxed{\frac{9\pi r_0^2 Y}{10}}$$

General Problems

55 • [SSM] A standard bowling ball weighs 16 pounds. You wish to hold a bowling ball in front of you, with the elbow bent at a right angle. Assume that your biceps attaches to your forearm at 2.5 cm out from the elbow joint, and that your biceps muscle pulls vertically upward, that is, it acts at right angles to the forearm. Also assume that the ball is held 38 cm out from the elbow joint. Let the mass of your forearm be 5.0 kg and assume its center of gravity is located 19 cm out from the elbow joint. How much force must your biceps muscle apply to forearm in order to hold out the bowling ball at the desired angle?

Picture the Problem We can model the forearm as a cylinder of length L = 38 cm with the forces shown in the pictorial representation acting on it. Because the forearm is in both translational and rotational equilibrium under the influence of these forces, the forces in the diagram must add (vectorially) to zero and the net torque with respect to any axis must also be zero.



Apply $\sum \vec{\tau} = 0$ about an axis through the elbow and perpendicular to the plane of the diagram:

$$\ell F_{\text{bicep}} - \tfrac{1}{2} L m_{\text{forearm}} g - L m_{\text{ball}} g = 0$$

Solving for F_{bicep} and simplifying yields:

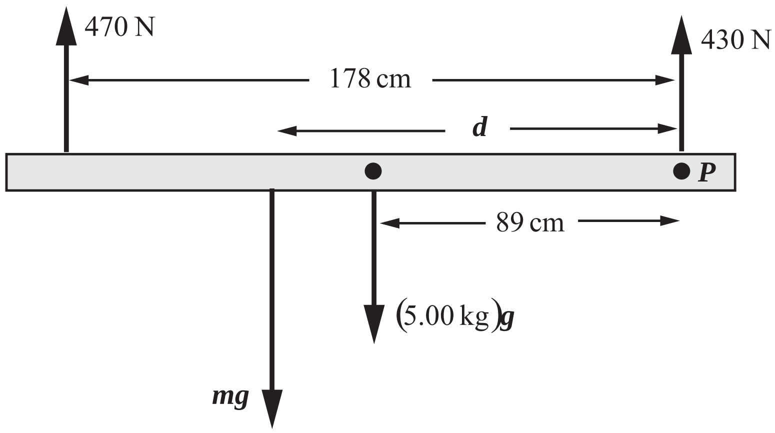
$$F_{\text{bicep}} = \frac{\tfrac{1}{2} L m_{\text{forearm}} g + L m_{\text{ball}} g}{\ell} = \frac{\left(\tfrac{1}{2} m_{\text{forearm}} + m_{\text{ball}}\right) L g}{\ell}$$

Substitute numerical values and evaluate F_{bicep}:

$$F_{\text{bicep}} = \frac{\left(\tfrac{1}{2}(5.0\,\text{kg}) + 16\,\text{lb} \times \dfrac{1\,\text{kg}}{2.205\,\text{lb}}\right)(38\,\text{cm})\left(9.81\,\text{m/s}^2\right)}{2.5\,\text{cm}} = \boxed{1.5\,\text{kN}}$$

56 •• A biology laboratory at your university is studying the location of a person's center of gravity as a function of their body weight. They pay well, and you decide to volunteer. The location of your center of gravity when standing erect is to be determined by having you lie on a uniform board (mass of 5.00 kg, length 2.00 m) supported by two scales as shown in Figure 12-48. If your height is 188 cm and the left scale reads 470 N while the right scale reads 430 N, where is your center of gravity relative to your feet? Assume the scales are both exactly the same distance from the two ends of the board, are separated by 178 cm, and are set to each read zero before you get on the platform.

Picture the Problem Because the you-board system is in equilibrium, we can apply the conditions for translational and rotational equilibrium to relate the forces exerted by the scales to the distance *d*, measured from your feet, to your center of mass and the distance to the center of gravity of the board. The following pictorial representation shows the forces acting on the board. mg represents your weight.



The forces responsible for a counterclockwise torque about an axis through your feet (point *P*) and perpendicular to the page are your weight and the weight of the board. The only force causing a clockwise torque about this axis is the 470 N force exerted by the scale under your head. Apply $\sum\vec{\tau}=0$ about an axis through your feet and perpendicular to the page:

$$m\left(9.81\,\text{m/s}^2\right)d-(5.0\,\text{kg})(0.89\,\text{m})-(1.78\,\text{m})(470\,\text{N})=0$$

where *m* is your mass.

Solve for *d* to obtain:

$$d=\frac{(5.00\,\text{kg})(0.89\,\text{m})+(1.78\,\text{m})(470\,\text{N})}{m\left(9.81\,\text{m/s}^2\right)} \qquad (1)$$

Let upward be the positive *y* direction and apply $\sum F_y=0$ to the plank to obtain:

$$470\,\text{N}+430\,\text{N}-m\left(9.81\,\text{m/s}^2\right)-(5.00\,\text{kg})\left(9.81\,\text{m/s}^2\right)=0$$

Solving for *m* yields: $m=86.74\,\text{kg}$

Substitute numerical values in equation (1) and evaluate *d*:

$$d=\frac{(5.0\,\text{kg})(0.89\,\text{m})+(1.78\,\text{m})(470\,\text{N})}{(86.74\,\text{kg})\left(9.81\,\text{m/s}^2\right)}=\boxed{99\,\text{cm}}$$

57 •• Figure 12-49 shows a mobile consisting of four objects hanging on three rods of negligible mass. Find the values of the unknown masses of the objects if the mobile is to balance. *Hint: Find the mass* m_1 *first*.

Picture the Problem We can apply the balance condition $\sum\vec{\tau}=0$ successively, starting with the lowest part of the mobile, to find the value of each of the unknown weights.

Apply $\sum\vec{\tau}=0$ about an axis through the point of suspension of the lowest part of the mobile:

$$(3.0\,\text{cm})(2.0\,\text{N})-(4.0\,\text{cm})m_1g=0$$

Solving for m_1 yields:

$$m_1=0.1529\,\text{kg}=\boxed{0.15\,\text{kg}}$$

Apply $\sum\vec{\tau}=0$ about an axis through the point of suspension of the middle part of the mobile:

$$(2.0\,\text{cm})m_2g-(4.0\,\text{cm})\left(\frac{2.0\,\text{N}}{g}+0.1529\,\text{kg}\right)g=0$$

Solving for m_2 yields:

$$m_2=0.7136\,\text{kg}=\boxed{0.71\,\text{kg}}$$

Apply $\sum\vec{\tau}=0$ about an axis through the point of suspension of the top part of the mobile:

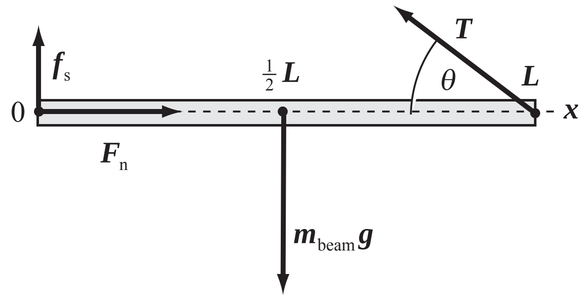
$$(2.0\,\text{cm})(2.0\,\text{N}+(0.7136\,\text{kg})g+(0.1529)g)-(6.0\,\text{cm})m_3g=0$$

Solving for m_3 yields:

$$m_3=0.3568\,\text{kg}=\boxed{0.36\,\text{kg}}$$

58 •• Steel construction beams, with an industry designation of ″W12 × 22,″ have a weight of 22 pounds per foot. A new business in town has hired you to place its sign on a 4.0 m long steel beam of this type. The design calls for the beam to extend outward horizontally from the front brick wall (Figure 12-50). It is to be held in place by a 5.0 m-long steel cable. The cable is attached to one end of the beam and to the wall above the point at which the beam is in contact with the wall. During the initial stage of construction, the beam is *not* to be bolted to the wall, but to be held in place solely by friction. (*a*) What is the minimum coefficient of friction between the beam and the wall for the beam to remain in static equilibrium? (*b*)What is the tension in the cable in this case?

Picture the Problem Because the beam is in both translational and rotational equilibrium under the influence of the forces shown below in the pictorial representation, we can apply $\sum\vec{F}=0$ and $\sum\vec{\tau}=0$ to it to find the coefficient of static friction and the tension in the supporting cable.



(*a*) Apply $\sum\vec{F}=0$ to the beam to obtain:

$$\sum F_x = F_n - T\cos\theta = 0 \quad (1)$$

and

$$\sum F_y = f_s - m_{beam}g + T\sin\theta = 0 \quad (2)$$

The mass of the beam m_{beam} is the product of its linear density λ and length L:

$$m_{beam} = \lambda L$$

Substituting for m_{beam} in equation (2) yields:

$$f_s - \lambda Lg + T\sin\theta = 0 \quad (3)$$

Relate the force of static friction to the normal force exerted by the wall:

$$f_s = \mu_s F_n$$

Substituting for f_s in equation (3) yields:

$$\mu_s F_n - \lambda Lg + T\sin\theta = 0$$

Solve for μ_s to obtain:

$$\mu_s = \frac{\lambda Lg - T\sin\theta}{F_n} \quad (4)$$

Solving equation (1) for F_n yields:

$$F_n = T\cos\theta$$

Substitute for F_n in equation (4) and simplify to obtain:

$$\mu_s = \frac{\lambda Lg - T\sin\theta}{T\cos\theta} = \frac{\lambda Lg}{T\cos\theta} - \tan\theta \quad (5)$$

Apply $\sum\vec{\tau}=0$ to the beam about an axis through the origin and normal to the page to obtain:

$$(T\sin\theta)L - m_{beam}g\left(\tfrac{1}{2}L\right) = 0$$

or

$$(T\sin\theta)L - \lambda Lg\left(\tfrac{1}{2}L\right) = 0$$

Solving for T yields:

$$T = \frac{\lambda L g}{2\sin\theta} \qquad (6)$$

Substitute for T in equation (5) and simplify to obtain:

$$\mu_s = \frac{\lambda L g}{\dfrac{\lambda L g}{2\sin\theta}\cos\theta} - \tan\theta = \tan\theta \qquad (7)$$

From Figure 12-50 we see that:

$$\tan\theta = \frac{3.0\ \text{m}}{4.0\ \text{m}} = \frac{3}{4}$$

Substituting for $\tan\theta$ in equation (7) yields:

$$\mu_s = \boxed{0.75}$$

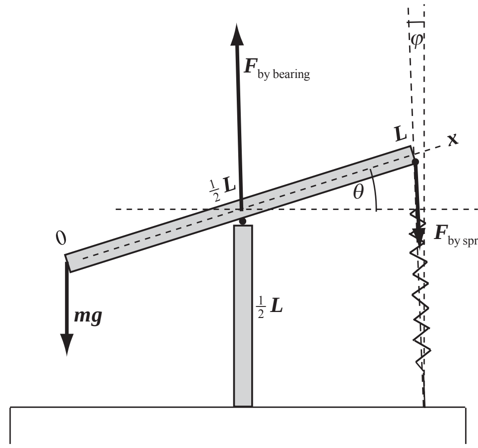
(b) Substitute numerical values in equation (6) and evaluate T:

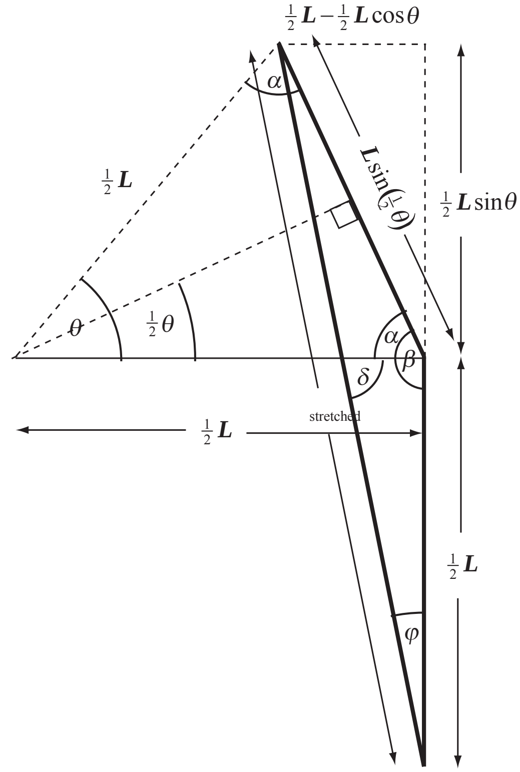
$$T = \frac{\left(22\,\dfrac{\text{lb}}{\text{ft}} \times \dfrac{1\ \text{kg}}{2.205\ \text{lb}} \times \dfrac{3.281\ \text{ft}}{\text{m}}\right)(4.0\ \text{m})\left(9.81\ \text{m/s}^2\right)}{2\sin\left(\tan^{-1}\left(\dfrac{3}{4}\right)\right)} = \boxed{1.1\ \text{kN}}$$

Remarks: 1.1 kN is approximately 240 lb.

59 ••• [SSM] Consider a rigid 2.5-m-long beam (Figure 12-51) that is supported by a fixed 1.25-m-high post through its center and pivots on a frictionless bearing at its center atop the vertical 1.25-m-high post. One end of the beam is connected to the floor by a spring that has a force constant k = 1250 N/m. When the beam is horizontal, the spring is vertical and unstressed. If an object is hung from the opposite end of the beam, the beam settles into an equilibrium position where it makes an angle of 17.5° with the horizontal. What is the mass of the object?

Picture the Problem Because the beam is in rotational equilibrium, we can apply $\sum\vec{\tau} = 0$ to it to determine the mass of the object suspended from its left end.





The pictorial representation directly above shows the forces acting on the beam when it is in static equilibrium. The pictorial representation to the right is an enlarged view of the right end of the beam. We'll use this diagram to determine the length of the stretched spring.

Apply $\sum\vec{\tau}=0$ to the beam about an axis through the bearing point to obtain:

$$mg\left(\tfrac{1}{2}L\cos\theta\right)-F_{\text{by spring}}\left(\tfrac{1}{2}L\sin\varphi\right)=0$$

or, because $F_{\text{by spring}}=k\Delta\ell_{\text{spring}}$,

$$mg\cos\theta-k\Delta\ell_{\text{spring}}\sin\varphi=0 \qquad (1)$$

Use the right-hand diagram above to relate the angles φ and θ:

$$\varphi=\tan^{-1}\left(\frac{\frac{1}{2}L-\frac{1}{2}L\cos\theta}{\frac{1}{2}L+\frac{1}{2}L\sin\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{1+\sin\theta}\right)$$

Substitute for φ in equation (1) to obtain:

$$mg\cos\theta-k\Delta\ell_{\text{spring}}\sin\left(\tan^{-1}\left(\frac{1-\cos\theta}{1+\sin\theta}\right)\right)=0$$

Solve for m to obtain:

$$m=\frac{k\Delta\ell_{\text{spring}}\sin\left(\tan^{-1}\left(\frac{1-\cos\theta}{1+\sin\theta}\right)\right)}{g\cos\theta} \qquad (2)$$

$\Delta\ell_{\text{spring}}$ is given by:

$$\Delta\ell_{\text{spring}} = \ell_{\text{stretched}} - \ell_{\text{unstretched}}$$

or, because $\ell_{\text{unstretched}} = \frac{1}{2}L$,

$$\Delta\ell_{\text{spring}} = \ell_{\text{stretched}} - \tfrac{1}{2}L \qquad (3)$$

To find the value of $\ell_{\text{stretched}}$, refer to the right-hand diagram and note that:

$$2\alpha + \theta = \pi \Rightarrow \alpha = \frac{\pi}{2} - \frac{1}{2}\theta$$

Again, referring to the diagram, relate β to α:

$$\beta = \alpha + \frac{\pi}{2}$$

Substituting for α yields:

$$\beta = \frac{\pi}{2} - \frac{1}{2}\theta + \frac{\pi}{2} = \pi - \frac{1}{2}\theta$$

Apply the law of cosines to the triangle defined with bold sides:

$$\ell^2_{\text{stretched}} = \left(\tfrac{1}{2}L\right)^2 + \left(L\sin\tfrac{1}{2}\theta\right)^2 - 2\left(\tfrac{1}{2}L\right)\left(L\sin\tfrac{1}{2}\theta\right)\cos\left(\pi - \tfrac{1}{2}\theta\right)$$

Use the formula for the cosine of the difference of two angles to obtain:

$$\cos\left(\pi - \tfrac{1}{2}\theta\right) = -\cos\tfrac{1}{2}\theta$$

Substituting for $\cos\left(\pi - \tfrac{1}{2}\theta\right)$ yields:

$$\begin{aligned}\ell^2_{\text{stretched}} &= \left(\tfrac{1}{2}L\right)^2 + \left(L\sin\tfrac{1}{2}\theta\right)^2 + 2\left(\tfrac{1}{2}L\right)\left(L\sin\tfrac{1}{2}\theta\right)\cos\tfrac{1}{2}\theta \\ &= \tfrac{1}{4}L^2 + L^2\sin^2\tfrac{1}{2}\theta + \tfrac{1}{2}L^2\left(2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\right)\end{aligned}$$

Use the trigonometric identities $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $\sin^2\frac{1}{2}\theta = \dfrac{1-\cos\theta}{2}$ to obtain:

$$\ell^2_{\text{stretched}} = \tfrac{1}{4}L^2 + L^2\left(\frac{1-\cos\theta}{2}\right) + \tfrac{1}{2}L^2(\sin\theta)$$

Simplifying yields:

$$\ell^2_{\text{stretched}} = \tfrac{1}{4}L^2\left[3 - 2(\cos\theta - \sin\theta)\right]$$

or

$$\ell_{\text{stretched}} = \tfrac{1}{2}L\sqrt{3 - 2(\cos\theta - \sin\theta)}$$

Substituting for $\ell_{\text{stretched}}$ in equation (3) yields:

$$\Delta\ell_{\text{spring}} = \tfrac{1}{2}L\sqrt{3-2(\cos\theta-\sin\theta)} - \tfrac{1}{2}L = \tfrac{1}{2}L\left(\sqrt{3-2(\cos\theta-\sin\theta)}-1\right)$$

Substitute for $\Delta\ell_{\text{spring}}$ in equation (2) to obtain:

$$m = \frac{\tfrac{1}{2}kL\left(\sqrt{3-2(\cos\theta-\sin\theta)}-1\right)\sin\left(\tan^{-1}\left(\frac{1-\cos\theta}{1+\sin\theta}\right)\right)}{g\cos\theta}$$

Substitute numerical values and evaluate m:

$$m = \frac{\tfrac{1}{2}(1250\text{ N/m})(2.5\text{ m})\left(\sqrt{3-2(\cos17.5°-\sin17.5°)}-1\right)\sin\left(\tan^{-1}\left(\frac{1-\cos17.5°}{1+\sin17.5°}\right)\right)}{(9.81\text{ m/s}^2)\cos17.5°}$$

$$= \boxed{1.8\text{ kg}}$$

60 •• A rope and pulley system, called a *block and tackle*, is used to raise an object of mass *M* (Figure 12-52) at constant speed. When the end of the rope moves downward through a distance *L,* the height of the lower pulley is increased by *h*. (*a*) What is the ratio *L*/*h*? (*b*) Assume that the mass of the block and tackle is negligible and that the pulley bearings are frictionless. Show that *FL* = *mgh* by applying the work–energy principle to the block–tackle object.

Picture the Problem We can determine the ratio of *L* to *h* by noting the number of ropes supporting the load whose mass is *M*.

(*a*) Noting that three ropes support the pulley to which the object whose mass is *M* is fastened we can conclude that:

$$\frac{L}{h} = \boxed{3}$$

(*b*) Apply the work-energy principle to the block-tackle object to obtain:

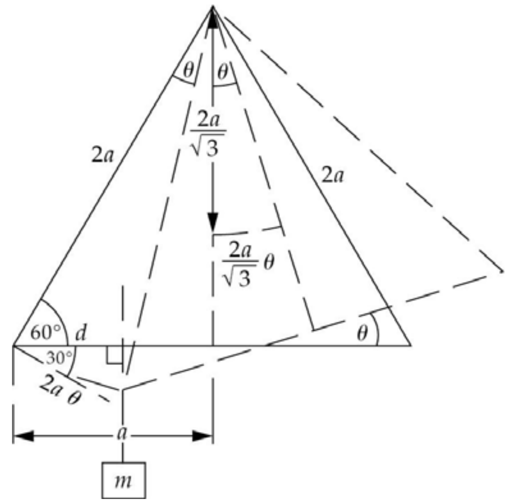
$$W_{\text{ext}} = \Delta E_{\text{system}} = \Delta U_{\text{block-tackle}}$$

or

$$FL = \boxed{mgh}$$

61 •• A plate of mass *M* in the shape of an equilateral triangle is suspended from one corner and a mass *m* is suspended from another of its corners. If the base of the triangle makes an angle of 6.0º with the horizontal, what is the ratio *m*/*M*?

Picture the Problem The figure shows the equilateral triangle without the mass m, and then the same triangle with the mass m and rotated through an angle θ. Let the side length of the triangle to be $2a$. Then the center of mass of the triangle is at a distance of $\dfrac{2a}{\sqrt{3}}$ from each vertex. As the triangle rotates, its center of mass shifts by $\dfrac{2a}{\sqrt{3}}\theta$, for θ << 1. Also, the vertex to which m is attached moves toward the plumb line by the distance $d = 2a\theta \cos 30° = \sqrt{3}a\theta$ (see the drawing).



Apply $\sum \vec{\tau} = 0$ about an axis through the point of suspension:

$$mg\left(a - \sqrt{3}a\theta\right) - Mg\frac{2a}{\sqrt{3}}\theta = 0$$

Solving for m/M yields:

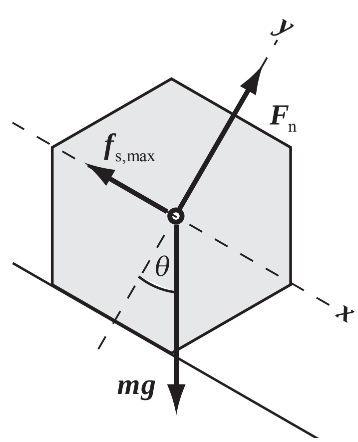
$$\frac{m}{M} = \frac{2\theta}{\sqrt{3}\left(1 - \sqrt{3}\theta\right)}$$

Substitute numerical values and evaluate m/M:

$$\frac{m}{M} = \frac{2(6.0°)\left(\dfrac{\pi\,\text{rad}}{180°}\right)}{\sqrt{3}\left[1 - \sqrt{3}(6.0°)\left(\dfrac{\pi\,\text{rad}}{180°}\right)\right]} = \boxed{0.15}$$

62 •• A standard six-sided pencil is placed on a pad of paper (Figure 12-53) Find the minimum coefficient of static friction μ_s such that, if the pad is inclined, the pencil rolls down the incline rather than sliding.

Picture the Problem If the hexagon is to roll rather than slide, the incline's angle must be such that the center of mass falls just beyond the support base. From the geometry of the hexagon, the critical angle is 30°. The free-body diagram shows the forces acting on the hexagonal pencil when it is on the verge of sliding. We can use Newton's second law to relate the coefficient of static friction to the angle of the incline for which rolling rather than sliding occurs.



Apply $\sum \vec{F} = 0$ to the pencil:

$$\sum F_x = mg\sin\theta - f_{s,max} = 0 \quad (1)$$

and

$$\sum F_y = F_n - mg\cos\theta = 0 \quad (2)$$

Substitute $f_{s,max} = \mu_s F_n$ in equation (1):

$$mg\sin\theta - \mu_s F_n = 0 \quad (3)$$

Divide equation (3) by equation (2) to obtain:

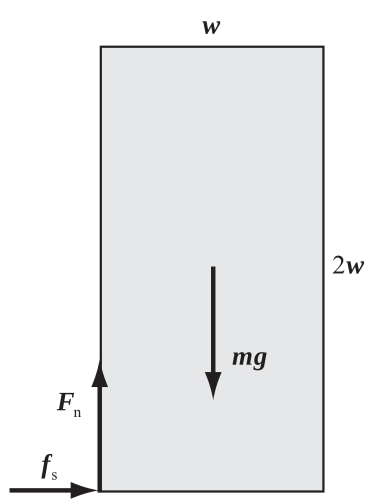
$$\tan\theta = \mu_s$$

Thus, if the pencil is to roll rather than slide when the pad is inclined:

$$\mu_s \geq \tan 30° = \boxed{0.58}$$

63 •• An 8.0-kg box that has a uniform density and is twice as tall as it is wide rests on the floor of a truck. What is the maximum coefficient of static friction between the box and floor so that the box will slide toward the rear of the truck rather than tip when the truck accelerates forward on a level road?

Picture the Problem The box and the forces acting on it are shown in the figure. The force accelerating the box is the static friction force. When the box is about to tip, F_n acts at its edge, as indicated in the drawing. We can use the definition of μ_s and apply the condition for rotational equilibrium in an accelerated frame to relate f_s to the weight of the box and, hence, to the normal force.



Using its definition, express μ_s:

$$\mu_s \geq \frac{f_s}{F_n}$$

Apply $\sum \vec{\tau} = 0$ about an axis through the box's center of mass:

$$wf_s - \tfrac{1}{2}wF_n = 0 \Rightarrow \frac{f_s}{F_n} = \frac{1}{2}$$

Substitute for f_s/F_n to obtain the condition for tipping:

$$\mu_s \geq 0.50$$

Therefore, if the box is to slide:

$$\mu_s < \boxed{0.50}$$

64 •• A balance scale has unequal arms. The scale is balanced with a 1.50-kg block on the left pan and a 1.95 kg block on the right pan (Figure 12-54). If the 1.95-kg block is removed from the right pan and the 1.50-kg block is then moved to the right pan, what mass on the left pan will balance the scale?

Picture the Problem Because the balance is in equilibrium, we can use the condition for rotational equilibrium to relate the masses of the blocks to the lever arms of the balance in the two configurations described in the problem statement.

Representing the mass of the 1.50-kg block by m_1 and the mass of the 1.95-kg block by m_2, apply $\sum\vec{\tau} = 0$ about an axis through the fulcrum:

$$m_1gL_1 - m_2gL_2 = 0$$

or

$$m_1gL_1 = m_2gL_2 \qquad (1)$$

Denoting the unknown mass as m_3, apply $\sum\vec{\tau} = 0$ about an axis through the fulcrum when the block whose mass is m_1 has been moved to the pan to the right:

$$m_1gL_2 - m_3gL_1 = 0$$

or

$$m_1gL_2 = m_3gL_1 \qquad (2)$$

Dividing equation (1) by equation (2) gives:

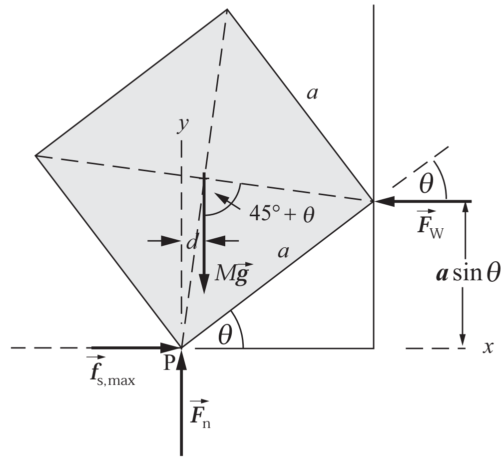
$$\frac{m_1gL_1}{m_3gL_1} = \frac{m_2gL_2}{m_1gL_2} \Rightarrow m_3 = \frac{m_1^2}{m_2}$$

Substitute numerical values and evaluate m_3:

$$M = \frac{(1.50\,\text{kg})^2}{1.95\,\text{kg}} = \boxed{1.15\,\text{kg}}$$

65 •• [SSM] A cube leans against a frictionless wall making an angle of θ with the floor as shown in Figure 12-55. Find the minimum coefficient of static friction μ_s between the cube and the floor that is needed to keep the cube from slipping.

Picture the Problem Let the mass of the cube be *M*. The figure shows the location of the cube's center of mass and the forces acting on the cube. The opposing couple is formed by the friction force $f_{s,max}$ and the force exerted by the wall. Because the cube is in equilibrium, we can use the condition for translational equilibrium to establish that $f_{s,max} = F_W$ and $F_n = Mg$ and the condition for rotational equilibrium to relate the opposing couples.



Apply $\sum\vec{F} = 0$ to the cube:

$$\sum F_y = F_n - Mg = 0 \Rightarrow F_n = Mg$$

and

$$\sum F_x = f_s - F_W = 0 \Rightarrow F_W = f_s$$

Noting that $\vec{f}_{s,max}$ and $\vec{F}_W$ form a couple (their magnitudes are equal), as do $\vec{F}_n$ and $M\vec{g}$, apply $\sum\vec{\tau} = 0$ about an axis though point P to obtain:

$$(F_W \sin\theta)a - Mg\cos(45° + \theta)\frac{a}{\sqrt{2}} = 0$$

Solving for F_W, expanding $\cos(45° + \theta)$, and simplifying gives:

$$F_W = \frac{Mg\cos(45° + \theta)\frac{a}{\sqrt{2}}}{a\sin\theta} = \frac{Mg[\cos 45°\cos\theta - \sin 45°\sin\theta]\frac{1}{\sqrt{2}}}{\sin\theta}$$

$$= \frac{Mg\left[\frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{2}}\right]\frac{1}{\sqrt{2}}}{\sin\theta} = \frac{Mg[\cos\theta - \sin\theta]}{2\sin\theta} = \frac{Mg}{2}(\cot\theta - 1)$$

Because $F_W = f_s$ and $f_s = \mu_s F_n = \mu_s Mg$:

$$\mu_s Mg = \frac{Mg}{2}(\cot\theta - 1)$$

or, solving for μ_s,

$$\mu_s = \boxed{\tfrac{1}{2}(\cot\theta - 1)}$$

66 •• Figure 12-56 shows a 5.00-kg, 1.00-m-long rod hinged to a vertical wall and supported by a thin wire. The wire and rod each make angles of 45º with the vertical. When a 10.0-kg block is suspended from the midpoint of the rod, the tension T in the supporting wire is 52.0 N. If the wire will break when the tension exceeds 75 N, what is the maximum distance from the hinge at which the block can be suspended?

Picture the Problem Because the rod is in equilibrium, we can apply the condition for rotational equilibrium to find the maximum distance from the hinge at which the block can be suspended.

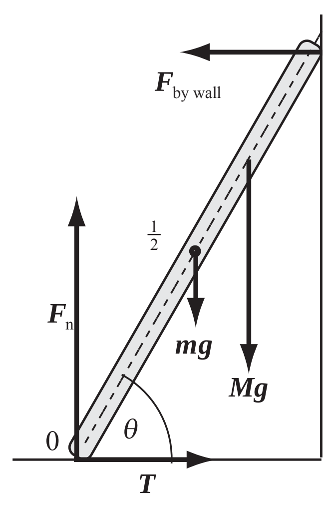
Apply $\sum\vec{\tau} = 0$ about an axis through the hinge to obtain:

$$\begin{gathered}(1.00\,\text{m})(75\,\text{N}) - (0.50\,\text{m})(5.00\,\text{kg})\left(9.81\,\text{m/s}^2\right)\cos 45° \\ -d(10.0\,\text{kg})\left(9.81\,\text{m/s}^2\right)\cos 45° = 0\end{gathered}$$

Solving for d yields: $d = \boxed{83\,\text{cm}}$

67 •• **[SSM]** Figure 12-57 shows a 20.0-kg ladder leaning against a frictionless wall and resting on a frictionless horizontal surface. To keep the ladder from slipping, the bottom of the ladder is tied to the wall with a thin wire. When no one is on the ladder, the tension in the wire is 29.4 N. (The wire will break if the tension exceeds 200 N.) (*a*) If an 80.0-kg person climbs halfway up the ladder, what force will be exerted by the ladder against the wall? (*b*) How far from the bottom end of the ladder can an 80.0-kg person climb?

Picture the Problem Let m represent the mass of the ladder and M the mass of the person. The force diagram shows the forces acting on the ladder for Part (*b*). From the condition for translational equilibrium, we can conclude that $T = F_{\text{by wall}}$, a result we'll need in Part (*b*). Because the ladder is also in rotational equilibrium, summing the torques about the bottom of the ladder will eliminate both F_n and T.



(*a*) Apply $\sum\vec{\tau} = 0$ about an axis through the bottom of the ladder:

$$F_{\text{by wall}}(\ell\sin\theta) - mg\left(\tfrac{1}{2}\ell\cos\theta\right) - Mg\left(\tfrac{1}{2}\ell\cos\theta\right) = 0$$

Solve for $F_{\text{by wall}}$ and simplify to obtain:

$$F_{\text{by wall}} = \frac{(m+M)g\cos\theta}{2\sin\theta} = \frac{(m+M)g}{2\tan\theta}$$

Refer to Figure 12-57 to determine $\tan\theta$:

$$\tan\theta = \frac{5.0\text{ m}}{1.5\text{ m}} = \frac{10}{3}$$

Substitute numerical values and evaluate $F_{\text{by wall}}$:

$$F_{\text{by wall}} = \frac{(20\text{ kg}+80\text{ kg})(9.81\text{ m/s}^2)}{2\left(\frac{10}{3}\right)} = \boxed{0.15\text{ kN}}$$

(*b*) Apply $\sum F_x = 0$ to the ladder to obtain:

$$T - F_{\text{by wall}} = 0 \Rightarrow T = F_{\text{by wall}}$$

Apply $\sum\vec{\tau} = 0$ about an axis through the bottom of the ladder subject to the condition that $F_{\text{by wall}} = T_{\text{max}}$:

$$T_{\text{max}}(\ell\sin\theta) - mg\left(\tfrac{1}{2}\ell\cos\theta\right) - Mg(L\cos\theta) = 0$$

where *L* is the maximum distance along the ladder that the person can climb without exceeding the maximum tension in the wire.

Solving for *L* and simplifying yields:

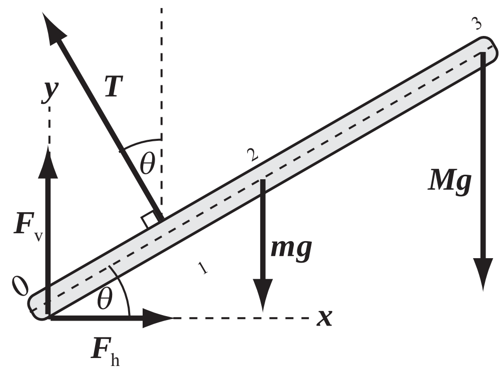
$$L = \frac{T_{\text{max}}\ell\sin\theta - \frac{1}{2}mg\ell\cos\theta}{Mg\cos\theta}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{(200\text{ N})(5.0\text{ m}) - \frac{1}{2}(20\text{ kg})(9.81\text{ m/s}^2)(1.5\text{ m})}{(80\text{ kg})(9.81\text{ m/s}^2)\cos 73.3°} = \boxed{3.8\text{ m}}$$

68 •• A 360-kg object is supported on a wire attached to a 15-m-long steel bar that is pivoted at a vertical wall and supported by a cable as shown in Figure 12-58. The mass of the bar is 85 kg. With the cable attached to the bar 5.0 m from the lower end as shown, find the tension in the cable and the force exerted by the wall on the steel bar.

Picture the Problem Let m represent the mass of the bar, M the mass of the suspended object, F_v the vertical component of the force the wall exerts on the bar, F_h the horizontal component of the force the wall exerts on the bar, and T the tension in the cable. The force diagram shows these forces and their points of application on the bar. Because the bar is in equilibrium, we can apply the conditions for translational and rotational equilibrium to relate the various forces and distances.



Apply $\sum \vec{\tau} = 0$ to the bar about an axis through the hinge:

$$T\ell_1 - mg\ell_2 \cos\theta - Mg\ell_3 \cos\theta = 0$$

Solving for T yields:

$$T = \frac{(m\ell_2 + M\ell_3)g\cos\theta}{\ell_1}$$

Substitute numerical values and evaluate T:

$$T = \frac{((85\,\text{kg})(7.5\,\text{m}) + (360\,\text{kg})(15\,\text{m}))(9.81\,\text{m/s}^2)\cos 30°}{5.0\,\text{m}} = 10.3\,\text{kN} = \boxed{10\,\text{kN}}$$

The magnitude and direction of the force exerted by the wall are given by:

$$F_{\text{by wall}} = \sqrt{F_v^2 + F_h^2} \quad (1)$$

and

$$\phi = \tan^{-1}\left(\frac{F_v}{F_h}\right) \quad (2)$$

Apply $\sum \vec{F} = 0$ to the bar to obtain:

$$\sum F_y = F_v + T\cos 30° - mg - Mg = 0$$

and

$$\sum F_x = F_h - T\sin 30° = 0$$

Solving the y equation for F_v yields:

$$F_v = (m + M)g - T\cos 30°$$

Solve the x equation for F_h to obtain:

$$F_h = T\sin 30°$$

Substituting for F_v and F_h in equation (1) yields:

$$F_{\text{by wall}} = \sqrt{\left((m+M)g - T\cos 30°\right)^2 + \left(T\sin 30°\right)^2}$$

Substitute numerical values and evaluate $F_{\text{by wall}}$:

$$\begin{aligned}F_{\text{by wall}} &= \sqrt{\left((85\text{ kg}+360\text{ kg})(9.81\text{ m/s}^2) - (10.3\text{ kN})\cos 30°\right)^2 + \left((10.3\text{ kN})\sin 30°\right)^2}\\ &= \boxed{6.9\text{ kN}}\end{aligned}$$

Substituting for F_v and F_h in equation (2) yields:

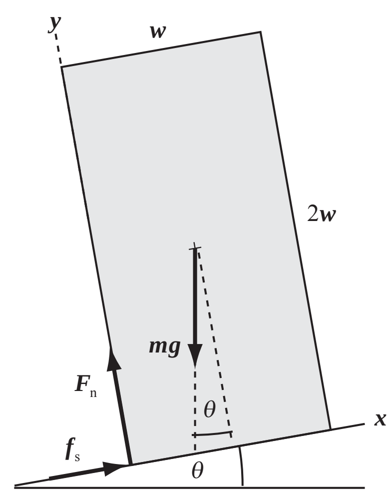
$$\phi = \tan^{-1}\left(\frac{(m+M)g - T\cos 30°}{T\sin 30°}\right)$$

Substitute numerical values and evaluate ϕ:

$$\begin{aligned}\phi &= \tan^{-1}\left(\frac{(85\text{ kg}+360\text{ kg})(9.81\text{ m/s}^2) - (10.3\text{ kN})\cos 30°}{(10.3\text{ kN})\sin 30°}\right) = -41.49°\\ &= \boxed{-41°}\text{ That is, }41°\text{ below the horizontal.}\end{aligned}$$

69 •• Repeat Problem 63 if the truck accelerates up a hill that makes an angle of 9.0º with the horizontal.

Picture the Problem The box and the forces acting on it are shown in the figure. When the box is about to tip, F_n acts at its edge, as indicated in the drawing. We can use the definition of μ_s and apply the condition for rotational equilibrium in an accelerated frame to relate f_s to the weight of the box and, hence, to the normal force.



Using its definition, express μ_s:

$$\mu_s \geq \frac{f_s}{F_n}$$

Apply $\sum \vec{\tau} = 0$ about an axis through the box's center of mass:

$$wf_s - \tfrac{1}{2}wF_n = 0 \Rightarrow \frac{f_s}{F_n} = \frac{1}{2}$$

Substitute for $\dfrac{f_s}{F_n}$ to obtain the condition for tipping:

$$\mu_s \geq 0.50$$

Therefore, if the box is to slide:

$$\mu_s < \boxed{0.50}\text{, as in Problem 63.}$$

Remarks: The difference between problems 63 and 69 is that in 63 the maximum acceleration before slipping is 0.5*g*, whereas in 69 it is (0.5 cos9.0°– sin9.0°)*g* = 0.337*g*.

70 •• A thin uniform rod 60 cm long is balanced 20 cm from one end when an object whose mass is (2*m* + 2.0 grams) is at the end nearest the pivot and an object of mass *m* is at the opposite end (Figure 12-59*a*). Balance is again achieved if the object whose mass is (2*m* + 2.0 grams) is replaced by the object of mass *m* and no object is placed at the other end (Figure 12-59*b*). Determine the mass of the rod.

Picture the Problem Let the mass of the rod be represented by *M*. Because the rod is in equilibrium, we can apply the condition for rotational equilibrium to relate the masses of the objects placed on the rod to its mass.

Apply $\sum \vec{\tau} = 0$ about an axis through the pivot for the initial condition:

$$(20\,\text{cm})(2m + 2.0\,\text{g}) - (40\,\text{cm})m - (10\,\text{cm})M = 0$$

Simplifying yields:

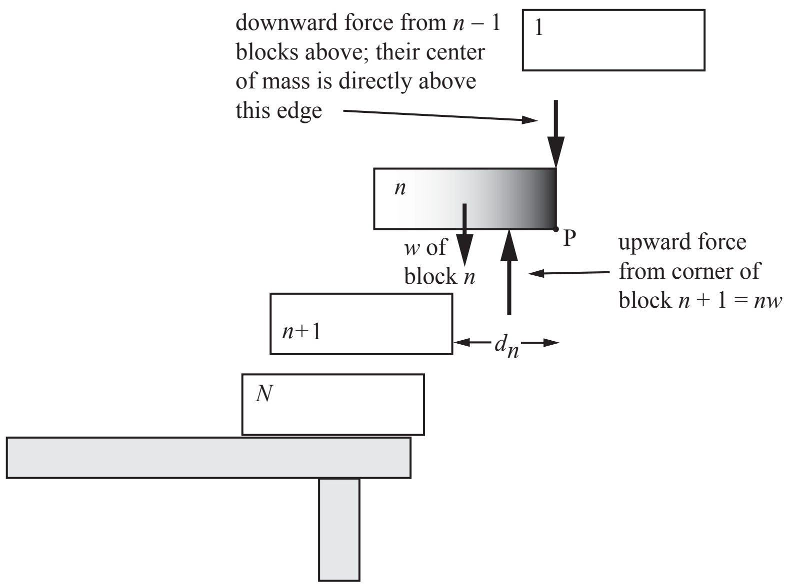
$$2(2m + 2.0\,\text{g}) - 4m - M = 0$$

Solve for *M* to obtain:

$$M = \boxed{4.0\,\text{g}}$$

71 ••• [SSM] There are a large number of identical uniform bricks, each of length *L*. If they are stacked one on top of another lengthwise (see Figure 12-63), the maximum offset that will allow the top brick to rest on the bottom brick is *L*/2. (*a*) Show that if this two-brick stack is placed on top of a third brick, the maximum offset of the second brick on the third brick is *L*/4. (*b*) Show that, in general, if you build a stack of *N* bricks, the maximum overhang of the (*n* – 1)th brick (counting down from the top) on the *n*th brick is *L*/2*n*. (*c*) Write a **spreadsheet** program to calculate total offset (the sum of the individual offsets) for a stack of *N* bricks, and calculate this for *L* = 20 cm and *N* = 5, 10, and 100. (*d*) Does the sum of the individual offsets approach a finite limit as $N \to \infty$? If so, what is that limit?

Picture the Problem Let the weight of each uniform brick be w. The downward force of all the bricks above the nth brick *must act at its corner,* because the upward reaction force points through the center of mass of all the bricks above the nth one. Because there is no vertical acceleration, the upward force exerted by the $(n + 1)$th brick on the nth brick must equal the total weight of the bricks above it. Thus this force is just nw. Note that it is convenient to develop the general relationship of Part (b) initially and then extract the answer for Part (a) from this general result.



(a) and (b) Noting that the line of action of the downward forces exerted by the blocks above the nth block passes through the point P, resulting in a lever arm of zero, apply $\sum \tau_{\text{P}} = 0$ to the nth brick to obtain:

$$w\left(\tfrac{1}{2}L\right) - nwd_n = 0$$

where d_n is the overhang of the nth brick beyond the edge of the $(n + 1)$th brick.

Solving for d_n yields:

$$d_n = \boxed{\frac{L}{2n}} \text{ where } n = 1,2,3,\ldots$$

For block number 2:

$$d_2 = \boxed{\frac{L}{4}}$$

(c) A spreadsheet program to calculate the sum of the offsets as a function of n is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B5	B4+1	$n+1$
C5	C4+$B$1/(2*B5)	$d_n + \dfrac{L}{2n}$

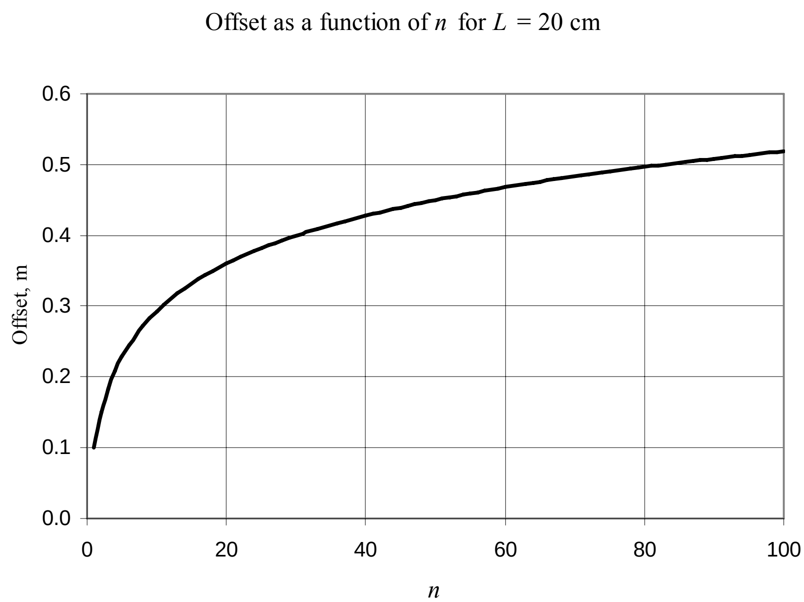
	A	B	C	D
1	$L=$	0.20	m	
2				
3		n	offset	
4		1	0.100	
5		2	0.150	
6		3	0.183	
7		4	0.208	
8		5	0.228	
9		6	0.245	
10		7	0.259	
11		8	0.272	
12		9	0.283	
13		10	0.293	
98		95	0.514	
99		96	0.515	
100		97	0.516	
101		98	0.517	
102		99	0.518	
103		100	0.519	

From the table we see that $d_5 = \boxed{15\,\text{cm},}$ $d_{10} = \boxed{26\,\text{cm},}$ and $d_{100} = \boxed{52\,\text{cm}.}$

(*d*) The sum of the individual offsets S is given by:

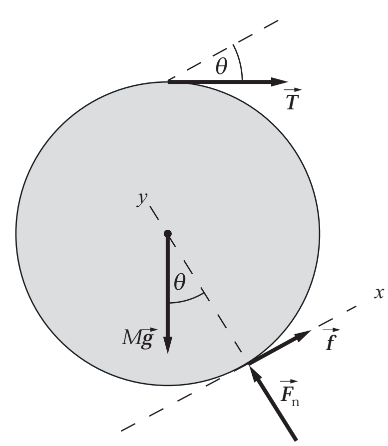
$$S = \sum_{n=1}^{N} d_n = \frac{L}{2}\sum_{n=1}^{N}\frac{1}{n}$$

No. Because this series is a harmonic series, S approaches infinity as the number of blocks N grows without bound. The following graph, plotted using a spreadsheet program, suggests that S has no limit.



72 •• A uniform sphere of radius R and mass M is held at rest on an inclined plane of angle θ by a horizontal string, as shown in Figure 12-61. Let R = 20 cm, M = 3.0 kg, and θ = 30º. (*a*) Find the tension in the string. (*b*) What is the normal force exerted on the sphere by the inclined plane? (*c*) What is the frictional force acting on the sphere?

Picture the Problem The four forces acting on the sphere: its weight, *mg*; the normal force of the plane, F_n; the frictional force, *f*, acting parallel to the plane; and the tension in the string, *T*, are shown in the figure. Because the sphere is in equilibrium, we can apply the conditions for translational and rotational equilibrium to find *f*, F_n, and *T*.



(*a*) Apply $\sum\vec{\tau} = 0$ about an axis through the center of the sphere:

$$fR - TR = 0 \Rightarrow T = f$$

Apply $\sum F_x = 0$ to the sphere:

$$f + T\cos\theta - Mg\sin\theta = 0$$

Substituting for f and solving for T yields:

$$T = \frac{Mg\sin\theta}{1+\cos\theta}$$

Substitute numerical values and evaluate T:

$$T = \frac{(3.0\,\text{kg})(9.81\,\text{m/s}^2)\sin 30°}{1+\cos 30°} = 7.89\,\text{N}$$
$$= \boxed{7.9\,\text{N}}$$

(b) Apply $\sum F_y = 0$ to the sphere:

$$F_n - T\sin\theta - Mg\cos\theta = 0$$

Solve for F_n:

$$F_n = T\sin\theta + Mg\cos\theta$$

Substitute numerical values and evaluate F_n:

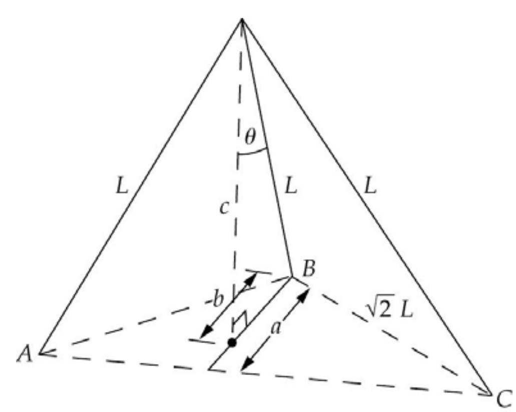
$$F_n = (7.89\,\text{N})\sin 30° + (3.0\,\text{kg})(9.81\,\text{m/s}^2)\cos 30° = \boxed{29\,\text{N}}$$

(c) In Part (a) we showed that $f = T$:

$$f = \boxed{7.9\,\text{N}}$$

73 ••• The legs of a tripod make equal angles of 90º with each other at the apex, where they join together. A 100-kg block hangs from the apex. What are the compressional forces in the three legs?

Picture the Problem Let L be the length of each leg of the tripod. Applying the Pythagorean theorem leads us to conclude that the distance a shown in the figure is $\sqrt{3/2}L$ and the distance b, the distance to the centroid of the triangle ABC is $\frac{2}{3}\sqrt{3/2}L$, and the distance c is $L/\sqrt{3}$. These results allow us to conclude that $\cos\theta = L/\sqrt{3}$. Because the tripod is in equilibrium, we can apply the condition for translational equilibrium to find the compressional forces in each leg.



Letting F_C represent the compressional force in a leg of the tripod, apply $\sum \vec{F} = 0$ to the apex of the tripod:

$$3F_C\cos\theta - mg = 0 \Rightarrow F_C = \frac{mg}{3\cos\theta}$$

Substitute for $\cos\theta$ and simplify to obtain:

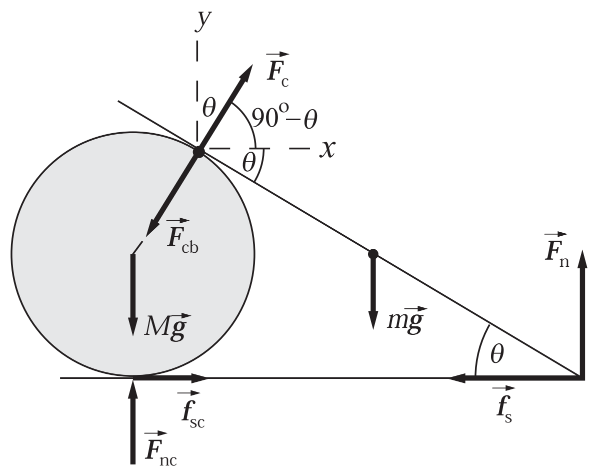
$$F_C = \frac{mg}{3\left(\frac{1}{\sqrt{3}}\right)} = \frac{mg}{\sqrt{3}}$$

Substitute numerical values and evaluate F_C:

$$F_C = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{\sqrt{3}} = \boxed{566\,\text{N}}$$

74 ••• Figure 12-63 shows a 20-cm-long uniform beam resting on a cylinder that has a radius of 4.0-cm. The mass of the beam is 5.0 kg and that of the cylinder is 8.0 kg. The coefficient of static friction between beam and cylinder is zero, whereas the coefficients of static friction between the cylinder and the floor, and between the beam and the floor, are *not* zero. Are there any values for these coefficients of static friction such that the system is in static equilibrium? If so, what are these values? If not, explain why none exist.

Picture the Problem The forces that act on the beam are its weight, mg; the force of the cylinder, F_c, acting along the radius of the cylinder; the normal force of the ground, F_n; and the friction force $f_s = \mu_s F_n$. The forces acting on the cylinder are its weight, Mg; the force of the beam on the cylinder, $F_{cb} = F_c$ in magnitude, acting radially inward; the normal force of the ground on the cylinder, F_{nc}; and the force of friction, $f_{sc} = \mu_{sc} F_{nc}$. Choose the coordinate system shown in the figure and apply the conditions for rotational and translational equilibrium.



Express $\mu_{s,\text{beam-floor}}$ in terms of f_s and F_n:

$$\mu_{s,\text{beam-floor}} = \frac{f_s}{F_n} \quad (1)$$

Express $\mu_{s,\text{cylinder-floor}}$ in terms of f_{sc} and F_{nc}:

$$\mu_{s,\text{cylinder-floor}} = \frac{f_{sc}}{F_{nc}} \quad (2)$$

Apply $\sum \vec{\tau} = 0$ about an axis through the right end of the beam:

$$[(10\,\text{cm})\cos\theta]mg - (15\,\text{cm})F_\text{c} = 0$$

Solve for F_c to obtain:

$$F_\text{c} = \frac{[(10\,\text{cm})\cos\theta]mg}{15\,\text{cm}}$$

Substitute numerical values and evaluate F_c:

$$F_\text{c} = \frac{[10\cos 30°](5.0\,\text{kg})(9.81\,\text{m/s}^2)}{15} = 28.3\,\text{N}$$

Apply $\sum F_y = 0$ to the beam:

$$F_\text{n} + F_\text{c}\cos\theta - mg = 0$$

Solving for F_n yields:

$$F_\text{n} = mg - F_\text{c}\cos\theta$$

Substitute numerical values and evaluate F_n:

$$F_\text{n} = (5.0\,\text{kg})(9.81\,\text{m/s}^2) - (28.3\,\text{N})\cos 30° = 24.5\,\text{N}$$

Apply $\sum F_x = 0$ to the beam:

$$-f_\text{s} + F_\text{c}\cos(90° - \theta) = 0$$

Solve for f_s to obtain:

$$f_\text{s} = F_\text{c}\cos(90° - \theta)$$

Substitute numerical values and evaluate f_s:

$$f_\text{s} = F_\text{c}\cos(90° - \theta) = (28.3\,\text{N})\cos 60° = 14.2\,\text{N}$$

$\vec{F}_\text{cb}$ is the reaction force to $\vec{F}_\text{c}$:

$F_\text{cb} = F_\text{c} = 28.3\,\text{N}$ radially inward.

Apply $\sum F_y = 0$ to the cylinder:

$$F_\text{nc} - F_\text{cb}\cos\theta - Mg = 0$$

Solve for F_nc to obtain:

$$F_\text{nc} = F_\text{cb}\cos\theta + Mg$$

Substitute numerical values and evaluate F_nc:

$$F_\text{nc} = (28.3\,\text{N})\cos 30° + (8.0\,\text{kg})(9.81\,\text{m/s}^2) = 103\,\text{N}$$

Apply $\sum F_x = 0$ to the cylinder:

$$f_\text{sc} - F_\text{cb}\cos(90° - \theta) = 0$$

Solve for and evaluate f_sc:

$$f_\text{sc} = F_\text{cb}\cos(90° - \theta) = (28.3\,\text{N})\cos 60° = 14.2\,\text{N}$$

Substitute numerical values in equations (1) and (2) and evaluate $\mu_{\text{s,beam-floor}}$ and $\mu_{\text{s,cylinder-floor}}$:

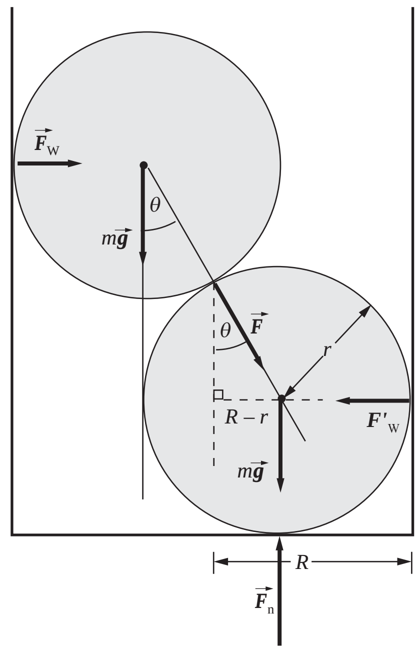
$$\mu_{\text{s,beam-floor}} = \frac{14.2\ \text{N}}{24.5\ \text{N}} = \boxed{0.58}$$

and

$$\mu_{\text{s,cylinder-floor}} = \frac{14.2\ \text{N}}{103\ \text{N}} = \boxed{0.14}$$

75 ••• [SSM] Two solid smooth (frictionless) spheres of radius r are placed inside a cylinder of radius R, as in Figure 12-62. The mass of each sphere is m. Find the force exerted by the bottom of the cylinder on the bottom sphere, the force exerted by the wall of the cylinder on each sphere, and the force exerted by one sphere on the other. All forces should be expressed in terms of m, R, and r.

Picture the Problem The geometry of the system is shown in the drawing. Let upward be the positive y direction and to the right be the $+x$ direction. Let the angle between the vertical center line and the line joining the two centers be θ. Then $\sin\theta = \dfrac{R-r}{r}$ and $\tan\theta = \dfrac{R-r}{\sqrt{R(2r-R)}}$. The force exerted by the bottom of the cylinder is just $2mg$. Let F be the force that the top sphere exerts on the lower sphere. Because the spheres are in equilibrium, we can apply the condition for translational equilibrium.



Apply $\sum F_y = 0$ to the spheres:

$$F_n - mg - mg = 0 \Rightarrow F_n = \boxed{2mg}$$

Because the cylinder wall is smooth, $F\cos\theta = mg$, and:

$$F = \frac{mg}{\cos\theta} = \boxed{mg\frac{r}{\sqrt{R(2r-R)}}}$$

Express the x component of F:

$$F_x = F\sin\theta = mg\tan\theta$$

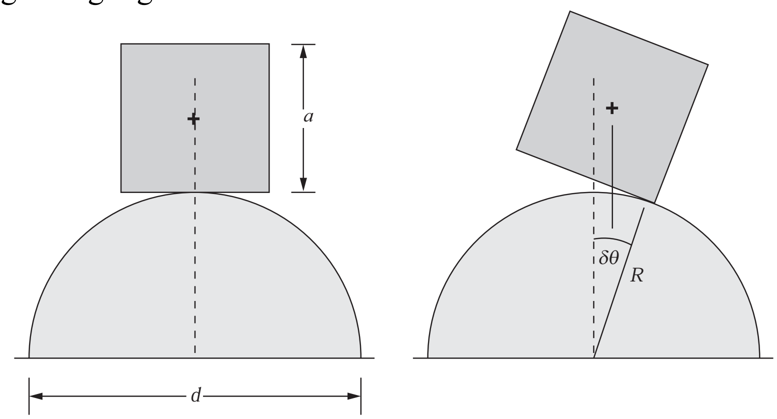
Express the force that the wall of the cylinder exerts:

$$F_W = \boxed{mg\frac{R-r}{\sqrt{R(2r-R)}}}$$

Remarks: Note that as *r* approaches *R*/2, $F_w \rightarrow \infty$.

76 ••• A solid cube of side length a balanced atop a cylinder of diameter d is in unstable equilibrium if $d << a$ (Figure 12-64) and is in stable equilibrium if $d >> a$. The cube does not slip on the cylinder. Determine the minimum value of the ratio d/a for which the cube is in stable equilibrium.

Picture the Problem Consider a small rotational displacement, $\delta\theta$ of the cube from equilibrium. This shifts the point of contact between cube and cylinder by $R\delta\theta,$ where $R = d/2$. As a result of that motion, the cube itself is rotated through the same angle $\delta\theta,$ and so its center is shifted in the same direction by the amount $(a/2)\,\delta\theta,$ neglecting higher order terms in $\delta\theta$.



If the displacement of the cube's center of mass is less than that of the point of contact, the torque about the point of contact is a restoring torque, and the cube will return to its equilibrium position. If, on the other hand, $(a/2)\,\delta\theta > (d/2)\,\delta\theta,$ then the torque about the point of contact due to mg is in the direction of $\delta\theta,$ and will cause the displacement from equilibrium to increase. We see that the minimum value of d/a for stable equilibrium is $d/a = 1$.

Chapter 13
Fluids

Conceptual Problems

1 • If the gauge pressure is doubled, the absolute pressure will be (*a*) halved, (*b*) doubled, (*c*) unchanged, (*d*) increased by a factor greater than 2, (*e*) increased by a factor less than 2.

Determine the Concept The absolute pressure is related to the gauge pressure according to $P = P_{\text{gauge}} + P_{\text{at}}$.

Prior to doubling the gauge pressure, the absolute pressure is given by:

$$P = P_{\text{gauge}} + P_{\text{at}} \Rightarrow P_{\text{gauge}} = P - P_{\text{at}}$$

Doubling the gauge pressure results in an absolute pressure given by:

$$P' = 2P_{\text{gauge}} + P_{\text{at}}$$

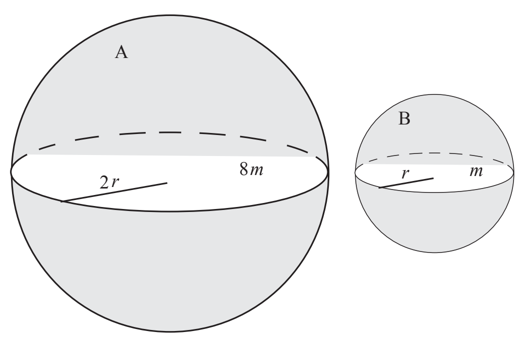
Substituting for P_{gauge} yields:

$$P' = 2(P - P_{\text{at}}) + P_{\text{at}} = 2P - P_{\text{at}}$$

Because $P' = 2P - P_{\text{at}}$, doubling the gauge pressure results in an increase in the absolute pressure by a factor less than 2. $\boxed{(e)}$ is correct.

2 • Two spherical objects differ in size and mass. Object A has a mass that is eight times the mass of object B. The radius of object A is twice the radius of object B. How do their densities compare? (*a*) $\rho_{\text{A}} > \rho_{\text{B}}$, (*b*) $\rho_{\text{A}} < \rho_{\text{B}}$, (*c*) $\rho_{\text{A}} = \rho_{\text{B}}$, (*d*) not enough information is given to compare their densities.

Determine the Concept The density of an object is its mass per unit volume. The pictorial representation shown to the right summarizes the information concerning the radii and masses of the two spheres. We can determine the relationship between the densities of A and B by examining their ratio.



Express the densities of the two spheres:

$$\rho_{\text{A}} = \frac{m_{\text{A}}}{V_{\text{A}}} = \frac{m_{\text{A}}}{\frac{4}{3}\pi r_{\text{A}}^3}$$

and

$$\rho_{\text{B}} = \frac{m_{\text{B}}}{V_{\text{B}}} = \frac{m_{\text{B}}}{\frac{4}{3}\pi r_{\text{B}}^3}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{\rho_A}{\rho_B}=\frac{\dfrac{m_A}{\frac{4}{3}\pi r_A^3}}{\dfrac{m_B}{\frac{4}{3}\pi r_B^3}}=\frac{m_A}{m_B}\frac{r_B^3}{r_A^3}$$

Substituting for the masses and radii and simplifying yields:

$$\frac{\rho_A}{\rho_B}=\frac{8m}{m}\frac{r^3}{(2r)^3}=1\Rightarrow\rho_A=\rho_B$$

and $\boxed{(c)}$ is correct.

3 • **[SSM]** Two objects differ in density and mass. Object A has a mass that is eight times the mass of object B. The density of object A is four times the density of object B. How do their volumes compare? (*a*) $V_A=\frac{1}{2}V_B$, (*b*) $V_A=V_B$, (*c*) $V_A=2V_B$, (*d*) not enough information is given to compare their volumes.

Determine the Concept The density of an object is its mass per unit volume. We can determine the relationship between the volumes of A and B by examining their ratio.

Express the volumes of the two objects:

$$V_A=\frac{m_A}{\rho_A}\text{ and }V_B=\frac{m_B}{\rho_B}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{V_A}{V_B}=\frac{\dfrac{m_A}{\rho_A}}{\dfrac{m_B}{\rho_B}}=\frac{\rho_B}{\rho_A}\frac{m_A}{m_B}$$

Substituting for the masses and densities and simplifying yields:

$$\frac{V_A}{V_B}=\frac{\rho_B}{4\rho_B}\frac{8m_B}{m_B}=2\Rightarrow V_A=2V_B$$

and $\boxed{(c)}$ is correct.

4 • A sphere is constructed by gluing together two hemispheres of different density materials. The density of each hemisphere is uniform, but the density of one is greater than the density of the other. True or false: The average density of the sphere is the numerical average of the two different densities. Clearly explain your reasoning.

Determine the Concept True. This is a special case. Because the volumes are equal, the average density is the numerical average of the two densities. ***This is not a general result.***

5 • In several jungle adventure movies, the hero and heroine escape the bad guys by hiding underwater for extended periods of time. To do this, they breathe through long vertical hollow reeds. Imagine that in one movie, the water is so clear that to be safely hidden the two are at a depth of 15 m. As a science consultant to the movie producers, you tell them that this is not a realistic depth and the knowledgeable viewer will laugh during this scene. Explain why this is so.

Determine the Concept Pressure increases approximately 1 atm every 10 m in fresh water. To breathe requires creating a pressure of less than 1 atm in your lungs. At the surface you can do this easily, but not at a depth of 10 m.

6 •• Two objects are balanced as in Figure 13-28. The objects have identical volumes but different masses. Assume all the objects in the figure are denser than water and thus none will float. Will the equilibrium be disturbed if the entire system is completely immersed in water? Explain your reasoning.

Determine the Concept Yes. Because the volumes of the two objects are equal, the downward force on each side is reduced by the same amount (the buoyant force acting on them) when they are submerged. The buoyant force is independent of their masses. That is, if $m_1L_1 = m_2L_2$ and $L_1 \neq L_2$, then $(m_1 - c)L_1 \neq (m_2 - c)L_2$.

7 •• **[SSM]** A solid 200-g block of lead and a solid 200-g block of copper are completely submerged in an aquarium filled with water. Each block is suspended just above the bottom of the aquarium by a thread. Which of the following is true?

(*a*) The buoyant force on the lead block is greater than the buoyant force on the copper block.
(*b*) The buoyant force on the copper block is greater than the buoyant force on the lead block.
(*c*) The buoyant force is the same on both blocks.
(*d*) More information is needed to choose the correct answer.

Determine the Concept The buoyant forces acting on these submerged objects are equal to the weight of the water each displaces. The weight of the displaced water, in turn, is directly proportional to the volume of the submerged object. Because $\rho_{Pb} > \rho_{Cu}$, the volume of the copper must be greater than that of the lead and, hence, the buoyant force on the copper is greater than that on the lead. $\boxed{(b)}$ is correct.

8 •• A 20-cm^3 block of lead and a 20-cm^3 block of copper are completely submerged in an aquarium filled with water. Each is suspended just above the bottom of the aquarium by a thread. Which of the following is true?

(*a*) The buoyant force on the lead block is greater than the buoyant force on the copper block..
(*b*) The buoyant force on the copper block is greater than the buoyant force on the lead block.
(*c*) The buoyant force is the same on both blocks.
(*d*) More information is needed to choose the correct answer.

Determine the Concept The buoyant forces acting on these submerged objects are equal to the weight of the water each displaces. The weight of the displaced water, in turn, is directly proportional to the volume of the submerged object. Because their volumes are the same, the buoyant forces on them must be the same. $\boxed{(c)}$ is correct.

9 •• Two bricks are completely submerged in water. Brick 1 is made of lead and has rectangular dimensions of 2″ × 4″ × 8″. Brick 2 is made of wood and has rectangular dimensions of 1″ × 8″ × 8″. *True or false*: The buoyant force on brick 2 is larger than the buoyant force on brick 1.

Determine the Concept False. The buoyant force on a submerged object depends on the weight of the displaced fluid which, in turn, depends on the volume of the displaced fluid. Because the bricks have the same volume, they will displace the same volume of water and the buoyant force will be the same on both of them.

10 •• Figure 13-29 shows an object called a ″Cartesian diver.″ The diver consists of a small tube, open at the bottom, with an air bubble at the top, inside a closed plastic soda bottle that is partly filled with water. The diver normally floats, but sinks when the bottle is squeezed hard. (*a*) Explain why this happens. (*b*) Explain the physics behind how a submarine can ″silently″ sink vertically simply by allowing water to flow into empty tanks near its keel. (*c*) Explain why a floating person will oscillate up and down on the water surface as he or she breathes in and out.

Determine the Concept

(*a*) When the bottle is squeezed, the force is transmitted equally through the fluid, leading to a pressure increase on the air bubble in the diver. The air bubble shrinks, and the loss in buoyancy is enough to sink the diver.

(*b*) As water enters its tanks, the weight of the submarine increases. When the submarine is completely submerged, the volume of the displaced water and, hence, the buoyant force acting on the submarine become constant. Because the weight of the submarine is now greater than the buoyant force acting on it, the submarine will start to sink.

(*c*) Breathing in lowers one's average density and breathing out increases one's average density. Because denser objects float lower on the surface than do less dense objects, a floating person will oscillate up and down on the water surface as he or she breathes in and out.

11 •• A certain object has a density just slightly less than that of water so that it floats almost completely submerged. However, the object is more compressible than water. What happens if the floating object is given a slight downward push? Explain.

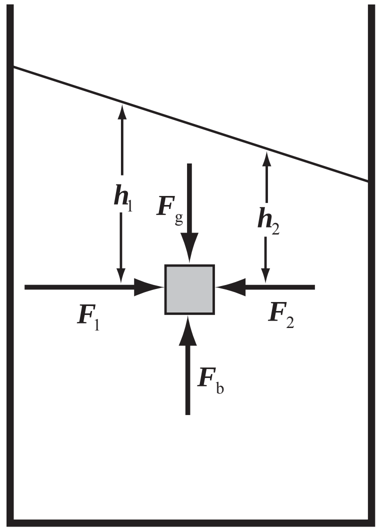
Determine the Concept Because the pressure increases with depth, the object will be compressed and its density will increase as its volume decreases. Thus, the object will sink to the bottom.

12 •• In Example 13.11 the fluid is accelerated to a greater speed as it enters the narrow part of the pipe. Identify the forces that act on the fluid at the entrance to the narrow region to produce this acceleration.

Determine the Concept The acceleration-producing force acting on the fluid is the product of the difference in pressure between the wide and narrow parts of the pipe and the area of the narrow part of the pipe.

13 •• **[SSM]** An upright glass of water is accelerating to the right along a flat, horizontal surface. What is the origin of the force that produces the acceleration on a small element of water in the middle of the glass? Explain by using a diagram. *Hint: The water surface will not remain level as long as the glass of water is accelerating. Draw a free body diagram of the small element of water.*

Determine the Concept The pictorial representation shows the glass and an element of water in the middle of the glass. As is readily established by a simple demonstration, the surface of the water is not level while the glass is accelerated, showing that there is a pressure gradient (a difference in pressure) due to the differing depths ($h_1 > h_2$ and hence $F_1 > F_2$) of water on the two sides of the element of water. This pressure gradient results in a net force on the element as shown in the figure. The upward buoyant force is equal in magnitude to the downward gravitational force.



14 •• You are sitting in a boat floating on a very small pond. You take the anchor out of the boat and drop it into the water. Does the water level in the pond rise, fall, or remain the same? Explain your answer.

Determine the Concept The water level in the pond will fall slightly. When the anchor is in the boat, the boat displaces enough water so that the buoyant force on it equals the sum of the weight of the boat, your weight, and the weight of the anchor. When you drop the anchor into the water, it displaces just its volume of water (rather than its weight as it did while in the boat). The total weight of the boat becomes less and the boat displaces less water as a consequence.

15 •• A horizontal pipe narrows from a diameter of 10 cm at location A to 5.0 cm at location B. For a nonviscous incompressible fluid flowing without turbulence from location A to location B, how do the flow speeds v (in m/s) at the two locations compare? (*a*) $v_A = v_B$, (*b*) $v_A = \frac{1}{2}v_B$, (*c*) $v_A = \frac{1}{4}v_B$, (*d*) $v_A = 2v_B$, (*e*) $v_A = 4v_B$

Determine the Concept We can use the equation of continuity to compare the flow rates at the two locations.

Apply the equation of continuity at locations A and B to obtain:

$$A_A v_A = A_B v_B \Rightarrow v_A = \frac{A_B}{A_A} v_B$$

Substitute for A_B and A_A and simplify to obtain:

$$v_A = \frac{\frac{1}{4}\pi d_B^2}{\frac{1}{4}\pi d_A^2} v_B = \left(\frac{d_B}{d_A}\right)^2 v_B$$

Substitute numerical values and evaluate v_A:

$$v_A = \left(\frac{5\text{ cm}}{10\text{ cm}}\right)^2 v_B = \tfrac{1}{4}v_B$$

and $\boxed{(c)}$ is correct.

16 •• A horizontal pipe narrows from a diameter of 10 cm at location A to 5.0 cm at location B. For a nonviscous incompressible fluid flowing without turbulence from location A to location B, how do the pressures P (in N/m^2) at the two locations compare? (*a*) $P_A = P_B$, (*b*) $P_A = \frac{1}{2}P_B$, (*c*) $P_A = \frac{1}{4}P_B$, (*d*) $P_A = 2P_B$, (*e*) $P_A = 4P_B$, (*f*) There is not enough information to compare the pressures quantitatively.

Determine the Concept We can use the equation of continuity to compare the flow rates at the two locations.

Apply Bernoulli's equation for constant elevations at locations A and B to obtain:

$$P_A + \tfrac{1}{2}\rho v_A^2 = P_B + \tfrac{1}{2}\rho v_B^2 \qquad (1)$$

Apply the equation of continuity at locations A and B to obtain:

$$A_A v_A = A_B v_B \Rightarrow v_A = \frac{A_B}{A_A}v_B$$

Substitute for A_B and A_A and simplify to obtain:

$$v_A = \frac{\frac{1}{4}\pi d_B^2}{\frac{1}{4}\pi d_A^2}v_B = \left(\frac{d_B}{d_A}\right)^2 v_B$$

Substituting for v_A in equation (1) yields:

$$P_A + \tfrac{1}{2}\rho\left(\left(\frac{d_B}{d_A}\right)^2 v_B\right)^2 = P_B + \tfrac{1}{2}\rho v_B^2$$

While the values of d_B, d_A, and ρ are known to us, we need a value for v_B (or v_A) in order to compare P_A and P_B. Hence $\boxed{(f)}$ is correct.

17 •• **[SSM]** Figure 13-30 is a diagram of a prairie dog tunnel. The geometry of the two entrances are such that entrance 1 is surrounded by a mound and entrance 2 is surrounded by flat ground. Explain how the tunnel remains ventilated, and indicate in which direction air will flow through the tunnel.

Determine the Concept The mounding around entrance 1 will cause the streamlines to curve concave downward over the entrance. An upward pressure gradient produces the downward centripetal force. This means there is a lowering of the pressure at entrance 1. No such lowering occurs aver entrance 2, so the pressure there is higher than the pressure at entrance 1. The air circulates in entrance 2 and out entrance 1. It has been demonstrated that enough air will circulate inside the tunnel even with slightest breeze outside.

Estimation and Approximation

18 •• Your undergraduate research project involves atmospheric sampling. The sampling device has a mass of 25.0 kg. Estimate the diameter of a helium-filled balloon required to lift the device off the ground. Neglect the mass of the balloon ″skin″ and the small buoyancy force on the device itself.

Picture the Problem We can use Archimedes' principle and the condition for vertical equilibrium to estimate the diameter of the helium-filled balloon that would just lift the sampling device.

Express the equilibrium condition that must be satisfied if the balloon-payload is to ″just lift off″:

$$F_{\text{bouyant}} = B = F_{\text{gravitational}} = mg \qquad (1)$$

Using Archimedes' principle, express the buoyant force B:

$$B = w_{\text{displaced fluid}} = m_{\text{displaced fluid}}g$$
$$= \rho_{\text{air}}V_{\text{balloon}}g$$

Substituting for B in equation (1) yields:

$$\rho_{\text{air}}V_{\text{balloon}}g = mg$$
or
$$\rho_{\text{air}}V_{\text{balloon}} = m \qquad (2)$$

The total mass m to be lifted is the sum of the mass of the payload m_{p} and the mass of the helium m_{He}:

$$m = m_{\text{p}} + m_{\text{He}} = m_{\text{p}} + \rho_{\text{He}}V_{\text{balloon}}$$

Substitute for m in equation (2) to obtain:

$$\rho_{\text{air}}V_{\text{balloon}} = m_{\text{p}} + \rho_{\text{He}}V_{\text{balloon}}$$

Solving for V_{balloon} yields:

$$V_{\text{balloon}} = \frac{m_{\text{p}}}{\rho_{\text{air}} - \rho_{\text{He}}}$$

The volume of the balloon is given by:

$$V_{\text{balloon}} = \tfrac{4}{3}\pi r^3 = \tfrac{1}{6}\pi d^3$$

Substituting for V_{balloon} yields:

$$\tfrac{1}{6}\pi d^3 = \frac{m_\text{p}}{\rho_\text{air} - \rho_\text{He}} \Rightarrow d = \sqrt[3]{\frac{6m_\text{p}}{\pi(\rho_\text{air} - \rho_\text{He})}}$$

Substitute numerical values and evaluate d:

$$d = \sqrt[3]{\frac{6(25.0\text{ kg})}{\pi(1.293\text{ kg/m}^3 - 0.179\text{ kg/m}^3)}} = \boxed{3.50\text{ m}}$$

19 •• Your friend wants to start a business giving hot-air balloon rides. The empty balloon, the basket and the occupants have a total maximum mass of 1000 kg. If the balloon has a diameter of 22.0 m when fully inflated with hot air, estimate the required density of the hot air. Neglect the buoyancy force on the basket and people.

Picture the Problem We can use Archimedes' principle and the condition for vertical equilibrium to estimate the density of the hot air that would enable the balloon and its payload to lift off.

Express the equilibrium condition that must be satisfied if the balloon-payload is to "just lift off":

$$F_\text{bouyant} = B = F_\text{gravitational} = mg \qquad (1)$$

Using Archimedes' principle, express the buoyant force B:

$$B = w_\text{displaced fluid} = m_\text{displaced fluid}\,g = \rho_\text{air} V_\text{balloon}\, g$$

Substituting for B in equation (1) yields:

$$\rho_\text{air} V_\text{balloon}\, g = mg \Rightarrow \rho_\text{air} V_\text{balloon} = m \qquad (2)$$

The total mass m to be lifted is the sum of the mass of the payload m_p and the mass of the hot air:

$$m = m_\text{p} + m_\text{hot air} = m_\text{p} + \rho_\text{hot air} V_\text{balloon}$$

Substitute for m in equation (2) to obtain:

$$\rho_\text{air} V_\text{balloon} = m_\text{p} + \rho_\text{hot air} V_\text{balloon}$$

Solving for $\rho_\text{hot air}$ and simplifying yields:

$$\rho_\text{hot air} = \frac{\rho_\text{air} V_\text{balloon} - m_\text{p}}{V_\text{balloon}} = \rho_\text{air} - \frac{m_\text{p}}{V_\text{balloon}}$$

The volume of the balloon is given by:

$$V_\text{balloon} = \tfrac{4}{3}\pi r^3 = \tfrac{1}{6}\pi d^3$$

Substituting for V_balloon yields:

$$\rho_\text{hot air} = \rho_\text{air} - \frac{m_\text{p}}{\tfrac{1}{6}\pi d^3} = \rho_\text{air} - \frac{6m_\text{p}}{\pi d^3}$$

Substitute numerical values and evaluate $\rho_{\text{hot air}}$:

$$\rho_{\text{hot air}} = 1.293\ \text{kg/m}^3 - \frac{6(1000\ \text{kg})}{\pi(22.0\ \text{m})^3} = \boxed{1.11\ \text{kg/m}^3}$$

Remarks: As expected, the density of the hot air is considerably less than the density of the surrounding cooler air.

Density

20 • Find the mass of a solid lead sphere with a radius equal to 2.00 cm.

Picture the Problem The mass of the sphere is the product of its density and volume. The density of lead can be found in Figure 13-1.

Using the definition of density, express the mass of the sphere:

$$m = \rho V = \rho\left(\tfrac{4}{3}\pi R^3\right)$$

Substitute numerical values and evaluate m:

$$m = \tfrac{4}{3}\pi\left(11.3\times10^3\ \text{kg/m}^3\right)\left(2.00\times10^{-2}\ \text{m}\right)^3 = \boxed{0.379\ \text{kg}}$$

21 • **[SSM]** Consider a room measuring 4.0 m × 5.0 m × 4.0 m. Under normal atmospheric conditions at Earth's surface, what would be the mass of the air in the room?

Picture the Problem The mass of the air in the room is the product of its density and volume. The density of air can be found in Figure 13-1.

Use the definition of density to express the mass of the air in the room:

$$m = \rho V = \rho LWH$$

Substitute numerical values and evaluate m:

$$m = \left(1.293\ \text{kg/m}^3\right)(4.0\ \text{m})(5.0\ \text{m})(4.0\ \text{m}) = \boxed{1.0\times10^2\ \text{kg}}$$

22 •• An average neutron star has approximately the same mass as the Sun, but is compressed into a sphere of radius roughly 10 km. What would be the approximate mass of a teaspoonful of matter that dense?

Picture the Problem We can use the definition of density to find the approximate mass of a teaspoonful of matter from a neutron star. Assume that the volume of a teaspoon is about 5 mL.

Use the definition of density to express the mass of a teaspoonful of matter whose density is ρ:

$$m = \rho V_{\text{teaspoon}} \qquad (1)$$

The density of the neutron star is given by:

$$\rho_{\text{NS}} = \frac{m_{\text{NS}}}{V_{\text{NS}}}$$

Substituting for m_{NS} and V_{NS} yields:

$$\rho_{\text{NS}} = \frac{m_{\text{Sun}}}{\frac{4}{3}\pi r_{\text{NS}}^3}$$

Let $\rho = \rho_{\text{NS}}$ in equation (1) to obtain:

$$m = \frac{3m_{\text{Sun}}V_{\text{teaspoon}}}{4\pi r_{\text{NS}}^3}$$

Substitute numerical values and evaluate m:

$$m = \frac{3\left(1.99\times10^{30}\ \text{kg}\right)\left(5\times10^{-6}\,\text{m}^3\right)}{4\pi\left(10\times10^3\ \text{m}\right)^3} \approx \boxed{2\ \text{Tg}}$$

23 •• A 50.0-g ball consists of a plastic spherical shell and a water-filled core. The shell has an outside diameter equal to 50.0 mm and an inside diameter equal to 20.0 mm. What is the density of the plastic?

Picture the Problem We can use the definition of density to find the density of the plastic of which the spherical shell is constructed.

The density of the plastic is given by:

$$\rho_{\text{plastic}} = \frac{m_{\text{plastic}}}{V_{\text{plastic}}} \qquad (1)$$

The mass of the plastic is the difference between the mass of the ball and the mass of its water-filled core:

$$m_{\text{plastic}} = m_{\text{ball}} - m_{\text{water}}$$

Use the definition of density to express the mass of the water:

$$m_{\text{water}} = \rho_{\text{water}}V_{\text{water}}$$

Substituting for m_{water} yields:

$$m_{\text{plastic}} = m_{\text{ball}} - \rho_{\text{water}}V_{\text{water}}$$

Substitute for m_{plastic} in equation (1) to obtain:

$$\rho_{\text{plastic}} = \frac{m_{\text{ball}} - \rho_{\text{water}}V_{\text{water}}}{V_{\text{plastic}}}$$

Because $V_{\text{water}} = \frac{4}{3}\pi R_{\text{inside}}^3$ and $V_{\text{ball}} = \frac{4}{3}\pi\left(R_{\text{outside}}^3 - R_{\text{inside}}^3\right)$:

$$\rho_{\text{plastic}} = \frac{m_{\text{ball}} - \rho_{\text{water}}\left(\frac{4}{3}\pi R_{\text{inside}}^3\right)}{\frac{4}{3}\pi\left(R_{\text{outside}}^3 - R_{\text{inside}}^3\right)}$$

Substitute numerical values and evaluate ρ_{plsstic}:

$$\rho_{\text{plastic}} = \frac{50.0\text{ g} - \left(1.00\,\frac{\text{g}}{\text{cm}^3}\right)\left(\frac{4}{3}\pi(10.0\text{ mm})^3\right)}{\frac{4}{3}\pi\left((25.0\text{ mm})^3 - (10.0\text{ mm})^3\right)} = \boxed{0.748\text{ g/cm}^3}$$

24 •• A 60.0-mL flask is filled with mercury at 0ºC (Figure 13-31). When the temperature rises to 80ºC, 1.47 g of mercury spills out of the flask. Assuming that the volume of the flask stays constant, find the change in density of mercury at 80ºC if its density at 0ºC is 13 645 kg/m^3.

Picture the Problem We can use the definition of density to relate the change in the density of the mercury to the amount spilled during the heating process.

The change in the density of the mercury as it is warmed is given by:

$$\Delta\rho = \rho_0 - \rho \qquad (1)$$

where ρ_0 is the density of the mercury before it is warmed.

The density of the mercury before it is warmed is the ratio of its mass to the volume it occupies:

$$\rho_0 = \frac{m_0}{V_0} \qquad (2)$$

The volume of the mercury that spills is the ratio of its mass to its density at the higher temperature:

$$V_{\text{spilled}} = \frac{m_{\text{spilled}}}{\rho}$$

The density ρ of the mercury at the higher temperature is given by:

$$\rho = \frac{m_0}{V_0 + V_{\text{spilled}}} = \frac{m_0}{V_0 + \dfrac{m_{\text{spilled}}}{\rho}}$$

Solving for ρ yields:

$$\rho = \frac{m_0 - m_{\text{spilled}}}{V_0} = \rho_0 - \frac{m_{\text{spilled}}}{V_0} \qquad (3)$$

Substituting equations (2) and (3) in equation (1) yields:

$$\Delta\rho = \rho_0 - \left(\rho_0 - \frac{m_{\text{spilled}}}{V_0}\right) = \frac{m_{\text{spilled}}}{V_0}$$

Substitute numerical values and evaluate $\Delta\rho$:

$$\Delta\rho = \frac{1.47\times10^{-3}\,\text{kg}}{60.0\times10^{-6}\,\text{m}^3} = \boxed{24.5\,\text{kg/m}^3}$$

25 •• One sphere is made of gold and has a radius r_{Au} and another sphere is made of copper and has a radius r_{Cu}. If the spheres have equal mass, what is the ratio of the radii, $r_{\text{Au}}/r_{\text{Cu}}$?

Picture the Problem We can use the definition of density to find the ratio of the radii of the two spheres. See Table 13-1 for the densities of gold and copper.

Use the definition of density to express the mass of the gold sphere:

$$m_{\text{Au}} = \rho_{\text{Au}}V_{\text{Au}} = \tfrac{4}{3}\pi\rho_{\text{Au}}r_{\text{Au}}^3$$

The mass of the copper sphere is given by:

$$m_{\text{Cu}} = \rho_{\text{Cu}}V_{\text{Cu}} = \tfrac{4}{3}\pi\rho_{\text{Cu}}r_{\text{Cu}}^3$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{m_{\text{Au}}}{m_{\text{Cu}}} = \frac{\tfrac{4}{3}\pi\rho_{\text{Au}}r_{\text{Au}}^3}{\tfrac{4}{3}\pi\rho_{\text{Cu}}r_{\text{Cu}}^3} = \frac{\rho_{\text{Au}}}{\rho_{\text{Cu}}}\left(\frac{r_{\text{Au}}}{r_{\text{Cu}}}\right)^3$$

Solve for $r_{\text{Au}}/r_{\text{Cu}}$ to obtain:

$$\frac{r_{\text{Au}}}{r_{\text{Cu}}} = \sqrt[3]{\frac{m_{\text{Au}}}{m_{\text{Cu}}}\frac{\rho_{\text{Cu}}}{\rho_{\text{Au}}}}$$

Because the spheres have the same mass:

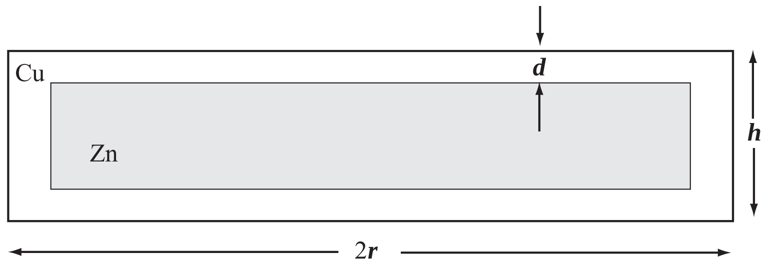
$$\frac{r_{\text{Au}}}{r_{\text{Cu}}} = \sqrt[3]{\frac{\rho_{\text{Cu}}}{\rho_{\text{Au}}}}$$

Substitute numerical values and evaluate $r_{\text{Au}}/r_{\text{Cu}}$:

$$\frac{r_{\text{Au}}}{r_{\text{Cu}}} = \sqrt[3]{\frac{8.93\,\text{kg/m}^3}{19.3\,\text{kg/m}^3}} = \boxed{0.773}$$

26 ••• Since 1983, the US Mint has coined pennies that are made out of zinc with a copper cladding. The mass of this type of penny is 2.50 g. Model the penny as a uniform cylinder of height 1.23 mm and radius 9.50 mm. Assume the copper cladding is uniformly thick on all surfaces. If the density of zinc is 7140 kg/m^3 and that of copper is 8930 kg/m^3, what is the thickness of the copper cladding?

Picture the Problem The pictorial representation shows a zinc penny with its copper cladding. We can use the definition of density to relate the difference between the mass of an all-copper penny and the mass of a copper-zinc penny to the thickness d of the copper cladding.



Express the mass 0f the cladded penny:

$$m = m_{\text{Cu}} + m_{\text{Zn}}$$

In terms of the densities of copper and zinc, our equation becomes:

$$m = \rho_{\text{Cu}} V_{\text{Cu}} + \rho_{\text{Zn}} V_{\text{Zn}}$$

Substituting for the volumes of zinc and copper gives:

$$m = \rho_{\text{Cu}}\left[2\pi r^2 d + 2\pi r(h-2d)d\right] + \rho_{\text{Zn}}\,\pi(r-d)^2(h-2d)$$

Factor h and r from $(h - 2d)$ and $(r - d)^2$ to obtain:

$$m = \rho_{\text{Cu}}\left[2\pi r^2 d + 2\pi r h\left(1-\frac{2d}{h}\right)d\right] + \rho_{\text{Zn}}\,\pi r^2\left(1-\frac{d}{r}\right)^2 h\left(1-\frac{2d}{h}\right)$$

Assuming that $d \ll h$ and $d \ll r$:

$$m \approx \rho_{\text{Cu}}\left[2\pi r^2 d + 2\pi r h d\right] + \rho_{\text{Zn}}\,\pi r^2 h$$

Solving for d yields:

$$d = \frac{m - \rho_{\text{Zn}}\pi r^2 h}{2\pi r \rho_{\text{Cu}}(r+h)}$$

Substitute numerical values and evaluate d:

$$d = \frac{2.50\times10^{-3}\ \text{kg} - \pi\left(7140\ \text{kg/m}^3\right)(9.50\ \text{mm})^2(1.23\ \text{mm})}{2\pi(9.50\ \text{mm})\left(8930\ \text{kg/m}^3\right)(9.50\ \text{mm}+1.23\ \text{mm})} \approx \boxed{1.7\ \mu\text{m}}$$

Remarks: This small value for *d* (approximately 2 μm) justifies our assumptions that *d* is much smaller than both *r* and *h*.

Pressure

27 • Barometer readings are commonly given in inches of mercury (inHg). Find the pressure in inches of mercury equal to 101 kPa.

Picture the Problem The pressure due to a column of height h of a liquid of density ρ is given by $P = \rho g h$.

Letting h represent the height of the column of mercury, express the pressure at its base:

$$\rho_{Hg}gh = 101\,\text{kPa} \Rightarrow h = \frac{101\,\text{kPa}}{\rho_{Hg}g}$$

Substitute numerical values and evaluate h:

$$h = \frac{101\,\text{kPa}}{(13.6\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)}$$
$$= 0.7570\,\text{m} \times \frac{1\,\text{in}}{2.54\times10^{-2}\ \text{m}}$$
$$= \boxed{29.8\ \text{inHg}}$$

28 • The pressure on the surface of a lake is P_{at} = 101 kPa. (*a*) At what depth is the pressure $2P_{at}$? (*b*) If the pressure at the top of a deep pool of mercury is P_{at}, at what depth is the pressure $2P_{at}$?

Picture the Problem The pressure due to a column of height h of a liquid of density ρ is given by $P = \rho gh$. The pressure at any depth in a liquid is the sum of the pressure at the surface of the liquid and the pressure due to the liquid at a given depth in the liquid.

(*a*) Express the pressure as a function of depth in the lake:

$$P = P_{at} + \rho_{water}gh \Rightarrow h = \frac{P - P_{at}}{\rho_{water}g}$$

Substitute for P and simplify to obtain:

$$h = \frac{2P_{at} - P_{at}}{\rho_{water}g} = \frac{P_{at}}{\rho_{water}g}$$

Substitute numerical values and evaluate h:

$$h = \frac{101\,\text{kPa}}{(1.00\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)}$$
$$= \boxed{10.3\,\text{m}}$$

(*b*) Proceed as in (*a*) with ρ_{water} replaced by ρ_{Hg} to obtain:

$$h = \frac{2P_{at} - P_{at}}{\rho_{Hg}g} = \frac{P_{at}}{\rho_{Hg}g}$$

Substitute numerical values and evaluate h:

$$h = \frac{101\,\text{kPa}}{(13.6\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)}$$
$$= \boxed{75.7\,\text{cm}}$$

29 • When at cruising altitude, a typical airplane cabin will have an air pressure equivalent to an altitude of about 2400 m. During the flight, ears often equilibrate, so that the air pressure inside the inner ear equalizes with the air

pressure outside the plane. The Eustachian tubes allow for this equalization, but can become clogged. If an Eustachian tube is clogged, pressure equalization may not occur on descent and the air pressure inside an inner ear may remain equal to the pressure at 2400 m. In that case, by the time the plane lands and the cabin is repressurized to sea-level air pressure, what is the net force on one ear drum, due to this pressure difference, assuming the ear drum has an area of 0.50 cm^2?

Picture the Problem Assuming the density of the air to be constant, we can use the definition of pressure and the expression for the variation of pressure with depth in a fluid to find the net force on one's ear drums.

The net force acting on one ear drum is given by:

$$F = \Delta PA$$

where A is the area of an ear drum.

Relate the pressure difference to the pressure at 2400 m and the pressure at sea level:

$$\Delta P = P_{\text{sea level}} - P_{2400\,\text{m}} = \rho g \Delta h$$

Substituting for ΔP in the expression for F yields:

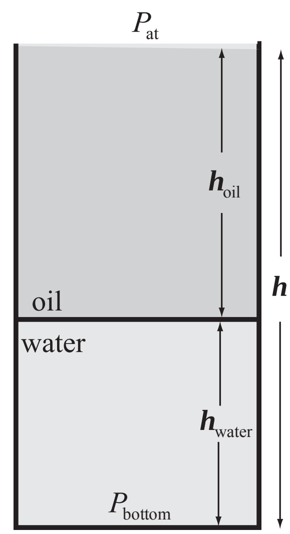
$$F = \rho\, g \Delta h A$$

Substitute numerical values and evaluate F:

$$F = \left(1.293\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)\left(2400\,\text{m}\right)\left(0.50\,\text{cm}^2\right) = \boxed{1.5\,\text{N}}$$

30 • The axis of a cylindrical container is vertical. The container is filled with equal masses of water and oil. The oil floats on top of the water, and the open surface of the oil is at a height h above the bottom of the container. What is the height h, if the pressure at the bottom of the water is 10 kPa greater than the pressure at the top of the oil? Assume the oil density is 875 kg/m^3.

Picture the Problem The pictorial representation represents oil floating on water in a cylindrical container that is open to the atmosphere. We can express the height of the cylinder as the sum of the heights of the cylinders of oil and water and then use the fact that the masses of oil and water are equal to obtain a relationship between h_{oil} and h_{water}. This relationship, together with the equation for the pressure as function of depth in a liquid will lead us to expressions for h_{oil} and h_{water}.



Express h as the sum of the heights of the cylinders of oil and water:

$$h = h_{\text{oil}} + h_{\text{water}} \tag{1}$$

The difference in pressure between the bottom of the water and the top of oil is:

$$\Delta P = P_{\text{bottom}} - P_{\text{at}} = \frac{m_{\text{water}}g + m_{\text{oil}}g}{A}$$

where A is the cross-sectional area of the cylinder.

Substituting for m_{water} and m_{oil} and simplifying gives:

$$\frac{\Delta P}{g} = \rho_{\text{water}}h_{\text{water}} + \rho_{\text{oil}}h_{\text{oil}} \tag{2}$$

Because the masses of oil and water are equal:

$$\frac{m_{\text{water}}}{m_{\text{oil}}} = \frac{\rho_{\text{water}}h_{\text{water}}A}{\rho_{\text{oil}}h_{\text{oil}}A} = \frac{\rho_{\text{water}}h_{\text{water}}}{\rho_{\text{oil}}h_{\text{oil}}} = 1$$

and

$$\rho_{\text{water}}h_{\text{water}} = \rho_{\text{oil}}h_{\text{oil}}$$

Substituting for $\rho_{\text{oil}}h_{\text{oil}}$ in equation (2) yields:

$$\frac{\Delta P}{g} = \rho_{\text{water}}h_{\text{water}} + \rho_{\text{water}}h_{\text{water}}$$

$$= 2\rho_{\text{water}}h_{\text{water}}$$

and

$$h_{\text{water}} = \frac{\Delta P}{2\rho_{\text{water}}g}$$

Proceed similarly to obtain:

$$h_{\text{oil}} = \frac{\Delta P}{2\rho_{\text{oil}}g}$$

Substituting for h_{water} and h_{oil} in equation (1) gives:

$$h = \frac{\Delta P}{2\rho_{\text{oil}}g} + \frac{\Delta P}{2\rho_{\text{water}}g} = \frac{\Delta P}{2g}\left(\frac{1}{\rho_{\text{oil}}} + \frac{1}{\rho_{\text{water}}}\right)$$

Substitute numerical values and evaluate h:

$$h = \frac{10\text{ kPa}}{2(9.81\text{ m/s}^2)}\left(\frac{1}{875\text{ kg/m}^3} + \frac{1}{1.00\times10^3\text{ kg/m}^3}\right) = \boxed{1.1\text{ m}}$$

31 • [SSM] A hydraulic lift is used to raise a 1500-kg automobile. The radius of the shaft of the lift is 8.00 cm and the radius of the compressor's piston is 1.00 cm. How much force must be applied to the piston to raise the automobile? *Hint*: *The shaft of the lift is the other piston.*

Picture the Problem The pressure applied to an enclosed liquid is transmitted undiminished to every point in the fluid and to the walls of the container. Hence we can equate the pressure produced by the force applied to the piston to the pressure due to the weight of the automobile and solve for F.

Express the pressure the weight of the automobile exerts on the shaft of the lift:

$$P_{\text{auto}} = \frac{w_{\text{auto}}}{A_{\text{shaft}}}$$

Express the pressure the force applied to the piston produces:

$$P = \frac{F}{A_{\text{piston}}}$$

Because the pressures are the same, we can equate them to obtain:

$$\frac{w_{\text{auto}}}{A_{\text{shaft}}} = \frac{F}{A_{\text{piston}}}$$

Solving for F yields:

$$F = w_{\text{auto}}\frac{A_{\text{piston}}}{A_{\text{shaft}}} = m_{\text{auto}}g\frac{A_{\text{piston}}}{A_{\text{shaft}}}$$

Substitute numerical values and evaluate F:

$$F = (1500\text{ kg})(9.81\text{ m/s}^2)\left(\frac{1.00\text{ cm}}{8.00\text{ cm}}\right)^2 = \boxed{230\text{ N}}$$

32 • A 1500-kg car rests on four tires, each of which is inflated to a gauge pressure of 200 kPa. If the four tires support the car's weight equally, what is the area of contact of each tire with the road?

Picture the Problem The area of contact of each tire with the road is related to the weight on each tire and the pressure in the tire through the definition of pressure.

Using the definition of gauge pressure, relate the area of contact to the pressure and the weight of the car:

$$A = \frac{\frac{1}{4}w}{P_{\text{gauge}}} = \frac{mg}{4P_{\text{gauge}}}$$

Substitute numerical values and evaluate A:

$$A = \frac{(1500\,\text{kg})(9.81\,\text{m/s}^2)}{4(200\,\text{kPa})} = \boxed{184\,\text{cm}^2}$$

33 •• What pressure increase is required to compress the volume of 1.00 kg of water from 1.00 L to 0.99 L? Could this compression occur in our oceans where the maximum depth is about 11 km? Explain.

Picture the Problem The required pressure ΔP is related to the change in volume ΔV and the initial volume V through the definition of the bulk modulus B; $B = -\dfrac{\Delta P}{\Delta V/V}$.

Using the definition of the bulk modulus, relate the change in volume to the initial volume and the required pressure:

$$B = -\frac{\Delta P}{\Delta V/V} \Rightarrow \Delta P = -B\frac{\Delta V}{V}$$

Substitute numerical values and evaluate ΔP:

$$\Delta P = -2.00\times10^{9}\ \text{Pa}\times\left(\frac{-0.01\,\text{L}}{1.00\,\text{L}}\right)$$

$$= 2.00\times10^{7}\ \text{Pa}\times\frac{1\,\text{atm}}{101.325\,\text{kPa}}$$

$$= 197.4\ \text{atm} = \boxed{197\,\text{atm}}$$

Express the pressure at a depth h in the ocean:

$$P(h) = P_{\text{at}} + \rho_{\text{seawater}}gh$$

or

$$\Delta P = \rho_{\text{seawater}}gh \Rightarrow h = \frac{\Delta P}{\rho_{\text{seawater}}g}$$

Substitute numerical values and evaluate h:

$$h = \frac{197.4\,\text{atm} \times \dfrac{101.325\,\text{kPa}}{\text{atm}}}{\left(1025\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)} \approx 2.0\,\text{km}$$

Because a depth of only 2 km is required to produce a one-percent compression, this does not occur in our oceans.

34 •• When a woman in high-heeled shoes takes a step, she momentarily places her entire weight on one heel of her shoe. If her mass is 56.0 kg and if the area of the heel is 1.00 cm^2, what is the pressure exerted on the floor by the heel? Compare your answer to the pressure exerted by the one foot of an elephant on a flat floor. Assume the elephant's mass is 5000 kg, that he has all four feet equally distributed on the floor, and that each foot has an area of 400 cm^2.

Picture the Problem The pressure exerted by the woman's heel on the floor is her weight divided by the area of her heel.

Using its definition, express the pressure exerted on the floor by the woman's heel:

$$P = \frac{F}{A} = \frac{w}{A} = \frac{mg}{A}$$

Substitute numerical values and evaluate P_{woman}:

$$P_{\text{woman}} = \frac{(56.0\,\text{kg})\left(9.81\,\text{m/s}^2\right)}{1.00 \times 10^{-4}\,\text{m}^2}$$

$$= 5.49 \times 10^6\ \text{N/m}^2 \times \frac{1\,\text{atm}}{101.325\,\text{kPa}}$$

$$= \boxed{54.2\,\text{atm}}$$

Express the pressure exerted by one of the elephant's feet:

$$P_{\text{elephant}} = \frac{\frac{1}{4}(5000\,\text{kg})\left(9.81\,\text{m/s}^2\right)}{400 \times 10^{-4}\,\text{m}^2} = 3.066 \times 10^5\ \text{N/m}^2 \times \frac{1\,\text{atm}}{101.325\,\text{kPa}} = 3.03\,\text{atm}$$

Express the ratio of the pressure exerted by the woman's heel to the pressure exerted by the elephant's foot:

$$\frac{P_{\text{woman}}}{P_{\text{elephant}}} = \frac{54.2\,\text{atm}}{3.03\,\text{atm}} \approx 18$$

Thus $P_{\text{woman}} \approx \boxed{18 P_{\text{elephant}}}.$

35 •• In the seventeenth century, Blaise Pascal performed the experiment shown in Figure 13-32. A wine barrel filled with water was coupled to a long tube. Water was added to the tube until the barrel burst. The radius of the lid was 20 cm and the height of the water in the tube was 12 m. (*a*) Calculate the force exerted on the lid due to the pressure increase. (*b*) If the tube had an inner radius

of 3.0 mm, what mass of water in the tube caused the pressure that burst the barrel?

Picture the Problem The force on the lid is related to pressure exerted by the water and the cross-sectional area of the column of water through the definition of density. We can find the mass of the water from the product of its density and volume.

(*a*) Using the definition of pressure, express the force exerted on the lid:

$$F = PA$$

Express the pressure due to a column of water of height *h*:

$$P = \rho_{\text{water}} gh$$

Substitute for *P* and *A* to obtain:

$$F = \rho_{\text{water}} gh\pi r^2$$

Substitute numerical values and evaluate *F*:

$$F = \left(1.00\times10^3\ \text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)\times\left(12\,\text{m}\right)\pi\left(0.20\,\text{m}\right)^2 = \boxed{15\,\text{kN}}$$

(*b*) Relate the mass of the water to its density and volume:

$$m = \rho_{\text{water}} V = \rho_{\text{water}} h\pi r^2$$

Substitute numerical values and evaluate *m*:

$$m = \left(1.00\times10^3\ \text{kg/m}^3\right)\left(12\,\text{m}\right)\pi\left(3.0\times10^{-3}\ \text{m}\right)^2 = \boxed{0.34\,\text{kg}}$$

36 •• Blood plasma flows from a bag through a tube into a patient's vein, where the blood pressure is 12 mmHg. The specific gravity of blood plasma at 37ºC is 1.03. What is the minimum elevation of the bag so that the plasma flows into the vein?

Picture the Problem The minimum elevation of the bag *h* that will produce a pressure of at least 12 mmHg is related to this pressure and the density of the blood plasma through $P = \rho_{\text{blood}} gh$.

Using the definition of the pressure due to a column of liquid, relate the pressure at its base to its height:

$$P = \rho_{\text{blood}} gh \Rightarrow h = \frac{P}{\rho_{\text{blood}} g}$$

Substitute numerical values and evaluate h:

$$h = \frac{12\,\text{mmHg} \times \dfrac{133.32\,\text{Pa}}{1\,\text{mmHg}}}{\left(1.03\times10^3\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)} = \boxed{16\,\text{cm}}$$

37 •• Many people have imagined that if they were to float the top of a flexible snorkel tube out of the water, they would be able to breathe through it while walking underwater (Figure 13-33). However, they generally do not take into account just how much water pressure opposes the expansion of the chest and the inflation of the lungs. Suppose you can just breathe while lying on the floor with a 400-N (90-lb) weight on your chest. How far below the surface of the water could your chest be for you still to be able to breathe, assuming your chest has a frontal area of 0.090 m^2?

Picture the Problem The depth h below the surface at which you would be able to breath is related to the pressure at that depth and the density of water ρ_w through $P = \rho_w gh$.

Express the pressure due to a column of water of height h:

$$P = \rho_w gh \Rightarrow h = \frac{P}{\rho_w g}$$

Express the pressure at depth h in terms of the weight on your chest:

$$P = \frac{w_{\text{on your chest}}}{A_{\text{of your chest}}}$$

Substituting for P yields:

$$h = \frac{w_{\text{on your chest}}}{A_{\text{of your chest}}\rho_w g}$$

Substitute numerical values and evaluate h:

$$h = \frac{400\,\text{N}}{\left(0.090\,\text{m}^2\right)\left(1.00\times10^3\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)} = \boxed{45\,\text{cm}}$$

38 •• In Example 13-3 a 150-N force is applied to a small piston to lift a car that weighs 15 000 N. Demonstrate that this does not violate the law of conservation of energy by showing that, when the car is lifted some distance h, the work done by the 150-N force acting on the small piston equals the work done on the car by the large piston.

Picture the Problem Let A_1 and A_2 represent the cross-sectional areas of the large piston and the small piston, and F_1 and F_2 the forces exerted by the large and on the small piston, respectively. The work done by the large piston is $W_1 = F_1h_1$ and

that done on the small piston is $W_2 = F_2h_2$. We'll use Pascal's principle and the equality of the volume of the displaced liquid in both pistons to show that W_1 and W_2 are equal.

Express the work done in lifting the car a distance h:

$$W_1 = F_1h_1$$

where F_1 is the weight of the car.

Using the definition of pressure, relate the forces F_1 (= w) and F_2 to the areas A_1 and A_2:

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \Rightarrow F_1 = F_2\frac{A_1}{A_2}$$

Equate the volumes of the displaced fluid in the two pistons:

$$h_1A_1 = h_2A_2 \Rightarrow h_1 = h_2\frac{A_2}{A_1}$$

Substitute in the expression for W_1 and simplify to obtain:

$$W_1 = F_2\frac{A_1}{A_2}h_2\frac{A_2}{A_1} = F_2h_2 = \boxed{W_2}$$

39 •• A 5.00-kg lead sinker is accidentally dropped overboard by fishermen in a boat directly above the deepest portion of the Marianas trench, near the Philippines. By what percentage does the volume of the sinker change by the time it settles to the trench bottom, which is 10.9 km below the surface?

Picture the Problem This problem is an application of the definition of the bulk modulus. The change in volume of the sinker is related to the pressure change and the bulk modulus by $B = -\frac{\Delta P}{\Delta V/V}$.

From the definition of bulk modulus:

$$\frac{\Delta V}{V} = -\frac{\Delta P}{B} \qquad (1)$$

Express the pressure at a depth h in the ocean:

$$P(h) = P_{at} + \rho gh$$

The difference in pressure between the pressure at depth h and the surface of the water is:

$$\Delta P = P(h) - P_{at} = \rho gh$$

Substitute for ΔP in equation (1) to obtain:

$$\frac{\Delta V}{V} = -\frac{\rho gh}{B}$$

Substitute numerical values and evaluate the fractional change in volume of the sinker:

$$\frac{\Delta V}{V}=-\frac{(1025\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(10.9\ \text{km})}{7.7\ \text{GPa}}=\boxed{-1.4\%}$$

40 ••• The volume of a cone of height h and base radius r is $V = \pi r^2h/3$. A jar in the shape of a cone of height 25 cm has a base with a radius equal to 15 cm. The jar is filled with water. Then its lid (the base of the cone) is screwed on and the jar is turned over so its lid is horizontal. (*a*) Find the volume and weight of the water in the jar. (*b*) Assuming the pressure inside the jar at the top of the cone is equal to 1 atm, find the excess force exerted by the water on the base of the jar. Explain how this force can be greater than the weight of the water in the jar.

Picture the Problem The weight of the water in the jar is the product of its mass and the gravitational field. Its mass, in turn, is related to its volume through the definition of density. The force the water exerts on the base of the jar can be determined from the product of the pressure it creates and the area of the base.

(*a*) The volume of the water is:

$$V=\tfrac{1}{3}\pi\left(15\times10^{-2}\ \text{m}\right)^2\left(25\times10^{-2}\ \text{m}\right)=5.890\times10^{-3}\ \text{m}^3=\boxed{5.9\ \text{L}}$$

Using the definition of density, relate the weight of the water to the volume it occupies:

$$w=mg=\rho Vg$$

$$w=\left(1.00\times10^3\ \frac{\text{kg}}{\text{m}^3}\right)\left(5.890\times10^{-3}\ \text{m}^3\right)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)=57.79\ \text{N}=\boxed{58\ \text{N}}$$

(*b*) Using the definition of pressure, relate the force exerted by the water on the base of the jar to the pressure it exerts and the area of the base:

$$F=PA=\rho gh\pi r^2$$

Substitute numerical values and evaluate F:

$$F=\left(1.00\times10^3\ \text{kg/m}^3\right)\left(9.81\text{m/s}^2\right)\left(25\times10^{-2}\ \text{m}\right)\pi\left(15\times10^{-2}\ \text{m}\right)^2=\boxed{0.17\text{kN}}$$

This occurs in the same way that the force on Pascal's barrel is much greater than the weight of the water in the tube. The downward force on the base is also the result of the downward component of the force exerted by the slanting walls of the cone on the water.

Buoyancy

41 • A 500-g piece of copper with specific gravity 8.96, is suspended from a spring scale and is submerged in water (Figure 13-34). What force does the spring scale read?

Picture the Problem The scale's reading is the difference between the weight of the piece of copper in air and the buoyant force acting on it.

Express the apparent weight w' of the piece of copper:

$$w' = w - B$$

Using the definition of density and Archimedes' principle, substitute for w and B to obtain:

$$w' = \rho_{\text{Cu}}Vg - \rho_{\text{w}}Vg = (\rho_{\text{Cu}} - \rho_{\text{w}})Vg$$

Express w in terms of ρ_{Cu} and V and solve for Vg:

$$w = \rho_{\text{Cu}}Vg \Rightarrow Vg = \frac{w}{\rho_{\text{Cu}}}$$

Substitute for Vg in the expression for w' to obtain:

$$w' = (\rho_{\text{Cu}} - \rho_{\text{w}})\frac{w}{\rho_{\text{Cu}}} = \left(1 - \frac{\rho_{\text{w}}}{\rho_{\text{Cu}}}\right)w$$

Substitute numerical values and evaluate w':

$$w' = \left(1 - \frac{1}{8.96}\right)(0.500\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{4.36\,\text{N}}$$

42 • When a certain rock is suspended from a spring scale the scale-display reads 60 N. However, when the suspended rock is submerged in water, the display reads 40 N. What is the density of the rock?

Picture the Problem We can use the definition of density and Archimedes' principle to find the density of the rock. The difference between the weight of the rock in air and the weight of the rock in water is the buoyant force acting on the rock.

Using its definition, express the density of the rock:

$$\rho_{\text{rock}} = \frac{m_{\text{rock}}}{V_{\text{rock}}} \qquad (1)$$

Apply Archimedes' principle to obtain:

$$B = w_{\substack{\text{displaced}\\\text{fluid}}} = m_{\substack{\text{displaced}\\\text{fluid}}} g = \rho_{\substack{\text{displaced}\\\text{fluid}}} V_{\substack{\text{displaced}\\\text{fluid}}} g$$

Solve for $V_{\substack{\text{displaced}\\\text{fluid}}}$:

$$V_{\substack{\text{displaced}\\\text{fluid}}} = \frac{B}{\rho_{\substack{\text{displaced}\\\text{fluid}}} g}$$

Because $V_{\substack{\text{displaced}\\\text{fluid}}} = V_{\text{rock}}$ and $\rho_{\substack{\text{displaced}\\\text{fluid}}} = \rho_{\text{water}}$:

$$V_{\text{rock}} = \frac{B}{\rho_{\text{water}} g}$$

Substituting in equation (1) and simplifying yields:

$$\rho_{\text{rock}} = \frac{m_{\text{rock}} g}{B} \rho_{\text{water}} = \frac{w_{\text{rock}}}{B} \rho_{\text{water}}$$

Substitute numerical values and evaluate ρ_{rock}:

$$\rho_{\text{rock}} = \frac{60\,\text{N}}{60\,\text{N} - 40\,\text{N}} \left(1.00 \times 10^3\ \text{kg/m}^3\right) = \boxed{3.0 \times 10^3\ \text{kg/m}^3}$$

43 • **[SSM]** A block of an unknown material weighs 5.00 N in air and 4.55 N when submerged in water. (*a*) What is the density of the material? (*b*) From what material is the block likely to have been made?

Picture the Problem We can use the definition of density and Archimedes' principle to find the density of the unknown object. The difference between the weight of the object in air and in water is the buoyant force acting on the object.

(*a*) Using its definition, express the density of the object:

$$\rho_{\text{object}} = \frac{m_{\text{object}}}{V_{\text{object}}} \qquad (1)$$

Apply Archimedes' principle to obtain:

$$B = w_{\substack{\text{displaced}\\\text{fluid}}} = m_{\substack{\text{displaced}\\\text{fluid}}} g = \rho_{\substack{\text{displaced}\\\text{fluid}}} V_{\substack{\text{displaced}\\\text{fluid}}} g$$

Solve for $V_{\substack{\text{displaced}\\\text{fluid}}}$:

$$V_{\substack{\text{displaced}\\\text{fluid}}} = \frac{B}{\rho_{\substack{\text{displaced}\\\text{fluid}}} g}$$

Because $V_{\substack{\text{displaced}\\ \text{fluid}}} = V_{\text{object}}$ and $\rho_{\substack{\text{displaced}\\ \text{fluid}}} = \rho_{\text{water}}$:

$$V_{\text{object}} = \frac{B}{\rho_{\text{water}} g}$$

Substitute in equation (1) and simplify to obtain:

$$\rho_{\text{object}} = \frac{m_{\text{object}} g}{B}\rho_{\text{water}} = \frac{w_{\text{object}}}{B}\rho_{\text{water}}$$

Substitute numerical values and evaluate ρ_{object}:

$$\rho_{\text{object}} = \frac{5.00\,\text{N}}{5.00\,\text{N} - 4.55\,\text{N}}\left(1.00\times10^3\ \text{kg/m}^3\right) = \boxed{11\times10^3\ \text{kg/m}^3}$$

(*b*) From Table 13-1, we see that the density of the unknown material is close to that of lead.

44 • A solid piece of metal weighs 90.0 N in air and 56.6 N when submerged in water. Determine the density of this metal.

Picture the Problem We can use the definition of density and Archimedes' principle to find the density of the unknown object. The difference between the weight of the object in air and the weight of the object in water is the buoyant force acting on the object.

Using its definition, express the density of the metal:

$$\rho_{\text{metal}} = \frac{m_{\text{metal}}}{V_{\text{metal}}} \qquad (1)$$

Apply Archimedes' principle to obtain:

$$B = w_{\substack{\text{displaced}\\ \text{fluid}}} = m_{\substack{\text{displaced}\\ \text{fluid}}} g = \rho_{\substack{\text{displaced}\\ \text{fluid}}} V_{\substack{\text{displaced}\\ \text{fluid}}} g$$

Solve for $V_{\substack{\text{displaced}\\ \text{fluid}}}$:

$$V_{\substack{\text{displaced}\\ \text{fluid}}} = \frac{B}{\rho_{\substack{\text{displaced}\\ \text{fluid}}} g}$$

Because $V_{\substack{\text{displaced}\\ \text{fluid}}} = V_{\text{metal}}$ and $\rho_{\substack{\text{displaced}\\ \text{fluid}}} = \rho_{\text{water}}$:

$$V_{\text{metal}} = \frac{B}{\rho_{\text{water}} g}$$

Substitute in equation (1) and simplify to obtain:

$$\rho_{\text{metal}} = \frac{m_{\text{metal}} g}{B}\rho_{\text{water}} = \frac{w_{\text{metal}}}{B}\rho_{\text{water}}$$

Substitute numerical values and evaluate ρ_{metal}:

$$\rho_{\text{metal}} = \frac{90.0\,\text{N}}{90.0\,\text{N} - 56.6\,\text{N}}\left(1.00\times10^3\ \text{kg/m}^3\right) = \boxed{2.69\times10^3\ \text{kg/m}^3}$$

45 •• A homogeneous solid object floats on water, with 80.0 percent of its volume below the surface. When placed in a second liquid, the same object floats on that liquid with 72.0 percent of its volume below the surface. Determine the density of the object and the specific gravity of the liquid.

Picture the Problem Let V be the volume of the object and V' be the volume that is submerged when it floats. The weight of the object is ρVg and the buoyant force due to the water is $\rho_w V'g$. Because the floating object is in translational equilibrium, we can use $\sum F_y = 0$ to relate the buoyant forces acting on the object in the two liquids to its weight.

Apply $\sum F_y = 0$ to the object floating in water:

$$\rho_w V'g - mg = \rho_w V'g - \rho Vg = 0 \quad (1)$$

Solving for ρ yields:

$$\rho = \rho_w \frac{V'}{V}$$

Substitute numerical values and evaluate ρ:

$$\rho = \left(1.00\times10^3\ \text{kg/m}^3\right)\frac{0.800V}{V} = \boxed{800\,\text{kg/m}^3}$$

Apply $\sum F_y = 0$ to the object floating in the second liquid and solve for mg:

$$mg = 0.720\,V\rho_L g$$

Solve equation (1) for mg:

$$mg = 0.800\,\rho_w Vg$$

Equate these two expressions to obtain:

$$0.720\,\rho_L = 0.800\,\rho_w$$

Substitute in the definition of specific gravity to obtain:

$$\text{specific gravity} = \frac{\rho_L}{\rho_w} = \frac{0.800}{0.720} = \boxed{1.11}$$

46 •• A 5.00-kg iron block is suspended from a spring scale and is submerged in a fluid of unknown density. The spring scale reads 6.16 N. What is the density of the fluid?

Picture the Problem We can use Archimedes' principle to find the density of the unknown fluid. The difference between the weight of the block in air and the weight of the block in the fluid is the buoyant force acting on the block.

Apply Archimedes' principle to obtain:

$$B = w_{\substack{\text{displaced}\\\text{fluid}}} = m_{\substack{\text{displaced}\\\text{fluid}}} g = \rho_{\substack{\text{displaced}\\\text{fluid}}} V_{\substack{\text{displaced}\\\text{fluid}}} g$$

Solve for $\rho_{\substack{\text{displaced}\\\text{fluid}}}$:

$$\rho_{\substack{\text{displaced}\\\text{fluid}}} = \frac{B}{V_{\substack{\text{displaced}\\\text{fluid}}} g}$$

Because $V_{\substack{\text{displaced}\\\text{fluid}}} = V_{\text{Fe block}}$:

$$\rho_{\text{f}} = \frac{B}{V_{\text{Fe block}} g} = \frac{B}{m_{\text{Fe block}} g} \rho_{\text{Fe}}$$

Substitute numerical values and evaluate ρ_{f}:

$$\rho_{\text{f}} = \frac{(5.00\,\text{kg})(9.81\,\text{m/s}^2) - 6.16\,\text{N}}{(5.00\,\text{kg})(9.81\,\text{m/s}^2)}(7.96\times10^3\,\text{kg/m}^3) = \boxed{7.0\times10^3\,\text{kg/m}^3}$$

47 •• A large piece of cork weighs 0.285 N in air. When held submerged underwater by a spring scale as shown in Figure 13-35, the spring scale reads 0.855 N. Find the density of the cork.

Picture the Problem The forces acting on the cork are B, the upward force due to the displacement of water, mg, the weight of the piece of cork, and F_s, the force exerted by the spring. The piece of cork is in equilibrium under the influence of these forces.

Apply $\sum F_y = 0$ to the piece of cork:

$$B - w - F_s = 0 \quad (1)$$

or

$$B - \rho_{\text{cork}} Vg - F_s = 0 \quad (2)$$

Express the buoyant force as a function of the density of water:

$$B = w_{\substack{\text{displaced}\\\text{fluid}}} = \rho_{\text{w}} Vg \Rightarrow Vg = \frac{B}{\rho_{\text{w}}}$$

Substitute for Vg in equation (2) to obtain:

$$B - \rho_{\text{cork}} \frac{B}{\rho_{\text{w}}} - F_s = 0 \quad (3)$$

Solve equation (1) for B:

$$B = w + F_s$$

Substitute in equation (3) to obtain:

$$w+F_s-\rho_{cork}\frac{w+F_s}{\rho_w}-F_s=0$$

or

$$w-\rho_{cork}\frac{w+F_s}{\rho_w}=0$$

Solving for ρ_{cork} yields:

$$\rho_{cork}=\rho_w\frac{w}{w+F_s}$$

Substitute numerical values and evaluate ρ_{cork}:

$$\rho_{cork}=\left(1.00\times10^3\ \text{kg/m}^3\right)\frac{0.285\,\text{N}}{0.285\,\text{N}+0.855\,\text{N}}=\boxed{250\,\text{kg/m}^3}$$

48 •• A helium balloon lifts a basket and cargo of total weight 2000 N under standard conditions, at which the density of air is 1.29 kg/m^3 and the density of helium is 0.178 kg/m^3. What is the minimum volume of the balloon?

Picture the Problem Under minimum-volume conditions, the balloon will be in equilibrium. Let B represent the buoyant force acting on the balloon, w_{tot} represent its total weight, and V its volume. The total weight is the sum of the weights of its basket, cargo, and helium in its balloon.

Apply $\sum F_y=0$ to the balloon:

$$B-w_{tot}=0$$

Express the total weight of the balloon:

$$w_{tot}=2000\,\text{N}+\rho_{He}Vg$$

Express the buoyant force due to the displaced air:

$$B=w_{\substack{\text{displaced}\\\text{fluid}}}=\rho_{air}Vg$$

Substitute for B and w_{tot} to obtain:

$$\rho_{air}Vg-2000\,\text{N}-\rho_{He}Vg=0$$

Solving for V yields:

$$V=\frac{2000\,\text{N}}{\left(\rho_{air}-\rho_{He}\right)g}$$

Substitute numerical values and evaluate V:

$$V=\frac{2000\,\text{N}}{\left(1.29\,\text{kg/m}^3-0.178\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)}=\boxed{183\,\text{m}^3}$$

49 •• **[SSM]** An object has ″neutral buoyancy″ when its density equals that of the liquid in which it is submerged, which means that it neither floats nor sinks. If the average density of an 85-kg diver is 0.96 kg/L, what mass of lead should the dive master suggest be added to give the diver neutral buoyancy?

Picture the Problem Let V = volume of diver, ρ_D the density of the diver, V_{Pb} the volume of added lead, and m_{Pb} the mass of lead. The diver is in equilibrium under the influence of his weight, the weight of the lead, and the buoyant force of the water.

Apply $\sum F_y = 0$ to the diver:

$$B - w_D - w_{Pb} = 0$$

Substitute to obtain:

$$\rho_w V_{D+Pb} g - \rho_D V_D g - m_{Pb} g = 0$$

or

$$\rho_w V_D + \rho_w V_{Pb} - \rho_D V_D - m_{Pb} = 0$$

Rewrite this expression in terms of masses and densities:

$$\rho_w \frac{m_D}{\rho_D} + \rho_w \frac{m_{Pb}}{\rho_{Pb}} - \rho_D \frac{m_D}{\rho_D} - m_{Pb} = 0$$

Solving for m_{Pb} yields:

$$m_{Pb} = \frac{\rho_{Pb}(\rho_w - \rho_D) m_D}{\rho_D(\rho_{Pb} - \rho_w)}$$

Substitute numerical values and evaluate m_{Pb}:

$$m_{Pb} = \frac{(11.3\times10^3\ \text{kg/m}^3)(1.00\times10^3\ \text{kg/m}^3 - 0.96\times10^3\ \text{kg/m}^3)(85\,\text{kg})}{(0.96\times10^3\ \text{kg/m}^3)(11.3\times10^3\ \text{kg/m}^3 - 1.00\times10^3\ \text{kg/m}^3)} = \boxed{3.9\,\text{kg}}$$

50 •• A 1.00-kg beaker containing 2.00 kg of water rests on a scale. A 2.00-kg block of aluminum (density 2.70×10^3 kg/m^3) suspended from a spring scale is submerged in the water, as in Figure 13-36. Find the readings of both scales.

Picture the Problem The hanging scale's reading w' is the difference between the weight of the aluminum block in air w and the buoyant force acting on it. The buoyant force is equal to the weight of the displaced fluid, which, in turn, is the product of its density and mass. We can apply a condition for equilibrium to relate the reading of the bottom scale to the weight of the beaker and its contents and the buoyant force acting on the block.

Express the apparent weight w' of the aluminum block. This apparent weight is the reading on the hanging scale and is equal to the tension in the string:

$$w' = w - B \quad (1)$$

Letting F_s be the reading of the bottom scale and choosing upward to be the $+y$ direction, apply $\sum F_y = 0$ to the system consisting of the beaker and the water to obtain:

$$F_s - (M_b + M_w)g - B = 0$$

or

$$F_s = (M_b + M_w)g + B = (M_b + M_w)g + w - w' \quad (2)$$

Using Archimedes' principle, substitute for B in equation (1) to obtain:

$$w' = w - \rho_w Vg \quad (3)$$

Express w in terms of ρ_{Al} and V and solve for Vg:

$$w = \rho_{Al} Vg \Rightarrow Vg = \frac{w}{\rho_{Al}}$$

Substitute for Vg in equation (3) and simplify to obtain:

$$w' = w - \rho_w \frac{w}{\rho_{Al}} = \left(1 - \frac{\rho_w}{\rho_{Al}}\right) w$$

Substitute numerical values and evaluate w':

$$w' = \left(1 - \frac{1.00\times10^3 \text{ kg/m}^3}{2.70\times10^3 \text{ kg/m}^3}\right)(2.00\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{12.4\,\text{N}}$$

Substitute numerical values in equation (2) and evaluate F_s:

$$F_s = (1.00\text{ kg} + 2.00\text{ kg})(9.81\text{ m/s}^2) + (2.00\text{ kg})(9.81\text{ m/s}^2) - 12.4\text{ N} = \boxed{36.7\text{ N}}$$

51 •• When cracks form at the base of a dam, the water seeping into the cracks exerts a buoyant force that tends to lift the dam. As a result, the dam can topple. Estimate the buoyant force exerted on a 2.0-m thick by 5.0-m wide dam wall by water seeping into cracks at its base. The water level in the lake is 5.0 m above the cracks.

Picture the Problem We can apply Archimedes' principle to estimate the buoyant force exerted by the water on the dam. We can determine whether the buoyant force is likely to lift the dam by comparing the buoyant and gravitational forces acting on the dam.

The buoyant force acting on the dam is given by Archimedes' principle:

$$B = w_{\substack{\text{displaced}\\\text{water}}} = m_{\substack{\text{displaced}\\\text{water}}} g = \rho_{\text{water}} V_{\substack{\text{displaced}\\\text{water}}} g$$

The volume of the displaced water is the same as the volume of the dam:

$$B = \rho_{\text{water}} V_{\text{dam}} g = \rho_{\text{water}} LWHg$$

where *L, W,* and *H* are the length and width of the dam, and *H* is the height of the water above the cracks.

Substitute numerical values and evaluate *B*:

$$B = \left(1.00\times10^3\ \text{kg/m}^3\right)(2.0\ \text{m})(5.0\ \text{m})(5.0\ \text{m})\left(9.81\,\text{m/s}^2\right) = \boxed{4.9\times10^5\ \text{N}}$$

52 •• Your team is in charge of launching a large helium weather balloon that is spherical in shape, and whose radius is 2.5 m and total mass is 15 kg (balloon plus helium plus equipment). (*a*) What is the initial upward acceleration of the balloon when it is released from sea level? (*b*) If the drag force on the balloon is given by $F_{\text{D}} = \frac{1}{2}\pi r^2 \rho v^2$, where *r* is the balloon radius, ρ is the density of air, and *v* the balloon's ascension speed, calculate the terminal speed of the ascending balloon.

Picture the Problem The forces acting on the balloon are the buoyant force *B*, its weight *mg*, and a drag force F_{D}. We can find the initial upward acceleration of the balloon by applying Newton's second law at the instant it is released. We can find the terminal speed of the balloon by recognizing that when $a_y = 0$, the net force acting on the balloon will be zero.

(*a*) Apply $\sum F_y = ma_y$ to the balloon at the instant of its release to obtain:

$$B - m_{\text{balloon}} g = m_{\text{balloon}} a_y \qquad (1)$$

Using Archimedes principle, express the buoyant force B acting on the balloon:

$$\begin{aligned} B &= w_{\substack{\text{displaced}\\ \text{fluid}}} = m_{\substack{\text{displaced}\\ \text{fluid}}} g \\ &= \rho_{\substack{\text{displaced}\\ \text{fluid}}} V_{\substack{\text{displaced}\\ \text{fluid}}} g \\ &= \rho_{\text{air}} V_{\text{balloon}} g = \tfrac{4}{3}\pi\rho_{\text{air}} r^3 g \end{aligned}$$

Substitute in equation (1) to obtain:

$$\tfrac{4}{3}\pi\rho_{\text{air}} r^3 g - m_{\text{balloon}} g = m_{\text{balloon}} a_y$$

Solving for a_y yields:

$$a_y = \left(\frac{\tfrac{4}{3}\pi\rho_{\text{air}} r^3}{m_{\text{balloon}}} - 1\right) g$$

Substitute numerical values and evaluate a_y:

$$a_y = \left[\frac{\tfrac{4}{3}\pi\left(1.29\,\text{kg/m}^3\right)\left(2.5\,\text{m}\right)^3}{15\,\text{kg}} - 1\right]\left(9.81\,\text{m/s}^2\right) = \boxed{45\,\text{m/s}^2}$$

(*b*) Apply $\sum F_y = ma_y$ to the balloon under terminal-speed conditions to obtain:

$$B - mg - \tfrac{1}{2}\pi r^2 \rho v_t^2 = 0$$

Substitute for B:

$$\tfrac{4}{3}\pi\rho_{\text{air}} r^3 g - mg - \tfrac{1}{2}\pi r^2 \rho v_t^2 = 0$$

Solving for v_t yields:

$$v_t = \sqrt{\frac{2\left(\tfrac{4}{3}\pi\rho_{\text{air}} r^3 - m\right) g}{\pi r^2 \rho}}$$

Substitute numerical values and evaluate v:

$$v_t = \sqrt{\frac{2\left[\tfrac{4}{3}\pi\left(1.29\,\text{kg/m}^3\right)\left(2.5\,\text{m}\right)^3 - 15\,\text{kg}\right]\left(9.81\,\text{m/s}^2\right)}{\pi\left(2.5\,\text{m}\right)^2\left(1.29\,\text{kg/m}^3\right)}} = 7.33\,\text{m/s} = \boxed{7.3\,\text{m/s}}$$

53 ••• [SSM] A ship sails from seawater (specific gravity 1.025) into freshwater, and therefore sinks slightly. When its 600,000-kg load is removed, it returns to its original level. Assuming that the sides of the ship are vertical at the water line, find the mass of the ship before it was unloaded.

Picture the Problem Let V = displacement of ship in the two cases, m be the mass of ship without load, and Δm be the load. The ship is in equilibrium under the influence of the buoyant force exerted by the water and its weight. We'll apply the condition for floating in the two cases and solve the equations simultaneously to determine the loaded mass of the ship.

Apply $\sum F_y = 0$ to the ship in fresh water:

$$\rho_{\text{w}} Vg - mg = 0 \quad (1)$$

Apply $\sum F_y = 0$ to the ship in salt water:

$$\rho_{\text{sw}} Vg - (m + \Delta m)g = 0 \quad (2)$$

Solve equation (1) for Vg:

$$Vg = \frac{mg}{\rho_{\text{w}}}$$

Substitute in equation (2) to obtain:

$$\rho_{\text{sw}} \frac{mg}{\rho_{\text{w}}} - (m + \Delta m)g = 0$$

Solving for m yields:

$$m = \frac{\rho_{\text{w}} \Delta m}{\rho_{\text{sw}} - \rho_{\text{w}}}$$

Add Δm to both sides of the equation and simplify to obtain:

$$\begin{aligned} m + \Delta m &= \frac{\rho_{\text{w}} \Delta m}{\rho_{\text{sw}} - \rho_{\text{w}}} + \Delta m \\ &= \Delta m \left(\frac{\rho_{\text{w}}}{\rho_{\text{sw}} - \rho_{\text{w}}} + 1 \right) \\ &= \frac{\Delta m \rho_{\text{sw}}}{\rho_{\text{sw}} - \rho_{\text{w}}} \end{aligned}$$

Substitute numerical values and evaluate $m + \Delta m$:

$$\begin{aligned} m + \Delta m &= \frac{(6.00 \times 10^5\ \text{kg})(1.025\rho_{\text{w}})}{1.025\rho_{\text{w}} - \rho_{\text{w}}} \\ &= \frac{(6.00 \times 10^5\ \text{kg})(1.025)}{1.025 - 1} \\ &= \boxed{2.5 \times 10^7\ \text{kg}} \end{aligned}$$

Continuity and Bernoulli's Equation

54 • Water flows at 0.65 m/s through a 3.0-cm-diameter hose that terminates in a 0.30-cm-diameter nozzle. Assume laminar nonviscous steady-state flow. (*a*) At what speed does the water pass through the nozzle? (*b*) If the pump at one end of the hose and the nozzle at the other end are at the same height, and if the pressure at the nozzle is 1.0 atm, what is the pressure at the pump outlet?

Picture the Problem Let A_1 represent the cross-sectional area of the hose, A_2 the cross-sectional area of the nozzle, v_1 the speed of the water in the hose, and v_2 the speed of the water as it passes through the nozzle. We can use the continuity

equation to find v_2 and Bernoulli's equation for constant elevation to find the pressure at the pump outlet.

(*a*) Using the continuity equation, relate the speeds of the water to the diameter of the hose and the diameter of the nozzle:

$$A_1v_1 = A_2v_2$$

or

$$\frac{\pi d_1^2}{4}v_1 = \frac{\pi d_2^2}{4}v_2 \Rightarrow v_2 = \frac{d_1^2}{d_2^2}v_1$$

Substitute numerical values and evaluate v_2:

$$v_2 = \left(\frac{3.0\,\text{cm}}{0.30\,\text{cm}}\right)^2(0.65\,\text{m/s}) = 65.0\,\text{m/s}$$

$$= \boxed{65\,\text{m/s}}$$

(*b*) Using Bernoulli's equation for constant elevation, relate the pressure at the pump P_P to the atmospheric pressure and the velocities of the water in the hose and the nozzle:

$$P_P + \tfrac{1}{2}\rho v_1^2 = P_{at} + \tfrac{1}{2}\rho v_2^2$$

Solve for the pressure at the pump:

$$P_P = P_{at} + \tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

Substitute numerical values and evaluate P_P:

$$P_P = 101.325\,\text{kPa} + \tfrac{1}{2}\left(1.00\times10^3\,\text{kg/m}^3\right)\left[(65.0\,\text{m/s})^2 - (0.65\,\text{m/s})^2\right]$$

$$= 2.21\times10^6\,\text{Pa}\times\frac{1\,\text{atm}}{101.325\,\text{kPa}} = \boxed{22\,\text{atm}}$$

55 • [SSM] Water is flowing at 3.00 m/s in a horizontal pipe under a pressure of 200 kPa. The pipe narrows to half its original diameter. (*a*) What is the speed of flow in the narrow section? (*b*) What is the pressure in the narrow section? (*c*) How do the volume flow rates in the two sections compare?

Picture the Problem Let A_1 represent the cross-sectional area of the larger-diameter pipe, A_2 the cross-sectional area of the smaller-diameter pipe, v_1 the speed of the water in the larger-diameter pipe, and v_2 the velocity of the water in the smaller-diameter pipe. We can use the continuity equation to find v_2 and Bernoulli's equation for constant elevation to find the pressure in the smaller-diameter pipe.

(*a*) Using the continuity equation, relate the velocities of the water to the diameters of the pipe:

$$A_1v_1 = A_2v_2$$

or

$$\frac{\pi d_1^2}{4}v_1 = \frac{\pi d_2^2}{4}v_2 \Rightarrow v_2 = \frac{d_1^2}{d_2^2}v_1$$

Substitute numerical values and evaluate v_2:

$$v_2 = \left(\frac{d_1}{\frac{1}{2}d_1}\right)^2(3.00\,\text{m/s}) = \boxed{12.0\,\text{m/s}}$$

(*b*) Using Bernoulli's equation for constant elevation, relate the pressures in the two segments of the pipe to the velocities of the water in these segments:

$$P_1 + \tfrac{1}{2}\rho_\text{w}v_1^2 = P_2 + \tfrac{1}{2}\rho_\text{w}v_2^2$$

Solving for P_2 yields:

$$\begin{aligned} P_2 &= P_1 + \tfrac{1}{2}\rho_\text{w}v_1^2 - \tfrac{1}{2}\rho_\text{w}v_2^2 \\ &= P_1 + \tfrac{1}{2}\rho_\text{w}\left(v_1^2 - v_2^2\right) \end{aligned}$$

Substitute numerical values and evaluate P_2:

$$P_2 = 200\,\text{kPa} + \tfrac{1}{2}\left(1.00\times10^3\ \text{kg/m}^3\right)\left[(3.00\,\text{m/s})^2 - (12.0\,\text{m/s})^2\right] = \boxed{133\,\text{kPa}}$$

(*c*) From the continuity equation; $\boxed{I_{\text{V}1} = I_{\text{V}2}}$.

56 •• The pressure in a section of horizontal pipe with a diameter of 2.00 cm is 142 kPa. Water flows through the pipe at 2.80 L/s. If the pressure at a certain point is to be reduced to 101 kPa by constricting a section of the pipe, what should the diameter of the constricted section be?

Picture the Problem Let A_1 represent the cross-sectional area of the 2.00-cm diameter pipe, A_2 the cross-sectional area of the constricted pipe, v_1 the speed of the water in the 2.00-cm diameter pipe, and v_2 the speed of the water in the constricted pipe. We can use the continuity equation to express d_2 in terms of d_1 and to find v_1 and Bernoulli's equation for constant elevation to find the speed of the water in the constricted pipe.

Using the continuity equation, relate the volume flow rate in the 2.00-cm diameter pipe to the volume flow rate in the constricted pipe:

$$A_1v_1 = A_2v_2$$

or

$$\frac{\pi d_1^2}{4}v_1 = \frac{\pi d_2^2}{4}v_2 \Rightarrow d_2 = d_1\sqrt{\frac{v_1}{v_2}} \quad (1)$$

Using the continuity equation, relate v_1 to the volume flow rate I_V:

$$v_1 = \frac{I_V}{A_1} = \frac{I_V}{\frac{1}{4}\pi d_1^2} = \frac{4I_V}{\pi d_1^2}$$

Using Bernoulli's equation for constant elevation, relate the pressures in the two segments of the pipe to the velocities of the water in these segments:

$$P_1 + \tfrac{1}{2}\rho_w v_1^2 = P_2 + \tfrac{1}{2}\rho_w v_2^2$$

Solve for v_2 to obtain:

$$v_2 = \sqrt{\frac{2(P_1 - P)}{\rho_w} + v_1^2}$$

Substituting for v_2 in equation (1) and simplifying yields:

$$d_2 = d_1\sqrt{\frac{v_1}{\sqrt{\frac{2(P_1-P)}{\rho_w} + v_1^2}}} = d_1\sqrt{\frac{1}{\sqrt{\frac{2(P_1-P)}{v_1^2\rho_w} + 1}}} = \frac{d_1}{\sqrt[4]{\frac{2(P_1-P)}{v_1^2\rho_w} + 1}}$$

Substitute for v_1 and simplify to obtain:

$$d_2 = \frac{d_1}{\sqrt[4]{\frac{2(P_1-P)}{\left(\frac{4I_V}{\pi d_1^2}\right)^2\rho_w} + 1}} = \frac{d_1}{\sqrt[4]{\frac{\pi^2 d_1^4(P_1-P)}{8I_V^2\rho_w} + 1}}$$

Substitute numerical values and evaluate d_2:

$$d_2 = \frac{2.00\,\text{cm}}{\sqrt[4]{\frac{\pi^2(0.0200\,\text{m})^4(142\,\text{kPa} - 101.325\,\text{kPa})}{8(2.80\,\text{L/s})^2(1.00\times10^3\,\text{kg/m}^3)} + 1}} = \boxed{1.68\,\text{cm}}$$

57 •• [SSM] Blood flows at at 30 cm/s in an aorta of radius 9.0 mm. (*a*) Calculate the volume flow rate in liters per minute. (*b*) Although the cross-sectional area of a capillary is much smaller than that of the aorta, there are many capillaries, so their total cross-sectional area is much larger. If all the blood from the aorta flows into the capillaries and the speed of flow through the capillaries is 1.0 mm/s, calculate the total cross-sectional area of the capillaries. Assume laminar nonviscous, steady-state flow.

Picture the Problem We can use the definition of the volume flow rate to find the volume flow rate of blood in an aorta and to find the total cross-sectional area of the capillaries.

(*a*) Use the definition of the volume flow rate to find the volume flow rate through an aorta:

$$I_V = Av$$

Substitute numerical values and evaluate I_V:

$$I_V = \pi\left(9.0\times10^{-3}\ \text{m}\right)^2 (0.30\ \text{m/s})$$
$$= 7.634\times10^{-5}\ \frac{\text{m}^3}{\text{s}}\times\frac{60\ \text{s}}{\text{min}}\times\frac{1\ \text{L}}{10^{-3}\ \text{m}^3}$$
$$= 4.58\ \text{L/min} = \boxed{4.6\ \text{L/min}}$$

(*b*) Use the definition of the volume flow rate to express the volume flow rate through the capillaries:

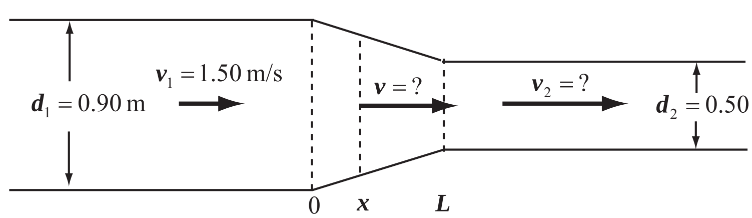
$$I_V = A_{\text{cap}}v_{\text{cap}} \Rightarrow A_{\text{cap}} = \frac{I_V}{v_{\text{cap}}}$$

Substitute numerical values and evaluate A_{cap}:

$$A_{\text{cap}} = \frac{7.63\times10^{-5}\ \text{m}^3/\text{s}}{0.0010\ \text{m/s}} = \boxed{7.6\times10^{-2}\ \text{m}^2}$$

58 •• Water flows through a 1.0 m-long conical section of pipe that joins a cylindrical pipe of radius 0.45 m, on the left, to a cylindrical pipe of radius 0.25 m, on the right. If the water is flowing into the 0.45 m pipe with a speed of 1.50 m/s, and if we assume laminar nonviscous steady-state flow, (*a*) what is the speed of flow in the 0.25 m pipe? (*b*) What is the speed of flow at a position *x* in the conical section, if *x* is the distance measured from the left-hand end of the conical section of pipe?

Picture the Problem The pictorial representation summarizes what we know about the two pipes and the connecting conical section. We can apply the continuity equation to find v_2. We can also use the continuity equation in (*b*) provided we first express the cross-sectional area of the transitional conical section as a function of *x*.



(*a*) Apply the continuity equation to the flow in the two sections of cylindrical pipes:

$$v_1A_1 = v_2A_2$$

Solving for v_2, expressing the cross-sectional areas in terms of their diameters, and simplifying yields:

$$v_2 = v_1 \frac{A_1}{A_2} = v_1 \frac{\frac{1}{4}\pi d_1^2}{\frac{1}{4}\pi d_2^2} = v_1 \left(\frac{d_1}{d_2}\right)^2$$

Substitute numerical values and evaluate v_2:

$$v_2 = (1.50\text{ m/s})\left(\frac{0.90\text{ m}}{0.50\text{ m}}\right)^2 = \boxed{4.9\text{ m/s}}$$

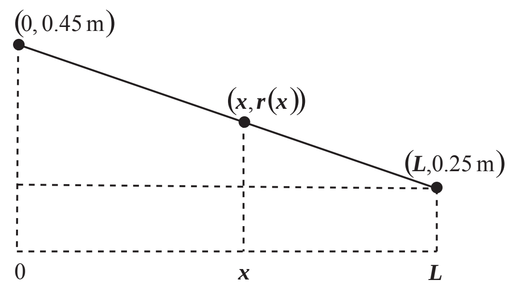
(*b*) Continuity of flow requires that:

$$v(0)A(0) = v(x)A(x)$$

Solve for $v(x)$ to obtain:

$$v(x) = v(0)\frac{A(0)}{A(x)} \qquad (1)$$

The diagram to the right is an enlargement of the upper half of the transitional conical section. Use the coordinates shown on the line to establish the following proportion:



$$\frac{r(x) - 0.45\text{ m}}{x - 0} = \frac{0.25\text{ m} - 0.45\text{ m}}{L - 0}$$

Solving the proportion for $r(x)$ yields:

$$r(x) = 0.45\text{ m} - (0.20\text{ m})\frac{x}{L}$$

where L is the length of the conical section.

The cross-sectional area of the conical section varies with x according to:

$$A(x) = \pi (r(x))^2 = \pi\left(0.45\text{ m} - (0.20\text{ m})\frac{x}{L}\right)^2$$

Substituting for $A(x)$ in equation (1) and simplifying yields:

$$v(x) = v(0)\frac{\pi r_1^2}{\pi\left(0.45\text{ m} - (0.20\text{ m})\frac{x}{L}\right)^2} = \frac{v(0) r_1^2}{\left(0.45\text{ m} - (0.20\text{ m})\frac{x}{L}\right)^2}$$

Substitute numerical values and simplify further to obtain:

$$v(x)=\frac{(1.50\ \text{m/s})(0.45\ \text{m})^2}{\left(0.45\ \text{m}-(0.20\ \text{m})\dfrac{x}{1.0\ \text{m}}\right)^2}=\boxed{\frac{0.304\ \text{m}^3/\text{s}}{(0.45\ \text{m}-0.20x)^2}}$$

To confirm the correctness of this equation, evaluate $v(0)$ and $v(1.0\ \text{m})$:

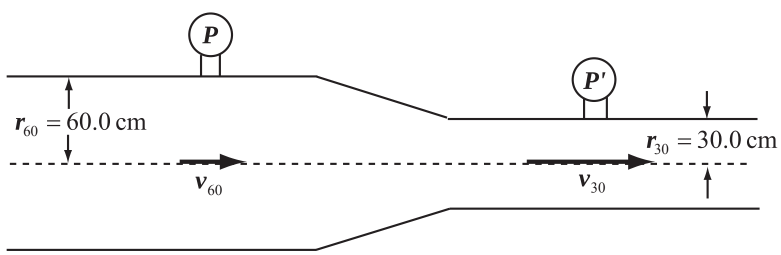
$$v(0)=\frac{0.304\ \text{m}^3/\text{s}}{(0.45\ \text{m})^2}=1.5\ \text{m/s}$$

and

$$v(1.0\ \text{m})=\frac{0.304\ \text{m}^3/\text{s}}{(0.45\ \text{m}-0.20(1.0\ \text{m}))^2}=4.9\ \text{m/s}$$

59 •• The \$8-billion, 800-mile-long Alaskan Pipeline has a maximum volume flow rate of 240,000 m^3 of oil per day. Most of the pipeline has a radius of60.0 cm. Find the pressure P' at a point where the pipe has a 30.0-cm radius. Take the pressure in the 60.0-cm-radius sections to be P = 180 kPa and the density of oil to be 800 kg/m^3. Assume laminar nonviscous steady-state flow.

Picture the Problem Let the subscript 60 denote the 60.0-cm-radius pipe and the subscript 30 denote the 30.0-cm-radius pipe. We can use Bernoulli's equation for constant elevation to express P' in terms of v_{60} and v_{30}, the definition of volume flow rate to find v_{60}, and the continuity equation to find v_{30}.



Using Bernoulli's equation for constant elevation, relate the pressures in the two pipes to the velocities of the oil:

$$P+\tfrac{1}{2}\rho v_{60}^2=P'+\tfrac{1}{2}\rho v_{30}^2$$

Solving for P' yields:

$$P'=P+\tfrac{1}{2}\rho\left(v_{60}^2-v_{30}^2\right) \qquad (1)$$

Use the definition of volume flow rate to express v_{60}:

$$v_{60}=\frac{I_V}{A_{60}}=\frac{I_V}{\pi r_{60}^2}$$

Using the continuity equation, relate the speed of the oil in the half-standard pipe to its speed in the standard pipe:

$$A_{60}v_{60} = A_{30}v_{30} \Rightarrow v_{30} = \frac{A_{60}}{A_{30}}v_{60}$$

Substituting for v_{60} and A_{30} yields:

$$v_{30} = \frac{A_{60}}{A_{30}}\frac{I_V}{A_{60}} = \frac{I_V}{A_{30}} = \frac{I_V}{\pi r_{30}^2}$$

Substitute for v_{60} and v_{30} and simplify to obtain:

$$P' = P + \tfrac{1}{2}\rho\left(\left(\frac{I_V}{\pi r_{60}^2}\right)^2 - \left(\frac{I_V}{\pi r_{30}^2}\right)^2\right) = P + \frac{\rho I_V^2}{2\pi^2}\left(\frac{1}{r_{60}^4} - \frac{1}{r_{30}^4}\right)$$

Substitute numerical values in equation (1) and evaluate P':

$$P' = 180\,\text{kPa} + \frac{\left(800\,\text{kg/m}^3\right)\left(2.40\times10^5\,\frac{\text{m}^3}{\text{d}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right)^2}{2\pi^2}$$
$$\times\left(\frac{1}{(0.600\,\text{m})^4} - \frac{1}{(0.300\,\text{m})^4}\right)$$
$$= \boxed{144\,\text{kPa}}$$

60 •• Water flows through a Venturi meter like that in Example 13-11 with a pipe diameter of 9.50 cm and a constriction diameter of 5.60 cm. The U-tube manometer is partially filled with mercury. Find the volume flow rate of the water if the difference in the mercury level in the U-tube is 2.40 cm.

Picture the Problem We'll use its definition to relate the volume flow rate in the pipe to the speed of the water and the result of Example 13-11 to find the speed of the water.

Using its definition, express the volume flow rate:

$$I_V = A_1 v_1 = \pi r^2 v_1$$

Using the result of Example 13-11, find the speed of the water upstream from the Venturi meter:

$$v_1 = \sqrt{\frac{2\rho_{Hg}gh}{\rho_w\left(\frac{R_1^2}{R_2^2} - 1\right)}}$$

Substituting for v_1 yields:

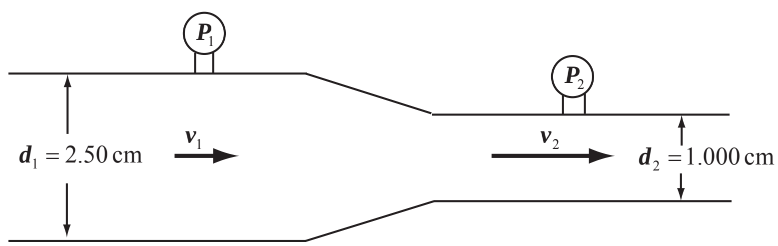
$$I_V = \pi r^2 \sqrt{\frac{2\rho_{Hg} g h}{\rho_w \left(\frac{R_1^2}{R_2^2} - 1\right)}}$$

Substitute numerical values and evaluate I_V:

$$I_V = \frac{\pi}{4}(0.0950\,\text{m})^2 \sqrt{\frac{2(13.6\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)(0.0240\,\text{m})}{(1.00\times10^3\ \text{kg/m}^3)\left[\left(\frac{0.0950\,\text{m}}{0.0560\,\text{m}}\right)^2 - 1\right]}} = \boxed{13.1\,\text{L/s}}$$

61 •• [SSM] Horizontal flexible tubing for carrying cooling water extends through a large electromagnet used in your physics experiment at Fermi National Accelerator Laboratory. A minimum volume flow rate of 0.050 L/s through the tubing is necessary in order to keep your magnet cool. Within the magnet volume, the tubing has a circular cross section of radius 0.500 cm. In regions outside the magnet, the tubing widens to a radius of 1.25 cm. You have attached pressure sensors to measure differences in pressure between the 0.500 cm and 1.25 cm sections. The lab technicians tell you that if the flow rate in the system drops below 0.050 L/s, the magnet is in danger of overheating and that you should install an alarm to sound a warning when the flow rate drops below that level. What is the critical pressure difference at which you should program the sensors to send the alarm signal (and is this a minimum, or maximum, pressure difference)? Assume laminar nonviscous steady-state flow.

Picture the Problem The pictorial representation shows the narrowing of the cold-water supply tubes as they enter the magnet. We can apply Bernoulli's equation and the continuity equation to derive an expression for the pressure difference $P_1 - P_2$.



Apply Bernoulli's equation to the two sections of tubing to obtain:

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2$$

Solving for the pressure difference $P_1 - P_2$ yields:

$$\Delta P = P_1 - P_2 = \tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 = \tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

Factor v_1^2 from the parentheses to obtain:

$$\Delta P = \tfrac{1}{2}\rho v_1^2\left(\frac{v_2^2}{v_1^2}-1\right) = \tfrac{1}{2}\rho v_1^2\left(\left(\frac{v_2}{v_1}\right)^2-1\right)$$

From the continuity equation we have:

$$A_1 v_1 = A_2 v_2$$

Solving for the ratio of v_2 to v_1, expressing the areas in terms of the diameters, and simplifying yields:

$$\frac{v_2}{v_1} = \frac{A_1}{A_2} = \frac{\frac{1}{4}\pi d_1^2}{\frac{1}{4}\pi d_2^2} = \left(\frac{d_1}{d_2}\right)^2$$

Use the expression for the volume flow rate to express v_1:

$$v_1 = \frac{I_V}{A_1} = \frac{I_V}{\frac{1}{4}\pi d_1^2} = \frac{4I_V}{\pi d_1^2}$$

Substituting for v_1 and v_2/v_1 in the expression for ΔP yields:

$$\Delta P = \tfrac{1}{2}\rho\left(\frac{4I_V}{\pi d_1^2}\right)^2\left(\left(\frac{d_1}{d_2}\right)^4-1\right)$$

Substitute numerical values and evaluate ΔP:

$$\Delta P = \tfrac{1}{2}\left(1.00\times10^3\ \text{kg/m}^3\right)\left(\frac{4\left(0.050\ \frac{\text{L}}{\text{s}}\times\frac{10^{-3}\ \text{m}^3}{\text{L}}\right)}{\pi(0.0250\ \text{m})^2}\right)^2\left(\left(\frac{2.50\ \text{cm}}{1.000\ \text{cm}}\right)^4-1\right) = \boxed{0.20\ \text{kPa}}$$

Because $\Delta P \propto I_V^2$ (ΔP as a function of I_V is a parabola that opens upward), this pressure difference is the minimum pressure difference.

62 •• Figure 13-37 shows a Pitot-static tube, a device used for measuring the speed of a gas. The inner pipe faces the incoming fluid, while the ring of holes in the outer tube is parallel to the gas flow. Show that the speed of the gas is given by $v = \sqrt{\frac{2gh(\rho_L - \rho_g)}{\rho_g}}$, where ρ_L is the density of the liquid used in the manometer and ρ_g is the density of the gas.

Picture the Problem Let the numeral 1 denote the opening in the end of the inner pipe and the numeral 2 to one of the holes in the outer tube. We can apply Bernoulli's principle at these locations and solve for the pressure difference between them. By equating this pressure difference to the pressure difference due to the height h of the liquid column we can express the speed of the gas as a function of ρ_L, ρ_g, g, and h.

Apply Bernoulli's principle at locations 1 and 2 to obtain:

$P_1 + \frac{1}{2}\rho_g v_1^2 = P_2 + \frac{1}{2}\rho_g v_2^2$
where we've ignored the difference in elevation between the two openings.

Solve for the pressure difference $\Delta P = P_1 - P_2$:

$\Delta P = P_1 - P_2 = \frac{1}{2}\rho_g v_2^2 - \frac{1}{2}\rho_g v_1^2$

Express the speed of the gas at 1:

Because the gas is brought to a halt (that is, is stagnant) at the opening to the inner pipe, $v_1 = 0$.

Express the speed of the gas at 2:

Because the gas flows freely past the holes in the outer ring, $v_2 = v$.

Substitute to obtain:

$\Delta P = \frac{1}{2}\rho_g v^2$

Letting A be the cross-sectional area of the tube, express the pressure at a depth h in the column of liquid whose density is ρ_l:

$$P_1 = P_2 + \frac{w_{\text{displaced liquid}}}{A} - \frac{B}{A}$$
where $B = \rho_g Ahg$ is the buoyant force acting on the column of liquid of height h.

Substitute to obtain:

$$P_1 = P_2 + \frac{\rho_L ghA}{A} - \frac{\rho_g ghA}{A}$$
$$= P_2 + (\rho_L - \rho_g)gh$$
or
$$\Delta P = P_1 - P_2 = (\rho_L - \rho_g)gh$$

Equate these two expressions for ΔP:

$\frac{1}{2}\rho_g v^2 = (\rho_L - \rho_g)gh$

Solving for v yields:

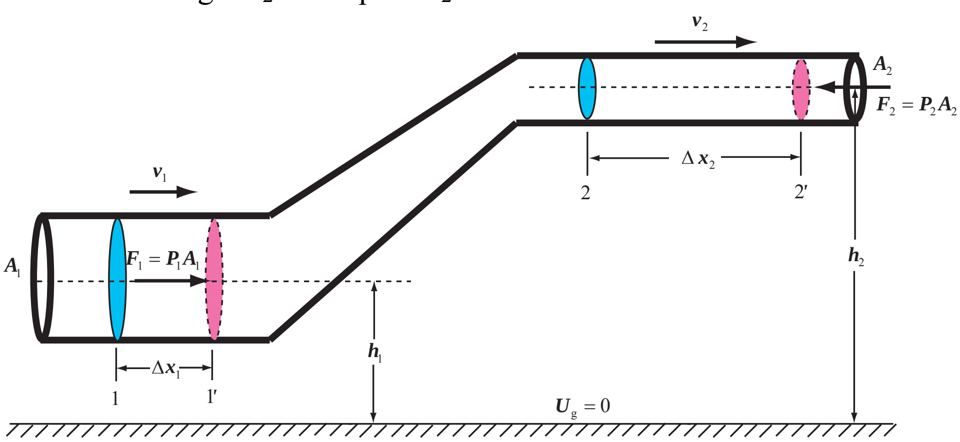
$$v = \boxed{\sqrt{\frac{2gh(\rho_L - \rho_g)}{\rho_g}}}$$

Note that the correction for buoyant force due to the displaced gas is very small and that, to a good approximation, $v = \sqrt{\frac{2gh\rho_L}{\rho_g}}$.

Remarks: Pitot tubes are used to measure the airspeed of airplanes.

63 ••• **[SSM]** Derive the Bernoulli Equation in more generality than done in the text, that is, allow for the fluid to change elevation during its movement. Using the work-energy theorem, show that when changes in elevation are allowed, Equation 13-16 becomes $P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$ (Equation 13-17).

Picture the Problem Consider a fluid flowing in a tube that varies in elevation as well as in cross-sectional area, as shown in the pictorial representation below. We can apply the work-energy theorem to a parcel of fluid that initially is contained between points 1 and 2. During time Δt this parcel moves along the tube to the region between point 1′ and 2′. Let ΔV be the volume of fluid passing point 1′ during time Δt. The same volume passes point 2 during the same time. Also, let $\Delta m = \rho\Delta V$ be the mass of the fluid with volume ΔV. The net effect on the parcel during time Δt is that mass Δm initially at height h_1 moving with speed v_1 is "transferred" to height h_2 with speed v_2.



Express the work-energy theorem:

$$W_{\text{total}} = \Delta U + \Delta K \qquad (1)$$

The change in the potential energy of the parcel is given by:

$$\begin{aligned}\Delta U &= (\Delta m)gh_2 - (\Delta m)gh_1 \\ &= (\Delta m)g(h_2 - h) \\ &= \rho\Delta V g(h_2 - h_1)\end{aligned}$$

The change in the kinetic energy of the parcel is given by:

$$\begin{aligned}\Delta K &= \tfrac{1}{2}(\Delta m)v_2^2 - \tfrac{1}{2}(\Delta m)v_1^2 \\ &= \tfrac{1}{2}(\Delta m)(v_2^2 - v_1^2) \\ &= \tfrac{1}{2}\rho\Delta V(v_2^2 - v_1^2)\end{aligned}$$

Express the work done by the fluid behind the parcel (to the parcel's left in the diagram) as it pushes on the parcel with a force of magnitude $F_1 = P_1A_1$, where P_1 is the pressure at point 1:

$$W_1 = F_1\Delta x_1 = P_1A_1\Delta x_1 = P_1\Delta V$$

Express the work done by the fluid in front of the parcel (to the parcel's right in the diagram) as it pushes on the parcel with a force of magnitude $F_2 = P_2 A_2$, where P_2 is the pressure at point 2:

$$W_2 = -F_2\Delta x_2 = -P_2 A_2 \Delta x_2 = -P_2 \Delta V$$

where the work is negative because the applied force and the displacement are in opposite directions.

The total work done on the parcel is:

$$W_{\text{total}} = P_1\Delta V - P_2 \Delta V = (P_1 - P_2)\Delta V$$

Substitute for ΔU, ΔK, and W_{total} in equation (1) to obtain:

$$(P_1 - P_2)\Delta V = \rho \Delta V g(h_2 - h_1) + \tfrac{1}{2}\rho \Delta V\left(v_2^2 - v_1^2\right)$$

Simplifying this expression by dividing out ΔV yields:

$$P_1 - P_2 = \rho g(h_2 - h_1) + \tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

Collect all the quantities having a subscript 1 on one side and those having a subscript 2 on the other to obtain:

$$\boxed{P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2}$$

Remarks: This equation is known as Bernoulli's equation for the steady, nonviscous flow of an incompressible fluid.

64 ••• A large keg of height H and cross-sectional area A_1 is filled with root beer. The top is open to the atmosphere. There is a spigot opening of area A_2, which is much smaller than A_1, at the bottom of the keg. (*a*) Show that when the height of the root beer is h, the speed of the root beer leaving the spigot is approximately $\sqrt{2gh}$. (*b*) Show that if $A_2 << A_1$, the rate of change of the height h of the root beer is given by $dh/dt = -(A_2/A_1)(2gh)^{1/2}$. (*c*) Find h as a function of time if $h = H$ at $t = 0$. (*d*) Find the total time needed to drain the keg if $H = 2.00$ m, $A_1 = 0.800$ m^2, and $A_2 = 1.00 \times 10^{-4}\,A_1$. Assume laminar nonviscous flow.

Picture the Problem We can apply Bernoulli's equation to the top of the keg and to the spigot opening to determine the rate at which the root beer exits the tank. Because the area of the spigot is much smaller than that of the keg, we can neglect the speed of the root beer at the top of the keg. We'll use the continuity equation to obtain an expression for the rate of change of the height of the root beer in the keg as a function of the height and integrate this function to find h as a function of time.

(a) Apply Bernoulli's equation to the beer at the top of the keg and at the spigot:

$$P_1+\rho_{\text{beer}}gh_1+\tfrac{1}{2}\rho_{\text{beer}}v_1^2=P_2+\rho_{\text{beer}}gh_2+\tfrac{1}{2}\rho_{\text{beer}}v_2^2$$

or, because $v_1 \approx 0$, $h_2 = 0$, $P_1 = P_2 = P_{\text{at}}$, and $h_1 = h$,

$$gh=\tfrac{1}{2}v_2^2 \Rightarrow v_2=\boxed{\sqrt{2gh}}$$

(b) Use the continuity equation to relate v_1 and v_2:

$$A_1v_1=A_2v_2$$

Substitute $-dh/dt$ for v_1 and $\sqrt{2gh}$ for v_2 to obtain:

$$-A_1\frac{dh}{dt}=A_2\sqrt{2gh}$$

Solving for dh/dt yields:

$$\frac{dh}{dt}=\boxed{-\frac{A_2}{A_1}\sqrt{2gh}}$$

(c) Separate the variables in the differential equation to obtain:

$$-\frac{A_1/A_2}{\sqrt{2g}}\frac{dh}{\sqrt{h}}=dt$$

Express the integral from H to h and 0 to t:

$$-\frac{A_1/A_2}{\sqrt{2g}}\int_H^h\frac{dh}{\sqrt{h}}=\int_0^t dt$$

Evaluate the integral to obtain:

$$-\frac{2A_1/A_2}{\sqrt{2g}}\left(\sqrt{H}-\sqrt{h}\right)=t$$

Solving for h gives:

$$h=\boxed{\left(\sqrt{H}-\frac{A_2}{2A_1}\sqrt{2g}\,t\right)^2}$$

(d) Solve $h(t)$ for the time-to-drain t':

$$t'=\frac{A_1}{A_2}\sqrt{\frac{2H}{g}}$$

Substitute numerical values and evaluate t'

$$t'=\frac{A_1}{1.00\times10^{-4}A_1}\sqrt{\frac{2(2.00\,\text{m})}{9.81\,\text{m/s}^2}}=6.39\times10^3\,\text{s}=\boxed{1\,\text{h}\ 46\,\text{min}}$$

65 •• A siphon is a device for transferring a liquid from container to container. The tube shown in Figure 13-38 must be filled to start the siphon, but once this has been done, fluid will flow through the tube until the liquid surfaces in the containers are at the same level. (a) Using Bernoulli's equation, show that

the speed of water in the tube is $v = \sqrt{2gd}$. (*b*) What is the pressure at the highest part of the tube?

Picture the Problem Let the letter ″a″ denote the entrance to the siphon tube and the letter ″b″ denote its exit and assume that ″b″ in at the surface of the liquid in the container to the right. Assuming streamline flow between these points, we can apply Bernoulli's equation to relate the entrance and exit speeds of the water flowing in the siphon to the pressures at either end, the density of the water, and the difference in elevation between the entrance and exit points. We'll also use the equation of continuity to argue that, provided the surface area of the beaker is large compared to the area of the opening of the tube, the entrance speed of the water is approximately zero.

(*a*) Assuming that the height of the containers is *H,* apply Bernoulli's equation at the surface of the left container and at point b:

$$P_{\text{at}} + \tfrac{1}{2}\rho v_{\text{surface}}^2 + \rho g(H-h) = P_{\text{at}} + \tfrac{1}{2}\rho v_{\text{b}}^2 + \rho g(H-h-d) \qquad (1)$$

Apply the continuity equation to a point at the surface of the liquid in the container to the left and to point b:

$$v_{\text{b}}A_{\text{b}} = v_{\text{surface}}A_{\text{surface}} \Rightarrow v_{\text{surface}} = v_{\text{b}}\frac{A_{\text{b}}}{A_{\text{surface}}}$$

or, because $A_{\text{surface}} >> A_{\text{b}}$,

$v_{\text{surface}} = 0$

With $v_{\text{a}} = 0$, equation (1) becomes:

$$\rho g(H-h) = \tfrac{1}{2}\rho v_{\text{b}}^2 + \rho g(H-h-d)$$

Solving for v_{b} gives:

$$v_{\text{b}} = \boxed{\sqrt{2gd}}$$

(*b*) Relate the pressure at the highest part of the tube P_{top} to the pressure at point b:

$$P_{\text{top}} + \rho g(H-h) + \tfrac{1}{2}\rho v_{\text{top}}^2 = P_{\text{at}} + \rho g(H-h-d) + \tfrac{1}{2}\rho v_{\text{b}}^2$$

The equation of continuity requires that $v_{\text{top}} = v_{\text{b}} = v_{\text{a}}$:

$$P_{\text{top}} + \rho g(H-h) + \tfrac{1}{2}\rho v_{\text{top}}^2 = P_{\text{at}} + \rho g(H-h-d) + \tfrac{1}{2}\rho v_{\text{top}}^2$$

or, simplifying,

$$P_{\text{top}} + \rho g(H-h) = P_{\text{at}} + \rho g(H-h-d)$$

Solve for P_{top} to obtain:

$$P_{\text{top}} = \boxed{P_{\text{at}} - \rho g d}$$

Remarks: If we let P_{top} = 0, we can use this result to find the maximum theoretical height a siphon can lift water.

66 •• A fountain designed to spray a column of water 12 m into the air has a 1.0-cm-diameter nozzle at ground level. The water pump is 3.0 m below the ground. The pipe to the nozzle has a diameter of 2.0 cm. Find pump pressure necessary if the fountain is to operate as designed. (Assume laminar nonviscous steady-state flow.)

Picture the Problem Let the letter P denote the pump and the 2.0-cm diameter pipe and the letter N the 1-cm diameter nozzle. We'll use Bernoulli's equation to express the necessary pump pressure, the continuity equation to relate the speed of the water coming out of the pump to its speed at the nozzle, and a constant-acceleration equation to relate its speed at the nozzle to the height to which the water rises.

Using Bernoulli's equation, relate the pressures, areas, and velocities in the pipe and nozzle:

$$P_P + \rho_w g h_P + \tfrac{1}{2}\rho_w v_P^2 = P_N + \rho_w g h_N + \tfrac{1}{2}\rho_w v_N^2$$

or, because $P_N = P_{at}$ and $h_P = 0$,

$$P_P + \tfrac{1}{2}\rho_w v_P^2 = P_N + \rho_w g h_N + \tfrac{1}{2}\rho_w v_N^2$$

Solve for the pump pressure:

$$P_P = P_{at} + \rho_w g h_N + \tfrac{1}{2}\rho_w\left(v_N^2 - v_P^2\right) \quad (1)$$

Use the continuity equation to relate v_P and v_N to the cross-sectional areas of the pipe from the pump and the nozzle:

$$A_P v_P = A_N v_N$$

and

$$v_P = \frac{A_N}{A_P} v_N = \frac{\frac{1}{4}\pi d_N^2}{\frac{1}{4}\pi d_P^2} v_N = \left(\frac{1.0\,\text{cm}}{2.0\,\text{cm}}\right)^2 v_N = \tfrac{1}{4} v_N$$

Use a constant-acceleration equation to express the speed of the water at the nozzle in terms of the desired height Δh:

$$v^2 = v_N^2 - 2g\Delta h$$

or, because $v = 0$,

$$v_N^2 = 2g\Delta h$$

Substitute for v_N and v_P in equation (1) and simplify to obtain:

$$P_P = P_{at} + \rho_w g h_N + \tfrac{1}{2}\rho_w\left[2g\Delta h - \tfrac{1}{16}(2g\Delta h)\right] = P_{at} + \rho_w g h_N + \tfrac{1}{2}\rho_w\left(\tfrac{15}{8} g\Delta h\right)$$
$$= P_{at} + \rho_w g\left(h_N + \tfrac{15}{16}\Delta h\right)$$

Substitute numerical values and evaluate P_P:

$$P_P = 101.325\,\text{kPa} + \left(1.00\times10^3\ \text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)\left[3.0\,\text{m} + \tfrac{15}{16}(12\,\text{m})\right] = \boxed{2.4\times10^5\,\text{Pa}}$$

67 •• Water at 20°C exits a circular tap moving straight down with a flow rate of 10.5 cm^3/s. (*a*) If the diameter of the tap is 1.20 cm, what is the speed of the water? (*b*) As the fluid falls from the tap, the stream of water narrows. Find the new diameter of the stream at a point 7.50 cm below the tap. Assume that the stream still has a circular cross section and neglect any effects of drag forces acting on the water. (*c*) If turbulent flows are characterized by Reynolds numbers above 2300 or so, how far does the water have to fall before it becomes turbulent? Does this match your everyday observations?

Picture the Problem Let I represent the flow rate of the water. Then we can use $I = Av$ to relate the flow rate to the cross-sectional area of the circular tap and the speed of the water. In (*b*) we can use the equation of continuity to express the diameter of the stream 7.50 cm below the tap and a constant-acceleration equation to find the speed of the water at this distance. In (*c*) we can use a constant-acceleration equation to express the distance-to-turbulence in terms of the speed of the water at turbulence v_t and the definition of Reynolds number N_R to relate v_t to N_R.

(*a*) Express the flow rate of the water in terms of the cross-sectional area A of the circular tap and the speed v_i of the water:

$$I = Av_i = \pi r^2 v_i = \tfrac{1}{4}\pi d^2 v_i \qquad (1)$$

Solving for v_i yields:

$$v_i = \frac{I}{\frac{1}{4}\pi d^2}$$

Substitute numerical values and evaluate v_i:

$$v_i = \frac{10.5\,\text{cm}^3/\text{s}}{\frac{1}{4}\pi(1.20\,\text{cm})^2} = \boxed{9.28\,\text{cm/s}}$$

(*b*) Apply the equation of continuity to the stream of water:

$$v_f A_f = v_i A_i$$

or

$$v_f \frac{\pi}{4} d_f^2 = v_i \frac{\pi}{4} d_i^2 \Rightarrow d_f = \sqrt{\frac{v_i}{v_f}}\, d_i \qquad (2)$$

Use a constant-acceleration equation to relate v_f and v_i to the distance Δh fallen by the water:

$$v_f^2 = v_i^2 + 2g\Delta h \Rightarrow v_f = \sqrt{v_i^2 + 2g\Delta h}$$

Substitute numerical values and evaluate v_f:

$$v_f = \sqrt{(9.28\,\text{cm/s})^2 + 2(981\,\text{cm/s}^2)(7.50\,\text{cm})} = 122\,\text{cm/s}$$

Substitute numerical values in equation (2) and evaluate d_f:

$$d_f = (1.20\,\text{cm})\sqrt{\frac{9.28\,\text{cm/s}}{122\,\text{cm/s}}} = \boxed{0.331\,\text{cm}}$$

(*c*) Use a constant-acceleration equation to relate the fall-distance-to-turbulence Δd to its initial speed v_i and its speed v_t when its flow becomes turbulent:

$$v_t^2 = v_i^2 + 2g\Delta d \Rightarrow \Delta d = \frac{v_f^2 - v_i^2}{2g} \quad (3)$$

Express Reynolds number N_R for turbulent flow:

$$N_R = \frac{2r\rho v_t}{\eta} \quad (4)$$

The volume flow rate equals Av. Express the volume flow rate at the speed v_t:

$$I = \pi r^2 v_t \quad (5)$$

Eliminate r from equations (4) and (5) and solve for v_t (see Table 13-1 for the coefficient of viscosity for water):

$$v_t = \frac{\pi N_R^2 \eta^2}{4\rho^2 I}$$

Substitute numerical values (see Figure 13-1 for the density of water and Table 13-3 for the coefficient of viscosity for water) and evaluate v_t:

$$v_t = \frac{\pi(2300)^2\left(1.00\times10^{-3}\ \text{Pa}\cdot\text{s}\right)^2}{4\left(1.00\times10^3\ \text{kg/m}^3\right)^2\left(10.5\,\text{cm}^3/\text{s}\right)} = 0.396\,\text{m/s}$$

Substitute numerical values in equation (3) and evaluate the fall-distance-to turbulence:

$$\Delta d = \frac{(39.6\,\text{cm/s})^2 - (9.28\,\text{cm/s})^2}{2\left(9.81\,\text{m/s}^2\right)} \approx \boxed{0.76\,\text{cm}}$$

This result seems to be too small.

68 •• To better fight fires in your seaside community, the local fire brigade has asked you to set up a pump system to draw seawater from the ocean to the top of a steep cliff adjacent to the water where most of the homes are. If the cliff is 12.0 m high, and the pump is capable of producing a gauge pressure of 150 kPa, how much water (in L/s) can be pumped using a hose with a radius of 4.00 cm?

Picture the Problem The water being drawn from the ocean is at rest initially and is pumped upward to a height of 12.0 m. Let the variables subscripted 1 correspond to the intake and those subscripted 2 to the outflow and use the volume flow rate equation and Bernoulli's equation.

Express the volume flow rate in the hose:

$$I_V = Av = \pi r^2 v_2 \quad (1)$$

Apply Bernoulli's equation to the intake and outflow to obtain:

$$P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2$$

Rewriting this equation to isolate the speed terms yields:

$$\tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right) = P_1 - P_2 + \rho g h_1 - \rho g h_2$$

Because $v_1 = 0$ and $h_1 = 0$ and $P_1 - P_2 = \Delta P$:

$$\tfrac{1}{2}\rho v_2^2 = \Delta P - \rho g h_2$$

Solving for v_2 yields:

$$v_2 = \sqrt{\frac{2\left(\Delta P - \rho g h_2\right)}{\rho}}$$

Substitute for v_2 in equation (1) to obtain:

$$I_{\rm V} = \pi r^2 \sqrt{\frac{2\left(\Delta P - \rho g h_2\right)}{\rho}}$$

Substitute numerical values and evaluate $I_{\rm V}$:

$$I_{\rm V} = \pi\left(0.0400\ \text{m}\right)^2 \sqrt{\frac{2\left(150\ \text{kPa} - \left(1.00\times10^3\ \text{kg/m}^3\right)\left(9.81\ \text{m/s}^2\right)\left(12.0\ \text{m}\right)\right)}{1.00\times10^3\ \text{kg/m}^3}}$$

$$= 4.039\times10^{-2}\ \frac{\text{m}^3}{\text{s}}\times\frac{1\ \text{L}}{10^{-3}\ \text{m}^3} = \boxed{40.4\ \text{L/s}}$$

69 ••• In Figure 13-39, H is the depth of the liquid and h is the distance from the surface of the liquid to the pipe inserted in the tank's side. (*a*) Find the distance x at which the water strikes the ground as a function of h and H. (*b*) Show that, for a given value of H, there are two values of h (whose average value is $\tfrac{1}{2}H$) that give the same distance x. (*c*) Show that, for a given value of H, x is a maximum when $h = \tfrac{1}{2}H$. Find the maximum value for x as a function of H.

Picture the Problem We can apply Bernoulli's equation to points a and b to determine the rate at which the water exits the tank. Because the diameter of the small pipe is much smaller than the diameter of the tank, we can neglect the speed of the water at the point a. The distance the water travels once it exits the pipe is the product of its speed and the time required to fall the distance $H - h$. That there are two values of h that are equidistant from the point $h = \tfrac{1}{2}H$ can be shown by solving the quadratic equation that relates x to h and H. That x is a maximum for this value of h can be established by treating $x = f(h)$ as an extreme-value problem.

(*a*) Express the distance x as a function of the exit speed of the water and the time to fall the distance $H - h$:

$$x = v_b \Delta t \qquad (1)$$

Apply Bernoulli's equation to the water at points a and b:

$$P_a + \rho_{\mathrm{w}} gH + \tfrac{1}{2}\rho_{\mathrm{w}} v_a^2 = P_{\mathrm{b}} + \rho_{\mathrm{w}} g(H-h) + \tfrac{1}{2}\rho_{\mathrm{w}} v_b^2$$

or, because $v_a \approx 0$ and $P_a = P_b = P_{\mathrm{at}}$,

$$gH = g(H-h) + \tfrac{1}{2}v_b^2 \Rightarrow v_b = \sqrt{2gh}$$

Using a constant-acceleration equation, relate the time of fall to the distance of fall:

$$\Delta y = v_{0y}\Delta t + \tfrac{1}{2}a(\Delta t)^2$$

or, because $v_{0y} = 0$,

$$H - h = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2(H-h)}{g}}$$

Substitute for Δt in equation (1) to obtain:

$$x = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}} = \boxed{2\sqrt{h(H-h)}}$$

(b) Square both sides of this equation and simplify to obtain:

$$x^2 = 4hH - 4h^2 \text{ or } 4h^2 - 4Hh + x^2 = 0$$

Solving this quadratic equation yields:

$$h = \boxed{\tfrac{1}{2}H \pm \tfrac{1}{2}\sqrt{H^2 - x^2}}$$

Find the average of these two values for h:

$$h_{\mathrm{av}} = \frac{\tfrac{1}{2}H + \sqrt{H^2 - x^2} + \tfrac{1}{2}H - \sqrt{H^2 - x^2}}{2} = \boxed{\tfrac{1}{2}H}$$

(c) Differentiate $x = 2\sqrt{h(H-h)}$ with respect to h:

$$\frac{dx}{dh} = 2\left(\tfrac{1}{2}\right)[h(H-h)]^{-\frac{1}{2}}(H-2h) = \frac{H-2h}{\sqrt{h(H-h)}}$$

Set dx/dh equal to zero for extrema:

$$\frac{H-2h}{\sqrt{h(H-h)}} = 0$$

Solve for h to obtain:

$$h = \boxed{\tfrac{1}{2}H}$$

Evaluate $x = 2\sqrt{h(H-h)}$ with $h = \tfrac{1}{2}H$:

$$x_{\max} = 2\sqrt{\tfrac{1}{2}H\left(H - \tfrac{1}{2}H\right)} = \boxed{H}$$

Remarks: To show that this value for *h* corresponds to a *maximum*, one can either show that $\frac{d^2x}{dh^2} < 0$ at $h = \frac{1}{2}H$ or confirm that the graph of *f*(*h*) at $h = \frac{1}{2}H$ is concave downward.

*Viscous Flow

70 • Water flows through a horizontal 25.0-cm-long tube with an inside diameter of 1.20 mm at 0.300 mL/s. Find the pressure difference required to drive this flow if the viscosity of water is 1.00 mPa·s. Assume laminar flow.

Picture the Problem The required pressure difference can be found by applying Poiseuille's law to the viscous flow of water through the horizontal tube.

Using Poiseuille's law, relate the pressure difference between the two ends of the tube to its length, radius, and the volume flow rate of the water:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_\text{V}$$

Substitute numerical values and evaluate ΔP:

$$\Delta P = \frac{8(1.00\,\text{mPa}\cdot\text{s})(0.250\,\text{m})}{\pi\left(\dfrac{1.20\times10^{-3}\,\text{m}}{2}\right)^4}(0.300\,\text{mL/s}) = \boxed{1.47\,\text{kPa}}$$

71 • Find the diameter of a tube that would give double the flow rate for the pressure difference in Problem 70.

Picture the Problem Because the pressure difference is unchanged, we can equate the expressions of Poiseuille's law for the two tubes and solve for the diameter of the tube that would double the flow rate.

Using Poiseuille's law, express the pressure difference required for the radius and volume flow rate of Problem 70:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_\text{V}$$

Express the pressure difference required for the radius r' that would double the volume flow rate of Problem 70:

$$\Delta P = \frac{8\eta L}{\pi r'^4}(2I_\text{V})$$

Equate these equations and simplify to obtain:

$$\frac{8\eta L}{\pi r'^4}(2I_\text{V})=\frac{8\eta L}{\pi r^4}I_\text{V}$$

or

$$\frac{2}{r'^4}=\frac{1}{r^4}\Rightarrow r'=\sqrt[4]{2}r$$

Because d' is two times r':

$$d'=2r'=2\sqrt[4]{2}r=\sqrt[4]{2}d$$

Substitute numerical values and evaluate d':

$$d'=\sqrt[4]{2}(1.20\,\text{mm})=\boxed{1.43\,\text{mm}}$$

72 • Blood takes about 1.00 s to pass through a 1.00-mm-long capillary in the human circulatory system. If the diameter of the capillary is 7.00 μm and the pressure drop is 2.60 kPa, find the viscosity of blood. Assume laminar flow.

Picture the Problem We can apply Poiseuille's law to relate the pressure drop across the capillary tube to the radius and length of the tube, the rate at which blood is flowing through it, and the viscosity of blood.

Using Poiseuille's law, relate the pressure drop to the length and diameter of the capillary tube, the volume flow rate of the blood, and the viscosity of the blood:

$$\Delta P=\frac{8\eta L}{\pi r^4}I_\text{V}\Rightarrow\eta=\frac{\pi r^4\Delta P}{8LI_\text{V}}$$

Using its definition, express the volume flow rate of the blood:

$$I_\text{V}=A_\text{cap}v=\pi r^2v$$

Substitute for I_V and simplify to obtain:

$$\eta=\frac{r^2\Delta P}{8Lv}$$

Substitute numerical values to obtain:

$$\eta=\frac{(3.50\times10^{-6}\,\text{m})^2(2.60\,\text{kPa})}{8(1.00\times10^{-3}\,\text{m})\left(\dfrac{1.00\times10^{-3}\,\text{m}}{1.00\,\text{s}}\right)}$$

$$=\boxed{3.98\,\text{mPa}\cdot\text{s}}$$

73 • **[SSM]** An abrupt transition occurs at Reynolds numbers of about 3×10^5, where the drag on a sphere moving through a fluid abruptly decreases. Estimate the speed at which this transition occurs for a baseball, and comment on whether it should play a role in the physics of the game.

Picture the Problem We can use the definition of Reynolds number to find the speed of a baseball at which the drag crisis occurs.

Using its definition, relate Reynolds number to the speed v of the baseball:

$$N_R = \frac{2r\rho v}{\eta} \Rightarrow v = \frac{\eta N_R}{2r\rho}$$

Substitute numerical values (see Figure 13-1 for the density of air and Table 13-3 for the coefficient of viscosity for air) and evaluate v:

$$v = \frac{(0.018\,\text{mPa}\cdot\text{s})(3\times10^5)}{2(0.05\,\text{m})(1.293\,\text{kg/m}^3)} = 41.8\,\text{m/s}\times\frac{1\,\text{mi/h}}{0.447\,\text{m/s}} \approx \boxed{90\,\text{mi/h}}$$

Because most major league pitchers can throw a fastball in the low-to-mid-90s, this drag crisis may very well play a role in the game.

Remarks: This is a topic that has been fiercely debated by people who study the physics of baseball.

74 •• A horizontal pipe of radius 1.5 cm and length 25 m is connected to the output that can sustain an output gauge pressure of 10 kPa. What is the speed of 20°C water flowing through the pipe? If the temperature of the water is 60°C, what is the speed of the water in the pipe?

Picture the Problem Assuming laminar flow of the water, we can apply Poiseuille's law to find the speed of the water coming out the outflow end of the pipe.

Use Poiseuille's law to relate the pressure difference to the volume flow rate in the pipe:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_V$$

Substituting for I_V yields:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_V = \frac{8\eta L}{\pi r^4} Av$$

Substitute for the cross-sectional area A of the pipe and simplify to obtain:

$$\Delta P = \frac{8\eta L}{\pi r^4}\pi r^2 v = \frac{8\eta L}{r^2} v$$

Solving for v yields:

$$v = \frac{r^2 \Delta P}{8\eta L}$$

Substitute numerical values and evaluate v(20°C):

$$v(20°\text{C}) = \frac{(0.015\,\text{m})^2(10\,\text{kPa})}{8(1.00\,\text{mPa}\cdot\text{s})(25\,\text{m})} = \boxed{11\,\text{m/s}}$$

Substitute numerical values and evaluate $v(60°\text{C})$:

$$v(60°\text{C}) = \frac{(0.015\ \text{m})^2(10\ \text{kPa})}{8(0.65\ \text{mPa}\cdot\text{s})(25\ \text{m})} = \boxed{17\ \text{m/s}}$$

75 •• A very large tank is filled to a depth of 250 cm with oil that has a density of 860 kg/m^3 and a viscosity of 180 mPa·s. If the container walls are 5.00 cm thick, and a cylindrical hole of radius 0.750 cm is bored through the base of the container, what is the initial volume flow rate (in L/s) of the oil through the hole?

Picture the Problem Assuming laminar flow, we can apply Poiseuille's law to relate the pressure difference between the inside and the outside of the container at its base to the volume flow rate of oil out of the hole. We can find the pressure difference from the expression for the pressure as a function of depth in a fluid.

Use Poiseuille's law to relate the pressure difference to the volume flow rate of oil out of the hole:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_\text{V} \Rightarrow I_\text{V} = \frac{\pi r^4 \Delta P}{8\eta L} \quad (1)$$

The pressure at the bottom of the container of oil is given by:

$$P = P_0 + \rho_\text{oil} g h$$

The pressure difference between the inside and outside of the container is:

$$\Delta P = P - P_0 = \rho_\text{oil} g h$$

Substituting for ΔP in equation (1) yields:

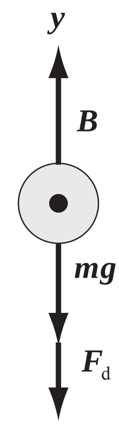
$$I_\text{V} = \frac{\pi r^4 \rho_\text{oil} g h}{8\eta L}$$

Substitute numerical values and evaluate I_V:

$$I_\text{V} = \frac{\pi(0.0075\ \text{m})^4(860\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2.50\ \text{m})}{8(180\ \text{mPa}\cdot\text{s})(0.0500\ \text{m})} = 2.91\times10^{-3}\ \text{m}^3/\text{s} = \boxed{2.91\ \text{L/s}}$$

76 ••• The drag force on a moving sphere at very low Reynolds number is given by $F_\text{D} = 6\pi\eta a v$, where η is the viscosity of the surrounding fluid and a is the radius of the sphere. (This relation is called Stokes' law.) Using this information, find the terminal speed of ascent for a spherical 1.0-mm diameter carbon dioxide bubble of 1.0-mm diameter rising in a carbonated beverage (ρ = 1.1 kg/L and η = 1.8 mPa·s). How long should it take for this bubble to rise 20 cm (the height of a drinking glass)? Is this length of time consistent with your observations?

Picture the Problem Let the subscripts ″f″ refer to ″displaced fluid", ″s″ to ″soda″, and ″g″ to the ″gas″ in the bubble. The free-body diagram shows the forces acting on the bubble prior to reaching its terminal speed. We can apply Newton's second law, Stokes′ law, and Archimedes principle to express the terminal speed of the bubble in terms of its radius, and the viscosity and density of water.



Apply $\sum F_y = ma_y$ to the bubble to obtain:

$$B - m_g g - F_D = ma_y$$

Under terminal speed conditions:

$$B - m_g g - F_D = 0$$

Using Archimedes principle, express the buoyant force B acting on the bubble:

$$B = w_f = m_f g = \rho_f V_f g = \rho_s V_{bubble} g$$

Express the mass of the gas bubble:

$$m_g = \rho_g V_g = \rho_g V_{bubble}$$

Substitute to obtain:

$$\rho_s V_{bubble} g - \rho_g V_{bubble} g - 6\pi\eta a v_t = 0$$

Solving for v_t gives:

$$v_t = \frac{V_{bubble} g(\rho_s - \rho_g)}{6\pi\eta a}$$

Substitute for V_{bubble} and simplify:

$$v_t = \frac{\frac{4}{3}\pi a^3 g(\rho_s - \rho_g)}{6\pi\eta a} = \frac{2a^2 g(\rho_s - \rho_g)}{9\eta} \approx \frac{2a^2 g\rho_s}{9\eta}, \text{since } \rho_s >> \rho_g.$$

Substitute numerical values and evaluate v_t:

$$v_t = \frac{2(0.50\times10^{-3}\text{ m})^2(9.81\text{m/s}^2)(1.1\times10^3\text{ kg/m}^3)}{9(1.8\times10^{-3}\text{ Pa}\cdot\text{s})} = 0.333\text{m/s} = \boxed{0.33\text{m/s}}$$

Express the rise time Δt in terms of the height of the soda glass h and the terminal speed of the bubble:

$$\Delta t = \frac{h}{v_t}$$

Assuming that a "typical" soda glass has a height of about 15 cm, evaluate Δt:

$$\Delta t = \frac{0.15\,\text{m}}{0.333\,\text{m/s}} = \boxed{0.45\,\text{s}}$$

Remarks: About half a second seems reasonable for the rise time of the bubble.

General Problems

77 • **[SSM]** Several teenagers swim toward a rectangular, wooden raft that is 3.00 m wide and 2.00 m long. If the raft is 9.00 cm thick, how many 75.0-kg teenage boys can stand on top of the raft without the raft becoming submerged? Assume the wood density is 650 kg/m^3.

Picture the Problem If the raft is to be just barely submerged, then the buoyant force on it will be equal in magnitude to the weight of the raft plus the weight of the boys. We can apply Archimedes' principle to find the buoyant force on the raft.

The buoyant force acting on the raft is the sum of the weights of the raft and the boys:

$$B = w_{\text{raft}} + w_{\text{boys}} \qquad (1)$$

Express the buoyant force acting on the raft:

$$B = w_{\text{displaced water}} = m_{\text{displaced water}}\,g = \rho_{\text{water}} V_{\text{displaced water}}\,g = \rho_{\text{water}} V_{\text{raft}}\,g$$

Express the weight of the raft:

$$w_{\text{raft}} = m_{\text{raft}}\,g = \rho_{\text{raft}} V_{\text{raft}}\,g$$

Express the weight of the boys:

$$w_{\text{boys}} = m_{\text{boys}}\,g = N m_{1\,\text{boy}}\,g$$

Substituting for B, w_{raft}, and w_{boys} in equation (1) yields:

$$\rho_{\text{water}} V_{\text{raft}}\,g = \rho_{\text{raft}} V_{\text{raft}}\,g + N m_{1\,\text{boy}}\,g$$

Solve for N and simplify to obtain:

$$N = \frac{\rho_{\text{water}} V_{\text{raft}}\,g - \rho_{\text{raft}} V_{\text{raft}}\,g}{m_{1\,\text{boy}}\,g} = \frac{(\rho_{\text{water}} - \rho_{\text{raft}})V_{\text{raft}}}{m_{1\,\text{boy}}}$$

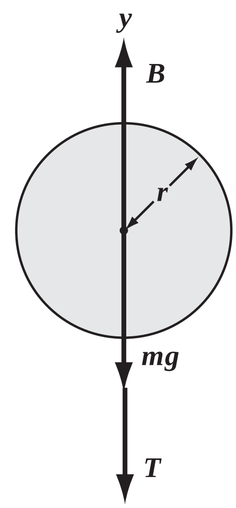
Substitute numerical values and evaluate N:

$$N = \frac{\left(1.00\times10^{3}\ \text{kg/m}^3 - 650\,\text{kg/m}^3\right)(3.00\,\text{m})(2.00\,\text{m})(0.0900\,\text{m})}{75.0\,\text{kg}} = 2.52$$

Hence, a maximum of $\boxed{2}$ boys can be on the raft under these circumstances.

78 • A thread attaches a 2.7-g Ping-Pong ball to the bottom of a beaker. When the beaker is filled with water so that the ball is totally submerged, the tension in the thread is 7.0 mN. Determine the diameter of the ball.

Picture the Problem The forces acting on the ball, shown in the free-body diagram, are the buoyant force, the weight of the ball, and the tension in the string. Because the ball is in equilibrium under the influence of these forces, we can apply the condition for translational equilibrium to establish the relationship between them. We can also apply Archimedes' principle to relate the buoyant force on the ball to its diameter.



Apply $\sum F_y = 0$ to the ball:

$$B - mg - T = 0$$

Using Archimedes' principle, relate the buoyant force on the ball to its diameter:

$$B = w_{\substack{\text{displaced}\\ \text{water}}} = m_{\substack{\text{displaced}\\ \text{water}}} g = \rho_{\text{water}} V_{\text{ball}} g = \tfrac{4}{3}\pi\rho_{\text{water}} g r^3$$

Substituting for B yields:

$$\tfrac{4}{3}\pi\rho_{\text{water}} g r^3 - mg - T = 0$$

Solve for r:

$$r = \sqrt[3]{\frac{3(T+mg)}{4\pi\rho_{\text{water}} g}} \Rightarrow d = 2\left(\sqrt[3]{\frac{3(T+mg)}{4\pi\rho_{\text{water}} g}}\right)$$

Substitute numerical values and evaluate d:

$$d = 2\left(\sqrt[3]{\frac{3\left[7.0\ \text{mN} + (0.0027\,\text{kg})(9.81\,\text{m/s}^2)\right]}{4\pi(1.00\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)}}\right) = \boxed{1.9\,\text{cm}}$$

79 • Seawater has a bulk modulus of 2.30×10^9 N/m^2. Find the difference in density of seawater at a depth where the pressure is 800 atm as compared to the density at the surface which is 1025 kg/m^3. Neglect any effects due to either temperature or salinity.

Picture the Problem Let ρ_0 represent the density of seawater at the surface. We can use the definition of density and the fact that mass is constant to relate the fractional change in the density of water to its fractional change in volume. We can also use the definition of bulk modulus to relate the fractional change in density to the increase in pressure with depth and solve the resulting equation for the change in density at the depth at which the pressure is 800 atm.

Using the definition of density, relate the mass of a given volume of seawater to its volume:

$$m = \rho V$$

Noting that the mass does not vary with depth, evaluate its differential:

$$\rho dV + V d\rho = 0$$

Solve for $d\rho/\rho$:

$$\frac{d\rho}{\rho} = -\frac{dV}{V} \text{ or } \frac{\Delta\rho}{\rho} \approx -\frac{\Delta V}{V}$$

Using the definition of the bulk modulus, relate ΔP to $\Delta\rho/\rho_0$:

$$B = -\frac{\Delta P}{\Delta V/V} = \frac{\Delta P}{\Delta\rho/\rho_0}$$

Solving for $\Delta\rho$ yields:

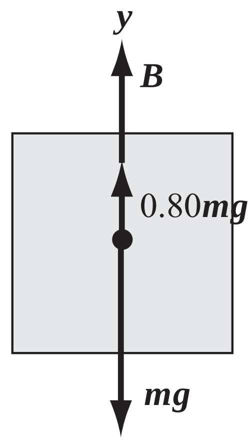
$$\Delta\rho = \rho - \rho_0 = \frac{\rho_0 \Delta P}{B}$$

Substitute numerical values and evaluate $\Delta\rho$:

$$\Delta\rho = \frac{\left(1025\,\text{kg/m}^3\right)\left(799\,\text{atm} \times \dfrac{101.325\,\text{kPa}}{1\,\text{atm}}\right)}{2.30\times10^9\ \text{N/m}^2} = \boxed{36.1\ \text{kg/m}^3}$$

80 • A solid cube with 0.60-m edge length is suspended from a spring balance. When the cube is submerged in water, the spring balance reads 80 percent of the reading when the cube is in air. Determine the density of the cube.

Picture the Problem When it is submerged, the block is in equilibrium under the influence of the buoyant force due to the water, the force exerted by the spring balance, and its weight. We can use the condition for translational equilibrium to relate the buoyant force to the weight of the block and the definition of density to express the weight of the block in terms of its density.



Apply $\sum F_y = 0$ to the block:

$$B + 0.80mg - mg = 0 \Rightarrow B = 0.20mg$$

Substitute for B and m to obtain:

$$\rho_{\text{w}} V_{\text{block}} g = 0.20 \rho_{\text{block}} V_{\text{block}} g$$

Solve for and evaluate ρ_{block}:

$$\rho_{\text{block}} = \frac{\rho_{\text{w}}}{0.20} = 5.0\left(1.00\times10^3\ \text{kg/m}^3\right) = \boxed{5.0\times10^3\ \text{kg/m}^3}$$

81 •• [SSM] A 1.5-kg block of wood floats on water with 68 percent of its volume submerged. A lead block is placed on the wood, fully submerging the wood to a depth where the lead remains entirely out of the water. Find the mass of the lead block.

Picture the Problem Let m and V represent the mass and volume of the block of wood. Because the block is in equilibrium when it is floating, we can apply the condition for translational equilibrium and Archimedes' principle to express the dependence of the volume of water it displaces when it is fully submerged on its weight. We'll repeat this process for the situation in which the lead block is resting on the wood block with the latter fully submerged. Let the upward direction be the $+y$ direction.

Apply $\sum F_y = 0$ to floating block:

$$B - mg = 0 \qquad (1)$$

Use Archimedes' principle to relate the density of water to the volume of the block of wood:

$$B = w_{\substack{\text{displaced}\\ \text{water}}} = m_{\substack{\text{displaced}\\ \text{water}}} g = \rho_{\text{water}} V_{\substack{\text{displaced}\\ \text{water}}} g = \rho_{\text{water}}\left(0.68V\right)g$$

Using the definition of density, express the weight of the block in terms of its density:

$$mg = \rho_{\text{wood}} V g$$

Substitute for B and mg in equation (1) to obtain:

$$\rho_{\text{water}}(0.68V)g - \rho_{\text{wood}}Vg = 0$$

Solving for ρ_{wood} yields:

$$\rho_{\text{wood}} = 0.68\rho_{\text{water}}$$

Use the definition of density to express the volume of the wood:

$$V = \frac{m}{\rho_{\text{wood}}}$$

Apply $\sum F_y = 0$ to the floating block when the lead block is placed on it:

$B' - m'g = 0$, where B' is the new buoyant force on the block and m' is the combined mass of the wood block and the lead block.

Use Archimedes' principle and the definition of density to obtain:

$$\rho_{\text{water}}Vg - (m_{\text{Pb}} + m)g = 0$$

Solve for the mass of the lead block to obtain:

$$m_{\text{Pb}} = \rho_{\text{water}}V - m$$

Substituting for V and ρ_{water} yields:

$$m_{\text{Pb}} = \frac{\rho_{\text{wood}}}{0.68}\frac{m}{\rho_{\text{wood}}} - m = \left(\frac{1}{0.68} - 1\right)m$$

Substitute numerical values and evaluate m_{Pb}:

$$m_{\text{Pb}} = \left(\frac{1}{0.68} - 1\right)(1.5\text{ kg}) = \boxed{0.71\text{ kg}}$$

82 •• A Styrofoam cube, 25 cm on an edge, is placed on one pan of a balance. The balance is in equilibrium when a 20-g mass of brass is placed on the other pan. Find the mass of the Styrofoam cube. Neglect the buoyant force of the air on the brass mass, but do not neglect the buoyant force of the air on the Styrofoam cube.

Picture the Problem The true mass of the Styrofoam cube is greater than that indicated by the balance due to the buoyant force acting on it. The balance is in rotational equilibrium under the influence of the buoyant and gravitational forces acting on the Styrofoam cube and the brass masses. Let m and V represent the mass and volume of the cube and L the lever arm of the balance.

Apply $\sum \vec{\tau} = 0$ to the balance:

$$(mg - B)L - m_{\text{brass}}gL = 0 \qquad (1)$$

Use Archimedes' principle to express the buoyant force on the Styrofoam cube as a function of volume and density of the air it displaces:

$$B = w_{\substack{\text{displaced}\\ \text{fluid}}} = m_{\substack{\text{displaced}\\ \text{fluid}}}\, g = \rho_{\substack{\text{displaced}\\ \text{fluid}}} V_{\substack{\text{displaced}\\ \text{fluid}}}\, g = \rho_{\text{air}} V g$$

Substitute for B in equation (1) and simplify to obtain:

$$m - \rho_{\text{air}} V - m_{\text{brass}} = 0$$

Solving for m yields:

$$m = \rho_{\text{air}} V + m_{\text{brass}}$$

Substitute numerical values and evaluate m:

$$m = \left(1.293\,\text{kg/m}^3\right)\left(0.25\,\text{m}\right)^3 + 20\times10^{-3}\,\text{kg} = \boxed{40\,\text{g}}$$

83 •• A spherical shell of copper with an outer diameter of 12.0 cm floats on water with half its volume above the water's surface. Determine the inner diameter of the shell. The cavity inside the spherical shell is empty.

Picture the Problem Let d_{in} and d_{out} represent the inner and outer diameters of the copper shell and V' the volume of the spherical shell that is submerged. Because the spherical shell is floating, it is in equilibrium and we can apply a condition for translational equilibrium to relate the buoyant force B due to the displaced water and its weight w.

Apply $\sum F_y = 0$ to the spherical shell:

$$B - w = 0$$

Using Archimedes' principle and the definition of w, substitute to obtain:

$$\rho_{\text{w}} V' g - mg = 0 \Rightarrow \rho_{\text{w}} V' - m = 0 \quad (1)$$

Express V' as a function d_{out}:

$$V' = \frac{1}{2}\frac{\pi}{6} d_{\text{out}}^3 = \frac{\pi}{12} d_{\text{out}}^3$$

Express m in terms of d_{in} and d_{out}:

$$m = \rho_{\text{Cu}}\left(V_{\text{out}} - V_{\text{in}}\right) = \rho_{\text{Cu}}\left(\frac{\pi}{6} d_{\text{out}}^3 - \frac{\pi}{6} d_{\text{in}}^3\right)$$

Substitute in equation (1) to obtain:

$$\rho_{\text{w}} \frac{\pi}{12} d_{\text{out}}^3 - \rho_{\text{Cu}}\left(\frac{\pi}{6} d_{\text{out}}^3 - \frac{\pi}{6} d_{\text{in}}^3\right) = 0$$

Simplifying yields:

$$\frac{1}{2}\rho_{\rm w}d_{\rm out}^3-\rho_{\rm Cu}\left(d_{\rm out}^3-d_{\rm in}^3\right)=0$$

Solve for $d_{\rm in}$ to obtain:

$$d_{\rm in}=d_{\rm out}\sqrt[3]{1-\frac{\rho_{\rm w}}{2\rho_{\rm Cu}}}$$

Substitute numerical values and evaluate $d_{\rm in}$:

$$d_{\rm in}=(12.0\,{\rm cm})\sqrt[3]{1-\frac{1}{2(8.93)}}=\boxed{11.8\,{\rm cm}}$$

84 •• A 200-mL beaker that is half-filled with water is on the left pan of a balance, and a sufficient amount of sand is placed on the right pan to bring the balance to equilibrium. A cube 4.0 cm on an edge that is attached to a string is then lowered into the water so that it is completely submerged, but not touching the bottom of the beaker. A piece of brass of mass m is then added to the right pan to restore equilibrium. What is m?

Determine the Concept Taking the water in the beaker and the beaker as the system, the additional weight on the right pan is needed to balance the reaction force to the buoyant force the water in the beaker exerts on the cube.

The buoyant force exerted on the cube by the water is given by:

$$B=w_{\rm displaced\ water}=m_{\rm water}g=\rho_{\rm water}V_{\rm cube}g$$

The weight of the piece of brass is the product of its mass and the gravitational field:

$$w=mg$$

Equating these forces yields:

$$\rho_{\rm water}V_{\rm cube}g=mg\Rightarrow m=\rho_{\rm water}V_{\rm cube}$$

Substitute numerical values and evaluate m:

$$m=\left(1.00\ {\rm g/cm^3}\right)(4\ {\rm cm})^3=\boxed{64\ {\rm g}}$$

85 •• **[SSM]** Crude oil has a viscosity of about 0.800 Pa·s at normal temperature. You are the chief design engineer in charge of constructing a 50.0-km horizontal pipeline that connects an oil field to a tanker terminal. The pipeline is to deliver oil at the terminal at a rate of 500 L/s and the flow through the pipeline is to be laminar. Assuming that the density of crude oil is 700 kg/m^3, estimate the diameter of the pipeline that should be used.

Picture the Problem We can use the definition of Reynolds number and assume a value for N_R of 1000 (well within the laminar-flow range) to obtain a trial value for the radius of the pipe. We'll then use Poiseuille's law to determine the pressure difference between the ends of the pipe that would be required to maintain a volume flow rate of 500 L/s.

Use the definition of Reynolds number to relate N_R to the radius of the pipe:

$$N_R = \frac{2r\rho v}{\eta}$$

Use the definition of I_V to relate the volume flow rate of the pipe to its radius:

$$I_V = Av = \pi r^2 v \Rightarrow v = \frac{I_V}{\pi r^2}$$

Substitute to obtain:

$$N_R = \frac{2\rho I_V}{\eta\pi r} \Rightarrow r = \frac{2\rho I_V}{\eta\pi N_R}$$

Substitute numerical values and evaluate r:

$$r = \frac{2(700\,\text{kg/m}^3)(0.500\,\text{m}^3/\text{s})}{\pi(0.800\,\text{Pa}\cdot\text{s})(1000)} = 27.9\,\text{cm}$$

Using Poiseuille's law, relate the pressure difference between the ends of the pipe to its radius:

$$\Delta P = \frac{8\eta L}{\pi r^4} I_V$$

Substitute numerical values and evaluate ΔP:

$$\Delta P = \frac{8(0.800\,\text{Pa}\cdot\text{s})(50\,\text{km})}{\pi(0.279\,\text{m})^4}(0.500\text{m}^3/\text{s})$$
$$= 8.41\times10^6\ \text{Pa} = 83.0\,\text{atm}$$

This pressure is too large to maintain in the pipe. Evaluate ΔP for a pipe of 50 cm radius:

$$\Delta P = \frac{8(0.800\,\text{Pa}\cdot\text{s})(50\,\text{km})}{\pi(0.50\,\text{m})^4}(0.500\,\text{m}^3/\text{s})$$
$$= 8.15\times10^5\ \text{Pa} = 8.04\,\text{atm}$$

1 m is a reasonable diameter for the pipeline. This larger diameter makes the Reynolds number still smaller, so the flow is still laminar.

86 •• Water flows through the pipe in Figure 13-40 and exits to the atmosphere at the right end of section C. The diameter of the pipe is 2.00 cm at A, 1.00 cm at B, and 0.800 cm at C. The gauge pressure in the pipe at the center of section A is 1.22 atm and the flow rate is 0.800 L/s. The vertical pipes are open to the air. Find the level (above the flow mid-line as shown) of the liquid–air interfaces in the two vertical pipes. Assume laminar nonviscous flow.

Picture the Problem We'll measure the height of the liquid–air interfaces relative to the centerline of the pipe. We can use the definition of the volume flow rate in a pipe to find the speed of the water at point A and the relationship between the gauge pressures at points A and C to determine the level of the liquid-air interface at A. We can use the continuity equation to express the speed of the water at B in terms of its speed at A and Bernoulli's equation for constant elevation to find the gauge pressure at B. Finally, we can use the relationship between the gauge pressures at points A and B to find the level of the liquid-air interface at B.

Relate the gauge pressure in the pipe at A to the height of the liquid-air interface at A:

$$P_{\text{gauge,A}} = \rho g h_{\text{A}} \Rightarrow h_{\text{A}} = \frac{P_{\text{gauge,A}}}{\rho g}$$

where h_{A} is measured from the center of the pipe.

Substitute numerical values and evaluate h_{A}:

$$h_{\text{A}} = \frac{(1.22\,\text{atm})(1.01\times10^5\ \text{Pa/atm})}{(1.00\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)} = \boxed{12.6\,\text{m}}$$

Determine the speed of the water at A:

$$v_{\text{A}} = \frac{I_{\text{V}}}{A_{\text{A}}} = \frac{0.800\times10^{-3}\ \text{m}^3/\text{s}}{\frac{\pi}{4}(0.0200\,\text{m})^2} = 2.55\,\text{m/s}$$

Apply Bernoulli's equation for constant elevation to relate P_{B} and P_{A}:

$$P_{\text{A}} + \tfrac{1}{2}\rho v_{\text{A}}^2 = P_{\text{B}} + \tfrac{1}{2}\rho v_{\text{B}}^2 \qquad (1)$$

Use the continuity equation to relate v_{B} and v_{A}:

$$A_{\text{A}} v_{\text{A}} = A_{\text{B}} v_{\text{B}} \Rightarrow v_{\text{B}} = \frac{A_{\text{A}}}{A_{\text{B}}} v_{\text{A}}$$

Substitute numerical values and evaluate v_{B}:

$$v_{\text{B}} = \frac{(2.00\,\text{cm})^2}{(1.00\,\text{cm})^2} v_{\text{A}} = 4 v_{\text{A}}$$

Substitute in equation (1) to obtain:

$$P_{\text{A}} + \tfrac{1}{2}\rho v_{\text{A}}^2 = P_{\text{B}} + 8\rho v_{\text{A}}^2$$

Solving for P_{B} yields:

$$P_{\text{B}} = P_{\text{A}} - \tfrac{15}{2}\rho v_{\text{A}}^2 = P_{\text{gauge, A}} + P_{\text{at}} - \tfrac{15}{2}\rho v_{\text{A}}^2$$

Substitute numerical values and evaluate P_B:

$$P_B = (1.22\,\text{atm} + 1.00\,\text{atm})(101.325\,\text{kPa/atm}) - \tfrac{15}{2}(1.00\times10^3\ \text{kg/m}^3)(2.55\,\text{m/s})^2$$
$$= 1.762\times10^5\ \text{Pa}\times\frac{1\,\text{atm}}{101.325\,\text{kPa}} = 1.739\,\text{atm}$$

Relate the gauge pressure in the pipe at B to the height of the liquid-air interface at B:

$$P_{\text{gauge,B}} = \rho g h_B$$

Solve for h_B:

$$h_B = \frac{P_{\text{gauge,B}}}{\rho g} = \frac{P_B - P_{\text{at}}}{\rho g}$$

Substitute numerical values and evaluate h_B:

$$h_B = \frac{(1.739\,\text{atm} - 1.00\,\text{atm})\left(101.325\dfrac{\text{kPa}}{\text{atm}}\right)}{(1.00\times10^3\ \text{kg/m}^3)(9.81\,\text{m/s}^2)} = \boxed{7.63\,\text{m}}$$

87 •• [SSM] You are employed as a tanker truck driver for the summer. Heating oil is delivered to customers for winter usage by your large tanker truck. The delivery hose has a radius 1.00 cm. The specific gravity of the oil is 0.875, and its coefficient of viscosity is 200 mPa·s. What is the minimum time it will take you fill a customer's 55-gal oil drum if laminar flow through the hose must be maintained?

Picture the Problem We can use the volume of the drum and the volume flow rate equation to express the time to fill the customer's oil drum. The Reynolds number equation relates Reynolds number to the speed with which oil flows through the hose to the volume flow rate. Because the upper limit on the Reynolds number for laminar flow is approximately 2000, we'll use this value in our calculation of the fill time.

The time t_{fill} is related to the volume flow rate in the hose:

$$t_{\text{fill}} = \frac{V}{I_V} = \frac{V}{Av} \qquad (1)$$

where A is the cross-sectional area of the hose and V is the volume of the oil drum.

Reynolds number is defined by the equation:

$$N_R = \frac{2r\rho v}{\eta} \Rightarrow v = \frac{\eta N_R}{2r\rho}$$

Substitute for v and A in equation (1) to obtain:

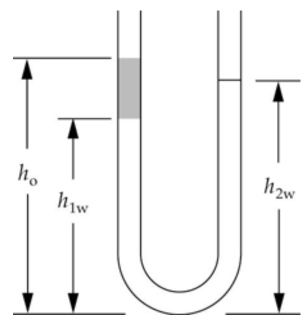
$$t_{\text{fill}} = \frac{V}{\pi r^2\left(\frac{\eta N_{\text{R}}}{2r\rho}\right)} = \frac{2\rho V}{\pi r\eta N_{\text{R}}}$$

Substitute numerical values and evaluate t_{fill}:

$$t_{\text{fill}} = \frac{2\left(875\ \text{kg/m}^3\right)\left(55\ \text{gal}\times\frac{3.785\ \text{L}}{\text{gal}}\right)}{\pi(0.010\ \text{m})(200\ \text{mPa}\cdot\text{s})(2000)} = \boxed{29\ \text{s}}$$

88 •• A U-tube is filled with water until the liquid level is 28 cm above the bottom of the tube (Figure 13-41*a*). Oil, which has a specific gravity 0.78, is now poured into one arm of the U-tube until the level of the water in the other arm of the tube is 34 cm above the bottom of the tube (Figure 13-41*b*). Find the levels of the oil–water and oil–air interfaces in the other arm of the tube.

Picture the Problem We can use the equality of the pressure at the bottom of the U-tube due to the water on one side and that due to the oil and water on the other to relate the various heights. Let h represent the height of the oil above the water. Then $h_{\text{o}} = h_{1\text{w}} + h$.



Using the constancy of the amount of water, express the relationship between $h_{1\text{w}}$ and $h_{2\text{w}}$:

$$h_{1\text{w}} + h_{2\text{w}} = 56\ \text{cm}$$

Find the height of the oil-water interface:

$$h_{1\text{w}} = 56\,\text{cm} - 34\,\text{cm} = \boxed{22\,\text{cm}}$$

Express the equality of the pressure at the bottom of the two arms of the U tube:

$$\rho_{\text{w}} g(34\,\text{cm}) = \rho_{\text{w}} g(22\,\text{cm}) + 0.78\rho_{\text{w}} g h_{\text{oil}}$$

Solve for and evaluate h_{oil}:

$$h_{\text{oil}} = \frac{\rho_{\text{w}} g(34\,\text{cm}) - \rho_{\text{w}} g(22\,\text{cm})}{0.78\rho_{\text{w}} g} = \frac{(34\,\text{cm}) - (22\,\text{cm})}{0.78} = 15.4\,\text{cm}$$

The height of the air-oil interface h_{o} is:

$$h_{\text{o}} = 22\text{cm} + 15.4\,\text{cm} = \boxed{37\,\text{cm}}$$

89 •• [SSM] A helium balloon can just lift a load that weighs 750 N and has a negligible volume. The skin of the balloon has a mass of 1.5 kg. (*a*) What is the volume of the balloon? (*b*) If the volume of the balloon were twice that found in Part (*a*), what would be the initial acceleration of the balloon when released at sea level carrying a load weighing 900 N?

Picture the Problem The balloon is in equilibrium under the influence of the buoyant force exerted by the air, the weight of its basket and load *w*, the weight of the skin of the balloon, and the weight of the helium. Choose upward to be the +*y* direction and apply the condition for translational equilibrium to relate these forces. Archimedes' principle relates the buoyant force on the balloon to the density of the air it displaces and the volume of the balloon.

(*a*) Apply $\sum F_y = 0$ to the balloon:

$$B - m_{\text{skin}}g - m_{\text{He}}g - w = 0$$

Letting *V* represent the volume of the balloon, use Archimedes' principle to express the buoyant force:

$$\rho_{\text{air}}Vg - m_{\text{skin}}g - m_{\text{He}}g - w = 0$$

Substituting for m_{He} yields:

$$\rho_{\text{air}}Vg - m_{\text{skin}}g - \rho_{\text{He}}Vg - w = 0$$

Solve for *V* to obtain:

$$V = \frac{m_{\text{skin}}g + w}{(\rho_{\text{air}} - \rho_{\text{He}})g}$$

Substitute numerical values and evaluate *V*:

$$V = \frac{(1.5\,\text{kg})(9.81\,\text{m/s}^2) + 750\,\text{N}}{(1.293\,\text{kg/m}^3 - 0.1786\,\text{kg/m}^3)(9.81\,\text{m/s}^2)} = \boxed{70\,\text{m}^3}$$

(*b*) Apply $\sum F_y = ma$ to the balloon:

$$B - m_{\text{tot}}g = m_{\text{tot}}a \Rightarrow a = \frac{B}{m_{\text{tot}}} - g \quad (1)$$

Assuming that the mass of the skin has not changed and letting *V'* represent the doubled volume of the balloon, express m_{tot}:

$$m_{\text{tot}} = m_{\text{load}} + m_{\text{He}} + m_{\text{skin}} = \frac{w_{\text{load}}}{g} + \rho_{\text{He}}V' + m_{\text{skin}}$$

Express the buoyant force acting on the balloon:

$$B = w_{\text{displaced fluid}} = \rho_{\text{air}}V'g$$

Substituting for m_{tot} and B in equation (1) yields:

$$a = \frac{\rho_{air} V' g}{\frac{w_{load}}{g} + \rho_{He} V' + m_{skin}} - g$$

Substitute numerical values and evaluate a:

$$a = \frac{(1.293\,\text{kg/m}^3)(140\,\text{m}^3)(9.81\,\text{m/s}^2)}{\frac{900\text{ N}}{9.81\,\text{m/s}^2} + (0.1786\,\text{kg/m}^3)(140\,\text{m}^3) + 1.5\,\text{kg}} - 9.81\,\text{m/s}^2 = \boxed{5.2\text{ m/s}^2}$$

90 •• A hollow sphere with an inner radius R and an outer radius $2R$. It is made of material of density ρ_0 and is floating in a liquid of density $2\rho_0$. The interior is now completely filled with material of density ρ' such that that the sphere just floats completely submerged. Find ρ'.

Picture the Problem When the hollow sphere is completely submerged but floating, it is in translational equilibrium under the influence of a buoyant force and its weight. The buoyant force is given by Archimedes' principle and the weight of the sphere is the sum of the weights of the hollow sphere and the material filling its center.

Apply $\sum F_y = 0$ to the hollow sphere:

$$B - w = 0 \qquad (1)$$

Express the buoyant force acting on the completely submerged sphere:

$$B = 2\rho_0 V_{sphere} g = 2\rho_0 \left[\tfrac{4}{3}\pi(2R)^3\right] g = \tfrac{64}{3}\rho_0 \pi R^3 g$$

Express the weight of the sphere when the hollow in it is filled with a material of density ρ':

$$\begin{aligned} w &= \rho_0 V_{\text{hollow sphere}} g + \rho' V_{\text{hollow}} g \\ &= \rho_0\left[\tfrac{4}{3}\pi\left\{(2R)^3 - R^3\right\}\right] g + \rho'\left[\tfrac{4}{3}\pi R^3\right] g \\ &= \tfrac{28}{3}\rho_0 \pi R^3 g + \tfrac{4}{3}\rho' \pi R^3 g \end{aligned}$$

Substitute for B and w in equation (1) to obtain:

$$\tfrac{64}{3}\rho_0 \pi R^3 g - \tfrac{28}{3}\rho_0 \pi R^3 g - \tfrac{4}{3}\rho' \pi R^3 g = 0$$

Solving for ρ' yields:

$$\rho' = \boxed{9\rho_0}$$

91 •• According to *the law of atmospheres*, the fractional decrease in atmospheric pressure is proportional to the change in altitude. This law can be expressed as the differential equation $dP/P = -Cdh$, where C is a positive constant. (a) Show that $P(h) = P_0 e^{-Ch}$ where P_0 is the pressure at $h = 0$, is a

solution of the differential equation. (*b*) Given that the pressure 5.5 km above sea level is half that at sea level, find the constant *C*.

Picture the Problem We can differentiate the function $P(h)$ to show that it satisfies the differential equation $dP/P = -C\,dh$ and in Part (*b*) we can use the approximation $e^{-x} \approx 1 - x$ and $\Delta h << h_0$ to establish the given result.

(*a*) Differentiate $P(h) = P_0\,e^{-Ch}$:

$$\frac{dP}{dh} = -CP_0e^{-Ch} = -CP$$

Separating variables yields:

$$\frac{dP}{P} = \boxed{-C\,dh}$$

(*b*) Take the logarithm of both sides of the function $P(h)$:

$$\ln P = \ln\left(P_0e^{-Ch}\right) = \ln P_0 + \ln e^{-Ch} = \ln\left(P_0\right) - Ch$$

Solving for C yields:

$$C = \frac{1}{h}\ln\left(\frac{P_0}{P}\right)$$

Substitute numerical values and evaluate C:

$$C = \frac{1}{5.5\,\text{km}}\ln\left(\frac{P_0}{\frac{1}{2}P_0}\right) = \frac{1}{5.5\,\text{km}}\ln 2 = \boxed{0.13\,\text{km}^{-1}}$$

92 •• A submarine has a total mass of 2.40×10^6 kg, including crew and equipment. The vessel consists of two parts, the pressure hull, which has a volume of $2.00 \times 10^3\ \text{m}^3$, and the ballast tanks, which have a volume of $4.00 \times 10^2\ \text{m}^3$. When the boat cruises on the surface, the ballast tanks are filled with air at atmospheric pressure; to cruise below the surface, seawater must be admitted into the tanks. (*a*) What fraction of the submarine's volume is above the water surface when the tanks are filled with air? (*b*) How much water must be admitted into the tanks to give the submarine neutral buoyancy? Neglect the mass of any air in the tanks and use 1.025 as the specific gravity of seawater.

Picture the Problem Let V represent the volume of the submarine and V' the volume of seawater it displaces when it is on the surface. The submarine is in equilibrium in both parts of the problem. Hence we can apply the condition for translational equilibrium (neutral buoyancy) to the submarine to relate its weight to the buoyant force acting on it. We'll also use Archimedes' principle to connect the buoyant forces to the volume of seawater the submarine displaces. Let upward be the $+y$ direction.

(*a*) Express *f*, the fraction of the submarine's volume above the surface when the tanks are filled with air:

$$f = \frac{V - V'}{V} = 1 - \frac{V'}{V} \qquad (1)$$

Apply $\sum F_y = 0$ to the submarine when its tanks are full of air:

$$B - w = 0$$

Use Archimedes' principle to express the buoyant force on the submarine in terms of the volume of the displaced water:

$$B = \rho_{sw} V' g$$

Substituting for *B* and *w* yields:

$$\rho_{sw} V' g - mg = 0 \Rightarrow V' = \frac{m}{\rho_{sw}}$$

Substitute in equation (1) to obtain:

$$f = 1 - \frac{m}{\rho_{sw} V}$$

Substitute numerical values and evaluate *f*:

$$f = 1 - \frac{2.40 \times 10^6\ \text{kg}}{\left(1.025 \times 10^3\ \text{kg/m}^3\right)\left(2.40 \times 10^3\ \text{m}^3\right)} = \boxed{2.44\%}$$

(*b*) Express the volume of seawater in terms of its mass and density:

$$V_{sw} = \frac{m_{sw}}{\rho_{sw}} \qquad (2)$$

Apply $\sum F_y = 0$, the condition for neutral buoyancy, to the submarine:

$$B - w_{sub} - w_{sw} = 0$$

Use Archimedes' principle to express the buoyant force on the submarine in terms of the volume of the displaced water:

$$B = \rho_{sw} V g$$

Substituting for *B*, w_{sub}, and w_{sw} yields:

$$\rho_{sw} V g - m_{sub} g - m_{sw} g = 0$$

Solve for m_{sw} to obtain:

$$m_{sw} = \rho_{sw} V - m_{sub}$$

Substituting for V_{sw} in equation (2) yields:

$$V_{sw} = \frac{\rho_{sw}V - m_{sub}}{\rho_{sw}} = V - \frac{m_{sub}}{\rho_{sw}}$$

Substitute numerical values and evaluate V_{sw}:

$$V_{sw} = 2.40\times10^3\ \text{m}^3 - \frac{2.40\times10^6\ \text{kg}}{1025\,\text{kg/m}^3} = \boxed{60\,\text{m}^3}$$

93 ••• Most species of fish have expandable sacs, commonly known as "swim bladders," that enable fish to rise in the water by filling the bladders with oxygen collected by their gills and to sink by emptying the bladders into the surrounding water. A freshwater fish has an average density equal to 1.05 kg/L when its swim bladder is empty. How large must the volume of oxygen in the fish's swim bladder be if the fish is to have neutral buoyancy? The fish has a mass of 0.825 kg. Assume the density of oxygen in the bladder is equal to air density at standard temperature and pressure.

Picture the Problem For the fish to achieve neutral buoyancy, its overall density must be lowered to 1.00 kg/L. In order to lower its density, the buoyant force acting on the fish needs to be increased by increasing the fish's volume. It can accomplish this by filling its swim bladder.

Apply the condition for translational equilibrium to the fish:

$$\sum F_y = B - F_g = 0$$

Substitute for B and F_g to obtain:

$$\rho_{water}(V + \delta V)g - mg = 0$$

where V is the fish's volume and δV is the increase in the fish's volume resulting from the additional oxygen in its swim bladder.

Solving for δV yields:

$$\delta V = \frac{m}{\rho_{water}} - V$$

Express the density ρ of the fish:

$$\rho = \frac{m}{V} \Rightarrow V = \frac{m}{\rho}$$

Substitute for V to obtain:

$$\delta V = \frac{m}{\rho_{water}} - \frac{m}{\rho} = m\left(\frac{1}{\rho_{water}} - \frac{1}{\rho}\right)$$

The volume of oxygen transferred to the fish bladders is:

$$\delta V = (0.825\ \text{kg})\left(\frac{1}{1.00\ \frac{\text{kg}}{\text{L}}} - \frac{1}{1.05\ \frac{\text{kg}}{\text{L}}}\right)$$
$$= \boxed{39\ \text{cm}^3}$$

Remarks: In this solution, we've neglected the additional mass of oxygen that should be added to the fish's mass. Consider how small that mass is for a volume of 39 cm^3: $m_{\text{oxygen}} = \rho_{\text{air}}\delta V = (1.293\ \text{kg/m}^3)(39\times10^{-6}\ \text{m}^3) = 50\ \mu\text{g}$.

Chapter 14
Oscillations

Conceptual Problems

1 • True or false:

(*a*) For a simple harmonic oscillator, the period is proportional to the square of the amplitude.
(*b*) For a simple harmonic oscillator, the frequency does not depend on the amplitude.
(*c*) If the net force on a particle undergoing one-dimensional motion is proportional to, and oppositely directed from, the displacement from equilibrium, the motion is simple harmonic.

(*a*) False. In simple harmonic motion, the period is independent of the amplitude.

(*b*) True. In simple harmonic motion, the frequency is the reciprocal of the period which, in turn, is independent of the amplitude.

(*c*) True. This is the condition for simple harmonic motion

2 • If the amplitude of a simple harmonic oscillator is tripled, by what factor is the energy changed?

Determine the Concept The energy of a simple harmonic oscillator varies as the square of the amplitude of its motion. Hence, tripling the amplitude increases the energy by a factor of 9.

3 •• **[SSM]** An object attached to a spring exhibits simple harmonic motion with an amplitude of 4.0 cm. When the object is 2.0 cm from the equilibrium position, what percentage of its total mechanical energy is in the form of potential energy? (*a*) One-quarter. (*b*) One-third. (*c*) One-half. (*d*) Two-thirds. (*e*) Three-quarters.

Picture the Problem The total energy of an object undergoing simple harmonic motion is given by $E_{\text{tot}} = \frac{1}{2}kA^2$, where k is the force constant and A is the amplitude of the motion. The potential energy of the oscillator when it is a distance x from its equilibrium position is $U(x) = \frac{1}{2}kx^2$.

Express the ratio of the potential energy of the object when it is 2.0 cm from the equilibrium position to its total energy:

$$\frac{U(x)}{E_{\text{tot}}} = \frac{\frac{1}{2}kx^2}{\frac{1}{2}kA^2} = \frac{x^2}{A^2}$$

Evaluate this ratio for $x = 2.0$ cm and $A = 4.0$ cm:

$$\frac{U(2\,\text{cm})}{E_{\text{tot}}} = \frac{(2.0\,\text{cm})^2}{(4.0\,\text{cm})^2} = \frac{1}{4}$$

and $\boxed{(a)}$ is correct.

4 • An object attached to a spring exhibits simple harmonic motion with an amplitude of 10.0 cm. How far from equilibrium will the object be when the system's potential energy is equal to its kinetic energy? (*a*) 5.00 cm. (*b*) 7.07 cm. (*c*) 9.00 cm. (*d*) The distance can't be determined from the data given.

Determine the Concept Because the object's total energy is the sum of its kinetic and potential energies, when its potential energy equals its kinetic energy, its potential energy (and its kinetic energy) equals one-half its total energy.

Equate the object's potential energy to one-half its total energy:

$$U_{\text{s}} = \tfrac{1}{2}E_{\text{total}}$$

Substituting for U_{s} and E_{total} yields:

$$\tfrac{1}{2}kx^2 = \tfrac{1}{2}\left(\tfrac{1}{2}kA^2\right) \Rightarrow x = \frac{A}{\sqrt{2}}$$

Substitute the numerical value of A and evaluate x to obtain:

$$\boldsymbol{x} = \frac{10.0\,\text{cm}}{\sqrt{2}} = 7.07\,\text{cm}$$

and $\boxed{(b)}$ is correct.

5 • Two identical systems each consist of a spring with one end attached to a block and the other end attached to a wall. The springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions such that the amplitude of the motion of block A is four times as large as the amplitude of the motion of block B. How do their maximum speeds compare? (*a*) $v_{\text{A max}} = v_{\text{B max}}$, (*b*) $v_{\text{A max}} = 2v_{\text{B max}}$, (*c*) $v_{\text{A max}} = 4v_{\text{B max}}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to the amplitude of its motion:

$$v_{\text{A max}} = \omega_{\text{A}} A_{\text{A}}$$

Relate the maximum speed of system B to the amplitude of its motion:

$$v_{\text{B max}} = \omega_{\text{B}} A_{\text{B}}$$

Divide the first of these equations by the second to obtain:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{\omega_{\text{A}} A_{\text{A}}}{\omega_{\text{B}} A_{\text{B}}}$$

Because the systems differ only in amplitude, $\omega_{\text{A}} = \omega_{\text{B}}$, and:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{A_{\text{A}}}{A_{\text{B}}}$$

Substituting for A_{A} and simplifying yields:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{4A_{\text{B}}}{A_{\text{B}}} = 4 \Rightarrow v_{\text{A max}} = 4v_{\text{B max}}$$

$\boxed{(c)}$ is correct.

6 • Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions with equal amplitudes. However, the force constant of spring A is four times as large as the force constant of spring B. How do their maximum speeds compare? (*a*) $v_{\text{A max}} = v_{\text{B max}}$, (*b*) $v_{\text{A max}} = 2v_{\text{B max}}$, (*c*) $v_{\text{A max}} = 4v_{\text{B max}}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to its force constant:

$$v_{\text{A max}} = \omega_{\text{A}} A_{\text{A}} = \sqrt{\frac{k_{\text{A}}}{m_{\text{A}}}} A_{\text{A}}$$

Relate the maximum speed of system B to its force constant:

$$v_{\text{B max}} = \omega_{\text{B}} A_{\text{B}} = \sqrt{\frac{k_{\text{B}}}{m_{\text{B}}}} A_{\text{B}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{\sqrt{\frac{k_{\text{A}}}{m_{\text{A}}}} A_{\text{A}}}{\sqrt{\frac{k_{\text{B}}}{m_{\text{B}}}} A_{\text{B}}} = \sqrt{\frac{m_{\text{B}}}{m_{\text{A}}} \frac{k_{\text{A}}}{k_{\text{B}}}} \frac{A_{\text{A}}}{A_{\text{B}}}$$

Because the systems differ only in their force constants:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \sqrt{\frac{k_{\text{A}}}{k_{\text{B}}}}$$

Substituting for k_A and simplifying yields:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \sqrt{\frac{4k_B}{k_B}} = 2 \Rightarrow v_{\text{A max}} = 2v_{\text{B max}}$$

$\boxed{(b)}$ is correct.

7 •• **[SSM]** Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The identical springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions with equal amplitudes. However, the mass of block A is four times as large as the mass of block B. How do their maximum speeds compare? (*a*) $v_{\text{A max}} = v_{\text{B max}}$, (*b*) $v_{\text{A max}} = 2v_{\text{B max}}$, (*c*) $v_{\text{A max}} = \frac{1}{2}v_{\text{B max}}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to its force constant:

$$v_{\text{A max}} = \omega_A A_A = \sqrt{\frac{k_A}{m_A}} A_A$$

Relate the maximum speed of system B to its force constant:

$$v_{\text{B max}} = \omega_B A_B = \sqrt{\frac{k_B}{m_B}} A_B$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{\sqrt{\frac{k_A}{m_A}} A_A}{\sqrt{\frac{k_B}{m_B}} A_B} = \sqrt{\frac{m_B}{m_A}\frac{k_A}{k_B}}\frac{A_A}{A_B}$$

Because the systems differ only in the masses attached to the springs:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \sqrt{\frac{m_B}{m_A}}$$

Substituting for m_A and simplifying yields:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \sqrt{\frac{m_B}{4m_B}} = \tfrac{1}{2} \Rightarrow v_{\text{A max}} = \tfrac{1}{2}v_{\text{B max}}$$

$\boxed{(c)}$ is correct.

8 •• Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The identical springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are

oscillating in simple harmonic motions with equal amplitudes. However, the mass of block A is four times as large as the mass of block B. How do the magnitudes of their maximum acceleration compare? (*a*) $a_{\text{A max}} = a_{\text{B max}}$, (*b*) $a_{\text{A max}} = 2a_{\text{B max}}$, (*c*) $a_{\text{A max}} = \frac{1}{2}a_{\text{B max}}$, (*d*) $a_{\text{A max}} = \frac{1}{4}a_{\text{B max}}$, (*e*) This comparison cannot be done by using the data given.

Determine the Concept The maximum acceleration of a simple harmonic oscillator is the product of the square of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum acceleration of system A to its force constant:

$$a_{\text{A max}} = \omega_{\text{A}}^2 A_{\text{A}} = \frac{k_{\text{A}}}{m_{\text{A}}} A_{\text{A}}$$

Relate the maximum acceleration of system B to its force constant:

$$a_{\text{B max}} = \omega_{\text{B}}^2 A_{\text{B}} = \frac{k_{\text{B}}}{m_{\text{B}}} A_{\text{B}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{a_{\text{A max}}}{a_{,\text{max}}} = \frac{\frac{k_{\text{A}}}{m_{\text{A}}} A_{\text{A}}}{\frac{k_{\text{B}}}{m_{\text{B}}} A_{\text{B}}} = \frac{m_{\text{B}}}{m_{\text{A}}} \frac{k_{\text{A}}}{k_{\text{B}}} \frac{A_{\text{A}}}{A_{\text{B}}}$$

Because the systems differ only in the masses attached to the springs:

$$\frac{a_{\text{A max}}}{a_{\text{B max}}} = \frac{m_{\text{B}}}{m_{\text{A}}}$$

Substituting for m_{A} and simplifying yields:

$$\frac{a_{\text{A max}}}{a_{\text{B max}}} = \frac{m_{\text{B}}}{4m_{\text{B}}} = \frac{1}{4} \Rightarrow a_{\text{A max}} = \frac{1}{4} a_{\text{B max}}$$

$\boxed{(d)}$ is correct.

9 •• **[SSM]** In general physics courses, the mass of the spring in simple harmonic motion is usually neglected because its mass is usually much smaller than the mass of the object attached to it. However, this is not always the case. If you neglect the mass of the spring when it is not negligible, how will your calculation of the system's period, frequency and total energy compare to the actual values of these parameters? Explain.

Determine the Concept Neglecting the mass of the spring, the period of a simple harmonic oscillator is given by $T = 2\pi/\omega = 2\pi\sqrt{m/k}$ where m is the mass of the oscillating system (spring plus object) and its total energy is given by $E_{\text{total}} = \frac{1}{2}kA^2$.

Neglecting the mass of the spring results in your using a value for the mass of the oscillating system that is smaller than its actual value. Hence, your calculated value for the period will be smaller than the actual period of the system and the calculated value for the frequency, which is the reciprocal of the period, will be higher than the actual value.

Because the total energy of the oscillating system depends solely on the amplitude of its motion and on the stiffness of the spring, it is independent of the mass of the system, and so neglecting the mass of the spring would have no effect on your calculation of the system's total energy.

10 •• Two mass–spring systems oscillate with periods T_A and T_B. If $T_A = 2T_B$ and the systems' springs have identical force constants, it follows that the systems' masses are related by (*a*) $m_A = 4m_B$, (*b*) $m_A = m_B/\sqrt{2}$, (*c*) $m_A = m_B/2$, (*d*) $m_A = m_B/4$.

Picture the Problem We can use $T = 2\pi\sqrt{m/k}$ to express the periods of the two mass-spring systems in terms of their force constants. Dividing one of the equations by the other will allow us to express m_A in terms of m_B.

Express the period of system A:

$$T_A = 2\pi\sqrt{\frac{m_A}{k_A}} \Rightarrow m_A = \frac{k_A T_A^2}{4\pi^2}$$

Relate the mass of system B to its period:

$$m_B = \frac{k_B T_B^2}{4\pi^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{m_A}{m_B} = \frac{\dfrac{k_A T_A^2}{4\pi^2}}{\dfrac{k_B T_B^2}{4\pi^2}} = \frac{k_A T_A^2}{k_B T_B^2}$$

Because the force constants of the two systems are the same:

$$\frac{m_A}{m_B} = \frac{T_A^2}{T_B^2} = \left(\frac{T_A}{T_B}\right)^2$$

Substituting for T_A and simplifying yields:

$$\frac{m_A}{m_B} = \left(\frac{2T_B}{T_B}\right)^2 = 4 \Rightarrow m_A = 4m_B$$

$\boxed{(a)}$ is correct.

11 •• Two mass–spring systems oscillate at frequencies f_A and f_B. If $f_A = 2f_B$ and the systems' springs have identical force constants, it follows that the systems' masses are related by (*a*) $m_A = 4m_B$, (*b*) $m_A = m_B/\sqrt{2}$, (*c*) $m_A = m_B/2$, (*d*) $m_A = m_B/4$.

Picture the Problem We can use $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ to express the frequencies of the two mass-spring systems in terms of their masses. Dividing one of the equations by the other will allow us to express m_A in terms of m_B.

Express the frequency of mass-spring system A as a function of its mass:

$$f_A = \frac{1}{2\pi}\sqrt{\frac{k}{m_A}}$$

Express the frequency of mass-spring system B as a function of its mass:

$$f_B = \frac{1}{2\pi}\sqrt{\frac{k}{m_B}}$$

Divide the second of these equations by the first to obtain:

$$\frac{f_B}{f_A} = \sqrt{\frac{m_A}{m_B}}$$

Solve for m_A:

$$m_A = \left(\frac{f_B}{f_A}\right)^2 m_B = \left(\frac{f_B}{2f_B}\right)^2 m_B = \tfrac{1}{4}m_B$$

$\boxed{(d)}$ is correct.

12 •• Two mass–spring systems *A* and *B* oscillate so that their total mechanical energies are equal. If $m_A = 2m_B$, which expression best relates their amplitudes? (*a*) $A_A = A_B/4$, (*b*) $A_A = A_B/\sqrt{2}$, (*c*) $A_A = A_B$, (*d*) Not enough information is given to determine the ratio of the amplitudes.

Picture the Problem We can relate the energies of the two mass-spring systems through either $E = \frac{1}{2}kA^2$ or $E = \frac{1}{2}m\omega^2A^2$ and investigate the relationship between their amplitudes by equating the expressions, substituting for m_A, and expressing A_A in terms of A_B.

Express the energy of mass-spring system A: $E_A = \frac{1}{2}k_A A_A^2 = \frac{1}{2}m_A\omega_A^2 A_A^2$

Express the energy of mass-spring system B: $E_B = \frac{1}{2}k_B A_B^2 = \frac{1}{2}m_B\omega_B^2 A_B^2$

Divide the first of these equations by the second to obtain:

$$\frac{E_A}{E_B} = 1 = \frac{\frac{1}{2}m_A\omega_A^2 A_A^2}{\frac{1}{2}m_B\omega_B^2 A_B^2}$$

Substitute for m_A and simplify:

$$1 = \frac{2m_B\omega_A^2 A_A^2}{m_B\omega_B^2 A_B^2} = \frac{2\omega_A^2 A_A^2}{\omega_B^2 A_B^2}$$

Solve for A_A:

$$A_A = \frac{\omega_B}{\sqrt{2}\omega_A}A_B$$

Without knowing how ω_A and ω_B, or k_A and k_B, are related, we cannot simplify this expression further. $\boxed{(d)}$ is correct.

13 •• [SSM] Two mass–spring systems *A* and *B* oscillate so that their total mechanical energies are equal. If the force constant of spring A is two times the force constant of spring B, then which expression best relates their amplitudes? (*a*) $A_A = A_B/4$, (*b*) $A_A = A_B/\sqrt{2}$, (*c*) $A_A = A_B$, (*d*) Not enough information is given to determine the ratio of the amplitudes.

Picture the Problem We can express the energy of each system using $E = \frac{1}{2}kA^2$ and, because the energies are equal, equate them and solve for A_A.

Express the energy of mass-spring system A in terms of the amplitude of its motion: $E_A = \frac{1}{2}k_A A_A^2$

Express the energy of mass-spring system B in terms of the amplitude of its motion: $E_B = \frac{1}{2}k_B A_B^2$

Because the energies of the two systems are equal we can equate them to obtain:

$$\frac{1}{2}k_A A_A^2 = \frac{1}{2}k_B A_B^2 \Rightarrow A_A = \sqrt{\frac{k_B}{k_A}}A_B$$

Substitute for k_A and simplify to obtain:

$$A_A = \sqrt{\frac{k_B}{2k_B}}A_B = \frac{A_B}{\sqrt{2}} \Rightarrow \boxed{(b)} \text{ is correct.}$$

14 •• The length of the string or wire supporting a pendulum bob increases slightly when the temperature of the string or wire is raised. How does this affect a clock operated by a simple pendulum?

Determine the Concept The period of a simple pendulum depends on the square root of the length of the pendulum. Increasing the length of the pendulum will lengthen its period and, hence, the clock will run slow.

15 •• A lamp hanging from the ceiling of the club car in a train oscillates with period T_0 when the train is at rest. The period will be (match left and right columns)

1. greater than T_0 when	A. The train moves horizontally at constant velocity.
2. less than T_0 when	B. The train rounds a curve at constant speed.
3. equal to T_0 when	C. The train climbs a hill at constant speed.
	D. The train goes over the crest of a hill at constant speed.

Determine the Concept The period of the lamp varies inversely with the square root of the effective value of the local gravitational field.

1 matches up with B. The period will be greater than T_0 when the train rounds a curve of radius R with speed v.

2 matches up with D. The period will be less than T_0 when the train goes over the crest of a hill of radius of curvature R with constant speed.

3 matches up with A. The period will be equal to T_0 when the train moves horizontally with constant velocity.

16 •• Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B and a bob of mass m_B. If the period of A is twice that of B, then (*a*) $L_A = 2L_B$ and $m_A = 2m_B$,

(b) $L_A = 4L_B$ and $m_A = m_B$, (c) $L_A = 4L_B$ whatever the ratio m_A/m_B, (d) $L_A = \sqrt{2}L_B$ whatever the ratio m_A/m_B.

Picture the Problem The period of a simple pendulum is independent of the mass of its bob and is given by $T = 2\pi\sqrt{L/g}$.

Express the period of pendulum A:

$$T_A = 2\pi\sqrt{\frac{L_A}{g}}$$

Express the period of pendulum B:

$$T_B = 2\pi\sqrt{\frac{L_B}{g}}$$

Divide the first of these equations by the second and solve for L_A/L_B:

$$\frac{L_A}{L_B} = \left(\frac{T_A}{T_B}\right)^2$$

Substitute for T_A and solve for L_B to obtain:

$$L_A = \left(\frac{2T_B}{T_B}\right)^2 L_B = 4L_B$$

$\boxed{(c)}$ is correct.

17 •• [SSM] Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B and a bob of mass m_B. If the frequency of A is one-third that of B, then (a) $L_A = 3L_B$ and $m_A = 3m_B$, (b) $L_A = 9L_B$ and $m_A = m_B$, (c) $L_A = 9L_B$ regardless of the ratio m_A/m_B, (d) $L_A = \sqrt{3}L_B$ regardless of the ratio m_A/m_B.

Picture the Problem The frequency of a simple pendulum is independent of the mass of its bob and is given by $f = \frac{1}{2\pi}\sqrt{g/L}$.

Express the frequency of pendulum A:

$$f_A = \frac{1}{2\pi}\sqrt{\frac{g}{L_A}} \Rightarrow L_A = \frac{g}{4\pi^2 f_A^2}$$

Similarly, the length of pendulum B is given by:

$$L_B = \frac{g}{4\pi^2 f_B^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{L_A}{L_B}=\frac{\dfrac{g}{4\pi^2 f_A^2}}{\dfrac{g}{4\pi^2 f_B^2}}=\frac{f_B^2}{f_A^2}=\left(\frac{f_B}{f_A}\right)^2$$

Substitute for f_A to obtain:

$$\frac{L_A}{L_B}=\left(\frac{f_B}{\frac{1}{3}f_B}\right)^2=9\Rightarrow\boxed{(c)}\text{ is correct.}$$

18 •• Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B a bob of mass m_B. They have the same period. The only thing different between their motions is that the amplitude of A's motion is twice that of B's motion, then (*a*) $L_A = L_B$ and $m_A = m_B$, (*b*) $L_A = 2L_B$ and $m_A = m_B$, (*c*) $L_A = L_B$ whatever the ratio m_A/m_B, (*d*) $L_A = \frac{1}{2}L_B$ whatever the ratio m_A/m_B.

Picture the Problem The period of a simple pendulum is independent of the mass of its bob and is given by $T = 2\pi\sqrt{L/g}$. For small amplitudes, the period is independent of the amplitude.

Express the period of pendulum A:

$$T_A = 2\pi\sqrt{\frac{L_A}{g}}$$

Express the period of pendulum B:

$$T_B = 2\pi\sqrt{\frac{L_B}{g}}$$

Divide the first of these equations by the second and solve for L_A/L_B:

$$\frac{L_A}{L_B}=\left(\frac{T_A}{T_B}\right)^2$$

Because their periods are the same:

$$\frac{L_A}{L_B}=\left(\frac{T_B}{T_B}\right)^2=1\Rightarrow\boxed{(c)}\text{ is correct.}$$

19 •• True or false:

(*a*) The mechanical energy of a damped, undriven oscillator decreases exponentially with time.
(*b*) Resonance for a damped, driven oscillator occurs when the driving frequency exactly equals the natural frequency.
(*c*) If the Q factor of a damped oscillator is high, then its resonance curve will be narrow.

(*d*) The decay time τ for a spring-mass oscillator with linear damping is independent of its mass.
(*e*) The Q factor for a driven spring-mass oscillator with linear damping is independent of its mass.

(*a*) True. Because the energy of an oscillator is proportional to the square of its amplitude, and the amplitude of a damped, undriven oscillator decreases exponentially with time, so does its energy.

(*b*) False. For a damped driven oscillator, the resonant frequency ω' is given by $\omega' = \omega_0\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2}$, where ω_0 is the natural frequency of the oscillator.

(*c*) True. The ratio of the width of a resonance curve to the resonant frequency equals the reciprocal of the Q factor ($\Delta\boldsymbol{\omega}/\boldsymbol{\omega}_0 = 1/\boldsymbol{Q}$). Hence, the larger Q is, the narrower the resonance curve.

(*d*) False. The decay time for a damped but undriven spring-mass oscillator is directly proportional to its mass.

(*e*) True. From $\Delta\omega/\omega_0 = 1/Q$ one can see that Q is independent of m.

20 •• Two damped spring-mass oscillating systems have identical spring and damping constants. However, system A's mass m_A is four times system B's. How do their decay times compare? (*a*) $\tau_A = 4\tau_B$, (*b*) $\tau_A = 2\tau_B$, (*c*) $\tau_A = \tau_B$, (*d*) Their decay times cannot be compared, given the information provided.

Picture the Problem The decay time τ of a damped oscillator is related to the mass m of the oscillator and the damping constant b for the motion according to $\tau = m/b$.

Express the decay time of System A:

$$\tau_A = \frac{m_A}{b_A}$$

The decay time for System B is given by:

$$\tau_B = \frac{m_B}{b_B}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\tau_A}{\tau_B} = \frac{\frac{m_A}{b_A}}{\frac{m_B}{b_B}} = \frac{m_A}{m_B}\frac{b_B}{b_A}$$

Because their damping constants are the same:

$$\frac{\tau_A}{\tau_B} = \frac{m_A}{m_B}$$

Substituting for m_A yields:

$$\frac{\tau_A}{\tau_B} = \frac{4m_B}{m_B} = 4 \Rightarrow \tau_A = 4\tau_B$$

$\boxed{(a)}$ is correct.

21 •• Two damped spring-mass oscillating systems have identical spring constants and decay times. However, system A's mass m_A is twice system B's mass m_B. How do their damping constants, b, compare? (*a*) $b_A = 4b_B$, (*b*) $b_A = 2b_B$, (*c*) $b_A = b_B$, (*d*) $b_A = \frac{1}{2}b_B$, (*e*) Their decay times cannot be compared, given the information provided.

Picture the Problem The decay time τ of a damped oscillator is related to the mass m of the oscillator and the damping constant b for the motion according to $\tau = m/b$.

Express the damping constant of System A:

$$b_A = \frac{m_A}{\tau_A}$$

The damping constant for System B is given by:

$$b_B = \frac{m_B}{\tau_B}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{b_A}{b_B} = \frac{\dfrac{m_A}{\tau_A}}{\dfrac{m_B}{\tau_B}} = \frac{m_A}{m_B}\frac{\tau_B}{\tau_A}$$

Because their decay times are the same:

$$\frac{b_A}{b_B} = \frac{m_A}{m_B}$$

Substituting for m_A yields:

$$\frac{b_A}{b_B} = \frac{2m_B}{m_B} = 2 \Rightarrow b_A = 2b_B$$

$\boxed{(b)}$ is correct.

22 •• Two damped, driven spring-mass oscillating systems have identical driving forces as well as identical spring and damping constants. However, the mass of system A is four times the mass of system B. Assume both systems are very weakly damped. How do their resonant frequencies compare? (*a*) $\omega_A = \omega_B$, (*b*) $\omega_A = 2\omega_B$, (*c*) $\omega_A = \frac{1}{2}\omega_B$, (*d*) $\omega_A = \frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a spring-mass oscillator is the same as its natural frequency and is given by $\omega_0 = \sqrt{k/m}$, where m is the oscillator's mass and k is the force constant of the spring.

Express the resonant frequency of System A:

$$\omega_A = \sqrt{\frac{k_A}{m_A}}$$

The resonant frequency of System B is given by:

$$\omega_B = \sqrt{\frac{k_B}{m_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B} = \frac{\sqrt{\dfrac{k_A}{m_A}}}{\sqrt{\dfrac{k_B}{m_B}}} = \sqrt{\frac{k_A}{k_B}\frac{m_B}{m_A}}$$

Because their force constants are the same:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{m_B}{m_A}}$$

Substituting for m_A yields:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{m_B}{4m_B}} = \tfrac{1}{2} \Rightarrow \boxed{(c)} \text{ is correct.}$$

23 •• [SSM] Two damped, driven spring-mass oscillating systems have identical masses, driving forces, and damping constants. However, system A's force constant k_A is four times system B's force constant k_B. Assume they are both very weakly damped. How do their resonant frequencies compare? (*a*) $\omega_A = \omega_B$, (*b*) $\omega_A = 2\omega_B$, (*c*) $\omega_A = \frac{1}{2}\omega_B$, (*d*) $\omega_A = \frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a spring-mass oscillator is the same as its natural frequency and is given by $\omega_0 = \sqrt{k/m}$, where m is the oscillator's mass and k is the force constant of the spring.

Express the resonant frequency of System A:

$$\omega_A = \sqrt{\frac{k_A}{m_A}}$$

The resonant frequency of System B is given by:

$$\omega_B = \sqrt{\frac{k_B}{m_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B}=\frac{\sqrt{\frac{k_A}{m_A}}}{\sqrt{\frac{k_B}{m_B}}}=\sqrt{\frac{k_A}{k_B}\frac{m_B}{m_A}}$$

Because their masses are the same:

$$\frac{\omega_A}{\omega_B}=\sqrt{\frac{k_A}{k_B}}$$

Substituting for k_A yields:

$$\frac{\omega_A}{\omega_B}=\sqrt{\frac{4k_B}{k_B}}=2\Rightarrow\boxed{(b)}\text{ is correct.}$$

24 •• Two damped, driven simple-pendulum systems have identical masses, driving forces, and damping constants. However, system A's length is four times system B's length. Assume they are both very weakly damped. How do their resonant frequencies compare? (*a*) $\omega_A=\omega_B$, (*b*) $\omega_A=2\omega_B$, (*c*) $\omega_A=\frac{1}{2}\omega_B$, (*d*) $\omega_A=\frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a simple pendulum is the same as its natural frequency and is given by $\omega_0=\sqrt{g/L}$, where L is the length of the simple pendulum and g is the gravitational field.

Express the resonant frequency of System A:

$$\omega_A=\sqrt{\frac{g}{L_A}}$$

The resonant frequency of System B is given by:

$$\omega_B=\sqrt{\frac{g}{L_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B}=\frac{\sqrt{\frac{g}{L_A}}}{\sqrt{\frac{g}{L_B}}}=\sqrt{\frac{L_B}{L_A}}$$

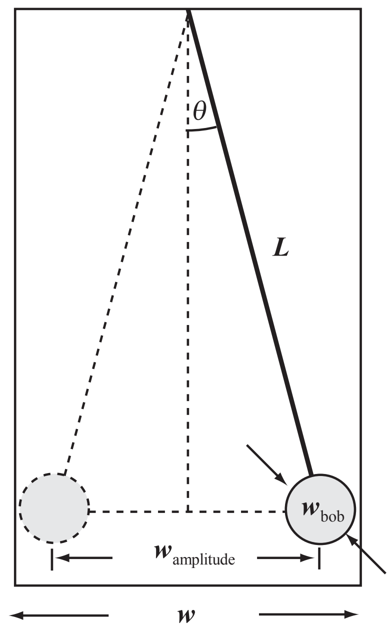
Substituting for L_A yields:

$$\frac{\omega_A}{\omega_B}=\sqrt{\frac{L_B}{4L_B}}=\tfrac{1}{2}\Rightarrow\boxed{(c)}\text{ is correct.}$$

Estimation and Approximation

25 • **[SSM]** Estimate the width of a typical grandfather clocks' cabinet relative to the width of the pendulum bob, presuming the desired motion of the pendulum is simple harmonic.

Picture the Problem If the motion of the pendulum in a grandfather clock is to be simple harmonic motion, then its period must be independent of the angular amplitude of its oscillations. The period of the motion for large-amplitude oscillations is given by Equation 14-30 and we can use this expression to obtain a maximum value for the amplitude of swinging pendulum in the clock. We can then use this value and an assumed value for the length of the pendulum to estimate the width of the grandfather clocks' cabinet.



Referring to the diagram, we see that the minimum width of the cabinet is determined by the width of the bob and the width required to accommodate the swinging pendulum:

$$w = w_{\text{bob}} + w_{\text{amplitude}}$$

and

$$\frac{w}{w_{\text{bob}}} = 1 + \frac{w_{\text{amplitude}}}{w_{\text{bob}}} \qquad (1)$$

Express $w_{\text{amplitude}}$ in terms of the angular amplitude θ and the length of the pendulum L:

$$w_{\text{amplitude}} = 2L\sin\theta$$

Substituting for $w_{\text{amplitude}}$ in equation (1) yields:

$$\frac{w}{w_{\text{bob}}} = 1 + \frac{2L\sin\theta}{w_{\text{bob}}} \qquad (2)$$

Equation 14-30 gives us the period of a simple pendulum as a function of its angular amplitude:

$$T = T_0\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\theta + \ldots\right]$$

If T is to be approximately equal to T_0, the second term in the brackets must be small compared to the first term. Suppose that:

$$\frac{1}{4}\sin^2\frac{1}{2}\theta \leq 0.001$$

Solving for θ yields:

$$\theta \leq 2\sin^{-1}(0.0632) \approx 7.25°$$

If we assume that the length of a grandfather clock's pendulum is about 1.5 m and that the width of the bob is about 10 cm, then equation (2) yields:

$$\frac{w}{w_{\text{bob}}} = 1 + \frac{2(1.5\text{ m})\sin 7.25°}{0.10\text{ m}} \approx \boxed{5}$$

26 • A small punching bag for boxing workouts is approximately the size and weight of a person's head and is suspended from a very short rope or chain. Estimate the natural frequency of oscillations of such a punching bag.

Picture the Problem For the purposes of this estimation, model the punching bag as a sphere of radius R and assume that the spindle about which it rotates to be 1.5 times the radius of the sphere. The natural frequency of oscillations of this physical pendulum is given by $f_0 = \omega_0 = \frac{1}{2\pi}\sqrt{\frac{MgD}{I}}$ where M is the mass of the pendulum, D is the distance from the point of support to the center of mass of the punching bag, and I is its moment of inertia about an axis through the spindle from which it is supported and about which it swivels.

Express the natural frequency of oscillation of the punching bag:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{MgD}{I_{\text{spindle}}}} \qquad (1)$$

From the parallel-axis theorem we have:

$$I_{\text{spindle}} = I_{\text{cm}} + Mh^2$$
where $h = 1.5R + 0.5R = 2R$

Substituting for I_{cm} and h yields:

$$I_{\text{spindle}} = \tfrac{2}{5}MR^2 + M(2R)^2 = 4.4MR^2$$

Substitute for I_{spindle} in equation (1) to obtain:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{Mg(2R)}{4.4MR^2}} = \frac{1}{2\pi}\sqrt{\frac{g}{2.2R}}$$

Assume that the radius of the punching bag is 10 cm, substitute numerical values and evaluate f_0:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\text{ m/s}^2}{2.2(0.10\text{ m})}} \approx \boxed{1\text{ Hz}}$$

27 •• For a child on a swing, the amplitude drops by a factor of $1/e$ in about eight periods if no additional mechanical energy is given to the system. Estimate the Q factor for this system.

Picture the Problem The Q factor for this system is related to the decay constant τ through $Q = \omega_0\tau = 2\pi\tau/T$ and the amplitude of the child's damped motion varies with time according to $A = A_0 e^{-t/2\tau}$. We can set the ratio of two displacements separated by eight periods equal to $1/e$ to determine τ in terms of T.

Express Q as a function of τ:

$$Q = \omega_0\tau = \frac{2\pi\tau}{T} \qquad (1)$$

The amplitude of the oscillations varies with time according to:

$$A = A_0 e^{-t/2\tau}$$

The amplitude after eight periods is:

$$A_8 = A_0 e^{-(t+8T)/2\tau}$$

Express and simplify the ratio A_8/A:

$$\frac{A_8}{A} = \frac{A_0 e^{-(t+8T)/2\tau}}{A_0 e^{-t/2\tau}} = e^{-4T/\tau}$$

Set this ratio equal to $1/e$ and solve for τ:

$$e^{-4T/\tau} = e^{-1} \Rightarrow \tau = 4T$$

Substitute in equation (1) and evaluate Q:

$$Q = \frac{2\pi(4T)}{T} = \boxed{8\pi}$$

28 •• (a) Estimate the natural period of oscillation for swinging your arms as you walk, when your hands are empty. (b) Now estimate the natural period of oscillation when you are carrying a heavy briefcase. (c) Observe other people as they walk. Do your estimates seem reasonable?

Picture the Problem Assume that an average length for an arm is about 80 cm, and that it can be treated as a uniform rod, pivoted at one end. We can use the expression for the period of a physical pendulum to derive an expression for the period of the swinging arm. When carrying a heavy briefcase, the mass is concentrated mostly at the end of the rod (that is, in the briefcase), so we can treat the arm-plus-briefcase system as a simple pendulum.

(*a*) Express the period of a uniform rod pivoted at one end:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

where I is the moment of inertia of the stick about an axis through one end, M is the mass of the stick, and D (= $L/2$) is the distance from the end of the stick to its center of mass.

Express the moment of inertia of a rod about an axis through its end:

$$I = \tfrac{1}{3}ML^2$$

Substitute the values for I and D in the expression for T and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\left(\frac{1}{2}L\right)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.80\,\text{m})}{3(9.81\,\text{m/s}^2)}} = \boxed{1.5\,\text{s}}$$

(*b*) Express the period of a simple pendulum:

$$T' = 2\pi\sqrt{\frac{L'}{g}}$$

where L' is slightly longer than the arm length due to the size of the briefcase.

Assuming $L' = 1.0$ m, evaluate the period of the simple pendulum:

$$T' = 2\pi\sqrt{\frac{1.0\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{2.0\,\text{s}}$$

(*c*) From observation of people as they walk, these estimates seem reasonable.

Simple Harmonic Motion

29 • The position of a particle is given by $x = (7.0\text{ cm})\cos 6\pi t$, where t is in seconds. What are (*a*) the frequency, (*b*) the period, and (*c*) the amplitude of the particle's motion? (*d*) What is the first time after $t = 0$ that the particle is at its equilibrium position? In what direction is it moving at that time?

Picture the Problem The position of the particle is given by $x = A\cos(\omega t + \delta)$ where A is the amplitude of the motion, ω is the angular frequency, and δ is a phase constant. The frequency of the motion is given by $f = \omega/2\pi$ and the period of the motion is the reciprocal of its frequency.

(*a*) Use the definition of ω to determine f:

$$f = \frac{\omega}{2\pi} = \frac{6\pi\,\text{s}^{-1}}{2\pi} = \boxed{3.00\,\text{Hz}}$$

(*b*) Evaluate the reciprocal of the frequency:

$$T = \frac{1}{f} = \frac{1}{3.00\,\text{Hz}} = \boxed{0.333\,\text{s}}$$

(*c*) Compare $x = (7.0\text{ cm})\cos 6\pi t$ to $x = A\cos(\omega t + \delta)$ to conclude that:

$$A = \boxed{7.0\,\text{cm}}$$

(*d*) Express the condition that must be satisfied when the particle is at its equilibrium position:

$$\cos\omega t = 0 \Rightarrow \omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega}$$

Substituting for ω yields:

$$t = \frac{\pi}{2(6\pi)} = \boxed{0.0833\,\text{s}}$$

Differentiate x to find $v(t)$:

$$v = \frac{d}{dt}\left[(7.0\,\text{cm})\cos 6\pi t\right] = -(42\pi\text{ cm/s})\sin 6\pi t$$

Evaluate $v(0.0833\text{ s})$:

$$v(0.0833\,\text{s}) = -(42\pi\text{ cm/s})\sin 6\pi(0.0833\,\text{s}) < 0$$

Because $v < 0$, the particle is moving in the $-x$ direction at $t = 0.0833$ s.

30 • What is the phase constant δ in $x = A\cos(\omega t + \delta)$ (Equation 14-4) if the position of the oscillating particle at time $t = 0$ is (*a*) 0, (*b*) $-A$, (*c*) A, (*d*) $A/2$?

Picture the Problem The initial position of the oscillating particle is related to the amplitude and phase constant of the motion by $x_0 = A\cos\delta$ where $0 \le \delta < 2\pi$.

(*a*) For $x_0 = 0$:

$$\cos\delta = 0 \Rightarrow \delta = \cos^{-1}(0) = \boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$$

(*b*) For $x_0 = -A$:

$$-A = A\cos\delta \Rightarrow \delta = \cos^{-1}(-1) = \boxed{\pi}$$

(*c*) For $x_0 = A$:

$$A = A\cos\delta \Rightarrow \delta = \cos^{-1}(1) = \boxed{0}$$

(*d*) When $x = A/2$:

$$\frac{A}{2} = A\cos\delta \Rightarrow \delta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$$

31 • **[SSM]** A particle of mass m begins at rest from x = +25 cm and oscillates about its equilibrium position at x = 0 with a period of 1.5 s. Write expressions for (a) the position x as a function of t, (b) the velocity v_x as a function of t, and (c) the acceleration a_x as a function of t.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. The velocity and acceleration of the particle can be found by differentiating the expression for the position of the particle with respect to time.

(a) The position of the particle is given by:

$$x = A\cos(\omega t + \delta)$$

For a particle that begins from rest from x = +25 cm, δ = 0 and A = +25 cm. Hence:

$$x = (+25\ \text{cm})\cos\omega t$$

The angular frequency of the particle's motion is:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\text{s}^{-1} = 4.19\,\text{s}^{-1}$$

Substituting for ω gives:

$$x = (25\,\text{cm})\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{(0.25\,\text{m})\cos\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

(b) Differentiate the result in Part (a) with respect to time to express the velocity of the particle:

$$v_x = \frac{dx}{dt} = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)\sin\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-(1.0\,\text{m/s})\sin\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

(c) Differentiate the result in Part (b) with respect to time to express the acceleration of the particle:

$$a_x = \frac{dv_x}{dt} = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)^2\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-\left(4.4\,\text{m/s}^2\right)\cos\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

32 •• Find (a) the maximum speed and (b) the maximum acceleration of the particle in Problem 31. (c) What is the first time that the particle is at x = 0 and moving to the right?

Picture the Problem The maximum speed and maximum acceleration of the particle in Problem 29 are given by $v_{max} = A\omega$ and $a_{max} = A\omega^2$. The particle's position is given by $x = A\cos(\omega t + \delta)$ where A = 7.0 cm, $\omega = 6\pi\ \text{s}^{-1}$, and $\delta = 0$, and its velocity is given by $v = -A\omega\sin(\omega t + \delta)$.

(*a*) Express v_{max} in terms of A and ω:

$$v_{max} = A\omega = (7.0\,\text{cm})(6\pi\,\text{s}^{-1}) = 42\pi\,\text{cm/s} = \boxed{1.3\,\text{m/s}}$$

(*b*) Express a_{max} in terms of A and ω:

$$a_{max} = A\omega^2 = (7.0\,\text{cm})(6\pi\,\text{s}^{-1})^2 = 252\pi^2\,\text{cm/s}^2 = \boxed{25\,\text{m/s}^2}$$

(*c*) When $x = 0$:

$$\cos\omega t = 0 \Rightarrow \omega t = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}$$

Evaluate v for $\omega t = \frac{\pi}{2}$:

$$v = -A\omega\sin\left(\frac{\pi}{2}\right) = -A\omega$$

That is, the particle is moving to the left.

Evaluate v for $\omega t = \frac{3\pi}{2}$:

$$v = -A\omega\sin\left(\frac{3\pi}{2}\right) = A\omega$$

That is, the particle is moving to the right.

Solve $\omega t = \frac{3\pi}{2}$ for t to obtain:

$$t = \frac{3\pi}{2\omega} = \frac{3\pi}{2(6\pi\,\text{s}^{-1})} = \boxed{0.25\,\text{s}}$$

33 •• Work Problem 31 for when the particle initially at x = 25 cm and moving with velocity v_0 = +50 cm/s.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. Its velocity as a function of time is given by $v = -A\omega\sin(\omega t + \delta)$ and its acceleration by $a = -A\omega^2\cos(\omega t + \delta)$. The initial position and velocity give us two equations from which to determine the amplitude A and phase constant δ.

(*a*) Express the position, velocity, and acceleration of the particle as functions of t:

$$x = A\cos(\omega t + \delta) \quad (1)$$

$$v_x = -A\omega\sin(\omega t + \delta) \quad (2)$$

$$a_x = -A\omega^2\cos(\omega t + \delta) \quad (3)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\mathrm{s}^{-1} = 4.19\,\mathrm{s}^{-1}$$

Relate the initial position and velocity to the amplitude and phase constant:

$$x_0 = A\cos\delta$$
and
$$v_0 = -\omega A\sin\delta$$

Divide these equations to eliminate A:

$$\frac{v_0}{x_0} = \frac{-\omega A\sin\delta}{A\cos\delta} = -\omega\tan\delta$$

Solving for δ yields:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right)$$

Substitute numerical values and evaluate δ:

$$\delta = \tan^{-1}\left(-\frac{50\,\mathrm{cm/s}}{(25\,\mathrm{cm})(4.192\,\mathrm{s}^{-1})}\right) = -0.445\,\mathrm{rad}$$

Use either the x_0 or v_0 equation (x_0 is used here) to find the amplitude:

$$A = \frac{x_0}{\cos\delta} = \frac{25\,\mathrm{cm}}{\cos(-0.445\,\mathrm{rad})} = 27.7\,\mathrm{cm}$$

Substitute in equation (1) to obtain:

$$x = \boxed{(0.28\,\mathrm{m})\cos\left[\left(4.2\,\mathrm{s}^{-1}\right)t - 0.45\right]}$$

(*b*) Substitute numerical values in equation (2) to obtain:

$$v_x = -(27.7\,\mathrm{cm})\left(\frac{4\pi}{3}\mathrm{s}^{-1}\right)\sin\left[\left(\frac{4\pi}{3}\mathrm{s}^{-1}\right)t - 0.445\right] = \boxed{-(1.2\,\mathrm{m/s})\sin\left[\left(4.2\,\mathrm{s}^{-1}\right)t - 0.45\right]}$$

(*c*) Substitute numerical values in equation (3) to obtain:

$$a_x = -(27.7\,\mathrm{cm})\left(\frac{4\pi}{3}\mathrm{s}^{-1}\right)^2\cos\left[\left(\frac{4\pi}{3}\mathrm{s}^{-1}\right)t - 0.445\right]$$
$$= \boxed{-\left(4.9\,\mathrm{m/s}^2\right)\cos\left[\left(4.2\,\mathrm{s}^{-1}\right)t - 0.45\right]}$$

34 •• The period of a particle that is oscillating in simple harmonic motion is 8.0 s and its amplitude is 12 cm. At $t = 0$ it is at its equilibrium position. Find the distance it travels during the intervals (*a*) $t = 0$ to $t = 2.0$ s, (*b*) $t = 2.0$ s to $t = 4.0$ s, (*c*) $t = 0$ to $t = 1.0$ s, and (*d*) $t = 1.0$ s to $t = 2.0$ s.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. We're given the amplitude A of the motion and can use the initial position of the particle to determine the phase constant δ. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of t:

$$x = (12\,\text{cm})\cos(\omega t + \delta) \quad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0\,\text{s}} = \frac{\pi}{4}\,\text{s}^{-1}$$

Relate the initial position of the particle to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

Solve for δ:

$$\delta = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(\frac{0}{A}\right) = \frac{\pi}{2}$$

Substitute in equation (1) to obtain:

$$x = (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t + \frac{\pi}{2}\right]$$

Express the distance the particle travels in terms of t_f and t_i:

$$\begin{aligned}\Delta x &= \left|(12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_\text{f} + \frac{\pi}{2}\right] - (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_\text{i} + \frac{\pi}{2}\right]\right| \\ &= \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_\text{f} + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_\text{i} + \frac{\pi}{2}\right]\right\}\right|\end{aligned}$$

(*a*) Evaluate Δx for $t_\text{f} = 2.0$ s, $t_\text{i} = 0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right| = \boxed{12\,\text{cm}}$$

(*b*) Evaluate Δx for $t_\text{f} = 4.0$ s, $t_\text{i} = 2.0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(4.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right]\right\}\right| = \boxed{12\,\text{cm}}$$

(*c*) Evaluate Δx for $t_f = 1.0$ s, $t_i = 0$:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(1.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right|$$

$$= \left|(12\,\text{cm})\{-0.7071 - 0\}\right| = \boxed{8.5\,\text{cm}}$$

(*d*) Evaluate Δx for $t_f = 2.0$ s, $t_i = 1.0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(1.0\,\text{s}) + \frac{\pi}{2}\right]\right\}\right| = \boxed{3.5\,\text{cm}}$$

35 •• The period of a particle oscillating in simple harmonic motion is 8.0 s. At $t = 0$, the particle is at rest at $x = A = 10$ cm. (*a*) Sketch x as a function of t. (*b*) Find the distance traveled in the first, second, third, and fourth second after $t = 0$.

Picture the Problem The position of the particle as a function of time is given by $x = (10\,\text{cm})\cos(\omega t + \delta)$. We can determine the angular frequency ω from the period of the motion and the phase constant δ from the initial position and velocity. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of t:

$$x = (10\,\text{cm})\cos(\omega t + \delta) \qquad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0\,\text{s}} = \frac{\pi}{4}\text{s}^{-1}$$

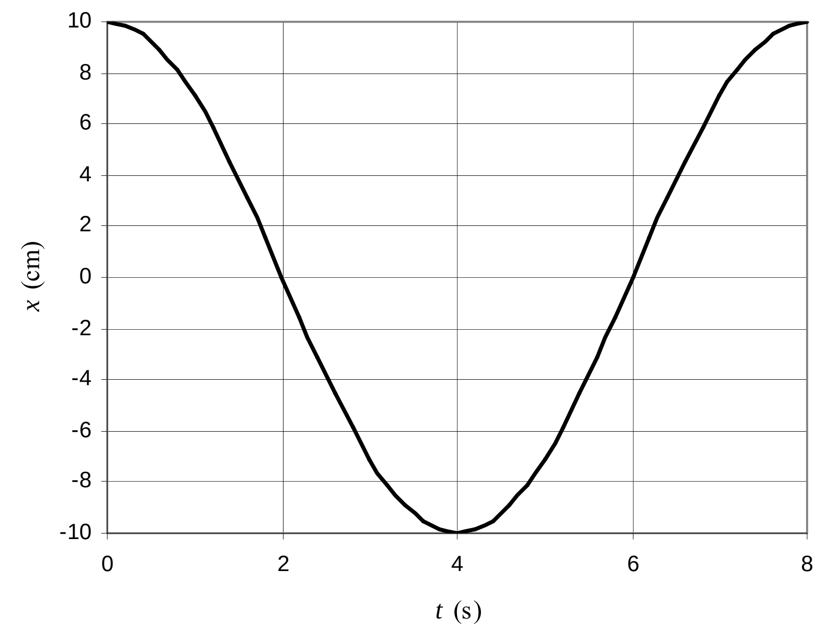
Find the phase constant of the motion:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right) = \tan^{-1}\left(-\frac{0}{x_0\omega}\right) = 0$$

Substitute in equation (1) to obtain:

$$x = (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t\right]$$

(a) A graph of $x=(10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t\right]$ follows:



(b) Express the distance the particle travels in terms of t_f and t_i:

$$\begin{aligned}\Delta x &= \left|(10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_\text{f}\right]-(10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_\text{i}\right]\right| \\ &= \left|(10\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_\text{f}\right]-\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_\text{i}\right]\right\}\right|\end{aligned} \qquad (2)$$

Substitute numerical values in equation (2) and evaluate Δx in each of the given time intervals to obtain:

t_f (s)	t_i (s)	Δx (cm)
1	0	$\boxed{2.9}$
2	1	$\boxed{7.1}$
3	2	$\boxed{7.1}$
4	3	$\boxed{2.9}$

36 •• Military specifications often call for electronic devices to be able to withstand accelerations of up to $10g$ ($10g = 98.1\ \text{m/s}^2$). To make sure that your company's products meet this specification, your manager has told you to use a "shaking table," which can vibrate a device at controlled and adjustable frequencies and amplitudes. If a device is placed on the table and made to

oscillate at an amplitude of 1.5 cm, what should you adjust the frequency to in order to test for compliance with the 10*g* military specification?

Picture the Problem We can use the expression for the maximum acceleration of an oscillator to relate the 10*g* military specification to the compliance frequency.

Express the maximum acceleration of an oscillator:

$$a_{\max} = A\omega^2$$

Express the relationship between the angular frequency and the frequency of the vibrations:

$$\omega = 2\pi f$$

Substitute for ω to obtain:

$$a_{\max} = 4\pi^2 A f^2 \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{a_{\max}}{A}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{98.1\,\text{m/s}^2}{1.5\times10^{-2}\,\text{m}}} = \boxed{13\,\text{Hz}}$$

37 •• [SSM] The position of a particle is given by $x = 2.5 \cos \pi t$, where x is in meters and t is in seconds. (*a*) Find the maximum speed and maximum acceleration of the particle. (*b*) Find the speed and acceleration of the particle when $x = 1.5$ m.

Picture the Problem The position of the particle is given by $x = A\cos\omega t$, where $A = 2.5$ m and $\omega = \pi$ rad/s. The velocity is the time derivative of the position and the acceleration is the time derivative of the velocity.

(*a*) The velocity is the time derivative of the position and the acceleration is the time derivative of the acceleration:

$$x = A\cos\omega t \Rightarrow v = \frac{dx}{dt} = -\omega A\sin\omega t$$

and $a = \dfrac{dv}{dt} = -\omega^2 A\cos\omega t$

The maximum value of $\sin\omega t$ is +1 and the minimum value of $\sin\omega t$ is −1. A and ω are positive constants:

$$v_{\max} = A\omega = (2.5\,\text{m})(\pi\,\text{s}^{-1}) = \boxed{7.9\,\text{m/s}}$$

The maximum value of $\cos\omega t$ is +1 and the minimum value of $\cos\omega t$ is −1:

$$a_{\max} = A\omega^2 = (2.5\,\text{m})(\pi\,\text{s}^{-1})^2 = \boxed{25\,\text{m/s}^2}$$

(*b*) Use the Pythagorean identity $\sin^2 \omega t + \cos^2 \omega t = 1$ to eliminate *t* from the equations for *x* and *v*:

$$\frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1 \Rightarrow |v| = \omega\sqrt{A^2 - x^2}$$

Substitute numerical values and evaluate $|v(1.5\text{ m})|$:

$$|v(1.5\text{ m})| = (\pi\text{ rad/s})\sqrt{(2.5\text{ m})^2 - (1.5\text{ m})^2} = \boxed{6.3\text{ m/s}}$$

Substitute *x* for $A\cos\omega t$ in the equation for *a* to obtain:

$$a = -\omega^2 x$$

Substitute numerical values and evaluate *a*:

$$a = -(\pi\text{ rad/s})^2(1.5\text{ m}) = \boxed{-15\text{ m/s}^2}$$

38 ••• (*a*) Show that $A_0 \cos(\omega t + \delta)$ can be written as $A_s \sin(\omega t) + A_c \cos(\omega t)$, and determine A_s and A_c in terms of A_0 and δ. (*b*) Relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.

Picture the Problem We can use the formula for the cosine of the sum of two angles to write $x = A_0 \cos(\omega t + \delta)$ in the desired form. We can then evaluate *x* and dx/dt at $t = 0$ to relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.

(*a*) Apply the trigonometric identity $\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$ to obtain:

$$\begin{aligned} x &= A_0\cos(\omega t + \delta) \\ &= A_0[\cos\omega t\cos\delta - \sin\omega t\sin\delta] \\ &= -A_0\sin\delta\sin\omega t + A_0\cos\delta\cos\omega t \\ &= \boxed{A_s\sin\omega t + A_c\cos\omega t} \end{aligned}$$

provided
$A_s = -A_0\sin\delta$ and $A_c = A_0\cos\delta$

(*b*) When $t = 0$:

$$x(0) = \boxed{A_0\cos\delta = A_c}$$

Evaluate dx/dt:

$$\begin{aligned} v &= \frac{dx}{dt} = \frac{d}{dt}[A_s\sin\omega t + A_c\cos\omega t] \\ &= A_s\omega\cos\omega t - A_c\omega\sin\omega t \end{aligned}$$

Evaluate $v(0)$ to obtain:

$$v(0) = \omega A_s = \boxed{-\omega A_0\sin\delta}$$

Simple Harmonic Motion as Related to Circular Motion

39 • **[SSM]** A particle moves at a constant speed of 80 cm/s in a circle of radius 40 cm centered at the origin. (*a*) Find the frequency and period of the *x* component of its position. (*b*) Write an expression for the *x* component of its position as a function of time *t*, assuming that the particle is located on the +*y*-axis at time $t = 0$.

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The frequency of the motion is the reciprocal of its period and the *x*-component of the particle's position is given by $x = A\cos(\omega t + \delta)$. We can use the initial position of the particle to determine the phase constant δ.

(*a*) Use the definition of speed to find the period of the motion:

$$T = \frac{2\pi r}{v} = \frac{2\pi(0.40\,\text{m})}{0.80\,\text{m/s}} = 3.14 = \boxed{3.1\,\text{s}}$$

Because the frequency and the period are reciprocals of each other:

$$f = \frac{1}{T} = \frac{1}{3.14\,\text{s}} = \boxed{0.32\,\text{Hz}}$$

(*b*) Express the *x* component of the position of the particle:

$$x = A\cos(\omega t + \delta) = A\cos(2\pi f t + \delta) \quad (1)$$

The initial condition on the particle's position is:

$$\boldsymbol{x}(0) = 0$$

Substitute in the expression for *x* to obtain:

$$0 = A\cos\delta \Rightarrow \delta = \cos^{-1}(0) = \frac{\pi}{2}$$

Substitute for *A*, ω, and δ in equation (1) to obtain:

$$x = \boxed{(40\,\text{cm})\cos\left[\left(2.0\,\text{s}^{-1}\right)t + \frac{\pi}{2}\right]}$$

40 • A particle moves in a 15-cm-radius circle centered at the origin and completes 1.0 rev every 3.0 s. (*a*) Find the speed of the particle. (*b*) Find its angular speed ω. (*c*) Write an equation for the *x* component of the position of the particle as a function of time *t*, assuming that the particle is on the –*x* axis at time $t = 0$.

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The angular frequency of the motion is 2π times the reciprocal of its period and the *x*-component of the particle's position is given by $x = A\cos(\omega t + \delta)$. We can use the initial position of the particle to determine the phase constant δ.

(*a*) Use the definition of speed to express and evaluate the speed of the particle:

$$v = \frac{2\pi r}{T} = \frac{2\pi(15\,\text{cm})}{3.0\,\text{s}} = \boxed{31\,\text{cm/s}}$$

(*b*) The angular speed of the particle is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.0\,\text{s}} = \boxed{\frac{2\pi}{3}\,\text{rad/s}}$$

(*c*) Express the *x* component of the position of the particle:

$$x = A\cos(\omega t + \delta) \qquad (1)$$

The initial condition on the particle's position is:

$$x(0) = -A$$

Substituting for *x* in equation (1) yields:

$$-A = A\cos\delta \Rightarrow \delta = \cos^{-1}(-1) = \pi$$

Substitute for *A*, ω, and δ in equation (1) to obtain:

$$x = \boxed{(15\,\text{cm})\cos\left(\left(\frac{2\pi}{3}\,\text{s}^{-1}\right)t + \pi\right)}$$

Energy in Simple Harmonic Motion

41 • A 2.4-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring of force constant k = 4.5 kN/m. The other end of the spring is held stationary. The spring is stretched 10 cm from equilibrium and released. Find the system's total mechanical energy.

Picture the Problem The total mechanical energy of the object is given by $E_{\text{tot}} = \frac{1}{2}kA^2$, where *A* is the amplitude of the object's motion.

The total mechanical energy of the system is given by:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E_{tot}:

$$E_{\text{tot}} = \tfrac{1}{2}(4.5\,\text{kN/m})(0.10\,\text{m})^2 = \boxed{23\,\text{J}}$$

42 • Find the total energy of a system consisting of a 3.0-kg object on a frictionless horizontal surface oscillating with an amplitude of 10 cm and a frequency of 2.4 Hz at the end of a horizontal spring.

Picture the Problem The total energy of an oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{tot} = \frac{1}{2}mv_{max}^2$. Its maximum speed, in turn, can be expressed in terms of its angular frequency and the amplitude of its motion.

Express the total energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{max}^2$$

The maximum speed v_{max} of the oscillating object is given by:

$$v_{max} = A\omega = 2\pi Af$$

Substitute for v_{max} to obtain:

$$E = \tfrac{1}{2}m(2\pi Af)^2 = 2mA^2\pi^2 f^2$$

Substitute numerical values and evaluate E:

$$E = 2(3.0\,\text{kg})(0.10\,\text{m})^2\pi^2\left(2.4\,\text{s}^{-1}\right)^2 = \boxed{3.4\,\text{J}}$$

43 • **[SSM]** A 1.50-kg object on a frictionless horizontal surface oscillates at the end of a spring of force constant k = 500 N/m. The object's maximum speed is 70.0 cm/s. (*a*) What is the system's total mechanical energy? (*b*) What is the amplitude of the motion?

Picture the Problem The total mechanical energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{tot} = \frac{1}{2}mv_{max}^2$. Its total energy is also given by $E_{tot} = \frac{1}{2}kA^2$. We can equate these expressions to obtain an expression for A.

(*a*) Express the total mechanical energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{max}^2$$

Substitute numerical values and evaluate E:

$$E = \tfrac{1}{2}(1.50\,\text{kg})(0.700\,\text{m/s})^2 = 0.3675\,\text{J} = \boxed{0.368\,\text{J}}$$

(*b*) Express the total mechanical energy of the object in terms of the amplitude of its motion:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E_{\text{tot}}}{k}}$$

Substitute numerical values and evaluate *A*:

$$A = \sqrt{\frac{2(0.3675\,\text{J})}{500\,\text{N/m}}} = \boxed{3.83\,\text{cm}}$$

44 • A 3.0-kg object on a frictionless horizontal surface is oscillating on the end of a spring that has a force constant equal to 2.0 kN/m and a total mechanical energy of 0.90 J. (*a*) What is the amplitude of the motion? (*b*) What is the maximum speed?

Picture the Problem The total mechanical energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2$. Its total energy is also given by $E_{\text{tot}} = \tfrac{1}{2}kA^2$. We can solve the latter equation to find *A* and solve the former equation for v_{max}.

(*a*) Express the total mechanical energy of the object as a function of the amplitude of its motion:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E_{\text{tot}}}{k}}$$

Substitute numerical values and evaluate *A*:

$$A = \sqrt{\frac{2(0.90\,\text{J})}{2000\,\text{N/m}}} = \boxed{3.0\,\text{cm}}$$

(*b*) Express the total mechanical energy of the object in terms of its maximum speed:

$$E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2 \Rightarrow v_{\text{max}} = \sqrt{\frac{2E_{\text{tot}}}{m}}$$

Substitute numerical values and evaluate v_{max}:

$$v_{\text{max}} = \sqrt{\frac{2(0.90\,\text{J})}{3.0\,\text{kg}}} = \boxed{77\,\text{cm/s}}$$

45 • An object on a frictionless horizontal surface oscillates at the end of a spring with an amplitude of 4.5 cm. Its total mechanical energy is 1.4 J. What is the force constant of the spring?

Picture the Problem The total mechanical energy of the object is given by $E_{\text{tot}} = \tfrac{1}{2}kA^2$. We can solve this equation for the force constant *k* and substitute the numerical data to determine its value.

Express the total mechanical energy of the oscillator as a function of the amplitude of its motion:

$$E_{tot} = \tfrac{1}{2}kA^2 \Rightarrow k = \frac{2E_{tot}}{A^2}$$

Substitute numerical values and evaluate k:

$$k = \frac{2(1.4\,\text{J})}{(0.045\,\text{m})^2} = \boxed{1.4\ \text{kN/m}}$$

46 •• A 3.0-kg object on a frictionless horizontal surface oscillates at the end of a spring with an amplitude of 8.0 cm. Its maximum acceleration is 3.5 m/s^2. Find the total mechanical energy.

Picture the Problem The total mechanical energy of the system is the sum of the potential and kinetic energies. That is, $E_{tot} = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$. Newton's second law relates the acceleration to the displacement. That is, $-kx = ma$. In addition, when $x = A$, $v = 0$. Use these equations to solve E_{tot} in terms of the given parameters m, A and a_{max}.

The total mechanical energy is the sum of the potential and kinetic energies. We don't know k so we need an equation relating k to one or more of the given parameters:

$$E_{tot} = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$$

The force exerted by the spring equals the mass of the object multiplied by its acceleration:

$$-kx = ma \Rightarrow k = -\frac{ma}{x}$$

When $x = -A$, $a = a_{max}$. Thus,

$$k = -\frac{ma_{max}}{-A} = \frac{ma_{max}}{A}$$

Substitute to obtain:

$$E_{tot} = \tfrac{1}{2}\frac{ma_{max}}{A}x^2 + \tfrac{1}{2}mv^2$$

When $x = A$, $v = 0$. Substitute to obtain:

$$E_{tot} = \tfrac{1}{2}\frac{ma_{max}}{A}A^2 + 0 = \tfrac{1}{2}ma_{max}A$$

Substitute numerical values and evaluate E_{tot}:

$$E_{tot} = \tfrac{1}{2}(3.0\,\text{kg})(3.5\,\text{m/s}^2)(0.080\,\text{m}) = \boxed{0.42\,\text{J}}$$

Simple Harmonic Motion and Springs

47 • A 2.4-kg object on a frictionless horizontal surface is attached to a horizontal spring that has a force constant 4.5 kN/m. The spring is stretched 10 cm from equilibrium and released. What are (*a*) the frequency of the motion, (*b*) the period, (*c*) the amplitude, (*d*) the maximum speed, and (*e*) the maximum acceleration? (*f*) When does the object first reach its equilibrium position? What is its acceleration at this time?

Picture the Problem The frequency of the object's motion is given by $f = \dfrac{1}{2\pi}\sqrt{k/m}$ and its period is the reciprocal of *f*. The maximum velocity and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{4.5\,\text{kN/m}}{2.4\,\text{kg}}} = 6.89\ \text{Hz} = \boxed{6.9\,\text{Hz}}$$

(*b*) The period of the motion to is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{6.89\,\text{s}^{-1}} = 0.145\ \text{s} = \boxed{0.15\,\text{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 10 cm:

$$A = \boxed{10\,\text{cm}}$$

(*d*) The object's maximum speed is given by:

$$v_{max} = A\omega = 2\pi f A$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = 2\pi\left(6.89\,\text{s}^{-1}\right)\left(0.10\,\text{m}\right) = 4.33\,\text{m/s} = \boxed{4.3\,\text{m/s}}$$

(*e*) The object's maximum acceleration is given by:

$$a_{max} = A\omega^2 = \omega v_{max} = 2\pi f v_{max}$$

Substitute numerical values and evaluate a_{max}:

$$a_{max} = 2\pi\left(6.89\,\text{s}^{-1}\right)\left(4.33\,\text{m/s}\right) = \boxed{1.9\times10^{2}\ \text{m/s}^{2}}$$

(*f*) The object first reaches its equilibrium when:

$$t = \tfrac{1}{4}T = \tfrac{1}{4}\left(0.145\,\text{s}\right) = \boxed{36\,\text{ms}}$$

Because the resultant force acting on the object as it passes through its equilibrium position is zero, the acceleration of the object is:

$$a = \boxed{0}$$

48 • A 5.00-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring that has a force constant k = 700 N/m. The spring is stretched 8.00 cm from equilibrium and released. What are (*a*) the frequency of the motion, (*b*) the period, (*c*) the amplitude, (*d*) the maximum speed, and (*e*) the maximum acceleration? (*f*) When does the object first reach its equilibrium position? What is its acceleration at this time?

Picture the Problem The frequency of the object's motion is given by $f = \frac{1}{2\pi}\sqrt{k/m}$ and its period is the reciprocal of *f*. The maximum speed and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{700\,\text{N/m}}{5.00\,\text{kg}}} = 1.883\,\text{Hz} = \boxed{1.88\,\text{Hz}}$$

(*b*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{1.883\,\text{s}^{-1}} = 0.5310\,\text{s} = \boxed{0.531\,\text{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 8.00 cm:

$$A = \boxed{8.00\,\text{cm}}$$

(*d*) The object's maximum speed is given by:

$$v_{max} = A\omega = 2\pi f A$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = 2\pi\left(1.883\,s^{-1}\right)\left(0.0800\,m\right) = 0.9465\,m/s = \boxed{0.947\,m/s}$$

(*e*) The object's maximum acceleration is given by:

$$a_{max} = A\omega^2 = \omega v_{max} = 2\pi f v_{max}$$

Substitute numerical values and evaluate a_{max}:

$$a_{max} = 2\pi\left(1.883\,s^{-1}\right)\left(0.9465\,m/s\right) = \boxed{11.2\,m/s^2}$$

(*f*) The object first reaches its equilibrium when:

$$t = \tfrac{1}{4}T = \tfrac{1}{4}\left(0.5310\,s\right) = \boxed{0.133\,s}$$

Because the resultant force acting on the object as it passes through its equilibrium point is zero, the acceleration of the object is $a = \boxed{0.}$

49 • [SSM] A 3.0-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring and oscillates with an amplitude A = 10 cm and a frequency 2.4 Hz. (*a*) What is the force constant of the spring? (*b*) What is the period of the motion? (*c*) What is the maximum speed of the object? (*d*) What is the maximum acceleration of the object?

Picture the Problem (*a*) The angular frequency of the motion is related to the force constant of the spring through $\omega^2 = k/m$. (*b*) The period of the motion is the reciprocal of its frequency. (*c*) and (*d*) The maximum speed and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) Relate the angular frequency of the motion to the force constant of the spring:

$$\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = 4\pi^2 f^2 m$$

Substitute numerical values to obtain:

$$k = 4\pi^2\left(2.4\,s^{-1}\right)^2\left(3.0\,kg\right) = 682\,N/m = \boxed{0.68\,kN/m}$$

(*b*) Relate the period of the motion to its frequency:

$$T = \frac{1}{f} = \frac{1}{2.4\,\text{s}^{-1}} = 0.417\,\text{s} = \boxed{0.42\,\text{s}}$$

(*c*) The maximum speed of the object is given by:

$$v_{\max} = A\omega = 2\pi f A$$

Substitute numerical values and evaluate $v_{\max}$:

$$v_{\max} = 2\pi\left(2.4\,\text{s}^{-1}\right)\left(0.10\,\text{m}\right) = 1.51\,\text{m/s} = \boxed{1.5\,\text{m/s}}$$

(*d*) The maximum acceleration of the object is given by:

$$a_{\max} = A\omega^2 = 4\pi^2 f^2 A$$

Substitute numerical values and evaluate $a_{\max}$:

$$a_{\max} = 4\pi^2\left(2.4\,\text{s}^{-1}\right)^2\left(0.10\,\text{m}\right) = \boxed{23\,\text{m/s}^2}$$

50 • An 85.0-kg person steps into a car of mass 2400 kg, causing it to sink 2.35 cm on its springs. If started into vertical oscillation, and assuming no damping, at what frequency will the car and passenger vibrate on these springs?

Picture the Problem We can find the frequency of vibration of the car-and-passenger system using $f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$, where M is the total mass of the system. The force constant of the spring can be determined from the compressing force and the amount of compression.

Express the frequency of the car-and-passenger system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

The force constant is given by:

$$k = \frac{F}{\Delta x} = \frac{mg}{\Delta x}$$

where m is the person's mass.

Substitute for k in the expression for f to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{M\Delta x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{\left(85.0\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{\left(2485\,\text{kg}\right)\left(2.35\times10^{-2}\,\text{m}\right)}} = \boxed{0.601\,\text{Hz}}$$

51 • A 4.50-kg object oscillates on a horizontal spring with an amplitude of 3.80 cm. The object's maximum acceleration is 26.0 m/s^2. Find (*a*) the force constant of the spring, (*b*) the frequency of the object, and (*c*) the period of the motion of the object.

Picture the Problem (*a*) We can relate the force constant *k* to the maximum acceleration by eliminating ω^2 between $\omega^2 = k/m$ and $a_{max} = A\omega^2$. (*b*) We can find the frequency *f* of the motion by substituting ma_{max}/A for *k* in $f = \frac{1}{2\pi}\sqrt{k/m}$. (*c*) The period of the motion is the reciprocal of its frequency. Assume that friction is negligible.

(*a*) Relate the angular frequency of the motion to the force constant and the mass of the oscillator:

$$\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 m$$

Relate the object's maximum acceleration to its angular frequency and amplitude and solve for the square of the angular frequency:

$$a_{max} = A\omega^2 \Rightarrow \omega^2 = \frac{a_{max}}{A} \qquad (1)$$

Substitute for ω^2 to obtain:

$$k = \frac{ma_{max}}{A}$$

Substitute numerical values and evaluate *k*:

$$k = \frac{(4.50\,\text{kg})(26.0\,\text{m/s}^2)}{3.80\times10^{-2}\,\text{m}} = \boxed{3.08\,\text{kN/m}}$$

(*b*) Replace ω in equation (1) by $2\pi f$ and solve for f to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{a_{max}}{A}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{26.0\,\text{m/s}^2}{3.80\times10^{-2}\,\text{m}}} = 4.163\,\text{Hz} = \boxed{4.16\,\text{Hz}}$$

(*c*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{4.163\,\text{s}^{-1}} = \boxed{0.240\,\text{s}}$$

52 •• An object of mass *m* is suspended from a vertical spring of force constant 1800 N/m. When the object is pulled down 2.50 cm from equilibrium and released from rest, the object oscillates at 5.50 Hz. (*a*) Find *m*. (*b*) Find the amount the spring is stretched from its unstressed length when the object is in

equilibrium. (*c*) Write expressions for the displacement x, the velocity v_x, and the acceleration a_x as functions of time t.

Picture the Problem Choose a coordinate system in which upward is the $+y$ direction. We can find the mass of the object using $m = k/\omega^2$. We can apply a condition for translational equilibrium to the object when it is at its equilibrium position to determine the amount the spring has stretched from its natural length. Finally, we can use the initial conditions to determine A and δ and express $x(t)$ and then differentiate this expression to obtain $v_x(t)$ and $a_x(t)$.

(*a*) Express the angular frequency of the system in terms of the mass of the object fastened to the vertical spring and solve for the mass of the object:

$$\omega^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega^2}$$

Express ω^2 in terms of f:

$$\omega^2 = 4\pi^2 f^2$$

Substitute for $\boldsymbol{\omega}^2$ to obtain:

$$m = \frac{k}{4\pi^2 f^2}$$

Substitute numerical values and evaluate m:

$$m = \frac{1800\,\text{N/m}}{4\pi^2\left(5.50\,\text{s}^{-1}\right)^2} = 1.507\text{ kg}$$

$$= \boxed{1.51\,\text{kg}}$$

(*b*) Letting Δx represent the amount the spring is stretched from its natural length when the object is in equilibrium, apply $\sum F_y = 0$ to the object when it is in equilibrium:

$$k\Delta x - mg = 0 \Rightarrow \Delta x = \frac{mg}{k}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{(1.507\text{ kg})\left(9.81\,\text{m/s}^2\right)}{1800\text{ N/m}} = \boxed{8.21\,\text{mm}}$$

(*c*) Express the position of the object as a function of time:

$$x = A\cos(\omega t + \delta)$$

Use the initial conditions $x_0 = -2.50$ cm and $v_0 = 0$ to find δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) = \tan^{-1}(0) = \pi$$

Evaluate ω:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1800\,\text{N/m}}{1.507\,\text{kg}}} = 34.56\,\text{rad/s}$$

Substituting numerical values in the expression for x gives:

$$x = (2.50\,\text{cm})\cos[(34.56\,\text{rad/s})t + \pi] = \boxed{-(2.50\,\text{cm})\cos[(34.6\,\text{rad/s})t]}$$

Differentiate $x(t)$ to obtain v_x:

$$v_x = (86.39\,\text{cm/s})\sin[(34.56\,\text{rad/s})t] = \boxed{(86.4\,\text{cm/s})\sin[(34.6\,\text{rad/s})t]}$$

Differentiate $v(t)$ to obtain a_x:

$$a_x = (29.86\,\text{m/s}^2)\cos[(34.56\,\text{rad/s})t] = \boxed{(29.9\,\text{m/s}^2)\cos[(34.6\,\text{rad/s})t]}$$

53 •• An object is hung on the end of a vertical spring and is released from rest with the spring unstressed. If the object falls 3.42 cm before first coming to rest, find the period of the resulting oscillatory motion.

Picture the Problem Let the system include the object and the spring. Then, the net external force acting on the system is zero. Choose E_i = 0 and apply the conservation of mechanical energy to the system.

Express the period of the motion in terms of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply conservation of energy to the system:

$$E_\text{i} = E_\text{f} \Rightarrow 0 = U_\text{g} + U_\text{spring}$$

Substituting for U_g and U_spring yields:

$$0 = -mg\Delta x + \tfrac{1}{2}k(\Delta x)^2 \Rightarrow \frac{k}{m} = \frac{2g}{\Delta x}$$

and $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2g}{\Delta x}}$

Substituting for ω in equation (1) and simplifying yields:

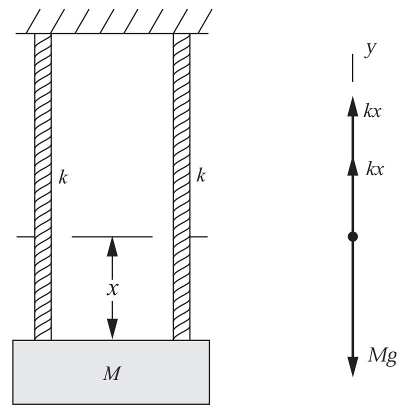
$$T = \frac{2\pi}{\sqrt{\frac{2g}{\Delta x}}} = 2\pi\sqrt{\frac{\Delta x}{2g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{3.42\,\text{cm}}{2(9.81\,\text{m/s}^2)}} = \boxed{0.262\,\text{s}}$$

54 •• A suitcase of mass 20 kg is hung from two bungee cords, as shown in Figure 14-27. Each cord is stretched 5.0 cm when the suitcase is in equilibrium. If the suitcase is pulled down a little and released, what will be its oscillation frequency?

Picture the Problem The diagram shows the stretched bungee cords supporting the suitcase under equilibrium conditions. We can use $f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$ to express the frequency of the suitcase in terms of the effective ″spring″ constant k_{eff} and apply the condition for translational equilibrium to the suitcase to find k_{eff}.



Express the frequency of the suitcase oscillator:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}} \qquad (1)$$

Apply $\sum F_y = 0$ to the suitcase to obtain:

$$kx + kx - Mg = 0$$

or

$$2kx - Mg = 0$$

or

$$k_{\text{eff}}x - Mg = 0 \Rightarrow k_{\text{eff}} = \frac{Mg}{x}$$

where $k_{\text{eff}} = 2k$

Substitute for k_{eff} in equation (1) to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{0.050\,\text{m}}} = \boxed{2.2\,\text{Hz}}$$

55 •• A 0.120-kg block is suspended from a spring. When a small pebble of mass 30 g is placed on the block, the spring stretches an additional 5.0 cm. With the pebble on the block, the spring oscillates with an amplitude of 12 cm. (*a*) What is the frequency of the motion? (*b*) How long does the block take to travel from its lowest point to its highest point? (*c*) What is the net force on the pebble when it is at the point of maximum upward displacement?

Picture the Problem (*a*) The frequency of the motion of the pebble and block depends on the force constant of the spring and the mass of the pebble plus block. The force constant can be determined from the equilibrium of the system when

the spring is stretched additionally by the addition of the pebble to the block.
(*b*) The time required for the block to travel from its lowest point to its highest point is half its period. (*c*) We can apply Newton's second law to the pebble to find the net force on the pebble when it is at the point of maximum upward displacement.

(*a*) Express the frequency of the motion in terms of k and m_{tot}:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{tot}}} \qquad (1)$$

where m_{tot} is the total mass suspended from the spring.

Apply $\sum F_y = 0$ to the pebble when it is at its equilibrium position:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

Substitute for k in equation (1) to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{\Delta y m_{tot}}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{(0.030\ \text{kg})(9.81\ \text{m/s}^2)}{(0.050\ \text{m})(0.15\ \text{kg})}} = 0.997\ \text{Hz} = \boxed{1.0\ \text{Hz}}$$

(*b*) The time it takes for the block to travel from its lowest point to its highest point is one-half its period:

$$t = \tfrac{1}{2}T = \frac{1}{2f} = \frac{1}{2(0.997\ \text{s}^{-1})} = \boxed{0.50\ \text{s}}$$

(*c*) Apply $\sum F_y = 0$ to the pebble when it is at the point of maximum upward displacement:

$$F_{net} = ma_{max} = -m(-\omega^2 A) = m\omega^2 A = m(2\pi f)^2 A = 4\pi^2 f^2 mA$$

Substituting for ω and simplifying gives:

$$F_{net} = m(2\pi f)^2 A = 4\pi^2 f^2 mA$$

Substitute numerical values and evaluate F_{net}:

$$F_{net} = 4\pi^2 (0.997\ \text{s}^{-1})^2 (0.030\ \text{kg})(0.12\ \text{m}) = \boxed{0.14\ \text{N}}$$

56 •• Referring to Problem 55, find the maximum amplitude of oscillation at which the pebble will remain in contact with the block.

Picture the Problem We can use the maximum acceleration of the oscillator to express a_{max} in terms of A, k, and m. k can be determined from the equilibrium of the system when the spring is stretched additionally by the addition of the stone to the mass. If the stone is to remain in contact with the block, the block's maximum downward acceleration must not exceed g.

Express the maximum acceleration of the oscillator in terms of its angular frequency and amplitude of the motion:

$$a_{max} = A\omega^2$$

Relate ω^2 to the force constant of the spring and the mass of the block-plus-stone:

$$\omega^2 = \frac{k}{m_{tot}}$$

Substitute for ω^2 to obtain:

$$a_{max} = A\frac{k}{m_{tot}} \qquad (1)$$

Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

where Δy is the additional distance the spring stretched when the stone was placed on the block.

Substitute for k in equation (1) to obtain:

$$a_{max} = A\left(\frac{mg}{\Delta y m_{tot}}\right)$$

Set $a_{max} = g$ and solve for A_{max}:

$$A_{max} = \frac{\Delta y m_{tot}}{mg}g = \frac{m_{tot}}{m}\Delta y$$

Substitute numerical values and evaluate A_{max}:

$$A_{max} = \left(\frac{0.15\text{ kg}}{0.030\text{ kg}}\right)(0.050\text{ m}) = \boxed{25\text{ cm}}$$

57 •• An object of mass 2.0 kg is attached to the top of a vertical spring that is anchored to the floor. The unstressed length of the spring is 8.0 cm and the length of the spring when the object is in equilibrium is 5.0 cm. When the object is resting at its equilibrium position, it is given a sharp downward blow with a hammer so that its initial speed is 0.30 m/s. (*a*) To what maximum height above the floor does the object eventually rise? (*b*) How long does it take for the object to reach its maximum height for the first time? (*c*) Does the spring ever become unstressed? What minimum initial speed must be given to the object for the spring to be unstressed at some time?

Picture the Problem (*a*) The maximum height above the floor to which the object rises is the sum of its initial distance from the floor and the amplitude of its motion. We can find the amplitude of its motion by relating it to the object's maximum speed. (*b*) Because the object initially travels downward, it will be three-fourths of the way through its cycle when it first reaches its maximum height. (*c*) We can find the minimum initial speed the object would need to be given in order for the spring to become uncompressed by applying conservation of mechanical energy.

(*a*) Relate *h*, the maximum height above the floor to which the object rises, to the amplitude of its motion:

$$h = A + 5.0\ \text{cm} \qquad (1)$$

Relate the maximum speed of the object to the angular frequency and amplitude of its motion and solve for the amplitude:

$$v_{\text{max}} = A\omega$$

or, because $\omega^2 = \dfrac{k}{m}$,

$$A = v_{\text{max}}\sqrt{\frac{m}{k}} \qquad (2)$$

Apply $\sum F_y = 0$ to the object when it is resting at its equilibrium position to obtain:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

Substitute for k in equation (2):

$$A = v_{\text{max}}\sqrt{\frac{m\Delta y}{mg}} = v_{\text{max}}\sqrt{\frac{\Delta y}{g}}$$

Substituting for A in equation (1) yields:

$$h = v_{\text{max}}\sqrt{\frac{\Delta y}{g}} + 5.0\ \text{cm}$$

Substitute numerical values and evaluate *h*:

$$h = 0.30\ \text{m/s}^2\sqrt{\frac{0.030\ \text{m}}{9.81\ \text{m/s}^2}} + 5.0\ \text{cm}$$

$$= \boxed{6.7\ \text{cm}}$$

(*b*) The time required for the object to reach its maximum height the first time is three-fourths its period:

$$t = \tfrac{3}{4}T$$

Express the period of the motion of the oscillator:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\frac{mg}{\Delta y}}} = 2\pi\sqrt{\frac{\Delta y}{g}}$$

Substitute for T in the expression for t to obtain:

$$t = \frac{3}{4}\left(2\pi\sqrt{\frac{\Delta y}{g}}\right) = \frac{3\pi}{2}\sqrt{\frac{\Delta y}{g}}$$

Substitute numerical values and evaluate t:

$$t = \frac{3\pi}{2}\sqrt{\frac{0.030\ \text{m}}{9.81\ \text{m/s}^2}} = \boxed{0.26\ \text{s}}$$

(c) Because $h < 8.0$ cm, the spring is never uncompressed.

Using conservation of energy and letting U_g be zero 5 cm above the floor, relate the height to which the object rises, Δy, to its initial kinetic energy:

$$\Delta K + \Delta U_g + \Delta U_s = 0$$

or, because $K_f = U_i = 0$,

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(L - y_i)^2 = 0$$

Because $\Delta y = L - y_i$:

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(\Delta y)^2 = 0$$

and

$$\tfrac{1}{2}mv_i^2 - mg\Delta y = 0 \Rightarrow v_i = \sqrt{2g\Delta y}$$

Substitute numerical values and evaluate v_i:

$$v_i = \sqrt{2(9.81\ \text{m/s}^2)(3.0\ \text{cm})} = 77\ \text{cm/s}$$

That is, the minimum initial speed that must be given to the object for the spring to be uncompressed at some time is $\boxed{77\ \text{cm/s}}$

58 ••• A winch cable has a cross-sectional area of 1.5 cm^2 and a length of 2.5 m. Young's modulus for the cable is 150 GN/m^2. A 950-kg engine block is hung from the end of the cable. (a) By what length does the cable stretch? (b) If we treat the cable as a simple spring, what is the oscillation frequency of the engine block at the end of the cable?

Picture the Problem We can relate the elongation of the cable to the load on it using the definition of Young's modulus and use the expression for the frequency of a spring-mass oscillator to find the oscillation frequency of the engine block at the end of the wire.

(*a*) Using the definition of Young's modulus, relate the elongation of the cable to the applied stress:

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta\ell/\ell} \Rightarrow \Delta\ell = \frac{F\ell}{AY} = \frac{Mg\ell}{AY}$$

Substitute numerical values and evaluate $\Delta\ell$:

$$\Delta\ell = \frac{(950\,\text{kg})(9.81\,\text{m/s}^2)(2.5\,\text{m})}{(1.5\,\text{cm}^2)(150\,\text{GN/m}^2)} = 1.0355\,\text{mm} = \boxed{1.0\,\text{mm}}$$

(*b*) Express the oscillation frequency of the wire-engine block system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$$

Express the effective ″spring″ constant of the cable:

$$k_{\text{eff}} = \frac{F}{\Delta\ell} = \frac{Mg}{\Delta\ell}$$

Substitute for k_{eff} to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta\ell}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{1.0355\,\text{mm}}} = \boxed{15\,\text{Hz}}$$

Simple Pendulum Systems

59 • **[SSM]** Find the length of a simple pendulum if its frequency for small amplitudes is 0.75 Hz.

Picture the Problem The frequency of a simple pendulum depends on its length and on the local gravitational field and is given by $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

The frequency of a simple pendulum oscillating with small amplitude is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \Rightarrow L = \frac{g}{4\pi^2 f^2}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{9.81\,\text{m/s}^2}{4\pi^2\left(0.75\,\text{s}^{-1}\right)^2} = \boxed{44\,\text{cm}}$$

60 • Find the length of a simple pendulum if its period for small amplitudes is 5.0 s.

Picture the Problem We can determine the required length of the pendulum from the expression for the period of a simple pendulum.

Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow L = \frac{T^2 g}{4\pi^2}$$

Substitute numerical values and evaluate L:

$$L = \frac{(5.0\,\text{s})^2(9.81\,\text{m/s}^2)}{4\pi^2} = \boxed{6.2\,\text{m}}$$

61 • What would be the period of the pendulum in Problem 60 if the pendulum were on the moon, where the acceleration due to gravity is one-sixth that on Earth?

Picture the Problem We can find the period of the pendulum from $T = 2\pi\sqrt{L/g_{\text{moon}}}$ where $g_{\text{moon}} = \frac{1}{6}g$ and L = 6.21 m.

Express the period of a simple pendulum on the moon:

$$T = 2\pi\sqrt{\frac{L}{g_{\text{moon}}}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{6.21\,\text{m}}{\frac{1}{6}(9.81\,\text{m/s}^2)}} = \boxed{12\,\text{s}}$$

62 • If the period of a 70.0-cm-long simple pendulum is 1.68 s, what is the value of g at the location of the pendulum?

Picture the Problem We can find the value of g at the location of the pendulum by solving the equation $T = 2\pi\sqrt{L/g}$ for g and evaluating it for the given length and period.

Express the period of a simple pendulum where the gravitational field is g:

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow g = \frac{4\pi^2 L}{T^2}$$

Substitute numerical values and evaluate g:

$$g = \frac{4\pi^2(0.700\,\text{m})}{(1.68\,\text{s})^2} = \boxed{9.79\,\text{m/s}^2}$$

63 • A simple pendulum set up in the stairwell of a 10-story building consists of a heavy weight suspended on a 34.0-m-long wire. What is the period of oscillation?

Picture the Problem We can use $T = 2\pi\sqrt{L/g}$ to find the period of this pendulum.

Express the period of a simple pendulum:

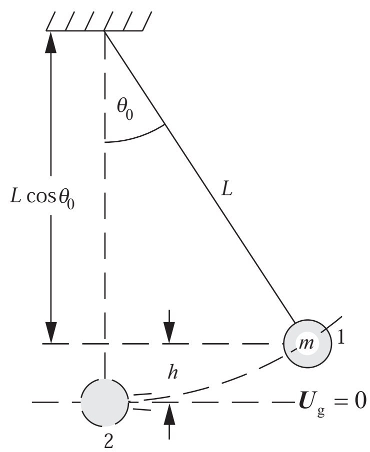
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{34.0\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{11.7\,\text{s}}$$

64 •• Show that the total energy of a simple pendulum undergoing oscillations of small amplitude ϕ_0 (in radians) is $E \approx \frac{1}{2}mgL\phi_0^2$. *Hint: Use the approximation* $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ *for small* ϕ.

Picture the Problem The figure shows the simple pendulum at maximum angular displacement ϕ_0. The total energy of the simple pendulum is equal to its initial gravitational potential energy. We can apply the definition of gravitational potential energy and use the small-angle approximation to show that $E \approx \frac{1}{2}mgL\phi_0^2$.



Express the total energy of the simple pendulum at maximum displacement:

$$E = U_{\text{max displacement}} = mgh$$

Referring to the diagram, express h in terms of L and ϕ_0:

$$h = L - L\cos\phi_0 = L(1 - \cos\phi_0)$$

Substituting for h yields:

$$E = mgL[1 - \cos\phi_0]$$

From the power series expansion for $\cos\phi$, for $\phi << 1$:

$$\cos\phi \approx 1 - \tfrac{1}{2}\phi^2$$

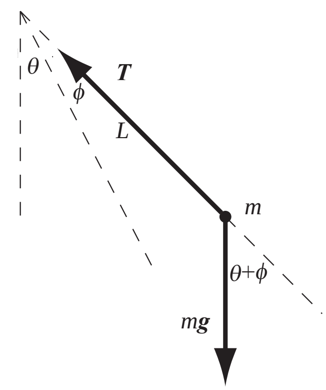
Substitute and simplify to obtain:

$$E = mgL\left[1 - \left(1 - \tfrac{1}{2}\phi_0^2\right)\right] = \boxed{\tfrac{1}{2}mgL\phi_0^2}$$

65 ••• **[SSM]** A simple pendulum of length L is attached to a massive cart that slides without friction down a plane inclined at angle θ with the horizontal, as shown in Figure 14-28. Find the period of oscillation for small oscillations of this pendulum.

Picture the Problem The cart accelerates down the ramp with a constant acceleration of $g\sin\theta$. This happens because the cart is much more massive than the bob, so the motion of the cart is unaffected by the motion of the bob oscillating back and forth. The path of the bob is quite complex in the reference frame of the ramp, but in the reference frame moving with the cart the path of the bob is much simpler—in this frame the bob moves back and forth along a circular arc. To solve this problem we first apply Newton's second law (to the bob) in the inertial reference frame of the ramp. Then we transform to the reference frame moving with the cart in order to exploit the simplicity of the motion in that frame.

Draw the free-body diagram for the bob. Let ϕ denote the angle that the string makes with the normal to the ramp. The forces on the bob are the tension force and the force of gravity:



Apply Newton's second law to the bob, labeling the acceleration of the bob relative to the ramp $\vec{a}_{\mathrm{BR}}$:

$$\vec{T} + m\vec{g} = m\vec{a}_{\mathrm{BR}}$$

The acceleration of the bob relative to the ramp is equal to the acceleration of the bob relative to the cart plus the acceleration of the cart relative to the ramp:

$$\vec{a}_{\mathrm{BR}} = \vec{a}_{\mathrm{BC}} + \vec{a}_{\mathrm{CR}}$$

Substitute for $\vec{a}_{\mathrm{BR}}$ in $\vec{T} + m\vec{g} = m\vec{a}_{\mathrm{BR}}$:

$$\vec{T} + m\vec{g} = m\left(\vec{a}_{\mathrm{BC}} + \vec{a}_{\mathrm{CR}}\right)$$

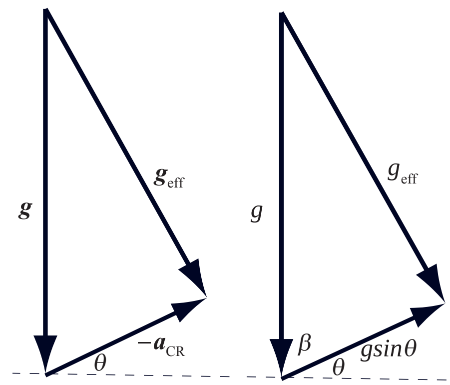
Rearrange terms and label $\vec{g} - \vec{a}_{\mathrm{CR}}$ as $\vec{g}_{\mathrm{eff}}$, where $\vec{g}_{\mathrm{eff}}$ is the acceleration, relative to the cart, of an object in free fall. (If the tension force is set to zero the bob is in free fall.):

$$\vec{T} + m\left(\vec{g} - \vec{a}_{\mathrm{CR}}\right) = m\vec{a}_{\mathrm{BC}}$$

Label $\vec{g} - \vec{a}_{\mathrm{CR}}$ as $\vec{g}_{\mathrm{eff}}$ to obtain

$$\vec{T} + m\vec{g}_{\mathrm{eff}} = m\vec{a}_{\mathrm{BC}} \qquad (1)$$

To find the magnitude of $\vec{g}_{\text{eff}}$, first draw the vector addition diagram representing the equation $\vec{g}_{\text{eff}} = \vec{g} - \vec{a}_{\text{CR}}$. Recall that $a_{\text{CR}} = g\sin\theta$:



From the diagram, find the magnitude of $\vec{g}_{\text{eff}}$. Use the law of cosines to obtain:

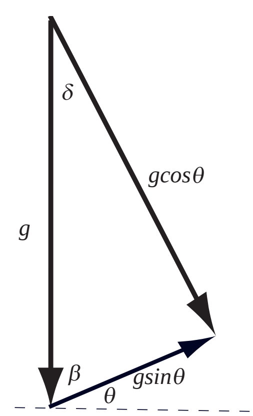
$$g_{\text{eff}}^2 = g^2 + g^2\sin^2\theta - 2g(g\sin\theta)\cos\beta$$

But $\cos\beta = \sin\theta$, so

$$\begin{aligned} g_{\text{eff}}^2 &= g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta \\ &= g^2(1-\sin^2\theta) = g^2\cos^2\theta \end{aligned}$$

Thus $g_{\text{eff}} = g\cos\theta$

To find the direction of $\vec{g}_{\text{eff}}$, first redraw the vector addition diagram as shown:

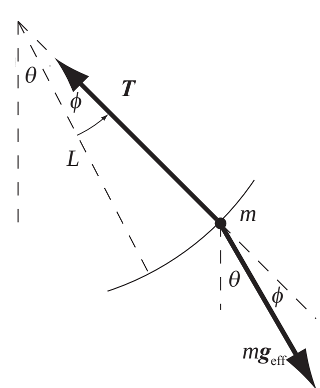


From the diagram find the direction of $\vec{g}_{\text{eff}}$. Use the law of cosines again and solve for δ:

$$g^2\sin^2\theta = g^2 + g^2\cos^2\theta - 2g^2\cos\theta\cos\delta$$

and so $\delta = \theta$

To find an equation for the motion of the bob draw the free-body diagram for the "forces" that appear in equation (1). Draw the path of the bob in the reference frame moving with the cart:



Take the tangential components of each vector in equation (1) in the frame of the cart yields. The tangential component of the acceleration is equal to the radius of the circle times the angular acceleration $(a_t = r\alpha)$:

$$0 - mg_{\text{eff}}\sin\phi = mL\frac{d^2\phi}{dt^2}$$

where L is the length of the string and $\frac{d^2\phi}{dt^2}$ is the angular acceleration of the bob. The positive tangential "direction" is counterclockwise.

Rearranging this equation yields:

$$mL\frac{d^2\phi}{dt^2} + mg_{\text{eff}}\sin\phi = 0 \qquad (2)$$

For small oscillations of the pendulum:

$|\phi| << 1$ and $\sin\phi \approx \phi$

Substituting for $\sin\phi$ in equation (2) yields:

$$mL\frac{d^2\phi}{dt^2} + mg_{\text{eff}}\phi = 0$$

or

$$\frac{d^2\phi}{dt^2} + \frac{g_{\text{eff}}}{L}\phi = 0 \qquad (3)$$

Equation (3) is the equation of motion for simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{g_{\text{eff}}}{L}}$$

where ω is the angular frequency of the oscillations (and not the angular speed of the bob).

The period of this motion is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} \qquad (4)$$

Substitute $g\cos\theta$ for g_{eff} in equation (4) to obtain:

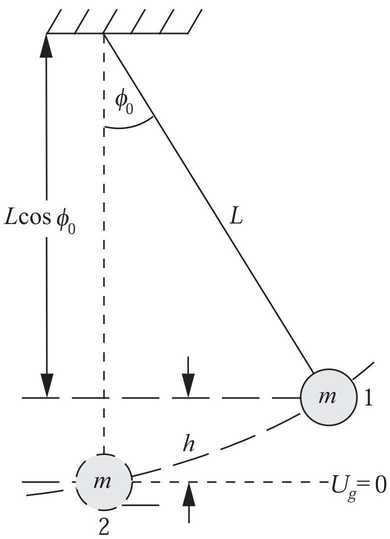
$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = \boxed{2\pi\sqrt{\frac{L}{g\cos\theta}}}$$

Remarks: Note that, in the limiting case $\theta = 0$, $T = 2\pi\sqrt{L/g}$. As $\theta \to 90°$, $T \to \infty$.

66 ••• The bob at the end of a simple pendulum of length L is released from rest from an angle ϕ_0. (*a*) Model the pendulum's motion as simple harmonic motion and find its speed as it passes through $\phi = 0$ by using the small angle approximation. (*b*) Using the conservation of energy, find this speed exactly for any angle (not just small angles). (*c*) Show that your result from Part (*b*) agrees

with the approximate answer in Part (*a*) when ϕ_0 is small. (*d*) Find the difference between the approximate and exact results for $\phi_0 = 0.20$ rad and $L = 1.0$ m.
(*e*) Find the difference between the approximate and exact results for $\phi_0 = 1.20$ rad and $L = 1.0$ m.

Picture the Problem The pictorial representation shows the simple pendulum at maximum angular displacement ϕ_0. We can express the angular position of the pendulum bob in terms of its initial angular position and time and differentiate this expression to find the maximum speed of the bob. We can use conservation of energy to find an exact value for v_{max} and the approximation $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ to show that this value reduces to the former value for small ϕ.



(*a*) Relate the speed of the pendulum bob to its angular speed:

$$v = L\frac{d\phi}{dt} \qquad (1)$$

The angular position of the pendulum as a function of time is given by:

$$\phi = \phi_0 \cos\omega t$$

Differentiate this expression with respect to time to express the angular speed of the pendulum:

$$\frac{d\phi}{dt} = -\phi_0\omega \sin\omega t$$

Substitute in equation (1) to obtain:

$$v = -L\phi_0\omega \sin\omega t = -v_{max}\sin\omega t$$

Simplify v_{max} to obtain:

$$v_{max} = L\phi_0\sqrt{\frac{g}{L}} = \boxed{\phi_0\sqrt{gL}}$$

(*b*) Use conservation of energy to relate the potential energy of the pendulum at point 1 to its kinetic energy at point 2:

$$\Delta K + \Delta U = 0$$

or, because $K_1 = U_2 = 0$,

$$K_2 - U_1 = 0$$

Substitute for K_2 and U_1:

$$\tfrac{1}{2}mv_2^2 - mgh = 0$$

Express h in terms of L and ϕ_0: $h = L(1-\cos\phi_0)$

Substituting for h yields: $\frac{1}{2}mv_2^2 - mgL(1-\cos\phi_0) = 0$

Solve for $v_2 = v_{max}$ to obtain:

$$v_{max} = \boxed{\sqrt{2gL(1-\cos\phi_0)}} \qquad (2)$$

(c) For $\phi_0 << 1$: $1-\cos\phi_0 \approx \frac{1}{2}\phi_0^2$

Substitute in equation (2) to obtain:

$$v_{max} = \sqrt{2gL\left(\tfrac{1}{2}\phi_0^2\right)} = \boxed{\phi_0\sqrt{gL}}$$

in agreement with our result in Part (a).

(d) Express the difference in the results from (a) and (b): $\Delta v = v_{max,a} - v_{max,b}$

Substitute for $v_{max,a}$ and $v_{max,b}$ and simplify to obtain:

$$\begin{aligned}\Delta v &= \phi_0\sqrt{gL} - \sqrt{2gL(1-\cos\phi_0)} \\ &= \sqrt{gL}\left(\phi_0 - \sqrt{2(1-\cos\phi_0)}\right)\end{aligned} \qquad (3)$$

The ratio of Δv to v_{max} is given by:

$$\begin{aligned}\frac{\Delta v}{v_{max}} &= \frac{\sqrt{gL}\left(\phi_0 - \sqrt{2(1-\cos\phi_0)}\right)}{\phi_0\sqrt{gL}} \\ &= 1 - \frac{\sqrt{2(1-\cos\phi_0)}}{\phi_0}\end{aligned}$$

Substitute numerical values and evaluate Δv:

$$\Delta v = \sqrt{(9.81\ \text{m/s}^2)(1.0\ \text{m})}\left(0.20\ \text{rad} - \sqrt{2(1-\cos(0.20\ \text{rad}))}\right) \approx \boxed{1\ \text{mm/s}}$$

To make the importance of these Δv differences clearer, evaluate $\Delta v/v_{max}$ to obtain:

$$\begin{aligned}\frac{\Delta v}{v_{max}} &= 1 - \frac{\sqrt{2(1-\cos(0.20\ \text{rad}))}}{0.20\ \text{rad}} \\ &\approx 1.7\times10^{-3}\end{aligned}$$

(e) Evaluate equation (3) for $\phi_0 = 1.20$ rad and $L = 1.0$ m:

$$\Delta v = \sqrt{(9.81\ \text{m/s}^2)(1.0\ \text{m})}\left(1.20\ \text{rad} - \sqrt{2(1-\cos(1.20\ \text{rad}))}\right) \approx \boxed{0.2\ \text{m/s}}$$

Evaluating $\Delta v/v_{max}$ gives:

$$\begin{aligned}\frac{\Delta v}{v_{max}} &= 1 - \frac{\sqrt{2(1-\cos(1.20\ \text{rad}))}}{1.20\ \text{rad}} \\ &\approx 5.9\times10^{-2}\end{aligned}$$

Remarks: Note that the ratio of Δv to v_{max} is approximately 35 times larger for ϕ_0 = 0.20 rad than for ϕ_0 = 0.20 rad.

*Physical Pendulums

67 • **[SSM]** A thin 5.0-kg disk with a 20-cm radius is free to rotate about a fixed horizontal axis perpendicular to the disk and passing through its rim. The disk is displaced slightly from equilibrium and released. Find the period of the subsequent simple harmonic motion.

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the thin disk about the fixed horizontal axis passing through its rim. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the disk about an axis through its center of mass and the distance from its center of mass to its pivot point.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the thin disk about an axis through the pivot point:

$$I = I_{\mathrm{cm}} + MR^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

Substituting for I and simplifying yields:

$$T = 2\pi\sqrt{\frac{\tfrac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{3(0.20\,\mathrm{m})}{2(9.81\,\mathrm{m/s^2})}} = \boxed{1.1\,\mathrm{s}}$$

68 • A circular hoop that has a 50-cm radius is hung on a narrow horizontal rod and allowed to swing in the plane of the hoop. What is the period of its oscillation, assuming that the amplitude is small?

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the circular hoop about an axis through its pivot point. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the hoop about an axis through its center of mass and the distance from its center of mass to its pivot point.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the circular hoop about an axis through the pivot point:

$$I = I_{\text{cm}} + MR^2 = MR^2 + MR^2 = 2MR^2$$

Substitute for I and simplify to obtain:

$$T = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.50\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{2.0\,\text{s}}$$

69 • A 3.0-kg plane figure is suspended at a point 10 cm from its center of mass. When it is oscillating with small amplitude, the period of oscillation is 2.6 s. Find the moment of inertia I about an axis perpendicular to the plane of the figure through the pivot point.

Picture the Problem The period of a physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is its moment of inertia about an axis through its pivot point. We can solve this equation for I and evaluate it using the given numerical data.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \Rightarrow I = \frac{MgDT^2}{4\pi^2}$$

Substitute numerical values and evaluate I:

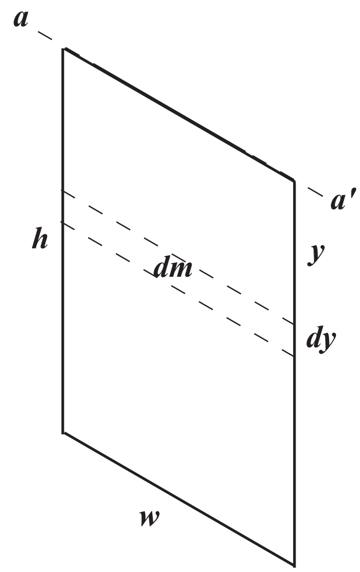
$$I = \frac{(3.0\,\text{kg})(9.81\,\text{m/s}^2)(0.10\,\text{m})(2.6\,\text{s})^2}{4\pi^2} = \boxed{0.50\,\text{kg}\cdot\text{m}^2}$$

70 •• You have designed a cat door that consists of a square piece of plywood that is 1.0 in. thick and 6.0 in. on a side, and is hinged at its top. To make sure the cat has enough time to get through it safely, the door should have a natural period of at least 1.0 s. Will your design work? If not, explain qualitatively what you would do to make it meet your requirements.

Picture the Problem The pictorial representation shows the cat door, of height h and width w, pivoted about an axis through a-a'. We can use

$$T = 2\pi\sqrt{\frac{I_{a-a'}}{mgD}}$$

to find the period of the door but first must find $I_{a-a'}$. The diagram also shows a differential strip of height dy and mass dm a distance y from the axis of rotation of the door. We can integrate the differential expression for the moment of inertia of this strip to determine the moment of inertia of the door.



The period of the cat door is given by:

$$T = 2\pi\sqrt{\frac{I_{a-a'}}{mgD}} \qquad (1)$$

where D is the distance from the center of mass of the door to the axis of rotation.

Express the moment of inertia, about the axis a-a', of the cat door:

$dI_{a-a'} = y^2 dm$

or, because $dm = \rho dV = \rho t dA = \rho w t dy$,

$dI_{a-a'} = \rho wt\, y^2 dy$

Integrating this expression between $y = 0$ and $y = h$ yields:

$$I_{a-a'} = \rho wt \int_0^h y^2 dy = \tfrac{1}{3}\rho wth^3$$

Because $\rho = \dfrac{m}{V} = \dfrac{m}{wht}$:

$$I_{a-a'} = \tfrac{1}{3}\left(\frac{m}{wht}\right)wth^3 = \tfrac{1}{3}mh^2$$

Substituting for D and $I_{a-a'}$ in equation (1) yields:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}mh^2}{mg\left(\frac{1}{2}h\right)}} = 2\pi\sqrt{\frac{2h}{3g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2\left(6.0\,\text{in}\times\dfrac{2.540\,\text{cm}}{\text{in}}\right)}{3\left(9.81\,\text{m/s}^2\right)}} = 0.64\,\text{s}$$

Thus the door's period is too short. The only way to increase it is to increase the height of the door.

71 •• You are given a meterstick and asked to drill a small diameter hole through it so that, when the stick is pivoted about a horizontal axis through the hole, the period of the pendulum will be a minimum. Where should you drill the hole?

Picture the Problem Let x be the distance of the pivot from the center of the meter stick, m the mass of the meter stick, and L its length. We'll express the period of the meter stick as a function of the distance x and then differentiate this expression with respect to x to determine where the hole should be drilled to minimize the period.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \qquad (1)$$

Express the moment of inertia of the meter stick about an axis through its center of mass:

$$I_{\text{cm}} = \tfrac{1}{12}mL^2$$

Using the parallel-axis theorem, express the moment of inertia of the meter stick about an axis through the pivot point:

$$I = I_{\text{cm}} + mx^2 = \tfrac{1}{12}mL^2 + mx^2$$

Substitute in equation (1) and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{12}mL^2 + mx^2}{mgx}} = \frac{2\pi}{\sqrt{g}}\sqrt{\frac{L^2 + 12x^2}{12x}}$$

The condition for an extreme value of T is that $\dfrac{d}{dx}\left(\sqrt{\dfrac{L^2+12x^2}{12x}}\right) = 0.$

Evaluate this derivative to obtain:

$$\frac{12x^2 - L^2}{24x^2\sqrt{\dfrac{L^2+12x^2}{12x}}} = 0 \Rightarrow 12x^2 - L^2 = 0$$

Noting that only the positive solution is physically meaningful, solve for x:

$$x = \frac{L}{\sqrt{12}} = \frac{100\,\text{cm}}{\sqrt{12}} = 28.9\,\text{cm}$$

The hole should be drilled at a distance:

$$d = 50.0\,\text{cm} - 28.9\,\text{cm} = \boxed{21.1\,\text{cm}}$$

from the center of the meter stick.

72 •• Figure 14-29 shows a uniform disk that has a radius $R = 0.80$ m, a mass of 6.00 kg, and a small hole a distance d from the disk's center that can serve as a pivot point. (*a*) What should be the distance d so that the period of this physical pendulum is 2.50 s? (*b*) What should be the distance d so that this physical pendulum will have the shortest possible period? What is this shortest possible period?

Picture the Problem (*a*) Let m represent the mass and R the radius of the uniform disk. The disk is a physical pendulum. We'll use the expression $T = 2\pi\sqrt{I/mgd}$ for the period of a physical pendulum. To find I we use the parallel-axis theorem ($I = I_{\text{cm}} + md^2$). (*b*) The period is a minimum when $dT/dx = 0$, where, to avoid notational difficulties, we have substituted x for d.

(*a*) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem ($I = I_{\text{cm}} + md^2$), relate the moment of inertia about the axis through the hole to the moment of inertia I_{cm} about the parallel axis through the center of mass. Obtain I_{cm} from Table 9-1:

$$I = I_{\text{cm}} + md^2 = \tfrac{1}{2}mR^2 + md^2$$

Substituting for I yields:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}mR^2 + md^2}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + d^2}{gd}} \qquad (1)$$

Square both sides of this equation, simplify, and substitute numerical values to obtain:

$$d^2 - \frac{gT^2}{4\pi^2}d + \frac{R^2}{2} = 0$$

or

$$d^2 - (1.553\,\text{m})d + 0.320\,\text{m}^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$d = 0.245\ \text{m} = \boxed{24\,\text{cm}}$$

The second root, $d = 1.31$ m, is greater than R, so it is too large to be physically meaningful.

(*b*) The period T is related to the distance d by equation (1). T will be a minimum when $\left(\frac{1}{2}R^2+d^2\right)/d$ is a minimum. Set the derivative of this expression equal to zero to find relative maxima and minima. We'll replace d with x to avoid the notational challenge of differentiating with respect to d. Evaluating $\frac{d}{dx}\left(\sqrt{\frac{\frac{1}{2}R^2+x^2}{x}}\right)=0$ yields:

$$\frac{2d^2-\left(\frac{1}{2}R^2+d^2\right)}{d^2}=0\Rightarrow 2d^2-\left(\frac{1}{2}R^2+d^2\right)=0$$

where we have changed x back to d.

Solving for d yields: $d=\boxed{\frac{R}{\sqrt{2}}}$

Evaluate equation (1) with $d=R/\sqrt{2}$ to obtain an expression for the shortest possible period:

$$T=2\pi\sqrt{\frac{\frac{1}{2}R^2+\frac{1}{2}R^2}{g\frac{R}{\sqrt{2}}}}=2\pi\sqrt{\frac{\sqrt{2}R}{g}}$$

Substitute numerical values and evaluate T:

$$T=2\pi\sqrt{\frac{\sqrt{2}(0.80\,\mathrm{m})}{9.81\,\mathrm{m/s^2}}}=\boxed{2.1\,\mathrm{s}}$$

Remarks: We've shown that $d=R/\sqrt{2}$ corresponds to an *extreme* value; that is, to a maximum, a minimum, or an inflection point. To verify that this value of d corresponds to a minimum, we can either (1) show that d^2T/dx^2 evaluated at $x=R/\sqrt{2}$ (where $x=d$) is positive, or (2) graph T as a function of d and note that the graph is a minimum at $d=R/\sqrt{2}$.

73 ••• [SSM] Points P_1 and P_2 on a plane object (Figure 14-30) are distances h_1 and h_2, respectively, from the center of mass. The object oscillates with the same period T when it is free to rotate about an axis through P_1 and when it is free to rotate about an axis through P_2. Both of these axes are perpendicular to the plane of the object. Show that $h_1+h_2=gT^2/(4\pi)^2$, where $h_1\neq h_2$.

Picture the Problem We can use the equation for the period of a physical pendulum and the parallel-axis theorem to show that $h_1+h_2=gT^2/4\pi^2$.

Express the period of the physical pendulum:

$$T=2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem, relate the moment of inertia about an axis through P_1 to the moment of inertia about an axis through the plane's center of mass:

$$I = I_{\text{cm}} + mh_1^2$$

Substitute for I to obtain:

$$T = 2\pi\sqrt{\frac{I_{\text{cm}} + mh_1^2}{mgh_1}}$$

Square both sides of this equation and rearrange terms to obtain:

$$\frac{mgT^2}{4\pi^2} = \frac{I_{\text{cm}}}{h_1} + mh_1 \qquad (1)$$

Because the period of oscillation is the same for point P_2:

$$\frac{I_{\text{cm}}}{h_1} + mh_1 = \frac{I_{\text{cm}}}{h_2} + mh_2$$

Combining like terms yields:

$$\left(\frac{1}{h_1} - \frac{1}{h_2}\right)I_{\text{cm}} = m(h_2 - h_1)$$

Provided $h_1 \neq h_2$:

$$I_{\text{cm}} = mh_1h_2$$

Substitute in equation (1) and simplify to obtain:

$$\frac{mgT^2}{4\pi^2} = \frac{mh_1h_2}{h_1} + mh_1$$

Solving for $h_1 + h_2$ gives:

$$h_1 + h_2 = \boxed{\frac{gT^2}{4\pi^2}}$$

74 ••• A physical pendulum consists of a spherical bob of radius r and mass m suspended from a rigid rod of negligible mass as in Figure 14-31. The distance from the center of the sphere to the point of support is L. When r is much less than L, such a pendulum is often treated as a simple pendulum of length L.
(*a*) Show that the period for small oscillations is given by $T = T_0\sqrt{1 + \frac{2r^2}{5L^2}}$ where $T_0 = 2\pi\sqrt{L/g}$ is the period of a simple pendulum of length L. (*b*) Show that when r is much smaller than L, the period can be approximated by $T \approx T_0\,(1 + r^2/5L^2)$. (*c*) If L = 1.00 m and r = 2.00 cm, find the error in the calculated value when the approximation $T = T_0$ is used for the period. How large must be the radius of the bob for the error to be 1.00 percent?

Picture the Problem (*a*) We can find the period of the physical pendulum in terms of the period of a simple pendulum by starting with $T = 2\pi\sqrt{I/mgL}$ and applying the parallel-axis theorem. (*b*) Performing a binomial expansion (with $r << L$) on the radicand of our expression for T will lead to $T \approx T_0(1 + r^2/5L^2)$.

(*a*) Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

Using the parallel-axis theorem, relate the moment of inertia of the pendulum about an axis through its center of mass to its moment of inertia about an axis through its point of support:

$$I = I_{\text{cm}} + mL^2 = \tfrac{2}{5}mr^2 + mL^2$$

Substitute for I and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{2}{5}mr^2 + mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}\sqrt{1+\frac{2r^2}{5L^2}} = \boxed{T_0\sqrt{1+\frac{2r^2}{5L^2}}}$$

(*b*) Expanding $\left(1+\frac{2r^2}{5L^2}\right)^{1/2}$ binomially yields:

$$\left(1+\frac{2r^2}{5L^2}\right)^{1/2} = 1+\frac{1}{2}\left(\frac{2r^2}{5L^2}\right)+\frac{1}{8}\left(\frac{2r^2}{5L^2}\right)^2 + \text{higher-order terms} \approx 1+\frac{r^2}{5L^2}$$

provided $r << L$

Substitute in our result from Part (*a*) to obtain:

$$T \approx \boxed{T_0\left(1+\frac{r^2}{5L^2}\right)}$$

(*c*) Express the fractional error when the approximation $T = T_0$ is used for this pendulum:

$$\frac{\Delta T}{T} \approx \frac{T - T_0}{T_0} = \frac{T}{T_0} - 1 = 1 + \frac{r^2}{5L^2} - 1 = \frac{r^2}{5L^2}$$

Substitute numerical values and evaluate $\Delta T/T$:

$$\frac{\Delta T}{T} \approx \frac{(2.00\,\text{cm})^2}{5(100\,\text{cm})^2} = \boxed{0.00800\%}$$

For an error of 1.00%:

$$\frac{r^2}{5L^2} = 0.0100 \Rightarrow r = L\sqrt{0.0500}$$

Substitute the numerical value of L and evaluate r to obtain:

$$r = (100\,\text{cm})\sqrt{0.0500} = \boxed{22.4\,\text{cm}}$$

75 ••• Figure 14-32 shows the pendulum of a clock in your grandmother's house. The uniform rod of length L = 2.00 m has a mass m = 0.800 kg. Attached to the rod is a uniform disk of mass M = 1.20 kg and radius 0.150 m. The clock is constructed to keep perfect time if the period of the pendulum is exactly 3.50 s. (*a*) What should the distance d be so that the period of this pendulum is 2.50 s? (*b*) Suppose that the pendulum clock loses 5.00 min/d. To make sure that your grandmother won't be late for her quilting parties, you decide to adjust the clock back to its proper period. How far and in what direction should you move the disk to ensure that the clock will keep perfect time?

Picture the Problem (*a*) The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$. We can express its period as a function of the distance d by using the definition of the center of mass of the pendulum to find D in terms of d and the parallel-axis theorem to express I in terms of d. Solving the resulting quadratic equation yields d. (*b*) Because the clock is losing 5 minutes per day, one would reposition the disk so that the clock runs faster; that is, so the pendulum has a shorter period. We can determine the appropriate correction to make in the position of the disk by relating the fractional time loss to the fractional change in its position.

(*a*) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{m_{\text{tot}}gx_{\text{cm}}}}$$

Solving for $\frac{I}{x_{\text{cm}}}$ yields:

$$\frac{I}{x_{\text{cm}}} = \frac{T^2 g m_{\text{tot}}}{4\pi^2} \qquad (1)$$

Express the moment of inertia of the physical pendulum, about an axis through the pivot point, as a function of d:

$$I = I_{\text{cm}} + Md^2 = \tfrac{1}{3}mL^2 + \tfrac{1}{2}Mr^2 + Md^2$$

Substitute numerical values and evaluate I:

$$\begin{aligned} I &= \tfrac{1}{3}(0.800\,\text{kg})(2.00\,\text{m})^2 \\ &\quad + \tfrac{1}{2}(1.20\,\text{kg})(0.150\,\text{m})^2 + (1.20\,\text{kg})d^2 \\ &= 1.0802\,\text{kg}\cdot\text{m}^2 + (1.20\,\text{kg})d^2 \end{aligned}$$

Locate the center of mass of the physical pendulum relative to the pivot point:

$$x_{cm} = \frac{(0.800\,\text{kg})(1.00\,\text{m}) + (1.20\,\text{kg})d}{2.00\,\text{kg}}$$

and

$$\boldsymbol{x}_{cm} = 0.400\,\text{m} + 0.600\boldsymbol{d}$$

Substitute in equation (1) to obtain:

$$\frac{1.0802\,\text{kg}\cdot\text{m}^2 + (1.20\,\text{kg})d^2}{0.400\,\text{m} + 0.600d} = \frac{T^2(9.81\,\text{m/s}^2)(2.00\,\text{kg})}{4\pi^2} = (0.49698\,\text{kg}\cdot\text{m/s}^2)T^2 \quad (2)$$

Setting $T = 2.50$ s and solving for d yields:

$$d = 1.63574\,\text{m} = \boxed{1.64\,\text{m}}$$

(b) There are 1440 minutes per day. If the clock loses 5.00 min per day, then the period of the clock is related to the perfect period of the clock by:

$$1435T = 1440T_{perfect} \Rightarrow T = \frac{1440}{1435}T_{perfect}$$

where $T_{perfect} = 3.50$ s.

Substitute numerical values and evaluate T:

$$T = \frac{1440}{1435}(3.50\,\text{s}) = 3.51220\,\text{s}$$

Substitute $T = 3.51220$ s in equation (2) and solve for d to obtain:

$$d = 3.40140\,\text{m}$$

Substitute $T = 3.50$ s in equation (2) and solve for d' to obtain:

$$d' = 3.37826\,\text{m}$$

Express the distance the disk needs to be moved upward to correct the period:

$$\Delta d = d - d' = 3.40140\,\text{m} - 3.37826\,\text{m} = \boxed{2.31\,\text{cm}}$$

Damped Oscillations

76 • A 2.00-kg object oscillates with an initial amplitude of 3.00 cm. The force constant of the spring is 400 N/m. Find (a) the period, and (b) the total initial energy. (c) If the energy decreases by 1.00 percent per period, find the linear damping constant b and the Q factor.

Picture the Problem (a) We can find the period of the oscillator from $T = 2\pi\sqrt{m/k}$. (b) The total initial energy of the spring-object system is given by $E_0 = \frac{1}{2}kA^2$. (c) The Q factor can be found from its definition $Q = 2\pi/\left(|\Delta E|/E\right)_{\text{cycle}}$ and the damping constant from $Q = \omega_0 m/b$.

(a) The period of the oscillator is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2.00\,\text{kg}}{400\,\text{N/m}}} = \boxed{0.444\,\text{s}}$$

(b) Relate the initial energy of the oscillator to its amplitude:

$$E_0 = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}(400\,\text{N/m})(0.0300\,\text{m})^2 = \boxed{0.180\,\text{J}}$$

(c) Relate the fractional rate at which the energy decreases to the Q value and evaluate Q:

$$Q = \frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}} = \frac{2\pi}{0.0100} = \boxed{628}$$

Express the Q value in terms of b:

$$Q = \frac{\omega_0 m}{b}$$

Solve for the damping constant b:

$$b = \frac{\omega_0 m}{Q} = \frac{2\pi m}{TQ} = \frac{2\pi m}{2\pi\sqrt{\frac{m}{k}}Q} = \frac{\sqrt{mk}}{Q}$$

Substitute numerical values and evaluate b:

$$b = \frac{\sqrt{(2.00\,\text{kg})(400\,\text{N/m})}}{628} = \boxed{0.0450\,\text{kg/s}}$$

77 •• [SSM] Show that the ratio of the amplitudes for two successive oscillations is constant for a linearly damped oscillator.

Picture the Problem The amplitude of the oscillation at time t is $A(t) = A_0 e^{-t/2\tau}$ where $\tau = m/b$ is the decay constant. We can express the amplitudes one period apart and then show that their ratio is constant.

Relate the amplitude of a given oscillation peak to the time at which the peak occurs:

$$A(t) = A_0 e^{-t/2\tau}$$

Express the amplitude of the oscillation peak at $t' = t + T$:

$$A(t+T) = A_0 e^{-(t+T)/2\tau}$$

Express the ratio of these consecutive peaks:

$$\frac{A(t)}{A(t+T)} = \frac{A_0 e^{-t/2\tau}}{A_0 e^{-(t+T)/2\tau}} = e^{-T/2\tau} = \boxed{\text{constant}}$$

78 •• An oscillator has a period of 3.00 s. Its amplitude decreases by 5.00 percent during each cycle. (*a*) By how much does its mechanical energy decrease during each cycle? (*b*) What is the time constant τ? (*c*) What is the *Q* factor?

Picture the Problem (*a*) We can relate the fractional change in the energy of the oscillator each cycle to the fractional change in its amplitude. (*b*) and (*c*) Both the *Q* value and the decay constant τ can be found from their definitions.

(*a*) Relate the energy of the oscillator to its amplitude:

$$E = \tfrac{1}{2}kA^2$$

Take the differential of this relationship to obtain:

$$dE = kAdA$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{dE}{E} = \frac{kAdA}{\frac{1}{2}kA^2} = 2\frac{dA}{A}$$

Approximate dE and dA by ΔE and ΔA and evaluate $\Delta E/E$:

$$\frac{\Delta E}{E} = 2(5.00\%) = \boxed{10.0\%}$$

(*b*) For small damping:

$$\frac{|\Delta E|}{E} = \frac{T}{\tau}$$

and

$$\tau = \frac{T}{|\Delta E|/E} = \frac{3.00\,\text{s}}{0.100} = \boxed{30.0\,\text{s}}$$

(*c*) The *Q* factor is given by:

$$Q = \omega_0\tau = \left(\frac{2\pi}{T}\right)\tau$$

Substitute numerical values and evaluate Q:

$$Q = \frac{2\pi}{3.00\,\text{s}}(30.0\,\text{s}) = \boxed{62.8}$$

79 •• A linearly damped oscillator has a Q factor of 20. (a) By what fraction does the energy decrease during each cycle? (b) Use Equation 14-40 to find the percentage difference between ω' and ω_0. *Hint: Use the approximation* $(1+x)^{1/2} \approx 1 + \frac{1}{2}x$ *for small x.*

Picture the Problem We can use the physical interpretation of Q for small damping $\left(Q = \dfrac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}}\right)$ to find the fractional decrease in the energy of the oscillator each cycle.

(a) Express the fractional decrease in energy each cycle as a function of the Q factor and evaluate $|\Delta E|/E$:

$$\frac{|\Delta E|}{E} = \frac{2\pi}{Q} = \frac{2\pi}{20} = 0.314 = \boxed{0.31}$$

(b) The percentage difference between ω' and ω_0 is given by:

$$\frac{\omega' - \omega_0}{\omega_0} = \frac{\omega'}{\omega_0} - 1$$

Using the definition of the Q factor, use Equation 14-40 to express the ratio of ω' to ω_0 as a function of Q:

$$\frac{\omega'}{\omega_0} = \left[1 - \frac{1}{4}\left(\frac{b^2}{m^2\omega_0^2}\right)\right]^{1/2} = \left[1 - \frac{1}{4Q^2}\right]^{1/2}$$

Substituting for $\dfrac{\omega'}{\omega_0}$ yields:

$$\frac{\omega' - \omega_0}{\omega_0} = \left[1 - \frac{1}{4Q^2}\right]^{1/2} - 1$$

Use the approximation

$$(1+x)^{1/2} \approx 1 + \tfrac{1}{2}x$$

for $x << 1$ to obtain:

$$\left[1 - \frac{1}{4Q^2}\right]^{1/2} \approx 1 - \frac{1}{8Q^2}$$

Substituting for $\left[1 - \dfrac{1}{4Q^2}\right]^{1/2}$ and simplifying yields:

$$\frac{\omega' - \omega_0}{\omega_0} = 1 - \frac{1}{8Q^2} - 1 = -\frac{1}{8Q^2}$$

Substitute the numerical value of Q and evaluate $\dfrac{\omega' - \omega_0}{\omega_0}$:

$$\frac{\omega' - \omega_0}{\omega_0} = -\frac{1}{8(20)^2} = \boxed{-3.1\times10^{-2}\%}$$

80 •• A linearly damped mass–spring system oscillates at 200 Hz. The time constant of the system is 2.0 s. At $t = 0$ the amplitude of oscillation is 6.0 cm and the energy of the oscillating system is 60 J. (*a*) What are the amplitudes of oscillation at t = 2.0 s and t = 4.0 s? (*b*) How much energy is dissipated in the first 2-s interval and in the second 2-s interval?

Picture the Problem The energy of the spring-and-mass oscillator varies with time according to $E = E_0 e^{-t/\tau}$ and its energy is proportional to the square of the amplitude.

(*a*) Using $E = E_0 e^{-t/\tau}$ and $E \propto A^2$, solve for the amplitude A as a function of time:

$E = E_0 e^{-t/\tau}$ and $E \propto A^2$ imply that $A^2 = A_0^2 e^{-t/\tau}$

Hence $A = A_0 e^{-t/2\tau}$

Express the amplitude of the oscillations as a function of time:

$$A = (6.0\text{ cm})e^{-t/4\text{ s}}$$

Evaluate the amplitude when t = 2.0 s:

$$A(2.0\text{ s}) = (6.0\text{ cm})e^{-2.0\text{ s}/4.0\text{ s}} = \boxed{3.6\text{ cm}}$$

Evaluate the amplitude when t = 4.0 s:

$$A(4.0\text{ s}) = (6.0\text{ cm})e^{-4.0\text{ s}/4.0\text{ s}} = \boxed{2.2\text{ cm}}$$

(*b*) Express the energy of the system at t = 0, t = 2.0 s, and t = 4.0 s:

$$E(0) = E_0 e^{-0/2.0\text{s}} = E_0$$
$$E(2.0\text{ s}) = E_0 e^{-2.0/2.0\text{s}} = E_0 e^{-1}$$
$$E(4.0\text{ s}) = E_0 e^{-4.0/2.0\text{s}} = E_0 e^{-2}$$

The energy dissipated in the first 2.0 s is equal to the negative of the change in mechanical energy:

$$-\Delta E_{0\to 2.0\text{ s}} = -E_0\left(e^{-2.0\text{ s}/2.0\text{ s}} - e^0\right) = (60\text{ J})\left(1 - e^{-1}\right) = \boxed{38\text{ J}}$$

The energy dissipated in the second 2.0-s interval is:

$$-\Delta E_{2.0\text{ s}\to 4.0\text{ s}} = -E_0\left(e^{-4.0\text{ s}/2.0\text{ s}} - e^{-2.0\text{ s}/2.0\text{ s}}\right) = (60\text{ J})e^{-1}\left(1 - e^{-1}\right) = \boxed{14\text{ J}}$$

81 •• **[SSM]** Seismologists and geophysicists have determined that the vibrating Earth has a resonance period of 54 min and a Q factor of about 400. After a large earthquake, Earth will "ring" (continue to vibrate) for up to 2 months. (*a*) Find the percentage of the energy of vibration lost to damping forces during each cycle. (*b*) Show that after n periods the vibrational energy is given by $E_n = (0.984)^n E_0$, where E_0 is the original energy. (*c*) If the original energy of vibration of an earthquake is E_0, what is the energy after 2.0 d?

Picture the Problem (*a*) We can find the fractional loss of energy per cycle from the physical interpretation of Q for small damping. (*b*) We will also find a general expression for Earth's vibrational energy as a function of the number of cycles it has completed. (*c*) We can then solve this equation for Earth's vibrational energy after any number of days.

(*a*) Express the fractional change in energy as a function of Q:

$$\frac{\Delta E}{E}=\frac{2\pi}{Q}=\frac{2\pi}{400}=\boxed{1.57\%}$$

(*b*) Express the energy of the damped oscillator after one cycle:

$$E_1=E_0\left(1-\frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2=E_1\left(1-\frac{\Delta E}{E}\right)=E_0\left(1-\frac{\Delta E}{E}\right)^2$$

Generalizing to n cycles:

$$E_n=E_0\left(1-\frac{\Delta E}{E}\right)^n=E_0(1-0.0157)^n$$
$$=\boxed{E_0(0.984)^n}$$

(*c*) Express 2.0 d in terms of the number of cycles; that is, the number of vibrations Earth will have experienced:

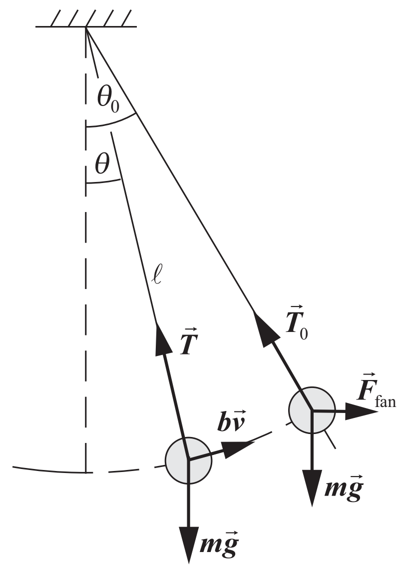
$$2.0\,\text{d}=2.0\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{60\,\text{m}}{\text{h}}$$
$$=2880\,\text{min}\times\frac{1T}{54\,\text{min}}$$
$$=53.3T$$

Evaluate E(2 d):

$$E(2\,\text{d})=E_0(0.9843)^{53.3}=\boxed{0.43E_0}$$

82 ••• A pendulum that is used in your physics laboratory experiment has a length of 75 cm and a compact bob with a mass equal to 15 g. To start the bob oscillating, you place a fan next to it that blows a horizontal stream of air on the bob. With the fan on, the bob is in equilibrium when the pendulum is displaced by an angle of 5.0º from the vertical. The speed of the air from the fan is 7.0 m/s. You turn the fan off, and allow the pendulum to oscillate. (*a*) Assuming that the drag force due to the air is of the form $-bv$, predict the decay time constant τ for this pendulum. (*b*) How long will it take for the pendulum's amplitude to reach 1.0º?

Picture the Problem The diagram shows 1) the pendulum bob displaced through an angle θ_0 and held in equilibrium by the force exerted on it by the air from the fan and 2) the bob accelerating, under the influence of gravity, tension force, and drag force, toward its equilibrium position. We can apply Newton's second law to the bob to obtain the equation of motion of the damped pendulum and then use its solution to find the decay time constant and the time required for the amplitude of oscillation to decay to 1°.



(*a*) Apply $\sum \tau = I\alpha$ to the pendulum to obtain:

$$-mg\ell\sin\theta + \ell F_{\text{d}} = I\frac{d^2\theta}{dt^2}$$

Express the moment of inertia of the pendulum about an axis through its point of support:

$$I = m\ell^2$$

Substitute for I and F_{d} to obtain:

$$m\ell^2\frac{d^2\theta}{dt^2} + \ell bv + mg\ell\sin\theta = 0$$

Because $\theta << 1$ and $v = \ell\omega = \ell d\theta/dt$:

$$m\ell^2\frac{d^2\theta}{dt^2} + \ell^2 b\frac{d\theta}{dt} + mg\ell\theta \approx 0$$

or

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{\ell}\theta \approx 0$$

The solution to this second-order homogeneous differential equation with constant coefficients is:

$$\theta = \theta_0 e^{-t/2\tau}\cos(\omega' t + \delta) \qquad (1)$$

where θ_0 is the maximum amplitude, $\tau = m/b$ is the time constant, and the frequency $\omega' = \omega_0\sqrt{1-(b/2m\omega_0)^2}$.

Apply $\sum \vec{F} = m\vec{a}$ to the bob when it is at its maximum angular displacement to obtain:

$$\sum F_x = F_{\text{fan}} - T\sin\theta_0 = 0$$

and

$$\sum F_y = T\cos\theta_0 - mg = 0$$

Divide the x-direction equation by the y-direction equation to obtain:

$$\frac{F_{\text{fan}}}{mg} = \frac{T\sin\theta_0}{T\cos\theta_0} = \tan\theta_0$$

or

$$F_{\text{fan}} = mg\tan\theta_0$$

When the bob is in equilibrium, the drag force on it equals F_{fan}:

$$bv = mg\tan\theta_0 \Rightarrow \tau = \frac{m}{b} = \frac{v}{g\tan\theta_0}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{7.0\,\text{m/s}}{\left(9.81\,\text{m/s}^2\right)\tan 5.0°} = 8.16\,\text{s} = \boxed{8.2\,\text{s}}$$

(b) From equation (1), the angular amplitude of the motion is given by:

$$\theta = \theta_0 e^{-t/2\tau}$$

When the amplitude has decreased to 1.0°:

$$1.0° = 5.0° e^{-t/2\tau} \text{ or } e^{-t/2\tau} = 0.20$$

Take the natural logarithm of both sides of the equation to obtain:

$$-\frac{t}{2\tau} = \ln(0.20) \Rightarrow t = -2\tau\ln(0.20)$$

Substitute the numerical value of τ and evaluate t:

$$t = -2(8.16\,\text{s})\ln(0.20) = \boxed{26\text{ s}}$$

83 ••• [SSM] You are in charge of monitoring the viscosity of oils at a manufacturing plant and you determine the viscosity of an oil by using the following method: The viscosity of a fluid can be measured by determining the decay time of oscillations for an oscillator that has known properties and operates while immersed in the fluid. As long as the speed of the oscillator through the fluid is relatively small, so that turbulence is not a factor, the drag force of the fluid on a sphere is proportional to the sphere's speed relative to the fluid: $F_\text{d} = 6\pi a\eta v$, where η is the viscosity of the fluid and a is the sphere's radius. Thus, the constant b is given by $6\pi a\eta$. Suppose your apparatus consists of a stiff spring that has a force constant equal to 350 N/cm and a gold sphere (radius 6.00 cm) hanging on the spring. (a) What viscosity of an oil do you measure if the decay time for this system is 2.80 s? (b) What is the Q factor for your system?

Picture the Problem (a) The decay time for a damped oscillator (with speed-dependent damping) system is defined as the ratio of the mass of the oscillator to the coefficient of v in the damping force expression. (b) The Q factor is the product of the resonance frequency and the damping time.

(a) From $F_\text{d} = 6\pi a\eta v$ and $F_\text{d} = -bv$, it follows that:

$$b = 6\pi a\eta \Rightarrow \eta = \frac{b}{6\pi a}$$

Because $\tau = m/b$, we can substitute for b to obtain:

$$\eta = \frac{m}{6\pi a \tau}$$

Substituting $m = \rho V$ and simplifying yields:

$$\eta = \frac{\rho V}{6\pi a \tau} = \frac{\frac{4}{3}\pi a^3 \rho}{6\pi a \tau} = \frac{2a^2 \rho}{9\tau}$$

Substitute numerical values and evaluate η (see Table 13-1 for the density of gold):

$$\eta = \frac{2(0.0600\ \text{m})^2\left(19.3\times10^3\ \text{kg/m}^3\right)}{9(2.8\ \text{s})} = \boxed{5.51\ \text{Pa}\cdot\text{s}}$$

(*b*) The Q factor is the product of the resonance frequency and the damping time:

$$Q = \omega_0\tau = \sqrt{\frac{k}{m}}\tau = \sqrt{\frac{k}{\rho V}}\tau = \sqrt{\frac{k}{\frac{4}{3}\pi a^3 \rho}}\tau$$

Substitute numerical values and evaluate Q:

$$Q = \sqrt{\frac{3\left(350\,\frac{\text{N}}{\text{cm}}\times\frac{100\ \text{cm}}{\text{m}}\right)}{4\pi(0.0600\ \text{m})^3\left(19.3\times10^3\ \text{kg/m}^3\right)}}(2.80\ \text{s}) \approx \boxed{125}$$

Driven Oscillations and Resonance

84 • A linearly damped oscillator loses 2.00 percent of its energy during each cycle. (*a*) What is its Q factor? (*b*) If its resonance frequency is 300 Hz, what is the width of the resonance curve $\Delta\omega$ when the oscillator is driven?

Picture the Problem (*a*) We can use the physical interpretation of Q for small damping to find the Q factor for this damped oscillator. (*b*) The width of the resonance curve depends on the Q factor according to $\Delta\omega = \omega_0/Q$.

(*a*) Using the physical interpretation of Q for small damping, relate Q to the fractional loss of energy of the damped oscillator per cycle:

$$Q = \frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}}$$

Evaluate this expression for $\left(|\Delta E|/E\right)_{\text{cycle}} = 2.00\%$:

$$Q = \frac{2\pi}{0.0200} = \boxed{314}$$

(*b*) Relate the width of the resonance curve to the Q value of the oscillatory system:

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{2\pi f_0}{Q}$$

Substitute numerical values and evaluate $\Delta\omega$:

$$\Delta\omega = \frac{2\pi\left(300\,\text{s}^{-1}\right)}{314} = \boxed{6.00\,\text{rad/s}}$$

85 • Find the resonance frequency for each of the three systems shown in Figure 14-33.

Picture the Problem The resonant frequency of a vibrating system depends on the mass m of the system and on its "stiffness" constant k according to $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or, in the case of a simple pendulum oscillating with small-amplitude vibrations, $f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

(*a*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{400.0\,\text{N/m}}{10\,\text{kg}}} = \boxed{1.0\,\text{Hz}}$$

(*b*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{800.0\,\text{N/m}}{5\,\text{kg}}} = \boxed{2\,\text{Hz}}$$

(*c*) For this simple pendulum we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{2.0\,\text{m}}} = \boxed{0.35\,\text{Hz}}$$

86 •• A damped oscillator loses 3.50 percent of its energy during each cycle. (*a*) How many cycles elapse before half of its original energy is dissipated? (*b*) What is its Q factor? (*c*) If the natural frequency is 100 Hz, what is the width of the resonance curve when the oscillator is driven by a sinusoidal force?

Picture the Problem (*a*) We'll find a general expression for the damped oscillator's energy as a function of the number of cycles it has completed. We can then solve this equation for the number of cycles corresponding to the loss of half the oscillator's energy. (*b*) The Q factor is related to the fractional energy loss per cycle through $\Delta E/E = 2\pi/Q$. (*c*) The width of the resonance curve is $\Delta\omega = \omega_0/Q$ where ω_0 is the oscillator's natural angular frequency.

(*a*) Express the energy of the damped oscillator after one cycle:

$$E_1 = E_0\left(1 - \frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2 = E_1\left(1 - \frac{\Delta E}{E}\right) = E_0\left(1 - \frac{\Delta E}{E}\right)^2$$

Generalizing to n cycles:

$$E_n = E_0\left(1-\frac{\Delta E}{E}\right)^n$$

Substituting numerical values yields:

$$0.50E_0 = E_0(1-0.035)^n$$

or

$$0.50 = (0.965)^n$$

Solving for n yields:

$$n = \frac{\ln 0.50}{\ln 0.965} = 19.46$$

$$\approx \boxed{19\text{ complete cycles.}}$$

(b) Apply the physical interpretation of Q for small damping to obtain:

$$Q = \frac{2\pi}{\Delta E/E} = \frac{2\pi}{0.0350} = \boxed{180}$$

(c) The width of the resonance curve is given by:

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{2\pi f_0(\Delta E/E)}{2\pi} = f_0(\Delta E/E)$$

Substitute numerical values and evaluate $\Delta\omega$:

$$\Delta\omega = (100\text{ Hz})(0.0350) = \boxed{3.50\text{ rad/s}}$$

87 •• [SSM] A 2.00-kg object oscillates on a spring of force constant 400 N/m. The linear damping constant has a value of 2.00 kg/s. The system is driven by a sinusoidal force of maximum value 10.0 N and angular frequency 10.0 rad/s. (a) What is the amplitude of the oscillations? (b) If the driving frequency is varied, at what frequency will resonance occur? (c) What is the amplitude of oscillation at resonance? (d) What is the width of the resonance curve $\Delta\omega$?

Picture the Problem (a) The amplitude of the damped oscillations is related to the damping constant, mass of the system, the amplitude of the driving force, and the natural and driving frequencies through $A = \dfrac{F_0}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)^2+b^2\omega^2}}$. ($b$) Resonance occurs when $\omega = \omega_0$. (c) At resonance, the amplitude of the oscillations is $A = F_0\big/\sqrt{b^2\omega^2}$. ($d$) The width of the resonance curve is related to the damping constant and the mass of the system according to $\Delta\omega = b/m$.

(a) Express the amplitude of the oscillations as a function of the driving frequency:

$$A = \frac{F_0}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)^2+b^2\omega^2}}$$

Because $\omega_0 = \sqrt{\dfrac{k}{m}}$:

$$A = \frac{F_0}{\sqrt{m^2\left(\dfrac{k}{m} - \omega^2\right)^2 + b^2\omega^2}}$$

Substitute numerical values and evaluate A:

$$A = \frac{10.0\ \text{N}}{\sqrt{(2.00\ \text{kg})^2\left(\dfrac{400\ \text{N/m}}{2.00\ \text{kg}} - (10.0\ \text{rad/s})^2\right)^2 + (2.00\ \text{kg/s})^2(10.0\ \text{rad/s})^2}} = \boxed{4.98\ \text{cm}}$$

(*b*) Resonance occurs when:

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{400\ \text{N/m}}{2.00\ \text{kg}}} = 14.14\ \text{rad/s}$$
$$= \boxed{14.1\ \text{rad/s}}$$

(*c*) The amplitude of the motion at resonance is given by:

$$A = \frac{F_0}{\sqrt{b^2\omega_0^2}} = \frac{F_0}{b\omega_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{10.0\ \text{N}}{(2.00\ \text{kg/s})(14.14\ \text{rad/s})}$$
$$= \boxed{35.4\ \text{cm}}$$

(*d*) The width of the resonance curve is:

$$\Delta\omega = \frac{b}{m} = \frac{2.00\ \text{kg/s}}{2.00\ \text{kg}} = \boxed{1.00\ \text{rad/s}}$$

88 •• Suppose you have the same apparatus described in Problem 83 and the same gold sphere hanging from a weaker spring that has a force constant of only 35.0 N/cm. You have studied the viscosity of ethylene glycol with this device, and found that ethylene glycol has a viscosity value of 19.9 mPa·s. Now you decide to drive this system with an external oscillating force. (*a*) If the magnitude of the driving force for the device is 0.110 N and the device is driven at resonance, how large would be the amplitude of the resulting oscillation? (*b*) If the system were not driven, but were allowed to oscillate, what percentage of its energy would it lose per cycle?

Picture the Problem (*a*) The amplitude of the steady-state oscillations when the system is in resonance is given by $A = F_0/b\omega$. (*b*) We can relate the fractional energy loss to the Q value of the oscillator.

(*a*) The amplitude of the steady-state oscillations when the system is in resonance is given by:

$$A = \frac{F_0}{b\omega}$$

Because $b = 6\pi a\eta$, and $\omega = \sqrt{k/m}$:

$$A = \frac{F_0}{6\pi a\eta\omega} = \frac{F_0}{6\pi a\eta}\sqrt{\frac{m}{k}}$$

Substituting $m = \rho V$ and simplifying yields:

$$A = \frac{F_0}{6\pi a\eta}\sqrt{\frac{\rho V}{k}} = \frac{F_0}{6\pi a\eta}\sqrt{\frac{\frac{4}{3}\pi a^3\rho}{k}} = \frac{F_0}{3\pi\eta}\sqrt{\frac{\pi a\rho}{3k}}$$

Substitute numerical values and evaluate A:

$$A = \frac{0.110\text{ N}}{3\pi(19.9\text{ mPa}\cdot\text{s})}\sqrt{\frac{\pi(0.0600\text{ m})(19.3\times10^3\text{ kg/m}^3)}{3(35.0\text{ N/cm})}} = \boxed{34.5\text{ cm}}$$

(*b*) This is a very weakly damped system and so we can relate the fractional energy loss per cycle to the system's Q value:

$$Q = \frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \omega_0\tau$$

Because $\tau = \dfrac{m}{b} = \dfrac{m}{6\pi a\eta}$:

$$\frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \frac{m\omega_0}{6\pi a\eta}$$

Substituting for m and ω_0 and simplifying yields:

$$\frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \frac{\rho V\sqrt{\frac{k}{m}}}{6\pi a\eta} = \frac{\frac{4}{3}\pi a^3\rho}{6\pi a\eta}\sqrt{\frac{k}{m}} = \frac{2a^2\rho}{9\eta}\sqrt{\frac{k}{m}}$$

Solve for $(\Delta E/E)_{\text{cycle}}$ to obtain:

$$(\Delta E/E)_{\text{cycle}} = \frac{9\pi\eta}{a^2\rho}\sqrt{\frac{m}{k}}$$

Substituting for m and then for V and simplifying gives:

$$(\Delta E/E)_{\text{cycle}} = \frac{9\pi\eta}{a^2\rho}\sqrt{\frac{\rho V}{k}} = \frac{9\pi\eta}{a^2\rho}\sqrt{\frac{\rho\frac{4}{3}\pi a^3}{k}} = 18\pi\eta\sqrt{\frac{\pi}{3a\rho k}}$$

Substitute numerical values and evaluate $(\Delta E/E)_{\text{cycle}}$:

$$(\Delta E/E)_{\text{cycle}} = 18\pi(19.9\text{ mPa}\cdot\text{s})\sqrt{\frac{\pi}{3(6.00\text{ cm})(19.3\times10^3\text{ kg/m}^3)(350\text{ N/cm})}} = \boxed{1.81\times10^{-4}}$$

General Problems

89 • A particle's displacement from equilibrium is given by $x(t) = 0.40\cos(3.0t + \pi/4)$, where x is in meters and t is in seconds. (a) Find the frequency and period of its motion. (b) Find an expression for the velocity of the particle as a function of time. (c) What is its maximum speed?

Picture the Problem (a) The particle's displacement from equilibrium is $x = A\cos(\omega t + \delta)$. Thus, we have $A = 0.40$ m, $\omega = 3.0$ rad/s, and $\delta = \pi/4$. We can find the frequency of the motion from its angular frequency and the period from the frequency. (b) The particle's velocity is the time derivative of its displacement. (c) The particle's maximum speed occurs when $\sin(\omega t + \delta) = -1$.

(a) The particle's displacement from equilibrium is of the form
$$x = A\cos(\omega t + \delta).$$
Comparing this to the given equation we see that:

$\omega = 3.0\text{ rad/s}$
and so
$$f = \frac{\omega}{2\pi} = \frac{3.0\text{ rad/s}}{2\pi} = 0.477\text{ Hz} = \boxed{0.48\text{ Hz}}$$

The period of the particle's motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{0.477\text{ s}^{-1}} = 2.09\text{ s} = \boxed{2.1\text{ s}}$$

(b) Differentiate $x = A\cos(\omega t + \delta)$ with respect to time to obtain an expression for the particle's velocity:

$$v_x = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \delta)] = -\omega A\sin(\omega t + \delta)$$

Substituting for A, ω, and δ yields:

$$v_x = -(3.0\text{ rad/s})(0.40\text{ m})\sin\left[(3.0\text{ rad/s})t + \frac{\pi}{4}\right] = \boxed{-(1.2\text{ m/s})\sin\left[(3.0\text{ rad/s})t + \frac{\pi}{4}\right]}$$

(*c*) The particle's maximum speed occurs when $\sin(\omega t + \delta) = \pm 1$:

$$|v_{x\,\max}| = |-(1.2\text{ m/s})(\pm 1)| = \boxed{1.2\text{ m/s}}$$

90 • An astronaut arrives at a new planet, and gets out his simple device to determine the gravitational acceleration there. Prior to his arrival, he noted that the radius of the planet was 7550 km. If his 0.500-m-long pendulum has a period of 1.0 s, what is the mass of the planet?

Picture the Problem We can apply Newton's second law and the law of gravity to an object at the surface of the new planet to obtain an expression for the mass of the planet as a function of the acceleration due to gravity at its surface. We can use the period of the astronaut's pendulum to obtain an expression for the acceleration of gravity a_g at the surface of the new planet.

Apply Newton's second law and the law of gravity to an object of mass m at the surface of the planet:

$$\frac{GM_{\text{planet}}m}{R_{\text{planet}}^2} = ma_g \Rightarrow M_{\text{planet}} = \frac{a_g R_{\text{planet}}^2}{G}$$

The period of the astronaut's simple pendulum is related to the gravitational field a_g at the surface of the new planet:

$$T = 2\pi\sqrt{\frac{L}{a_g}} \Rightarrow a_g = \frac{4\pi^2 L}{T^2}$$

Substituting for a_g and simplifying yields:

$$M_{\text{planet}} = \frac{4\pi^2 R_{\text{planet}}^2 L}{GT^2}$$

Substitute numerical values and evaluate M_{planet}:

$$M_{\text{planet}} = \frac{4\pi^2(7550\text{ km})^2(0.500\text{ m})}{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.0\text{ s})^2} = \boxed{1.7\times10^{25}\text{ kg}}$$

91 •• A pendulum clock keeps perfect time on Earth's surface. In which case will the error be greater: if the clock is placed in a mine of depth h or if the clock is elevated to a height h? Prove your answer and assume $h << R_E$.

Picture the Problem Assume that the density of Earth ρ is constant and let m represent the mass of the clock. We can decide the question of where the clock is more accurate by applying the law of gravitation to the clock at a depth h

below/above the surface of Earth and at Earth's surface and expressing the ratios of the acceleration due to gravity below/above the surface of Earth to its value at the surface of Earth.

Express the gravitational force acting on the clock when it is at a depth h in a mine:

$$mg' = \frac{GM'm}{(R_E - h)^2}$$

where M' is the mass between the location of the clock and the center of Earth.

Express the gravitational force acting on the clock at the surface of Earth:

$$mg = \frac{GM_E m}{R_E^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{g'}{g} = \frac{\dfrac{GM'}{(R_E - h)^2}}{\dfrac{GM_E}{R_E^2}} = \frac{M'}{M_E}\frac{R_E^2}{(R_E - h)^2}$$

Express M':

$$M' = \rho V' = \tfrac{4}{3}\pi\rho(R_E - h)^3$$

Express M_E:

$$M_E = \rho V = \tfrac{4}{3}\pi\rho R_E^3$$

Substitute for M' and M_E to obtain:

$$\frac{g'}{g} = \frac{\tfrac{4}{3}\pi\rho(R_E - h)^3}{\tfrac{4}{3}\pi\rho R_E^3}\frac{R_E^2}{(R_E - h)^2}$$

Simplifying and solving for g' yields:

$$g' = g\left(\frac{R_E - h}{R_E}\right) = g\left(1 - \frac{h}{R_E}\right)$$

or

$$g' = g\left(1 - \frac{h}{R_E}\right) \qquad (1)$$

Express the gravitational force acting on the clock when it is at an elevation h:

$$mg'' = \frac{GM_E m}{(R_E + h)^2}$$

Express the gravitational force acting on the clock at the surface of Earth:

$$mg = \frac{GM_E m}{R_E^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{g''}{g}=\frac{\dfrac{GM_E}{(R_E+h)^2}}{\dfrac{GM_E}{R_E^2}}=\frac{R_E^2}{(R_E-h)^2}$$

Factoring R_E^2 from the denominator yields:

$$\frac{g''}{g}=\frac{1}{\left(1+\dfrac{h}{R_E}\right)^2}$$

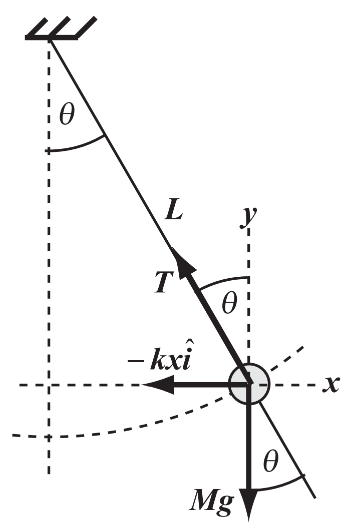
Solve for g'' to obtain:

$$g''=g\left(1+\frac{h}{R_E}\right)^{-2} \qquad (2)$$

Comparing equations (1) and (2), we see that g' is closer to g than is g''. Thus the error is greater if the clock is elevated.

92 •• Figure 14-34 shows a pendulum of length *L* with a bob of mass *M*. The bob is attached to a spring that has a force constant *k*. When the bob is directly below the pendulum support, the spring is unstressed. (*a*) Derive an expression for the period of this oscillating system for small-amplitude vibrations. (*b*) Suppose that M = 1.00 kg and *L* is such that in the absence of the spring the period is 2.00 s. What is the force constant *k* if the period of the oscillating system is 1.00 s?

Picture the Problem The pictorial representation shows this system when it has an angular displacement θ. The period of the system is related to its angular frequency according to $T = 2\pi/\omega$. We can find the equation of motion of the system by applying Newton's second law. By writing this equation in terms of θ and using a small-angle approximation, we'll find an expression for ω that we can use to express T.



(*a*) The period of the system in terms of its angular frequency is given by:

$$T=\frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum \vec{F} = m\vec{a}$ to the bob:

$\sum F_x = -kx - T\sin\theta = Ma_x$

and

$\sum F_y = T\cos\theta - Mg = 0$, assuming no deviation from horizontal motion.

Eliminate T between the two equations to obtain:

$-kx - Mg\tan\theta = Ma_x$

Noting that $x = L\theta$ and $a_x = L\alpha = L\dfrac{d^2\theta}{dt^2}$, eliminate the variable x in favor of θ:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\tan\theta$$

For $\theta << 1$, $\tan\theta \approx \theta$:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\theta$$
$$= -(kL + Mg)\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\left(\frac{k}{M} + \frac{g}{L}\right)\theta = -\omega^2\theta$$

where $\omega = \sqrt{\dfrac{k}{M} + \dfrac{g}{L}}$

Substitute in equation (1) to obtain:

$$T = \boxed{\frac{2\pi}{\sqrt{\dfrac{k}{M} + \dfrac{g}{L}}}}$$

(*b*) When $k = 0$ (no spring), $T = 2.00$ s, and $M = 1.00$ kg we have:

$$2.00\text{ s} = \frac{2\pi}{\sqrt{\dfrac{g}{L}}} \qquad (2)$$

With the spring present and $T = 1.00$ s we have:

$$1.00\text{ s} = \frac{2\pi}{\sqrt{k\text{ kg}^{-1} + \dfrac{g}{L}}} \qquad (3)$$

Solving equations (2) and (3) simultaneously yields:

$k = \boxed{29.6\text{ N/m}}$

93 •• **[SSM]** A block that has a mass equal to m_1 is supported from below by a frictionless horizontal surface. The block, which is attached to the end of a horizontal spring with a force constant k, oscillates with an amplitude A. When the spring is at its greatest extension and the block is instantaneously at rest, a second block of mass m_2 is placed on top of it. (*a*) What is the smallest value for the coefficient of static friction μ_s such that the second object does not slip on the first? (*b*) Explain how the total mechanical energy E, the amplitude A, the angular frequency ω, and the period T of the system are affected by the placing of m_2 on m_1, assuming that the coefficient of friction is great enough to prevent slippage.

Picture the Problem Applying Newton's second law to the first object as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the system which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(*a*) Apply $\sum F_x = ma_x$ to the second object as it is about to slip:

$$f_{s,\max} = m_2 a_{\max}$$

Apply $\sum F_y = 0$ to the second object:

$$F_n - m_2 g = 0$$

Use $f_{s,\max} = \mu_s F_n$ to eliminate $f_{s,\max}$ and F_n between the two equations and solve for μ_s:

$$\mu_s m_2 g = m_2 a_{\max} \Rightarrow \mu_s = \frac{a_{\max}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency and substitute for ω^2:

$$a_{\max} = A\omega^2 = A\frac{k}{m_1 + m_2}$$

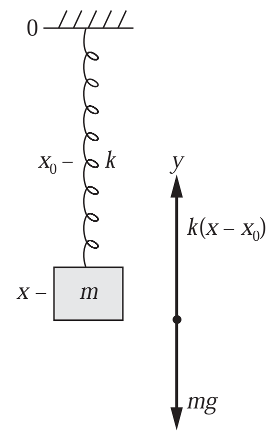
Finally, substitute for $a_{\max}$ to obtain:

$$\mu_s = \boxed{\frac{Ak}{(m_1 + m_2)g}}$$

(*b*) A is unchanged. E is unchanged because $E = \frac{1}{2}kA^2$. ω is reduced and T is increased by increasing the total mass of the system.

94 •• A 100-kg box hangs from the ceiling of a room−suspended from a spring with a force constant of 500 N/m. The unstressed length of the spring is 0.500 m. (*a*) Find the equilibrium position of the box. (*b*) An identical spring is stretched and attached to the ceiling and box and is parallel with the first spring. Find the frequency of the oscillations when the box is released. (*c*) What is the new equilibrium position of the box once it comes to rest?

Picture the Problem The pictorial representation shows the box hanging from the stretched spring and the free-body diagram when the box is in equilibrium. We can apply $\sum F_y = 0$ to the box to derive an expression for x. In (b) and (c), we can proceed similarly to obtain expressions for the effective force constant, the new equilibrium position of the box, and frequency of oscillations when the box is released.



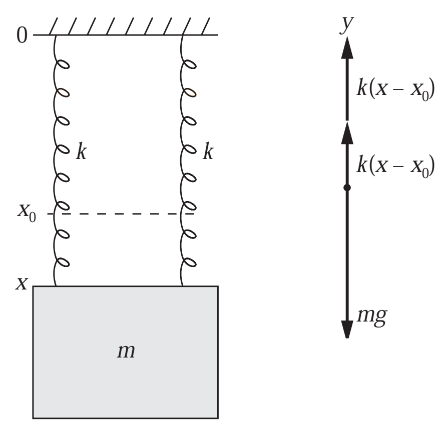
(a) Apply $\sum F_y = 0$ to the box to obtain:

$$k(x - x_0) - mg = 0 \Rightarrow x = \frac{mg}{k} + x_0$$

Substitute numerical values and evaluate x:

$$x = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{500\,\text{N/m}} + 0.500\,\text{m}$$
$$= \boxed{2.46\,\text{m}}$$

(b) Draw the free-body diagram for the block with the two springs exerting equal upward forces on it:



Apply $\sum F_y = 0$ to the box to obtain:

$$k(x - x_0) + k(x - x_0) - mg = 0$$

or

$$k_{\text{eff}}(x - x_0) - mg = 0 \qquad (1)$$

where $k_{\text{eff}} = 2k$

When the box is displaced from this equilibrium position and released, its motion is simple harmonic motion and its frequency is given by:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2k}{m}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{2(500\ \text{N/m})}{100\ \text{kg}}} = \boxed{3.16\ \text{rad/s}}$$

(*c*) Solve equation (1) for x to obtain:

$$x = \frac{mg}{2k} + x_0$$

Substitute numerical values and evaluate x:

$$x = \frac{(100\ \text{kg})(9.81\ \text{m/s}^2)}{2(500\ \text{N/m})} + 0.500\ \text{m} = \boxed{1.48\ \text{m}}$$

95 •• The acceleration due to gravity g varies with geographical location because of Earth's rotation and because Earth is not exactly spherical. This was first discovered in the seventeenth century, when it was noted that a pendulum clock carefully adjusted to keep correct time in Paris lost about 90 s/d near the equator. (*a*) Show by using the differential approximation that a small change in the acceleration of gravity Δg produces a small change in the period ΔT of a pendulum given by $\Delta T / T \approx -\frac{1}{2}\Delta g / g$. (*b*) How large a change in g is needed to account for a 90 s/d change in the period?

Picture the Problem We'll differentiate the expression for the period of simple pendulum $T = 2\pi\sqrt{L/g}$ with respect to g, separate the variables, and use a differential approximation to establish that $\dfrac{\Delta T}{T} \approx -\dfrac{1}{2}\dfrac{\Delta g}{g}$.

(*a*) Express the period of a simple pendulum in terms of its length and the local value of the acceleration due to gravity:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Differentiate this expression with respect to g to obtain:

$$\frac{dT}{dg} = \frac{d}{dg}\left[2\pi\sqrt{L}g^{-1/2}\right] = -\pi\sqrt{L}g^{-3/2} = -\frac{T}{2g}$$

Separate the variables to obtain:

$$\frac{dT}{T} = -\frac{1}{2}\frac{dg}{g}$$

For $\Delta g << g$ we can approximate dT and dg by ΔT and Δg:

$$\frac{\Delta T}{T} \approx \boxed{-\frac{1}{2}\frac{\Delta g}{g}}$$

(*b*) Solve the equation in Part (*a*) for Δg:

$$\Delta g = -2g\frac{\Delta T}{T}$$

Substitute numerical values and evaluate Δg for a 90 s/d change in the period:

$$\Delta g = -2\left(9.81\,\text{m/s}^2\right)\left(-90\frac{\text{s}}{\text{d}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right) = \boxed{2.0\,\text{cm/s}^2}$$

96 •• A small block that has a mass equal to m_1 rests on a piston that is vibrating vertically with simple harmonic motion described by the formula $y = A\sin\omega t$. (*a*) Show that the block will leave the piston if $\omega^2 A > g$. (*b*) If $\omega^2 A = 3g$ and $A = 15$ cm, at what time will the block leave the piston?

Picture the Problem If the displacement of the block is $y = A\sin\omega t$, its acceleration is $a = -\omega^2 A\sin\omega t$.

(*a*) At maximum upward extension, the block is momentarily at rest. Its downward acceleration is g. The downward acceleration of the piston is $\omega^2 A$. Therefore, if $\omega^2 A > g$, the block will separate from the piston.

(*b*) Express the acceleration of the small block:

$$a = -A\omega^2\sin\omega t$$

For $\omega^2 A = 3g$ and $A = 15$ cm:

$$a = -3g\sin\omega t = -g$$

Solving for t yields:

$$t = \frac{1}{\omega}\sin^{-1}\left(\frac{1}{3}\right) = \sqrt{\frac{A}{3g}}\sin^{-1}\left(\frac{1}{3}\right)$$

Substitute numerical values and evaluate t:

$$t = \sqrt{\frac{0.15\,\text{m}}{3\left(9.81\,\text{m/s}^2\right)}}\sin^{-1}\left(\frac{1}{3}\right) = \boxed{24\,\text{ms}}$$

97 •• **[SSM]** Show that for the situations in Figure 14-35*a* and Figure 14-35*b* the object oscillates with a frequency $f = (1/2\pi)\sqrt{k_{\text{eff}}/m}$, where k_{eff} is given by (*a*) $k_{\text{eff}} = k_1 + k_2$, and (*b*) $1/k_{\text{eff}} = 1/k_1 + 1/k_2$. *Hint: Find the magnitude of the net force F on the object for a small displacement x and write $F = -k_{\text{eff}}x$. Note that in Part (b) the springs stretch by different amounts, the sum of which is x.*

Picture the Problem Choose a coordinate system in which the $+x$ direction is to the right and assume that the object is displaced to the right. In case (*a*), note that the two springs undergo the same displacement whereas in (*b*) they experience the same force.

(a) Express the net force acting on the object:

$$F_{\text{net}} = -k_1 x - k_2 x = -(k_1 + k_2)x = -k_{\text{eff}} x$$
where $k_{\text{eff}} = \boxed{k_1 + k_2}$

(b) Express the force acting on each spring and solve for x_2:

$$F = -k_1 x_1 = -k_2 x_2 \Rightarrow x_2 = \frac{k_1}{k_2} x_1$$

Express the total extension of the springs:

$$x_1 + x_2 = -\frac{F}{k_{\text{eff}}}$$

Solving for k_{eff} yields:

$$k_{\text{eff}} = -\frac{F}{x_1 + x_2} = -\frac{-k_1 x_1}{x_1 + x_2} = \frac{k_1 x_1}{x_1 + \frac{k_1}{k_2} x_1} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Take the reciprocal of both sides of the equation to obtain:

$$\frac{1}{k_{\text{eff}}} = \boxed{\frac{1}{k_1} + \frac{1}{k_2}}$$

98 •• During an earthquake, a floor oscillates horizontally in approximately simple harmonic motion. Assume it oscillates at a single frequency with a period of 0.80 s. (a) After the earthquake, you are in charge of examining the video of the floor motion and discover that a box on the floor started to slip when the amplitude reached 10 cm. From your data, determine the coefficient of static friction between the box and the floor. (b) If the coefficient of friction between the box and floor were 0.40, what would be the maximum amplitude of vibration before the box would slip?

Picture the Problem Applying Newton's second law to the box as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the platform which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(a) Apply $\sum F_x = ma_x$ to the box as it is about to slip:

$$f_{\text{s,max}} = ma_{\text{max}}$$

Apply $\sum F_y = 0$ to the box:

$$F_{\text{n}} - mg = 0$$

Use $f_{\text{s,max}} = \mu_{\text{s}} F_{\text{n}}$ to eliminate $f_{\text{s,max}}$ and F_{n} between the two equations:

$$\mu_{\text{s}} mg = ma_{\text{max}} \text{ and } \mu_{\text{s}} = \frac{a_{\text{max}}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency:

$$a_{max} = A\omega^2$$

Substitute for a_{max} in the expression for μ_s:

$$\mu_s = \frac{A\omega^2}{g} = \frac{4\pi^2 A}{T^2 g}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{4\pi^2(0.10\,\text{m})}{(0.80\,\text{s})^2(9.81\,\text{m/s}^2)} = \boxed{0.63}$$

(*b*) Solve the equation derived above for A_{max}:

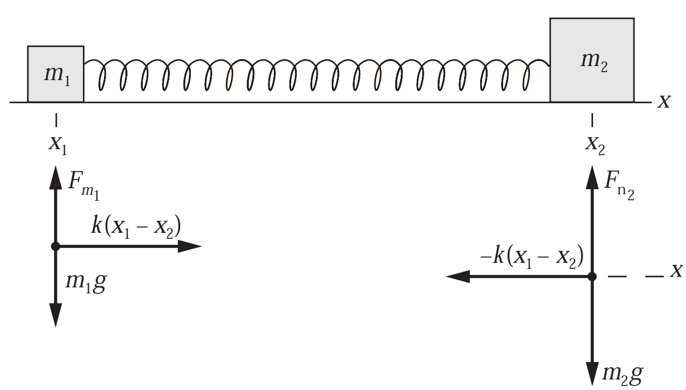
$$A_{max} = \frac{\mu_s g}{\omega^2} = \frac{\mu_s g T^2}{4\pi^2}$$

Substitute numerical values and evaluate A_{max}:

$$A_{max} = \frac{(0.40)(9.81\,\text{m/s}^2)(0.80\,\text{s})^2}{4\pi^2} = \boxed{6.4\,\text{cm}}$$

99 •• If we attach two blocks of masses m_1 and m_2 to either end of a spring of force constant k and set them into oscillation by releasing them from rest with the spring stretched, show that the oscillation frequency is given by $\omega = (k/\mu)^{1/2}$, where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of the system.

Picture the Problem The pictorial representation shows the two blocks connected by the spring and displaced from their equilibrium positions. We can apply Newton's second law to each of these coupled oscillators and solve the resulting equations simultaneously to obtain the equation of motion of the coupled oscillators. We can then compare this equation and its solution to the equation of motion of the simple harmonic oscillator and its solution to show that the oscillation frequency is $\omega = (k/\mu)^{1/2}$ where $\mu = m_1m_2/(m_1 + m_2)$.



Apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_1 and solve for its acceleration:

$$k(x_1 - x_2) = m_1 a_1 = m_1 \frac{d^2 x_1}{dt^2}$$

or

$$a_1 = \frac{d^2 x_1}{dt^2} = \frac{k}{m_1}(x_1 - x_2)$$

Apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_2 and solve for its acceleration:

$$-k(x_1 - x_2) = m_2 a_2 = m_2 \frac{d^2 x_2}{dt^2}$$

or

$$a_2 = \frac{d^2 x_2}{dt^2} = \frac{k}{m_2}(x_2 - x_1)$$

Subtract the first equation from the second to obtain:

$$\frac{d^2(x_2 - x_1)}{dt^2} = \frac{d^2 x}{dt^2} = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x$$

where $x = x_2 - x_1$

The reduced mass of the system is:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Substitute to obtain:

$$\frac{d^2 x}{dt^2} = -\frac{k}{\mu}x \qquad (1)$$

Compare this equation to the equation of the simple harmonic oscillator:

$$\frac{d^2 x}{dt^2} = -\frac{k}{m}x$$

The solution to this equation is:

$$x = x_0 \cos(\omega t + \delta)$$

where $\omega = \sqrt{\frac{k}{m}}$

Because of the similarity of the two differential equations, the solution to equation (1) must be:

$$x = x_0 \cos(\omega t + \delta)$$

where $\omega = \boxed{\sqrt{\frac{k}{\mu}}}$ and $\mu = \boxed{\frac{m_1 m_2}{m_1 + m_2}}$

100 •• In one of your chemistry labs you determine that one of the vibrational modes of the HCl molecule has a frequency of 8.969×10^{13} Hz. Using the result of Problem 99, find the effective ″spring constant″ between the H atom and the Cl atom in the HCl molecule.

Picture the Problem We can use $\omega = (k/\mu)^{1/2}$ and $\mu = m_1 m_2/(m_1 + m_2)$ from Problem 99 to find the spring constant for the HCl molecule.

Use the result of Problem 99 to relate the oscillation frequency to the force constant and reduced mass of the HCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}} \Rightarrow k = \mu\omega^2$$

Express the reduced mass of the HCl molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Substitute for μ to obtain:

$$k = \frac{m_1 m_2 \omega^2}{m_1 + m_2}$$

Express the masses of the hydrogen and Cl atoms:

$m_1 = 1\text{ amu} = 1.67\times10^{-27}\text{ kg}$
and
$m_2 = 35.45\text{ amu} = 5.92\times10^{-26}\text{ kg}$

Substitute numerical values and evaluate k:

$$k = \frac{\left(1.673\times10^{-27}\text{ kg}\right)\left(5.92\times10^{-26}\text{ kg}\right)\left(8.969\times10^{13}\text{ s}^{-1}\right)^2}{1.673\times10^{-27}\text{ kg} + 5.92\times10^{-26}\text{ kg}} = \boxed{13.1\text{ N/m}}$$

101 •• If a hydrogen atom in HCl were replaced by a deuterium atom (forming DCl) in Problem 100, what would be the new vibration frequency of the molecule? Deuterium consists of 1 proton and 1 neutron.

Picture the Problem In Problem 100, we derived an expression for the oscillation frequency of a spring-and-two-block system as a function of the force constant of the spring and the reduced mass of the two blocks. We can solve this problem, assuming that the "spring constant" does not change, by using the result of Problem 101 and the reduced mass of a deuterium atom and a Cl atom in the equation for the oscillation frequency.

Use the result of Problem 100 to relate the oscillation frequency to the force constant and reduced mass of the DCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

Express the reduced mass of the DCl molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

The masses of the deuterium and Cl atoms are:

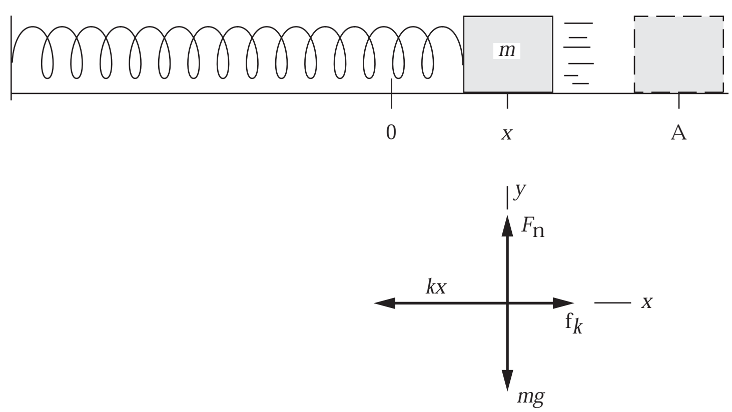
$m_1 = 2\text{ amu} = 3.34\times10^{-27}\text{ kg}$
and
$m_2 = 35.45\text{ amu} = 5.92\times10^{-26}\text{ kg}$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{13.1\,\text{N/m}}{\frac{(3.34\times10^{-27}\,\text{kg})(5.92\times10^{-26}\,\text{kg})}{3.34\times10^{-27}\,\text{kg}+5.92\times10^{-26}\,\text{kg}}}} = \boxed{6.44\times10^{13}\,\text{rad/s}}$$

102 ••• A block of mass m on a horizontal table is attached to a spring of force constant k, as shown in Figure 14-36. The coefficient of kinetic friction between the block and the table is μ_k. The spring is unstressed if the block is at the origin (x = 0), and the +x direction is to the right. The spring is stretched a distance A, where $kA > \mu_k mg$, and the block is released. (*a*) Apply Newton's second law to the block to obtain an equation for its acceleration d^2x/dt^2 for the first half-cycle, during which the block is moving to the left. Show that the resulting equation can be written as $d^2x'/dt^2 = -\omega^2 x'$, where $\omega = \sqrt{k/m}$ and $x' = x - x_0$, with $x_0 = \mu_k mg/k = \mu_k g/\omega^2$. (*b*) Repeat Part (*a*) for the second half-cycle as the block moves to the right, and show that $d^2x''/dt^2 = -\omega^2 x''$, where $x'' = x + x_0$ and x_0 has the same value. (*c*) Use a **spreadsheet program** to graph the first 5 half-cycles for $A = 10x_0$. Describe the motion, if any, after the fifth half-cycle.

Picture the Problem The pictorial representation shows the block moving from right to left with an instantaneous displacement x from its equilibrium position. The free-body diagram shows the forces acting on the block during the half-cycles that it moves from right to left. When the block is moving from left to right, the directions of the kinetic friction force and the restoring force exerted by the spring are reversed. We can apply Newton's second law to these motions to obtain the equations given in the problem statements and then use their solutions to plot the graph called for in (*c*).



(*a*) Apply $\sum F_x = ma_x$ to the block while it is moving to the left to obtain:

$$f_k - kx = m\frac{d^2x}{dt^2}$$

Using $f_k = \mu_k F_n = \mu_k mg$, eliminate f_k in the equation of motion:

$$m\frac{d^2x}{dt^2} = -kx + \mu_k mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x - \frac{\mu_k mg}{k}\right)$$

Let $x_0 = \dfrac{\mu_k mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x - x_0)$$

or

$$\boxed{\frac{d^2x'}{dt^2} = -\frac{k}{m}x' = -\omega^2 x'}$$

provided $x' = x - x_0$ and

$$x_0 = \boxed{\frac{\mu_k mg}{k} = \frac{\mu_k g}{\omega^2}}$$

The solution to the equation of motion is:

$$x' = x_0' \cos(\omega t + \delta)$$

and its derivative is

$$v' = -\omega x_0' \sin(\omega t + \delta)$$

The initial conditions are:

$$x'(0) = x - x_0 \text{ and } v'(0) = 0$$

Apply these conditions to obtain:

$$x'(0) = x_0' \cos\delta = x - x_0$$

and

$$v'(0) = -\omega x_0' \sin\delta = 0$$

Solve these equations simultaneously to obtain:

$$\delta = 0 \text{ and } x_0' = x - x_0$$

and

$$x' = (x - x_0)\cos\omega t$$

or

$$x = \boxed{(x - x_0)\cos\omega t + x_0} \qquad (1)$$

(*b*) Apply $\sum \vec{F} = m\vec{a}$ to the block while it is moving to the right to obtain:

$$-f_k - kx = m\frac{d^2x}{dt^2}$$

Using $f_k = \mu_k F_n = \mu_k mg$, eliminate f_k in the equation of motion:

$$m\frac{d^2x}{dt^2} = -kx - \mu_k mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x + \frac{\mu_k mg}{k}\right)$$

Let $x_0 = \dfrac{\mu_k mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x + x_0)$$

or

$$\boxed{\frac{d^2x''}{dt^2} = -\frac{k}{m}x'' = -\omega^2 x''}$$

provided $x'' = x + x_0$ and

$$x_0 = \boxed{\frac{\mu_k mg}{k} = \frac{\mu_k g}{\omega^2}}.$$

The solution to the equation of motion is:

$$x'' = x_0'' \cos(\omega t + \delta)$$

and its derivative is

$$v'' = -\omega x_0'' \sin(\omega t + \delta)$$

The initial conditions are:

$$x''(0) = x + x_0 \text{ and } v''(0) = 0$$

Apply these conditions to obtain:

$$x''(0) = x_0'' \cos\delta = x + x_0$$

and

$$v''(0) = -\omega x_0'' \sin\delta = 0$$

Solve these equations simultaneously to obtain:

$$\delta = 0 \text{ and } x_0'' = x + x_0$$

and

$$x'' = (x + x_0)\cos\omega t$$

or

$$\boxed{x = (x + x_0)\cos\omega t - x_0} \qquad (2)$$

(*c*) A spreadsheet program to calculate the position of the oscillator as a function of time (equations (1) and (2)) is shown below. The constants used in the position functions (x_0 = 1 m and T = 2 s were used for simplicity) and the formulas used to calculate the positions are shown in the table. *After each half-period, one must compute a new amplitude for the oscillation, using the final value of the position from the last half-period.*

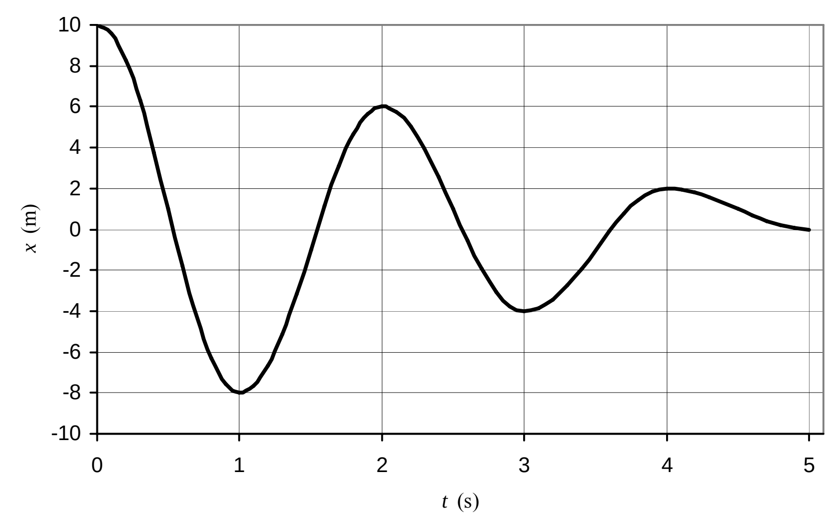
Cell	Content/Formula	Algebraic Form
B1	1	x_0
B2	10	A
C7	C6 + 0.1	$t + \Delta t$
D7	($B$2−$B$1)*COS(PI()*C7)+$B$1	$(A - x_0)\cos\pi t + x_0$
D17	(ABS($D$6+$B$1))*COS(PI()*C17)−$B$1	$\lvert x + x_0\rvert \cos\pi t - x_0$
D27	(ABS($D$6−$B$1))*COS(PI()*C27)+$B$1	$\lvert x - x_0\rvert \cos\pi t + x_0$
D37	(ABS($D$36+$B$1))*COS(PI()*C37)−$B$1	$\lvert x + x_0\rvert \cos\pi t - x_0$

D47	($D$46−$B$1)*COS(PI()*C47)+$B$1	$(x - x_0)\cos \pi t + x_0$

	A	B	C	D
1	x_0=	1	m	
2	A=	10		
3				
4			t	x
5			(s)	(m)
6			0.0	10.00
7			0.1	9.56
8			0.2	8.28
9			0.3	6.29
10			0.4	3.78
53			4.7	0.41
54			4.8	0.19
55			4.9	0.05
56			5.0	0.00

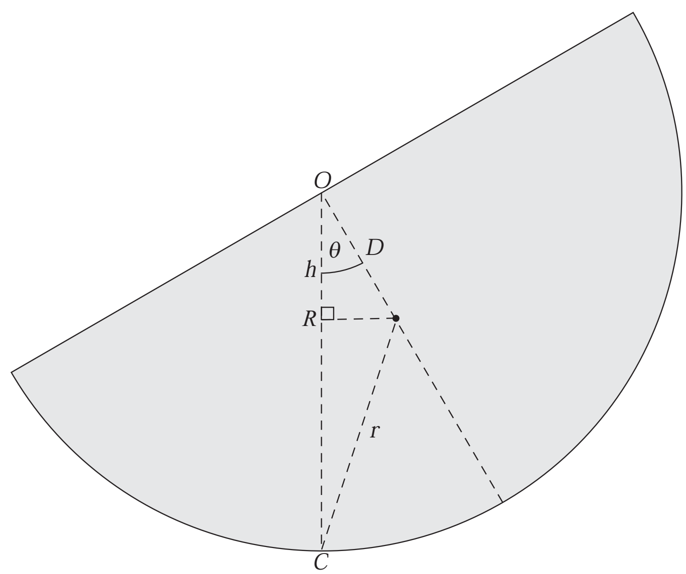
The following graph was plotted using the data from columns C (t) and D (x).

$\boxed{\text{Note that the motion of the block ceases after five half - cycles.}}$



103 ••• Figure 14-37 shows a uniform solid half-cylinder of mass M and radius R resting on a horizontal surface. If one side of this cylinder is pushed down slightly and then released, the half-cylinder will oscillate about its equilibrium position. Determine the period of this oscillation.

Picture the Problem The diagram shows the half-cylinder displaced from its equilibrium position through an angle θ. The frequency of its motion will be found by expressing the mechanical energy E in terms of θ and $d\theta/dt$. For small θ we will obtain an equation of the form $E = \frac{1}{2}\kappa\theta^2 + \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$. Differentiating both sides of this equation with respect to time will lead to $0 = \left(\kappa\theta + I\frac{d^2\theta}{dt^2}\right)\frac{d\theta}{dt}$, an equation that must be valid at all times. Because the situation of interest to us requires that $d\theta/dt$ is not always equal to zero, we have $0 = \kappa\theta + I\frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$, the equation of simple harmonic motion with $\omega^2 = \kappa/I$. We'll show that the distance from O to the center of mass D, is given by $D = \frac{4R}{3\pi}$, and let the distance from the contact point C to the center of mass be r. Finally, we'll take the potential energy to be zero where θ is zero and assume that there is no slipping.



Apply conservation of energy to obtain:

$$E = U + K$$

$$= Mg(h - D) + \frac{1}{2}I_C\left(\frac{d\theta}{dt}\right)^2 \quad (1)$$

From Table 9-1, the moment of inertia of a solid cylinder about an axis perpendicular to its face and through its center is given by:

$$I_{0,\,\text{solid cylinder}} = \frac{1}{2}(2M)R^2 = MR^2$$

where M is the mass of the half-cylinder.

Express the moment of inertia of the half-cylinder about the same axis:

$$I_{0,\text{ half cylinder}} = I_0 = \frac{1}{2}\left[MR^2\right] = \frac{1}{2}MR^2$$

Use the parallel-axis theorem to relate I_{cm} to I_0:

$$I_0 = I_{cm} + MD^2$$

Substitute for I_0 and solve for I_{cm}:

$$I_{cm} = I_0 - D^2M = \frac{1}{2}MR^2 - D^2M$$

Apply the parallel-axis theorem a second time to obtain an expression for I_C:

$$\begin{aligned} I_C &= \frac{1}{2}MR^2 - D^2M + Mr^2 \\ &= M\left(\frac{1}{2}R^2 - D^2 + r^2\right) \end{aligned} \qquad (2)$$

Apply the law of cosines to obtain:

$$r^2 = R^2 + D^2 - 2RD\cos\theta$$

Substitute for r^2 in equation (2) to obtain:

$$I_C = M\left(\frac{1}{2}R^2 - D^2 + R^2 + D^2 - 2RD\cos\theta\right) = MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)$$

Substitute for h and I_C in equation (1):

$$E = MgD(1-\cos\theta) + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)\left(\frac{d\theta}{dt}\right)^2$$

Use the small angle approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - \frac{D}{R}\left[2-\theta^2\right]\right)\left(\frac{d\theta}{dt}\right)^2$$

Because $\theta^2 << 2$, we can neglect the θ^2 in the square brackets to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d\theta}{dt}\right)^2$$

Differentiating both sides with respect to time and simplifying yields:

$$R^2\left(\frac{3}{2}-2\frac{D}{R}\right)\left(\frac{d^2\theta}{dt^2}\right)+gD\theta=0\,,$$

or

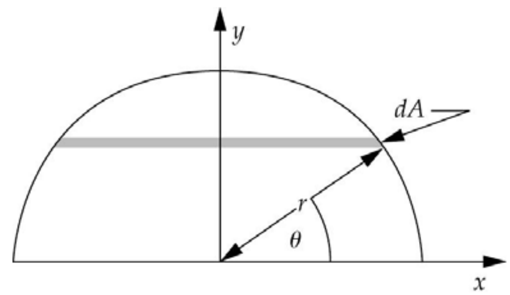
$$\frac{d^2\theta}{dt^2}+\frac{gD}{R^2\left(\frac{3}{2}-2\frac{D}{R}\right)}\theta=0\,,$$

the equation of simple harmonic motion with $\omega^2=\dfrac{gD}{R^2\left(\frac{3}{2}-2\frac{D}{R}\right)}$. (3)

D is the y coordinate of the center of mass of the semicircular disk shown. A surface element of area dA is shown in the diagram. Because the disk is a continuous object, we'll use

$$M\vec{r}_{\rm cm}=\int\vec{r}dm$$

to find $y_{\rm cm}=D$.



Express the coordinates of the center of mass of the semicircular disk:

$x_{\rm cm}=0$ by symmetry.

$$y_{\rm cm}=D=\frac{\int y\sigma\,dA}{M}$$

Express y as a function of r and θ:

$$y=r\sin\theta$$

Express dA in terms of r and θ:

$$dA=r\,d\theta\,dr$$

Substitute for y and dA and evaluate D:

$$D=\frac{\sigma\int_0^R\int_0^\pi r^2\sin\theta\,d\theta\,dr}{M}=\frac{2\sigma}{M}\int_0^R r^2dr$$

$$=\frac{2\sigma}{3M}R^3$$

Express M as a function of r and θ:

$$M=\sigma A_{\text{half disk}}=\tfrac{1}{2}\sigma\pi R^2$$

Substituting for M and simplifying yields:

$$D=\frac{2\sigma}{3\left(\frac{1}{2}\sigma\pi R^2\right)}R^3=\boxed{\frac{4}{3\pi}R}$$

Substitute for D in equation (3) and simplify to obtain:

$$\omega^2 = \frac{\frac{4}{3\pi}}{\left(\frac{3}{2} - \frac{8}{3\pi}\right)}\frac{g}{R} = \left(\frac{8}{9\pi - 16}\right)\frac{g}{R}$$

The period of the motion is given by:

$$T = \frac{2\pi}{\omega}$$

Substituting for ω and simplifying yields:

$$T = 2\pi\sqrt{\left(\frac{9\pi - 16}{8}\right)\frac{R}{g}} = \boxed{7.78\sqrt{\frac{R}{g}}}$$

104 ••• A straight tunnel is dug through Earth as shown in Figure 14-38. Assume that the walls of the tunnel are frictionless. (*a*) The gravitational force exerted by Earth on a particle of mass *m* at a distance *r* from the center of Earth when $r < R_E$ is $F_r = -\left(GmM_{\mathrm{E}} / R_{\mathrm{E}}^3\right)r$, where M_{E} is the mass of Earth and R_{E} is its radius. Show that the net force on a particle of mass *m* at a distance *x* from the middle of the tunnel is given by $F_x = -\left(GmM_{\mathrm{E}} / R_{\mathrm{E}}^3\right)x$, and that the motion of the particle is therefore simple harmonic motion. (*b*) Show that the period of the motion is independent of the length of the tunnel and is given by $T = 2\pi\sqrt{R_{\mathrm{E}}/g}$. (*c*) Find its numerical value in minutes.

Picture the Problem The net force acting on the particle as it moves in the tunnel is the *x*-component of the gravitational force acting on it. We can find the period of the particle from the angular frequency of its motion. We can apply Newton's second law to the particle in order to express ω in terms of the radius of Earth and the acceleration due to gravity at the surface of Earth.

(*a*) From the figure we see that:

$$F_x = F_r \sin\theta = -\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} r \frac{x}{r} = \boxed{-\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} x}$$

Because this force is a linear restoring force, the motion of the particle is simple harmonic motion.

(*b*) Express the period of the particle as a function of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum F_x = ma_x$ to the particle:

$$-\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} x = ma$$

Solving for a yields:

$$a = -\frac{GM_E}{R_E^3}x = -\omega^2 x$$

where $\omega = \sqrt{\frac{GM_E}{R_E^3}}$

Use $GM_E = gR_E^2$ to simplify ω:

$$\omega = \sqrt{\frac{gR_E^2}{R_E^3}} = \sqrt{\frac{g}{R_E}}$$

Substitute in equation (1) to obtain:

$$T = \frac{2\pi}{\sqrt{\frac{g}{R_E}}} = \boxed{2\pi\sqrt{\frac{R_E}{g}}}$$

(c) Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{6.37\times10^6\ \text{m}}{9.81\,\text{m/s}^2}} = 5.06\times10^3\ \text{s}$$

$$= \boxed{84.4\,\text{min}}$$

105 ••• [SSM] In this problem, derive the expression for the average power delivered by a driving force to a driven oscillator (Figure 14-39).
(a) Show that the instantaneous power input of the driving force is given by

$$P = Fv = -A\,\omega F_0 \cos\omega t \sin(\omega t - \delta).$$

(b) Use the identity $\sin(\theta_1 - \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$ to show that the equation in (a) can be written as

$$P = A\omega F_0 \sin\delta\cos^2\omega t - A\omega F_0\cos\delta\cos\omega t\sin\omega t$$

(c) Show that the average value of the second term in your result for (b) over one or more periods is zero, and that therefore $P_{av} = \frac{1}{2}A\omega F_0 \sin\delta$.
(d) From Equation 14-56 for $\tan\delta$, construct a right triangle in which the side opposite the angle δ is $b\omega$ and the side adjacent is $m(\omega_0^2 - \omega^2)$, and use this triangle to show that

$$\sin\delta = \frac{b\omega}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}} = \frac{b\omega A}{F_0}.$$

(e) Use your result for Part (d) to eliminate ωA from your result for Part (c) so that the average power input can be written as

$$P_{av} = \frac{1}{2}\frac{F_0^2}{b}\sin^2\delta = \frac{1}{2}\left[\frac{b\omega^2F_0^2}{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}\right].$$

Picture the Problem We can follow the step-by-step instructions provided in the problem statement to obtain the desired results.

(*a*) Express the average power delivered by a driving force to a driven oscillator:

$P = \vec{F}\cdot\vec{v} = Fv\cos\theta$
or, because θ is 0°,
$P = Fv$

Express F as a function of time:

$F = F_0\cos\omega t$

Express the position of the driven oscillator as a function of time:

$x = A\cos(\omega t - \delta)$

Differentiate this expression with respect to time to express the velocity of the oscillator as a function of time:

$v = -A\omega\sin(\omega t - \delta)$

Substitute to express the average power delivered to the driven oscillator:

$$P = (F_0\cos\omega t)[-A\omega\sin(\omega t - \delta)] = \boxed{-A\omega F_0\cos\omega t\sin(\omega t - \delta)}$$

(*b*) Expand $\sin(\omega t - \delta)$ to obtain:

$\sin(\omega t - \delta) = \sin\omega t\cos\delta - \cos\omega t\sin\delta$

Substitute in your result from (*a*) and simplify to obtain:

$$P = -A\omega F_0\cos\omega t(\sin\omega t\cos\delta - \cos\omega t\sin\delta) = \boxed{\begin{array}{l} A\omega F_0\sin\delta\cos^2\omega t \\ \quad - A\omega F_0\cos\delta\cos\omega t\sin\omega t \end{array}}$$

(*c*) Integrate $\sin\theta\cos\theta$ over one period to determine $\langle\sin\theta\cos\theta\rangle$:

$$\langle\sin\theta\cos\theta\rangle = \frac{1}{2\pi}\left[\int_0^{2\pi}\sin\theta\cos\theta d\theta\right] = \frac{1}{2\pi}\left[\frac{1}{2}\sin^2\theta\Big|_0^{2\pi}\right] = 0$$

Integrate $\cos^2\theta$ over one period to determine $\langle\cos^2\theta\rangle$:

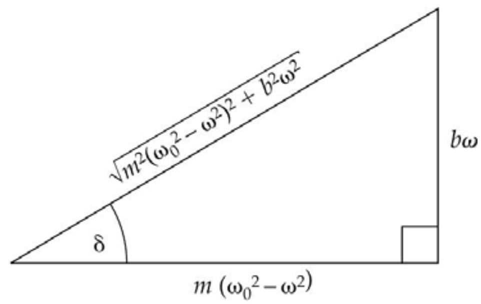
$$\langle\cos^2\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2\pi}\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta\right] = \frac{1}{2\pi}\left[\tfrac{1}{2}\int_0^{2\pi}d\theta + \tfrac{1}{2}\int_0^{2\pi}\cos 2\theta d\theta\right] = \frac{1}{2\pi}(\pi + 0) = \frac{1}{2}$$

Substitute and simplify to express P_{av}:

$$P_{\text{av}} = A\omega F_0 \sin\delta\left\langle\cos^2\omega t\right\rangle - A\omega F_0\cos\delta\left\langle\cos\omega t\sin\omega t\right\rangle = \tfrac{1}{2}A\omega F_0\sin\delta - A\omega F_0\cos\delta(0) = \boxed{\tfrac{1}{2}A\omega F_0\sin\delta}$$

(*d*) Construct a triangle that is consistent with

$$\tan\delta = \frac{b\omega}{m\left(\omega_0^2-\omega^2\right)}:$$



Using the triangle, express $\sin\delta$:

$$\sin\delta = \boxed{\frac{b\omega}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)^2+b^2\omega^2}}}$$

Using Equation 14-56, reduce this expression to the simpler form:

$$\sin\delta = \boxed{\frac{b\omega A}{F_0}}$$

(*e*) Solve $\sin\delta = \frac{b\omega A}{F_0}$ for ω:

$$\omega = \frac{F_0}{bA}\sin\delta$$

Substitute in the expression for P_{av} to eliminate ω:

$$P_{\text{av}} = \boxed{\frac{F_0^2}{2b}\sin^2\delta}$$

Substitute for $\sin\delta$ from (*d*) to obtain:

$$P_{\text{av}} = \boxed{\frac{1}{2}\left[\frac{b\omega^2F_0^2}{m^2\left(\omega_0^2-\omega^2\right)^2+b^2\omega^2}\right]}$$

106 ••• In this problem, you are to use the result of Problem 105 to derive Equation 14-51. At resonance, the denominator of the fraction in brackets in Problem 105(*e*) is $b^2\omega_0^2$ and P_{av} has its maximum value. For a sharp resonance, the variation in ω in the numerator in this equation can be neglected. Then the power input will be half its maximum value at the values of ω, for which the denominator is $2b^2\omega_0^2$.

(*a*) Show that ω then satisfies $m^2\left(\omega-\omega_0\right)^2\left(\omega+\omega_0\right)^2 \approx b^2\omega_0^2$.

(*b*) Using the approximation $\omega+\omega_2 \approx 2\omega_0$, show that $\omega-\omega_0 \approx \pm b/2m$.

(*c*) Express b in terms of Q.

(*d*) Combine the results of (*b*) and (*c*) to show that there are two values of ω for which the power input is half that at resonance and that they are given by

$$\omega_1 = \omega_0 - \frac{\omega_0}{2Q} \text{ and } \omega_2 = \omega_0 - \frac{\omega_0}{2Q}$$

Therefore, $\omega_2 - \omega_1 = \Delta\omega = \omega_0 / Q$, which is equivalent to Equation 14-51.

Picture the Problem We can follow the step-by-step instructions given in the problem statement to derive the given results.

(*a*) Express the condition on the denominator of Equation 14-56 when the power input is half its maximum value:

$$m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2 = 2b^2\omega_0^2$$

and, for a sharp resonance,

$$m^2\left(\omega_0^2 - \omega^2\right)^2 \approx b^2\omega_0^2$$

Factor the difference of two squares to obtain:

$$m^2\left[(\omega_0 - \omega)(\omega_0 + \omega)\right]^2 \approx b^2\omega_0^2$$

or

$$\boxed{m^2(\omega_0 - \omega)^2(\omega_0 + \omega)^2 \approx b^2\omega_0^2}$$

(*b*) Use the approximation $\omega + \omega_0 \approx 2\omega_0$ to obtain:

$$m^2(\omega_0 - \omega)^2(2\omega_0)^2 \approx b^2\omega_0^2$$

Solving for $\omega_0 - \omega$ yields:

$$\omega_0 - \omega = \boxed{\pm\frac{b}{2m}} \qquad (1)$$

(*c*) Using its definition, express *Q*:

$$Q = \frac{\omega_0 m}{b} \Rightarrow b = \boxed{\frac{\omega_0 m}{Q}}$$

(*d*) Substitute for *b* in equation (1) to obtain:

$$\omega_0 - \omega = \pm\frac{\omega_0}{2Q} \Rightarrow \omega = \omega_0 \pm \frac{\omega_0}{2Q}$$

Express the two values of ω:

$$\omega_+ = \boxed{\omega_0 + \frac{\omega_0}{2Q}} \text{ and } \omega_- = \boxed{\omega_0 - \frac{\omega_0}{2Q}}$$

Remarks: Note that the width of the resonance at half-power is $\Delta\omega = \omega_+ - \omega_- = \omega_0/Q$, in agreement with Equation 14-51.

107 ••• The Morse potential, which is often used to model interatomic forces, can be written in the form $U(r) = D\left(1 - e^{-\beta(r - r_0)}\right)^2$, where *r* is the distance between the two atomic nuclei. (*a*) Using a **spreadsheet program** or **graphing calculator**, make a graph of the Morse potential using D = 5.00 eV, β = 0.20 nm^{-1}, and

$r_0 = 0.750$ nm. (*b*) Determine the equilibrium separation and "spring constant" for small displacements from equilibrium for the Morse potential. (*c*) Determine an expression for the oscillation frequency for a *homonuclear* diatomic molecule (that is, two of the same atoms), where the atoms each have mass *m*.

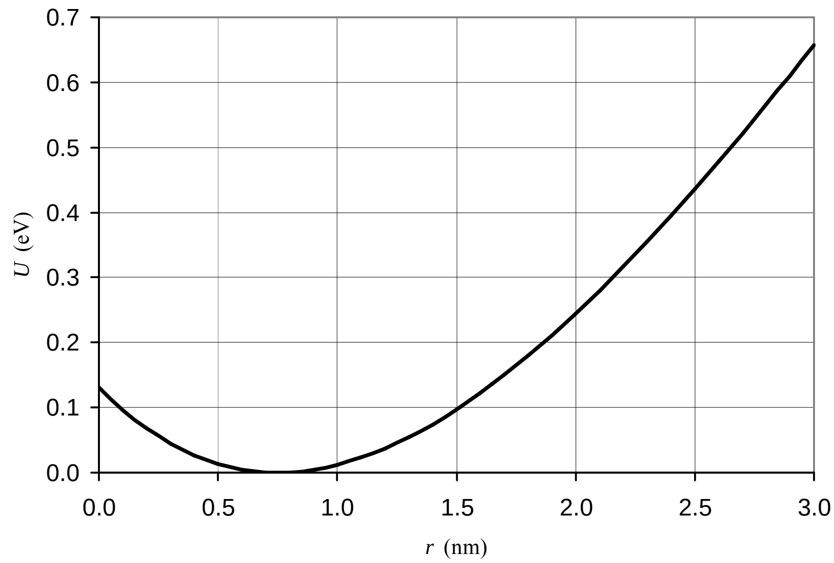
Picture the Problem We can find the equilibrium separation for the Morse potential by setting $dU/dr = 0$ and solving for *r*. The second derivative of *U* will give the "spring constant" for small displacements from equilibrium. In (*c*), we can use $\omega = \sqrt{k/\mu}$, where *k* is our result from (*b*) and μ is the reduced mass of a homonuclear diatomic molecule, to find the oscillation frequency of the molecule.

(*a*) A spreadsheet program to calculate the Morse potential as a function of *r* is shown below. The constants and cell formulas used to calculate the potential are shown in the table.

Cell	Content/Formula	Algebraic Form
B1	5	D
B2	0.2	β
C9	C8 + 0.1	$r + \Delta r$
D8	$B$1*(1−EXP(−$B$2*(C8−$B$3)))^2	$D\left[1-e^{-\beta(r-r_0)}\right]^2$

	A	B	C	D
1	D=	5	eV	
2	β=	0.2	nm^{-1}	
3	r_0=	0.75	nm	
4				
5				
6			r	$U(r)$
7			(nm)	(eV)
8			0.0	0.13095
9			0.1	0.09637
10			0.2	0.06760
11			0.3	0.04434
12			0.4	0.02629
235			22.7	4.87676
236			22.8	4.87919
237			22.9	4.88156
238			23.0	4.88390
239			23.1	4.88618

The following graph was plotted using the data from columns C (r) and D ($U(r)$).



(*b*) Differentiate the Morse potential with respect to r to obtain:

$$\frac{dU}{dr} = \frac{d}{dr}\left\{D\left[1-e^{-\beta(r-r_0)}\right]^2\right\} = -2\beta D\left[1-e^{-\beta(r-r_0)}\right]$$

This derivative is equal to zero for extrema:

$$-2\beta D\left[1-e^{-\beta(r-r_0)}\right] = 0 \Rightarrow r = \boxed{r_0}$$

Evaluate the second derivative of $U(r)$ to obtain:

$$\frac{d^2U}{dr^2} = \frac{d}{dr}\left\{-2\beta D\left[1-e^{-\beta(r-r_0)}\right]\right\} = 2\beta^2 D e^{-\beta(r-r_0)}$$

Evaluate this derivative at $r = r_0$:

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 2\beta^2 D \qquad (1)$$

Recall that the potential function for a simple harmonic oscillator is:

$$U = \tfrac{1}{2}kx^2$$

Differentiate this expression twice to obtain:

$$\frac{d^2U}{dx^2} = k$$

By comparison with equation (1) we have:

$$k = \boxed{2\beta^2 D}$$

(c) Express the oscillation frequency of the diatomic molecule:

$$\omega=\sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass of the molecule.

Express the reduced mass of the homonuclear diatomic molecule:

$$\mu=\frac{m_1 m_2}{m_1+m_2}=\frac{m^2}{2m}=\frac{m}{2}$$

Substitute for ω and simplify to obtain:

$$\omega=\sqrt{\frac{2\beta^2 D}{\frac{m}{2}}}=\boxed{2\beta\sqrt{\frac{D}{m}}}$$

Remarks: An alternative approach in (*b*) is to expand the Morse potential in a Taylor series

$$\boldsymbol{U(r)=U(r_0)+(r-r_0)U'(r_0)+\frac{1}{2!}(r-r_0)^2 U''(r_0)+\textbf{higher order terms}}$$

to obtain $\boldsymbol{U(r)\approx\beta^2 D(r-r_0)^2}$. Comparing this expression to the energy of a spring-and-mass oscillator we see that, as was obtained above, $\boldsymbol{k=2\beta^2 D}$.

Chapter 15
Traveling Waves

Conceptual Problems

1 • [SSM] A rope hangs vertically from the ceiling. A pulse is sent up the rope. Does the pulse travel faster, slower, or at a constant speed as it moves toward the ceiling? Explain your answer.

Determine the Concept The speed of a transverse wave on a uniform rope increases with increasing tension. The waves on the rope move faster as they move toward the ceiling because the tension increases due to the weight of the rope below the pulse.

2 • A pulse on a horizontal taut string travels to the right. If the rope's mass per unit length decreases to the right, what happens to the speed of the pulse as it travels to the right? (*a*) It slows down. (*b*) It speeds up. (*c*) Its speed is constant. (*d*) You cannot tell from the information given.

Determine the Concept The speed v of a pulse on the string varies with the tension F_T in the string and its mass per unit length μ according to $v=\sqrt{F_T/\mu}$. Because the rope's mass per unit length decreases to the right, the speed of the pulse increases. $\boxed{(b)}$ is correct.

3 • As a sinusoidal wave travels past a point on a taut string, the arrival time between successive crests is measured to be 0.20 s. Which of the following is true? (*a*) The wavelength of the wave is 5.0 m. (*b*) The frequency of the wave is 5.0 Hz. (*c*) The velocity of propagation of the wave is 5.0 m/s. (*d*) The wavelength of the wave is 0.20 m. (*e*) There is not enough information to justify any of these statements.

Determine the Concept The distance between successive crests is one wavelength and the time between successive crests is the period of the wave motion. Thus, T = 0.20 s and $f = 1/T = 5.0$ Hz. $\boxed{(b)}$ is correct.

4 • Two harmonic waves on identical strings differ only in amplitude. Wave A has an amplitude that is twice that of wave B's. How do the energies of these waves compare? (*a*) $E_A = E_B$. (*b*) $E_A = 2E_B$. (*c*) $E_A = 4E_B$. (*d*) There is not enough information to compare their energies.

Picture the Problem The average energy transmitted by a wave on a string is proportional to the square of its amplitude and is given by $(\Delta E)_{av} = \frac{1}{2}\mu\omega^2 A^2 \Delta x$ where A is the amplitude of the wave, μ is the linear density (mass per unit length)

of the string, ω is the angular frequency of the wave, and Δx is the length of the string.

Because the waves on the strings differ only in amplitude, the energy of the wave on string A is given by:

$$E_{\mathrm{A}} = cA_{\mathrm{A}}^2$$

Express the energy of the wave on string B:

$$E_{\mathrm{B}} = cA_{\mathrm{B}}^2$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{E_{\mathrm{A}}}{E_{\mathrm{B}}} = \frac{cA_{\mathrm{A}}^2}{cA_{\mathrm{B}}^2} = \left(\frac{A_{\mathrm{A}}}{A_{\mathrm{B}}}\right)^2$$

Because $A_{\mathrm{A}} = 2A_{\mathrm{B}}$:

$$\frac{E_{\mathrm{A}}}{E_{\mathrm{B}}} = \left(\frac{2A_{\mathrm{B}}}{A_{\mathrm{B}}}\right)^2 = 4 \Rightarrow \boxed{(c)} \text{ is correct.}$$

5 • [SSM] To keep all of the lengths of the treble strings (unwrapped steel wires) in a piano all about the same order of magnitude, wires of different linear mass densities are employed. Explain how this allows a piano manufacturer to use wires with lengths that are the same order of magnitude.

Determine the Concept The resonant (standing wave) frequencies on a string are inversely proportional to the square root of the linear density of the string $\left(f = \sqrt{T_{\mathrm{T}}/\mu}/\lambda\right)$. Thus extremely high frequencies (which might otherwise require very long strings) can be accommodated on relatively short strings if the strings are linearly less dense than the high frequency strings. High frequencies are not a problem as they use short strings anyway.

6 • Musical instruments produce sounds of widely varying frequencies. Which sounds waves have the longer wavelengths? (*a*) The lower frequencies. (*b*) The higher frequencies. (*c*) All frequencies have the same wavelength. (*d*) There is not enough information to compare the wavelengths of the different frequency sounds.

Determine the Concept Once the sound has been produced by a vibrating string, membrane, or air column, the speed with which it propagates is the product of its frequency and wavelength $(v = f\lambda)$. For a given medium and temperature, v is constant. Hence the wavelength and frequency for a given sound are inversely proportional and $\boxed{(a)}$ is correct.

7 • In Problem 6, which sound waves have the higher speeds? (*a*) The lower frequency sounds. (*b*) The higher frequency sounds. (*c*) All frequencies have the same wave speed. (*d*) There is not enough information to compare their speeds.

Determine the Concept Once the sound has been produced by a vibrating string, membrane, or air column, the wave speed with which it propagates depends on the properties of the medium in which it is propagating and is independent of the frequency and wavelength of the sound. $\boxed{(c)}$ is correct.

8 • Sound travels at 343 m/s in air and 1500 m/s in water. A sound of 256 Hz is made under water, but you hear the sound while walking along the side of the pool. In the air, the frequency is (*a*) the same, but the wavelength of the sound is shorter, (*b*) higher, but the wavelength of the sound stays the same, (*c*) lower, but the wavelength of the sound is longer, (*d*) lower, and the wavelength of the sound is shorter, (*e*) the same, and the wavelength of the sound stays the same.

Determine the Concept In any medium, the wavelength, frequency, and speed of a sound wave are related through $\lambda = v/f$. Because the frequency of a wave is determined by its source and is independent of the nature of the medium, if v is greater in water than in air, λ will be shorter in air than in water. $\boxed{(a)}$ is correct.

9 • While out on patrol, the battleship *Rodger Young* hits a mine, begins to burn, and ultimately explodes. Sailor Abel jumps into the water and begins swimming away from the doomed ship, while Sailor Baker gets into a life raft. Comparing their experiences later, Abel tells Baker, ″I was swimming underwater, and heard a big explosion from the ship. When I surfaced, I heard a second explosion. What do you think it could be?″ Baker says, ″I think it was your imagination—I only heard one explosion. ″ Explain why Baker only heard one explosion, while Abel heard two.

Determine the Concept There was only one explosion. Sound travels faster in water than air. Abel heard the sound wave in the water first, then, surfacing, heard the sound wave traveling through the air, which took longer to reach him.

10 • True or false: A 60-dB sound has twice the intensity of a 30-dB sound.

Picture the Problem The intensity level β, measured in decibels, is given by $\beta = (10\,\text{dB})\log(I/I_0)$ where $I_0 = 10^{-12}$ W/m^2 is defined to be the threshold of hearing.

Express the intensity level of the 60-dB sound:

$$60\,\text{dB} = (10\,\text{dB})\log\frac{I_{60}}{I_0} \Rightarrow I_{60} = 10^6 I_0$$

Express the intensity level of the 30-dB sound:

$$30\,\text{dB} = (10\,\text{dB})\log\frac{I_{30}}{I_0} \Rightarrow I_{30} = 10^3 I_0$$

Because $I_{60} = 10^3 I_{30}$, the statement is false.

11 • **[SSM]** At a given location, two harmonic sound waves have the same displacement amplitude, but the frequency of sound A is twice the frequency of sound B. How do their average energy densities compare? (*a*) The average energy density of A is twice the average energy density of B. (*b*) The average energy density of A is four times the average energy density of B. (*c*) The average energy density of A is 16 times the average energy density of B. (*d*) You cannot compare the average energy densities from the data given.

Determine the Concept The average energy density of a sound wave is given by $\eta_{\text{av}} = \frac{1}{2}\rho\omega^2 s_0^2$ where ρ is the average density of the medium, s_0 is the displacement amplitude of the molecules making up the medium, and ω is the angular frequency of the sound waves.

Express the average energy density of sound A:

$$\eta_{\text{av, A}} = \tfrac{1}{2}\rho_{\text{A}}\omega_{\text{A}}^2 s_{0,\text{A}}^2$$

The average energy density of sound B is given by:

$$\eta_{\text{av, B}} = \tfrac{1}{2}\rho_{\text{B}}\omega_{\text{B}}^2 s_{0,\text{B}}^2$$

Dividing the first of these equation by the second yields:

$$\frac{\eta_{\text{av, A}}}{\eta_{\text{av, B}}} = \frac{\tfrac{1}{2}\rho_{\text{A}}\omega_{\text{A}}^2 s_{0,\text{A}}^2}{\tfrac{1}{2}\rho_{\text{B}}\omega_{\text{B}}^2 s_{0,\text{B}}^2}$$

Because the sound waves are identical except for their frequencies:

$$\frac{\eta_{\text{av, A}}}{\eta_{\text{av, B}}} = \frac{\omega_{\text{A}}^2}{\omega_{\text{B}}^2} = \left(\frac{2\pi f_{\text{A}}}{2\pi f_{\text{B}}}\right)^2 = \left(\frac{f_{\text{A}}}{f_{\text{B}}}\right)^2$$

Because $f_{\text{A}} = 2f_{\text{B}}$:

$$\frac{\eta_{\text{av, A}}}{\eta_{\text{av, B}}} = \left(\frac{2f_{\text{B}}}{f_{\text{B}}}\right)^2 = 4 \Rightarrow \boxed{(b)} \text{ is correct.}$$

12 • At a given location, two harmonic sound waves have the same frequency but the amplitude of sound A is twice the amplitude of sound B. How do their average energy densities compare? (*a*) The average energy density of A is twice the average energy density of B. (*b*) The average energy density of A is four times the average energy density of B (*c*) The average energy density of A is 16

times the average energy density of B (*d*) You cannot compare the average energy densities from the data given.

Determine the Concept The average energy density of a sound wave is given by $\eta_{av} = \frac{1}{2}\rho\omega^2 s_0^2$ where ρ is the average density of the medium, s_0 is the displacement amplitude of the molecules making up the medium, and ω is the angular frequency of the sound waves.

Express the average energy density of sound A:

$$\eta_{av,A} = \tfrac{1}{2}\rho_A\omega_A^2 s_{0,A}^2$$

The average energy density of sound B is given by:

$$\eta_{av,B} = \tfrac{1}{2}\rho_B\omega_B^2 s_{0,B}^2$$

Dividing the first of these equation by the second yields:

$$\frac{\eta_{av,A}}{\eta_{av,B}} = \frac{\frac{1}{2}\rho_A\omega_A^2 s_{0,A}^2}{\frac{1}{2}\rho_B\omega_B^2 s_{0,B}^2}$$

Because the sound waves are identical except for their displacement amplitudes:

$$\frac{\eta_{av,A}}{\eta_{av,B}} = \frac{s_{0,A}^2}{s_{0,B}^2} = \left(\frac{s_{0,A}}{s_{0,B}}\right)^2$$

Because $s_{0,A} = 2s_{0,B}$

$$\frac{\eta_{av,A}}{\eta_{av,B}} = \left(\frac{2s_{0,B}}{s_{0,B}}\right)^2 = 4 \Rightarrow \boxed{(b)} \text{ is correct.}$$

13 • What is the ratio of the intensity of normal conversation to the sound intensity of a soft whisper (at a distance of 5.0 m)? (*a*) 10^3, (*b*) 2, (*c*) 10^{-3}, (*d*) 1/2. *Hint: See Table 15-1.*

Determine the Concept The problem is asking us for the ratio of the intensities corresponding to normal conversation and a soft whisper. Table 15-1 includes the quantities we need in order to find this ratio.

Use Table 15-1 to find the ratio of I/I_0, where I_0 is the threshold of hearing, for normal conversation:

$$\left(\frac{I}{I_0}\right)_{\substack{\text{normal}\\\text{conversation}}} = 10^6$$

Use Table 15-1 to find the ratio of I/I_0 for a soft whisper:

$$\left(\frac{I}{I_0}\right)_{\substack{\text{soft}\\\text{whisper}}} = 10^3$$

Dividing the first of these equation by the second yields:

$$\frac{\left(\frac{I}{I_0}\right)_{\substack{\text{normal}\\\text{conversation}}}}{\left(\frac{I}{I_0}\right)_{\substack{\text{soft}\\\text{whisper}}}}=\frac{10^6}{10^3}=10^3$$

$\boxed{(a)}$ is correct.

14 • What is the ratio of the intensity level of normal conversation to the sound intensity level of a soft whisper (at a distance of 5.0 m)? (*a*) 10^3, (*b*) 2, (*c*) 10^{-3}, (*d*) 1/2. *Hint: See Table 15-1.*

Determine the Concept The problem is asking us for the ratio of the sound intensity levels (decibel levels) corresponding to normal conversation and a soft whisper. Table 15-1 includes the quantities we need in order to find this ratio.

From Table 15-1, the sound intensity level corresponding to normal conversation is:

$$\beta_{\substack{\text{normal}\\\text{conversation}}}=60\text{ dB}$$

From Table 15-1, the sound intensity level corresponding to a soft whisper is:

$$\beta_{\substack{\text{soft}\\\text{whisper}}}=30\text{ dB}$$

Dividing the first of these equation by the second yields:

$$\frac{\beta_{\substack{\text{normal}\\\text{conversation}}}}{\beta_{\substack{\text{soft}\\\text{whisper}}}}=\frac{60\text{ dB}}{30\text{ dB}}=2$$

$\boxed{(b)}$ is correct.

15 • To increase the sound intensity level by 20 dB requires the sound intensity to increase by what factor? (*a*) 10, (*b*) 100, (*c*) 1000, (*d*) 2.

Determine the Concept The sound intensity level is given by $\beta=(10\text{ dB})\log(I/I_0)$ where I is the intensity of a given sound and I_0 is the intensity of the threshold of hearing.

The sound intensity level for a sound whose intensity is I_1 is given by:

$$\beta_1=(10\text{ dB})\log\left(\frac{I_1}{I_0}\right)$$

The sound intensity level for a sound whose intensity is I_2 is given by:

$$\beta_2=(10\text{ dB})\log\left(\frac{I_2}{I_0}\right)$$

Subtract the second equation from the first and simplify to obtain:

$$\beta_1 - \beta_2 = (10\text{ dB})\left(\log\left(\frac{I_1}{I_2}\right)\right)$$

Solving for I_1/I_2 yields:

$$\frac{I_1}{I_2} = 10^{\frac{\beta_1-\beta_2}{10\text{ dB}}}$$

For a 20-dB increase in the sound intensity level:

$$\frac{I_1}{I_2} = 10^{\frac{20\text{ dB}}{10\text{ dB}}} = 10^2 \Rightarrow \boxed{(b)} \text{ is correct.}$$

16 • You are using a hand-held sound level meter to measure the intensity level of the roars produced by a lion prowling in the high grass. To decrease the measured sound intensity level by 20 dB requires the lion move away from you until its distance from you has increased by what factor? (*a*) 10, (*b*) 100, (*c*) 1000, (*d*) You cannot tell the required distance from the data given.

Determine the Concept The sound intensity level is given by $\beta = (10\text{ dB})\log(I/I_0)$ where I is the intensity of a given sound and I_0 is the intensity of the threshold of hearing. The intensity of the a given sound varies with distance from its source according to $I = P_{\text{av}}/(4\pi r^2)$.

The sound intensity level for a sound whose intensity is I_1 is given by:

$$\beta_1 = (10\text{ dB})\log\left(\frac{I_1}{I_0}\right)$$

The sound intensity level for a sound whose intensity is I_2 is given by:

$$\beta_2 = (10\text{ dB})\log\left(\frac{I_2}{I_0}\right)$$

Subtract the second equation from the first and simplify to obtain:

$$\beta_1 - \beta_2 = (10\text{ dB})\left(\log\left(\frac{I_1}{I_2}\right)\right)$$

Solving for I_1/I_2 yields:

$$\frac{I_1}{I_2} = 10^{\frac{\beta_1-\beta_2}{10\text{ dB}}}$$

Letting r_1 be the lion's initial distance from you and r_2 be the lion's final distance from you, substitute for I_1 and I_2 and simplify to obtain:

$$\frac{\dfrac{P_{\text{av}}}{4\pi r_1^2}}{\dfrac{P_{\text{av}}}{4\pi r_2^2}} = \frac{r_2^2}{r_1^2} = 10^{\frac{\beta_1-\beta_2}{10\text{ dB}}} \Rightarrow \frac{r_2}{r_1} = \sqrt{10^{\frac{\beta_1-\beta_2}{10\text{ dB}}}}$$

For a 20-dB decrease in the sound intensity level:

$$\frac{r_2}{r_1} = \sqrt{10^{\frac{20\,\text{dB}}{10\,\text{dB}}}} = 10 \Rightarrow \boxed{(a)} \text{ is correct.}$$

17 • One end of a very light (but strong) thread is attached to an end of a thicker and denser cord. The other end of the thread is fastened to a sturdy post and you pull the other end of the cord so the thread and cord are taut. A pulse is sent down the thicker, denser cord. True or false:

(*a*) The pulse that is reflected back from the thread-cord attachment point is inverted compared to the initial incoming pulse.
(*b*) The pulse that continues past the thread-cord attachment point is not inverted compared to the initial incoming pulse.
(*c*) The pulse that continues past the thread-cord attachment point has an amplitude that is smaller than the pulse that is reflected.

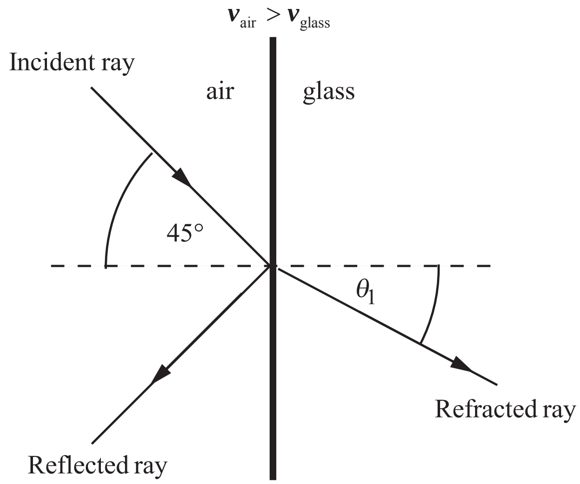
(*a*) False. Because the reflection medium (cord) in which the pulse travels initially has a greater linear density than the transmission medium (thread), there is no phase shift in the reflected pulse.

(*b*) True. Because the thread-cord attachment point of the media is more like a loose end than a fixed end, the pulse transmitted into the light thread is in phase with the incoming pulse in the thicker, denser cord.

(*c*) True. Because the string is attached to a thicker, denser cord, the reflected pulse behaves almost as though it was reflected from a fixed end and is, therefore, inverted. The pulse in the light thread is in phase with the incoming pulse in the thicker, denser cord. Because energy is conserved and there is some energy transmitted and some reflected, the transmitted wave cannot have an amplitude as large as the amplitude of the pulse in the thicker, denser cord.

18 • Light traveling in air strikes a glass surface at an incident angle of 45°. True or false: (*a*) The angle between the reflected light ray and the incident ray is 90°. (*b*) The angle between the reflected light ray and the refracted light ray is less than 90°.

Determine the Concept The ray diagram shows the relationship between the incident, reflected, and refracted rays.



(*a*) True. Because the angle of incidence is equal to the angle of reflection, the sum of these angles in this example is 90°. Note: this is a special case and is not generally true.

(*b*) False. Because light travels slower in glass than in air, the refracted ray makes an angle of less than 45° with the normal. Hence the angle it makes with the reflected ray is greater than 90°.

19 • **[SSM]** Sound waves in air encounter a 1.0-m wide door into a classroom. Due to the effects of diffraction, the sound of which frequency is least likely to be heard by all the students in the room, assuming the room is full? (*a*) 600 Hz, (*b*) 300 Hz, (*c*) 100 Hz, (*d*) All the sounds are equally likely to be heard in the room. (*e*) Diffraction depends on wavelength not frequency, so you cannot tell from the data given.

Determine the Concept If the wavelength is large relative to the door, the diffraction effects are large and the waves spread out as they pass through the door. Because we're interested in sounds that are least likely to be heard everywhere in the room, we want the wavelength to be short and the frequency to be high. Hence $\boxed{(a)}$ is correct.

20 • Microwave radiation in modern microwave ovens has a wavelength on the order of centimeters. Would you expect significant diffraction if this radiation was aimed at a 1.0-m wide door? Explain.

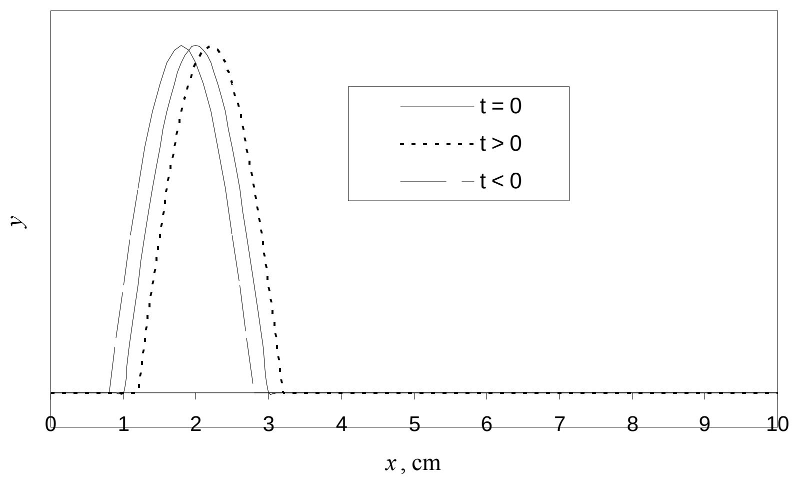
Determine the Concept No. Because the wavelength of the radiation is small relative to the door, the diffraction effects are small and the waves do not spread out significantly as they pass through the door.

21 •• [SSM] Stars often occur in pairs revolving around their common center of mass. If one of the stars is a black hole, it is invisible. Explain how the existence of such a black hole might be inferred by measuring the Doppler frequency shift of the light observed from the other, visible star.

Determine the Concept The light from the visible star will be shifted about its mean frequency periodically due to the relative approach toward and recession away from Earth as the star revolves about the common center of mass.

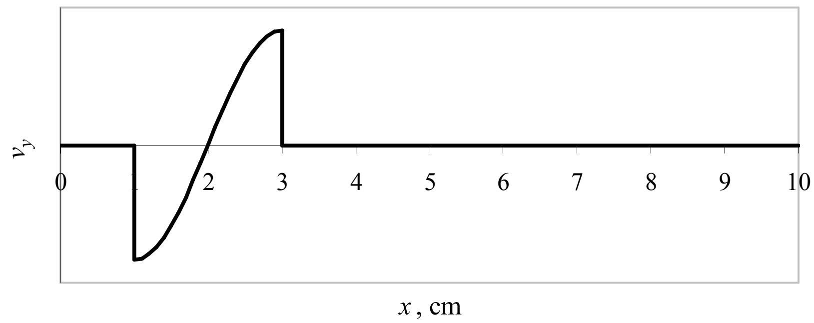
22 •• Figure 15-30 shows a wave pulse at time $t = 0$ moving to the right. (*a*) At this particular time, which segments of the string are moving up? (*b*) Which segments are moving down? (*c*) Is there any segment of the string at the pulse that is instantaneously at rest? Answer these questions by sketching the pulse at a slightly later time and a slightly earlier time to see how the segments of the string are moving.

Determine the Concept The graph shown below shows the pulse at an earlier time ($t < 0$) and later time ($t > 0$). (*a*) One can see that at $t = 0$, the portion of the string between 1 cm and 2 cm is moving down, (*b*) the portion between 2 cm and 3 cm is moving up, and (*c*) the string at $x = 2$ cm is instantaneously at rest.



23 •• Make a sketch of the velocity of each string segment versus position for the pulse shown in Figure 15-30.

Determine the Concept The velocity of the string at $t = 0$ is shown. Note that the velocity is negative for 1 cm < x < 2 cm and is positive for 2 cm < x < 3 cm.



24 •• An object of mass m hangs on a very light rope that is connected to the ceiling. You pluck the rope just above the object, and a wave pulse travels up to the ceiling and back. Compare the round-trip time for such a wave pulse to the round-trip time of a wave pulse on the same rope if an object of mass $9m$ is hung on the rope instead. (Assume the rope does not stretch, that is, the mass-to-ceiling distance is the same in each case.)

Determine the Concept If the mass providing the tension in the rope is increased by a factor of n, then the tension in the rope increases by the same factor and the speed of the pulse on the rope increases by a factor of $\sqrt{n}$ and the round trip time for the pulse is decreased by a factor of $1/\sqrt{n}$.

Let ℓ represent the distance the pulse travels. Then the time for the round trip is given by:

$$t = \frac{\ell}{v} \tag{1}$$

Express the speed of the wave when the object of mass m provides the tension in the rope:

$$v_m = \sqrt{\frac{F_{\mathrm{T},m}}{\mu}}$$

Express the speed of the wave when the object of mass $9m$ provides the tension in the rope:

$$v_{9m} = \sqrt{\frac{F_{\mathrm{T},9m}}{\mu}}$$

Substituting for v_{9m} in equation (1) yields:

$$t_{9m} = \frac{\ell}{\sqrt{\frac{F_{\mathrm{T},9m}}{\mu}}} = \ell\sqrt{\frac{\mu}{F_{\mathrm{T},9m}}} \tag{2}$$

Similarly, the travel time when the object whose mass is m is providing the tension is given by:

$$t_m = \frac{\ell}{\sqrt{\frac{F_{\mathrm{T},m}}{\mu}}} = \ell\sqrt{\frac{\mu}{F_{\mathrm{T},m}}} \tag{3}$$

Divide equation (2) by equation (3) and simplify to obtain:

$$\frac{t_{9m}}{t_m}=\sqrt{\frac{F_{\mathrm{T},m}}{F_{\mathrm{T},9m}}}$$

Because $F_{\mathrm{T},9m}=9F_{\mathrm{T},m}$:

$$\frac{t_{9m}}{t_m}=\sqrt{\frac{F_{\mathrm{T},m}}{9F_{\mathrm{T},m}}}=\tfrac{1}{3}\Rightarrow t_{9m}=\boxed{\tfrac{1}{3}t_m}$$

25 •• The explosion of a depth charge beneath the surface of the water is recorded by a helicopter hovering above its surface, as shown in Figure 15-31. Along which path—A, B, or C—will the sound wave take the least time to reach the helicopter? Explain why you chose the path you did.

Determine the Concept Path C. Because the wave speed is highest in the water, and more of path C is underwater than are paths A or B, the sound wave will spend the least time on path C.

26 •• Does a speed of Mach 2 at an altitude of 60,000 feet mean the same as a speed of Mach 2 near ground level? Explain clearly.

Determine the Concept No. The term Mach 2 means that the speed is twice the speed of sound at a given altitude. Because the speed of sound is lower at high altitudes than at ground level (due to lower density and colder temperature), Mach 2 means less than twice the speed of sound at sea level.

Estimation and Approximation

27 •• Many years ago, Olympic 100 m dashes were started by the sound from a starter's pistol, with the starter positioned several meters down the track, just on the inside of the track. (Today, the pistol that is used is often only a trigger, which is used to electronically activate speakers behind each sprinter's starting blocks. This method avoids the problem of one runner hearing sound before the other runners.) Estimate the time advantage the inside lane (relative to the runner at the outside lane of 8 runners) would have if all runners started when they heard the sound from the starter's pistol.

Picture the Problem Due to the differences in the distances from the starter to the inside- and outside-lane sprinters, the sound of the starting pistol took more time to reach the outside-lane sprinter than it did to reach the inside-lane sprinter.

Express the difference in time for the two sprinters in terms of the distances the sound must travel and the speed of sound:

$$\Delta t=t_{\substack{\text{ouside}\\\text{lane}}}-t_{\substack{\text{inside}\\\text{lane}}}=\frac{\ell_{\substack{\text{outside}\\\text{lane}}}}{v}-\frac{\ell_{\substack{\text{inside}\\\text{lane}}}}{v}$$

Assuming that the separation of the inside- and outside-lane sprinters is w:

$$\ell_{\substack{\text{outside}\\\text{lane}}} = \sqrt{\ell^2_{\substack{\text{inside}\\\text{lane}}} + w^2}$$

Substitute for $\ell_{\substack{\text{outside}\\\text{lane}}}$ to obtain:

$$\Delta t = \frac{\sqrt{\ell^2_{\substack{\text{inside}\\\text{lane}}} + w^2} - \ell_{\substack{\text{inside}\\\text{lane}}}}{v}$$

Assuming that the separation of the inside- and outside-lane sprinters is 7.0 m and that the starter is 5.0 m from the inside-lane sprinter yields:

$$\Delta t = \frac{\sqrt{(5.0\text{ m})^2 + (7.0\text{ m})^2} - 5.0\text{ m}}{343\text{ m/s}}$$
$$= \boxed{11\text{ ms}}$$

Remarks: Because a 100-m sprint is often decided by one or two hundredths of a second, this difference could be significant. This difference led to the invention of the electronic starter's sound.

28 •• Estimate the speed of the bullet as it passes through the helium balloon in Figure 15-32. *Hint: A protractor would be beneficial.*

Picture the Problem You can use a protractor to measure the angle of the shock cone and then estimate the speed of the bullet using $\sin\theta = v/u$. The speed of sound in helium at room temperature (293 K) is 977 m/s.

Relate the speed of the bullet u to the speed of sound v in helium and the angle of the shock cone θ:

$$\sin\theta = \frac{v}{u} \Rightarrow u = \frac{v}{\sin\theta}$$

Measuring θ yields:

$$\theta \approx 70°$$

Substitute numerical values and evaluate u:

$$u = \frac{977\text{ m/s}}{\sin 70°} = \boxed{1.0\text{ km/s}}$$

29 •• The new student townhouses at a local college are in the form of a semicircle half-enclosing the track field. To estimate the speed of sound in air, an ambitious physics student stood at the center of the semicircle and clapped his hands rhythmically at a frequency at which he could not hear the echo of the clap, because it reached him at the same time as his next clap. This frequency was about 2.5 claps/s. Once he established this frequency, he paced off the distance to the townhouses, which was 30 double strides. Assuming that the length of each stride is equal to half his height (5 ft 11 in), estimate the speed of sound in air using these data. How far off is your estimation from the commonly accepted value of 343 m/s?

Picture the Problem Let d be the distance to the townhouses. We can relate the speed of sound to the distance to the townhouses to the frequency of the clapping for which no echo is heard. 5 ft 11 in is equal to 1.8 m.

Relate the speed of sound to the distance it travels to the townhouses and back to the elapsed time:

$$v = \frac{2d}{\Delta t}$$

Express d in terms of the number of strides and distance covered per stride:

$$d = (30\,\text{strides})(1.8\,\text{m/stride}) = 54\,\text{m}$$

Relate the elapsed time Δt to the frequency f of the clapping:

$$\Delta t = \frac{1}{f} = \frac{1}{2.5\,\text{claps/s}} = 0.40\,\text{s}$$

Substitute numerical values and evaluate v:

$$v = \frac{2(54\,\text{m})}{0.40\,\text{s}} = 270\,\text{m/s} = \boxed{0.27\,\text{km/s}}$$

The percent difference between this result and 343 m/s is:

$$\%\,\text{diff} = \frac{343\,\text{m/s} - 270\,\text{m/s}}{343\,\text{m/s}} = \boxed{21\%}$$

Speed of Waves

30 • (a) The bulk modulus of water is 2.00×10^9 N/m^2. Use this value to find the speed of sound in water. (b) The speed of sound in mercury is 1410 m/s. What is the bulk modulus of mercury ($\rho = 13.6 \times 10^3$ kg/m^3)?

Picture the Problem The speed of sound in a fluid is given by $v = \sqrt{B/\rho}$ where B is the bulk modulus of the fluid and ρ is its density.

(a) Express the speed of sound in water in terms of its bulk modulus:

$$v = \sqrt{\frac{B}{\rho}}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{2.00\times10^9\ \text{N/m}^2}{1.00\times10^3\ \text{kg/m}^3}} = 1.414\,\text{km/s}$$
$$= \boxed{1.41\,\text{km/s}}$$

(b) Solving $v = \sqrt{B/\rho}$ for B yields:

$$B = \rho v^2$$

Substitute numerical values and evaluate B:

$$B = (13.6\times10^3\ \text{kg/m}^3)(1410\,\text{m/s})^2$$
$$= \boxed{2.70\times10^{10}\ \text{N/m}^2}$$

31 • Calculate the speed of sound waves in hydrogen gas (M = 2.00 g/mol and γ = 1.40) at T = 300 K.

Picture the Problem The speed of sound in a gas is given by $v = \sqrt{\gamma RT/M}$ where R is the gas constant, T is the absolute temperature, M is the molecular mass of the gas, and γ is a constant that is characteristic of the particular molecular structure of the gas. Because hydrogen gas is diatomic, γ = 1.4.

Express the dependence of the speed of sound in hydrogen gas on the absolute temperature:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{1.4(8.314\,\text{J/mol}\cdot\text{K})(300\,\text{K})}{2.00\times10^{-3}\,\text{kg/mol}}} = \boxed{1.32\,\text{km/s}}$$

32 • A 7.00-m-long string has a mass of 100 g and is under a tension of 900 N. What is the speed of a transverse wave pulse on this string?

Picture the Problem The speed of a transverse wave pulse on a string is given by $v = \sqrt{F_\text{T}/\mu}$ where F_T is the tension in the string, m is its mass, L is its length, and μ is its mass per unit length.

Express the dependence of the speed of the pulse on the tension in the string:

$v = \sqrt{\frac{F_\text{T}}{\mu}}$ where μ is the mass per unit length of the string.

Substitute numerical values and evaluate v:

$$v = \sqrt{\frac{900\,\text{N}}{\dfrac{0.100\,\text{kg}}{7.00\,\text{m}}}} = \boxed{251\,\text{m/s}}$$

33 •• **[SSM]** (*a*) Compute the derivative of the speed of a wave on a string with respect to the tension dv/dF_T, and show that the differentials dv and dF_T obey $dv/v = \frac{1}{2}dF_\text{T}/F_\text{T}$. (*b*) A wave moves with a speed of 300 m/s on a string that is under a tension of 500 N. Using the differential approximation, estimate how much the tension must be changed to increase the speed to 312 m/s. (*c*) Calculate ΔF_T exactly and compare it to the differential approximation result in Part (*b*). Assume that the string does not stretch with the increase in tension.

Picture the Problem (*a*) The speed of a transverse wave on a string is given by $v=\sqrt{F_T/\mu}$ where F_T is the tension in the wire and μ is its linear density. We can differentiate this expression with respect to F_T and then separate the variables to show that the differentials satisfy $dv/v=\frac{1}{2}dF_T/F_T$. (*b*) We'll approximate the differential quantities to determine by how much the tension must be changed to increase the speed of the wave to 312 m/s. (*c*) We can use $v=\sqrt{F_T/\mu}$ to obtain an exact expression for ΔF_T,

(*a*) Evaluate dv/dF_T:

$$\frac{dv}{dF}=\frac{d}{dF_T}\left[\sqrt{\frac{F_T}{\mu}}\right]=\frac{1}{2}\sqrt{\frac{1}{F_T\mu}}=\frac{1}{2}\cdot\frac{v}{F_T}$$

Separate the variables to obtain:

$$\frac{dv}{v}=\boxed{\frac{1}{2}\frac{dF_T}{F_T}}$$

(*b*) Solve the equation derived in Part (*a*) for dF_T:

$$dF_T=2F_T\frac{dv}{v}$$

Approximate dF_T with ΔF_T and dv with Δv to obtain:

$$\Delta F_T=2F_T\frac{\Delta v}{v}$$

Substitute numerical values and evaluate ΔF_T:

$$\Delta F_T=2(500\text{ N})\left(\frac{312\text{ m/s}-300\text{ m/s}}{300\text{ m/s}}\right)=\boxed{40\text{ N}}$$

(*c*) The exact value for $(\Delta F)_{\text{exact}}$ is given by:

$$(\Delta F)_{\text{exact}}=F_{T,2}-F_{T,1} \qquad (1)$$

Express the wave speeds for the two tensions:

$$v_1=\sqrt{\frac{F_{T,1}}{\mu}}\text{ and }v_2=\sqrt{\frac{F_{T,2}}{\mu}}$$

Dividing the second equation by the first and simplifying yields:

$$\frac{v_2}{v_1}=\frac{\sqrt{\dfrac{F_{T,2}}{\mu}}}{\sqrt{\dfrac{F_{T,1}}{\mu}}}=\sqrt{\frac{F_{T,2}}{F_{T,1}}}\Rightarrow F_{T,2}=F_{T,1}\left(\frac{v_2}{v_1}\right)^2$$

Substituting for $F_{T,2}$ in equation (1) yields:

$$\left(\Delta F_T\right)_{exact} = F_{F,1}\left(\frac{v_2}{v_1}\right)^2 - F_{T,1}$$
$$= F_{F,1}\left[\left(\frac{v_2}{v_1}\right)^2 - 1\right]$$

Substitute numerical values and evaluate $(\Delta F_T)_{exact}$:

$$\left(\Delta F_T\right)_{exact} = (500\ \text{N})\left[\left(\frac{312\ \text{m/s}}{300\ \text{m/s}}\right)^2 - 1\right]$$
$$= \boxed{40.8\ \text{N}}$$

The percent error between the exact and approximate values for ΔF_T is:

$$\frac{\left(\Delta F_T\right)_{exact} - \Delta F_T}{\left(\Delta F_T\right)_{exact}} = \frac{40.8\ \text{N} - 40.0\ \text{N}}{40.8\ \text{N}}$$
$$\approx \boxed{2\%}$$

34 •• (*a*) Compute the derivative of the speed of sound in air with respect to the absolute temperature, and show that the differentials dv and dT obey $dv/v = \frac{1}{2}dT/T$. (*b*) Use this result to estimate the percentage change in the speed of sound when the temperature changes from 0 to 27°C. (*c*) If the speed of sound is 331 m/s at 0°C, estimate its value at 27°C using the differential approximation. (*d*) How does this approximation compare with the result of an exact calculation?

Picture the Problem The speed of sound in a gas is given by $v = \sqrt{\gamma RT/M}$ where R is the gas constant, T is the absolute temperature, M is the molecular mass of the gas, and γ is a constant that is characteristic of the particular molecular structure of the gas. We can differentiate this expression with respect to T and then separate the variables to show that the differentials satisfy $dv/v = \frac{1}{2}dT/T$. We'll approximate the differential quantities to estimate the percentage change in the speed of sound when the temperature increases from 0 to 27°C. Lacking information regarding the nature of the gas, we can express the ratio of the speeds of sound at 300 K and 273 K to obtain an expression that involves just the temperatures.

(*a*) Evaluate dv/dT:

$$\frac{dv}{dT} = \frac{d}{dT}\left[\sqrt{\frac{\gamma RT}{M}}\right] = \frac{1}{2}\sqrt{\frac{M}{\gamma RT}}\left(\frac{\gamma R}{M}\right)$$
$$= \frac{1}{2}\frac{v}{T}$$

Separate the variables to obtain:

$$\frac{dv}{v}=\boxed{\frac{1}{2}\frac{dT}{T}}$$

(*b*) Approximate dT with ΔT and dv with Δv and substitute numerical values to obtain:

$$\frac{\Delta v}{v}=\frac{1}{2}\left(\frac{300\text{ K}-273\text{ K}}{273\text{ K}}\right)=\boxed{5.0\%}$$

(*c*) Using a differential approximation, approximate the speed of sound at 300 K:

$$v_{300\text{ K}}\approx v_{273\text{ K}}+v_{273\text{ K}}\frac{\Delta v}{v}$$
$$=v_{273\text{ K}}\left(1+\frac{\Delta v}{v}\right)$$

Substitute numerical values and evaluate $v_{300\text{ K}}$:

$$v_{300\text{ K}}=(331\text{ m/s})(1+0.0495)$$
$$=\boxed{347\text{ m/s}}$$

(*d*) Use $v=\sqrt{\gamma RT/M}$ to express the speed of sound at 300 K:

$$v_{300\text{ K}}=\sqrt{\frac{\gamma R(300\text{ K})}{M}}$$

Use $v=\sqrt{\gamma RT/M}$ to express the speed of sound at 273 K:

$$v_{273\text{ K}}=\sqrt{\frac{\gamma R(273\text{ K})}{M}}$$

Divide the first of these equations by the second and solve for and evaluate $v_{300\text{ K}}$:

$$\frac{v_{300\text{ K}}}{v_{273\text{ K}}}=\frac{\sqrt{\frac{\gamma R(300\text{ K})}{M}}}{\sqrt{\frac{\gamma R(273\text{ K})}{M}}}=\sqrt{\frac{300}{273}}$$

and

$$v_{300\text{ K}}=(331\text{ m/s})\sqrt{\frac{300}{273}}=\boxed{347\text{ m/s}}$$

Note that these two results agree to three significant figures.

35 ••• Derive a convenient formula for the speed of sound in air at temperature t in Celsius degrees. Begin by writing the temperature as $T = T_0 + \Delta T$, where $T_0 = 273$ K corresponds to 0°C and $\Delta T = t$, the Celsius temperature. The speed of sound is a function of T, $v(T)$. To a first-order approximation, you can write $v(T)\approx v(T_0)+(dv/dT)_{T_0}\,\Delta T$, where $(dv/\,dT)_{T_0}$ is the derivative evaluated at $T = T_0$. Compute this derivative, and show that the result leads to $v=(331\text{ m/s})\left(1+\dfrac{t}{2T_0}\right)=(331+0.606t)$ m/s.

Picture the Problem We can use the approximate expression for $v(T)$ given in the problem statement and the result of Problem 34 to show that $v = (331 + 0.606t)\,\text{m/s}$.

The speed of sound as a function of T is given by:

$$v(T) \approx v(T_0) + \left(\frac{dv}{dT}\right)_{T_0} \Delta T \qquad (1)$$

From Problem 44, dv/dT is given by:

$$\frac{dv}{dt} = \frac{1}{2}\frac{v(T)}{T}$$

Evaluating $\left.\frac{dv}{dT}\right]_{T=T_0}$ yields:

$$\left.\frac{dv}{dT}\right]_{T=T_0} = \frac{1}{2}\frac{v(T_0)}{T_0} \qquad (2)$$

Substitute equation (2) in equation (1) and simplify to obtain:

$$v(T) \approx v(T_0) + \left(\frac{1}{2}\frac{v(T_0)}{T_0}\right)\Delta T = v(T_0)\left(1 + \frac{\Delta T}{2T_0}\right)$$

For T_0 = 273° K, $v(T_0) = 331\,\text{m/s}$, and $\Delta T = t$:

$$v = (331\,\text{m/s})\left(1 + \frac{t}{2(273\,\text{K})}\right) = \boxed{(331 + 0.606t)\,\text{m/s}}$$

The Wave Equation

36 • Show explicitly that the following functions satisfy the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$: (*a*) $y(x,t) = k(x + vt)^3$, (*b*) $y(x, t) = Ae^{ik(x-vt)}$, where A and k are constants and $i = \sqrt{-1}$, and (*c*) $y(x,t) = \ln k(x - vt)$.

Picture the Problem To show that each of the functions satisfies the wave equation, we'll need to find their first and second derivatives with respect to x and t and then substitute these derivatives in the wave equation.

(*a*) Find the first two spatial derivatives of $y(x,t) = k(x + vt)^3$:

$$\frac{\partial y}{\partial x} = 3k(x + vt)^2$$

and

$$\frac{\partial^2 y}{\partial x^2} = 6k(x + vt) \qquad (1)$$

Find the first two temporal derivatives of $y(x,t)=k(x+vt)^3$:

$$\frac{\partial y}{\partial t}=3kv(x+vt)^2$$

and

$$\frac{\partial^2 y}{\partial t^2}=6kv^2(x+vt) \qquad (2)$$

Express the ratio of equation (1) to equation (2):

$$\frac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial^2 y}{\partial t^2}}=\frac{6k(x+vt)}{6kv^2(x+vt)}=\boxed{\frac{1}{v^2}}$$

confirming that $y(x,t)=k(x+vt)^3$ satisfies the general wave equation.

(*b*) Find the first two spatial derivatives of $y(x,t)=Ae^{ik(x-vt)}$:

$$\frac{\partial y}{\partial x}=ikAe^{ik(x-vt)},\ \frac{\partial^2 y}{\partial x^2}=i^2k^2Ae^{ik(x-vt)}$$

or

$$\frac{\partial^2 y}{\partial x^2}=-k^2Ae^{ik(x-vt)} \qquad (3)$$

Find the first two temporal derivatives of $y(x,t)=Ae^{ik(x-vt)}$:

$$\frac{\partial y}{\partial t}=-ikvAe^{ik(x-vt)},$$

$$\frac{\partial^2 y}{\partial t^2}=i^2k^2v^2Ae^{ik(x-vt)}$$

or

$$\frac{\partial^2 y}{\partial t^2}=-k^2v^2Ae^{ik(x-vt)} \qquad (4)$$

Express the ratio of equation (3) to equation (4):

$$\frac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial^2 y}{\partial t^2}}=\frac{-k^2Ae^{ik(x-vt)}}{-k^2v^2Ae^{ik(x-vt)}}=\boxed{\frac{1}{v^2}}$$

confirming that $y(x,t)=Ae^{ik(x-vt)}$ satisfies the general wave equation.

(*c*) Find the first two spatial derivatives of $y(x,\,t)=\ln k(x-vt)$:

$$\frac{\partial y}{\partial x}=\frac{k}{x-vt}$$

and

$$\frac{\partial^2 y}{\partial x^2}=-\frac{k^2}{(x-vt)^2} \qquad (5)$$

Find the first two temporal derivatives of $y(x, t) = \ln k(x - vt)$:

$$\frac{\partial y}{\partial t} = -\frac{vk}{x - vt}$$

and

$$\frac{\partial^2 y}{\partial t^2} = -\frac{v^2 k^2}{(x - vt)^2} \qquad (6)$$

Express the ratio of equation (5) to equation (6):

$$\frac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial^2 y}{\partial t^2}} = \frac{-\dfrac{k^2}{(x - vt)^2}}{-\dfrac{v^2 k^2}{(x - vt)^2}} = \boxed{\frac{1}{v^2}}$$

confirming that $y(x, t) = \ln k(x - vt)$ satisfies the general wave equation.

37 • Show that the function $y = A \sin kx \cos \omega t$ satisfies the wave equation.

Picture the Problem The wave equation is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$. To show that $y = A \sin kx \cos \omega t$ satisfies this equation, we'll need to find the first and second derivatives of y with respect to x and t and then substitute these derivatives in the wave equation.

Find the first two spatial derivatives of $y = A \sin kx \cos \omega t$:

$$\frac{\partial y}{\partial x} = Ak \cos kx \cos \omega t$$

and

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2 \sin kx \cos \omega t \qquad (1)$$

Find the first two temporal derivatives of $y = A \sin kx \cos \omega t$:

$$\frac{\partial y}{\partial t} = -\omega A \sin kx \sin \omega t$$

and

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin kx \cos \omega t \qquad (2)$$

Express the ratio of equation (1) to equation (2):

$$\frac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial^2 y}{\partial t^2}} = \frac{-Ak^2 \sin kx \cos \omega t}{-A\omega^2 \sin kx \cos \omega t} = \frac{k^2}{\omega^2}$$

$$= \boxed{\frac{1}{v^2}}$$

confirming that $y = A \sin kx \cos \omega t$ satisfies the general wave equation.

Harmonic Waves on a String

38 • One end of a string 6.0 m long is moved up and down with simple harmonic motion at a frequency of 60 Hz. If the wave crests travel the length of the string in 0.50 s, find the wavelength of the waves on the string.

Picture the Problem We can find the wavelength of the waves on the string using the relationship $v = f\lambda$.

Express the wavelength of the waves:

$$f\lambda = v \Rightarrow \lambda = \frac{v}{f}$$

The speed of the wave is equal to the distance divided by the time:

$$v = \frac{\Delta x}{\Delta t} = \frac{6.0\,\text{m}}{0.50\,\text{s}} = 12\,\text{m/s}$$

Substitute numerical values to obtain:

$$\lambda = \frac{12\,\text{m/s}}{60\,\text{s}^{-1}} = \boxed{20\,\text{cm}}$$

39 • **[SSM]** A harmonic wave on a string with a mass per unit length of 0.050 kg/m and a tension of 80 N, has an amplitude of 5.0 cm. Each point on the string moves with simple harmonic motion at a frequency of 10 Hz. What is the power carried by the wave propagating along the string?

Picture the Problem The average power propagated along the string by a harmonic wave is $P_{\text{av}} = \frac{1}{2}\mu\omega^2 A^2 v,$ where v is the speed of the wave, and $\mu,$ $\omega,$ and A are the linear density of the string, the angular frequency of the wave, and the amplitude of the wave, respectively.

Express and evaluate the power propagated along the string:

$$P_{\text{av}} = \frac{1}{2}\mu\omega^2 A^2 v$$

The speed of the wave on the string is given by:

$$v=\sqrt{\frac{F_{\mathrm{T}}}{\mu}}$$

Substitute for v and simplify to obtain:

$$P_{\mathrm{av}}=\tfrac{1}{2}\mu\omega^2A^2\sqrt{\frac{F_{\mathrm{T}}}{\mu}}=\tfrac{1}{2}(2\pi f)^2A^2\sqrt{F_{\mathrm{T}}\mu}$$
$$=2(\pi fA)^2\sqrt{F_{\mathrm{T}}\mu}$$

Substitute numerical values and evaluate P_{av}:

$$P_{\mathrm{av}}=2\left[\pi\left(10\,\mathrm{s}^{-1}\right)(0.050\,\mathrm{m})\right]^2\sqrt{(80\,\mathrm{N})(0.050\,\mathrm{kg/m})}=\boxed{9.9\,\mathrm{W}}$$

40 • A 2.00-m-long rope has a mass of 0.100 kg. The tension is 60.0 N. An oscillator at one end sends a harmonic wave with an amplitude of 1.00 cm down the rope. The other end of the rope is terminated so that all of the energy of the wave is absorbed and none is reflected. What is the frequency of the oscillator if the power transmitted is 100 W?

Picture the Problem The average power propagated along the rope by a harmonic wave is $P_{\mathrm{av}}=\tfrac{1}{2}\mu\omega^2A^2v,$ where v is the velocity of the wave, and μ, ω, and A are the linear density of the string, the angular frequency of the wave, and the amplitude of the wave, respectively.

Rewrite the power equation in terms of the frequency of the wave:

$$P_{\mathrm{av}}=\tfrac{1}{2}\mu\omega^2A^2v=2\pi^2\mu f^2A^2v$$

Solve for the frequency and simplify to obtain:

$$f=\sqrt{\frac{P_{\mathrm{av}}}{2\pi^2\mu A^2v}}=\frac{1}{2\pi A}\sqrt{\frac{2P_{\mathrm{av}}}{\mu v}}$$

The wave speed is given by:

$$v=\sqrt{\frac{F_{\mathrm{T}}}{\mu}}$$

Substitute for v and simplify to obtain:

$$f=\frac{1}{2\pi A}\sqrt{\frac{2P_{\mathrm{av}}}{\mu\sqrt{\frac{F_{\mathrm{T}}}{\mu}}}}=\frac{1}{2\pi A}\sqrt{\frac{2P_{\mathrm{av}}}{\sqrt{\mu F_{\mathrm{T}}}}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi(0.0100\text{ m})}\sqrt{\frac{2(100\text{ W})}{\sqrt{\left(\frac{0.100\text{ kg}}{2.00\text{ m}}\right)(60.0\text{ N})}}} = \boxed{171\text{ Hz}}$$

41 •• The wave function for a harmonic wave on a string is $y(x, t) = (1.00\text{ mm})\sin(62.8\text{ m}^{-1}x + 314\text{ s}^{-1}t)$. (*a*) In what direction does this wave travel, and what is the wave's speed? (*b*) Find the wavelength, frequency, and period of this wave. (*c*) What is the maximum speed of any point on the string?

Picture the Problem $y(x,t) = A\sin(kx - \omega t)$ describes a wave traveling in the $+x$ direction. For a wave traveling in the $-x$ direction, we have $y(x,t) = A\sin(kx + \omega t)$. We can determine A, k, and ω by examination of the wave function. The wavelength, frequency, and period of the wave can, in turn, be determined from k and ω.

(*a*) Because the sign between the kx and ωt terms is positive, the wave is traveling in the $-x$ direction.

The speed of the wave is:

$$v = \frac{\omega}{k} = \frac{314\text{ s}^{-1}}{62.8\text{ m}^{-1}} = \boxed{5.00\text{ m/s}}$$

(*b*) The coefficient of x is k and:

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{2\pi}{62.8\text{ m}^{-1}} = \boxed{10.0\text{ cm}}$$

The coefficient of t is ω and:

$$f = \frac{\omega}{2\pi} = \frac{314\text{ s}^{-1}}{2\pi} = \boxed{50.0\text{ Hz}}$$

The period of the wave motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{50.0\text{ s}^{-1}} = \boxed{0.0200\text{ s}}$$

(*c*) Express and evaluate the maximum speed of any string segment:

$$v_{max} = A\omega = (1.00\text{ mm})(314\text{ rad/s}) = \boxed{0.314\text{ m/s}}$$

42 •• A harmonic wave on a string with a frequency of 80 Hz and an amplitude of 0.025 m travels in the $+x$ direction with a speed of 12 m/s. (*a*) Write a suitable wave function for this wave. (*b*) Find the maximum speed of a point on the string. (*c*) Find the maximum acceleration of a point on the string.

Picture the Problem $y(x,t)=A\sin(kx-\omega t)$ describes a wave traveling in the $+x$ direction. We can find ω and k from the data included in the problem statement and substitute in the general equation. The maximum speed of a point on the string can be found from $v_{\max}=A\omega$ and the maximum acceleration from $a_{\max}=A\omega^2$.

(*a*) Express the general form of the equation of a harmonic wave traveling to the right:

$$y(x,t)=A\sin(kx-\omega t)$$

Evaluate ω:

$$\omega=2\pi f=2\pi\left(80\,\mathrm{s}^{-1}\right)=503\,\mathrm{s}^{-1}$$
$$=5.0\times10^{2}\,\mathrm{s}^{-1}$$

Determine k:

$$k=\frac{\omega}{v}=\frac{503\,\mathrm{s}^{-1}}{12\,\mathrm{m/s}}=42\,\mathrm{m}^{-1}$$

Substitute to obtain: $y(x,t)=\boxed{(0.025\,\mathrm{m})\sin\left[\left(42\,\mathrm{m}^{-1}\right)x-\left(5.0\times10^{2}\,\mathrm{s}^{-1}\right)t\right]}$

(*b*) The maximum speed of a point on the string is:

$$v_{\max}=A\omega=(0.025\,\mathrm{m})\left(503\,\mathrm{s}^{-1}\right)$$
$$=\boxed{13\,\mathrm{m/s}}$$

(*c*) Express the maximum acceleration of a point on the string:

$$a_{\max}=A\omega^2$$

Substitute numerical values and evaluate $a_{\max}$:

$$a_{\max}=(0.025\,\mathrm{m})\left(503\,\mathrm{s}^{-1}\right)^2=\boxed{6.3\,\mathrm{km/s^2}}$$

43 •• A 200-Hz harmonic wave with an amplitude equal to 1.2 cm moves along a 40-m-long string that has a mass of 0.120 kg and a tension of 50 N. (*a*) What is the average total energy of the waves on a 20-m-long segment of string? (*b*) What is the power transmitted past a given point on the string?

Picture the Problem The average power propagated along the string is $P_{\mathrm{av}}=\frac{1}{2}\mu\omega^2A^2v.$ The average energy on a segment of length L is the average power multiplied by the time for the wave to travel the distance L

(*a*) Express the power transmitted past a given point on the string. The average energy passing a point in time *t* is the average power multiplied by the time:

$$P_{av} = \tfrac{1}{2}\mu\omega^2 A^2 v$$

and

$$E_{av} = P_{av} t$$

The time for the wave to travel the length L is L/v:

$$t = \frac{L}{v}$$

Combine the previous steps and solve for the average energy on the string:

$$E_{av} = \tfrac{1}{2}\mu v\omega^2 A^2 \frac{L}{v} = \tfrac{1}{2}\mu L\omega^2 A^2 = \tfrac{1}{2}\mu L(2\pi f)^2 A^2 = 2\pi^2\mu L f^2 A^2$$

Substitute numerical values and evaluate E_{av}:

$$E_{av} = 2\pi^2\left(\frac{0.120\,\text{kg}}{40\,\text{m}}\right)\left(200\,\text{s}^{-1}\right)^2(20\,\text{m})(0.012\,\text{m})^2 = \boxed{6.8\,\text{J}}$$

(*b*) Express P_{av}:

$$P_{av} = \tfrac{1}{2}\mu v\omega^2 A^2$$

To calculate P_{av} we need an expression for the wave speed v:

$$v = \sqrt{\frac{F_T}{\mu}}$$

Substitute for v to obtain:

$$P_{av} = \tfrac{1}{2}\mu(2\pi f)^2 A^2\sqrt{\frac{F_T}{\mu}} = 2\pi^2\mu f^2 A^2\sqrt{\frac{F_T}{\mu}}$$

Substitute numerical values and evaluate P_{av}:

$$P_{av} = 2\pi^2\left(\frac{0.060\,\text{kg}}{20\,\text{m}}\right)\left(200\,\text{s}^{-1}\right)^2(0.012\,\text{m})^2\sqrt{\frac{50\,\text{N}}{\frac{0.060\,\text{kg}}{20\,\text{m}}}} = \boxed{44\,\text{W}}$$

44 •• On a real string, some of the energy of the wave dissipates as the wave travels down the string. Such a situation can be described by a wave function whose amplitude $A(x)$ depends on x: $y = A(x)\sin(kx - \omega t)$ where $A(x) = (A_0 e^{-bx})$. What is the power transported by the wave as a function of x, where $x > 0$?

Picture the Problem The power propagated along the string by a harmonic wave is $P = \frac{1}{2}\mu\omega^2 A^2 v$ where v is the velocity of the wave, and μ, ω, and A are the linear density of the string, the angular frequency of the wave, and the amplitude of the wave, respectively.

Express the power associated with the wave at the origin:

$$P = \frac{1}{2}\mu\omega^2 A^2 v \quad (1)$$

Express the amplitude of the wave at x:

$$A(x) = A_0 e^{-bx}$$

Substitute in equation (1) to obtain:

$$P(x) = \frac{1}{2}\mu\omega^2\left(A_0 e^{-bx}\right)^2 v = \boxed{\frac{1}{2}\mu\omega^2 A_0^2 v e^{-2bx}}$$

45 •• **[SSM]** Power is to be transmitted along a taut string by means of transverse harmonic waves. The wave speed is 10 m/s and the linear mass density of the string is 0.010 kg/m. The power source oscillates with an amplitude of 0.50 mm. (*a*) What average power is transmitted along the string if the frequency is 400 Hz? (*b*) The power transmitted can be increased by increasing the tension in the string, the frequency of the source, or the amplitude of the waves. By how much would each of these quantities have to increase to cause an increase in power by a factor of 100 if it is the only quantity changed?

Picture the Problem The average power propagated along a string by a harmonic wave is $P_{av} = \frac{1}{2}\mu\omega^2 A^2 v$ where v is the speed of the wave, and μ, ω, and A are the linear density of the string, the angular frequency of the wave, and the amplitude of the wave, respectively.

(*a*) Express the average power transmitted along the string:

$$P_{av} = \frac{1}{2}\mu\omega^2 A^2 v = 2\pi^2 \mu f^2 A^2 v$$

Substitute numerical values and evaluate P_{av}:

$$P_{av} = 2\pi^2 (0.010\,\text{kg/m})(400\,\text{s}^{-1})^2 \times (0.50\times10^{-3}\,\text{m})^2 (10\,\text{m/s}) = \boxed{79\,\text{mW}}$$

(*b*) Because $P_{av} \propto f^2$, increasing the frequency by a factor of 10 would increase the power by a factor of 100.

Because $P_{av} \propto A^2$, increasing the amplitude by a factor of 10 would increase the power by a factor of 100.

Because $P_{av} \propto v$ and $v \propto \sqrt{F}$, increasing the tension by a factor of 10^4 would increase v by a factor of 100 and the power by a factor of 100.

46 ••• Two very long strings are tied together at the point $x = 0$. In the region $x < 0$, the wave speed is v_1, while in the region $x > 0$, the speed is v_2. A sinusoidal wave is incident on the knot from the left ($x < 0$); part of the wave is reflected and part is transmitted. For $x < 0$, the displacement of the wave is describabed by $y(x,t) = A \sin(k_1x - \omega t) + B \sin(k_1x + \omega t)$, while for $x > 0$, $y(x,t) = C \sin(k_2x - \omega t)$, where $\omega/k_1 = v_1$ and $\omega/k_2 = v_2$. (*a*) If we assume that both the wave function y and its first spatial derivative $\partial y/\partial x$ must be continuous at $x = 0$, show that $C/A = 2v_2/(v_1 + v_2)$, and that $B/A = (v_1 - v_2)/(v_1 + v_2)$. (*b*) Show that $B^2 + (v_1/v_2)C^2 = A^2$.

Picture the Problem We can use the assumption that both the wave function and its first spatial derivative are continuous at $x = 0$ to establish equations relating A, B, C, k_1, and k_2. Then, we can solve these simultaneous equations to obtain expressions for B and C in terms of A, v_1, and v_2.

(*a*) Let $y_1(x, t)$ represent the wave function in the region $x < 0$, and $y_2(x, t)$ represent the wave function in the region $x > 0$. Express the continuity of the two wave functions at $x = 0$:

$$y_1(0,t) = y_2(0,t)$$
and
$$A\sin[k_1(0) - \omega t] + B\sin[k_1(0) + \omega t] = C\sin[k_2(0) - \omega t]$$
or
$$A\sin(-\omega t) + B\sin\omega t = C\sin(-\omega t)$$

Because the sine function is odd; that is, $\sin(-\theta) = -\sin\theta$:

$$-A\sin\omega t + B\sin\omega t = -C\sin\omega t$$
and
$$A - B = C \qquad (1)$$

Differentiate the wave functions with respect to x to obtain:

$$\frac{\partial y_1}{\partial x} = Ak_1\cos(k_1x - \omega t) + Bk_1\cos(k_1x + \omega t)$$
and
$$\frac{\partial y_2}{\partial x} = Ck_2\cos(k_2x - \omega t)$$

Express the continuity of the slopes of the two wave functions at $x = 0$:

$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0}$$

and

$$Ak_1\cos[k_1(0)-\omega t]+Bk_1\cos[k_1(0)+\omega t] = Ck_2\cos[k_2(0)-\omega t]$$

or

$$Ak_1\cos(-\omega t)+Bk_1\cos\omega t = Ck_2\cos(-\omega t)$$

Because the cosine function is even; that is, $\cos(-\theta)=\cos\theta$:

$$Ak_1\cos\omega t + Bk_1\cos\omega t = Ck_2\cos\omega t$$

and

$$k_1A + k_1B = k_2C \qquad (2)$$

Multiply equation (1) by k_1 and add it to equation (2) to obtain:

$$2k_1A = (k_1 + k_2)C$$

Solving for C yields:

$$C = \frac{2k_1}{k_1+k_2}A = \frac{2}{1+k_2/k_1}A$$

Solve for C/A and substitute ω/v_1 for k_1 and ω/v_2 for k_2 to obtain:

$$\frac{C}{A} = \frac{2}{1+k_2/k_1} = \frac{2}{1+v_1/v_2} = \boxed{\frac{2v_2}{v_1+v_2}}$$

Substitute in equation (1) to obtain:

$$A - B = \left(\frac{2v_2}{v_2+v_1}\right)A$$

Solving for B/A yields:

$$\frac{B}{A} = \boxed{\frac{v_1-v_2}{v_1+v_2}}$$

(b) We wish to show that

$$B^2 + (v_1/v_2)C^2 = A^2$$

Use the results of (a) to obtain the expressions $B = -[(1 - \alpha)/(1 + \alpha)]\,A$ and $C = 2A/(1 + \alpha)$, where $\alpha = v_1/v_2$.

Substitute these expressions into

$$B^2 + (v_1/v_2)C^2 = A^2$$

and check to see if the resulting equation is an identity:

$$B^2 + \frac{v_1}{v_2}C^2 = A^2$$

$$\left(\frac{1-\alpha}{1+\alpha}\right)^2 A^2 + \alpha\left(\frac{2}{1+\alpha}\right)^2 A^2 = A^2$$

$$\left(\frac{1-\alpha}{1+\alpha}\right)^2 + \alpha\left(\frac{2}{1+\alpha}\right)^2 = 1$$

$$\frac{(1-\alpha)^2 + 4\alpha}{(1+\alpha)^2} = 1$$

$$\frac{1-2\alpha+\alpha^2+4\alpha}{(1+\alpha)^2} = 1$$

$$\frac{1+2\alpha+\alpha^2}{(1+\alpha)^2} = 1$$

$$\frac{(1+\alpha)^2}{(1+\alpha)^2} = 1$$

$$1 = 1$$

The equation is an identity: Therefore, $\boxed{B^2 + \frac{v_1}{v_2}C^2 = A^2}$

Remarks: Our result in (*a*) can be checked by considering the limit of *B*/*A* as $v_2/v_1 \to 0$. This limit gives *B*/*A* = +1, telling us that the transmitted wave has zero amplitude and the incident and reflected waves superpose to give a standing wave with a node at *x* = 0.

Harmonic Sound Waves

47 • A sound wave in air produces a pressure variation given by $p(x,t) = 0.75\cos\frac{\pi}{2}(x - 343t)$, where p is in pascals, x is in meters, and t is in seconds. Find (a) the pressure amplitude, (b) the wavelength, (c) the frequency, and (d) the wave speed.

Picture the Problem The pressure variation is of the form $p(x,t) = p_0\cos k(x - vt)$ where k is the wave number, p_0 is the pressure amplitude, and v is the wave speed. We can find λ from k and f from ω and k.

(a) By inspection of the equation: $p_0 = \boxed{0.75\,\text{Pa}}$

(b) Because $k=\dfrac{2\pi}{\lambda}=\dfrac{\pi}{2}$: $\lambda=\boxed{4.00\,\text{m}}$

(c) Solve $v=\dfrac{\omega}{k}=\dfrac{2\pi f}{k}$ for f to obtain: $f=\dfrac{kv}{2\pi}$

Substitute numerical values and evaluate f:

$$f=\frac{\frac{\pi}{2}(343\,\text{m/s})}{2\pi}=\boxed{85.8\,\text{Hz}}$$

(d) By inspection of the equation: $v=\boxed{343\,\text{m/s}}$

48 • (a) Middle C on the musical scale has a frequency of 262 Hz. What is the wavelength of this note in air? (b) The frequency of the C an octave above middle C is twice that of middle C. What is the wavelength of this note in air?

Picture the Problem The frequency, wavelength, and speed of the sound waves are related by $v=f\lambda$.

(a) The wavelength of middle C is given by:

$$\lambda=\frac{v}{f}=\frac{340\,\text{m/s}}{262\,\text{s}^{-1}}=\boxed{1.30\,\text{m}}$$

(b) Evaluate λ for a frequency twice that of middle C:

$$\lambda=\frac{v}{f}=\frac{340\,\text{m/s}}{2(262\,\text{s}^{-1})}=\boxed{0.649\,\text{m}}$$

49 • **[SSM]** The density of air is 1.29 kg/m^3. (a) What is the displacement amplitude for a sound wave with a frequency of 100 Hz and a pressure amplitude of 1.00×10^{-4} atm? (b) The displacement amplitude of a sound wave of frequency 300 Hz is 1.00×10^{-7} m. What is the pressure amplitude of this wave?

Picture the Problem The pressure amplitude depends on the density of the medium ρ, the angular frequency of the sound wave μ, the speed of the wave v, and the displacement amplitude s_0 according to $p_0=\rho\omega v s_0$.

(a) Solve $p_0=\rho\omega v s_0$ for s_0:

$$s_0=\frac{p_0}{\rho\omega v}$$

Substitute numerical values and evaluate s_0:

$$s_0 = \frac{(1.00\times10^{-4}\text{ atm})(1.01325\times10^{5}\text{ Pa/atm})}{2\pi(1.29\text{ kg/m}^3)(100\text{ s}^{-1})(343\text{ m/s})} = 3.64\times10^{-5}\text{ m} = \boxed{36.4\ \mu\text{m}}$$

(*b*) Use $p_0 = \rho\omega v s_0$ to find p_0:

$$p_0 = 2\pi(1.29\text{ kg/m}^3)(300\text{ s}^{-1})(343\text{ m/s})(1.00\times10^{-7}\text{ m}) = \boxed{83.4\text{ mPa}}$$

50 • The density of air is 1.29 kg/m^3. (*a*) What is the displacement amplitude of a sound wave that has a frequency of 500 Hz at the pain-threshold pressure amplitude of 29.0 Pa? (*b*) What is the displacement amplitude of a sound wave that has the same pressure amplitude as the wave in Part (*a*), but has a frequency of 1.00 kHz?

Picture the Problem The pressure amplitude depends on the density of the medium ρ, the angular frequency of the sound wave μ, the wave speed v, and the displacement amplitude s_0 according to $p_0 = \rho\omega v s_0$.

(*a*) Solve $p_0 = \rho\omega v s_0$ for s_0:

$$s_0 = \frac{p_0}{\rho\omega v}$$

Substitute numerical values and evaluate s_0:

$$s_0 = \frac{29.0\text{ Pa}}{2\pi(1.29\text{ kg/m}^3)(500\text{ Hz})(343\text{ m/s})} = \boxed{20.9\ \mu\text{m}}$$

(*b*) Proceed as in (*a*) with $f = 1.00$ kHz:

$$s_0 = \frac{29.0\text{ Pa}}{2\pi(1.29\text{ kg/m}^3)(1.00\text{ kHz})(343\text{ m/s})} = \boxed{10.4\ \mu\text{m}}$$

51 • A typical loud sound wave that has a frequency of 1.00 kHz has a pressure amplitude of about 1.00×10^{-4} atm. (*a*) At $t = 0$, the pressure is a maximum at some point x_1. What is the displacement at that point at $t = 0$? (*b*) Assuming the density of air is 1.29 kg/m^3, what is the maximum value of the displacement at any time and place?

Picture the Problem The pressure or density wave is 90° out of phase with the displacement wave. When the displacement is zero, the pressure and density changes are either a maximum or a minimum. When the displacement is a

maximum or minimum, the pressure and density changes are zero. We can use $p_0 = \rho\omega v s_0$ to find the maximum value of the displacement at any time and place.

(*a*) If the pressure is a maximum at x_1 when $t = 0$, the displacement s is zero.

(*b*) Solve $p_0 = \rho\omega v s_0$ for s_0:

$$s_0 = \frac{p_0}{\rho\omega v}$$

Substitute numerical values and evaluate s_0:

$$s_0 = \frac{\left(1.00\times10^{-4}\text{ atm}\right)\left(1.01325\times10^{5}\text{ Pa/atm}\right)}{2\pi\left(1.29\text{ kg/m}^3\right)\left(1.00\text{ kHz}\right)\left(343\text{ m/s}\right)} = \boxed{3.64\ \mu\text{m}}$$

52 • An octave represents a change in frequency by a factor of two. Over how many octaves can a typical person hear?

Picture the Problem A human can hear sounds between roughly 20 Hz and 20 kHz; a factor of 1000. An octave represents a change in frequency by a factor of 2. We can evaluate $2^N = 1000$ to find the number of octaves heard by a person who can hear this range of frequencies.

Relate the number of octaves to the difference between 20 kHz and 20 Hz:

$$2^N = 1000$$

Take the logarithm of both sides of the equation to obtain:

$$\log 2^N = \log 10^3 \Rightarrow N\log 2 = 3$$

Solving for N yields:

$$N = \frac{3}{\log 2} = 9.97 \approx \boxed{10}$$

53 •• In the oceans, whales communicate by sound transmission through the water. A whale emits a sound of 50.0 Hz to tell a wayward calf to catch up to the pod. The speed of sound in water is about 1500 m/s. (*a*) How long does it take the sound to reach the calf if he is 1.20 km away? (*b*) What is the wavelength of this sound in the water? (*c*) If the whales are close to the surface, some of the sound energy might refract out into the air. What would be the frequency and wavelength of the sound in the air?

Picture the Problem (*a*) We can use the definition of average speed to find the time required for the sound to travel to the calf. (*b*) We can use the relationship between wavelength, frequency, and speed to find the wavelength of the sound in water. (*c*) The frequency of the sound does not change as it travels from water to air, but its wavelength changes because of the difference in the speed of sound in water and in air.

(*a*) Relate the time it takes the sound to reach the calf to the distance from the whale to the calf and the speed of sound in water:

$$\Delta t = \frac{d}{v} = \frac{1.20\text{ km}}{1500\text{ m/s}} = \boxed{0.80\text{ s}}$$

(*b*) The wavelength of this sound in water is the ratio of its speed in water to its frequency:

$$\lambda_{\text{water}} = \frac{v_{\text{water}}}{f} = \frac{1500\text{ m/s}}{50.0\text{ Hz}} = \boxed{30\text{ m}}$$

(*c*) Because the frequency does not change as it travels from water to air:

$$f = \boxed{50.0\text{ Hz}}$$

The wavelength of this sound in air is the ratio of its speed in air to its frequency:

$$\lambda_{\text{air}} = \frac{v_{\text{air}}}{f} = \frac{343\text{ m/s}}{50.0\text{ Hz}} = \boxed{6.86\text{ m}}$$

Waves in Three Dimensions: Intensity

54 • A spherical sinusoidal source radiates sound uniformly in all directions. At a distance of 10.0 m, the sound intensity level is 1.00×10^{-4} W/m^2. (*a*) At what distance from the source is the intensity 1.00×10^{-6} W/m^2? (*b*) What power is radiated by this source?

Picture the Problem The intensity of the sound from the spherical sinusoidal source varies inversely with the square of the distance from the source. The power radiated by the source is the product of the intensity of the radiation and the surface area over which it is distributed.

(*a*) Relate the intensity I_1 at a distance R_1 from the source to the energy per unit time (power) arriving at the point of interest:

$$I_1 = \frac{P_{\text{av},1}}{4\pi R_1^2} \Rightarrow P_{\text{av},1} = 4\pi R_1^2 I_1$$

At a distance R_2 from the source:

$$I_2 = \frac{P_{\text{av},2}}{4\pi R_2^2} \Rightarrow P_{\text{av},2} = 4\pi R_2^2 I_2$$

Because $P_{\text{av},1} = P_{\text{av},2}$:

$$4\pi R_1^2 I_1 = 4\pi R_2^2 I_2 \Rightarrow R_2 = R_1\sqrt{\frac{I_1}{I_2}}$$

Substituting numerical values and evaluating R_2 gives:

$$R_2 = (10.0\text{ m})\sqrt{\frac{1.00\times10^{-4}\text{ W/m}^2}{1.00\times10^{-6}\text{ W/m}^2}} = \boxed{100\text{ m}}$$

(*b*) Solve $I = \dfrac{P_{\text{av}}}{4\pi r^2}$ for P_{av}:

$$P_{\text{av}} = 4\pi r^2 I$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = 4\pi(10.0\text{ m})^2\left(1.00\times10^{-4}\text{ W/m}^2\right) = \boxed{126\text{ mW}}$$

55 • [SSM] A loudspeaker at a rock concert generates a sound that has an intensity level equal to 1.00×10^{-2} W/m^2 at 20.0 m and has a frequency of 1.00 kHz. Assume that the speaker spreads its energy uniformly in three dimensions. (*a*) What is the total acoustic power output of the speaker? (*b*) At what distance will the sound intensity be at the pain threshold of 1.00 W/m^2? (*c*) What is the sound intensity at 30.0 m?

Picture the Problem Because the power radiated by the loudspeaker is the product of the intensity of the sound and the area over which it is distributed, we can use this relationship to find the average power, the intensity of the radiation, or the distance to the speaker for a given intensity or average power.

(*a*) Use $P_{\text{av}} = 4\pi r^2 I$ to find the total acoustic power output of the speaker:

$$P_{\text{av}} = 4\pi(20.0\text{ m})^2\left(1.00\times10^{-2}\text{ W/m}^2\right) = 50.27\text{ W} = \boxed{50.3\text{ W}}$$

(*b*) Relate the intensity of the sound at 20 m to the distance from the speaker:

$$1.00\times10^{-2}\text{ W/m}^2 = \frac{P_{\text{av}}}{4\pi(20.0\text{ m})^2}$$

Relate the threshold-of-pain intensity to the distance from the speaker:

$$1.00\ \text{W/m}^2 = \frac{P_{\text{av}}}{4\pi r^2}$$

Divide the first of these equations by the second and solve for r:

$$r = \sqrt{\left(1.00\times10^{-2}\right)\left(20.0\ \text{m}\right)^2} = \boxed{2.00\ \text{m}}$$

(c) Use $I = \dfrac{P_{\text{av}}}{4\pi r^2}$ to find the intensity at 30.0 m:

$$I\left(30.0\ \text{m}\right) = \frac{50.3\ \text{W}}{4\pi\left(30.0\ \text{m}\right)^2} = \boxed{4.45\times10^{-3}\ \text{W/m}^2}$$

56 •• When a pin of mass 0.100 g is dropped from a height of 1.00 m, 0.050 percent of its energy is converted into a sound pulse that has a duration of 0.100 s. (a) Estimate how far away the dropped pin can be heard if the minimum audible intensity is 1.00×10^{-11} W/m^2. (b) Your result in Part (a) is much too large in practice because of background noise. If you assume that the intensity must be at least 1.00×10^{-8} W/m^2 for the sound to be heard, estimate how far away the dropped pin can be heard. (In both parts, assume that the intensity is $P/4\pi r^2$.)

Picture the Problem We can use conservation of energy to find the acoustical energy resulting from the dropping of the pin. The power developed can then be found from the given time during which the energy was transformed from mechanical to acoustical form. We can find the range at which the dropped pin can be heard from $I = P/4\pi r^2$.

(a) Assuming that $I = P/4\pi r^2$, express the distance at which one can hear the dropped pin:

$$r = \sqrt{\frac{P}{4\pi I}}$$

Use conservation of energy to determine the sound energy generated when the pin falls:

$$\begin{aligned} E &= \varepsilon mgh \\ &= \left(0.00050\right)\left(0.100\times10^{-3}\ \text{kg}\right) \\ &\qquad \times\left(9.81\ \text{m/s}^2\right)\left(1.00\ \text{m}\right) \\ &= 4.905\times10^{-7}\ \text{J} \end{aligned}$$

The power of the sound pulse is given by:

$$P = \frac{E}{\Delta t} = \frac{4.905\times10^{-7}\ \text{J}}{0.100\ \text{s}} = 4.905\times10^{-6}\ \text{W}$$

Substitute numerical values and evaluate r:

$$r=\sqrt{\frac{4.905\times10^{-6}\text{ W}}{4\pi\left(1.00\times10^{-11}\text{ W/m}^2\right)}}=\boxed{0.20\text{ km}}$$

(b) Repeat the last step in (a) with $I = 1.00\times10^{-8}$ W/m^2:

$$r=\sqrt{\frac{4.905\times10^{-6}\text{ W}}{4\pi\left(1.00\times10^{-8}\text{ W/m}^2\right)}}=\boxed{6.2\text{ m}}$$

*Intensity Level

57 • **[SSM]** What is the intensity level in decibels of a sound wave that has an intensity of (a) 1.00×10^{-10} W/m^2 and (b) 1.00×10^{-2} W/m^2?

Picture the Problem The intensity level β of a sound wave, measured in decibels, is given by $\beta=(10\text{ dB})\log(I/I_0)$ where $I_0 = 10^{-12}$ W/m^2 is defined to be the threshold of hearing.

(a) Using its definition, calculate the intensity level of a sound wave whose intensity is 1.00×10^{-10} W/m^2:

$$\beta=(10\text{ dB})\log\left(\frac{1.00\times10^{-10}\text{ W/m}^2}{10^{-12}\text{ W/m}^2}\right)$$
$$=10\log10^2=\boxed{20.0\text{ dB}}$$

(b) Proceed as in (a) with $I = 1.00\times10^{-2}$ W/m^2:

$$\beta=(10\text{ dB})\log\left(\frac{1.00\times10^{-2}\text{ W/m}^2}{10^{-12}\text{ W/m}^2}\right)$$
$$=10\log10^{10}=\boxed{100\text{ dB}}$$

58 • What is the intensity of a sound wave if, at a particular location, the intensity level is (a) β = 10 dB and (b) β = 3 dB?

Picture the Problem (a) and (b) The intensity level β of a sound wave, measured in decibels, is given by $\beta=(10\text{ dB})\log(I/I_0)$ where $I_0 = 10^{-12}$ W/m^2 is defined to be the threshold of hearing.

(a) Solve $\beta=(10\text{ dB})\log(I/I_0)$ for I to obtain:

$$I=10^{\beta/(10\text{ dB})}I_0$$

Evaluate I for β = 10 dB:

$$I=10^{(10\text{ dB})/(10\text{ dB})}I_0=10I_0$$
$$=10\left(10^{-12}\text{ W/m}^2\right)=\boxed{10^{-11}\text{ W/m}^2}$$

(*b*) Proceed as in (*a*) with $\beta = 3$ dB:

$$I = 10^{(3\,\text{dB})/(10\,\text{dB})} I_0 = 2I_0$$
$$= 2\left(10^{-12}\ \text{W/m}^2\right) = \boxed{2\times 10^{-12}\ \text{W/m}^2}$$

59 • At a certain distance, the sound intensity level of a dog's bark is 50 dB. At that same distance, the sound intensity of a rock concert is 10,000 times that of the dog's bark. What is the sound intensity level of the rock concert?

Picture the Problem The intensity level β of a sound wave, measured in decibels, is given by $\beta = (10\,\text{dB})\log(I/I_0)$ where $I_0 = 10^{-12}$ W/m^2 is defined to be the threshold of hearing.

Express the sound intensity level of the rock concert:

$$\beta_{\text{concert}} = (10\,\text{dB})\log\left(\frac{I_{\text{concert}}}{I_0}\right) \quad (1)$$

Express the sound intensity level of the dog's bark:

$$50\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{\text{dog}}}{I_0}\right)$$

Solve for the intensity of the dog's bark:

$$I_{\text{dog}} = 10^5 I_0 = 10^5\left(10^{-12}\ \text{W/m}^2\right)$$
$$= 10^{-7}\ \text{W/m}^2$$

Express the intensity of the rock concert in terms of the intensity of the dog's bark:

$$I_{\text{concert}} = 10^4 I_{\text{dog}} = 10^4\left(10^{-7}\ \text{W/m}^2\right)$$
$$= 10^{-3}\ \text{W/m}^2$$

Substitute in equation (1) and evaluate β_{concert}:

$$\beta_{\text{concert}} = (10\,\text{dB})\log\left(\frac{10^{-3}\ \text{W/m}^2}{10^{-12}\ \text{W/m}^2}\right)$$
$$= (10\,\text{dB})\log 10^9 = \boxed{90\,\text{dB}}$$

60 • What fraction of the acoustic power of a noise would have to be eliminated to lower its sound intensity level from 90 to 70 dB?

Picture the Problem We can express the intensity levels at both 90 dB and 70 dB in terms of the intensities of the sound at those levels. By subtracting the two expressions, we can solve for the ratio of the intensities at the two levels and then find the fractional change in the intensity that corresponds to a decrease in intensity level from 90 dB to 70 dB.

Express the intensity level at 90 dB:

$$90\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{90}}{I_0}\right)$$

Express the intensity level at 70 dB:

$$70\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{70}}{I_0}\right)$$

Express $\Delta\beta = \beta_{90} - \beta_{70}$:

$$\Delta\beta = 20\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{90}}{I_0}\right) - (10\,\text{dB})\log\left(\frac{I_{70}}{I_0}\right) = (10\,\text{dB})\log\left(\frac{I_{90}}{I_{70}}\right)$$

Solving for I_{90} yields:

$$I_{90} = 100I_{70}$$

Express the fractional change in the intensity from 90 dB to 70 dB:

$$\frac{I_{90} - I_{70}}{I_{90}} = \frac{100I_{70} - I_{70}}{100I_{70}} = 99\%$$

Because $P \propto I$, the fractional change in power is $\boxed{99\%}$.

61 •• A spherical source radiates sound uniformly in all directions. At a distance of 10 m, the sound intensity level is 80 dB. (*a*) At what distance from the source is the intensity level 60 dB? (*b*) What power is radiated by this source?

Picture the Problem The intensity at a distance r from a spherical source varies with distance from the source according to $I = P_{\text{av}}/4\pi r^2$. We can use this relationship to relate the intensities corresponding to an 80-dB intensity level (I_{80}) and the intensity corresponding to a 60-dB intensity level (I_{60}) to their distances from the source. We can relate the intensities to the intensity levels through $\beta = (10\,\text{dB})\log(I/I_0)$.

(*a*) Express the intensity of the sound where the intensity level is 80 dB:

$$I_{80} = \frac{P_{\text{av}}}{4\pi r_{80}^2}$$

Express the intensity of the sound where the intensity level is 60 dB:

$$I_{60} = \frac{P_{\text{av}}}{4\pi r_{60}^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{I_{80}}{I_{60}}=\frac{\dfrac{P_{\text{av}}}{4\pi(10\,\text{m})^2}}{\dfrac{P_{\text{av}}}{4\pi r_{60}^2}}=\frac{r_{60}^2}{100\,\text{m}^2}$$

Solving for r_{60} yields:

$$r_{60}=(10\,\text{m})\sqrt{\frac{I_{80}}{I_{60}}}$$

Find the intensity of the 80-dB sound level radiation:

$$80\,\text{dB}=(10\,\text{dB})\log\left(\frac{I_{80}}{I_0}\right)$$

and

$$I_{80}=10^8 I_0=10^{-4}\ \text{W/m}^2$$

Find the intensity of the 60-dB sound level radiation:

$$60\,\text{dB}=(10\,\text{dB})\log\left(\frac{I_{60}}{I_0}\right)$$

and

$$I_{60}=10^6 I_0=10^{-6}\ \text{W/m}^2$$

Substitute numerical values for I_{80} and I_{60} and evaluate r_{60}:

$$r_{60}=(10\,\text{m})\sqrt{\frac{10^{-4}\ \text{W/m}^2}{10^{-6}\ \text{W/m}^2}}=\boxed{0.10\,\text{km}}$$

(*b*) Using the intensity corresponding to an intensity level of 80 dB, express and evaluate the power radiated by this source:

$$P=I_{80}A=\left(10^{-4}\ \text{W/m}^2\right)\left[4\pi(10\,\text{m})^2\right]$$
$$=\boxed{0.13\,\text{W}}$$

62 •• Harry and Sally are sitting on opposite sides of a circus tent when an elephant trumpets a loud blast. If Harry experiences a sound intensity level of 65 dB and Sally experiences only 55 dB, what is the ratio of the distance between Sally and the elephant to the distance between Harry and the elephant?

Picture the Problem The intensity of the sound heard by Harry and Sally depends inversely on the square of the distance between the elephant and each of them. We can use the definition of sound intensity (decibel) level ($\beta=(10\text{ dB})\log(I/I_0)$) and the definition of intensity ($I=P_{\text{av}}/A$) to find the ratio of these distances.

Express the sound intensity level at Harry's location:

$$\beta_H = (10\text{ dB})\log\left(\frac{I_H}{I_0}\right) = (10\text{ dB})\log\left(\frac{P_{av}}{4\pi r_H^2 I_0}\right)$$

Similarly, the sound intensity level at Sally's location is:

$$\beta_S = (10\text{ dB})\log\left(\frac{P_{av}}{4\pi r_S^2 I_0}\right)$$

The difference in decibel level's at the two locations is given by:

$$\Delta\beta = \beta_H - \beta_S$$

Substituting for β_H, β_S, and $\Delta\beta$ yields:

$$\Delta\beta = (10\text{ dB})\log\left(\frac{P_{av}}{4\pi r_H^2 I_0}\right) - (10\text{ dB})\log\left(\frac{P_{av}}{4\pi r_S^2 I_0}\right) = 65\text{ dB} - 55\text{ dB} = 10\text{ dB}$$

Simplifying this expression yields:

$$\log\left(\frac{r_S^2}{r_H^2}\right) = 1 \Rightarrow \frac{r_S^2}{r_H^2} = 10^1$$

Solving for the ratio r_S/r_H yields:

$$\frac{r_S}{r_H} = \sqrt{10} = \boxed{3.2}$$

63 •• Three noise sources produce intensity levels of 70, 73, and 80 dB when acting separately. When the sources act together, the resultant intensity is the sum of the individual intensities. (*a*) Find the sound intensity level in decibels when the three sources act at the same time. (*b*) Discuss the effectiveness of eliminating the two least intense sources in reducing the intensity level of the noise.

Picture the Problem We can find the intensities of the three sources from their intensity levels and, because their intensities are additive, find the intensity level when all three sources are acting.

(*a*) Express the sound intensity level when the three sources act at the same time:

$$\beta_{3\,\text{sources}} = (10\,\text{dB})\log\left(\frac{I_{3\,\text{sources}}}{I_0}\right) = (10\,\text{dB})\log\left(\frac{I_{70} + I_{73} + I_{80}}{I_0}\right)$$

Find the intensities of each of the three sources:

$$70\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{70}}{I_0}\right) \Rightarrow I_{70} = 10^7 I_0$$

$$73\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{73}}{I_0}\right) \Rightarrow I_{73} = 10^{7.3} I_0$$

and

$$80\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{80}}{I_0}\right) \Rightarrow I_{80} = 10^8 I_0$$

Substituting in the expression for $\beta_{3\text{ sources}}$ yields:

$$\beta_{3\text{sources}} = (10\,\text{dB})\log\left(\frac{10^7 I_0 + 10^{7.3} I_0 + 10^8 I_0}{I_0}\right) = (10\,\text{dB})\log\left(10^7 + 10^{7.3} + 10^8\right)$$
$$= \boxed{81\,\text{dB}}$$

(*b*) Find the intensity level with the two least intense sources eliminated:

$$\beta_{80} = (10\,\text{dB})\log\left(\frac{10^8 I_0}{I_0}\right) = \boxed{80\,\text{dB}}$$

Eliminating the 70-dB and 73-dB sources does not reduce the intensity level significantly.

64 •• Show that if two people are different distances away from a sound source, the difference $\Delta\beta$ between the intensity levels reaching the people, in decibels, will always be the same, no matter the power radiated by the source.

Picture the Problem Let the two people be identified by the numerals 1 and 2 and use the definition of the intensity level, in decibels, to express $\Delta\beta$ as a function of the distances r_1 and r_2 of the two people from the source.

Express the difference in the sound intensity level heard by the two people:

$$\Delta\beta = \beta_2 - \beta_1$$

The sound level intensity (decibel level) heard by the second person is given by:

$$\beta_2 = (10\text{ dB})\log\left(\frac{I_2}{I_0}\right)$$
$$= (10\text{ dB})\log\left(\frac{P_{\text{av}}}{A_2 I_0}\right)$$
$$= (10\text{ dB})\log\left(\frac{P_{\text{av}}}{4\pi r_2^2 I_0}\right)$$

In like manner:

$$\beta_1 = (10\text{ dB})\log\left(\frac{P_{\text{av}}}{4\pi r_1^2 I_0}\right)$$

Substitute in the expression for $\Delta\beta$ to obtain:

$$\Delta\beta = (10\text{ dB})\log\left(\frac{P_{\text{av}}}{4\pi r_2^2 I_0}\right) - (10\text{ dB})\log\left(\frac{P_{\text{av}}}{4\pi r_1^2 I_0}\right)$$

Simplifying yields:

$$\Delta\beta = (10\text{ dB})\log\left(\frac{\dfrac{P_{\text{av}}}{4\pi r_2^2 I_0}}{\dfrac{P_{\text{av}}}{4\pi r_1^2 I_0}}\right)$$

$$= (10\text{ dB})\log\left(\frac{r_1^2}{r_2^2}\right)$$

This result shows that the difference in sound level ***intensities*** depends only on the distances to the source and not on the source's power output.

65 **•••** Everyone at a party is talking equally loudly. One person is talking to you and the sound intensity level at your location is 72 dB. Assuming that all 38 people at the party are at the same distance from you as the person who you are talking to, find the sound intensity level at your location.

Picture the Problem The sound intensity level can be found from the intensity of the sound due to the 38 people. When 38 people are talking, the intensities add.

Express the sound level when all 38 people are talking:

$$\beta_{38} = (10\,\text{dB})\log\left(\frac{38 I_1}{I_0}\right)$$

$$= (10\,\text{dB})\log 38 + (10\,\text{dB})\log\left(\frac{I_1}{I_0}\right)$$

$$= (10\,\text{dB})\log 38 + 72\,\text{dB}$$

$$= \boxed{88\,\text{dB}}$$

An equivalent but longer solution:

Express the sound intensity level when all 38 people are talking:

$$\beta_{38} = (10\,\text{dB})\log\left(\frac{38 I_1}{I_0}\right)$$

Express the sound intensity level when only one person is talking:

$$\beta_1 = 72\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_1}{I_0}\right)$$

Solving for I_1 yields:

$$I_1 = 10^{7.2} I_0 = 10^{7.2}\left(10^{-12}\ \text{W/m}^2\right)$$
$$= 1.58\times 10^{-5}\ \text{W/m}^2$$

Express the sound intensity when all 38 people are talking:

$$I_{38} = 38 I_1$$

The sound intensity level is:

$$\beta_{38} = (10\,\text{dB})\log\left[\frac{38\left(1.58\times 10^{-5}\ \text{W/m}^2\right)}{10^{-12}\ \text{W/m}^2}\right]$$
$$= \boxed{88\,\text{dB}}$$

66 ••• When a violinist pulls the bow across a string, the force with which the bow is pulled is fairly small, about 0.60 N. Suppose the bow travels across the A string, which vibrates at 440 Hz, at 0.50 m/s. A listener 35 m from the performer hears a sound of 60-dB intensity. Assuming that the sound radiates uniformly in all directions, with what efficiency is the mechanical energy of bowing converted to sound energy?

Picture the Problem Let η represent the efficiency with which mechanical energy is converted to sound energy. Because we're given information regarding the rate at which mechanical energy is delivered to the string and the rate at which sound energy arrives at the location of the listener, we'll take the efficiency to be the ratio of the sound power delivered to the listener divided by the power delivered to the string. We can calculate the power input directly from the given data. We'll calculate the intensity of the sound at 35 m from its intensity level at that distance and use this result to find the power output.

Express the efficiency of the conversion of mechanical energy to sound energy:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}$$

Find the power delivered by the bow to the string:

$$P_{\text{in}} = Fv = (0.60\,\text{N})(0.50\,\text{m/s}) = 0.30\,\text{W}$$

Using $\beta = (10\,\text{dB})\log(I/I_0)$, find the intensity of the sound at 35 m:

$$60\,\text{dB} = (10\,\text{dB})\log\left(\frac{I_{35\,\text{m}}}{I_0}\right)$$

and

$$I_{35\,\text{m}} = 10^6 I_0 = 1.00\times 10^{-6}\ \text{W/m}^2$$

The rate at which sound energy is emitted is given by:

$$P_{\text{out}} = IA = 4\pi\left(1.00\times10^{-6}\ \text{W/m}^2\right)\left(35\,\text{m}\right)^2 = 0.0154\,\text{W}$$

Substitute numerical values and evaluate η:

$$\eta = \frac{0.0154\,\text{W}}{0.30\,\text{W}} = \boxed{5.1\%}$$

67 ••• [SSM] The noise intensity level at some location in an empty examination hall is 40 dB. When 100 students are writing an exam, the noise level at that location increases to 60 dB. Assuming that the noise produced by each student contributes an equal amount of acoustic power, find the noise intensity level at that location when 50 students have left.

Picture the Problem Because the sound intensities are additive, we'll find the noise intensity level due to one student by subtracting the background noise intensity from the intensity due to the students and dividing by 100. Then, we'll use this result to calculate the intensity level due to 50 students.

Express the intensity level due to 50 students:

$$\beta_{50} = \left(10\,\text{dB}\right)\log\left(\frac{50I_1}{I_0}\right)$$

Find the sound intensity when 100 students are writing the exam:

$$60\,\text{dB} = \left(10\,\text{dB}\right)\log\left(\frac{I_{100}}{I_0}\right)$$

and

$$I_{100} = 10^6 I_0 = 10^{-6}\ \text{W/m}^2$$

Find the sound intensity due to the background noise:

$$40\,\text{dB} = \left(10\,\text{dB}\right)\log\left(\frac{I_{\text{background}}}{I_0}\right)$$

and

$$I_{\text{background}} = 10^4 I_0 = 10^{-8}\ \text{W/m}^2$$

Express the sound intensity due to the 100 students:

$$I_{100} - I_{\text{background}} = 10^{-6}\ \text{W/m}^2 - 10^{-8}\ \text{W/m}^2 \approx 10^{-6}\ \text{W/m}^2$$

Find the sound intensity due to 1 student:

$$\frac{I_{100} - I_{\text{background}}}{100} = 10^{-8}\ \text{W/m}^2$$

Substitute numerical values and evaluate the noise intensity level due to 50 students:

$$\beta_{50} = \left(10\,\text{dB}\right)\log\frac{50\left(1.00\times10^{-8}\ \text{W/m}^2\right)}{10^{-12}\ \text{W/m}^2} = \boxed{57\,\text{dB}}$$

String Waves Experiencing Speed Changes

68 • A 3.00-m-long piece of string with a mass of 25.0 g is tied to 4.00 m of heavy twine with a mass of 75.0 g and the combination is put under a tension of 100 N. If a transverse pulse is sent down the less dense string, determine the reflection and transmission coefficients at the junction point.

Picture the Problem Let the subscript ″s″ refer to the string and the subscript ″t″ to the heavy twine. We can use the definitions of the reflection and transmission coefficients and the expression for the speed of waves on a string (Equation 15-3) to find the speeds of the pulse on the string and the heavy twine.

Use their definitions to express the reflection and transmission coefficients:

$$r = \frac{v_t - v_s}{v_t + v_s} = \frac{1 - \frac{v_s}{v_t}}{1 + \frac{v_s}{v_t}} \qquad (1)$$

and

$$\tau = \frac{2v_t}{v_t + v_s} = \frac{2}{1 + \frac{v_s}{v_t}} \qquad (2)$$

Use Equation 15-3 to express v_t and v_s:

$$v_t = \sqrt{\frac{F_T}{\mu_t}} \text{ and } v_s = \sqrt{\frac{F_T}{\mu_s}}$$

Dividing the expression for v_s by the expression for v_t and simplifying yields:

$$\frac{v_s}{v_t} = \frac{\sqrt{\frac{F_T}{\mu_s}}}{\sqrt{\frac{F_T}{\mu_t}}} = \sqrt{\frac{\mu_t}{\mu_s}}$$

Substitute for v_s/v_t in equation (1) to obtain:

$$r = \frac{1 - \sqrt{\frac{\mu_t}{\mu_s}}}{1 + \sqrt{\frac{\mu_t}{\mu_s}}} \qquad (3)$$

Express the ratio of μ_t to μ_s:

$$\frac{\mu_t}{\mu_s} = \frac{\frac{m_t}{\ell_t}}{\frac{m_s}{\ell_s}} = \frac{m_t \ell_s}{m_s \ell_t}$$

Substituting in equation (3) yields:

$$r = \frac{1-\sqrt{\dfrac{m_t \ell_s}{m_s \ell_t}}}{1+\sqrt{\dfrac{m_t \ell_s}{m_s \ell_t}}}$$

Substitute numerical values and evaluate r:

$$r = \frac{1-\sqrt{\dfrac{(75.0\times10^{-3}\text{ kg})(3.00\text{ m})}{(25.0\times10^{-3}\text{ kg})(4.00\text{ m})}}}{1+\sqrt{\dfrac{(75.0\times10^{-3}\text{ kg})(3.00\text{ m})}{(25.0\times10^{-3}\text{ kg})(4.00\text{ m})}}} = \boxed{-0.20}$$

Substitute for v_s/v_t in equation (2) to obtain:

$$\tau = \frac{2}{1+\sqrt{\dfrac{\mu_t}{\mu_s}}} = \frac{2}{1+\sqrt{\dfrac{m_t \ell_s}{m_s \ell_t}}}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{2}{1+\sqrt{\dfrac{(75.0\times10^{-3}\text{ kg})(3.00\text{ m})}{(25.0\times10^{-3}\text{ kg})(4.00\text{ m})}}} = \boxed{0.80}$$

Remarks: Because $r+\tau=1$, we could have used this relationship to find τ.

69 • [SSM] Consider a taut string with a mass per unit length μ_1, carrying transverse wave pulses that are incident upon a point where the string connects to a second string with a mass per unit length μ_2. (*a*) Show that if $\mu_2 = \mu_1$, then the reflection coefficient r equals zero and the transmission coefficient τ equals +1. (*b*) Show that if $\mu_2 >> \mu_1$, then $r \approx -1$ and $\tau \approx 0$; and (*c*) if $\mu_2 << \mu_1$ then $r \approx +1$ and $\tau \approx +2$.

Picture the Problem We can use the definitions of the reflection and transmission coefficients and the expression for the speed of waves on a string (Equation 15-3) to r and t in terms of the linear densities of the strings.

(*a*) Use their definitions to express the reflection and transmission coefficients:

$$r = \frac{v_2 - v_1}{v_2 + v_1} = \frac{1 - \frac{v_1}{v_2}}{1 + \frac{v_1}{v_2}} \qquad (1)$$

and

$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2}{1 + \frac{v_1}{v_2}} \qquad (2)$$

Use Equation 15-3 to express v_2 and v_1:

$$v_2 = \sqrt{\frac{F_T}{\mu_2}} \text{ and } v_1 = \sqrt{\frac{F_T}{\mu_1}}$$

Dividing the expression for v_1 by the expression for v_2 and simplifying yields:

$$\frac{v_1}{v_2} = \frac{\sqrt{\frac{F_T}{\mu_1}}}{\sqrt{\frac{F_T}{\mu_2}}} = \sqrt{\frac{\mu_2}{\mu_1}}$$

Substitute for v_1/v_2 in equation (1) to obtain:

$$r = \frac{1 - \sqrt{\frac{\mu_2}{\mu_1}}}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} \qquad (3)$$

Substitute for v_1/v_2 in equation (2) to obtain:

$$\tau = \frac{2}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} \qquad (4)$$

If $\mu_2 = \mu_1$:

$$r = \frac{1 - \sqrt{1}}{1 + \sqrt{1}} = \boxed{0}$$

and

$$\tau = \frac{2}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} = \frac{2}{1 + \sqrt{1}} = \boxed{1}$$

(*b*) From equations (1) and (2), if $\mu_2 >> \mu_1$ then $v_1 >> v_2$:

$$r = \frac{v_2 - v_1}{v_2 + v_1} \approx \frac{-v_1}{v_1} = \boxed{-1}$$

and

$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2}{1 + \frac{v_1}{v_2}} \approx \boxed{0}$$

(*c*) If $\mu_2 << \mu_1$, then $v_2 >> v_1$:

$$r = \frac{v_2 - v_1}{v_2 + v_1} = \frac{1 - \frac{v_1}{v_2}}{1 + \frac{v_1}{v_2}} \approx \boxed{1}$$

and

$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2}{1 + \frac{v_1}{v_2}} \approx \boxed{2}$$

70 •• Verify the validity of $1 = r^2 + (v_1/v_2)\tau^2$ (Equation 15-36) by substituting the expressions for r and τ into it.

Picture the Problem Making the indicated substitutions will lead us to the identity 1 = 1.

The reflection and transmission coefficients are given by:

$$r = \frac{v_2 - v_1}{v_2 + v_1} \text{ and } \tau = \frac{2v_2}{v_2 + v_1}$$

Substituting into the equation given in the problem statement yields:

$$1 = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 + \frac{v_1}{v_2}\left(\frac{2v_2}{v_2 + v_1}\right)^2$$

Simplify this expression algebraically to obtain:

$$1 = \frac{(v_2 - v_1)^2 + 4v_2^2\left(\frac{v_1}{v_2}\right)}{(v_2 + v_1)^2} = \frac{v_2^2 - 2v_2v_1 + v_1^2 + 4v_2v_1}{(v_2 + v_1)^2} = \frac{v_2^2 + 2v_2v_1 + v_1^2}{(v_2 + v_1)^2} = \frac{(v_2 + v_1)^2}{(v_2 + v_1)^2}$$
$$= \boxed{1}$$

71 ••• Consider a taut string that has a mass per unit length μ_1 carrying transverse wave pulses of the form $y = f(x - v_1t)$ that are incident upon a point P where the string connects to a second string with mass per unit length μ_2. Derive $1 = r^2 + (v_1/v_2)\tau^2$ by equating the power incident on point P to the power reflected at P plus the power transmitted at P.

Picture the Problem Choose the direction of propagation of the incident pulse as the $+x$ direction and let $x = 0$ at point P. Energy is conserved as the incident pulse is partially reflected and partially transmitted at point P.

From the conservation of energy we have:

$$P_{\text{in}} + P_{\text{r}} = P_{\text{t}} \qquad (1)$$

From Equation 15-20, the power transmitted in the direction of increasing x is given by:

$$P = -F_{\text{T}}\frac{\partial y}{\partial x}\frac{\partial y}{\partial x}$$

Substituting in equation (1) yields:

$$\left(-F_{\text{T}}\frac{\partial y_{\text{in}}}{\partial x}\frac{\partial y_{\text{in}}}{\partial x}\right)+\left(-F_{\text{T}}\frac{\partial y_{\text{r}}}{\partial x}\frac{\partial y_{\text{r}}}{\partial x}\right)=\left(-F_{\text{T}}\frac{\partial y_{\text{t}}}{\partial x}\frac{\partial y_{\text{t}}}{\partial x}\right)$$

or, upon simplification,

$$\frac{\partial y_{\text{in}}}{\partial x}\frac{\partial y_{\text{in}}}{\partial x}+\frac{\partial y_{\text{r}}}{\partial x}\frac{\partial y_{\text{r}}}{\partial x}=\frac{\partial y_{\text{t}}}{\partial x}\frac{\partial y_{\text{t}}}{\partial x} \qquad (2)$$

Because the incident pulse is given by $y_{\text{in}} = f(x - v_1 t)$, the reflected and transmitted pulses are given by:

$$y_{\text{r}} = rf(-x - v_1 t)$$

and

$$y_{\text{t}} = \tau\, f\left(\frac{v_1}{v_2}[x - v_2 t]\right)$$

Evaluating $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$ using the chain rule yields:

$$\frac{\partial y}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} \text{ and } \frac{\partial y}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t}$$

where η is the argument of the wave function.

For the transmitted pulse:

$$\frac{\partial y_{\text{t}}}{\partial x} = \tau\frac{df}{d\eta}\frac{\partial}{\partial x}\left(\frac{v_1}{v_2}[x - v_2 t]\right) = \tau\frac{df}{d\eta}\frac{v_1}{v_2}$$

and

$$\frac{\partial y_{\text{t}}}{\partial t} = \tau\frac{df}{d\eta}\frac{\partial}{\partial t}\left(\frac{v_1}{v_2}[x - v_2 t]\right) = \tau\frac{df}{d\eta}(-v_1)$$

$$= -\tau v_1\frac{df}{d\eta}$$

For the reflected pulse:

$$\frac{\partial y_{\mathrm{r}}}{\partial x}=r\frac{df}{d\eta}\frac{\partial}{\partial x}(-x-v_1t)=-r\frac{df}{d\eta}$$

and

$$\frac{\partial y_{\mathrm{r}}}{\partial t}=r\frac{df}{d\eta}\frac{\partial}{\partial t}(-x-v_1t)=r\frac{df}{d\eta}(-v_1)$$

$$=-rv_1\frac{df}{d\eta}$$

For the incident pulse:

$$\frac{\partial y_{\mathrm{in}}}{\partial x}=\frac{df}{d\eta}\frac{\partial}{\partial x}(x-v_1t)=\frac{df}{d\eta}$$

and

$$\frac{\partial y_{\mathrm{in}}}{\partial t}=\frac{df}{d\eta}\frac{\partial}{\partial t}(x-v_1t)=\frac{df}{d\eta}(-v_1)$$

$$=-v_1\frac{df}{d\eta}$$

Substitute in equation (2) to obtain:

$$\frac{df}{d\eta}\left(-v_1\frac{df}{d\eta}\right)+\left(-r\frac{df}{d\eta}\right)\left(-rv_1\frac{df}{d\eta}\right)=\left(\tau\frac{df}{d\eta}\frac{v_1}{v_2}\right)\left(-\tau v_1\frac{df}{d\eta}\right)$$

Simplifying and rearranging terms yields:

$$\boxed{1=r^2+\frac{v_1}{v_2}\tau^2}$$

The Doppler Effect

72 • A sound source is moving at 80 m/s toward a stationary listener that is standing in still air. (*a*) Find the wavelength of the sound in the region between the source and the listener. (*b*) Find the frequency heard by the listener.

Picture the Problem We can use Equation 15-38 ($\lambda=\frac{v\pm u_{\mathrm{s}}}{f_{\mathrm{s}}}$) to find the wavelength of the sound between the source and the listener and Equation15-41*a* ($f_{\mathrm{r}}=\frac{v\pm u_{\mathrm{r}}}{v\pm u_{\mathrm{s}}}f_{\mathrm{s}}$) to find the frequency heard by the listener.

(*a*) Apply Equation 15-38 to find λ:

$$\lambda=\frac{v\pm u_{\mathrm{s}}}{f_{\mathrm{s}}}=\frac{v-u_{\mathrm{s}}}{f_{\mathrm{s}}}=\frac{343\,\mathrm{m/s}-80\,\mathrm{m/s}}{200\,\mathrm{s}^{-1}}$$

$$=\boxed{1.32\,\mathrm{m}}$$

(*b*) Apply Equation 15-41*a* to obtain f_r:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{v \pm 0}{v - u_s} f_s$$
$$= \frac{343\,\text{m/s}}{343\,\text{m/s} - 80\,\text{m/s}} \left(200\,\text{s}^{-1}\right)$$
$$= \boxed{261\,\text{Hz}}$$

73 • Consider the situation described in Problem 72 from the reference frame of the source. In this frame, the listener and the air are moving toward the source at 80 m/s and the source is at rest. (*a*) At what speed, relative to the source, is the sound traveling in the region between the source and the listener? (*b*) Find the wavelength of the sound in the region between the source and the listener. (*c*) Find the frequency heard by the listener.

Picture the Problem (*a*) In the reference frame of the source, the speed of sound from the source to the listener is reduced by the speed of the air. (*b*) We can find the wavelength of the sound in the region between the source and the listener from $v = f\lambda$. (*c*) Because the sound waves in the region between the source and the listener will be compressed by the motion of the listener, the frequency of the sound heard by the listener will be higher than the frequency emitted by the source and can be calculated using $f_r = \frac{v \pm u_r}{v \pm u_s} f_s$ (Equation 15-41*a*).

(*a*) The speed of sound in the reference frame of the source is:

$$v' = v - u_{\text{wind}} = 343\,\text{m/s} - 80\,\text{m/s}$$
$$= \boxed{263\,\text{m/s}}$$

(*b*) Noting that the frequency is unchanged, express the wavelength of the sound:

$$\lambda = \frac{v'}{f} = \frac{263\,\text{m/s}}{200\,\text{s}^{-1}} = \boxed{1.32\,\text{m}}$$

(*c*) Apply Equation 15-41*a* to obtain:

$$f_r = \frac{v' \pm u_r}{v' \pm u_s} f_s = \left(\frac{v' + u_r}{v' \pm 0}\right) f_s$$
$$= \left(\frac{263\,\text{m/s} + 80\,\text{m/s}}{263\,\text{m/s}}\right)\left(200\,\text{s}^{-1}\right)$$
$$= \boxed{261\,\text{Hz}}$$

74 • A sound source is moving away from the stationary listener at 80 m/s. (*a*) Find the wavelength of the sound waves in the region between the source and the listener. (*b*) Find the frequency heard by the listener.

Picture the Problem We can use $\lambda = (v \pm u_s)/f_s$ (Equation 15-38) to find the wavelength of the sound in the region between the source and the listener and $f_r = \dfrac{v \pm u_r}{v \pm u_s} f_s$ (Equation 15-41*a*) to find the frequency heard by the listener. Because the sound waves in the region between the source and the listener will be spread out by the motion of the listener, the frequency of the sound heard by the listener will be lower than the frequency emitted by the source.

(*a*) Because the source is moving away from the listener, use the positive sign in the numerator of Equation 15-38 to find the wavelength of the sound between the source and the listener:

$$\lambda = \frac{v \pm u_s}{f_s} = \frac{v + u_s}{f_s} = \frac{343\,\text{m/s} + 80\,\text{m/s}}{200\,\text{s}^{-1}} = \boxed{2.12\,\text{m}}$$

(*b*) Because the listener is at rest and the source is receding, $u_r = 0$ and the denominator of Equation 15-41*a* is the sum of the two speeds:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{v \pm 0}{v + u_s} f_s = \frac{343\,\text{m/s}}{343\,\text{m/s} + 80\,\text{m/s}}\left(200\,\text{s}^{-1}\right) = \boxed{162\,\text{Hz}}$$

75 • The listener is moving 80 m/s from the stationary source that is in rest relative to the air. Find the frequency heard by the listener.

Picture the Problem Because the listener is moving away from the source, we know that the frequency he/she will hear will be less than the frequency emitted by the source. We can use $f_r = \dfrac{v \pm u_r}{v \pm u_s} f_s$ (Equation 15-41a), with $u_s = 0$ and the minus sign in the numerator, to determine its value.

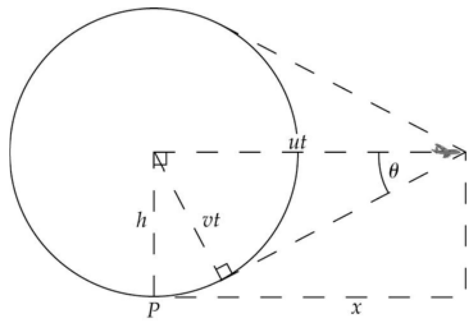
Relate the frequency heard by the listener to that of the source:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \left(\frac{v - u_r}{v \pm 0}\right) f_s = \left(\frac{343\,\text{m/s} - 80\,\text{m/s}}{343\,\text{m/s}}\right)\left(200\,\text{s}^{-1}\right) = \boxed{153\,\text{Hz}}$$

76 •• You have made the trek to observe a Space Shuttle landing. Near the end of its descent, the ship is traveling at Mach 2.50 at an altitude of 5000 m. (*a*) What is the angle that the shock wave makes with the line of flight of the shuttle? (*b*) How far are you from the shuttle by the time you hear its shock wave,

assuming the shuttle maintains both a constant heading and a constant 5000-m altitude after flying directly over your head?

Picture the Problem The diagram shows the position of the shuttle at time t after it was directly over your head (located at point P) Let u represent the speed of the shuttle and v the speed of sound. We can use trigonometry to determine the angle of the shock wave as well as the location of the shuttle x when you hear the shock wave.



(a) Referring to the diagram, express θ in terms of v, u, and t:

$$\sin\theta = \frac{vt}{ut} = \frac{1}{u/v} = \frac{1}{2.5}$$

Solving for θ yields:

$$\theta = \sin^{-1}\left(\frac{1}{2.50}\right) = 23.58° = \boxed{23.6°}$$

(b) Using the diagram, relate θ to the altitude h of the shuttle and the distance x:

$$\tan\theta = \frac{h}{x} \Rightarrow x = \frac{h}{\tan\theta}$$

Substitute numerical values and evaluate x:

$$x = \frac{5000\,\text{m}}{\tan 23.58°} = \boxed{11.5\,\text{km}}$$

77 •• The SuperKamiokande neutrino detector in Japan is a water tank the size of a 14-story building. When neutrinos collide with electrons in water, most of their energy is transferred to the electrons. As a consequence, the electrons then fly off at speeds that approach c. The neutrino is counted by detecting the shock wave, called Cerenkov radiation, that is produced when the high-speed electrons travel through the water at speeds greater than the speed of light in water. If the maximum angle of the Cerenkov shock-wave cone is 48.75°, what is the speed of light in water?

Picture the Problem The angle θ of the Cerenkov shock wave is related to the speed of light in water v and the speed of light in a vacuum c according to $\sin\theta = v/c$.

Relate the speed of light in water v to the angle of the Cerenkov cone:

$$\sin\theta = \frac{v}{c} \Rightarrow v = c\sin\theta$$

Substitute numerical values and evaluate v:

$$v = (2.998\times10^{8}\text{ m/s})\sin 48.75° = \boxed{2.254\times10^{8}\text{ m/s}}$$

78 •• You are in charge of calibrating the radar guns for a local police department. One such device emits microwaves at a frequency of 2.00 GHz. During the trials, you have it arranged so that these waves are reflected from a car moving directly away from the stationary emitter. In this situation, you detect a frequency difference (between the received microwaves and the ones sent out) of 293 Hz. Find the speed of the car.

Picture the Problem Because the car is moving away from the stationary emitter at a speed u_r, the frequency f_r it receives will be less than the frequency emitted by the emitter. The microwaves reflected from the car, moving away from a stationary detector, will be of a still lower frequency f_r'. We can use the Doppler shift equations to derive an expression for the speed of the car in terms of difference of these frequencies.

Express the frequency f_r received by the moving car in terms of f_s, u_r, and c:

$$f_r = \frac{c\pm u_r}{c\pm u_s}f_s = \left(\frac{c-u_r}{c\pm 0}\right)f_s \qquad (1)$$

The waves reflected by the car are like waves re-emitted by a source moving away from the radar gun:

$$f_r' = \frac{c\pm u_r}{c\pm u_s}f_r = \left(\frac{c}{c+u_s}\right)f_r \qquad (2)$$

Substitute equation (1) in equation (2) to eliminate f_r:

$$f_r' = \left(\frac{c}{c+u_s}\right)\left(\frac{c-u_r}{c}\right)f_s = \left(\frac{c-u_r}{c+u_s}\right)f_s$$

$$= \left(\frac{1-\frac{u_r}{c}}{1+\frac{u_r}{c}}\right)f_s = \left(1-\frac{u_r}{c}\right)\left(1+\frac{u_r}{c}\right)^{-1}f_s$$

Because $u_r << c$, $\left(1+\frac{u_r}{c}\right)^{-1} \approx 1-\frac{u_r}{c}$. Substituting for $\left(1+\frac{u_r}{c}\right)^{-1}$ and simplifying yields:

$$f_r' \approx \left(1-\frac{u_r}{c}\right)\left(1-\frac{u_r}{c}\right)f_s = \left(1-\frac{u_r}{c}\right)^{2}f_s$$

Because $u_r << c$, $\left(1-\frac{u_r}{c}\right)^2 \approx 1-\frac{2u_r}{c}$.

Substituting for $\left(1-\frac{u_r}{c}\right)^2$ gives:

$$f_r' \approx \left(1-\frac{2u_r}{c}\right)f_s$$

The frequency difference detected at the source is given by:

$$\Delta f = f_s - f_r' = f_s - \left(1-\frac{2u_r}{c}\right)f_s$$
$$= \frac{2u_r}{c}f_s$$

Solving for u_r yields:

$$u_r = \frac{c}{2f_s}\Delta f$$

Substitute numerical values and evaluate u_r:

$$u_r = \frac{2.998\times10^8\text{ m/s}}{2(2.00\,\text{GHz})}(293\,\text{Hz})$$
$$= 22.0\frac{\text{m}}{\text{s}}\times\frac{1\,\text{km}}{10^3\,\text{m}}\times\frac{3600\,\text{s}}{\text{h}}$$
$$= \boxed{79.1\,\text{km/h}}$$

79 •• [SSM] The Doppler effect is routinely used to measure the speed of winds in storm systems. As the manager of a weather monitoring station in the Midwest, you are using a Doppler radar system that has a frequency of 625 MHz to bounce a radar pulse off of the raindrops in a swirling thunderstorm system 50 km away. You measure the reflected radar pulse to be up-shifted in frequency by 325 Hz. Assuming the wind is headed directly toward you, how fast are the winds in the storm system moving? *Hint: The radar system can only measure the component of the wind velocity along its "line of sight."*

Picture the Problem Because the wind is moving toward the weather station (radar device), the frequency f_r the raindrops receive will be greater than the frequency emitted by the radar device. The radar waves reflected from the raindrops, moving toward the stationary detector at the weather station, will be of a still higher frequency f_r'. We can use the Doppler shift equations to derive an expression for the radial speed u of the wind in terms of difference of these frequencies.

Use Equation 15-41*a* to express the frequency f_r received by the raindrops in terms of f_s, u_r, and c:

$$f_r = \frac{c\pm u_r}{c\pm u_s}f_s = \left(\frac{c+u_r}{c}\right)f_s \qquad (1)$$

The waves reflected by the drops are like waves re-emitted by a source moving toward the source at the weather station:

$$f_r' = \frac{c \pm u_r}{c \pm u_s} f_r = \left(\frac{c}{c - u_s}\right) f_r \qquad (2)$$

Substitute equation (1) in equation (2) to eliminate f_r:

$$f_r' = \left(\frac{c}{c-u_s}\right)\left(\frac{c+u_r}{c}\right) f_s = \left(\frac{c+u_r}{c-u_s}\right) f_s$$

$$= \left(\frac{1+\frac{u_r}{c}}{1-\frac{u_r}{c}}\right) f_s = \left(1+\frac{u_r}{c}\right)\left(1-\frac{u_r}{c}\right)^{-1} f_s$$

Because $u_r << c$, $\left(1-\frac{u_r}{c}\right)^{-1} \approx 1+\frac{u_r}{c}$. Substituting for $\left(1-\frac{u_r}{c}\right)^{-1}$ and simplifying yields:

$$f_r' \approx \left(1+\frac{u_r}{c}\right)\left(1+\frac{u_r}{c}\right) f_s$$

$$= \left(1+\frac{u_r}{c}\right)^2 f_s$$

Because $u_r << c$, $\left(1+\frac{u_r}{c}\right)^2 \approx 1+\frac{2u_r}{c}$. Substituting for $\left(1+\frac{u_r}{c}\right)^2$ gives:

$$f_r' \approx \left(1+\frac{2u_r}{c}\right) f_s$$

The frequency difference detected at the source is:

$$\Delta f = f_r' - f_s = \left(1+\frac{2u_r}{c}\right) f_s - f_s$$

$$= \frac{2u_r}{c} f_s$$

Solving for u_r yields:

$$u_r = \frac{c}{2 f_s} \Delta f$$

Substitute numerical values and evaluate u_r:

$$u_r = \frac{2.998\times10^8 \text{ m/s}}{2(625\,\text{MHz})}(325\,\text{Hz})$$

$$= 77.95 \text{ m/s} \times \frac{1\,\text{mi/h}}{0.4470\,\text{m/s}}$$

$$= \boxed{174\,\text{mi/h}}$$

80 •• A stationary destroyer is equipped with sonar that sends out 40-MHz-pulses of sound. The destroyer receives pulses back from a submarine directly below with a time delay of 80 ms at a frequency of 39.958 MHz. If the speed of sound in seawater is 1.54 km/s, (*a*) what is the depth of the submarine? (*b*) What is its vertical speed?

Picture the Problem Let the depth of the submarine be represented by *D* and its vertical speed by *u*. The submarine acts as both a receiver and source. We can apply the definition of average speed to determine the depth of the submarine and use the Doppler shift equations to derive an expression for the vertical speed of the submarine in terms of the frequency difference. The frequency received is lower than the frequency of the source, so the sub is descending.

(*a*) Using the definition of average speed, relate the depth of the submarine to the time delay between the transmitted and reflected pulses:

$$2D = v\Delta t \Rightarrow D = \tfrac{1}{2}v\Delta t$$

Substitute numerical values and evaluate *D*:

$$D = \tfrac{1}{2}(1.54\,\text{km/s})(80\,\text{ms}) = \boxed{62\,\text{m}}$$

(*b*) Use $f_r = \left(\dfrac{v \pm u_r}{v \pm u_s}\right) f_s$ to express the frequency f_{sub} received by the submarine:

$$f_{sub} = \left(\frac{v-u}{v}\right) f_0 \qquad (1)$$

where *u* is the vertical speed of the submarine.

Use $f_r = \left(\dfrac{v \pm u_r}{v \pm u_s}\right) f_s$ to express the frequency f_r' received by the destroyer:

$$f_r' = \left(\frac{v}{v+u}\right) f_{sub} \qquad (2)$$

Substitute equation (1) in equation (2) to eliminate f_{sub}:

$$f_r' = \left(\frac{v-u}{v+u}\right) f_0 \Rightarrow u = \frac{f_0 - f_r'}{f_0 + f_r'} v$$

Substitute numerical values and evaluate *u*:

$$u = \left(\frac{40.000\,\text{MHz} - 39.958\,\text{MHz}}{40.000\,\text{MHz} + 39.958\,\text{MHz}}\right)(1.54\,\text{km/s}) = \boxed{0.81\,\text{m/s}}$$

and so, because *u* is positive, the sub is descending.

81 •• A police radar unit transmits microwaves of frequency 3.00×10^{10} Hz, and their speed in air is 3.00×10^{8} m/s. Suppose a car is receding from the stationary police car at a speed of 140 km/h. (*a*) What is the frequency difference between the transmitted signal and the signal received from the receding car? (*b*) Suppose the police car is, instead, moving at a speed of 60 km/h in the same direction as the other vehicle. What is the difference in frequency between the emitted and the reflected signals?

Picture the Problem The radar wave strikes the speeding car at frequency f_r . This frequency is less than f_s because the car is moving away from the source. The frequency shift is given by Equation 15-42 (the low-speed, relative to light, approximation). The car then acts as a moving source emitting waves of frequency f_r. The police car detects waves of frequency $f_r' < f_r$ because the source (the speeding car) is moving away from the police car. The total frequency shift is the sum of the two frequency shifts.

(*a*) Express the frequency difference Δf as the sum of the frequency difference $\Delta f_1 = f_r - f_s$ and the frequency difference $\Delta f_2 = f_r' - f_r$:

$$\Delta f = \Delta f_1 + \Delta f_2 \qquad (1)$$

Using Equation 15-42, substitute for the frequency differences in equation (1):

$$\Delta f = -\frac{u}{c} f_s - \frac{u}{c} f_r = -\frac{u}{c}\left(f_s + f_r\right) \qquad (2)$$

where $u = u_s \pm u_r$ is the speed of the source relative to the receiver.

Apply Equation 15-42 to Δf_1 to obtain:

$$\frac{\Delta f_1}{f_s} = \frac{f_r - f_s}{f_s} = -\frac{u}{c}$$

where we've used the minus sign because we know the frequency difference is a downshift.

Solving for f_r yields:

$$f_r = \left(1 - \frac{u}{c}\right) f_s$$

Substitute for f_r in equation (2) and simplify to obtain:

$$\Delta f = -\frac{u}{c}\left(f_s + \left(1 - \frac{u}{c}\right) f_s\right) = -\frac{u}{c}\left(2 - \frac{u}{c}\right) f_s$$

Because u/c is negligible compared to 2:

$$\Delta f \approx -2 f_s \frac{u}{c}$$

Substitute numerical values and evaluate Δf:

$$\Delta f \approx -2\left(3.00\times10^{10}\ \text{Hz}\right)\frac{\left(140\,\dfrac{\text{km}}{\text{h}}\times\dfrac{1\,\text{h}}{3600\,\text{s}}\right)}{3.00\times10^{8}\ \text{m/s}} = \boxed{-7.78\,\text{kHz}}$$

(b) Use the result derived in (a) with u equal to the difference in the speeds of the car and the police cruiser to obtain:

$$\Delta f \approx -2\left(3.00\times10^{10}\ \text{Hz}\right)\frac{\left(140\,\dfrac{\text{km}}{\text{h}}-60\,\dfrac{\text{km}}{\text{h}}\right)\left(\dfrac{1\,\text{h}}{3600\,\text{s}}\right)}{3.00\times10^{8}\ \text{m/s}} = \boxed{-4.4\,\text{kHz}}$$

82 •• In modern medicine, the Doppler effect is routinely used to measure the rate and direction of blood flow in arteries and veins. High frequency "ultrasound" (sound at frequencies above the human hearing range) is typically employed. Suppose you are in charge of measuring the blood flow in a vein (located in the lower leg of an older patient) that returns blood upward to the heart. Her varicose veins indicate that perhaps the one-way valves in the vein are not working properly and that the blood is "pooling" in the veins and perhaps even that the blood flow is backward toward her feet. Employing sound with a frequency of 50.0 kHz, you point the sound source from above her thigh region down towards her feet and measure the sound reflected from that vein area to be lower than 50.0 kHz. (a) Was your diagnosis of the valve condition correct? If so, explain. (b) Estimate the instrument's frequency difference capability to enable you to measure speeds down to 1.00 mm/s. Take the speed of sound in flesh to be the same as that in water, 1500 m/s.

Picture the Problem (a) Whether your diagnosis was correct depends on whether the signal reflected from the blood is upshifted or downshifted. (b) Applying the Doppler shift equations with the source stationary and the receiver moving initially and then a second time with the source moving and the receiver stationary will allow us to estimate the instrument's frequency difference capability.

(a) Because the received sound frequency is less than the frequency that was sent out, the blood it reflected from must have been moving away from the source, or toward the feet of the patient. The blood is flowing the wrong way and your diagnosis is correct.

(*b*) The blood receives a frequency that is downshifted according to:

$$f_r = \left(\frac{v \pm u_r}{v \pm u_s}\right) f_s = \left(\frac{v - u_r}{v \pm 0}\right) f_s = \left(1 - \frac{u_r}{v}\right) f_s \qquad (1)$$

The frequency given by equation (1) is the frequency emitted by the moving blood back toward the receiver. This results in a second downshift:

$$f_r' = \left(\frac{v \pm u_r}{v \pm u_s}\right) f_s' = \left(\frac{v \pm 0}{v + u_s}\right) f_s' = \left(1 + \frac{u_s}{v}\right)^{-1} f_s' \approx \left(1 - \frac{u_s}{v}\right) f_s' \qquad (2)$$

The source frequency in equation (2) is the received frequency given by equation (1). Note that the receiver and emitter speeds are the same … the speed of the blood. Let this speed be u to obtain:

$$f_r' \approx \left(1 - \frac{u}{v}\right)\left(1 - \frac{u}{v}\right) f_s \approx \left(1 - 2\frac{u}{v}\right) f_s$$

Express the magnitude of the difference in frequencies:

$$\Delta f = \left| f_s - f_r' \right| \approx \left| f_s - \left(1 - 2\frac{u}{v}\right) f_s \right| = 2\frac{u}{v} f_s$$

Substitute numerical values and evaluate Δf:

$$\Delta f \approx 2\left(\frac{1.00\text{ m/s}}{1500\text{ m/s}}\right)(50.0\text{ kHz}) = \boxed{0.033\text{ Hz}}$$

83 •• **[SSM]** A sound source of frequency f_s moves with speed u_s relative to still air toward a receiver who is moving away with speed u_r relative to still air away from the source. (*a*) Write an expression for the received frequency f'_r. (*b*) Use the result that $(1 - x)^{-1} \approx 1 + x$ to show that if both u_s and u_r are small compared to v, then the received frequency is approximately

$$f_r' = \left(1 + \frac{u_{rel}}{v}\right) f_s$$

where $u_{rel} = u_s - u_r$ is the velocity of the source relative to the receiver.

Picture the Problem The received and transmitted frequencies are related through $f_r = \dfrac{v \pm u_r}{v \pm u_s} f_s$ (Equation 15-41a), where the variables have the meanings given in the problem statement. Because the source and receiver are moving in the same direction, we use the minus signs in both the numerator and denominator.

(*a*) Relate the received frequency f_r to the frequency f_s of the source:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{1 - \dfrac{u_r}{v}}{1 - \dfrac{u_s}{v}} f_s$$

$$= \boxed{\left(1 - \frac{u_r}{v}\right)\left(1 - \frac{u_s}{v}\right)^{-1} f_s}$$

(*b*) Expand $(1 - u_s/v)^{-1}$ binomially and discard the higher-order terms:

$$\left(1 - \frac{u_s}{v}\right)^{-1} \approx 1 + \frac{u_s}{v}$$

Substitute to obtain:

$$f_r = \left(1 - \frac{u_r}{v}\right)\left(1 + \frac{u_s}{v}\right) f_s$$

$$= \left[1 + \frac{u_s}{v} - \frac{u_r}{v} - \left(\frac{u_r}{v}\right)\left(\frac{u_s}{v}\right)\right] f_s$$

$$\approx \left(1 + \frac{u_s - u_r}{v}\right) f_s$$

because both u_s and u_r are small compared to v.

Because $u_{rel} = u_s - u_r$

$$f_r' \approx \boxed{\left(1 + \frac{u_{rel}}{v}\right) f_s}$$

84 ••• To study the Doppler shift on your own, you take an electronic tone generating device that is set to a frequency of middle C (262Hz) to a campus wishing well known as ″The Abyss. ″ When you hold the device at arm's length (1.00 m), you measure its intensity level to be 80.0 dB. You then drop the tuner down the hole, listening to its sound as it falls. After the tuner has fallen for 5.50 s, what frequency do you hear?

Picture the Problem As the tuner falls, its speed increases and so the frequency you hear is Doppler shifted. Our concern, however, is with the frequency you hear 5.50 s after you've dropped the tuner. The sound that you hear at this time was emitted sometime before the tuner had fallen for 5.50 s. Hence we must answer the question ″At what time was the sound emitted that you heard 5.50 s after dropping the tuner?″

Apply Equation 15-41a to express the frequency you'll hear after 5.50 s:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{v}{v + v_{fall}} f_s \qquad (1)$$

Relate the elapsed time of 5.50 s to the time required for the tuner to fall and the time for the sound to return to you:

$$\Delta t_{tot} = \Delta t_{fall} + \Delta t_{return} \qquad (2)$$

Use a constant-acceleration equation to relate the time-of-fall to the fall distance:

$$\Delta y = \tfrac{1}{2} g (\Delta t_{fall})^2 \Rightarrow \Delta t_{fall} = \sqrt{\frac{2\Delta y}{g}}$$

The return time depends on how far the tuner has fallen and on the speed of sound in air:

$$\Delta t_{return} = \frac{\Delta y}{v} \Rightarrow \Delta y = v \Delta t_{return}$$

Substituting for Δy in the expression for Δt_{fall} yields:

$$\Delta t_{fall} = \sqrt{\frac{2v\Delta t_{return}}{g}}$$

Square both sides of the equation, use equation (2) to substitute for Δt_{return}, and write the result explicitly as a quadratic equation to obtain:

$$(\Delta t_{fall})^2 + \frac{2v}{g}(\Delta t_{fall}) - \frac{2v}{g}\Delta t_{tot} = 0$$

Substitute numerical values to obtain:

$$(\Delta t_{fall})^2 + \frac{2(343\ \text{m/s})}{9.81\ \text{m/s}^2}(\Delta t_{fall}) - \frac{2(343\ \text{m/s})(5.50\ \text{s})}{9.81\ \text{m/s}^2} = 0$$

or

$$(\Delta t_{fall})^2 + (69.93\ \text{s})(\Delta t_{fall}) - 384.6\ \text{s}^2 = 0$$

Use your graphing calculator or the quadratic formula to solve this equation for its positive root:

$$\Delta t_{\text{fall}} = 5.124\text{ s}$$

Because $v_{\text{fall}} = g\Delta t_{\text{fall}}$, equation (1) becomes:

$$f_{\text{r}} = \frac{v}{v + g\Delta t_{\text{fall}}} f_{\text{s}}$$

Substitute numerical values and evaluate f_{r}:

$$f_{\text{r}} = \frac{343\text{ m/s}}{343\text{ m/s} + (9.81\text{ m/s}^2)(5.124\text{ s})}(262\text{ Hz}) = \boxed{229\text{ Hz}}$$

85 •• You are in a hot-air balloon carried along by a 36-km/h wind and have a sound source with you that emits a sound of 800 Hz as it approaches a tall building. (*a*) What is the frequency of the sound heard by an observer at the window of this building? (*b*) What is the frequency of the reflected sound heard by you?

Picture the Problem The simplest way to approach this problem is to transform to a reference frame in which the balloon is at rest. In that reference frame, the speed of sound is v = 343 m/s, and u_{r} = 36 km/h = 10 m/s. Then, we can use the equations for a moving receiver and a moving source to find the frequencies heard at the window and on the balloon.

(*a*) Use Equation 15-41*a* to express the received frequency in terms of the frequency of the source:

$$f_{\text{r}} = \frac{v \pm u_{\text{r}}}{v \pm u_{\text{s}}} f_{\text{s}} = \frac{1 \pm \frac{u_{\text{r}}}{v}}{1 \pm \frac{u_{\text{s}}}{v}} f_{\text{s}} = \frac{1 + \frac{u_{\text{r}}}{v}}{1 \pm \frac{0}{v}} f_{\text{s}}$$

$$= \left(1 + \frac{u_{\text{r}}}{v}\right) f_{\text{s}}$$

Substitute numerical values and evaluate f_{r}:

$$f_{\text{r}} = \left(1 + \frac{10\text{ m/s}}{343\text{ m/s}}\right)(800\text{ Hz}) = 823\text{ Hz}$$

$$= \boxed{0.82\text{ kHz}}$$

(*b*) Treating the tall building as a moving source, express the frequency of the reflected sound heard by a person riding in the balloon:

$$f_r' = \frac{v \pm u_r}{v \pm u_s} f_r = \frac{1 \pm \frac{u_r}{v}}{1 \pm \frac{u_s}{v}} f_r = \frac{1 \pm \frac{0}{v}}{1 - \frac{u_s}{v}} f_r = \left(\frac{1}{1 - \frac{u_s}{v}} \right) f_r$$

Substitute numerical values and evaluate f_r':

$$f_r' = \left(\frac{1}{1 - \frac{10\,\text{m/s}}{343\,\text{m/s}}} \right) (823\,\text{Hz}) = \boxed{0.85\,\text{kHz}}$$

86 •• A car is approaching a reflecting wall. A stationary observer behind the car hears a sound of frequency 745 Hz from the car horn and a sound of frequency 863 Hz from the wall. (*a*) How fast is the car traveling? (*b*) What is the frequency of the car horn? (*c*) What frequency does the car driver hear reflected from the wall?

Picture the Problem We can relate the frequencies f_r and f_r' heard by the stationary observer behind the car to the speed of the car u_r and the frequency of the car's horn f_s. Dividing these equations will eliminate the frequency of the car's horn and allow us to solve for the speed of the car. We can then substitute to find the frequency of the car's horn. We can find the frequency heard by the driver as a moving receiver by relating this frequency to the frequency reflected from the wall.

(*a*) Use Equation 15-41*a* to relate the frequency heard by the observer directly from the car's horn to the speed of the car:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{1 \pm \frac{u_r}{v}}{1 \pm \frac{u_s}{v}} f_s = \frac{1 \pm \frac{0}{v}}{1 + \frac{u_s}{v}} f_s = \frac{1}{1 + \frac{u_s}{v}} f_s \qquad (1)$$

Relate the frequency reflected from the wall to the speed of the car:

$$f_r' = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{1 \pm \frac{0}{v}}{1 - \frac{u_s}{v}} f_s = \frac{1}{1 - \frac{u_s}{v}} f_s \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{f_r'}{f_r} = \frac{1 + \frac{u_s}{v}}{1 - \frac{u_s}{v}} \Rightarrow u = \frac{f_r' - f_r}{f_r' + f_r'} v$$

Substitute numerical values and evaluate u_s:

$$u_s = \frac{863\,\text{Hz} - 745\,\text{Hz}}{863\,\text{Hz} + 745\,\text{Hz}}(343\,\text{m/s}) = 25.17\,\frac{\text{m}}{\text{s}} \times \frac{1\,\text{km}}{10^3\,\text{m}} \times \frac{3600\,\text{s}}{\text{h}} = \boxed{90.6\,\text{km/h}}$$

(*b*) Solve equation (1) for f_s to obtain:

$$f_s = \left(1 + \frac{u_r}{v}\right) f_r$$

Substitute numerical values and evaluate f_s:

$$f_s = \left(1 + \frac{25.17\,\text{m/s}}{343\,\text{m/s}}\right)(745\,\text{Hz}) = \boxed{800\,\text{Hz}}$$

(*c*) The driver is a moving receiver and so we can relate the frequency heard by the driver to the frequency reflected by the wall (the frequency heard by the stationary observer):

$$f_{\text{driver}} = \left(1 + \frac{u_r}{v}\right) f_r'$$

Substitute numerical values and evaluate f_{driver}:

$$f_{\text{driver}} = \left(1 + \frac{25.17\,\text{m/s}}{343\,\text{m/s}}\right)(863\,\text{H}z) = \boxed{926\,\text{Hz}}$$

87 •• The driver of a car traveling at 100 km/h toward a vertical wall briefly sounds the horn. Exactly 1.00 s later she hears the echo and notes that its frequency is 840 Hz. How far from the wall was the car when the driver sounded the horn and what is the frequency of the horn?

Picture the Problem Let $t = 0$ when the driver sounds her horn and let the distance to the wall at that instant be d. The received and transmitted frequencies are related through $f_{\text{r}} = \dfrac{v \pm u_{\text{r}}}{v \pm u_{\text{s}}} f_{\text{s}}$ (Equation 15-41*a*). Solving this equation for f_{s} will allow us to determine the frequency of the car horn. We can use the total distance the sound travels (car-to-wall plus wall-back to-car … now closer to the wall) to determine the distance to the wall when the horn was briefly sounded.

Use Equation 15-41*a* to relate the frequency heard by the driver to her speed and to the frequency of her horn:

$$f_{\text{r}} = \frac{v \pm u_{\text{r}}}{v \pm u_{\text{s}}} f_{\text{s}} = \frac{1 \pm \dfrac{u_{\text{r}}}{v}}{1 \pm \dfrac{u_{\text{s}}}{v}} f_{\text{s}} = \frac{1 + \dfrac{u_{\text{r}}}{v}}{1 - \dfrac{u_{\text{s}}}{v}} f_{\text{s}}$$

Solving for f_{s} yields:

$$f_{\text{s}} = \frac{1 - \dfrac{u_{\text{s}}}{v}}{1 + \dfrac{u_{\text{r}}}{v}} f_{\text{r}}$$

Substitute numerical values and evaluate f_{s}:

$$f_{\text{s}} = \frac{1 - \dfrac{100 \dfrac{\text{km}}{\text{h}} \times \dfrac{1\,\text{h}}{3600\,\text{s}}}{343\,\text{m/s}}}{1 + \dfrac{100 \dfrac{\text{km}}{\text{h}} \times \dfrac{1\,\text{h}}{3600\,\text{s}}}{343\,\text{m/s}}} (840\,\text{Hz}) = \frac{1 - \dfrac{27.78\,\text{m/s}}{343\,\text{m/s}}}{1 + \dfrac{27.78\,\text{m/s}}{343\,\text{m/s}}} (840\,\text{Hz}) = \boxed{714\,\text{Hz}}$$

Relate the distance d to the wall at $t = 0$ to the distance she travels in time $\Delta t = 1$ s, her speed u_{s}, and the speed of sound v:

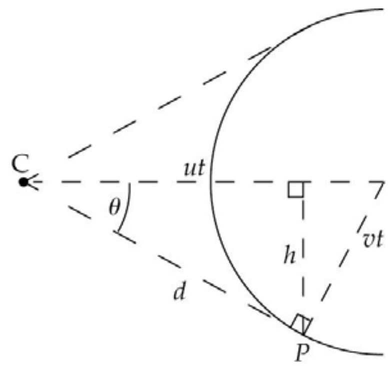
$$d + (d - u_{\text{s}}\Delta t) = v\Delta t \Rightarrow d = \tfrac{1}{2}(u_{\text{s}} + v)\Delta t$$

Substitute numerical values and evaluate d:

$$d = \tfrac{1}{2}(27.78\,\text{m/s} + 343\,\text{m/s})(1.00\,\text{s}) = \boxed{185\,\text{m}}$$

88 •• You are on a transatlantic flight traveling due west at 800 km/h. An experimental plane flying at Mach 1.6 and 3.0 km to the north of your plane is also on an east-to-west course. What is the distance between the two planes when you hear the sonic boom from the experimental plane?

Picture the Problem You'll hear the sonic boom when the surface of its cone reaches your plane. In the following diagram, the experimental plane is at C and your plane is at P. The distance h = 3.0 km. The distance between the planes when you hear the sonic boom is d. We can use trigonometry to determine the angle of the shock wave as well as the separation of the planes when you hear the sonic boom.



Using the Pythagorean theorem, relate the separation of the planes d, to the distance h and the angle θ:

$$d^2 = h^2 + d^2\cos^2\theta \Rightarrow d = h\sqrt{\frac{1}{1-\cos^2\theta}}$$

Express θ in terms of v, u, and t:

$$\sin\theta = \frac{vt}{ut} = \frac{1}{u/v} = \frac{1}{1.6}$$

so

$$\theta = \sin^{-1}\left(\frac{1}{1.6}\right) = 38.7°$$

Substitute numerical values and evaluate d:

$$d = (3.0\,\text{km})\sqrt{\frac{1}{1-\cos^2 38.7°}} = \boxed{4.8\,\text{km}}$$

89 ••• The Hubble space telescope has been used to determine the existence of planets orbiting distant stars. The planet orbiting the star will cause the star to ″wobble″ with the same period as the planet's orbit. Because of wobble, light from the star will be Doppler-shifted up and down periodically. Estimate the maximum and minimum wavelengths of light of nominal wavelength 500 nm emitted by the Sun that is Doppler-shifted by the motion of the Sun due to the planet Jupiter.

Picture the Problem The Sun and Jupiter orbit about their effective mass located at their common center of mass. We can apply Newton's second law to the Sun to obtain an expression for its orbital speed about the Sun-Jupiter center of mass and then use this speed in the Doppler shift equation to estimate the maximum and minimum wavelengths resulting from the Jupiter-induced motion of the Sun.

Letting v be the orbital speed of the sun about the center of mass of the Sun-Jupiter system, express the Doppler shift of the light due to this motion when the Sun is approaching Earth:

$$f' = \frac{c}{\lambda'} = f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \frac{c}{\lambda}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Solving for λ' and simplifying yields:

$$\lambda' = \lambda\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = \lambda\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)^{-1}}$$

$$= \lambda\left(1-\frac{v}{c}\right)^{1/2}\left(1+\frac{v}{c}\right)^{-1/2}$$

Because $v << c,$ we can expand $\left(1-\frac{v}{c}\right)^{1/2}$ and $\left(1+\frac{v}{c}\right)^{-1/2}$ binomially to obtain:

$$\left(1-\frac{v}{c}\right)^{1/2} \approx 1-\frac{v}{2c}$$

and

$$\left(1+\frac{v}{c}\right)^{-1/2} \approx 1-\frac{v}{2c}$$

Substitute for $\left(1-\frac{v}{c}\right)^{1/2}$ and $\left(1+\frac{v}{c}\right)^{-1/2}$ to obtain:

$$\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = \left(1-\frac{v}{2c}\right)^2 \approx 1-\frac{v}{c}$$

When the Sun is receding from Earth:

$$\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \left(1+\frac{v}{2c}\right)^2 \approx 1+\frac{v}{c}$$

Hence the motion of the Sun will give an observed Doppler shift of:

$$\lambda' \approx \lambda\left(1\pm\frac{v}{c}\right) \qquad (1)$$

Apply Newton's second law to the Sun:

$$\frac{GM_S M_{eff}}{r_{cm}^2} = M_S \frac{v^2}{r_{cm}} \Rightarrow v = \sqrt{\frac{GM_{eff}}{r_{cm}}}$$

Measured from the center of the Sun, the distance to the center of mass of the Sun-Jupiter system is:

$$r_{cm} = \frac{(0)M_S + r_{s\text{-}J} M_J}{M_s + M_J} = \frac{r_{s\text{-}J} M_J}{M_s + M_J}$$

The effective mass is related to the masses of the Sun and Jupiter according to:

$$\frac{1}{M_{eff}} = \frac{1}{M_s} + \frac{1}{M_J} \Rightarrow M_{eff} = \frac{M_s M_J}{M_s + M_J}$$

Substitute for M_{eff} and r_{cm} to obtain:

$$v = \sqrt{\frac{G\dfrac{M_s M_J}{M_s + M_J}}{\dfrac{r_{s\text{-}J} M_J}{M_s + M_J}}} = \sqrt{\frac{GM_s}{r_{s\text{-}J}}}$$

Using $r_{s\text{-}J} = 7.78 \times 10^{11}$ m as the mean orbital radius of Jupiter, substitute numerical values and evaluate v:

$$v = \sqrt{\frac{(6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\ \text{kg})}{7.78 \times 10^{11}\ \text{m}}} = 1.306 \times 10^4\ \text{m/s}$$

Substitute numerical values in equation (1) to obtain:

$$\lambda' \approx (500\,\text{nm})\left(1 \pm \frac{1.306 \times 10^4\ \text{m/s}}{2.998 \times 10^8\ \text{m/s}}\right)$$
$$= (500\,\text{nm})(1 \pm 4.36 \times 10^{-5})$$

The maximum and minimum wavelengths are:

$$\lambda_{max} = \boxed{500.02\ \text{nm}}$$

and

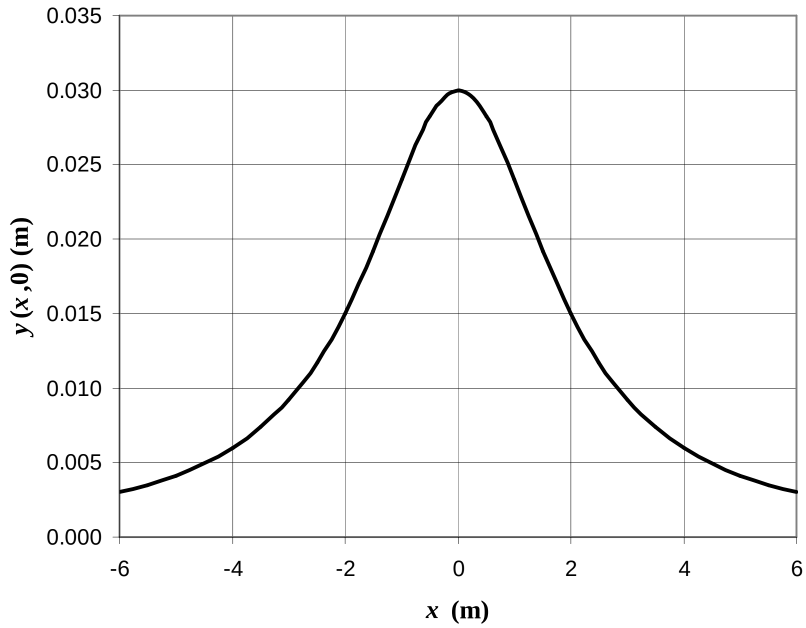
$$\lambda_{min} = \boxed{499.98\ \text{nm}}$$

General Problems

90 • At time $t = 0$, the shape of a wave pulse on a string is given by the function $y(x,0) = 0.120\ \text{m}^3/\left((2.00\ \text{m})^2 + x^2\right)$, where x is in meters. (*a*) Sketch $y(x, 0)$ versus x. (*b*) Give the wave function $y(x,t)$ at a general time t if the pulse is moving in the $+x$ direction with a speed of 10.0 m/s and if the pulse is moving in the $-x$ direction with a speed of 10.0 m/s.

Picture the Problem The equation of a wave traveling in the +x direction is of the form $y(x,t)=f(x-vt)$ and that of a wave traveling in the –x direction is $y(x,t)=f(x+vt)$.

(a) The pulse at t = 0 shown below was plotted using a spreadsheet program:



(b) If the pulse is moving in the +x direction, the wave function must be of the form:

$$y(x,t)=f(x-vt)=f[x-(10.0\,\text{m/s})t]$$

because v = 10.0 m/s

Replace x with $x-(10.0\,\text{m/s})t$ to obtain:

$$y(x,t)=\boxed{\frac{0.120\,\text{m}^3}{(2.00\,\text{m})^2+[x-(10.0\,\text{m/s})t]^2}}$$

If the pulse is moving in the –x direction, the wave function must be of the form:

$$y(x,t)=f(x+vt)=f[x+(10.0\,\text{m/s})t]$$

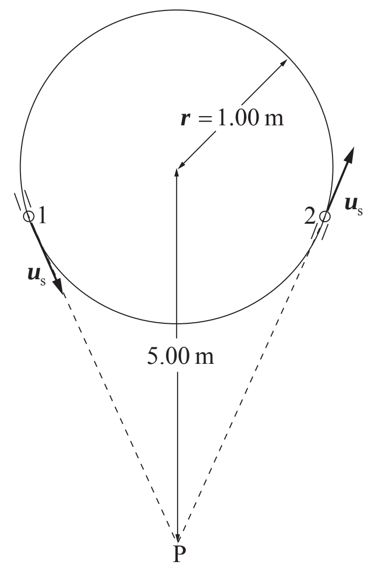
because v = 10.0 m/s

Replace x with $x+(10.0\,\text{m/s})t$ to obtain:

$$y(x,t)=\boxed{\frac{0.120\,\text{m}^3}{(2.00\,\text{m})^2+[x+(10.0\,\text{m/s})t]^2}}$$

91 • [SSM] A whistle that has a frequency of 500 Hz moves in a circle of radius 1.00 m at 3.00 rev/s. What are the maximum and minimum frequencies heard by a stationary listener in the plane of the circle and 5.00 m away from its center?

Picture the Problem The pictorial representation depicts the whistle traveling in a circular path of radius r = 1.00 m. The stationary listener will hear the maximum frequency when the whistle is at point 1 and the minimum frequency when it is at point 2. These maximum and minimum frequencies are determined by f_0 and the tangential speed $u_s = 2\pi\, r/T$. We can relate the frequencies heard at point P to the speed of the approaching whistle at point 1 and the speed of the receding whistle at point 2.



Relate the frequency heard at point P to the speed of the approaching whistle at point 1:

$$f_{max} = \frac{1}{1 - \dfrac{u_s}{v}} f_s$$

Because $u_s = r\omega$:

$$f_{max} = \frac{1}{1 - \dfrac{r\omega}{v}} f_s$$

Substitute numerical values and evaluate f_{max}:

$$f_{max} = \frac{1}{1 - \dfrac{(1.00\,\text{m})\left(3.00\,\dfrac{\text{rev}}{\text{s}} \times \dfrac{2\pi\,\text{rad}}{\text{rev}}\right)}{343\,\text{m/s}}}(500\,\text{Hz}) = \boxed{529\,\text{Hz}}$$

Relate the frequency heard at point P to the speed of the receding whistle at point 2:

$$f_{min} = \frac{1}{1 + \dfrac{u_s}{v}} f_s$$

Substitute numerical values and evaluate $f_{\min}$:

$$f_{\min}=\frac{1}{1+\dfrac{(1.00\,\text{m})\left(3.00\,\dfrac{\text{rev}}{\text{s}}\times\dfrac{2\pi\,\text{rad}}{\text{rev}}\right)}{343\,\text{m/s}}}(500\,\text{Hz})=\boxed{474\,\text{Hz}}$$

92 • Ocean waves move toward the beach with a speed of 8.90 m/s and a crest-to-crest separation of 15.0 m. You are in a small boat anchored off shore. (*a*) At what frequency do the wave crests reach you boat? (*b*) You now lift anchor and head out to sea at a speed of 15.0 m/s. At what frequency do the wave crests reach your boat now?

Picture the Problem (*a*) The crest-to-crest separation of the waves is their wavelength. We can find the frequency of the waves from $v = f\lambda$. (*b*) When you lift anchor and head out to sea you'll become a moving receiver and we can apply $f_{\text{r}}=\dfrac{v\pm u_{\text{r}}}{v\pm u_{\text{s}}}f_{\text{s}}$ to calculate the frequency you'll observe.

(*a*) The frequency of the ocean waves is the ratio of their speed to their wavelength:

$$f_0=\frac{v}{\lambda}=\frac{8.90\,\text{m/s}}{15.0\,\text{m}}=\boxed{0.593\,\text{Hz}}$$

(*b*) Express the frequency of the waves in terms of their speed and the speed of a moving receiver:

$$f_{\text{r}}=\frac{v\pm u_{\text{r}}}{v\pm u_{\text{s}}}f_{\text{s}}=\frac{v+u_{\text{r}}}{v}f_{\text{s}}=\left(1+\frac{u_{\text{r}}}{v}\right)f_{\text{s}}$$

Substitute numerical values and evaluate f_{r}:

$$f_{\text{r}}=\left(1+\frac{15.0\,\text{m/s}}{8.90\,\text{m/s}}\right)(0.593\,\text{Hz})$$

$$=\boxed{1.59\,\text{Hz}}$$

93 •• A 12.0-m-long wire that has an 85.0-g mass is stretched under a tension of 180 N. A pulse is generated at the left end of the wire, and 25.0 ms later a second pulse is generated at the right end of the wire. Where do the pulses first meet?

Picture the Problem Let t be the time of travel of the left-hand pulse and the subscripts L and R refer to the pulse coming from the left and right, respectively. Because the pulse traveling from the right starts later than the pulse from the left, its travel time is $t - \Delta t$, where Δt = 25.0 ms. Both pulses travel at the same speed and the sum of the distances they travel is 12.0 m.

Express the total distance the two pulses travel:

$$d = d_L + d_R = vt + v(t - \Delta t)$$

Solve for vt to obtain:

$$vt = \tfrac{1}{2}(d + v\Delta t)$$

The speed of the pulse is given by:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{F_T}{m/L}}$$

Substituting for v yields:

$$vt = \tfrac{1}{2}\left(d + \sqrt{\frac{F_T}{m/L}}\Delta t\right)$$

Substitute numerical values and evaluate vt:

$$vt = \tfrac{1}{2}\left[12.0\,\text{m} + \sqrt{\frac{(180\,\text{N})(12.0\,\text{m})}{0.085\,\text{kg}}}\left(25.0\times10^{-3}\,\text{s}\right)\right] = \boxed{7.99\,\text{m}}$$

from the left end of the wire.

94 •• You are parked on the shoulder of a highway. Find the speed of a car in which the tone of the car's horn drops by 10 percent as it passes you. (In other words, the total drop in frequency between the ″approach″ value and the ″recession″ value is 10%.)

Picture the Problem Let the frequency of the car's horn be f_s, the frequency you hear as the car approaches be f_r, and the frequency you hear as the car recedes be f_r'. We can use $f_r = \frac{v \pm u_r}{v \pm u_s} f_s$ to express the frequencies heard as the car approaches and recedes and then use these frequencies to express the fractional change in frequency as the car passes you.

Express the fractional change in frequency as the car passes you:

$$\frac{\Delta f}{f_r} = 0.10$$

Relate the frequency heard as the car approaches to the speed of the car:

$$f_r = \frac{v \pm u_r}{v \pm u_s} f_s = \frac{v \pm 0}{v - u_s} f_s = \frac{1}{1 - \frac{u_s}{v}} f_s$$

Express the frequency heard as the car recedes in terms of the speed of the car:

$$f_r' = \frac{1}{1+\frac{u_s}{v}} f_s$$

Divide the second of these frequency equations by the first to obtain:

$$\frac{f_r'}{f_r} = \frac{1-\frac{u_s}{v}}{1+\frac{u_s}{v}}$$

and

$$\frac{f_r}{f_r} - \frac{f_r'}{f_r} = \frac{\Delta f}{f_r} = 1 - \frac{1-\frac{u_s}{v}}{1+\frac{u_s}{v}} = 0.10$$

Solving for u_s yields:

$$u_s = \frac{0.10}{1.9} v$$

Substitute numerical values and evaluate u_s:

$$u_s = \frac{0.10}{1.9}(343\,\text{m/s})$$
$$= 18.05\frac{\text{m}}{\text{s}} \times \frac{1\,\text{km}}{10^3} \times \frac{3600\,\text{s}}{\text{h}}$$
$$= \boxed{65\,\text{km/h}}$$

95 •• [SSM] A loudspeaker driver 20.0 cm in diameter is vibrating at 800 Hz with an amplitude of 0.0250 mm. Assuming that the air molecules in the vicinity have the same amplitude of vibration, find (*a*) the pressure amplitude immediately in front of the driver, (*b*) the sound intensity, and (*c*) the acoustic power being radiated by the front surface of the driver.

Picture the Problem (*a*) and (*b*) The pressure amplitude can be calculated directly from $p_0 = \rho\omega v s_0$, and the intensity from $I = \frac{1}{2}\rho\omega^2 s_0^2 v$. (*c*) The power radiated is the intensity times the area of the driver.

(*a*) Relate the pressure amplitude to the displacement amplitude, angular frequency, wave velocity, and air density:

$$p_0 = \rho\omega v s_0$$

Substitute numerical values and evaluate p_0:

$$p_0 = (1.29\,\text{kg/m}^3)\left[2\pi(800\,\text{s}^{-1})\right] \times (343\,\text{m/s})(0.0250\times10^{-3}\,\text{m})$$
$$= \boxed{55.6\,\text{N/m}^2}$$

(*b*) Relate the intensity to these same quantities:

$$I = \tfrac{1}{2}\rho\omega^2 s_0^2 v$$

Substitute numerical values and evaluate *I*:

$$I = \tfrac{1}{2}\left(1.29\,\text{kg/m}^3\right)\left[2\pi\left(800\,\text{s}^{-1}\right)\right]^2 \times\left(0.0250\times10^{-3}\,\text{m}\right)^2\left(343\,\text{m/s}\right) = 3.494\,\text{W/m}^2 = \boxed{3.49\,\text{W/m}^2}$$

(*c*) Express the power in terms of the intensity and the area of the driver:

$$P = IA = \pi r^2 I$$

Substitute numerical values and evaluate *P*:

$$P = \pi\left(0.100\,\text{m}\right)^2\left(3.494\,\text{W/m}^2\right) = \boxed{0.110\,\text{W}}$$

96 •• A plane, harmonic, sound wave in air has an amplitude of 1.00 μm and an intensity of 10.0 mW/m^2. What is the frequency of the wave?

Picture the Problem The frequency of the wave is related to the density of the air, displacement amplitude, and velocity by $I = \tfrac{1}{2}\rho\omega^2 s_0^2 v$.

Relate the intensity of the wave to the density of the air, displacement amplitude, velocity, and angular frequency:

$$I = \tfrac{1}{2}\rho\omega^2 s_0^2 v \Rightarrow \omega = \frac{1}{s_0}\sqrt{\frac{2I}{\rho v}}$$

Because $\omega = 2\pi f$:

$$f = \frac{1}{2\pi s_0}\sqrt{\frac{2I}{\rho v}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi\left(1.00\times10^{-6}\,\text{m}\right)}\sqrt{\frac{2\left(1.00\times10^{-2}\,\text{W/m}^2\right)}{\left(1.29\,\text{kg/m}^3\right)\left(343\,\text{m/s}\right)}} = \boxed{1.07\,\text{kHz}}$$

97 •• Water flows at 7.0 m/s in a pipe of radius 5.0 cm. A plate with area equal to the cross-sectional area of the pipe is suddenly inserted to stop the flow. Find the force exerted on the plate. Take the speed of sound in water to be 1.4 km/s. *Hint: When the plate is inserted, a pressure wave propagates through the water at the speed of sound v_s. The mass of water brought to a stop in time Δt is the water in a length of pipe equal to $v_s\Delta t$.*

Picture the Problem The force exerted on the plate is due to the change in momentum of the water. We can use Newton's second law in the form $F = \Delta p/\Delta t$ to relate F to the mass of water in a length of pipe equal to $v_s\Delta t$ and to the speed of the water. This mass of water, in turn, is given by the product of its density and the volume of water in a length of pipe equal to $v_s\Delta t$.

Relate the force exerted on the plate to the change in momentum of the water:

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta m v_w}{\Delta t}$$

Express Δm in terms of the mass of water in a length of pipe equal to $v_s\Delta t$:

$$\Delta m = \rho \Delta V = \rho v_s A \Delta t$$

Substitute for Δm to obtain:

$$F = \rho v_s A v_w$$

Substitute numerical values and evaluate F:

$$F = \left(1.00\times10^3\ \text{kg/m}^3\right)\left(1.4\,\text{km/s}\right)\left[\pi\left(0.050\,\text{m}\right)^2\right]\left(7.0\,\text{m/s}\right) = \boxed{77\,\text{kN}}$$

98 •• A high-speed flash photography setup to capture a picture of a bullet exploding a soap bubble is shown in Figure 15-33. The shock wave from the bullet is to be detected by a microphone that will trigger the flash. The microphone is placed on a track that is parallel to, and 0.350 m below, the path of the bullet. The track is used to adjust the position of the microphone. If the bullet is traveling at 1.25 times the speed of sound, how far back from the soap bubble must the microphone be set to trigger the flash? (Assume that the flash itself is instantaneous once the microphone is triggered.)

Picture the Problem Let d be the horizontal distance from the soap bubble to the position of the microphone. The angle θ of the shock wave is related to the speed of sound in air u and the speed of the bullet v according to $\sin\theta = u/v$. We can determine θ from the given information and then use this angle to find d.

Express d in terms of the angle of the shock wave and the distance from the soap bubble to the laboratory bench:

$$d = \frac{0.350\,\text{m}}{\tan\theta} \qquad (1)$$

Relate the speed of the bullet to the angle of the shock-wave cone:

$$\sin\theta = \frac{u}{v} \Rightarrow \theta = \sin^{-1}\left(\frac{u}{v}\right)$$

Substitute for θ in equation (1) to obtain:

$$d = \frac{0.350\,\text{m}}{\tan\left[\sin^{-1}\left(\frac{u}{v}\right)\right]}$$

Substitute numerical values and evaluate d:

$$d = \frac{0.350\,\text{m}}{\tan\left[\sin^{-1}\left(\frac{u}{1.25u}\right)\right]} = \boxed{26.3\,\text{cm}}$$

99 •• A column of precision marchers keeps in step by listening to the band positioned at the head of the column. The beat of the music is for 100 paces/min. A television camera shows that only the marchers at the front and the rear of the column are actually in step. The marchers in the middle section are striding forward with the left foot when those at the front and rear are striding forward with the right foot. The marchers are so well trained, however, that they are all certain that they are in proper step with the music. How long is the column?

Picture the Problem The source of the problem is that it takes a finite time for the sound to travel from the front of the line of marchers to the back. We can use the given data to determine the time required for the beat to reach the marchers in the back of the column and then use this time and the speed of sound to find the length of the column.

Express the length of the column in terms of the speed of sound and the time required for the beat to travel the length of the column:

$$L = v\Delta t$$

Calculate the time for the sound to travel the length of the column:

$$\Delta t = \frac{1}{100}\,\text{min} = 0.600\,\text{s}$$

Substitute numerical values and evaluate L:

$$L = (343\,\text{m/s})(0.600\,\text{s}) = \boxed{206\,\text{m}}$$

100 •• A bat flying toward a stationary obstacle at 12.0 m/s emits brief, high-frequency sound pulses at a repetition frequency of 80.0 Hz. What is the interval between the arrival times of the reflected pulses heard by the bat?

Picture the Problem The interval between the arrival times of the reflected pulses heard by the bat is the reciprocal of the frequency of the reflected pulses. We can use $f_\text{r} = \frac{v \pm u_\text{r}}{v \pm u_\text{s}} f_\text{s}$ to relate the frequency of the reflected pulses to the speed of the bat and the frequency it emits.

Relate the interval between the arrival times of the echo pulses heard by the bat to frequency of the reflected pulses:

$$\Delta t = \frac{1}{f_{\text{r}}}$$

Relate the frequency of the pulses received by the bat to its speed and the frequency it emits:

$$f_{\text{r}} = \frac{v \pm u_{\text{r}}}{v \pm u_{\text{s}}} f_{\text{s}} = \frac{v + u_{\text{r}}}{v - u_{\text{s}}} f_{\text{s}} = \frac{1 + \frac{u_{\text{r}}}{v}}{1 - \frac{u_{\text{s}}}{v}} f_{\text{s}}$$

Substitute for f_{r} to obtain:

$$\Delta t = \frac{1 - \frac{u_{\text{s}}}{v}}{\left(1 + \frac{u_{\text{r}}}{v}\right) f_{\text{s}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1 - \frac{12.0\,\text{m/s}}{343\,\text{m/s}}}{\left(1 + \frac{12.0\,\text{m/s}}{343\,\text{m/s}}\right)\left(80.0\,\text{s}^{-1}\right)} = \boxed{11.7\,\text{ms}}$$

101 •• Laser ranging to the moon is done routinely to accurately determine the Earth–moon distance. However, to determine the distance accurately, corrections must be made for the average speed of light in Earth's atmosphere, which is 99.997 percent of the speed of light in vacuum. Assuming that Earth's atmosphere is 8.00 km high, estimate the length of the correction.

Picture the Problem Let d be the distance to the moon, h be the height of Earth's atmosphere, and v be the average speed of light in Earth's atmosphere. We can express d', the distance measured when Earth's atmosphere is ignored, in terms of the time for a pulse of light to make a round-trip from Earth to the moon and solve this equation for the length of correction $d' - d$.

Express the roundtrip time for a pulse of light to reach the moon and return:

$$t = t_{\text{Earth's atmosphere}} + t_{\text{out of Earth's atmosphere}} = 2\frac{h}{v} + 2\frac{d-h}{c}$$

Express the "measured" distance d' when we do not account for the atmosphere:

$$d' = \frac{1}{2}ct = \frac{1}{2}c\left(2\frac{h}{v} + 2\frac{d-h}{c}\right) = \frac{c}{v}h + d - h$$

Solve for the length of correction $d' - d$:

$$d' - d = h\left(\frac{c}{v} - 1\right)$$

Substitute numerical values and evaluate $d' - d$:

$$d' - d = (8.00\,\text{km})\left(\frac{c}{0.99997c} - 1\right)$$
$$\approx \boxed{0.2\,\text{m}}$$

Remarks: This is larger than the accuracy of the measurements, which is about 3 to 4 cm.

102 •• A tuning fork attached to a taut string generates transverse waves. The vibration of the fork is perpendicular to the string. Its frequency is 400 Hz and the amplitude of its oscillation is 0.50 mm. The string has a linear mass density of 0.010 kg/m and is under a tension of 1.0 kN. Assume that there are no waves reflected at the far end of the string. (*a*) What are the period and frequency of waves on the string? (*b*) What is the speed of the waves? (*c*) What are the wavelength and wave number? (*d*) What is a suitable wave function for the waves on the string? (*e*) What is the maximum speed and acceleration of a point on the string? (*f*) At what minimum average rate must energy be supplied to the fork to keep it oscillating at a steady amplitude?

Picture the Problem (*a*) The frequency of the waves on the string is the same as the frequency of the tuning fork and their period is the reciprocal of the frequency. (*b*) We can find the speed of the waves from the tension in the string and its linear density. (*c*) The wavelength can be determined from the frequency and the speed of the waves and the wave number from its definition. (*d*) The general form of the wave function for waves on a string is $y(x,t) = A\sin(kx \pm \omega t)$, so, once we know k and ω, because A is given, we can write a suitable wave function for the waves on this string. (*e*) The maximum speed and acceleration of a point on the string can be found from the angular frequency and amplitude of the waves. (*f*) Finally, we can use $P_{\text{av}} = \frac{1}{2}\mu\omega^2 A^2 v$ to find the minimum average rate at which energy must be supplied to the tuning fork to keep it oscillating with a steady amplitude.

(*a*) The frequency of the waves on the string is the same as the frequency of the tuning fork:

$$f = \boxed{400\,\text{Hz}}$$

The period of the waves on the wire is the reciprocal of their frequency:

$$T = \frac{1}{f} = \frac{1}{400\,\text{s}^{-1}} = \boxed{2.50\,\text{ms}}$$

(*b*) Relate the speed of the waves to the tension in the string and its linear density:

$$v=\sqrt{\frac{F_{\mathrm{T}}}{\mu}}=\sqrt{\frac{1.0\,\mathrm{kN}}{0.010\,\mathrm{kg/m}}}=316.2\,\mathrm{m/s}$$
$$=\boxed{0.32\,\mathrm{km/s}}$$

(*c*) Use the relationship between the wavelength, speed and frequency of a wave to find λ:

$$\lambda=\frac{v}{f}=\frac{316.2\,\mathrm{m/s}}{400\,\mathrm{s^{-1}}}=79.05\,\mathrm{cm}$$
$$=\boxed{79\,\mathrm{cm}}$$

Using its definition, express and evaluate the wave number:

$$k=\frac{2\pi}{\lambda}=\frac{2\pi}{79.05\times10^{-2}\,\mathrm{m}}=7.95\,\mathrm{m^{-1}}$$
$$=\boxed{7.9\,\mathrm{m^{-1}}}$$

(*d*) Determine the angular frequency of the waves:

$$\omega=2\pi f=2\pi\left(400\,\mathrm{s^{-1}}\right)=2.51\times10^{3}\,\mathrm{s^{-1}}$$

Substitute for A, k, and ω in the general form of the wave function to obtain:

$$y(x,t)=\boxed{(0.50\,\mathrm{mm})\sin\left[\left(7.9\,\mathrm{m^{-1}}\right)x-\left(2.51\times10^{3}\,\mathrm{s^{-1}}\right)t\right]}$$

(*e*) Relate the maximum speed of a point on the string to the amplitude of the waves and the angular frequency of the tuning fork:

$$v_{\mathrm{max}}=A\omega$$
$$=\left(0.50\times10^{-3}\,\mathrm{m}\right)\left(2.51\times10^{3}\,\mathrm{s^{-1}}\right)$$
$$=\boxed{1.3\,\mathrm{m/s}}$$

Express the maximum acceleration of a point on the string in terms of the amplitude of the waves and the angular frequency of the tuning fork:

$$a_{\mathrm{max}}=A\omega^{2}$$
$$=\left(0.50\times10^{-3}\,\mathrm{m}\right)\left(2.51\times10^{3}\,\mathrm{s^{-1}}\right)^{2}$$
$$=\boxed{3.2\,\mathrm{km/s^{2}}}$$

(*f*) Express the minimum average power required to keep the tuning fork oscillating at a steady amplitude in terms of the linear density of the string, the amplitude of its vibrations, and the speed of the waves on the string:

$$P_{\mathrm{av}}=\tfrac{1}{2}\mu\omega^{2}A^{2}v$$

Substitute numerical values and evaluate P_{av}:

$$P_{av} = \tfrac{1}{2}(0.010\,\text{kg/m})(2.51\times10^{3}\,\text{s}^{-1})^{2}(0.50\times10^{-3}\,\text{m})^{2}(316\,\text{m/s}) = \boxed{2.5\,\text{W}}$$

103 ••• A long rope with a mass per unit length of 0.100 kg/m is under a constant tension of 10.0 N. A motor drives one end of the rope with transverse simple harmonic motion at 5.00 cycles per second and an amplitude of 40.0 mm. (*a*) What is the wave speed? (*b*) What is the wavelength? (*c*) What is the maximum transverse linear momentum of a 1.00-mm segment of the rope? (*d*) What is the maximum net force on a 1.00-mm segment of the rope?

Picture the Problem Let Δm represent the mass of the segment of length Δx = 1.00 mm. We can find the wave speed from the given data for the tension in the rope and its linear density. The wavelength can be found from $v = f\lambda$. We'll use the definition of linear momentum to find the maximum transverse linear momentum of the 1-mm segment and apply Newton's second law to the segment to find the maximum net force on it.

(*a*) Find the wave speed from the tension and linear density:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{10.0\,\text{N}}{0.100\,\text{kg/m}}} = \boxed{10.0\,\text{m/s}}$$

(*b*) Express the wavelength in terms of the speed and frequency of the wave:

$$\lambda = \frac{v}{f} = \frac{10.0\,\text{m/s}}{5.00\,\text{s}^{-1}} = \boxed{2.00\,\text{m}}$$

(*c*) Relate the maximum transverse linear momentum of the 1.00-mm segment to the maximum transverse speed of the wave:

$$p_{max} = \Delta m v_{max} = \mu\Delta x A\omega = 2\pi f\mu\Delta x A$$

Substitute numerical values and evaluate p_{max}:

$$\begin{aligned} p_{max} &= 2\pi(5.00\,\text{s}^{-1})(0.100\,\text{kg/m}) \\ &\quad\times(1.00\times10^{-3}\,\text{m})(0.0400\,\text{m}) \\ &= 1.257\times10^{-4}\,\text{kg}\cdot\text{m/s} \\ &= \boxed{1.26\times10^{-4}\,\text{kg}\cdot\text{m/s}} \end{aligned}$$

(*d*) The maximum net force acting on the segment is the product of the mass of the segment and its maximum acceleration:

$$\begin{aligned} F_{max} &= \Delta m a_{max} = \mu\Delta x A\omega^{2} = \omega p_{max} \\ &= 2\pi f p_{max} \end{aligned}$$

Substitute numerical values and evaluate F_{max}:

$$F_{max} = 2\pi\left(5.00\,\mathrm{s}^{-1}\right)\left(1.257\times10^{-4}\,\mathrm{kg\cdot m/s}\right) = \boxed{3.95\,\mathrm{mN}}$$

104 ••• In this problem, you will derive an expression for the potential energy of a segment of a string carrying a traveling wave (Figure 15-34). The potential energy of a segment equals the work done by the tension in stretching the string, which is $\Delta U = F_T(\Delta\ell - \Delta x)$, where F_T is the tension, $\Delta\ell$ is the length of the stretched segment, and Δx is its original length. (*a*) Use the binomial expansion to show that $\Delta\ell - \Delta x \approx \frac{1}{2}(\Delta y/\Delta x)^2\Delta x$, and therefore $\Delta U \approx \frac{1}{2}F_T\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x$.
(*b*) Compute $\partial y/\partial x$ from the wave function $y(x,t) = A\sin(kx-\omega t)$ (Equation 15-15) and show that $\Delta U \approx \frac{1}{2}F_T A^2k^2\cos^2(kx-\omega t)\Delta x$.

Picture the Problem We can follow the step-by-step instructions outlined above to obtain the given expressions for ΔU.

(*a*) Express the potential energy of a segment of the string:

$$\Delta U = F_T(\Delta\ell - \Delta x)$$

For $\Delta y/\Delta x \ll 1$:

$$\Delta\ell = \Delta x\left[1+\tfrac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

and

$$\Delta\ell - \Delta x = \Delta x\left[1+\tfrac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right] - \Delta x = \boxed{\tfrac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x}$$

Substitute to obtain:

$$\Delta U = F_T\left[\tfrac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right] = \boxed{\tfrac{1}{2}F_T\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x}$$

(*b*) Differentiate $y(x,t) = A\sin(kx-\omega t)$ to obtain:

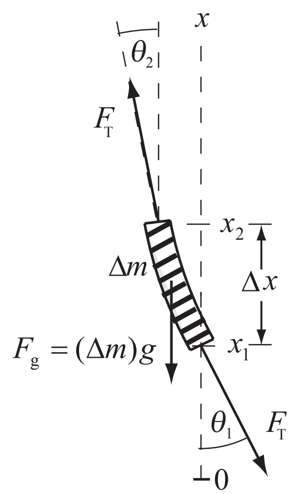
$$\frac{\partial y}{\partial x} = kA\cos(kx-\omega t)$$

Approximate $\Delta y/\Delta x$ by $\partial y/\partial x$ and substitute in our result from Part (*a*):

$$\Delta U = \tfrac{1}{2}F_T\left(kA\cos(kx-\omega t)\right)^2\Delta x = \boxed{\tfrac{1}{2}F_T A^2k^2\Delta x\cos^2(kx-\omega t)}$$

105 ••• One end of a heavy, 3.00-m-long rope is attached to a high ceiling and the rest of the rope is allowed to hang freely. Show that transverse waves on the rope must follow the equation $\dfrac{\partial}{\partial x}\left(x\dfrac{\partial y}{\partial x}\right)=\dfrac{1}{g}\left(\dfrac{\partial^2 y}{\partial t^2}\right)$ instead of $\dfrac{\partial^2 y}{\partial x^2}=\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$.

Picture the Problem Let the $+x$ direction be straight upward and choose the origin 3.00 m below the ceiling. Let y be the transverse direction in which the rope is displaced. Applying Newton's second law to an element of length of the rope will lead to the given equation, which relates the spatial derivatives of $y(x,t)$ to its time derivatives. We'll consider only small values of the angles θ_1 and θ_2 (see the diagram to the right) between the tangent to the rope and the x axis.



Apply $\sum F_x = ma_x$ to the segment whose mass is Δm and whose length is Δx to obtain:

$$F_T(x_2)\cos\theta_2 - F_T(x_1)\cos\theta_1 - (\Delta m)g = 0$$

Note that the segment is not accelerated vertically.

The mass of the segment is the product of its linear density and length:

$$\Delta m = \mu\Delta x$$

Substituting for Δm yields:

$$F_T(x_2)\cos\theta_2 - F_T(x_1)\cos\theta_1 - g\mu\Delta x = 0 \qquad (1)$$

Apply the small-angle approximation ($\cos\theta = 1$ for $|\theta| << 1$) and let $x_1 = x$ and $x_2 = x_1 + \Delta x$ to obtain:

$$F_T(x_2) - F_T(x_1) - g\mu\Delta x = 0$$

or

$$F_T(x+\Delta x) - F_T(x) = g\mu\Delta x$$

Noting that the tension in the rope is zero at $x = 0$, solve for the tension at x_2:

$$F_T(\Delta x) = g\mu\Delta x \Rightarrow F_T(x_2) = g\mu x_2$$

Using the differential approximation we have:

$$F_T(x_2) = F_T(x_1) + \left(\left. \frac{\partial F_T}{\partial x} \right|_{x=x_1} \right)\Delta x = F_T(x_1) + \frac{\partial F_T}{\partial x_1}\Delta x \qquad (2)$$

where we've written $\frac{\partial F_T}{\partial x_1}$ in place of $\left. \frac{\partial F_T}{\partial x} \right|_{x=x_1}$ in order to be more concise.

Substituting equation (2) in equation (1) yields:

$$F_T(x_1) + \frac{\partial F_T}{\partial x_1}\Delta x - F_T(x_1) - g\mu\Delta x = 0$$

Simplify this equation to obtain:

$$\frac{\partial F_T}{\partial x_1} - g\mu = 0 \qquad (3)$$

The segment moves horizontally, and the net force acting in this direction is:

$$\sum F_y = F_T(x_2)\sin\theta_2 - F_T(x_1)\sin\theta_1$$

Using the small-angle approximation $\sin\theta \approx \tan\theta$ yields:

$$\sum F_y \approx F_T(x_2)\tan\theta_2 - F_T(x_1)\tan\theta_1 = F_T(x_2)\frac{\partial y}{\partial x_2} - F_T(x_1)\frac{\partial y}{\partial x_1} \qquad (4)$$

where we've also used $\tan\theta = \text{slope} = \frac{\partial \text{y}}{\partial \text{x}}$.

Applying the differential approximation gives:

$$\frac{\partial y}{\partial x_2} = \frac{\partial y}{\partial x_1} + \left(\frac{\partial}{\partial x}\frac{\partial y}{\partial x_1} \right)\Delta x = \frac{\partial y}{\partial x_1} + \frac{\partial^2 y}{\partial x_1^2}\Delta x$$

Substituting for $F_T(x_2)$ and $\frac{\partial y}{\partial x_2}$ in equation (4) yields:

$$\sum F_y = \left(F_T(x_1) + \frac{\partial F_T}{\partial x_1}\Delta x \right)\left(\frac{\partial y}{\partial x_1} + \frac{\partial^2 y}{\partial x_1^2}\Delta x \right) - F_T(x_1)\frac{\partial y}{\partial x_1}$$

Expand and simplify this expression to obtain:

$$\sum F_y = F_T(x_1)\frac{\partial y}{\partial x_1} + F_T(x_1)\frac{\partial^2 y}{\partial x_1^2}\Delta x + \frac{\partial F_T}{\partial x_1}\frac{\partial y}{\partial x_1}\Delta x + \frac{\partial F_T}{\partial x_1}\frac{\partial^2 y}{\partial x_1^2}(\Delta x)^2 - F_T(x_1)\frac{\partial y}{\partial x_1}$$
$$\approx F_T(x_1)\frac{\partial^2 y}{\partial x_1^2}\Delta x + \frac{\partial F_T}{\partial x_1}\frac{\partial y}{\partial x_1}\Delta x$$

where we have neglected the term with $(\Delta x)^2$.

Applying $\sum F_y = ma_y$ to our length element yields:

$$F_T(x_1)\frac{\partial^2 y}{\partial x_1^2}\Delta x + \frac{\partial F_T}{\partial x_1}\frac{\partial y}{\partial x_1}\Delta x = \mu\Delta x\frac{\partial^2 y}{\partial t^2}$$

Divide both sides of this equation by Δx to obtain:

$$F_T(x_1)\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial F_T}{\partial x_1}\frac{\partial y}{\partial x_1} = \mu\frac{\partial^2 y}{\partial t^2} \quad (5)$$

Substituting $\mu g x$ for F_T yields:

$$\mu g x\frac{\partial^2 y}{\partial x^2} + \frac{\partial(\mu g x)}{\partial x}\frac{\partial y}{\partial x} = \mu\frac{\partial^2 y}{\partial t^2}$$

where we have dropped the subscript 1 by replacing x_1 with x. This can be done without loss of generality because all the x's in equation (5) are x_1's.

Divide both sides of the equation by μg to obtain:

$$x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} = \frac{1}{g}\frac{\partial^2 y}{\partial t^2} \quad (6)$$

Note that the expression on the left-hand side of the equation can be written as:

$x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right)$, a result you can verify by using the chain rule to differentiate the right-hand side of this equation.

Substituting in equation (6) yields:

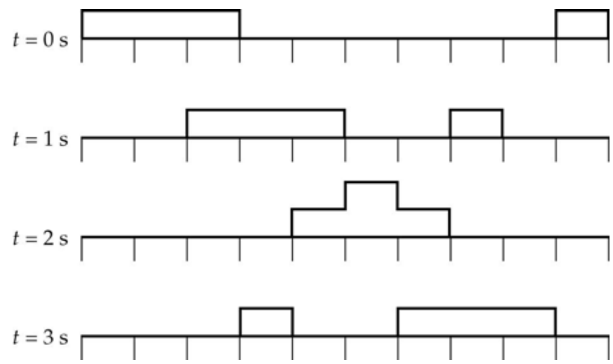
$$\boxed{\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = \frac{1}{g}\frac{\partial^2 y}{\partial t^2}}$$

Chapter 16
Superposition and Standing Waves

Conceptual Problems

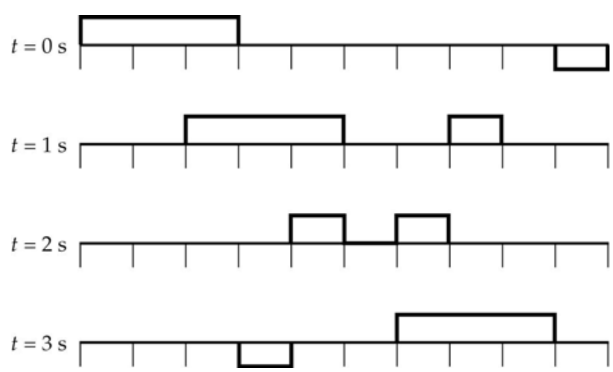
1 • **[SSM]** Two rectangular wave pulses are traveling in opposite directions along a string. At $t = 0$, the two pulses are as shown in Figure 16-29. Sketch the wave functions for $t = 1.0$, 2.0, and 3.0 s.

Picture the Problem We can use the speeds of the pulses to determine their positions at the given times.



2 • Repeat Problem 1 for the case in which the pulse on the right is inverted.

Picture the Problem We can use the speeds of the pulses to determine their positions at the given times.



3 • Beats are produced by the superposition of two harmonic waves if (*a*) their amplitudes and frequencies are equal, (*b*) their amplitudes are the same but their frequencies differ slightly, (*c*) their frequencies differ slightly even if their amplitudes are not equal, (*d*) their frequencies are equal but their amplitudes differ slightly.

Determine the Concept Beats are a consequence of the alternating constructive and destructive interference of waves due to slightly different frequencies. The amplitudes of the waves play no role in producing the beats. $\boxed{(b)\text{ and }(c)}$ are correct.

4 • Two tuning forks are struck and the sounds from each reach your ears at the same time. One sound has a frequency of 256 Hz, and the second sound has a frequency of 258 Hz. The underlying ″hum″ frequency that you hear is (*a*) 2.0 Hz, (*b*) 256 Hz, (*c*) 258 Hz, (*d*) 257 Hz.

Determine the Concept The tone you hear is the average of the frequencies emitted by the vibrating tuning forks; $f_{\text{av}} = \frac{1}{2}(f_1 + f_2) = \frac{1}{2}(256\text{ Hz} + 258\text{ Hz})$ or 257 Hz. Hence $\boxed{(d)}$ is correct.

5 • In Problem 4, the beat frequency is (*a*) 2.0 Hz, (*b*) 256 Hz, (*c*) 258 Hz, (*d*) 257 Hz.

Determine the Concept The beat frequency is the difference between the two frequencies; $f_{\text{beat}} = \Delta f = 258\text{ Hz} - 256\text{ Hz} = 2\text{ Hz}$. Hence $\boxed{(a)}$ is correct.

6 • As a graduate student, you are teaching your first physics lecture while the professor is away. To demonstrate interference of sound waves, you have set up two speakers that are driven coherently and in phase by the same frequency generator on the front desk. Each speaker generates sound with a 2.4–m wavelength. One student in the front row says she hears a very low volume (loudness) of the sound from the speakers compared to the volume of the sound she hears when only one speaker is generating sound. What could be the difference in the distance between her and each of the two speakers? (*a*) 1.2 m, (*b*) 2.4 m, (*c*) 4.8 m, (*d*) You cannot determine the difference in distances from the data given.

Determine the Concept Because the sound reaching her from the two speakers is very low, the sound waves must be interfering destructively (or nearly destructively) and the difference in distance between her position and the two speakers must be an odd multiple of a half wavelength. That is, it must be 1.2 m, 3.6 m, 6.0 m, etc. Hence $\boxed{(a)}$ is correct.

7 • In Problem 6, determine the longest wavelength for which a student would hear ″extra loud″ sound due to constructive interference, assuming this student is located so that one speaker is 3.0 m further from her than the other speaker.

Determine the Concept Because the sound reaching the student from the two speakers is extra loud, the sound waves must be interfering constructively (or nearly constructively) and the difference in distance between the student's position and the two speakers must be an integer multiple of a wavelength. Hence the wavelength of the sound is $\boxed{3.0\text{ m}}$.

8 • Consider standing waves in an organ pipe. True or False:

(*a*) In a pipe open at both ends, the frequency of the third harmonic is three times that of the first harmonic.
(*b*) In a pipe open at both ends, the frequency of the fifth harmonic is five times that of the fundamental.
(*c*) In a pipe that is open at one end and stopped at the other, the even harmonics are not excited.

Explain your choices.

(*a*) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by $f_n = n\frac{v}{2\ell}$, where n = 1, 2, 3… Hence the frequency of the third harmonic is three times that of the first harmonic.

(*b*) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by $f_n = n\frac{v}{2\ell}$, where n = 1, 2, 3… Hence the frequency of the fifth harmonic is five times that of the first harmonic.

(*c*) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by $f_n = n\frac{v}{4\ell}$, where n = 1, 3, 5…

9 • Standing waves result from the superposition of two waves that have (*a*) the same amplitude, frequency, and direction of propagation, (*b*) the same amplitude and frequency and opposite directions of propagation, (*c*) the same amplitude, slightly different frequencies, and the same direction of propagation, (*d*) the same amplitude, slightly different frequencies, and opposite directions of propagation.

Determine the Concept Standing waves are the consequence of the constructive interference of waves that have the same amplitude and frequency but are traveling in opposite directions. $\boxed{(b)}$ is correct.

10 • If you blow air over the top of a fairly large drinking straw you can hear a fundamental frequency due to a standing wave being set up in the straw. What happens to the fundamental frequency, (*a*) if while blowing, you cover the bottom of the straw with your fingertip? (*b*) if while blowing you cut the straw in half with a pair of scissors? (*c*) Explain your answers to Parts (*a*) and (*b*).

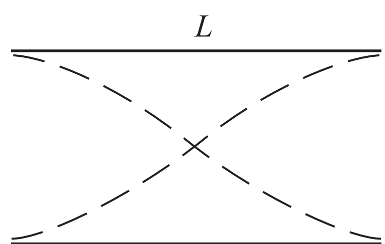
Determine the Concept
(*a*) The fundamental frequency decreases.

(*b*) The fundamental frequency increases.

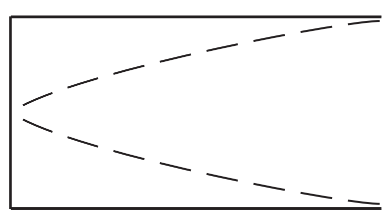
(*c*) **Part (*a*):** The fundamental frequency in a closed-open pipe is half that of an open-open pipe, so the frequency you hear with the bottom covered is half that you hear before you cover the bottom. **Part (*b*):** The fundamental frequencies of all pipes, independently of whether they are open-open or open-closed, varies inversely with the length of the pipe. Hence halving the length of the pipe doubles the fundamental frequency.

11 • **[SSM]** An organ pipe that is open at both ends has a fundamental frequency of 400 Hz. If one end of this pipe is now stopped, the fundamental frequency is (*a*) 200 Hz, (*b*) 400 Hz, (*c*) 546 Hz, (*d*) 800 Hz.

Picture the Problem The first harmonic displacement-wave pattern in an organ pipe open at both ends (open-open) and vibrating in its fundamental mode is represented in part (*a*) of the diagram. Part (*b*) of the diagram shows the wave pattern corresponding to the fundamental frequency for a pipe of the same length L that is stopped. Letting unprimed quantities refer to the open pipe and primed quantities refer to the stopped pipe, we can relate the wavelength and, hence, the frequency of the fundamental modes using $v = f\lambda$.



(*a*)



(*b*)

Express the frequency of the first harmonic in the open-open pipe in terms of the speed and wavelength of the waves:

$$f_1 = \frac{v}{\lambda_1}$$

Relate the length of the open pipe to the wavelength of the fundamental mode:

$$\lambda_1 = 2L$$

Substitute for λ_1 to obtain:

$$f_1 = \frac{v}{2L}$$

Express the frequency of the first harmonic in the closed pipe in terms of the speed and wavelength of the waves:

$$f_1' = \frac{v}{\lambda_1'}$$

Relate the length of the open-closed pipe to the wavelength of it's fundamental mode:

$$\lambda_1' = 4L$$

Substitute for λ_1' to obtain:

$$f_1' = \frac{v}{4L} = \frac{1}{2}\left(\frac{v}{2L}\right) = \frac{1}{2} f_1$$

Substitute numerical values and evaluate f_1':

$$f_1' = \frac{1}{2}(400\,\text{Hz}) = 200\,\text{Hz}$$

and $\boxed{(a)}$ is correct.

12 •• A string fixed at both ends resonates at a fundamental frequency of 180 Hz. Which of the following will reduce the fundamental frequency to 90 Hz? (*a*) Double the tension and double the length. (*b*) Halve the tension and keep the length and the mass per unit length fixed. (*c*) Keep the tension and the mass per unit length fixed and double the length. (*d*) Keep the tension and the mass per unit length fixed and halve the length.

Picture the Problem The frequency of the fundamental mode of vibration is directly proportional to the speed of waves on the string and inversely proportional to the wavelength which, in turn, is directly proportional to the length of the string. By expressing the fundamental frequency in terms of the length L of the string and the tension F in it we can examine the various changes in lengths and tension to determine which would halve it.

Express the dependence of the frequency of the fundamental mode of vibration of the string on its wavelength:

$$f_1 = \frac{v}{\lambda_1}$$

Relate the length of the string to the wavelength of the fundamental mode:

$$\lambda_1 = 2L$$

Substitute for λ_1 to obtain:

$$f_1 = \frac{v}{2L}$$

Express the dependence of the speed of waves on the string on the tension in the string:

$$v = \sqrt{\frac{F_T}{\mu}}$$

Substitute for v in the expression for f_1 to obtain:

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$

(*a*) Doubling the tension and the length would increase the frequency by a factor of $\sqrt{2}/2$.

(*b*) Halving the tension and keeping the length and the mass per unit length fixed would decrease the frequency by a factor of $1/\sqrt{2}$.

(*c*) Keeping the tension and the mass per unit length fixed and doubling the length will have the fundamental frequency. $\boxed{(c)}$ is correct.

(*d*) Keeping the tension and the mass per unit length fixed and halving the length would double the frequency.

13 •• [SSM] Explain how you might use the resonance frequencies of an organ pipe to estimate the temperature of the air in the pipe.

Determine the Concept You could measure the lowest resonant frequency f and the length L of the pipe. Assuming the end corrections are negligible, the wavelength equals $4L$ if the pipe is stopped at one end, and is $2L$ if the pipe is open at both ends. Then use $v = f\lambda$ to find the speed of sound at the ambient temperature. Finally, use $v = \sqrt{\gamma RT/M}$ (Equation 15-5), where γ = 1.4 for a diatomic gas such as air, M is the molar mass of air, R is the universal gas constant, and T is the absolute temperature, to estimate the temperature of the air.

14 •• In the fundamental standing-wave pattern of a organ pipe stopped at one end, what happens to the wavelength, frequency, and speed of the sound needed to create the pattern if the air in the pipe becomes significantly colder? Explain your reasoning.

Determine the Concept The pipe will contract as the air in it becomes significantly colder, and so the wavelength (equal to $4L$) will decrease as well. This effect, however, is negligible compared to the decrease in the speed of sound (recall that the speed of sound in a gas depends on the square root of the absolute temperature). Because $v = f\lambda$ and v decreases with λ remaining approximately constant, f must decrease.

15 •• [SSM] (*a*) When a guitar string is vibrating in its fundamental mode, is the wavelength of the sound it produces in air typically the same as the wavelength of the standing wave on the string? Explain. (*b*) When an organ pipe is in any one of its standing wave modes, is the wavelength of the traveling sound wave it produces in air typically the same as the wavelength of the standing sound wave in the pipe? Explain.

Determine the Concept

(*a*) No; the wavelength of a wave is related to its frequency and speed of propagation ($\lambda = v/f$). The frequency of the plucked string will be the same as the frequency of the wave it produces in air, but the speeds of the waves depend on the media in which they are propagating. Because the velocities of propagation differ, the wavelengths will not be the same.

(*b*) Yes. Because both the standing waves in the pipe and the traveling waves have the same speed and frequency, they must have the same wavelength.

16 •• Figure 16-30 is a photograph of two pieces of very finely woven silk placed one on top of the other. Where the pieces overlap, a series of light and dark lines are seen. This moiré pattern can also be seen when a scanner is used to copy photos from a book or newspaper. What causes the moiré pattern, and how is it similar to the phenomenon of interference?

Determine the Concept The light is being projected up from underneath the silk, so you will see light where there is a gap and darkness where two threads overlap. Because the two weaves have almost the same spatial period but not exactly identical (because the two are stretched unequally), there will be places where, for large sections of the cloth, the two weaves overlap in phase, leading to brightness, and large sections where the two overlap 90° out of phase (that is, thread on gap and vice versa) leading to darkness. This is exactly the same idea as in the interference of two waves.

17 •• When a musical instrument consisting of drinking glasses, each partially filled to a different height with water, is struck with a small mallet, each glass produces a different frequency of sound wave. Explain how this instrument works.

Determine the Concept Standing sound waves are produced in the air columns above the water. The resonance frequency of the air columns depends on the length of the air column, which depends on how much water is in the glass.

18 •• During an organ recital, the air compressor that drives the organ pipes suddenly fails. An enterprising physics student in the audience tries to help by replacing the compressor with a tank of a pressurized tank of nitrogen gas. What effect, if any, will the nitrogen gas have on the frequency output of the organ pipes? What effect, if any, would helium gas have on the frequency output of the organ pipes?

Picture the Problem We can use $v = f\lambda$ to relate the frequency of the sound waves in the organ pipes to the speed of sound in air, nitrogen, and helium. We can use $v = \sqrt{\gamma RT/M}$ to relate the speed of sound, and hence its frequency, to the properties of the three gases.

Express the frequency of a given note as a function of its wavelength and the speed of sound:

$$f = \frac{v}{\lambda}$$

Relate the speed of sound to the absolute temperature and the molar mass of the gas used in the organ:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

where γ depends on the kind of gas, R is a constant, T is the absolute temperature, and M is the molar mass.

Substitute for v to obtain:

$$f = \frac{1}{\lambda}\sqrt{\frac{\gamma RT}{M}}$$

For air in the organ pipes we have:

$$f_{\text{air}} = \frac{1}{\lambda}\sqrt{\frac{\gamma_{\text{air}} RT}{M_{\text{air}}}} \quad (1)$$

When nitrogen is in the organ pipes:

$$f_{N_2} = \frac{1}{\lambda}\sqrt{\frac{\gamma_{N_2} RT}{M_{N_2}}} \quad (2)$$

Express the ratio of equation (2) to equation (1) and solve for f_{N_2}:

$$\frac{f_{N_2}}{f_{air}} = \sqrt{\frac{\gamma_{N_2}}{\gamma_{air}}\frac{M_{air}}{M_{N_2}}}$$

and

$$f_{N_2} = f_{air}\sqrt{\frac{\gamma_{N_2}}{\gamma_{air}}\frac{M_{air}}{M_{N_2}}}$$

Because $\gamma_{N_2} = \gamma_{air}$ and $M_{air} > M_{N_2}$:

$$f_{N_2} > f_{air}$$

That is,

$\boxed{f \text{ will increase for each organ pipe.}}$

Note, however, that because M_{air} is not very much greater than M_{N_2}, the change in frequency will not be very great.

If helium were used, we'd have:

$$f_{He} = f_{air}\sqrt{\frac{\gamma_{He}}{\gamma_{air}}\frac{M_{air}}{M_{He}}}$$

Because $\gamma_{He} > \gamma_{air}$ and $M_{air} >> M_{He}$, $f_{He} >> f_{air}$ and the effect will be even more pronounced.

19 •• The constant γ for helium (and all monatomic gases) is 1.67. If a man inhales helium and then speaks, his voice has a high-pitch and becomes cartoon-like. Why?

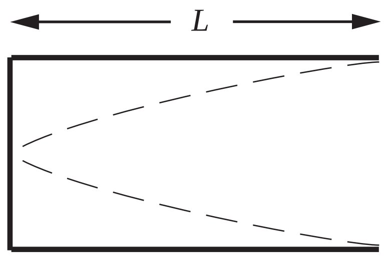
Determine the Concept The wavelength is determined mostly by the resonant cavity of the mouth; the frequency of sounds he makes is equal to the wave speed divided by the wavelength. Because $v_{He} > v_{air}$ (see Equation 15-5), the resonance frequency is higher if helium is the gas in the cavity.

Estimation and Approximation

20 • It is said that a powerful opera singer can hit a high note with sufficient intensity to shatter an empty wine glass by causing the air in it to resonate at the frequency of her voice. Estimate the frequency necessary to obtain a standing wave in an 8.0-cm-high glass. (The 8.0 cm does not include the height of the stem.) Approximately how many octaves above middle C (262 Hz) is this? *Hint: To go up one octave means to double the frequency.*

Picture the Problem If you model the wine glass as a half-open (closed-open) cylinder (shown below on its side), then, knowing the speed of sound in air and

the relationship between the height of the wine glass and the wavelength of its fundamental (1st harmonic) frequency, you can find the fundamental frequency with which it resonates using the relationship $v = f\lambda$. The diagram shows the displacement-amplitude pattern for the 1st harmonic wave pattern. Note that there is a displacement node at the bottom of the wine glass and a displacement antinode at the top of the wine glass. Note further that, in reality, the displacement antinode is a short distance to the right of the open end of the wine glass.



The fundamental frequency is related to the wavelength of the sound in the wine glass according to:

$$f_1 = \frac{v}{\lambda_1}$$

Using the diagram, determine the wavelength of the fundamental mode:

$$L = \tfrac{1}{4}\lambda_1 \Rightarrow \lambda_1 = 4L$$

Substituting for λ_1 yields:

$$f_1 = \frac{v}{4L}$$

Substitute numerical values and evaluate f_1:

$$f_1 = \frac{343\ \text{m/s}}{4(8.0\ \text{cm})} = 1.072\ \text{kHz} = \boxed{1.1\ \text{kHz}}$$

Because the frequencies 262 Hz, 524 Hz, and 1024 Hz are 2^0, 2^1, and 2^2 times 262 Hz, 1.1 kHz is approximately $\boxed{\text{2 octaves}}$ above 262 Hz.

21 • Estimate how accurately you can tune a piano string to a tuning fork of known frequency using only your ears, the tuning fork and a wrench. Explain your answer.

Determine the Concept If you do not hear beats for the entire time the string and the tuning fork are vibrating, you can be sure that their frequencies, while not exactly the same, are very close. If the sounds of the vibrating string and the tuning fork last for 10 s, it follows that the beat frequency is less than 0.1 Hz. Hence, the frequencies of the vibrating string and the tuning fork are within 0.1 Hz of each other.

22 •• The shortest pipes used in organs are about 7.5 cm long. (*a*) Estimate the fundamental frequency of a pipe this long that is open at both ends. (*b*) For such a pipe, estimate the harmonic number *n* of the highest-frequency harmonic that is within the audible range. (The audible range of human hearing is about 20 to 20,000 Hz.)

Picture the Problem We can use $v = f_1\lambda_1$ to express the resonance frequencies in the organ pipes in terms of their wavelengths and $L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots$ to relate the length of the pipes to the resonance wavelengths.

(*a*) Relate the fundamental frequency of the pipe to its wavelength and the speed of sound:

$$f_1 = \frac{v}{\lambda_1}$$

Express the condition for constructive interference in a pipe that is open at both ends:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots \qquad (1)$$

Solve for λ_1:

$$\lambda_1 = 2L \Rightarrow f_1 = \frac{v}{2L}$$

Substitute numerical values and evaluate f_1:

$$f_1 = \frac{343\,\text{m/s}}{2\left(7.5\times10^{-2}\,\text{m}\right)} = 2.29\,\text{kHz} = \boxed{2.3\,\text{kHz}}$$

(*b*) Relate the resonance frequencies of the pipe to their wavelengths and the speed of sound:

$$f_n = \frac{v}{\lambda_n}$$

Solve equation (1) for λ_n:

$$\lambda_n = \frac{2L}{n} \Rightarrow f_n = n\frac{v}{2L}$$

Substitute numerical values to obtain:

$$f_n = n\frac{343\,\text{m/s}}{2\left(7.5\times10^{-2}\,\text{m}\right)} = n\left(2.29\,\text{kHz}\right)$$

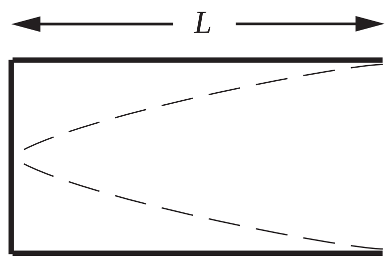
Set $f_n = 20$ kHz and evaluate *n*:

$$n = \frac{20\,\text{kHz}}{2.29\,\text{kHz}} = 8.7$$

The eighth harmonic is within the range defined as audible. The ninth harmonic might be heard by a person with very good hearing.

23 •• Estimate the resonant frequencies that are in the audible range of human hearing of the human ear canal. Treat the canal as an air column open at one end, stopped at the other end, and with a length of 1.00 in. How many resonant frequencies lie in this range? Human hearing has been found experimentally to be the most sensitive at frequencies of about 3, 9 and 15 kHz. How do these frequencies compare to your calculations?

Picture the Problem If you model the human ear canal as an open-stopped column, then, knowing the speed of sound in air and the relationship between the depth of the ear canal and the wavelength of its fundamental (1st harmonic) frequency, you can find the fundamental frequency with which it resonates using the relationship $v = f\lambda$. The diagram shows the displacement-amplitude pattern for the 1st harmonic wave pattern.



The frequencies at which our model ear will resonate are given by:

$$f_n = n\frac{v}{4L}, n = 1,3,5,...$$

Solving for n yields:

$$n = \frac{4Lf_n}{v}$$

The approximate upper limit for a human ear is 20 kHz. Setting f_n equal to 20 kHz and assuming that the temperature of the air in the ear canal is 20°C yields :

$$n = \frac{4\left(1.00\text{ in}\times\frac{2.54\text{ cm}}{\text{in}}\right)(20\text{ kHz})}{343\text{ m/s}} = 5.92 \approx 5$$

and

$\boxed{3}$ resonant frequencies lie within the range of normal human hearing.

The frequencies that lie within the range of human hearing correspond to $n = 1, 3,$ and 5 are:

$$f_1 = \frac{343\ \text{m/s}}{4(2.54\ \text{cm})} \approx 3.38\ \text{kHz},$$

$$f_3 = 3\left(\frac{343\ \text{m/s}}{4(2.54\ \text{cm})}\right) \approx 10.1\ \text{kHz},$$

and

$$f_5 = 5\left(\frac{343\ \text{m/s}}{4(2.54\ \text{cm})}\right) \approx 16.9\ \text{kHz}$$

The calculated frequencies agree with the observed frequencies to within 14%.

Superposition and Interference

24 • Two harmonic waves traveling on a string in the same direction both have a frequency of 100 Hz, a wavelength of 2.0 cm, and an amplitude of 0.020 m. In addition, they overlap each other. What is the amplitude of the resultant wave if the original waves differ in phase by (*a*) $\pi/6$ and (*b*) $\pi/3$?

Picture the Problem We can use $A = 2y_0 \cos\frac{1}{2}\delta$ to find the amplitude of the resultant wave.

(*a*) Evaluate the amplitude of the resultant wave when $\delta = \pi/6$:

$$A = 2y_0 \cos\tfrac{1}{2}\delta = 2(0.020\ \text{m})\cos\frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \boxed{3.9\ \text{cm}}$$

(*b*) Proceed as in (*a*) with $\delta = \pi/3$:

$$A = 2y_0 \cos\tfrac{1}{2}\delta = 2(0.020\ \text{m})\cos\frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$= \boxed{3.5\ \text{cm}}$$

25 • **[SSM]** Two harmonic waves having the same frequency, wave speed and amplitude are traveling in the same direction and in the same propagating medium. In addition, they overlap each other. If they differ in phase by $\pi/2$ and each has an amplitude of 0.050 m, what is the amplitude of the resultant wave?

Picture the Problem We can use $A = 2y_0 \cos\frac{1}{2}\delta$ to find the amplitude of the resultant wave.

Evaluate the amplitude of the resultant wave when $\delta = \pi/2$:

$$A = 2y_0 \cos\tfrac{1}{2}\delta = 2(0.050\ \text{m})\cos\frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \boxed{7.1\ \text{cm}}$$

26 • Two audio speakers facing in the same direction oscillate in phase at the same frequency. They are separated by a distance equal to one-third of a wavelength. Point *P* is in front of both speakers, on the line that passes through their centers. The amplitude of the sound at *P* due to either speaker acting alone is *A*. What is the amplitude (in terms of *A*) of the resultant wave at that point?

Picture the Problem The phase shift in the waves generated by these two sources is due to their separation of $\lambda/3$. We can find the phase difference due to the path difference from $\delta = 2\pi\frac{\Delta x}{\lambda}$ and then the amplitude of the resultant wave using $A = 2y_0\cos\frac{1}{2}\delta$.

Evaluate the phase difference δ:

$$\delta = 2\pi\frac{\Delta x}{\lambda} = 2\pi\frac{\lambda/3}{\lambda} = \frac{2}{3}\pi$$

Find the amplitude of the resultant wave:

$$A_{\text{res}} = 2y_0\cos\tfrac{1}{2}\delta = 2A\cos\frac{1}{2}\left(\frac{2}{3}\pi\right)$$

$$= 2A\cos\frac{\pi}{3} = \boxed{A}$$

27 • Two compact sources of sound oscillate in phase with a frequency of 100 Hz. At a point 5.00 m from one source and 5.85 m from the other, the amplitude of the sound from each source separately is *A*. (*a*) What is the phase difference of the two waves at that point? (*b*) What is the amplitude (in terms of *A*) of the resultant wave at that point?

Picture the Problem The phase shift in the waves generated by these two sources is due to a path difference Δx = 5.85 m – 5.00 m = 0.85 m. We can find the phase difference due to this path difference from $\delta = 2\pi\frac{\Delta x}{\lambda}$ and then the amplitude of the resultant wave using $A = 2y_0\cos\frac{1}{2}\delta$.

(*a*) Find the phase difference due to the path difference:

$$\delta = 2\pi\frac{\Delta x}{\lambda}$$

Use $v = f\lambda$ to eliminate λ:

$$\delta = 2\pi\frac{\Delta x}{\frac{v}{f}} = 2\pi f\,\frac{\Delta x}{v}$$

Substitute numerical values and evaluate δ:

$$\delta = 2\pi\left(100\,\text{s}^{-1}\right)\frac{0.85\,\text{m}}{343\,\text{m/s}} = 1.557\,\text{rad}$$

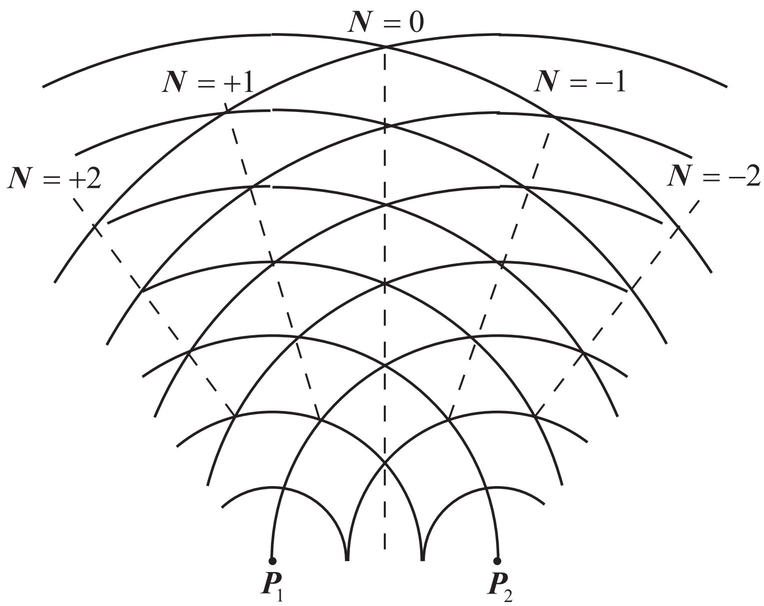
$$= \boxed{89^\circ}$$

(*b*) Relate the amplitude of the resultant wave to the amplitudes of the interfering waves and the phase difference between them:

$$A = 2y_0 \cos\tfrac{1}{2}\delta = 2A\cos\tfrac{1}{2}(1.557)$$
$$= \boxed{1.4A}$$

28 • With a drawing program or a compass, draw circular arcs of radius 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm and 7 cm centered at each of two points (P_1 and P_2) a distance d = 3.0 cm apart. Draw smooth curves through the intersections corresponding to points N centimeters farther from P_1 than from P_2 for N = +2, +1, 0, –1 and –2, and label each curve with the corresponding value of N. There are two additional such curves you can draw, one for N = +3 and one for N = –3. If identical sources of coherent in-phase 1.0-cm wavelength waves were placed at points P_1 and P_2, the waves would interfere constructively along each of the smooth curves.

Picture the Problem The following diagram was constructed using a spreadsheet program.



29 • **[SSM]** Two speakers separated by some distance emit sound waves of the same frequency. At some point P, the intensity due to each speaker separately is I_0. The distance from P to one of the speakers is $\frac{1}{2}\lambda$ longer than that from P to the other speaker. What is the intensity at P if (*a*) the speakers are coherent and in phase, (*b*) the speakers are incoherent, and (*c*) the speakers are coherent and 180° out of phase?

Picture the Problem The intensity at the point of interest is dependent on whether the speakers are coherent and on the total phase difference in the waves arriving at the given point. We can use $\delta = 2\pi\dfrac{\Delta x}{\lambda}$ to determine the phase

difference δ, $A=\left|2p_0\cos\frac{1}{2}\delta\right|$ to find the amplitude of the resultant wave, and the fact that the intensity I is proportional to the square of the amplitude to find the intensity at P for the given conditions.

(*a*) Find the phase difference δ:

$$\delta=2\pi\frac{\frac{1}{2}\lambda}{\lambda}=\pi$$

Find the amplitude of the resultant wave:

$$A=\left|2p_0\cos\tfrac{1}{2}\pi\right|=0$$

Because the intensity is proportional to A^2:

$$I=\boxed{0}$$

(*b*) The sources are incoherent and the intensities add:

$$I=\boxed{2I_0}$$

(*c*) The total phase difference is the sum of the phase difference of the sources and the phase difference due to the path difference:

$$\begin{aligned}\delta_{\text{tot}}&=\delta_{\text{sources}}+\delta_{\text{path difference}}\\&=\pi+2\pi\frac{\Delta x}{\lambda}=\pi+2\pi\left(\frac{1}{2}\right)\\&=2\pi\end{aligned}$$

Find the amplitude of the resultant wave:

$$A=\left|2p_0\cos\tfrac{1}{2}(2\pi)\right|=2p_0$$

Because the intensity is proportional to A^2:

$$I=\frac{A^2}{p_0^2}I_0=\frac{(2p_0)^2}{p_0^2}I_0=\boxed{4I_0}$$

30 • Two speakers separated by some distance emit sound waves of the same frequency. At some point P', the intensity due to each speaker separately is I_0. The distance from P' to one of the speakers is one wavelength longer than that from P' to the other speaker. What is the intensity at P if (*a*) the speakers are coherent and in phase, (*b*) the speakers are incoherent, and (*c*) the speakers are coherent and out of phase?

Picture the Problem The intensity at the point of interest is dependent on whether the speakers are coherent and on the total phase difference in the waves arriving at the given point. We can use $\delta = 2\pi \frac{\Delta x}{\lambda}$ to determine the phase difference δ, $A = \left|2p_0 \cos\frac{1}{2}\delta\right|$ to find the amplitude of the resultant wave, and the fact that the intensity is proportional to the square of the amplitude to find the intensity at P for the given conditions.

(*a*) Find the phase difference δ:

$$\delta = 2\pi \frac{\lambda}{\lambda} = 2\pi$$

Find the amplitude of the resultant wave:

$$A = \left|2p_0 \cos\tfrac{1}{2}(2\pi)\right| = 2p_0$$

Because the intensity is proportional to A^2:

$$I = \frac{A^2}{p_0^2} I_0 = \frac{(2p_0)^2}{p_0^2} I_0 = \boxed{4I_0}$$

(*b*) The sources are incoherent and the intensities add:

$$I = \boxed{2I_0}$$

(*c*) The total phase difference is the sum of the phase difference of the sources and the phase difference due to the path difference:

$$\begin{aligned}\delta_{\text{tot}} &= \delta_{\text{sources}} + \delta_{\text{path difference}} \\ &= \pi + 2\pi \frac{\Delta x}{\lambda} = \pi + 2\pi\left(\frac{\lambda}{\lambda}\right) \\ &= 3\pi\end{aligned}$$

Find the amplitude of the resultant wave:

$$A = \left|2p_0 \cos\tfrac{1}{2}(3\pi)\right| = 0$$

Because the intensity is proportional to A^2:

$$I = \boxed{0}$$

31 •• A transverse harmonic wave with a frequency equal to 40.0 Hz propagates along a taut string. Two points 5.00 cm apart are out of phase by $\pi/6$. (*a*) What is the wavelength of the wave? (*b*) At a given point on the string, how much does the phase change in 5.00 ms? (*c*) What is the wave speed?

Picture the Problem (*a*) Let the +*x* direction be the direction of propagation of the wave. We can express the phase difference in terms of the separation of the two points and the wavelength of the wave and solve for λ. (*b*) We can find the phase difference by relating the time between displacements to the period of the wave. (*c*) We can use the relationship between the speed, frequency, and wavelength of a wave to find its velocity.

(*a*) Relate the phase difference to the wavelength of the wave:

$$\delta = 2\pi\frac{\Delta x}{\lambda}$$

Solve for and evaluate λ:

$$\lambda = 2\pi\frac{\Delta x}{\delta} = 2\pi\frac{5.00\,\text{cm}}{\frac{1}{6}\pi} = \boxed{60.0\,\text{cm}}$$

(*b*) The period of the wave is given by:

$$T = \frac{1}{f} = \frac{1}{40.0\,\text{s}^{-1}} = 25.0\,\text{ms}$$

Relate the time between the two displacements to the period of the wave:

$$5.00\,\text{ms} = \frac{1}{5}T$$

The phase difference corresponding to one-fifth of a period is:

$$\delta = \boxed{\frac{2\pi}{5}}$$

(*c*) The wave speed is the product of its frequency and wavelength:

$$v = f\lambda = \left(40.0\,\text{s}^{-1}\right)\left(0.600\,\text{m}\right) = \boxed{24.0\,\text{m/s}}$$

32 •• It is thought that the brain determines the direction of the source of a sound by sensing the phase difference between the sound waves striking the eardrums. A distant source emits sound of frequency 680 Hz. When you are directly facing a sound source there is no phase difference. Estimate the phase difference between the sounds received by your ears when you are facing 90° away from the direction of the source.

Picture the Problem Assume a distance of about 20 cm between your ears. When you rotate your head through 90°, you introduce a path difference of 20 cm. We can apply the equation for the phase difference due to a path difference to determine the change in phase between the sounds received by your ears as you rotate your head through 90°.

Express the phase difference due to the rotation of your head through 90°:

$$\delta = 2\pi \frac{20\,\text{cm}}{\lambda}$$

Because $\lambda = v/f$:

$$\delta = 2\pi f \frac{20\,\text{cm}}{v}$$

Substitute numerical values and evaluate δ:

$$\delta = 2\pi\left(680\,\text{s}^{-1}\right)\frac{20\,\text{cm}}{343\,\text{m/s}} = \boxed{0.79\pi\,\text{rad}}$$

33 •• [SSM] Sound source A is located at $x = 0$, $y = 0$, and sound source B is located at $x = 0$, $y = 2.4$ m. The two sources radiate coherently and in phase. An observer at $x = 15$ m, $y = 0$ notes that as he takes a few steps from $y = 0$ in either the $+y$ or $-y$ direction, the sound intensity diminishes. What is the lowest frequency and the next to lowest frequency of the sources that can account for that observation?

Picture the Problem Because the sound intensity diminishes as the observer moves, parallel to a line through the sources, away from his initial position, we can conclude that his initial position is one at which there is constructive interference of the sound coming from the two sources. We can apply the condition for constructive interference to relate the wavelength of the sound to the path difference at his initial position and the relationship between the velocity, frequency, and wavelength of the waves to express this path difference in terms of the frequency of the sources.

Express the condition for constructive interference at (15 m, 0):

$$\Delta r = n\lambda,\ n = 1, 2, 3, \ldots \qquad (1)$$

The path difference Δr is given by:

$$\Delta r = r_B - r_A$$

Using the Pythagorean theorem, express r_B:

$$r_B = \sqrt{(15\,\text{m})^2 + (2.4\,\text{m})^2}$$

Substitute for r_B to obtain:

$$\Delta r = \sqrt{(15\,\text{m})^2 + (2.4\,\text{m})^2} - 15\,\text{m}$$

Using $v = f\lambda$ and equation (1), express f_n in terms of Δr and n:

$$f_n = n\frac{v}{\Delta r},\ n = 1,2,3,\ldots$$

Substituting numerical values yields:

$$f_n = n\frac{343\text{ m/s}}{\sqrt{(15\text{ m})^2 + (2.4\text{ m})^2} - 15\text{ m}}$$
$$= (1.798\text{ kHz})n = (2\text{ kHz})n$$

Evaluate f_1 and f_2 to obtain:

$$f_1 = \boxed{2\text{ kHz}} \text{ and } f_2 = \boxed{4\text{ kHz}}$$

34 •• Suppose that the observer in Problem 33 finds himself at a point of minimum intensity at $x = 15$ m, $y = 0$. What is then the lowest frequency and the next to lowest frequency of the sources consistent with this observation?

Picture the Problem Because the sound intensity increases as the observer moves, parallel to a line through the sources, away from his initial position, we can conclude that his initial position is one at which there is destructive interference of the sound coming from the two sources. We can apply the condition for destructive interference to relate the wavelength of the sound to the path difference at his initial position and the relationship between the velocity, frequency, and wavelength of the waves to express this path difference in terms of the frequency of the sources.

Express the condition for destructive interference at (15.0 m, 0):

$$\Delta r = n\frac{\lambda}{2},\ n = 1,3,5,... \qquad (1)$$

Express the path difference Δr:

$$\Delta r = r_B - r_A$$

Using the Pythagorean theorem, find r_B:

$$r_B = \sqrt{(15\text{ m})^2 + (2.4\text{ m})^2}$$

Substitute for r_B to obtain:

$$\Delta r = \sqrt{(15\text{ m})^2 + (2.4\text{ m})^2} - 15\text{ m}$$

Using $v = f\lambda$ and equation (1), express f_n in terms of Δr and n:

$$f_n = n\frac{v}{2\Delta r},\ n = 1,3,5,...$$

Substituting numerical values yields:

$$f_n = n\frac{343\text{ m/s}}{2\left(\sqrt{(15\text{ m})^2 + (2.4\text{ m})^2} - 15\text{ m}\right)}$$
$$= (0.8989\text{ kHz})n = (1\text{ kHz})n$$

Evaluate f_1 and f_3 to obtain:

$$f_1 = \boxed{1\text{ kHz}} \text{ and } f_3 = \boxed{3\text{ kHz}}$$

35 •• **[SSM]** Two harmonic water waves of equal amplitudes but different frequencies, wave numbers, and speeds are traveling in the same direction. In addition, they are superposed on each other. The total displacement of the wave can be written as $y(x,t) = A[\cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t)]$, where $\omega_1/k_1 = v_1$ (the speed of the first wave) and $\omega_2/k_2 = v_2$ (the speed of the second wave). (*a*) Show that $y(x,t)$ can be written in the form $y(x,t) = Y(x, t)\cos(k_{av}x - \omega_{av}t)$, where $\omega_{av} = (\omega_1 + \omega_2)/2$, $k_{av} = (k_1 + k_2)/2$, $Y(x, t) = 2A\cos[(\Delta k/2)x - (\Delta\omega/2)t]$, $\Delta\omega = \omega_1 - \omega_2$, and $\Delta k = k_1 - k_2$. The factor $Y(x, t)$ is called the *envelope* of the wave. (*b*) Let $A = 1.00$ cm, $\omega_1 = 1.00$ rad/s, $k_1 = 1.00\ \text{m}^{-1}$, $\omega_2 = 0.900$ rad/s, and $k_2 = 0.800\text{m}^{-1}$. Using a **spreadsheet program** or **graphing calculator**, make a plot of $y(x,t)$ versus x at $t = 0.00$ s for $0 < x < 5.00$ m. (*c*) Using a **spreadsheet program** or **graphing calculator**, make three plots of $Y(x,t)$ versus x for $-5.00\ \text{m} < x < 5.00$ m on the same graph. Make one plot for $t = 0.00$ s, the second for $t = 5.00$ s, and the third for $t = 10.00$ s. Estimate the speed at which the envelope moves from the three plots, and compare this estimate with the speed obtained using $v_{\text{envelope}} = \Delta\omega/\Delta k$.

Picture the Problem We can use the trigonometric identity

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

to derive the expression given in (*a*) and the speed of the envelope can be found from the second factor in this expression; i.e., from $\cos[(\Delta k/2)x - (\Delta\omega/2)t]$.

(*a*) Express the amplitude of the resultant wave function $y(x,t)$:

$$y(x,t) = A(\cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t))$$

Use the trigonometric identity $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ to obtain:

$$y(x,t) = 2A\left[\cos\frac{k_1x - \omega_1t + k_2x - \omega_2t}{2}\cos\frac{k_1x - \omega_1t - k_2x + \omega_2t}{2}\right]$$

$$= 2A\left[\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{k_1 - k_2}{2}x + \frac{\omega_2 - \omega_1}{2}t\right)\right]$$

Substitute $\omega_{av} = (\omega_1 + \omega_2)/2$, $k_{av} = (k_1 + k_2)/2$, $\Delta\omega = \omega_1 - \omega_2$ and $\Delta k = k_1 - k_2$ to obtain:

$$y(x,t) = 2A\left[\cos\left(k_{ave}x - \omega_{ave}t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\right]$$

$$= \boxed{Y(x,t)\left[\cos\left(k_{ave}x - \omega_{ave}t\right)\right.}$$

where

$$Y(x,t) = \boxed{2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)}$$

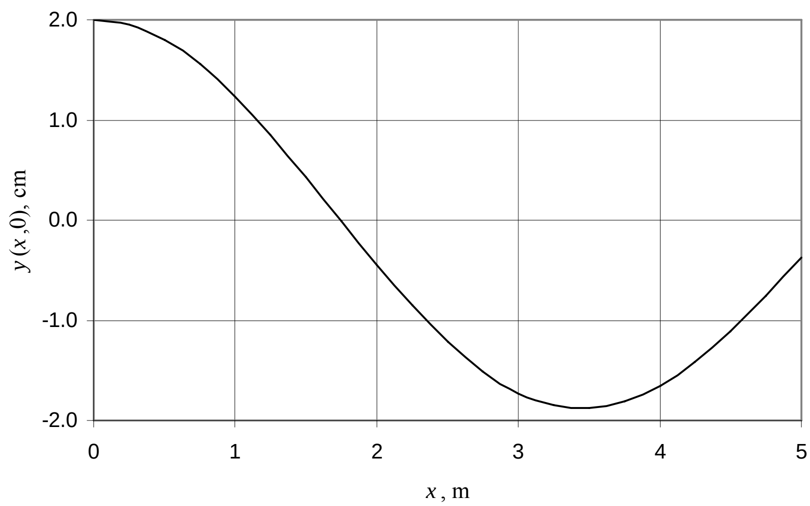
(*b*) A spreadsheet program to calculate $y(x,t)$ between 0 m and 5.00 m at $t = 0.00$ s, follows. The constants and cell formulas used are shown in the table.

Cell	Content/Formula	Algebraic Form
A15	0	
A16	A15+0.25	$x + \Delta x$
B15	2*\$B\$2*COS(0.5*(\$B\$3 −\$B\$4)*A15 −0.5*(\$B\$5−\$B\$6)*\$B\$8)	$Y(x,0.00\text{ s})$
C15	B15*COS(0.5*(\$B\$3+\$B\$4)*A15 −0.5*(\$B\$5+\$B\$6)*\$B\$7)	$y(x,0.00\text{ s})$

	A	B	C
1			
2	A=	1	cm
3	k_1=	1	m^{-1}
4	k_2=	0.8	m^{-1}
5	ω_1=	1	rad/s
6	ω_2=	0.9	rad/s
7	t =	0.00	s
8			
9			
10			
11			
12	x	$Y(x,0)$	$y(x,0)$
13	(m)	(cm)	(cm)
14			
15	0.00	2.000	2.000
16	0.25	1.999	1.949
17	0.50	1.998	1.799
18	0.75	1.994	1.557
31	4.00	1.842	−1.652
32	4.25	1.822	−1.413

33	4.50	1.801	−1.108
34	4.75	1.779	−0.753
35	5.00	1.755	−0.370

A graph of $y(x,0)$ follows:



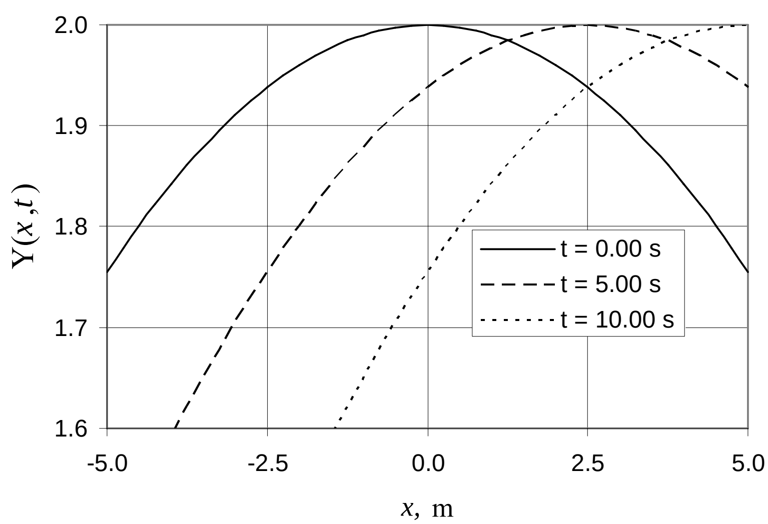
(c) A spreadsheet program to calculate $Y(x,t)$ for $-5.00\text{ m} < x < 5.00\text{ m}$ and $t = 0.00$ s, $t = 5.00$ s and $t = 10.00$ s follow: The constants and cell formulas used are shown in the table.

Cell	Content/Formula	Algebraic Form
A15	0	
A16	A15+0.25	$x + \Delta x$
B15	2*$B$2*COS(0.5*($B$3 −$B$4)*A15 −0.5*($B$5−$B$6)*$B$7)	$\boldsymbol{Y(x,}0.00\text{ s})$
C15	2*$B$2*COS(0.5*($B$3 −$B$4)*A15 −0.5*($B$5−$B$6)*$B$8)	$Y(x,5.00\text{ s})$
D15	2*$B$2*COS(0.5*($B$3 −$B$4)*A15 −0.5*($B$5−$B$6)*$B$9)	$Y(x,10.00\text{ s})$

	A	B	C	D
1				
2	A=	1	cm	
3	k_1=	1	m^{-1}	
4	k_2=	0.8	m^{-1}	
5	ω_1=	1	rad/s	
6	ω_2=	0.9	rad/s	
7	t=	0	s	
8	t=	5	s	
9	t=	10	s	
10				

11				
12	x	$Y(x,0)$	$Y(x,5\text{ s})$	$Y(x,10\text{ s})$
13	(m)	(cm)	(cm)	(cm)
14				
15	–5.00	1.755	1.463	1.081
16	–4.75	1.779	1.497	1.122
17	–4.50	1.801	1.530	1.163
18	–4.25	1.822	1.561	1.204
19	–4.00	1.842	1.592	1.243
51	4.00	1.842	1.978	1.990
52	4.25	1.822	1.969	1.994
53	4.50	1.801	1.960	1.998
54	4.75	1.779	1.950	1.999
55	5.00	1.755	1.938	2.000

(c) Graphs of $Y(x,t)$ versus x for $-5.00\text{ m} < x < 5.00\text{ m}$ and $t = 0.00$ s, $t = 5.00$ s and $t = 10.00$ s follow:



To estimate the speed of the envelope, we can use its horizontal displacement between $t = 0.00$ s and $t = 5.00$ s:

$$v_{\text{est}} = \frac{\Delta x}{\Delta t} \quad (1)$$

From the graph we note that the wave traveled 2.5 m in 5.00 s:

$$v_{\text{est}} = \frac{2.50\text{ m}}{5.00\text{ s}} = \boxed{50\text{ cm/s}}$$

The speed of the envelope is given by:

$$v_{\text{envelope}} = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

Substitute numerical values and evaluate v_{envelope}:

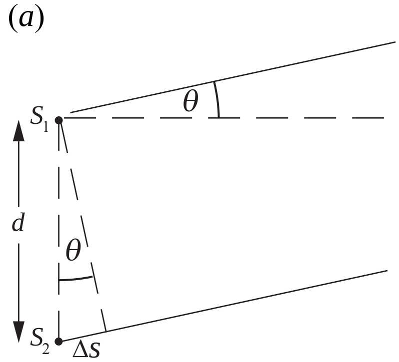
$$v_{\text{envelope}} = \frac{1.00\,\text{rad/s} - 0.900\,\text{rad/s}}{1.00\,\text{m}^{-1} - 0.800\,\text{m}^{-1}} = 50\,\text{cm/s}$$

in agreement with our graphical estimate.

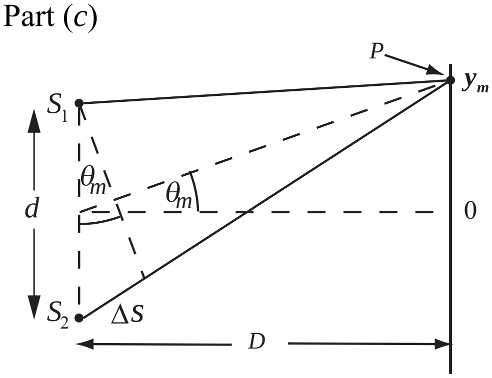
36 •• Two coherent point sources are in phase and are separated by a distance *d*. An interference pattern is detected along a line parallel to the line through the sources and a large distance *D* from the sources, as shown in Figure 16-31. (*a*) Show that the path difference Δs from the two sources to some point on the line at an angle θ is given, approximately, by $\Delta s \approx d\sin\theta$. *Hint: Assume that D >> d, so the lines from the sources to P are approximately parallel* (Figure 16-31*b*). (*b*) Show that the two waves interfere constructively at *P* if $\Delta s = m\lambda$, where $m = 0, 1, 2,...$ (That is, show there is an interference maximum at *P* if $\Delta s = m\lambda$, where $m = 0, 1, 2, ...$) (*c*) Show that the distance y_m from the central maximum (at $y = 0$) to the *m*th interference maximum at *P* is given by $y_m = D\tan\theta_m$, where $d\sin\theta_m = m\lambda$.

Picture the Problem The following diagram shows the two sources separated by a distance *d* and the path difference Δs between the two sources and point *P*. Because the lines from the sources to the distant point are approximately parallel, the triangle shown in the diagram is approximately a right triangle and we can use trigonometry to express Δs in terms of *d* and θ. In Part (*b*), we can use the relationship giving the phase difference due to a path difference to show that the two waves interfere constructively at *P* if $\Delta s = m\lambda$. In Part (*c*) we use the geometry of Figure 16-31*a* to relate y_m to *D* and θ_m.

Part (*a*)



Part (*c*)



(*a*) Using the diagram for Part (*a*), relate Δs to the separation of the sources and the angle θ:

$$\sin\theta \approx \frac{\Delta s}{d} \Rightarrow \Delta s \approx \boxed{d\sin\theta}$$

(*b*) δ is related to Δs through the proportion:

$$\frac{\delta}{2\pi} = \frac{\Delta s}{\lambda} \Rightarrow \Delta s = \frac{\delta\lambda}{2\pi}$$

There will be constructive interference at P provided:

$$\delta = 2\pi m,\ m = 0,1,2,,...$$

Substituting for δ and simplifying yields:

$$\Delta s = \frac{(2\pi m)\lambda}{2\pi} = \boxed{m\lambda,\ m = 0,1,2,,...}$$

(c) Referring to the diagram for Part (c), note that, if $\theta_m << 1$ (equivalently, $d << D$), then $d\sin\theta_m = m\lambda$ and:

$$\tan\theta_m = \frac{y_m}{D} \Rightarrow y_m = \boxed{D\tan\theta_m}$$

37 •• Two sound sources radiating in phase at a frequency of 480 Hz interfere such that maxima are heard at angles of 0º and 23º from a line perpendicular to that joining the two sources. The listener is at a large distance from the line through both sources, and no additional maxima are heard at angles in the range 0° < θ < 23°. Find the separation d between the two sources, and any other angles at which intensity maxima will be heard. (Use the result of Problem 36.)

Picture the Problem Because a maximum is heard at 0° and the sources are in phase, we can conclude that the path difference is 0. Because the next maximum is heard at 23°, the path difference to that position must be one wavelength. We can use the result of Part (a) of Problem 36 to relate the separation of the sources to the path difference and the angle θ. We'll apply the condition for constructive interference to determine the angular locations of other points of maximum intensity in the interference pattern.

Using the result of Part (a) of Problem 36, express the separation of the sources in terms of Δs and θ:

$$d = \frac{\Delta s}{\sin\theta}$$

Because $\Delta s = \lambda$ and $v = f\lambda$:

$$d = \frac{\lambda}{\sin\theta} = \frac{v}{f\sin\theta}$$

Evaluate d with $\Delta s = \lambda$ and $\theta = 23°$:

$$d = \frac{343\ \text{m/s}}{(480\ \text{s}^{-1})\sin 23°} = 1.83\ \text{m} = \boxed{1.8\ \text{m}}$$

Express the condition for additional intensity maxima:

$$d\sin\theta_m = m\lambda$$

where m = 1, 2, 3, …, or

$$\theta_m = \sin^{-1}\left[\frac{m\lambda}{d}\right]$$

Evaluate this expression for $m = 2$:

$$\theta_2 = \sin^{-1}\left[\frac{2(343\,\text{m/s})}{(480\,\text{s}^{-1})(1.83\,\text{m})}\right] = \boxed{51°}$$

Remarks: It is easy to show that, for *m* > 2, the inverse sine function is undefined and that, therefore, there are no additional relative maxima at angles larger than 51°.

38 •• Two loudspeakers are driven in phase by an audio amplifier at a frequency of 600 Hz. The speakers are on the y axis, one at $y = +1.00$ m and the other at $y = -1.00$ m. A listener, starting at $(x, y) = (D, 0)$, where $D >> 2.00$ m, walks in the $+y$ direction along the line $x = D$. (See Problem 36.) (*a*) At what angle θ will she first hear a minimum in the sound intensity? (θ is the angle between the positive x axis and the line from the origin to the listener.) (*b*) At what angle will she first hear a maximum in the sound intensity (after $\theta = 0$)? (*c*) How many maxima can she possibly hear if she keeps walking in the same direction?

Picture the Problem Because the speakers are driven in phase and the path difference is 0 at her initial position, the listener will hear a maximum at (*D*, 0). As she walks along a line parallel to the *y* axis she will hear a minimum wherever it is true that the path difference is an odd multiple of a half wavelength. She will hear an intensity maximum wherever the path difference is an integral multiple of a wavelength. We'll apply the condition for destructive interference in Part (*a*) to determine the angular location of the first minimum and, in Part (*b*), the condition for constructive interference find the angle at which she'll hear the first maximum after the one at 0°. In Part (*c*), we can apply the condition for constructive interference to determine the number of maxima she can hear as keeps walking parallel to the *y* axis.

(*a*) Express the condition for destructive interference:

$$d\sin\theta_m = m\frac{\lambda}{2}$$

where $m = 1, 3, 5, \ldots$, or

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{2d}\right) = \sin^{-1}\left(\frac{mv}{2fd}\right)$$

Evaluate this expression for $m = 1$:

$$\theta_1 = \sin^{-1}\left[\frac{343\,\text{m/s}}{2(600\,\text{s}^{-1})(2.00\,\text{m})}\right] = \boxed{8.22°}$$

(*b*) Express the condition for additional intensity maxima:

$$d\sin\theta_m = m\lambda$$

where m = 0, 1, 2, 3,…, or

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right) = \sin^{-1}\left(\frac{mv}{fd}\right)$$

Evaluate this expression for $m = 1$:

$$\theta_1 = \sin^{-1}\left[\frac{343\,\text{m/s}}{(600\,\text{s}^{-1})(2.00\,\text{m})}\right] = \boxed{16.6°}$$

(*c*) Express the limiting condition on $\sin\theta$:

$$\sin\theta_m = m\frac{\lambda}{d} \leq 1 \Rightarrow m \leq \frac{d}{\lambda} = \frac{fd}{v}$$

Substitute numerical values and evaluate m:

$$m \leq \frac{(600\,\text{s}^{-1})(2.00\,\text{m})}{343\,\text{m/s}} = 3.50$$

Because m must be an integer:

$$m = \boxed{3}$$

39 ••• [SSM] Two sound sources driven in phase by the same amplifier are 2.00 m apart on the y axis, one at $y = +1.00$ m and the other at $y = -1.00$ m. At points large distances from the y axis, constructive interference is heard at at angles with the x axis of $\theta_0 = 0.000$ rad, $\theta_1 = 0.140$ rad and $\theta_2 = 0.283$ rad, and at no angles in between (see Figure 16-31). (*a*) What is the wavelength of the sound waves from the sources? (*b*) What is the frequency of the sources? (*c*) At what other angles is constructive interference heard? (*d*) What is the smallest angle for which the sound waves cancel?

Picture the Problem (*a*) Let d be the separation of the two sound sources. We can express the wavelength of the sound in terms of the d and either of the angles at which intensity maxima are heard. (*b*) We can find the frequency of the sources from its relationship to the speed of the waves and their wavelengths. (*c*) Using the condition for constructive interference, we can find the angles at which intensity maxima are heard. (*d*) We can use the condition for destructive interference to find the smallest angle for which the sound waves cancel.

(*a*) Express the condition for constructive interference:

$$d\sin\theta_m = m\lambda \Rightarrow \lambda = \frac{d\sin\theta_m}{m} \quad (1)$$

where m = 0, 1, 2, 3,…

Evaluate λ for $m = 1$:

$$\lambda = (2.00\,\text{m})\sin(0.140\,\text{rad}) = \boxed{0.279\,\text{m}}$$

(*b*) The frequency of the sound is given by:

$$f=\frac{v}{\lambda}=\frac{343\,\text{m/s}}{0.279\,\text{m}}=\boxed{1.23\,\text{kHz}}$$

(*c*) Solve equation (1) for θ_m:

$$\theta_m=\sin^{-1}\left(\frac{m\lambda}{d}\right)=\sin^{-1}\left[\frac{m(0.279\,\text{m})}{2.00\,\text{m}}\right]$$
$$=\sin^{-1}[(0.1395)m]$$

The table shows the values for θ as a function of m:

m	θ_m
	(rad)
3	$\boxed{0.432}$
4	$\boxed{0.592}$
5	$\boxed{0.772}$
6	$\boxed{0.992}$
7	$\boxed{1.35}$
8	undefined

(*d*) Express the condition for destructive interference:

$$d\sin\theta_m=m\frac{\lambda}{2}$$

where m = 1, 3, 5,…

Solving for θ_m yields:

$$\theta_m=\sin^{-1}\left(m\frac{\lambda}{2d}\right)$$

Evaluate this expression for m = 1:

$$\theta_1=\sin^{-1}\left[\frac{0.279\,\text{m}}{2(2.00\,\text{m})}\right]=\boxed{0.0698\,\text{rad}}$$

40 ••• The two sound sources from Problem 39 are now driven 90º out-of-phase, but at the same frequency as in Problem 39. At what angles are constructive and destructive interference heard?

Picture the Problem The total phase shift in the waves arriving at the points of interest is the sum of the phase shift due to the difference in path lengths from the two sources to a given point and the phase shift due to the sources being out of phase by 90°. From Problem 39 we know that λ = 0.279 m. Using the conditions on the path difference Δx for constructive and destructive interference, we can find the angles at which intensity maxima are heard.

Letting the subscript ″pd″ denote ″path difference″ and the subscript ″s″ the ″sources″, express the total phase shift δ:

$$\delta = \delta_{\text{pd}} + \delta_{\text{s}} = 2\pi\frac{\Delta x}{\lambda} + \frac{\pi}{4}$$

where Δx is the path difference between the two sources and the points at which constructive or destructive interference is heard.

Express the condition for constructive interference:

$$\delta = 2\pi\frac{\Delta x}{\lambda} + \frac{\pi}{4} = 2\pi, 4\pi, 6\pi, \ldots$$

Solve for Δx to obtain:

$$\Delta x = \frac{7}{8}\lambda, \frac{15}{8}\lambda, \frac{23}{8}\lambda, \ldots = \frac{(8m-1)}{8}\lambda$$

where $m = 1, 2, 3, \ldots$

Relate Δx to d to obtain:

$$\Delta x = \frac{(8m-1)}{8}\lambda = d\sin\theta_{\text{c}}$$

where the ″c″ denotes constructive interference.

Solving for θ_{c} yields:

$$\theta_{\text{c}} = \sin^{-1}\left[\frac{(8m-1)\lambda}{8d}\right], m = 1, 2, 3, \ldots$$

The table shows the values for θ_{c} for $m = 1$ to 5:

m	θ_{c}
1	$\boxed{7.01°}$
2	$\boxed{15.2°}$
3	$\boxed{23.6°}$
4	$\boxed{32.7°}$
5	$\boxed{42.8°}$

Express the condition for destructive interference:

$$\delta = 2\pi\frac{\Delta x}{\lambda} + \frac{\pi}{4} = \pi, 3\pi, 5\pi, \ldots$$

Solve for Δx to obtain:

$$\Delta x = \frac{3}{8}\lambda, \frac{11}{8}\lambda, \frac{19}{8}\lambda, \ldots = \frac{(8m-5)}{8}\lambda$$

where $m = 1, 2, 3, \ldots$

Letting ″d″ denotes destructive interference, relate Δx to d to obtain:

$$\Delta x = \frac{(8m-5)}{8}\lambda = d\sin\theta_{\text{d}}$$

Solving for θ_d yields:

$$\theta_d = \sin^{-1}\left[\frac{(8m-5)\lambda}{8d}\right],\ m = 1, 2, 3, \ldots$$

The table shows the values for θ_d for $m = 1$ to 5:

m	θ_d
1	$\boxed{3.00°}$
2	$\boxed{11.1°}$
3	$\boxed{19.3°}$
4	$\boxed{28.1°}$
5	$\boxed{37.6°}$

41 ••• An astronomical radio telescope consists of two antennas separated by a distance of 200 m. Both antennas are tuned to the frequency of 20 MHz. The signals from each antenna are fed into a common amplifier, but one signal first passes through a phase selector that delays its phase by a chosen amount so that the telescope can "look" in different directions (Figure 16-32). When the phase delay is zero, plane radio waves that are incident vertically on the antennas produce signals that add constructively at the amplifier. What should the phase delay be so that signals coming from an angle $\theta = 10°$ with the vertical (in the plane formed by the vertical and the line joining the antennas) will add constructively at the amplifier? *Hint: Radio waves travel at* 3.00×10^8 m/s.

Picture the Problem We can calculate the required phase shift from the path difference and the wavelength of the radio waves using $\delta = 2\pi\frac{\Delta s}{\lambda}$.

Express the phase delay as a function of the path difference and the wavelength of the radio waves:

$$\delta = 2\pi\frac{\Delta s}{\lambda} \quad (1)$$

Find the wavelength of the radio waves:

$$\lambda = \frac{v}{f} = \frac{2.998\times10^8\ \text{m/s}}{20\times10^6\ \text{s}^{-1}} = 15.0\ \text{m}$$

Express the path difference for the signals coming from an angle θ with the vertical:

$$\Delta s = d\sin\theta$$

Substitute numerical values and evaluate Δs:

$$\Delta s = (200\ \text{m})\sin 10° = 34.73\ \text{m}\times\frac{1\lambda}{15.0\ \text{m}}$$
$$= 2.315\lambda = 2\lambda + 0.315\lambda$$

Substitute in equation (1) and evaluate δ:

$$\delta = 2\pi \frac{0.315\lambda}{\lambda} \approx \boxed{2.0\,\text{rad}}$$

Beats

42 • When two tuning forks are struck simultaneously, 4.0 beats per second are heard. The frequency of one fork is 500 Hz. (*a*) What are the possible values for the frequency of the other fork? (*b*) A piece of wax is placed on the 500-Hz fork to lower its frequency slightly. Explain how the measurement of the new beat frequency can be used to determine which of your answers to Part (*a*) is the correct frequency of the second fork.

Picture the Problem The beat frequency is the difference between the frequencies of the two tuning forks. Let $f_1 = 500$ Hz.

(*a*) Express the relationship between the beat frequency of the frequencies of the two tuning forks:

$$f_2 = f_1 \pm \Delta f$$
$$= 500\,\text{Hz} \pm 4\,\text{Hz}$$

Solving for f_2 yields:

$$f_2 = \boxed{504\,\text{Hz or } 496\,\text{Hz}}$$

(*b*) If the beat frequency increases, then $f_2 = 504$ Hz; if it decreases, $f_2 = 496$ Hz.

43 ••• **[SSM]** A stationary police radar gun emits microwaves at 5.00 GHz. When the gun is aimed at a car, it superimposes the transmitted and reflected waves. Because the frequencies of these two waves differ, beats are generated, with the speed of the car proportional to the beat frequency. The speed of the car, 83 mi/h, appears on the display of the radar gun. Assuming the car is moving along the line-of-sight of the police officer, and using the Doppler shift equations, (*a*) show that, for a fixed radar gun frequency, the beat frequency is proportional to the speed of the car. *HINT: Car speeds are tiny compared to the speed of light.* (*b*) What is the beat frequency in this case? (*c*) What is the calibration factor for this radar gun? That is, what is the beat frequency generated per mi/h of speed?

Picture the Problem The microwaves strike the speeding car at frequency f_r .This frequency will be less than f_s if the car is moving away from the radar gun and greater than f_s if the car is moving toward the radar gun. The frequency shift is given by Equation 15-42 (the low-speed, relative to light, approximation). The car then acts as a moving source emitting waves of frequency f_r. The radar gun detects waves of frequency f_r' that are either greater than or less than f_r depending on the direction the car is moving. The total frequency shift is the sum of the two frequency shifts.

(*a*) Express the frequency difference Δf as the sum of the frequency difference $\Delta f_1 = f_\text{r} - f_\text{s}$ and the frequency difference $\Delta f_2 = f_\text{r}' - f_\text{r}$:

$$\Delta f = \Delta f_1 + \Delta f_2 \qquad (1)$$

Using Equation 15-42, substitute for the frequency differences in equation (1):

$$\Delta f = -\frac{u}{c} f_\text{s} - \frac{u}{c} f_\text{r} = -\frac{u}{c}\left(f_\text{s} + f_\text{r}\right) \quad (2)$$

where $u = u_\text{s} \pm u_\text{r} = u_\text{r}$ is the speed of the source relative to the receiver.

Apply Equation 15-42 to Δf_1 to obtain:

$$\frac{\Delta f_1}{f_\text{s}} = \frac{f_\text{r} - f_\text{s}}{f_\text{s}} = -\frac{u_\text{r}}{c}$$

where we've used the minus sign because we know the frequency difference is a downshift.

Solving for f_r yields:

$$f_\text{r} = \left(1 - \frac{u_\text{r}}{c}\right) f_\text{s}$$

Substitute for f_r in equation (2) and simplify to obtain:

$$\begin{aligned}\Delta f &= -\frac{u_\text{r}}{c}\left(f_\text{s} + \left(1 - \frac{u_\text{r}}{c}\right) f_\text{s}\right)\\ &= -\frac{u_\text{r}}{c}\left(2 - \frac{u_\text{r}}{c}\right) f_\text{s}\\ &= -2\left(\frac{u_\text{r}}{c}\right) f_\text{s} + \left(\frac{u_\text{r}}{c}\right)^2 f_\text{s}\end{aligned}$$

Because $\frac{u_\text{r}}{c} << 1$:

$$\Delta f \approx -2 f_\text{s}\frac{u_\text{r}}{c} \Rightarrow \Delta f \propto \boxed{u_\text{r}}$$

(*b*) Substitute numerical values and evaluate $|\Delta f|$:

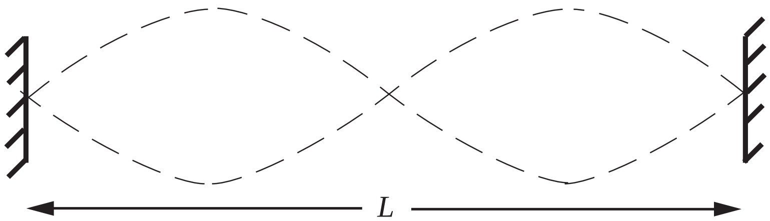
$$|\Delta f| \approx 2\left(5.00\times10^{9}\ \text{Hz}\right)\frac{\left(83\,\text{mi/h}\times\dfrac{0.4470\ \text{m/s}}{1\,\text{mi/h}}\right)}{2.998\times10^{8}\ \text{m/s}} = \boxed{1.2\ \text{kHz}}$$

(*c*) The calibration factor is $\frac{1.24\ \text{kHz}}{83\ \text{mi/h}} = \boxed{15\ \text{Hz/mi/h}}$.

Standing Waves

44 • A string fixed at both ends is 3.00 m long. It resonates in its second harmonic at a frequency of 60.0 Hz. What is the speed of transverse waves on the string?

Picture the Problem The pictorial representation shows the fixed string vibrating in its second harmonic. We can use $v = f\lambda$ to relate the second-harmonic frequency to the wavelength of the standing wave for the second harmonic.



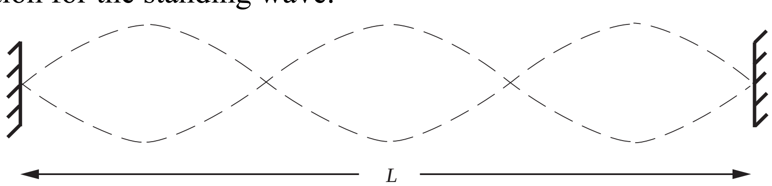
Relate the speed of transverse waves on the string to their frequency and wavelength: $v = f_2\lambda_2$

Express λ_2 in terms of the length L of the string: $\lambda_2 = L$

Substitute for λ_2 and evaluate v: $v = f_2L = (60.0\,\text{s}^{-1})(3.00\,\text{m}) = \boxed{180\,\text{m/s}}$

45 • A string 3.00 m long and fixed at both ends is vibrating in its third harmonic. The maximum displacement of any point on the string is 4.00 mm. The speed of transverse waves on this string is 50.0 m/s. (*a*) What are the wavelength and frequency of this standing wave? (*b*) Write the wave function for this standing wave.

Picture the Problem The pictorial representation shows the string fixed at both ends vibrating in its third harmonic. (*a*) We can find the wavelength of this standing wave from the standing-wave condition for a string fixed at both ends and its frequency from $v = f_3\lambda_3$. (*b*) We can use the wave function for a standing wave on a string fixed at both ends ($y_n(x,t) = A_n \sin k_n x \cos \omega_n t$) to write the wave function for the standing wave.



(*a*) Using the standing-wave condition for a string fixed at both ends, relate the length of the string to the wavelength of the harmonic mode in which it is vibrating:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots \Rightarrow \lambda_3 = \frac{2}{3}L$$

Substitute the numerical value of L and evaluate λ_3:

$$\lambda_3 = \frac{2}{3}(3.00\,\text{m}) = \boxed{2.00\,\text{m}}$$

Express the frequency of the third harmonic in terms of the speed of transverse waves on the string and their wavelength:

$$f_3 = \frac{v}{\lambda_3} = \frac{50.0\,\text{m/s}}{2.00\,\text{m}} = \boxed{25.0\,\text{Hz}}$$

(*b*) Write the equation for a standing wave, fixed at both ends, in its third harmonic:

$$y_3(x,t) = A_3 \sin k_3 x \cos \omega_3 t$$

Evaluate k_3:

$$k_3 = \frac{2\pi}{\lambda_3} = \frac{2\pi}{2.00\,\text{m}} = \pi\,\text{m}^{-1}$$

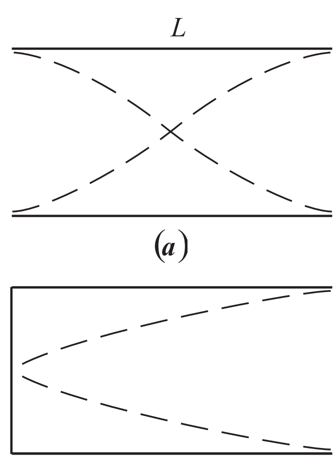
Evaluate ω_3:

$$\omega_3 = 2\pi f_3 = 2\pi\left(25.0\,\text{s}^{-1}\right) = 50.0\pi\,\text{s}^{-1}$$

Substitute to obtain: $\boxed{y_3(x,t) = (4.00\,\text{mm})\sin\left(\pi\,\text{m}^{-1}\right)x\cos\left(50.0\pi\,\text{s}^{-1}\right)t}$

46 • Calculate the fundamental frequency for an organ pipe, with an effective length equal to 10 m, that is (*a*) open at both ends and (*b*) stopped at one end.

Picture the Problem The first harmonic displacement-wave pattern in an organ pipe open at both ends and vibrating in its fundamental mode is represented in Part (*a*) of the diagram. Part (*b*) of the diagram shows the wave pattern corresponding to the fundamental frequency for a pipe of the same length L that is stopped at one end. We can relate the wavelength to the frequency of the fundamental modes using $v = f\lambda$.



(*a*)

(*b*)

(*a*) Express the dependence of the frequency of the fundamental mode of vibration in the open pipe on its wavelength:

$$f_{1,\text{open}}=\frac{v}{\lambda_{1,\text{open}}}$$

Relate the length of the open pipe to the wavelength of the fundamental mode:

$$\lambda_{1,\text{open}}=2L$$

Substitute and evaluate $f_{1,\text{open}}$:

$$f_{1,\text{open}}=\frac{v}{2L}=\frac{343\,\text{m/s}}{2(10\,\text{m})}=\boxed{17\,\text{Hz}}$$

(*b*) Express the dependence of the frequency of the fundamental mode of vibration in the closed pipe on its wavelength:

$$f_{1,\text{closed}}=\frac{v}{\lambda_{1,\text{closed}}}$$

Relate the length of the closed pipe to the wavelength of the fundamental mode:

$$\lambda_{1,\text{closed}}=4L$$

Substitute for $\lambda_{1,\text{closed}}$ to obtain:

$$f_{1,\text{closed}}=\frac{v}{4L}$$

Substitute numerical values and evaluate $f_{1,\text{closed}}$:

$$f_{1,\text{closed}}=\frac{343\,\text{m/s}}{4(10\,\text{m})}=\boxed{8.6\,\text{Hz}}$$

47 • [SSM] A 5.00-g, 1.40-m long flexible wire has a tension of 968 N and is fixed at both ends. (*a*) Find the speed of transverse waves on the wire. (*b*) Find the wavelength and frequency of the fundamental. (*c*) Find the frequencies of the second and third harmonics.

Picture the Problem We can find the speed of transverse waves on the wire using $v=\sqrt{F_\text{T}/\mu}$ and the wavelengths of any harmonic from $L=n\frac{\lambda_n}{2}$, where n = 1, 2, 3, … We can use $v=f\lambda$ to find the frequency of the fundamental. For a wire fixed at both ends, the higher harmonics are integer multiples of the first harmonic (fundamental).

(*a*) Relate the speed of transverse waves on the wire to the tension in the wire and its linear density:

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{F_T L}{m}}$$

Substitute numerical values and evaluate *v*:

$$v=\sqrt{\frac{968\,\text{N}}{(0.00500\,\text{kg})/(1.40\,\text{m})}}=\boxed{521\,\text{m/s}}$$

(*b*) Using the standing-wave condition for a wire fixed at both ends, relate the length of the wire to the wavelength of the harmonic mode in which it is vibrating:

$$L=n\frac{\lambda_n}{2},\ n=1,2,3,\ldots$$

Solve for λ_1:

$$\lambda_1=2L=2(1.40\,\text{m})=\boxed{2.80\,\text{m}}$$

Express the frequency of the first harmonic in terms of the speed and wavelength of the waves:

$$f_1=\frac{v}{\lambda_1}=\frac{521\,\text{m/s}}{2.80\,\text{m}}=\boxed{186\,\text{Hz}}$$

(*c*) Because, for a wire fixed at both ends, the higher harmonics are integer multiples of the first harmonic:

$$f_2=2\,f_1=2(186\,\text{Hz})=\boxed{372\,\text{Hz}}$$

and

$$f_3=3\,f_1=3(186\,\text{Hz})=\boxed{558\,\text{Hz}}$$

48 • A taut, 4.00-m-long rope has one end fixed and the other end free. (The free end is attached to a long, light string.) The speed of waves on the rope is 20.0 m/s. (*a*) Find the frequency of the fundamental. (*b*) Find the second harmonic. (*c*) Find the third harmonic.

Picture the Problem We can use Equation 16-13, $f_n=n\frac{v}{4L}=nf_1, n=1,3,5,\ldots,$ to find the resonance frequencies for a rope that is fixed at one end.

(*a*) Using the resonance-frequency condition for a rope fixed at one end, relate the resonance frequencies to the speed of the waves and the length of the rope:

$$f_n=n\frac{v}{4L}=nf_1, n=1,3,5,\ldots$$

Solving for f_1 yields:

$$f_1=\frac{20.0\,\text{m/s}}{4(4.00\,\text{m})}=\boxed{1.25\,\text{Hz}}$$

(*b*) Because this rope is fixed at just one end, it does not support a second harmonic.

(*c*) For the third harmonic, $n = 3$: $$f_3 = 3f_1 = 3(1.25\,\text{Hz}) = \boxed{3.75\,\text{Hz}}$$

49 • A steel piano wire without windings has a fundamental frequency of 200 Hz. When it is wound with copper wire, its linear mass density is doubled. What is its new fundamental frequency, assuming that the tension is unchanged?

Picture the Problem We can find the fundamental frequency of the piano wire using the general expression for the resonance frequencies of a wire fixed at both ends, $f_n = n\dfrac{v}{2L} = nf_1,\ n = 1, 2, 3, \ldots$, with $n = 1$. We can use $v = \sqrt{F_\text{T}/\mu}$ to express the frequencies of the fundamentals of the two wires in terms of their linear densities.

Relate the fundamental frequency of the piano wire to the speed of transverse waves on it and its linear density:

$$f_1 = \frac{v}{2L}$$

Express the dependence of the speed of transverse waves on the tension and linear density:

$$v = \sqrt{\frac{F_\text{T}}{\mu}}$$

Substituting for v yields:

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_\text{T}}{\mu}}$$

Doubling the linear density results in a new fundamental frequency f' given by:

$$f_1' = \frac{1}{2L}\sqrt{\frac{F_\text{T}}{2\mu}} = \frac{1}{\sqrt{2}}\left(\frac{1}{2L}\sqrt{\frac{F_\text{T}}{\mu}}\right) = \frac{1}{\sqrt{2}}f_1$$

Substitute the numerical value of f_1 to obtain:

$$f_1' = \frac{1}{\sqrt{2}}(200\,\text{Hz}) = \boxed{141\,\text{Hz}}$$

50 • What is the greatest length that an organ pipe can have in order that its fundamental note be in the audible range (20 to 20,000 Hz) if (*a*) the pipe is stopped at one end and (*b*) it is open at both ends?

Picture the Problem Because the frequency and wavelength of sound waves are inversely proportional, the greatest length of the organ pipe corresponds to the lowest frequency in the normal hearing range. We can relate wavelengths to the length of the pipes using the expressions for the resonance frequencies for pipes that are open at both ends and stopped at one end.

Find the wavelength of a 20-Hz note:

$$\lambda_{\text{max}} = \frac{v}{f_{\text{lowest}}} = \frac{343\,\text{m/s}}{20\,\text{s}^{-1}} = 17.2\,\text{m}$$

(*a*) Relate the length L of a stopped-at-one-end organ pipe to the wavelengths of its standing waves:

$$L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, ...$$

For $n = 1$:

$$L = \frac{\lambda_{\text{max}}}{4} = \frac{17.2\,\text{m}}{4} = \boxed{4.3\,\text{m}}$$

(*b*) Relate the length L of an open-at-both-ends organ pipe to the wavelengths of its standing waves:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, ...$$

For $n = 2$:

$$L = \frac{\lambda_{\text{max}}}{2} = \frac{17.2\,\text{m}}{2} = \boxed{8.6\,\text{m}}$$

51 •• **[SSM]** The wave function $y(x,t)$ for a certain standing wave on a string fixed at both ends is given by $y(x,t) = 4.20\sin(0.200x)\cos(300t)$, where y and x are in centimeters and t is in seconds. (*a*) A standing wave can be considered as the superposition of two traveling waves. What are the wavelength and frequency of the two traveling waves that make up the specified standing wave? (*b*) What is the speed of these waves on this string? (*c*) If the string is vibrating in its fourth harmonic, how long is it?

Picture the Problem We can find λ and f by comparing the given wave function to the general wave function for a string fixed at both ends. The speed of the waves can then be found from $v = f\lambda$. We can find the length of the string from its fourth harmonic wavelength.

(*a*) Using the wave function, relate k and λ:

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{2\pi}{0.200\,\text{cm}^{-1}} = 10\pi\,\text{cm} = \boxed{31.4\,\text{cm}}$$

Using the wave function, relate f and ω:

$$\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}$$

Substitute numerical values and evaluate f:

$$f = \frac{300\,\mathrm{s}^{-1}}{2\pi} = \boxed{47.7\,\mathrm{Hz}}$$

(b) The speed of the traveling waves is the ratio of their angular frequency and wave number:

$$v = \frac{\omega}{k} = \frac{300\,\mathrm{s}^{-1}}{0.200\,\mathrm{cm}^{-1}} = \boxed{15.0\,\mathrm{m/s}}$$

(c) Relate the length of the string to the wavelengths of its standing-wave patterns:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots$$

Solve for L when $n = 4$:

$$L = 2\lambda_4 = 2(31.4\,\mathrm{cm}) = \boxed{62.8\,\mathrm{cm}}$$

52 •• The wave function $y(x,t)$ for a certain standing wave on a string that is fixed at both ends is given by $y(x,t) = (0.0500\ \mathrm{m})\sin(2.50\ \mathrm{m}^{-1}\ x)\cos(500\ \mathrm{s}^{-1}\ t)$. A standing wave can be considered as the superposition of two traveling waves. (a) What are the speed and amplitude of the two traveling waves that result in the specified standing wave? (b) What is the distance between successive nodes on the string? (c) What is the shortest possible length of the string?

Picture the Problem (a) v, ω and k are related according to $\omega = kv$. ω and k can be found from the given wave function. (b) In a standing wave pattern, the nodes are separated by one-half wavelength. (c) Because there is a standing wave on the string, the shortest possible length is one-half the wavelength of the waves interfering to produce the standing wave.

(a) The speed of a traveling wave is the ratio of its angular frequency and wave number:

$$v = \frac{\omega}{k}$$

Substitute numerical values and evaluate v:

$$v = \frac{500\,\mathrm{s}^{-1}}{2.50\,\mathrm{m}^{-1}} = \boxed{200\,\mathrm{m/s}}$$

Express the amplitude of the standing wave A_{SW} in terms of the amplitude of the two traveling waves that result in the standing wave:

$$A_{\mathrm{SW}} = 2A \Rightarrow A = \tfrac{1}{2}A_{\mathrm{SW}}$$

Substitute for A_{SW} and evaluate A:

$$A = \tfrac{1}{2}(0.0500\,\text{m}) = \boxed{2.50\,\text{cm}}$$

(*b*) The distance between nodes is half the wavelength:

$$d = \tfrac{1}{2}\lambda = \tfrac{1}{2}\left(\frac{v}{f}\right) = \tfrac{1}{2}\left(\frac{2\pi v}{\omega}\right) = \frac{\pi v}{\omega}$$

Substitute numerical values and evaluate d:

$$d = \frac{\pi(200\,\text{m/s})}{500\,\text{s}^{-1}} = \boxed{1.26\,\text{m}}$$

(*c*) Because there is a standing wave on the string, the shortest possible length is:

$$L_{\text{min}} = \tfrac{1}{2}\lambda = \boxed{1.26\,\text{m}}$$

53 •• A 1.20-m-long pipe is stopped at one end. Near the open end, there is a loudspeaker that is driven by an audio oscillator whose frequency can be varied from 10.0 to 5000 Hz. (Neglect any end corrections.) (*a*) What is the lowest frequency of the oscillator that will produce resonance within the tube? (*b*) What is the highest frequency that will produce resonance? (*c*) How many different frequencies of the oscillator will produce resonance?

Picture the Problem (*a*) The lowest resonant frequency in this closed-at-one-end tube is its fundamental frequency. This frequency is related to its wavelength through $v = f_{\text{min}}\lambda_{\text{max}}$. (*b*) We can use the relationship between the nth harmonic and the fundamental frequency, $f_{\text{n}} = (2n+1)f_1, n = 1, 2, 3, \ldots$, to find the highest frequency less than or equal to 5000 Hz that will produce resonance.

(*a*) The wavelengths of the resonant frequencies in a stopped pipe of length L are given by:

$$\lambda_n = \frac{4L}{n}, n = 1, 3, 5, \ldots$$

λ_{max} corresponds to $n = 1$:

$$\lambda_{\text{max}} = 4L$$

Use $v = f_{\text{min}}\lambda_{\text{max}}$ to express f_{min}:

$$f_{\text{min}} = \frac{v}{\lambda_{\text{max}}} = \frac{v}{4L}$$

Substitute numerical values and evaluate f_{min}:

$$f_{\text{min}} = \frac{343\,\text{m/s}}{4(1.20\,\text{m})} = 71.46\,\text{Hz}$$
$$= \boxed{71.5\,\text{Hz}}$$

(*b*) Express the nth harmonic in terms of the fundamental frequency (first harmonic):

$$f_{\text{n}} = nf_1, n = 1, 3, 5, \ldots$$

To find the highest harmonic below 5000 Hz, let f_n = 5000 Hz:

$$5000\,\text{Hz} = n_{\text{highest}}(71.46\,\text{Hz})$$

Solve for n (an odd integer) to obtain (because there are only odd harmonics for a pipe stopped at one end):

$$n = 69$$

Evaluate f_{69}:

$$f_{69} = 69 f_1 = 69(71.46\,\text{Hz}) = \boxed{4.93\text{ kHz}}$$

(c) There are 69 odd harmonics higher than the fundamental frequency, so the total number resonant frequencies is $\boxed{35}$.

54 •• A 460-Hz tuning fork causes resonance in the tube depicted in Figure 16-33 when the length L of the air column above the water is 18.3 and 55.8 cm. (a) Find the speed of sound in air. (b) What is the end correction to adjust for the fact that the antinode does not occur exactly at the end of the open tube?

Picture the Problem Sound waves of frequency 460 Hz are excited in the tube, whose length L can be adjusted. Resonance occurs when the effective length of the tube $L_{\text{eff}} = L + \Delta L$ equals $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, $\frac{5}{4}\lambda$, and so on, where λ is the wavelength of the sound. Even though the pressure node is not exactly at the end of the tube, the wavelength can be found from the fact that the distance between water levels for successive resonances is half the wavelength. We can find the speed from $v = f\lambda$ and the end correction from the fact that, for the fundamental, $L_{\text{eff}} = \frac{1}{4}\lambda = L_1 + \Delta L$, where L_1 is the distance from the top of the tube to the location of the first resonance.

(a) Relate the speed of sound in air to its wavelength and the frequency of the tuning fork:

$$v = f\lambda$$

Using the fact that nodes are separated by one-half wavelength, find the wavelength of the sound waves:

$$\lambda = 2(55.8\,\text{cm} - 18.3\,\text{cm}) = 75.0\,\text{cm}$$

Substitute numerical values and evaluate v:

$$v = (460\,\text{s}^{-1})(0.750\,\text{m}) = \boxed{345\,\text{m/s}}$$

(*b*) Relate the end correction ΔL to the wavelength of the sound and effective length of the tube:

$$L_{\text{eff}} = \tfrac{1}{4}\lambda = L_1 + \Delta L \Rightarrow \Delta L = \tfrac{1}{4}\lambda - L_1$$

Substitute numerical values and evaluate ΔL:

$$\Delta L = \tfrac{1}{4}(75.0\,\text{cm}) - 18.3\,\text{cm} = \boxed{0.5\,\text{cm}}$$

55 •• [SSM] An organ pipe has a fundamental frequency of 440.0 Hz at 16.00ºC. What will be the fundamental frequency of the pipe if the temperature increases to 32.00ºC (assuming the length of the pipe remains constant)? Would it be better to construct organ pipes from a material that expands substantially as the temperature increases or, should the pipes be made of material that maintains the same length at all normal temperatures?

Picture the Problem We can use $v = f\lambda$ to express the fundamental frequency of the organ pipe in terms of the speed of sound and $v = \sqrt{\dfrac{\gamma RT}{M}}$ to relate the speed of sound and the fundamental frequency to the absolute temperature.

Express the fundamental frequency of the organ pipe in terms of the speed of sound:

$$f = \frac{v}{\lambda}$$

Relate the speed of sound to the temperature:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

where γ and R are constants, M is the molar mass, and T is the absolute temperature.

Substitute for v to obtain:

$$f = \frac{1}{\lambda}\sqrt{\frac{\gamma RT}{M}}$$

Using primed quantities to represent the higher temperature, express the new frequency as a function of T:

$$f' = \frac{1}{\lambda'}\sqrt{\frac{\gamma RT'}{M}}$$

As we have seen, λ is proportional to the length of the pipe. For the first question, we assume the length of the pipe does not change, so $\lambda = \lambda'$. Then the ratio of f' to f is:

$$\frac{f'}{f} = \sqrt{\frac{T'}{T}} \Rightarrow f' = f\sqrt{\frac{T'}{T}}$$

Evaluate f' for $T' = 305$ K and $T = 289$ K:

$$f' = f_{305\,\mathrm{K}} = f_{289\,\mathrm{K}}\sqrt{\frac{305\,\mathrm{K}}{289\,\mathrm{K}}}$$

$$= (440.0\,\mathrm{Hz})\sqrt{\frac{305\,\mathrm{K}}{289\,\mathrm{K}}} = \boxed{452\,\mathrm{Hz}}$$

Ideally, the pipe should expand so that v/L, where L is the length of the pipe, is independent of temperature.

56 •• According to theory, the end correction for a pipe is approximately $\Delta L = 0.3186D$, where D is the pipe diameter. Find the actual length of a pipe open at both ends that will produce a middle C (256 Hz) as its fundamental mode for pipes of diameter $D = 1.00$ cm, 10.0 cm, and 30.0 cm.

Picture the Problem We can express the wavelength of the fundamental in a pipe open at both ends in terms of the effective length of the pipe using $\lambda = 2L_{\mathrm{eff}} = 2(L + \Delta L)$, where L is the actual length of the pipe and $\lambda = v/f$. Solving these equations simultaneously will lead us to an expression for L as a function of D.

Express the wavelength of the fundamental in a pipe open at both ends in terms of the pipe's effective length L_{eff}:

$$\lambda = 2L_{\mathrm{eff}} = 2(L + \Delta L)$$

where L is its actual length.

Solve for L to obtain:

$$L = \frac{\lambda}{2} - \Delta L = \frac{\lambda}{2} - 0.3186D$$

Express the wavelength of middle C in terms of its frequency f and the speed of sound v:

$$\lambda = \frac{v}{f}$$

Substitute for λ to obtain:

$$L = \frac{v}{2f} - 0.3186D$$

Substitute numerical values to express L as a function of D:

$$L = \frac{343\,\mathrm{m/s}}{2(256\,\mathrm{s}^{-1})} - 0.3186D$$

$$= 0.670\,\mathrm{m} - 0.3186D$$

Evaluate L for $D = 1.00$ cm:

$$L = 0.670\,\mathrm{m} - 0.3186(0.0100\,\mathrm{m})$$

$$= \boxed{66.7\,\mathrm{cm}}$$

Evaluate L for $D = 10.0$ cm:

$$L = 0.670\,\text{m} - 0.3186(0.100\,\text{m}) = \boxed{63.8\,\text{cm}}$$

Evaluate L for $D = 30.0$ cm:

$$L = 0.670\,\text{m} - 0.3186(0.300\,\text{m}) = \boxed{57.4\,\text{cm}}$$

57 •• Assume a 40.0-cm-long violin string has a mass of 1.20 g and is vibrating in its fundamental mode at a frequency of 500 Hz. (*a*) What is the wavelength of the standing wave on the string? (*b*) What is the tension in the string? (*c*) Where should you place your finger to increase the fundamental frequency to 650 Hz?

Picture the Problem (*a*) We know that, when a string is vibrating in its fundamental mode, its ends are one-half wavelength apart. (*b*) We can use $v = f\lambda$ to express the fundamental frequency of the violin string in terms of the speed of waves in the string and $v = \sqrt{F_\text{T}/\mu}$ to relate the speed of waves in the string and the fundamental frequency to the tension in the string. (*c*) We can use this relationship between f and L, the length of the string, to find the length of string when it vibrates with a frequency of 650 Hz.

(*a*) Express the wavelength of the standing wave, vibrating in its fundamental mode, to the length L of the string:

$$\lambda = 2L = 2(40.0\,\text{cm}) = \boxed{80.0\,\text{cm}}$$

(*b*) Relate the speed of the waves combining to form the standing wave to its frequency and wavelength:

$$v = f\lambda$$

Express the speed of transverse waves as a function of the tension in the string:

$$v = \sqrt{\frac{F_\text{T}}{\mu}} = \sqrt{\frac{F_\text{T}L}{m}}$$

where m is the mass of the string and L is its length.

Substitute for v to obtain:

$$\sqrt{\frac{F_\text{T}L}{m}} = f\lambda \Rightarrow F_\text{T} = f^2\lambda^2\frac{m}{L}$$

Substitute numerical values and evaluate F:

$$F = \left(500\,\text{s}^{-1}\right)^2 (0.800\,\text{m})^2 \left(\frac{1.20\times10^{-3}\,\text{kg}}{0.400\,\text{m}}\right) = \boxed{480\,\text{N}}$$

(c) Using $v = f\lambda$ and assuming that the string is still vibrating in its fundamental mode, express its frequency in terms of its length:

$$f = \frac{v}{\lambda} = \frac{v}{2L} \Rightarrow L = \frac{v}{2f}$$

Letting primed quantities refer to a second length and frequency, express L' in terms of f':

$$L' = \frac{v}{2f'}$$

Express the ratio of L' to L and solve for L':

$$\frac{L'}{L} = \frac{f}{f'} \Rightarrow L' = \frac{f}{f'}L$$

Evaluate $L_{650\,\text{Hz}}$:

$$L_{650\,\text{Hz}} = \frac{500\,\text{Hz}}{650\,\text{Hz}} L_{500\,\text{Hz}} = \frac{500\,\text{Hz}}{650\,\text{Hz}}(40.0\,\text{cm}) = 30.77\,\text{cm}$$

You should place your finger 9.2 cm from the fixed end of the string.

58 •• The G string on a violin is 30.0 cm long. When played without fingering, it vibrates in its fundamental mode at a frequency of 196 Hz. The next higher notes on its C-major scale are A (220 Hz), B (247 Hz), C (262 Hz), and D (294 Hz). How far from the end of the string must a finger be placed to play each of these notes?

Picture the Problem Let f' represent the frequencies corresponding to the A, B, C, and D notes and $x(f')$ represent the distances from the end of the string that a finger must be placed to play each of these notes. Then, the distances at which the finger must be placed are given by $x(f') = L(f_G) - L(f')$.

Express the distances at which the finger must be placed in terms of the lengths of the G string and the frequencies f' of the A, B, C, and D notes:

$$x(f') = L(f_G) - L(f') \qquad (1)$$

Assuming that it vibrates in its fundamental mode, express the frequency of the G string in terms of its length:

$$f_G = \frac{v}{\lambda_G} = \frac{v}{2L_G} \Rightarrow L_G = \frac{v}{2f_G}$$

Letting primed quantities refer to the string lengths and frequencies of the A, B, C, and D notes, express L' in terms of f':

$$L' = \frac{v}{2f'}$$

Express the ratio of L' to L and solve for L':

$$\frac{L'}{L_G} = \frac{f_G}{f'} \Rightarrow L' = \frac{f_G}{f'} L_G$$

Evaluate $L' = L(f')$ for the notes A, B, C and D to complete the table:

Note	Frequency (Hz)	$L(f')$ (cm)
A	220	26.73
B	247	23.81
C	262	22.44
D	294	20.00

Use equation (1) to evaluate $x(f')$ and complete the table to the right:

Note	Frequency (Hz)	$L(f')$ (cm)	$x(f')$ (cm)
A	220	26.73	$\boxed{3.3}$
B	247	23.81	$\boxed{6.2}$
C	262	22.44	$\boxed{7.6}$
D	294	20.00	$\boxed{10}$

59 •• A string that has a linear mass density of 4.00×10^{-3} kg/m is under a tension of 360 N and is fixed at both ends. One of its resonance frequencies is 375 Hz. The next higher resonance frequency is 450 Hz. (*a*) What is the fundamental frequency of this string? (*b*) Which harmonics have the given frequencies? (*c*) What is the length of the string?

Picture the Problem We can use the fact that the resonance frequencies are multiples of the fundamental frequency to find both the fundamental frequency and the harmonic numbers corresponding to 375 Hz and 450 Hz. We can find the length of the string by relating it to the wavelength of the waves on it and the wavelength to the speed and frequency of the waves. The speed of the waves is, in turn, a function of the tension in the string and its linear density, both of which we are given.

(*a*) Express 375 Hz as an integer multiple of the fundamental frequency of the string:

$$nf_1 = 375\,\text{Hz} \qquad (1)$$

Express 450 Hz as an integer multiple of the fundamental frequency of the string:

$$(n+1)f_1 = 450\,\text{Hz} \qquad (2)$$

Solve equations (1) and (2) simultaneously for f_1:

$$f_1 = \boxed{75\,\text{Hz}}$$

(*b*) Substitute in equation (1) to obtain:

$n = 5 \Rightarrow$ the harmonics are the $\boxed{5^{\text{th}}}$ and $\boxed{6^{\text{th}}}$.

(*c*) Express the length of the string as a function of the speed of transverse waves on it and its fundamental frequency:

$$L = \frac{\lambda}{2} = \frac{v}{2f_1}$$

Express the speed of transverse waves on the string in terms of the tension in the string and its linear density:

$$v = \sqrt{\frac{F_\text{T}}{\mu}}$$

Substitute for v to obtain:

$$L = \frac{1}{2f_1}\sqrt{\frac{F_\text{T}}{\mu}}$$

Substitute numerical values and evaluate L:

$$L = \frac{1}{2(75\,\text{s}^{-1})}\sqrt{\frac{360\,\text{N}}{4.00\times10^{-3}\,\text{kg/m}}} = \boxed{2.0\,\text{m}}$$

60 •• A string fixed at both ends has successive resonances with wavelengths of 0.54 m for the *n*th harmonic and 0.48 m for the (*n* + 1)th harmonic. (*a*) Which harmonics are these? (*b*) What is the length of the string?

Picture the Problem (*a*) We can use the fact that the resonance frequencies are multiples of the fundamental frequency and are expressible in terms of the speed of the waves and their wavelengths to find the harmonic numbers corresponding to wavelengths of 0.54 m and 0.48 m. (*b*) We can find the length of the string by using the standing-wave condition for a string fixed at both ends.

(*a*) Express the frequency of the *n*th harmonic in terms of its wavelength:

$$nf_1 = \frac{v}{\lambda_n} = \frac{v}{0.54\,\text{m}}$$

Express the frequency of the (*n* + 1)th harmonic in terms of its wavelength:

$$(n+1)f_1 = \frac{v}{\lambda_{n+1}} = \frac{v}{0.48\,\text{m}}$$

Solve these equations simultaneously for *n*:

$n = 8 \Rightarrow$ the harmonics are the $\boxed{8^{\text{th}}}$ and $\boxed{9^{\text{th}}}$.

(*b*) Using the standing-wave condition, both ends fixed, relate the length of the string to the wavelength of its *n*th harmonic:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots$$

Evaluate *L* for the eighth harmonic:

$$L = 8\left(\frac{0.54\,\text{m}}{2}\right) = \boxed{2.2\,\text{m}}$$

61 •• **[SSM]** The strings of a violin are tuned to the tones G, D, A, and E, which are separated by a fifth from one another. That is, $f(\text{D}) = 1.5f(\text{G})$, $f(\text{A}) = 1.5f(\text{D}) = 440$ Hz, and $f(\text{E}) = 1.5f(\text{A})$. The distance between the bridge at the scroll and the bridge over the body, the two fixed points on each string, is 30.0 cm. The tension on the E string is 90.0 N. (*a*) What is the linear mass density of the E string? (*b*) To prevent distortion of the instrument over time, it is important that the tension on all strings be the same. Find the linear mass densities of the other strings.

Picture the Problem (*a*) The mass densities of the strings are related to the transverse wave speed and tension through $v = \sqrt{F_\text{T}/\mu}$. (*b*) We can use $v = f\lambda = 2fL$ to relate the frequencies of the violin strings to their lengths and linear densities.

(*a*) Relate the speed of transverse waves on a string to the tension in the string and solve for the string's linear density:

$$v = \sqrt{\frac{F_\text{T}}{\mu}} \Rightarrow \mu = \frac{F_\text{T}}{v^2}$$

Express the dependence of the speed of the transverse waves on their frequency and wavelength:

$$v = f_\text{E}\lambda = 2f_\text{E}L$$

Substituting for v gives:

$$\mu_E = \frac{F_{T,E}}{4f_E^2L^2}$$

Substitute numerical values and evaluate μ_E:

$$\mu_E = \frac{90.0\,\text{N}}{4\left[1.5\left(440\,\text{s}^{-1}\right)\right]^2\left(0.300\,\text{m}\right)^2} = 5.74\times10^{-4}\,\text{kg/m} = \boxed{0.574\,\text{g/m}}$$

(*b*) Evaluate μ_A:

$$\mu_A = \frac{90.0\,\text{N}}{4\left(440\,\text{s}^{-1}\right)^2\left(0.300\,\text{m}\right)^2} = 1.29\times10^{-3}\,\text{kg/m} = \boxed{1.29\,\text{g/m}}$$

Evaluate μ_D:

$$\mu_D = \frac{90.0\,\text{N}}{4\left(293\,\text{s}^{-1}\right)^2\left(0.300\,\text{m}\right)^2} = 2.91\times10^{-3}\,\text{kg/m} = \boxed{2.91\,\text{g/m}}$$

Evaluate μ_G:

$$\mu_G = \frac{90.0\,\text{N}}{4\left(195\,\text{s}^{-1}\right)^2\left(0.300\,\text{m}\right)^2} = \boxed{6.57\,\text{g/m}}$$

62 •• On a cello, like most other stringed instruments, the positioning of the fingers by the player determines the fundamental frequencies of the strings. Suppose that one of the strings on a cello is tuned to play a middle C (262 Hz) when played at its full length. By what fraction must that string be shortened in order to play a note that is the interval of a third higher (namely, an E, (330 Hz)? How about a fifth higher or a G (392 Hz)?

Picture the Problem The speed of a wave on a string is the product of its wavelength and frequency. In this problem, the standing waves are at the fundamental frequency; that is, the only nodes are at the ends of the strings and the wavelength is twice the length of the string. The speed of a wave on a string is determined by the tension in the string and its mass density. Pressing the string against the neck of the cello does not change the tension in the string appreciably and so we can ignore this very small increase in tension in our solution of the problem. Because the length of the string is half the wavelength of the standing wave on it we compare the lengths of the string for the various notes by comparing the wavelengths corresponding to these frequencies.

The wavelength of a standing wave on a string is given by:

$$\lambda = \frac{v}{f} = \frac{1}{f}\sqrt{\frac{F_T}{\mu}}$$

Express the wavelength corresponding to middle C:

$$\lambda_C = \frac{1}{f_C}\sqrt{\frac{F_T}{\mu}} \quad (1)$$

Express he wavelength corresponding to an E:

$$\lambda_E = \frac{1}{f_E}\sqrt{\frac{F_T}{\mu}} \quad (2)$$

Express he wavelength corresponding to a G:

$$\lambda_G = \frac{1}{f_G}\sqrt{\frac{F_T}{\mu}} \quad (3)$$

Divide equation (2) by equation (1) and simplify to obtain:

$$\frac{\lambda_E}{\lambda_C} = \frac{\frac{1}{f_E}\sqrt{\frac{F_T}{\mu}}}{\frac{1}{f_C}\sqrt{\frac{F_T}{\mu}}} = \frac{f_C}{f_E}$$

Substitute numerical values to obtain:

$$\frac{\lambda_E}{\lambda_C} = \frac{262\text{ Hz}}{330\text{ Hz}} = 0.79 \approx \frac{4}{5}$$

Dividing equation (3) by equation (1) and simplifying yields:

$$\frac{\lambda_G}{\lambda_C} = \frac{f_C}{f_G}$$

Substitute numerical values to obtain:

$$\frac{\lambda_E}{\lambda_C} = \frac{262\text{ Hz}}{392\text{ Hz}} = 0.67 \approx \frac{2}{3}$$

Thus, to play and E, the string is shortened by one-fifth and, to play a G, the string is shortened by one-third.

63 •• To tune your violin, you first tune the A string to the correct pitch of 440 Hz and then you bow both it and an adjoining string simultaneously, all the while listening for beats. While bowing the A and E strings, you hear a beat frequency of 3.00 Hz and note that the beat frequency increases as the tension on the E string is increased. (The E string is to be tuned to 660 Hz.) (*a*) Why are beats produced by these two strings when bowed simultaneously? (*b*) What is the frequency of the E string vibration when the beat frequency is 3.00 Hz?

Picture the Problem (*a*) and (*b*) Beat frequencies are heard when the strings are vibrating with slightly different frequencies. To understand the beat frequency heard when the A and E strings are bowed simultaneously, we need to consider the harmonics of both strings. In Part (*c*) we'll relate the tension in the string to the frequency of its vibration and set up a proportion involving the frequencies corresponding to the two tensions that we can solve for the tension when the E string is perfectly tuned.

(*a*) The two sounds produce a beat because the third harmonic of the A string is the same as the second harmonic of the E string, and the original frequency of the E string is slightly greater than 660 Hz. If $f_E = (660 + \Delta f)\text{Hz}$, a beat of $2\Delta f$ will be heard.

(*b*) Because f_{beat} increases with increasing tension, the frequency of the E string is greater than 660 Hz. Thus the frequency of the E string is:

$$f_E = 660\,\text{Hz} + \tfrac{1}{2}(3.00\,\text{Hz}) = 661.5\,\text{Hz} = \boxed{662\,\text{Hz}}$$

64 •• A 2.00-m-long string fixed at one end and free at the other (the free end is fastened to the end of a long, light thread) is vibrating in its third harmonic with a maximum amplitude of 3.00 cm and a frequency 100 Hz. (*a*) Write the wave function for this vibration. (*b*) Write a function for the kinetic energy of a segment of the string of length *dx*, at a point a distance *x* from the fixed end, as a function of time *t*. At what times is this kinetic energy maximum? What is the shape of the string at these times? (*c*) Find the maximum kinetic energy of the string by integrating your expression for Part (*b*) over the total length of the string.

Picture the Problem ω_3 and k_3 can be expressed in terms of the given information and then substituted to find the wave function for the third harmonic. We can use the time-derivative of this expression (the transverse speed) to express the kinetic energy of a segment of mass *dm* and length *dx* of the string. Integrating this expression will give us the maximum kinetic energy of the string in terms of its mass.

(*a*) Write the general form of the wave function for the third harmonic:

$$y_3(x,t) = A_3 \sin k_3 x \cos \omega_3 t \qquad (1)$$

ω_3 is given by:

$$\omega_3 = 2\pi f_3$$

Using the standing-wave condition for a string fixed at one end, relate the length of the string to its third harmonic wavelength:

$$L = 3\frac{\lambda_3}{4} \Rightarrow \lambda_3 = \frac{4}{3}L$$

k_3 is given by:

$$k_3 = \frac{2\pi}{\lambda_3} = \frac{2\pi}{\frac{4}{3}L} = \frac{3\pi}{2L}$$

Substituting for ω_3 and k_3 in equation (1) yields:

$$y_3(x,t) = A_3 \sin\frac{3\pi}{2L}x \cos 2\pi f_3 t$$

Substitute numerical values to obtain:

$$y_3(x,t) = \boxed{(0.0300\,\text{m})\sin\left[\left(\frac{3\pi}{4}\,\text{m}^{-1}\right)x\right]\cos\left(200\pi\,\text{s}^{-1}\right)t}$$

where x is measured from the fixed end and $0 \le \boldsymbol{x} \le 2.00\,\text{m}$

(*b*) Express the kinetic energy of a segment of string of mass dm:

$$dK = \tfrac{1}{2}dmv_y^2$$

Express the mass of the segment in terms of its length dx and the linear density of the string:

$$dm = \mu dx$$

Using our result in (*a*), evaluate v_y:

$$\begin{aligned} v_y &= \frac{\partial}{\partial t}\left[(0.0300\,\text{m})\sin\left[\left(\frac{3\pi}{4}\,\text{m}^{-1}\right)x\right]\cos\left(200\pi\,\text{s}^{-1}\right)t\right] \\ &= -\left(200\pi\,\text{s}^{-1}\right)(0.03\,\text{m})\sin\left[\left(\frac{3\pi}{4}\,\text{m}^{-1}\right)x\right]\sin\left(200\pi\,\text{s}^{-1}\right)t \\ &= -(6\pi\,\text{m/s})\sin\left[\left(\frac{3\pi}{4}\,\text{m}^{-1}\right)x\right]\sin\left(200\pi\,\text{s}^{-1}\right)t \end{aligned}$$

Substitute in the expression for dK to obtain:

$$dK = \boxed{\tfrac{1}{2}\left[(6\pi\,\text{m/s})\sin\left[\left(\frac{3\pi}{4}\,\text{m}^{-1}\right)x\right]\sin\left(200\pi\,\text{s}^{-1}\right)t\right]^2 \mu dx}$$

Express the condition on the time that makes dK a maximum:

$$\sin\left(200\pi\,\text{s}^{-1}\right)t = 1$$

or

$$\left(200\pi\,\text{s}^{-1}\right)t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

Solve for and evaluate t:

$$\begin{aligned} t &= \frac{1}{200\pi\,\text{s}^{-1}}\frac{\pi}{2}, \frac{1}{200\pi\,\text{s}^{-1}}\frac{3\pi}{2}, \ldots \\ &= \boxed{2.50\,\text{ms}, 7.50\,\text{ms}, \ldots} \end{aligned}$$

Because the string's maximum kinetic energy occurs when $y(x,t) = 0$. Thus the string is a straight line.

(*c*) Integrate dK from (*b*) over the length of the string to obtain:

$$K_{\max} = \int_0^L \tfrac{1}{2}\left[\omega A \sin kx \sin \omega t\right]^2 \mu dx$$
$$= \tfrac{1}{2}\mu\omega^2 A^2 \int_0^L \sin^2 kx dx$$
$$= \tfrac{1}{2}\mu\omega^2 A^2 \frac{1}{k}\left[\tfrac{1}{2}kx - \tfrac{1}{4}\sin 2kx\right]_0^L$$
$$= \tfrac{1}{4}m\omega^2 A^2$$

where m is the mass of the string.

Substitute numerical values and evaluate $K_{\max}$:

$$K_{\max} = \tfrac{1}{4}m\left(200\pi\,\mathrm{s}^{-1}\right)^2 (0.0300\,\mathrm{m})^2$$
$$= \boxed{(88.8\,\mathrm{J/kg})m}$$

65 •• [SSM] A commonly used physics experiment that examines resonances of transverse waves on a string is shown in Figure 16-34. A weight is attached to the end of a string draped over a pulley; the other end of the string is attached to a mechanical oscillator that moves up and down at a frequency f that remains fixed throughout the demonstration. The length L between the oscillator and the pulley is fixed, and the tension is equal to the gravitational force on the weight. For certain values of the tension, the string resonates. Assume the string does not stretch or shrink as the tension is varied. You are in charge of setting up this apparatus for a lecture demonstration. (*a*) Explain why only certain discrete values of the tension result in standing waves on the string. (*b*) Do you need to increase or decrease the tension to produce a standing wave with an additional antinode? Explain. (*c*) Prove your reasoning in Part (*b*) by showing that the values for the tension $F_{\mathrm{T}n}$ for the nth standing-wave mode are given by $F_{\mathrm{T}n} = 4L^2 f^2 \mu / n^2$, and thus that $F_{\mathrm{T}n}$ is inversely proportional to n^2. (*d*) For your particular setup to fit onto the lecture table, you chose $L = 1.00$ m, $f = 80.0$ Hz, and $\mu = 0.750$ g/m. Calculate how much tension is needed to produce each of the first three modes (standing waves) of the string.

Picture the Problem (*c*) and (*d*) We can equate the expression for the velocity of a wave on a string and the expression for the velocity of a wave in terms of its frequency and wavelength to obtain an expression for the weight that must be suspended from the end of the string in order to produce a given standing wave pattern. By using the condition on the wavelength that must be satisfied at resonance, we can express the weight on the end of the string in terms of μ, f, L, and an integer n and then evaluate this expression for $n = 1$, 2, and 3 for the first three standing wave patterns.

(*a*) Because the frequency is fixed, the wavelength depends only on the tension on the string. This is true because the only parameter that can affect the wave speed on the string is the tension on the string. The tension on the string is provided by the weight hanging from its end. Given that the length of the string is fixed, only

certain wavelengths can resonate on the string. Thus, because only certain wavelengths are allowed, only certain wave speeds will work. This, in turn, means that only certain tensions, and therefore weights, will work.

(*b*) Higher frequency modes on the same length of string results in shorter wavelengths. To accomplish this without changing frequency, you need to reduce the wave speed. This is accomplished by reducing the tension in the string. Because the tension is provided by the weight on the end of the string, you must reduce the weight.

(*c*) Express the velocity of a wave on the string in terms of the tension F_T in the string and its linear density μ:

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{w}{\mu}}$$

where w is the weight of the object suspended from the end of the string.

Express the wave speed in terms of its wavelength λ and frequency f:

$$v=f\lambda$$

Equate these expressions for v to obtain:

$$f\lambda=\sqrt{\frac{w}{\mu}}\Rightarrow w=\mu f^2\lambda^2$$

Express the condition on λ that corresponds to resonance:

$$\lambda=\frac{2L}{n}, n=1, 2, 3, \ldots$$

Substitute to obtain:

$$w_n=\mu f^2\left(\frac{2L}{n}\right)^2, n=1, 2, 3, \ldots$$

or

$$w_n=\boxed{\frac{4\mu f^2L^2}{n^2}}, n=1, 2, 3, \ldots$$

(*d*) Substitute numerical values for L, f, and μ to obtain:

$$w_n=\frac{4(0.750\,\text{g/m})(80.0\,\text{s}^{-1})^2(1.00\,\text{m})^2}{n^2}=\frac{19.20\,\text{N}}{n^2}$$

Evaluate w_n for $n = 1$:

$$w_1=\frac{19.20\,\text{N}}{(1)^2}=\boxed{19.2\,\text{N}}$$

Evaluate w_n for $n = 2$:

$$w_2=\frac{19.20\,\text{N}}{(2)^2}=\boxed{4.80\,\text{N}}$$

Evaluate w_n for $n = 3$:

$$w_3 = \frac{19.20\ \text{N}}{(3)^2} = \boxed{2.13\,\text{N}}$$

*Harmonic Analysis

66 • A guitar string is given a light pluck at its midpoint. A microphone on your computer detects the sound and a program on the computer determines that most of the subsequent sound consists of a 100-Hz tone accompanied by a bit of sound with a 300-Hz tone. What are the two dominant standing-wave modes on the string?.

Picture the Problem Plucking a string that is fixed at both ends in the middle results in an antinode at the midpoint of the string. Thus the primary modes of vibration will be those that have an antinode at the midpoint of the string. These modes are the odd harmonics and are given by $f_n = n\left(\frac{v}{2L}\right) = nf_1$ where $n = 1, 3, 5,$ …

The odd harmonics of a string that is plucked in the middle are given by:

$$f_n = n\left(\frac{v}{2L}\right) = nf_1 \text{ where } n = 1, 3, 5, \ldots$$

From the information given in the problem statement $f_1 = 100\ \text{Hz}$. Hence the first dominant standing wave mode is:

$$n = \boxed{1}$$

Noting that 300 Hz = 3(100 Hz), it follows that the second dominant standing wave mode is:

$$n = \boxed{3}$$

*Wave Packets

67 • **[SSM]** A tuning fork with natural frequency f_0 begins vibrating at time $t = 0$ and is stopped after a time interval Δt. The waveform of the sound at some later time is shown (Figure 16-35) as a function of x. Let N be an estimate of the number of cycles in this waveform. (*a*) If Δx is the length in space of this wave packet, what is the range in wave numbers Δk of the packet? (*b*) Estimate the average value of the wavelength λ in terms of N and Δx. (*c*) Estimate the average wave number k in terms of N and Δx. (*d*) If Δt is the time it takes the wave packet to pass a point in space, what is the range in angular frequencies $\Delta\omega$ of the packet? (*e*) Express f_0 in terms of N and Δt. (*f*) The number N is uncertain

by about ±1 cycle. Use Figure 16-35 to explain why. (*g*) Show that the uncertainty in the wave number due to the uncertainty in *N* is $2\pi/\Delta x$.

Picture the Problem We can approximate the duration of the pulse from the product of the number of cycles in the interval and the period of each cycle and the wavelength from the number of complete wavelengths in Δx. We can use its definition to find the average wave number from the average wavelength.

(*a*) Relate the duration of the pulse to the number of cycles in the interval and the period of each cycle:

$$\Delta t \approx NT = \boxed{\frac{N}{f_0}}$$

(*b*) There are about *N* complete wavelengths in Δx; hence:

$$\lambda \approx \boxed{\frac{\Delta x}{N}}$$

(*c*) Use its definition to express the wave number *k*:

$$k = \frac{2\pi}{\lambda} = \boxed{\frac{2\pi N}{\Delta x}}$$

(*d*) *N* is uncertain because the waveform dies out gradually rather than stopping abruptly at some time; hence, where the pulse starts and stops is not well defined.

(*e*) Using our result in part (*c*), express the uncertainty in *k*:

$$\Delta k = \frac{2\pi \Delta N}{\Delta x} = \boxed{\frac{2\pi}{\Delta x}}$$

because $\Delta N = \pm 1$.

General Problems

68 •• A 35-m-long string has a linear mass density of 0.0085 kg/m and is under a tension of 18 N. Find the frequencies of the lowest four harmonics if (*a*) the string is fixed at both ends, and if (*b*) the string is fixed at one end and free at the other. (That is, if the free end is attached to a long string of negligible mass.)

Picture the Problem We can use $v = f_n \lambda_n$ to express the resonance frequencies of the string in terms of their wavelengths and $L = n\frac{\lambda_n}{2}, n = 1, 2, 3, \ldots$ to relate the length of the string to the resonance wavelengths for a string fixed at both ends. Our strategy for part (*b*) will be the same … except that we'll use the standing-wave condition $L = n\frac{\lambda_n}{4}, n = 1, 3, 5, \ldots$ for strings with one end free.

(*a*) Relate the frequencies of the harmonics to their wavelengths and the speed of transverse waves on the string:

$$f_n = \frac{v}{\lambda_n}$$

Express the standing-wave condition for a string with both ends fixed:

$$L = n\frac{\lambda_n}{2},\ n = 1, 2, 3, \ldots \Rightarrow \lambda_n = \frac{2L}{n}$$

Substitute for λ_n to obtain:

$$f_n = n\frac{v}{2L}$$

Express the speed of the transverse waves as a function of the tension in the string:

$$v = \sqrt{\frac{F_\mathrm{T}}{\mu}}$$

Substituting for v in the expression for f_n yields:

$$f_n = n\frac{1}{2L}\sqrt{\frac{F_\mathrm{T}}{\mu}}$$

Substitute numerical values and simplify to obtain:

$$f_n = n\frac{1}{2(35\,\mathrm{m})}\sqrt{\frac{18\,\mathrm{N}}{0.0085\,\mathrm{kg/m}}} = n(0.657\,\mathrm{Hz})$$

Use this equation to calculate the 1st four harmonics:

$$f_1 = \boxed{0.66\,\mathrm{Hz}}$$

$$f_2 = 2(0.657\,\mathrm{Hz}) = \boxed{1.3\,\mathrm{Hz}}$$

$$f_3 = 3(0.657\,\mathrm{Hz}) = \boxed{2.0\,\mathrm{Hz}}$$

and

$$f_4 = 4(0.657\,\mathrm{Hz}) = \boxed{2.6\,\mathrm{Hz}}$$

(*b*) Express the standing-wave condition for a string fixed at one end:

$$L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, \ldots \Rightarrow \lambda_n = \frac{4L}{n}$$

The resonance frequencies equation becomes:

$$f_n = n\frac{1}{4L}\sqrt{\frac{F_\mathrm{T}}{\mu}} = n\frac{1}{4(35\,\mathrm{m})}\sqrt{\frac{18\,\mathrm{N}}{0.0085\,\mathrm{kg/m}}} = n(0.3287\,\mathrm{Hz})$$

Calculate the 1st four harmonics:

$$f_1 = \boxed{0.33\,\text{Hz}}$$

$$f_3 = 3(0.3287\,\text{Hz}) = \boxed{0.99\,\text{Hz}}$$

$$f_5 = 5(0.3287\,\text{Hz}) = \boxed{1.6\,\text{Hz}}$$

and

$$f_7 = 7(0.3287\,\text{Hz}) = \boxed{2.3\,\text{Hz}}$$

69 •• Working for a small gold mining company, you stumble across an abandoned mine shaft that, because of decaying wood shoring, looks too dangerous to explore in person. To measure its depth, you employ an audio oscillator of variable frequency. You determine that successive resonances are produced at frequencies of 63.58 and 89.25 Hz. Estimate the depth of the shaft.

Picture the Problem We'll model the shaft as a pipe of length L with one end open. We can relate the frequencies of the harmonics to their wavelengths and the speed of sound using $v = f_n\lambda_n$ and the depth of the mine shaft to the resonance wavelengths using the standing-wave condition for a pipe with one end open; $L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, \ldots$.

Relate the frequencies of the harmonics to their wavelengths and the speed of sound:

$$f_n = \frac{v}{\lambda_n}$$

Express the standing-wave condition for a pipe with one end open:

$$L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, \ldots \Rightarrow \lambda_n = \frac{4L}{n}$$

Substitute for λ_n to obtain:

$$f_n = n\frac{v}{4L}$$

For f_n = 63.58 Hz:

$$63.58\,\text{Hz} = n\frac{v}{4L}$$

For f_{n+2} = 89.25 Hz:

$$89.25\,\text{Hz} = (n+2)\frac{v}{4L}$$

Divide either of these equations by the other and solve for n to obtain:

$$n = 4.95 \approx 5$$

Substitute in the equation for $f_n = f_5$ = 63.58 Hz:

$$f_5 = \frac{5v}{4L}$$

Solve for and evaluate L:

$$L = \frac{5v}{4f_5} = \frac{5(343\,\text{m/s})}{4(63.58\,\text{s}^{-1})} = \boxed{6.74\,\text{m}}$$

70 •• A 5.00-m-long string that is fixed at one end and attached to a long string of negligible mass at the other end is vibrating in its fifth harmonic, which has a frequency of 400 Hz. The amplitude of the motion at each antinode is 3.00 cm. (*a*) What is the wavelength of this wave? (*b*) What is the wave number? (*c*) What is the angular frequency? (*d*) Write the wave function for this standing wave.

Picture the Problem We can use the standing-wave condition for a string with one end free to find the wavelength of the 5th harmonic and the definitions of the wave number and angular frequency to calculate these quantitities. We can then substitute in the wave function for a wave in the *n*th harmonic to find the wave function for this standing wave.

(*a*) Express the standing-wave condition for a string with one end free:

$$L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, \ldots$$

Solve for and evaluate λ_5:

$$\lambda_5 = \frac{4L}{5} = \frac{4(5.00\,\text{m})}{5} = \boxed{4.00\,\text{m}}$$

(*b*) Use its definition to calculate the wave number:

$$k_5 = \frac{2\pi}{\lambda_5} = \frac{2\pi}{4.00\,\text{m}} = \boxed{\frac{\pi}{2}\,\text{m}^{-1}}$$

(*c*) Using its definition, calculate the angular frequency:

$$\omega_5 = 2\pi f_5 = 2\pi(400\,\text{s}^{-1}) = \boxed{800\pi\,\text{s}^{-1}}$$

(*d*) Write the wave function for a standing wave in the *n*th harmonic:

$$y_n(x,t) = A\sin k_n x \cos \omega_n t$$

Substitute for A, k_5, and ω_5 to obtain:

$$y_5(x,t) = A\sin(k_5 x)\cos(\omega_5 t) = \boxed{(0.0300\,\text{m})\sin\left[\left(\frac{\pi}{2}\,\text{m}^{-1}\right)x\right]\cos\left(800\pi\,\text{s}^{-1}\right)t}$$

71 •• **[SSM]** The wave function for a standing wave on a string is described by $y(x,t) = (0.020)\sin(4\pi x)\cos(60\pi t)$, where y and x are in meters and t is in seconds. Determine the maximum displacement and maximum speed of a point on the string at (*a*) $x = 0.10$ m, (*b*) $x = 0.25$ m, (*c*) $x = 0.30$ m, and (*d*) $x = 0.50$ m.

Picture the Problem The coefficient of the factor containing the time dependence in the wave function is the maximum displacement of any point on the string. The time derivative of the wave function is the instantaneous speed of any point on the string and the coefficient of the factor containing the time dependence is the maximum speed of any point on the string.

Differentiate the wave function with respect to t to find the speed of any point on the string:

$$\begin{aligned} v_y &= \frac{\partial}{\partial t}\left[(0.020)\sin 4\pi x \cos 60\pi t\right] \\ &= -(0.020)(60\pi)\sin 4\pi x \sin 60\pi t \\ &= -1.2\pi \sin 4\pi x \sin 60\pi t \end{aligned}$$

(a) Referring to the wave function, express the maximum displacement of the standing wave:

$$y_{\text{max}}(x) = (0.020\,\text{m})\sin\left[\left(4\pi\,\text{m}^{-1}\right)x\right] \quad (1)$$

Evaluate equation (1) at $x = 0.10$ m:

$$\begin{aligned} y_{\text{max}}(0.10\,\text{m}) &= (0.020\,\text{m}) \\ &\quad \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.10\,\text{m})\right] \\ &= \boxed{1.9\,\text{cm}} \end{aligned}$$

Referring to the derivative of the wave function with respect to t, express the maximum speed of the standing wave:

$$v_{y,\text{max}}(x) = (1.2\pi\,\text{m/s})\sin\left[\left(4\pi\,\text{m}^{-1}\right)x\right] \quad (2)$$

Evaluate equation (2) at $x = 0.10$ m:

$$\begin{aligned} v_{y,\text{max}}(0.10\,\text{m}) &= (1.2\pi\,\text{m/s}) \\ &\quad \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.10\,\text{m})\right] \\ &= \boxed{3.6\,\text{m/s}} \end{aligned}$$

(b) Evaluate equation (1) at $x = 0.25$ m:

$$\begin{aligned} y_{\text{max}}(0.25\,\text{m}) &= (0.020\,\text{m}) \\ &\quad \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.25\,\text{m})\right] \\ &= \boxed{0} \end{aligned}$$

Evaluate equation (2) at $x = 0.25$ m:

$$\begin{aligned} v_{y,\text{max}}(0.25\,\text{m}) &= (1.2\pi\,\text{m/s}) \\ &\quad \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.25\,\text{m})\right] \\ &= \boxed{0} \end{aligned}$$

(*c*) Evaluate equation (1) at $x = 0.30$ m:

$$y_{\max}(0.30\,\text{m}) = (0.020\,\text{m}) \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.30\,\text{m})\right] = \boxed{-1.2\,\text{cm}}$$

Evaluate equation (2) at $x = 0.30$ m:

$$v_{y,\max}(0.30\,\text{m}) = \left|(1.2\pi\,\text{m/s}) \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.30\,\text{m})\right]\right| = \boxed{2.2\,\text{m/s}}$$

(*d*) Evaluate equation (1) at $x = 0.50$ m:

$$y_{\max}(0.50\,\text{m}) = (0.020\,\text{m}) \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.50\,\text{m})\right] = \boxed{0}$$

Evaluate equation (2) at $x = 0.50$ m:

$$v_{y,\max}(0.50\,\text{m}) = (1.2\pi\,\text{m/s}) \times \sin\left[\left(4\pi\,\text{m}^{-1}\right)(0.50\,\text{m})\right] = \boxed{0}$$

72 •• A 2.5-m-long string that has a mass of 0.10 kg is fixed at both ends and is under tension of 30 N. When the *n*th harmonic is excited, there is a node 0.50 m from one end. (*a*) What is *n*? (*b*) What are the frequencies of the first three harmonics of this string?

Picture the Problem In Part (*a*) we can use the standing-wave condition for a wire fixed at both ends and the fact that nodes are separated by one-half wavelength to find the harmonic number. In Part (*b*) we can relate the resonance frequencies to their wavelengths and the speed of transverse waves and express the speed of the transverse waves in terms of the tension in the wire and its linear density.

(*a*) Express the standing-wave condition for a wire fixed at both ends:

$$L = n\frac{\lambda_n}{2},\; n = 1, 2, 3, \ldots \qquad (1)$$

Solve for and evaluate λ_1:

$$\lambda_1 = 2L = 2(2.5\,\text{m}) = 5.0\,\text{m}$$

Relate the distance between nodes to the distance of the node closest to one end and solve for λ_n:

$$\tfrac{1}{2}\lambda_n = 0.50\,\text{m} \Rightarrow \lambda_n = 1.0\,\text{m}$$

Solving equation (1) for n yields:

$$n = \frac{2L}{\lambda_n}$$

Substitute for λ_n and L and evaluate n:

$$n = \frac{2(2.5\,\text{m})}{1.0\,\text{m}} = \boxed{5}$$

(*b*) Express the resonance frequencies in terms of the their wavelengths and the speed of transverse waves on the wire:

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{\lambda_1}$$

Relate the speed of transverse waves on the wire to the tension in the wire:

$$v = \sqrt{\frac{F_\text{T}}{\mu}}$$

Substitute for v and simplify to obtain:

$$\begin{aligned} f_n &= n\frac{1}{\lambda_1}\sqrt{\frac{F_\text{T}L}{m}} \\ &= n\frac{1}{5.0\,\text{m}}\sqrt{\frac{(30\,\text{N})(2.5\,\text{m})}{0.10\,\text{kg}}} \\ &= n(5.48\,\text{Hz}) \end{aligned}$$

Evaluate f_n for $n = 1$, 2, and 3:

$$f_1 = \boxed{5.5\,\text{Hz}}$$

$$f_2 = 2(5.48\,\text{Hz}) = \boxed{11\,\text{Hz}}$$

and

$$f_3 = 3(5.48\,\text{Hz}) = \boxed{16\,\text{Hz}}$$

73 •• An organ pipe is such that its fundamental frequency is 220 Hz. It is placed in an atmosphere of sulfur hexafluoride (SF_6) at the same temperature and pressure. The molar mass of air is 29.0×10^{-3} kg/mol and the molar mass of SF_6 is 146×10^{-3} kg/mol. What is the fundamental frequency of the organ pipe when it is in an atmosphere of SF_6?

Picture the Problem Because sulfur hexafluoride is a heavy gas, we should expect the fundamental frequency (which is a function of the speed of sound) to be lower in an atmosphere of SF_6 than it is in air under normal conditions. We can use the relationship between the speed of a sound wave and its wavelength and frequency, together with $v = \sqrt{\gamma RT/M}$, where M is the molar mass, to express the fundamental frequency of the organ pipe in SF_6 in terms of its fundamental frequency in air.

The fundamental frequency of the organ pipe in SF_6 is given by:

$$f_{1,\mathrm{SF_6}} = \frac{v_{\mathrm{SF_6}}}{\lambda_1} = \frac{\sqrt{\dfrac{\gamma RT}{M_{\mathrm{SF_6}}}}}{\lambda_1} = \sqrt{\frac{\gamma RT}{\lambda_1^2 M_{\mathrm{SF_6}}}}$$

The fundamental frequency of the organ pipe in air is given by:

$$f_{1,\mathrm{air}} = \frac{v_{\mathrm{air}}}{\lambda_1} = \frac{\sqrt{\dfrac{\gamma RT}{M_{\mathrm{air}}}}}{\lambda_1} = \sqrt{\frac{\gamma RT}{\lambda_1^2 M_{\mathrm{air}}}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{f_{1,\mathrm{SF_6}}}{f_{1,\mathrm{air}}} = \frac{\sqrt{\dfrac{\gamma RT}{\lambda_1^2 M_{\mathrm{SF_6}}}}}{\sqrt{\dfrac{\gamma RT}{\lambda_1^2 M_{\mathrm{air}}}}} = \sqrt{\frac{M_{\mathrm{air}}}{M_{\mathrm{SF_6}}}}$$

Solving for $f_{1,\mathrm{SF_6}}$ yields:

$$f_{1,\mathrm{SF_6}} = \sqrt{\frac{M_{\mathrm{air}}}{M_{\mathrm{SF_6}}}}\, f_{1,\mathrm{air}}$$

Substitute numerical values and evaluate $f_{1,\mathrm{SF_6}}$:

$$f_{1,\mathrm{SF_6}} = \sqrt{\frac{29.0\times10^{-3}\ \mathrm{kg/mol}}{146\times10^{-3}\ \mathrm{kg/mol}}}\left(220\ \mathrm{Hz}\right) = \boxed{98.0\ \mathrm{Hz}}$$

74 •• During a lecture demonstration of standing waves, one end of a string is attached to a device that vibrates at 60 Hz and produces transverse waves of that frequency on the string. The other end of the string passes over a pulley, and the tension is varied by attaching weights to that end. The string has approximate nodes next to both the vibrating device and the pulley. (*a*) If the string has a linear mass density of 8.0 g/m and is 2.5 m long (from the vibrating device to the pulley), what must be the tension for the string to vibrate in its fundamental mode? (*b*) Find the tension necessary for the string to vibrate in its second, third, and fourth harmonic.

Picture the Problem We can use $v = \sqrt{F_{\mathrm{T}}/\mu}$ to express F as a function of v and $v = f\lambda$ to relate v to the frequency and wavelength of the string's fundamental mode. Because, for a string fixed at both ends, $f_n = nf_1$, we can extend our result in Part (*a*) to Part (*b*).

(*a*) Relate the speed of the transverse waves on the string to the tension in it:

$$v = \sqrt{\frac{F_{\mathrm{T}}}{\mu}} \Rightarrow F_{\mathrm{T}} = \mu v^2 \qquad (1)$$

Relate the speed of the transverse waves on the string to their frequency and wavelength:

$$v = f_1\lambda_1$$

Express the wavelength of the fundamental mode to the length of the string:

$$\lambda_1 = 2L$$

Substitute for λ_1 to obtain:

$$v = 2fL$$

Substitute for v in equation (1) to obtain:

$$F_T = 4f^2L^2\mu \qquad (2)$$

Substitute numerical values and evaluate F:

$$F_T = 4\left(60\,\text{s}^{-1}\right)^2(2.5\,\text{m})^2\left(8.0\times10^{-3}\,\text{kg/m}\right) = \boxed{0.72\,\text{kN}}$$

(*b*) For the *n*th harmonic, equation (2) becomes:

$$F_n = f_n^2L^2\mu = n^2f_1^2L^2\mu = n^2(720\,\text{N})$$

Evaluate this expression for $n = 2$, 3, and 4:

$$F_2 = 4(720\,\text{N}) = \boxed{2.9\,\text{kN}}$$

$$F_3 = 9(720\,\text{N}) = \boxed{6.5\,\text{kN}}$$

and

$$F_4 = 16(720\,\text{N}) = \boxed{12\,\text{kN}}$$

75 •• [SSM] Three successive resonance frequencies in an organ pipe are 1310, 1834, and 2358 Hz. (*a*) Is the pipe closed at one end or open at both ends? (*b*) What is the fundamental frequency? (*c*) What is the effective length of the pipe?

Picture the Problem (*a*) We can use the conditions $\Delta f = f_1$ and $f_n = nf_1$, where n is an integer, which must be satisfied if the pipe is open at both ends to decide whether the pipe is closed at one end or open at both ends. (*b*) Once we have decided this question, we can use the condition relating Δf and the fundamental frequency to determine the latter. In Part (*c*) we can use the standing-wave condition for the appropriate pipe to relate its length to its resonance wavelengths.

(*a*) Express the conditions on the frequencies for a pipe that is open at both ends:

$$\Delta f = f_1$$

and

$$f_n = nf_1$$

Evaluate $\Delta f = f_1$: $\Delta f = 1834\,\text{Hz} - 1310\,\text{Hz} = 524\,\text{Hz}$

Using the 2nd condition, find n:

$$n = \frac{f_n}{f_1} = \frac{1310\,\text{Hz}}{524\,\text{Hz}} = 2.5$$

$\boxed{\text{The pipe is closed at one end.}}$

(*b*) Express the condition on the frequencies for a pipe that is open at both ends:

$$\Delta f = 2f_1 \Rightarrow f_1 = \tfrac{1}{2}\Delta f$$

Substitute numerical values and evaluate f_1:

$$f_1 = \tfrac{1}{2}(524\,\text{Hz}) = \boxed{262\,\text{Hz}}$$

(*c*) Using the standing-wave condition for a pipe open at one end, relate the effective length of the pipe to its resonance wavelengths:

$$L = n\frac{\lambda_n}{4},\ n = 1, 3, 5, \ldots$$

For $n = 1$ we have:

$$\lambda_1 = \frac{v}{f_1} \text{ and } L = \frac{\lambda_1}{4} = \frac{v}{4f_1}$$

Substitute numerical values and evaluate L:

$$L = \frac{343\,\text{m/s}}{4\left(262\,\text{s}^{-1}\right)} = \boxed{32.7\,\text{cm}}$$

76 •• During an experiment studying the speed of sound in air using an audio oscillator and a tube open at one end and stopped at the other, a particular resonant frequency is found to have nodes roughly 6.94 cm apart. The oscillator's frequency is increased, and the next resonant frequency found has nodes 5.40 cm apart. (*a*) What are the two resonant frequencies? (*b*) What is the fundamental frequency? (*c*) Which harmonics are these two modes? The speed of sound is 343 m/s.

Picture the Problem (*a*) Because adjacent nodes are separated by one-half wavelength, we can find the frequencies from our knowledge of the speed of sound in air the wavelengths of the standing-wave patterns. (*b*) These frequencies are consecutive odd-multiples (the tube is half-open) of the fundamental frequency. (*c*) The frequencies found in Part (*a*) are integer multiples of the fundamental frequency.

(*a*) The frequencies are determined by the speed of sound in air and by the wavelengths of the standing-wave patterns:

$$f_n = \frac{v}{\lambda_n}$$

Because the nodes are a half-wavelength apart:

$$\lambda_n = 2d_{\text{node-to-node}}$$

Substitute for λ_n to obtain:

$$f_n = \frac{v}{2d_{\text{node-to-node}}}$$

Substitute numerical values and evaluate the two frequencies:

$$f_n = \frac{343\text{ m/s}}{2(6.94\text{ cm})} = 2471\text{ Hz} = \boxed{2.47\text{ kHz}}$$

and

$$f_{n'} = \frac{343\text{ m/s}}{2(5.40\text{ cm})} = 3176\text{ Hz} = \boxed{3.18\text{ kHz}}$$

(*b*) Assuming the two frequencies are adjacent resonant frequencies, they are odd multiples (because the tube is half-open) of the fundamental frequency:

$$nf_1 = 2471\text{ Hz}$$

and

$$(n+2)f_1 = 3176\text{ Hz}$$

Subtract the first of these equations from the second to obtain:

$$(n+2)f_1 - nf_1 = 3176\text{ Hz} - 2471\text{ Hz} = 705\text{ Hz}$$

Solving for f_1 yields:

$$f_1 = \tfrac{1}{2}(705\text{ Hz}) = \boxed{353\text{ Hz}}$$

(*c*) Divide f_n and $f_{n'}$ by f_1 to obtain:

$$n = \frac{2471\text{ Hz}}{f_1} = \frac{2471\text{ Hz}}{353\text{ Hz}} = \boxed{7}$$

and

$$n' = \frac{3176\text{ Hz}}{f_1} = \frac{3176\text{ Hz}}{353\text{ Hz}} = \boxed{9}$$

77 •• A standing wave on a rope is represented by the wave function $y(x,t) = (0.020\ ) \sin(\tfrac{1}{2}\pi x) \cos(40\pi t)$, where x and y are in meters and t is in seconds. (*a*) Write wave functions for two traveling waves that, when

superimposed, will produce this standing-wave pattern. (*b*) What is the distance between the nodes of the standing wave? (*c*) What is the maximum speed of the rope at $x = 1.0$ m? (*d*) What is the maximum acceleration of the rope at $x = 1.0$ m?

Picture the Problem We know that the superimposed traveling waves have the same wave number and angular frequency as the standing-wave function, have equal amplitudes that are half that of the standing-wave function, and travel in opposite directions. From inspection of the standing-wave function we note that $k = \frac{1}{2}\pi\,\text{m}^{-1}$ and $\omega = 40\pi\,\text{s}^{-1}$. We can express the speed of a segment of the rope by differentiating the standing-wave function with respect to time and the acceleration by differentiating the velocity function with respect to time.

(*a*) Write the wave function for the wave traveling in the +*x* direction:

$$y_1(x,t) = \boxed{(0.010\,\text{m})\sin\left[\left(\frac{\pi}{2}\text{m}^{-1}\right)x - \left(40\pi\,\text{s}^{-1}\right)t\right]}$$

Write the wave function for the wave traveling in the −*x* direction:

$$y_2(x,t) = \boxed{(0.010\,\text{m})\sin\left[\left(\frac{\pi}{2}\text{m}^{-1}\right)x + \left(40\pi\,\text{s}^{-1}\right)t\right]}$$

(*b*) Express the distance *d* between adjacent nodes in terms of the wavelength of the standing wave:

$$d = \tfrac{1}{2}\lambda$$

Use the wave number to find the wavelength:

$$k = \tfrac{1}{2}\pi\,\text{m}^{-1} = \frac{2\pi}{\lambda} \text{ and } \lambda = 4.00\,\text{m}$$

Substitute for λ and evaluate *d*:

$$d = \tfrac{1}{2}(4.00\,\text{m}) = \boxed{2.00\,\text{m}}$$

(*c*) Differentiate the given wave function with respect to *t* to express the speed of any segment of the rope:

$$\begin{aligned} v_y(x,t) &= \frac{\partial}{\partial t}\left[(0.020\,\text{m})\sin\left(\frac{\pi}{2}\text{m}^{-1}\right)x\cos\left(40\pi\,\text{s}^{-1}\right)t\right] \\ &= -(0.80\pi\,\text{m/s})\sin\left(\frac{\pi}{2}\text{m}^{-1}\right)x\sin\left(40\pi\,\text{s}^{-1}\right)t \end{aligned}$$

Evaluate $\left|v_y(1.0\,\text{m},t)\right|$:

$$\begin{aligned}\left|v_y(1.0\,\text{m},t)\right| &= \left|-(0.80\pi\,\text{m/s})\sin\left(\frac{\pi}{2}\,\text{m}^{-1}\right)(1\,\text{m})\sin\left(40\pi\,\text{s}^{-1}\right)t\right| \\ &= (0.80\pi\,\text{m/s})\sin\left(40\pi\,\text{s}^{-1}\right)t \\ &= (2.5\,\text{m/s})\sin\left(40\pi\,\text{s}^{-1}\right)t\end{aligned}$$

The maximum speed of the rope at $x = 1.0$ m occurs when $\sin\left(40\pi\,\text{s}^{-1}\right)t = 1$:

$$v_{\text{max}} = \boxed{2.5\,\text{m/s}}$$

(*d*) Differentiate $v_y(x,t)$ with respect to time to obtain $a_y(x,t)$:

$$\begin{aligned}a_y(x,t) &= \frac{\partial}{\partial t}\left[-(0.80\pi\,\text{m/s})\sin\left(\frac{\pi}{2}\,\text{m}^{-1}\right)x\sin\left(40\pi\,\text{s}^{-1}\right)t\right] \\ &= -\left(32\pi^2\,\text{m/s}^2\right)\sin\left(\frac{\pi}{2}\,\text{m}^{-1}\right)x\cos\left(40\pi\,\text{s}^{-1}\right)t\end{aligned}$$

Evaluate $a_y(1.0\,\text{m},t)$:

$$\begin{aligned}a_y(1.0\,\text{m},t) &= \pm\left(32\pi^2\,\text{m/s}^2\right)\sin\left(\frac{\pi}{2}\,\text{m}^{-1}\right)(1\,\text{m})\cos\left(40\pi\,\text{s}^{-1}\right)t \\ &= \pm\left(32\pi^2\,\text{m/s}^2\right)\cos\left(40\pi\,\text{s}^{-1}\right)t \\ &= \pm\left(0.32\,\text{km/s}^2\right)\cos\left(40\pi\,\text{s}^{-1}\right)t\end{aligned}$$

The maximum acceleration of the rope at $x = 1.0$ m occurs when $\cos\left(40\pi\,\text{s}^{-1}\right)t = 1$:

$$v_{\text{max}} = \boxed{\pm 0.32\,\text{km/s}^2}$$

78 •• Two traveling wave pulses on a string are represented by the wave functions $y_1(x,t) = \dfrac{0.020}{2.0 + (x - 2.0t)^2}$ and $y_2(x,t) = \dfrac{-0.020}{2.0 + (x + 2.0t)^2}$, where x is in meters and t is in seconds. (*a*) Using a **spreadsheet program** or **graphing calculator**, make a graph of each wave function separately as a function of x at $t = 0$ and again at $t = 1.0$ s and describe the behavior of each as time increases. For each graph make your plot for –5.0 m < x < +5.0 m. (*b*) Graph the resultant wave function at $t = -1.0$ s, at $t = 0.0$ s and at $t = 1.0$ s.

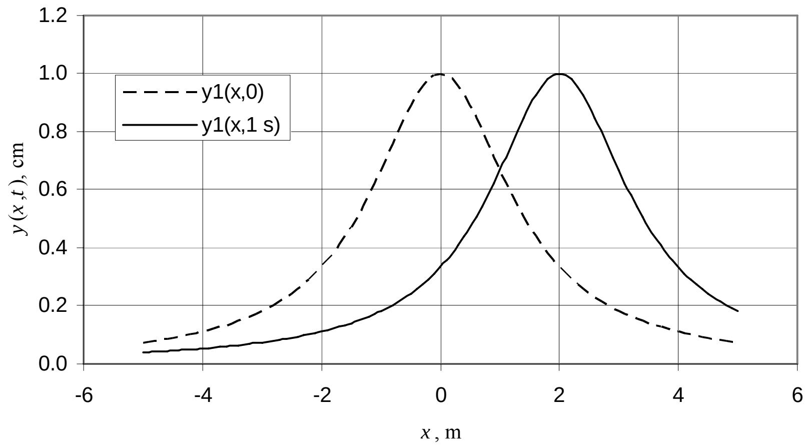
Picture the Problem We'll use a spreadsheet program to graph the wave functions and their sum as functions of x at $t = 0$ and at $t = 1.0$ s. In (*b*) we can add the wave functions algebraically to find the result wave function at $t = 0$ and at $t = 1.0$ s.

(*a*) and (*b*) Part of the spreadsheet program to calculate values for $y_1(x,t)$ and $y_2(x,t)$ in the interval $-5.0\text{ m} < x < +5.0\text{ m}$ for the given times follows. The constants and cell formulas used are shown in the table.

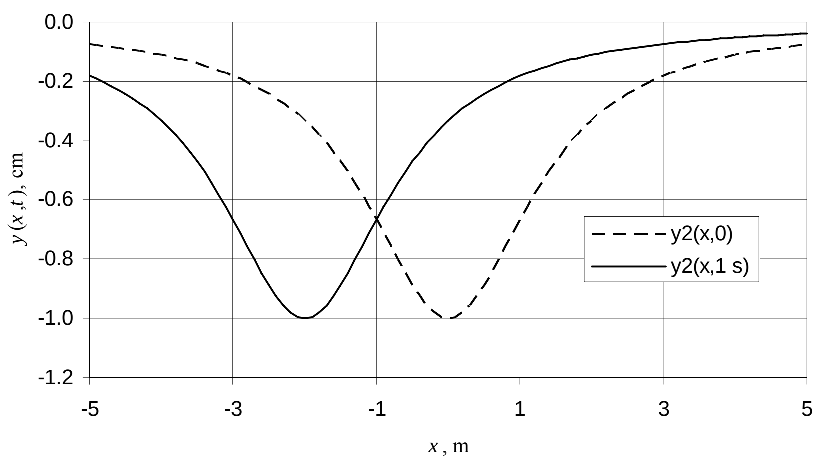
Cell	Content/Formula	Algebraic Form
A5	−5.0	x
A6	A5+0.1	$x+\Delta x$
B1	−1	
B2	0	
B3	1	
B5	0.02/(2+(A5−2*$B$2)^2)	$y_1(x,0)$
C5	0.02/(2+(A5−2*$B$3)^2)	$y_1(x,1.0\text{ s})$
D5	−0.02/(2+(A5+2*$B$2)^2)	$y_2(x,0)$
E5	−0.02/(2+(A5+2*$B$3)^2)	$y_2(x,1.0\text{ s})$
F5	0.02/(2+(A5−2*$B$1)^2) −0.02/(2+(A5+2*$B$1)^2)	$y_1(x,-1)+y_2(x,-1)$
G5	B5+D5	$y_1(x,0)+y_2(x,0)$
H5	0.02/(2+(A5−2*$B$3)^2) −0.02/(2+(A5+2*$B$3)^2)	$y_1(x,1)+y_2(x,1)$

	A	B	C	D	E
1	$t=$	−1	s		
2	$t=$	0			
3	$t=$	1	s		
4	x	$y_1(x,0)$	$y_1(x,1\text{ s})$	$y_2(x,0)$	$y_2(x,1\text{ s})$
5	−5.0	0.001	0.000	−0.001	−0.002
6	−4.9	0.001	0.000	−0.001	−0.002
7	−4.8	0.001	0.000	−0.001	−0.002
8	−4.7	0.001	0.000	−0.001	−0.002
9	−4.6	0.001	0.000	−0.001	−0.002
10	−4.5	0.001	0.000	−0.001	−0.002
110	4.7	0.001	0.002	−0.001	0.000
111	4.8	0.001	0.002	−0.001	0.000
112	4.9	0.001	0.002	−0.001	0.000
113	5.0	0.001	0.002	−0.001	0.000

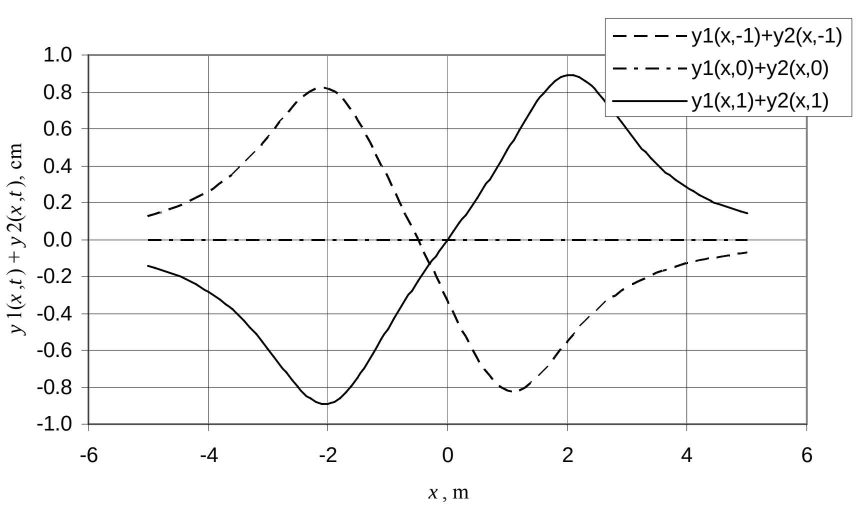
The following graph of $y_1(x,t)$ shows it traveling from left to right.



The following graph of $y_2(x,t)$ shows it traveling from right to left.



(*b*) The following graph shows the resultant wave function at $t = -1.0$ s, at $t = 0.0$ s and at $t = 1.0$ s.

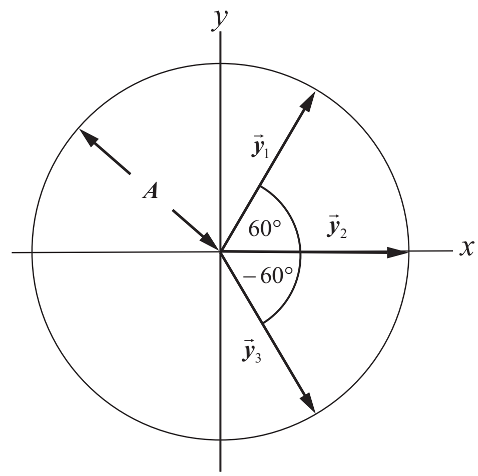


79 ••• Three waves that have the same frequency, wavelength, and amplitude are traveling along the x axis. The three waves are described by the following wave functions: $y_1(x,t) = (5.00\text{ cm})\sin\left(kx - \omega t - \frac{\pi}{3}\right)$,

$y_2(x,t) = (5.00\text{ cm})\sin(kx - \omega t)$, and $y_3(x,t) = (5.00\text{ cm})\sin\left(kx - \omega t + \frac{\pi}{3}\right)$,

where x is in meters and t is in seconds. The resultant wave function is given by $y(x,t) = A\sin(kx - \omega t + \delta)$. What are the values of A and δ?

Picture the Problem A harmonic function can be represented by a vector rotating at the angular frequency ω (see Chapter 14). The simplest way to do this problem is to use that representation. The vectors, of equal magnitude, are shown in the diagram. We can find the resultant wave function by finding the magnitude and direction of the resultant vector.



From the diagram it is evident that:

$$\sum y_y = 0$$

Find the sum of the x components of the vectors:

$$\sum y_x = (5.00\text{ cm})\cos 60° + (5.00\text{ cm})\cos 60° = 10.0\text{ cm}$$

Relate the magnitude of the resultant vector to the sum of its x and y components:

$$A = \sqrt{\left(\sum y_x\right)^2 + \left(\sum y_y\right)^2} = \sqrt{(10.0\text{ cm})^2 + (0)^2} = \boxed{10.0\text{ cm}}$$

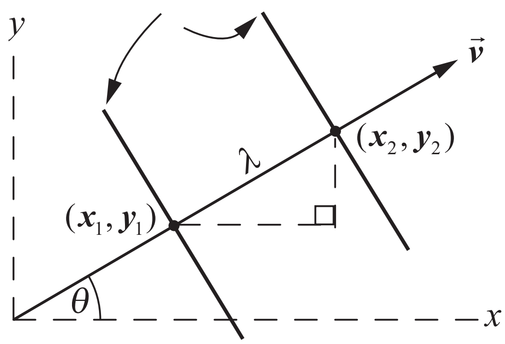
The direction of the resultant vector is δ:

$$\delta = \tan^{-1}\left(\frac{\sum y_y}{\sum y_x}\right) = \tan^{-1}\left(\frac{0}{10.0\text{ cm}}\right) = \boxed{0}$$

80 •• A harmonic pressure wave produced by a distant source is traveling through your vicinity, and the wave fronts that travel through your vicinity are vertical planes. Let the $+x$ direction be to the east and the $+y$ direction be toward the north. The wave function for the wave is $p(x, y, t) = A\cos(k_x x + k_y y - \omega t)$.

Show that the direction in which the wave is traveling makes an angle $\theta = \tan^{-1}(k_y/k_x)$ with the $+x$ direction and that the wave speed v is given by $v = \omega \Big/ \sqrt{k_x^2 + k_y^2}$.

Picture the Problem The diagram shows a snapshot of a two dimensional plane wave propagating at an angle θ with respect to the $+x$ axis. The view is along the $-z$ axis. The wave itself moves in a direction perpendicular to the wavefront. Choose two points (x_1, y_1) and (x_2, y_2) that have a separation of exactly 1 wavelength along the wave propagation direction. Let the snapshot be taken at some fixed time t.



Applying the Pythagorean theorem to the right triangle in the diagram yields:

$$\lambda = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (1)$$

Because k_x and k_y are the x and y components of $\vec{k}$:

$$k_x = k\cos\theta$$

and

$$k_y = k\sin\theta$$

Divide the second of these equations by the first to obtain:

$$\tan\theta = \frac{k_y}{k_x} \Rightarrow \theta = \boxed{\tan^{-1}\left(\frac{k_y}{k_x}\right)}$$

The wave speed v is the ratio of the angular frequency ω and wave number k:

$$v = \frac{\omega}{k} = \omega\frac{\lambda}{2\pi} \quad (2)$$

At time t, the phase at point 1 is $k_x x_1 + k_y y_1 - \omega t$ and the phase at point 2 is $k_x x_2 + k_y y_2 - \omega t$, so the phase difference is:

$$\delta = k_x x_2 + k_y y_2 - \omega t - (k_x x_1 + k_y y_1 - \omega t) = k_x \Delta x + k_y \Delta y$$

or, because the waves are separated by one wavelength, $\delta = 2\pi$ and we have $k_x \Delta x + k_y \Delta y = 2\pi$

Because $\tan\theta$ is also given by $\dfrac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x} = \frac{k_y}{k_x} \Rightarrow \Delta y = \frac{k_y}{k_x}\Delta x$$

Substituting for Δy yields:

$$k_x \Delta x + k_y \frac{k_y}{k_x}\Delta x = 2\pi$$

Solve for Δx to obtain:

$$\Delta x = \frac{2\pi k_x}{k_x^2 + k_y^2}$$

Similarly:

$$\Delta y = \frac{2\pi k_y}{k_x^2 + k_y^2}$$

Substitute for Δx and Δy in equation (1) to obtain:

$$\lambda = \sqrt{\left(\frac{2\pi k_x}{k_x^2 + k_y^2}\right)^2 + \left(\frac{2\pi k_y}{k_x^2 + k_y^2}\right)^2} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2}}$$

Substituting for λ in equation (2) yields:

$$v = \frac{\omega}{2\pi}\frac{2\pi}{\sqrt{k_x^2 + k_y^2}} = \boxed{\frac{\omega}{\sqrt{k_x^2 + k_y^2}}}$$

81 •• [SSM] The speed of sound in air is proportional to the square root of the absolute temperature T (Equation 15-5). (*a*) Show that if the air temperature changes by a small amount, the fractional change in the fundamental frequency of an organ pipe is approximately equal to half the fractional change in the absolute temperature. That is, show that $\Delta f/f \approx \frac{1}{2}\Delta T/T$, where f is the frequency at absolute temperature T and Δf is the change in frequency when the temperature changes by ΔT. (Ignore any change in the length of the pipe due to thermal expansion of the organ pipe.) (*b*) Suppose that an organ pipe that is stopped at one end has a fundamental frequency of 200.0 Hz when the temperature is 20.00ºC. Use the approximate result from Part (*a*) to determine its fundamental frequency when the temperature is 30.00ºC. (*c*) Compare your Part (*b*) result to what you would get

using exact calculations. (Ignore any change in the length of the pipe due to thermal expansion.)

Picture the Problem We can express the fundamental frequency of the organ pipe as a function of the air temperature and differentiate this expression with respect to the temperature to express the rate at which the frequency changes with respect to temperature. For changes in temperature that are small compared to the temperature, we can approximate the differential changes in frequency and temperature with finite changes to complete the derivation of $\Delta f/f = \frac{1}{2}\Delta T/T$. In Part (*b*) we'll use this relationship and the data for the frequency at 20.00°C to find the frequency of the fundamental at 30.00°C.

(*a*) Express the fundamental frequency of an organ pipe in terms of its wavelength and the speed of sound:

$$f = \frac{v}{\lambda}$$

Relate the speed of sound in air to the absolute temperature:

$$v = \sqrt{\frac{\gamma RT}{M}} = C\sqrt{T}$$

where

$$C = \sqrt{\frac{\gamma R}{M}} = \text{constant}$$

Defining a new constant C', substitute to obtain:

$$f = \frac{C}{\lambda}\sqrt{T} = C'\sqrt{T}$$

because λ is constant for the fundamental frequency we ignore any change in the length of the pipe.

Differentiate this expression with respect to T:

$$\frac{df}{dT} = \frac{1}{2}C'T^{-1/2} = \frac{f}{2T}$$

Separate the variables to obtain:

$$\frac{df}{f} = \frac{1}{2}\frac{dT}{T}$$

For $\Delta T << T$, we can approximate df by Δf and dT by ΔT to obtain:

$$\frac{\Delta f}{f} = \boxed{\frac{1}{2}\frac{\Delta T}{T}}$$

(*b*) Express the fundamental frequency at 30.00°C in terms of its frequency at 20.00°C:

$$f_{30} = f_{20} + \Delta f$$

Solve the result in (*a*) for Δf:

$$\Delta f = \tfrac{1}{2} f \frac{\Delta T}{T}$$

Substitute for Δf to obtain:

$$f_{30} = f_{20} + \tfrac{1}{2} f_{20} \frac{\Delta T}{T}$$

Substitute numerical values and evaluate f_{30}:

$$f_{30} = 200.0\,\text{Hz} + \tfrac{1}{2}(200.0\,\text{Hz})\frac{303.15\,\text{K} - 293.15\,\text{K}}{293.15\,\text{K}} = \boxed{203.4\,\text{Hz}}$$

(*c*) The exact expression for f_{30} is:

$$f_{30} = \frac{v_{30}}{\lambda} = \frac{\sqrt{\dfrac{\gamma R T_{30}}{M}}}{\lambda} = \sqrt{\frac{\gamma R T_{30}}{\lambda^2 M}}$$

The exact expression for f_{20} is:

$$f_{20} = \frac{v_{20}}{\lambda} = \frac{\sqrt{\dfrac{\gamma R T_{20}}{M}}}{\lambda} = \sqrt{\frac{\gamma R T_{20}}{\lambda^2 M}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{f_{30}}{f_{20}} = \frac{\sqrt{\dfrac{\gamma R T_{30}}{\lambda^2 M}}}{\sqrt{\dfrac{\gamma R T_{20}}{\lambda^2 M}}} = \sqrt{\frac{T_{30}}{T_{20}}}$$

Solve for f_{30} to obtain:

$$f_{30} = \sqrt{\frac{T_{30}}{T_{20}}}\, f_{20}$$

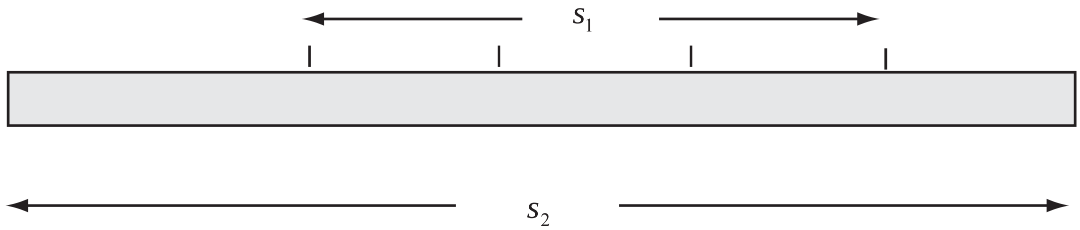
Substitute numerical values and evaluate f_{30}:

$$f_{30} = \sqrt{\frac{303.15\,\text{K}}{293.15\,\text{K}}}(200.0\,\text{Hz}) = \boxed{203.4\,\text{Hz}}$$

82 •• The pipe in Figure 16-36 is kept filled with natural gas (methane, CH_4). The pipe is punctured with a line of small holes 1.00 cm apart down its entire 2.20 m length. A speaker forms the closure on one end of the pipe, and a solid piece of metal closes the other end. What frequency is being played in this picture? The speed of sound in low pressure methane at room temperature is about 460 m/s.

Picture the Problem The frequency of the sound being played is given by $f = v/\lambda$. Because the holes are not visible in the photograph, we can not use their

separation to determine the wavelength of the sound being played. Instead, we can use the fact that the separation of the leftmost and rightmost flame maxima is 6 quarter wavelengths (drawing a standing wave pattern that is consistent with the flame pattern shown in Figure 16-36 will help you see this). Using any convenient scale to measure s_1 and s_2 (scaled distances on the photograph) and setting up a proportion involving these distances, the actual length L of the pipe, and the number of wavelengths in the distance s_1 will yield λ and, hence, f.



The frequency of the sound is given by:

$$f = \frac{v}{\lambda}$$

Letting L represent the length of the pipe yields:

$$\frac{\frac{6}{4}\lambda}{s_1} = \frac{L}{s_2} \Rightarrow \lambda = \frac{2Ls_1}{3s_2}$$

Substituting for λ yields:

$$f = \frac{v}{\frac{2Ls_1}{3s_2}} = \frac{3vs_2}{2Ls_1}$$

If s_1 = 10.5 cm and s_2 = 14.5 cm, then:

$$f = \frac{3(460\ \text{m/s})(14.5\ \text{cm})}{2(2.20\ \text{m})(10.5\ \text{cm})} = \boxed{433\ \text{Hz}}$$

83 •• Assume that your clarinet is entirely filled with helium and that before you start to play you fill your lungs with helium.. You pick up the clarinet and play it as though you were trying to play a B-flat, which has a frequency of 277 Hz. The frequency of 277 Hz is the natural resonance frequency of this clarinet with all finger holes closed and when filled with air. What frequency do you actually hear?

Picture the Problem The resonance frequency of the clarinet depends on the nature of the gas with which it is filled. We can express this frequency for both air and helium and express their ratio to eliminate the constant factors λ, R, and T. See Appendix C for the molar masses of helium and air.

Express the frequency of B-flat in helium:

$$f_{\text{B-flat, He}} = \frac{v_{\text{He}}}{\lambda} = \frac{1}{\lambda}\sqrt{\frac{\gamma_{\text{He}}RT}{M_{\text{He}}}}$$

Divide the second these equations by the first and simplify to obtain:

$$\frac{f_{\text{B-flat, He}}}{f_{\text{B-flat, air}}} = \frac{\frac{1}{\lambda}\sqrt{\frac{\gamma_{\text{He}}RT}{M_{\text{He}}}}}{\frac{1}{\lambda}\sqrt{\frac{\gamma_{\text{air}}RT}{M_{\text{air}}}}} = \sqrt{\frac{M_{\text{air}}\gamma_{\text{He}}}{M_{\text{He}}\gamma_{\text{air}}}}$$

Solving for $f_{\text{B-flat, He}}$ yields:

$$f_{\text{B-flat, He}} = \sqrt{\frac{M_{\text{air}}\gamma_{\text{He}}}{M_{\text{He}}\gamma_{\text{air}}}}\, f_{\text{B-flat, air}}$$

Substitute numerical values and evaluate $f_{\text{B-flat, He}}$:

$$f_{\text{B-flat, He}} = \sqrt{\frac{(28.81\text{ g/mol})(1.67)}{(4.003\text{ g/mol})(1.4)}}(277\text{ Hz}) = \boxed{812\text{ Hz}}$$

84 ••• A 2.00-m-long wire that is fixed at both ends is vibrating in its fundamental mode. The tension in the wire is 40.0 N and the mass of the wire is 0.100 kg. At the midpoint of the wire, the amplitude is 2.00 cm. (*a*) Find the maximum kinetic energy of the wire. (*b*) What is the kinetic energy of the wire at the instant the transverse displacement is given by $y = 0.0200\sin\left(\frac{\pi}{2}x\right)$, where y is in meters if x is in meters, for $0.00\text{ m} \le x \le 2.00\text{ m}$? (*c*) For what value of x is the average value of the kinetic energy per unit length the greatest? (*d*) For what value of x does the elastic potential energy per unit length have its maximum value?

Picture the Problem The maximum kinetic energy of the wire is given by $K_{\text{max}} = \frac{1}{4}m\omega^2A^2$. We can use $v = f\lambda$ and $v = \sqrt{F_{\text{T}}/\mu}$ to find an expression for ω. In Part (*d*) we'll use $\frac{\Delta U}{\Delta x} \approx \frac{1}{2}F_{\text{T}}\left(\frac{\partial y}{\partial x}\right)^2$ (Problem 15-104) to determine where the potential energy per unit length has its maximum value.

(*a*) The maximum kinetic energy of the wire is given by:

$$K_{\text{max}} = \frac{1}{4}m\omega^2A^2 \qquad (1)$$

Express ω_1 in terms of f_1:

$$\omega_1 = 2\pi f_1$$

Relate f_1 to the speed of transverse waves on the wire and the wavelength of the fundamental mode:

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

where L is the length of the wire.

Express the speed of the transverse waves on the wire in terms of the tension in the wire:

$$v=\sqrt{\frac{F_{\mathrm{T}}}{\mu}}=\sqrt{\frac{F_{\mathrm{T}}L}{m}}$$

Substitute and simplify to obtain:

$$f_1=\frac{1}{2L}\sqrt{\frac{F_{\mathrm{T}}L}{m}}=\sqrt{\frac{F_{\mathrm{T}}}{4mL}}$$

Substitute for ω_1 and f_1 in equation (1) to obtain:

$$K_{\max}=\tfrac{1}{4}m\left[2\pi\sqrt{\frac{F_{\mathrm{T}}}{4mL}}\right]^2A^2=\frac{\pi^2F_{\mathrm{T}}}{4L}A^2$$

Substitute numerical values and evaluate $K_{\max}$:

$$K_{\max}=\frac{\pi^2(40.0\,\mathrm{N})}{4(2.00\,\mathrm{m})}\left(2.00\times10^{-2}\,\mathrm{m}\right)^2=\boxed{19.7\,\mathrm{mJ}}$$

(*b*) Express the wave function for a standing wave in its first harmonic:

$$y_1(x,t)=A_1\sin k_1x\cos\omega_1t \qquad (2)$$

At the instant the transverse displacement is given by (0.02 m) sin ($\pi x/2$):

$$\cos\omega_1t=1\Rightarrow\omega_1t=0$$

and

$$K=\boxed{0}$$

(*c*) *dK* is a maximum where the displacement of the wire is greatest; that is, at its midpoint:

$$x=\tfrac{1}{2}L=\tfrac{1}{2}(2.00\,\mathrm{m})=\boxed{1.00\,\mathrm{m}}$$

(*d*) From Problem 15-104:

$$\frac{\Delta U}{\Delta x}\approx\tfrac{1}{2}F_{\mathrm{T}}\left(\frac{\partial y}{\partial x}\right)^2$$

Express the condition on $\partial y/\partial x$ that maximizes $\Delta U/\Delta x$:

$$\frac{\partial y}{\partial x}=\left(\frac{\partial y}{\partial x}\right)_{\max}$$

Differentiate $y_1(x,t)=A_1\sin k_1x\cos\omega_1t$ with respect to x and set the derivative equal to zero for extrema:

$$\frac{\partial y_1}{\partial x}=\frac{\partial}{\partial x}\left(A_1\sin k_1x\cos\omega_1t\right)=k_1A_1\cos k_1x\cos\omega_1t=0$$

or

$$\cos k_1x=0$$

Solve for k_1x and then x:

$$k_1 x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2k_1} = \frac{\pi\lambda}{2(2\pi)} = \tfrac{1}{4}\lambda$$

Because $\lambda = 2L$:

$$x = \tfrac{1}{2}L \Rightarrow x = \tfrac{1}{2}(2.00\,\text{m}) = \boxed{1.00\,\text{m}}$$

That is, the potential energy per unit length is a maximum at the midpoint of the wire.

Remarks: In Part (*d*) we've shown that Δ*U*/Δ*x* has an extreme value at *x* = 1 m. To show that Δ*U*/Δ*x* is a *maximum* at this location, you need to examine the sign of the 2nd derivative of $y_1(x,t)$ at this point.

85 ••• [SSM] In principle, a wave with almost any arbitrary shape can be expressed as a sum of harmonic waves of different frequencies. (*a*) Consider the function defined by

$$f(x) = \frac{4}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \ldots\right) = \frac{4}{\pi}\sum_{n=0}(-1)^n \frac{\cos[(2n+1)x]}{2n+1}$$

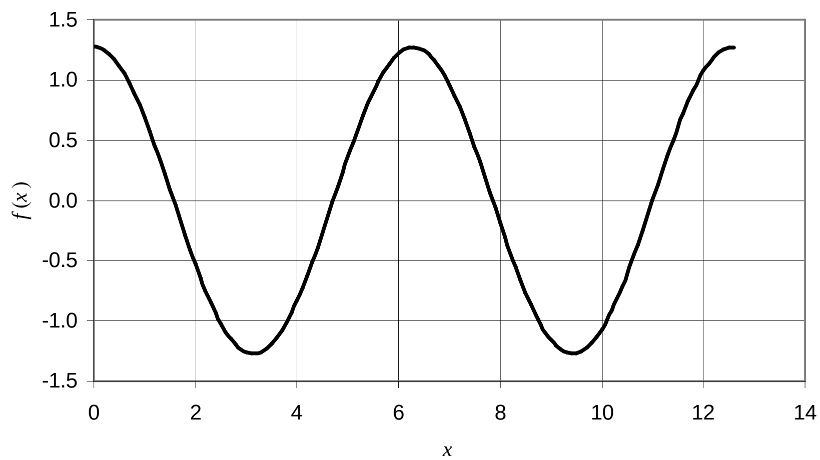
Write a **spreadsheet program** to calculate this series using a finite number of terms, and make three graphs of the function in the range $x = 0$ to $x = 4\pi$.. To create the first graph, for each value of x that you plot, approximate the sum from $n = 0$ to $n = \infty$ with the first term of the sum. To create the second and third graphs, use only the first five term and the first ten terms, respectively. This function is sometimes called the *square wave*. (*b*) What is the relation between this function and Liebnitz' series for π, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$?

A spreadsheet program to evaluate $f(x)$ is shown below. Typical cell formulas used are shown in the table.

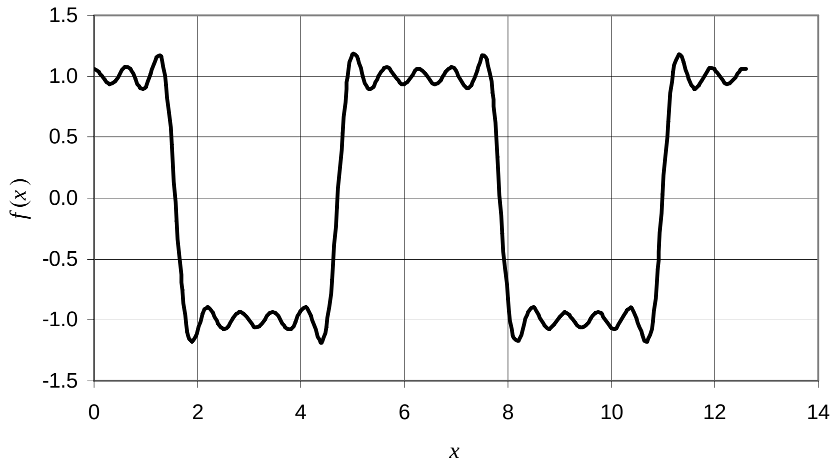
Cell	Content/Formula	Algebraic Form
A6	A5+0.1	$x + \Delta x$
B4	2*B3+1	$2n+1$
B5	(−1)^B$3*COS(B$4*$A5) /B$4*4/PI()	$\frac{4}{\pi}\frac{(-1)^0\cos((1)(0.0))}{1}$
C5	B5+(−1)^C$3*COS(C$4*$A5) /C$4*4/PI()	$1.2732 + \frac{4}{\pi}\frac{(-1)^1\cos((3)(0.0))}{3}$

	A	B	C	D		K	L
1							
2							
3	n =	0	1	2		9	10
4	2n+1=	1	3	5		19	21
5	0.0	1.2732	0.8488	1.1035		0.9682	1.0289
6	0.1	1.2669	0.8614	1.0849		1.0134	0.9828
7	0.2	1.2479	0.8976	1.0352		1.0209	0.9912
8	0.3	1.2164	0.9526	0.9706		0.9680	1.0286
9	0.4	1.1727	1.0189	0.9130		1.0057	0.9742
10	0.5	1.1174	1.0874	0.8833		1.0298	1.0010
130	12.5	1.2704	0.8544	1.0952		0.9924	1.0031
131	12.6	1.2725	0.8503	1.1013		0.9752	1.0213
132	12.7	1.2619	0.8711	1.0710		1.0287	0.9714
133	12.8	1.2386	0.9143	1.0141		1.0009	1.0126
134	12.9	1.2030	0.9740	0.9493		0.9691	1.0146
135	13.0	1.1554	1.0422	0.8990		1.0261	0.9685

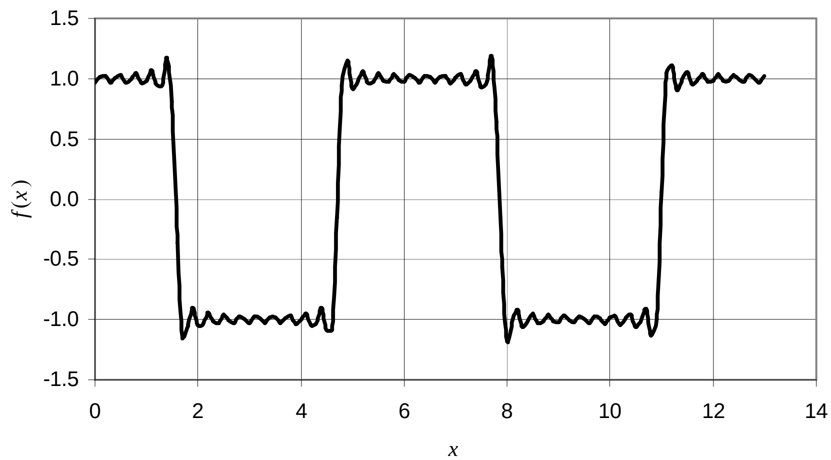
The graph of $f(x)$ versus x for $n = 1$ follows:



The graph of $f(x)$ versus x for $n = 5$ follows:



The graph of $f(x)$ versus x for $n = 10$ follows:



Evaluate $f(2\pi)$ to obtain:

$$\begin{aligned} f(2\pi) &= \frac{4}{\pi}\left(\frac{\cos 2\pi}{1} - \frac{\cos 3(2\pi)}{3} + \frac{\cos 5(2\pi)}{5} - \ldots\right) \\ &= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \\ &= 1 \end{aligned}$$

which is equivalent to the Liebnitz formula.

86 ••• Write a **spreadsheet program** to calculate and graph the function

$$y(x)=\frac{4}{\pi}\left(\sin x-\frac{\sin 3x}{9}+\frac{\sin 5x}{25}-...\right)=\frac{4}{\pi}\sum_n\frac{(-1)^n\sin(2n+1)x}{(2n+1)^2}$$

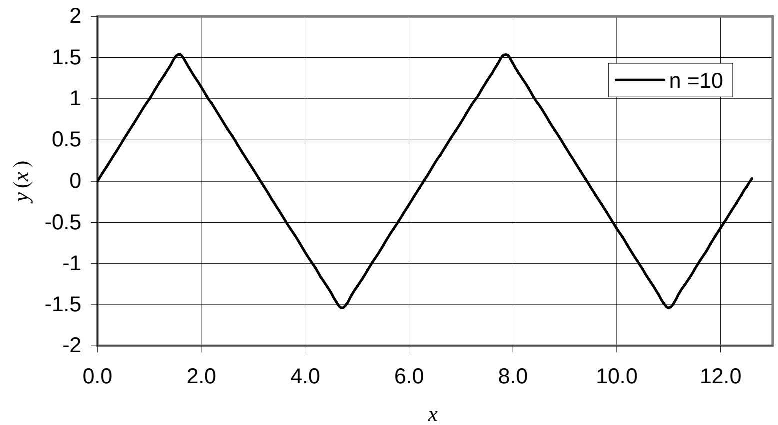
for $0\le x\le 4\pi$. Use only the first 25 terms in the sum for each value of x that you plot.

A spreadsheet program to evaluate $y(x)$ is shown below. Typical cell formulas used are shown in the table.

Cell	Content/Formula	Algebraic Form
A6	A5+0.1	$x+\Delta x$
B4	2*B3+1	$2n+1$
B5	((−1)^$B$3*sin($B$4*A5)/($B$4)^2)*4/PI()	$\frac{4}{\pi}\frac{(-1)^0\sin(1)(0.0)}{(1)^2}$
B6	((−1)^$B$3*sin($B$4*A6)/($B$4)^2)*4/PI()	$\frac{4}{\pi}\frac{(-1)^0\sin(1)(0.1)}{(1)^2}$
C5	B5+((−1)^$C$3*sin($C$4*A5)/($C$4)^2)*4/PI()	$0+\frac{4}{\pi}\frac{(-1)^1\sin(1)(0.0)}{(1)^2}$
C6	B6+((−1)^$C$3*sin($C$4*A6)/($C$4)^2)*4/PI()	$0.1271+\frac{4}{\pi}\frac{(-1)^1\sin(3)(0.1)}{(3)^2}$

	A	B	C	D		K	L
1							
2							
3	n =	0	1	2		9	10
4	$2n$+1=	1	3	5		19	21
5	0.0	0.0000	0.0000	0.0000		0.0000	0.0000
6	0.1	0.1271	0.0853	0.1097		0.0986	0.1011
7	0.2	0.2530	0.1731	0.2159		0.2012	0.1987
8	0.3	0.3763	0.2654	0.3163		0.3004	0.3005
9	0.4	0.4958	0.3640	0.4103		0.3983	0.4008
10	0.5	0.6104	0.4693	0.4998		0.5011	0.4985
72	6.7	0.5155	0.3812	0.4256		0.4153	0.4171
73	6.8	0.6291	0.4877	0.5146		0.5183	0.5154
74	6.9	0.7365	0.6005	0.6034		0.6171	0.6182
75	7.0	0.8365	0.7181	0.6963		0.7148	0.7166
76	7.1	0.9282	0.8380	0.7968		0.8183	0.8155

The following graph was obtain using $n = 10$.

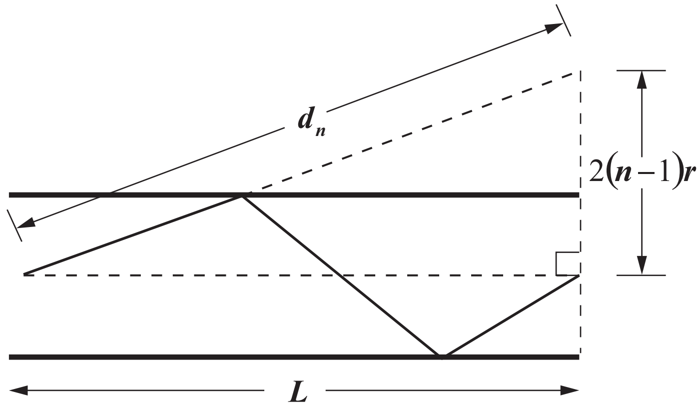


87 ••• If you clap your hands at the end of a long, cylindrical tube, the echo you hear back will not sound like the handclap; instead, you will hear what sounds like a whistle, initially at a very high frequency, but descending rapidly down to almost nothing. This "culvert whistler" is easily explained if you think of the sound from the clap as a single compression radiating outward from the hands. The echoes of the handclap arriving at your ear have traveled along different paths through the tube, as shown in Figure 16-37. The first echo to arrive travels straight down and straight back along the tube, while the second echo reflects once off of the center of the tube going out, and again going back, the third echo reflects twice at points 1/4 and 3/4 of the distance, etc. The tone of the sound you hear reflects the frequency at which these echoes reach your ears. (*a*) Show that the time delay between the n_{th} echo and the $n+1_{\text{th}}$ is

$$\Delta t_n = \frac{2}{v}\left(\sqrt{(2n)^2 r^2 + L^2} - \sqrt{\left[2(n-1)\right]^2 r^2 + L^2}\right)$$

where v is the speed of sound, L is the length of the tube and r is its radius.
(*b*) Using a **spreadsheet program** or **graphing calculator**, graph Δt_n versus n for $L = 90.0$ m, $r = 1.00$ m. (These values are the approximate length and radius of the long tube in the San Francisco Exploratorium.) Go to at least $n = 100$.
(*c*) From your graph, explain why the frequency decreases over time. What are the highest and lowest frequencies you will hear in the whistler?

Picture the Problem From the diagram above, the nth echo will reflect $n - 1$ times going out, and the same number of times going back. If we "unfold" the ray into a straight line, we get the representation shown below. Using this figure we can express the distance d_n traveled by the nth echo and then use this result to express the time delay between the n_{th} and $n + 1_{\text{th}}$ echoes. The reciprocal of this time delay is the frequency corresponding to the nth echo.



(*a*) Apply the Pythagorean theorem to the right triangle whose base is *L*, whose height is 2(*n* – 1), and whose hypotenuse is d_n to obtain:

$$d_n = 2\sqrt{4(n-1)^2 r^2 + L^2}$$

Express the time delay between the n_{th} and $n + 1_{\text{th}}$ echoes:

$$\Delta t_n = \frac{d_n}{v}$$

Substitute to obtain:

$$\Delta t_n = \frac{2}{v}\left(\sqrt{(2n)^2 r^2 + L^2} - \sqrt{[2(n-1)]^2 r^2 + L^2}\right)$$

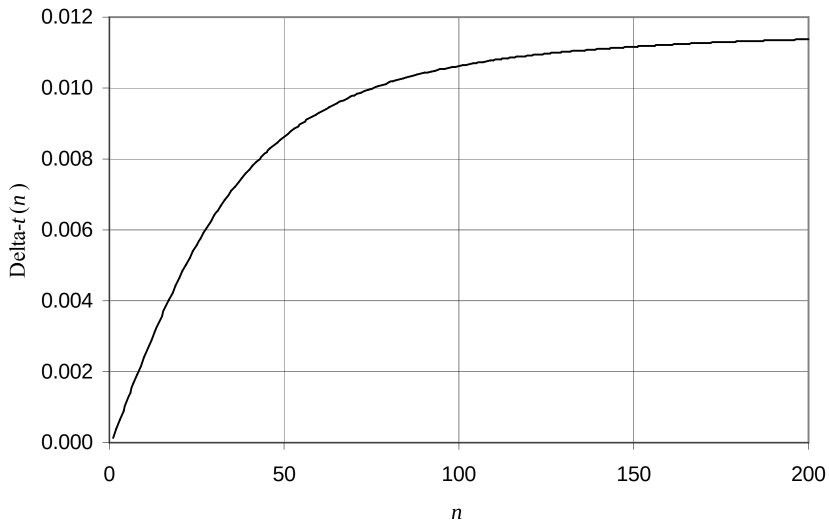
(*b*) A spreadsheet program to calculate Δt_n as a function of *n* is shown below. The constants and cell formulas used are shown in the table.

Cell	Content/Formula	Algebraic Form
B1	90	L
B2	1	r
B3	343	c
B8	B7+1	$n + 1$
C7	2/$B$3*((2*(B7−1) *$B$2)^2+$B$1^2)^0.5	Δt_n

	A	B	C	D
1	*L*=	90	m	
2	*r*=	1	m	
3	*c*=	343	m/s	
4				
5				
6		*n*	*t*(*n*)	delta *t*(*n*)
7		1	0.5248	0.0001
8		2	0.5249	0.0004

9		3	0.5253	0.0006
10		4	0.5259	0.0009
11		5	0.5269	0.0012
202		196	2.3338	0.0114
203		197	2.3452	0.0114
204		198	2.3566	0.0114
205		199	2.3679	0.0114
206		200	2.3793	0.0114

The graph of Δt_n as a function of n shown below was plotted using the data from columns B and D.



(c) The frequency heard at any time is $1/\Delta t_n$, so because Δt_n increases over time, the frequency of the culvert whistler decreases.

The highest frequency corresponds to $n = 1$ and is given by:

$$f_{\text{highest}} = \frac{1}{\Delta t_1}$$

Substitute for Δt_1 to obtain:

$$f_{\text{highest}} = \frac{1}{\Delta t_1} = \frac{v}{2\left(\sqrt{(2)^2 r^2 + L^2} - \sqrt{L^2}\right)}$$

Substitute numerical values and evaluate f_{highest}:

$$f_{\text{highest}} = \frac{343\,\text{m/s}}{2\left(\sqrt{4(1.00\,\text{m})^2 + (90.0\,\text{m})^2} - 90.0\,\text{m}\right)} = \boxed{7.72\,\text{kHz}}$$

The lowest frequency end can be found by examining the limit of Δt_n as $n \to \infty$:

$$\lim_{n\to\infty} \Delta t_n = \lim_{n\to\infty}\left[\frac{2}{v}\left((2n)\sqrt{r^2+\frac{L^2}{(2n)^2}}-2(n-1)\sqrt{r^2+\frac{L^2}{(2(n-1))^2}}\right)\right]$$
$$=\frac{2r}{v}(2n-2n+2)=\frac{4r}{v}$$

Express f_{lowest} in terms of Δt_∞:

$$f_{\text{lowest}}=\frac{1}{\Delta t_\infty}=\frac{v}{4r}$$

Substitute numerical values and evaluate f_{lowest}:

$$f_{\text{lowest}}=\frac{343\,\text{m/s}}{4(1.00\,\text{m})}=\boxed{85.8\,\text{Hz}}$$

Chapter 17
Temperature and the Kinetic Theory of Gases

Conceptual Problems

1 • True or false:

(*a*) The zeroth law of thermodynamics states that two objects in thermal equilibrium with each other must be in thermal equilibrium with a third object.
(*b*) The Fahrenheit and Celsius temperature scales differ only in the choice of the ice-point temperature.
(*c*) The Celsius degree and the kelvin are the same size.

(*a*) False. If two objects are in thermal equilibrium with a third, then they are in thermal equilibrium with each other.

(*b*) False. The Fahrenheit and Celsius temperature scales also differ in the number of intervals between the ice-point temperature and the steam-point temperature.

(*c*) True.

2 • How can you determine if two objects are in thermal equilibrium with each other when putting them into physical contact with each other would have undesirable effects? (For example, if you put a piece of sodium in contact with water there would be a violent chemical reaction.)

Determine the Concept Put each in thermal equilibrium with a third body; that is, a thermometer. If each body is in thermal equilibrium with the third, then they are in thermal equilibrium with each other.

3 • **[SSM]** ″Yesterday I woke up and it was 20ºF in my bedroom,″ said Mert to his old friend Mort. ″That's nothing,″ replied Mort. ″My room was –5.0ºC.″ Who had the colder room, Mert or Mort?

Picture the Problem We can decide which room was colder by converting 20°F to the equivalent Celsius temperature.

Using the Fahrenheit-Celsius conversion, convert 20°F to the equivalent Celsius temperature:

$$t_C = \tfrac{5}{9}(t_F - 32°F) = \tfrac{5}{9}(20°F - 32°F) = -6.7°C$$

so $\boxed{\text{Mert's room was colder.}}$

4 • Two identical vessels contain different ideal gases at the same pressure and temperature. It follows that (*a*) the number of gas molecules is the same in both vessels, (*b*) the total mass of gas is the same in both vessels, (*c*) the average speed of the gas molecules is the same in both vessels, (*d*) None of the above.

Determine the Concept Because the vessels are identical (have the same volume) and the two ideal gases are at the same pressure and temperature, the ideal-gas law ($PV = NkT$) tells us that the number of gas molecules must be the same in both vessels. $\boxed{(a)}$ is correct.

5 • **[SSM]** Figure 17-18 shows a plot of volume versus absolute temperature for a process that takes a fixed amount of an ideal gas from point A to point B. What happens to the pressure of the gas during this process?

Determine the Concept From the ideal-gas law, the equation of the line from the origin to the point (*T*,*V*) is $V = \frac{nR}{P}T$. The slope of this line is *nR*/*P*. During the process A $\rightarrow$ B the slope of the line from the origin to (*T*,*V*) continuously decreases, so the pressure continuously increases.

6 • Figure 17-19 shows a plot of pressure versus absolute temperature for a process that takes a sample of an ideal gas from point A to point B. What happens to the volume of the gas during this process?

Determine the Concept From the ideal-gas law, the equation of the line from the origin to the point (*T*,*P*) is $P = \frac{nR}{V}T$. The slope of this line is *nR*/*V*. During the process A $\rightarrow$ B the slope of the line from the origin to (*T*,*V*) continuously increases, so the volume continuously decreases.

7 • If a vessel contains equal amounts, by mass, of helium and argon, which of the following are true?

(*a*) The partial pressure exerted by each of the two gases on the walls of the container is the same.
(*b*) The average speed of a helium atom is the same as that of an argon atom.
(*c*) The number of helium atoms and argon atoms in the vessel are equal.
(*d*) None of the above.

Determine the Concept
(*a*) False. The gases occupy the same volume and they are at the same temperature. Because the masses of the gases are the same, the number of moles of each must be different. Because there are more moles of He, there are more

molecules of He. From the ideal-gas law, $P = nRT/V$, and so the partial pressure of He is greater.

(*b*) False. The average kinetic energy of both types of molecules is the same. Because He molecules have smaller masses, their average speed is larger.

(*c*) False. See (*a*).

(*d*) Because none of the above is true, $\boxed{(d)}$ is true.

8 • By what factor must the absolute temperature of a gas be increased to double the rms speed of its molecules?

Determine the Concept We can use $v_{\text{rms}} = \sqrt{3RT/M}$ to relate the temperature of a gas to the rms speed of its molecules.

Express the dependence of the rms speed of the molecules of a gas on their absolute temperature:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

where R is the gas constant, M is the molar mass, and T is the absolute temperature.

Because $v_{\text{rms}} \propto \sqrt{T}$, the temperature must be ***quadrupled*** in order to double the rms speed of the molecules.

9 • Two different gases are at the same temperature. What can be said about the average translational kinetic energies of the molecules? What can be said about the rms speeds of the gas molecules?

Determine the Concept The average kinetic energies are equal. The ratio of their rms speeds is equal to the square root of the reciprocal of the ratio of their molecular masses.

10 • A vessel holds a mixture of helium (He) and methane (CH_4). The ratio of the rms speed of the He atoms to that of the CH_4 molecules is (*a*) 1, (*b*) 2, (*c*) 4, (*d*) 16

Picture the Problem We can express the rms speeds of the helium atoms and the methane molecules using $v_{\text{rms}} = \sqrt{3RT/M}$.

Express the rms speed of the helium atoms:

$$v_{\text{rms, He}} = \sqrt{\frac{3RT}{M_{\text{He}}}}$$

Express the rms speed of the methane molecules:

$$v_{\text{rms, CH}_4} = \sqrt{\frac{3RT}{M_{\text{CH}_4}}}$$

Divide the first of these equations by the second to obtain:

$$\frac{v_{\text{rms, He}}}{v_{\text{rms, CH}_4}} = \sqrt{\frac{M_{\text{CH}_4}}{M_{\text{He}}}}$$

Use Appendix C to find the molar masses of helium and methane:

$$\frac{v_{\text{rms, He}}}{v_{\text{rms, CH}_4}} = \sqrt{\frac{16\,\text{g/mol}}{4\,\text{g/mol}}} = 2$$

and $\boxed{(b)}$ is correct.

11 • True or false: If the pressure of a fixed amount of gas increases, the temperature of the gas must increase.

False. Whether the pressure changes also depends on whether and how the volume changes. In an isothermal process, the pressure can increase while the volume decreases and the temperature is constant.

12 • Why might the Celsius and Fahrenheit scales be more convenient than the absolute scale for ordinary, nonscientific purposes?

Determine the Concept For the Celsius scale, the ice point (0°C) and the boiling point of water at 1 atm (100°C) are more convenient than 273 K and 373 K; temperatures in roughly this range are normally encountered. On the Fahrenheit scale, the temperature of warm-blooded animals is roughly 100°F; this may be a more convenient reference than approximately 300 K. Throughout most of the world, the Celsius scale is the standard for nonscientific purposes.

13 • An astronomer claims that the temperature at the center of the Sun is about 10^7 degrees. Do you think that this temperature is in kelvins, degrees Celsius, or doesn't it matter?

Determine the Concept Because $T = t_C + 273.15$ K and $10^7 >> 273$, it doesn't matter.

14 • Imagine that you have a fixed amount of ideal gas in a container that expands to maintain constant pressure. If you double the absolute temperature of the gas, the average speed of the molecules (*a*) remains constant, (*b*) doubles, (*c*) quadruples, (*d*) increases by a factor of $\sqrt{2}$.

Determine the Concept The average speed of the molecules in an ideal gas depends on the square root of the kelvin temperature. Because $v_{\text{av}} \propto \sqrt{T}$, doubling the temperature while maintaining constant pressure increases the average speed by a factor of $\sqrt{2}$. $\boxed{(d)}$ is correct.

15 • **[SSM]** Suppose that you compress an ideal gas to half its original volume, while also halving its absolute temperature. During this process, the pressure of the gas (*a*) halves, (*b*) remains constant, (*c*) doubles, (*d*) quadruples.

Determine the Concept From the ideal-gas law, $PV = nRT$, halving both the temperature and volume of the gas leaves the pressure unchanged. $\boxed{(b)}$ is correct.

16 • The average translational kinetic energy of the molecules of a gas depends on (*a*) the number of moles and the temperature, (*b*) the pressure and the temperature, (*c*) the pressure only, (*d*) the temperature only.

Determine the Concept The average translational kinetic energy of the molecules of an ideal gas K_{av} depends on its temperature T according to $K_{\text{av}} = \frac{3}{2}kT$. $\boxed{(d)}$ is correct.

17 •• **[SSM]** Which speed is greater, the speed of sound in a gas or the rms speed of the molecules of the gas? Justify your answer, using the appropriate formulas, and explain why your answer is intuitively plausible.

Determine the Concept The rms speed of molecules of an ideal gas is given by $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$ and the speed of sound in a gas is given by $v_{\text{sound}} = \sqrt{\frac{\gamma RT}{M}}$.

The rms speed of the molecules of an ideal-gas is given by:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

The speed of sound in a gas is given by:

$$v_{\text{sound}} = \sqrt{\frac{\gamma RT}{M}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{v_{\text{rms}}}{v_{\text{sound}}}=\frac{\sqrt{\frac{3RT}{M}}}{\sqrt{\frac{\gamma RT}{M}}}=\sqrt{\frac{3}{\gamma}}$$

For a monatomic gas, $\gamma = 1.67$ and:

$$\frac{v_{\text{rms, monatomic}}}{v_{\text{sound}}}=\sqrt{\frac{3}{1.67}}=1.34$$

For a diatomic gas, $\gamma = 1.40$ and:

$$\frac{v_{\text{rms, diatomic}}}{v_{\text{sound}}}=\sqrt{\frac{3}{1.40}}=1.46$$

The rms speed is always somewhat greater than the speed of sound. However, it is only the component of the molecular velocities in the direction of propagation that is relevant to this issue. In addition, in a gas the mean free path is greater than the average intermolecular distance.

18 •• Imagine that you increase the temperature of a gas while holding its volume fixed. Explain in terms of molecular motion why the pressure of the gas on the walls of its container increases.

Determine the Concept The pressure is a measure of the average change in momentum per second of gas molecules colliding with the walls of the container. When the gas is heated, the average velocity and the average momentum of the molecules increase and, as a consequence, the pressure exerted by the molecules increases.

19 •• Imagine that you compress a gas while holding it at a fixed temperature (perhaps by immersing the container in cool water). Explain in terms of molecular motion why the pressure of the gas on the walls of its container increases.

Determine the Concept If the volume decreases the pressure increases because more molecules hit a unit of area of the walls in a given time. The fact that the temperature does not change tells us the molecular speed does not change with volume.

20 •• Oxygen has a molar mass of 32 g/mol and nitrogen has a molar mass of 28 g/mol. The oxygen and nitrogen molecules in a room have

(*a*) equal average translational kinetic energies, but the oxygen molecules have a larger average speed than the nitrogen molecules have.
(*b*) equal average translational kinetic energies, but the oxygen molecules have a smaller average speed than the nitrogen molecules have.

(*c*) equal average translational kinetic energies and equal average speeds.
(*d*) equal average speeds, but the oxygen molecules have a larger average translational kinetic energy than the nitrogen molecules have.
(*e*) equal average speeds, but the oxygen molecules have a smaller average translational kinetic energy than the nitrogen molecules have.
(*f*) None of the above.

Picture the Problem The average kinetic energies of the molecules are given by $K_{av} = \left(\frac{1}{2}mv^2\right)_{av} = \frac{3}{2}kT$. Assuming that the room's temperature distribution is uniform, we can conclude that the oxygen and nitrogen molecules have equal average kinetic energies. Because the oxygen molecules are more massive, they must be moving slower than the nitrogen molecules. $\boxed{(b)}$ is correct.

21 •• [SSM] Liquid nitrogen is relatively cheap, while liquid helium is relatively expensive. One reason for the difference in price is that while nitrogen is the most common constituent of the atmosphere, only small traces of helium can be found in the atmosphere. Use ideas from this chapter to explain why it is that only small traces of helium can be found in the atmosphere.

Determine the Concept The average molecular speed of He gas at 300 K is about 1.4 km/s, so a significant fraction of He molecules have speeds in excess of Earth's escape velocity (11.2 km/s). Thus, they "leak" away into space. Over time, the He content of the atmosphere decreases to almost nothing.

Estimation and Approximation

22 • Estimate the total number of air molecules in your classroom.

Picture the Problem The number of air molecules in your classroom is given by the ideal-gas law.

From the ideal-gas law, the number of molecules N in a given volume V at pressure P and temperature T is given by:

$$N = \frac{PV}{kT}$$

where k is Boltzmann's constant.

Assuming that a typical classroom is a rectangular parallelepiped, its volume is given by:

$$V = \ell wh$$

Substituting for V yields:

$$N = \frac{P\ell wh}{kT}$$

Assume atmospheric pressure, room temperature and that the room is 15 m square and 5 m high to obtain:

$$N = \frac{\left(1\,\text{atm}\times\dfrac{101.325\,\text{kPa}}{\text{atm}}\right)(15\,\text{m})(15\,\text{m})(5\,\text{m})}{\left(1.381\times10^{-23}\ \text{J/K}\right)(293\,\text{K})} \approx \boxed{3\times10^{28}}$$

23 •• Estimate the density of dry air at sea level on a warm summer day.

Picture the Problem We can use the definition of mass density, the definition of the molar mass of a gas, and the ideal-gas law to estimate the density of air at sea level on a warm day.

The density of air at sea level is given by:

$$\rho = \frac{m}{V}$$

The mass of the air molecules occupying a given volume is the product of the number of moles *n* and the molar mass *M* of the molecules. Substitute for *m* to obtain:

$$\rho = \frac{nM}{V}$$

From the ideal-gas law, the number of moles in a given volume depends on the pressure and temperature according to:

$$n = \frac{PV}{RT}$$

Substitute in the expression for ρ and simplify to obtain:

$$\rho = \frac{M}{V}\left(\frac{PV}{RT}\right) = \frac{PM}{RT}$$

Assuming atmospheric pressure, a temperature of 27°C (93°F), and 29 g/mol for the molar mass of air, substitute numerical values and evaluate ρ:

$$\rho = \frac{\left(1\,\text{atm}\times\dfrac{101.325\,\text{kPa}}{\text{atm}}\right)(29\ \text{g/mol})}{(8.314\ \text{J/mol}\cdot\text{K})(300\,\text{K})} = \boxed{1.2\ \text{kg/m}^3}$$

24 •• A stoppered test tube that has a volume of 10.0 mL has 1.00 mL of water at its bottom. The water has a temperature of 100ºC and is initially at a pressure of 1.00 atm. The test tube is held over a flame until the water has completely boiled away. Estimate the final pressure inside the test tube.

Picture the Problem Assuming the steam to be an ideal gas at a temperature of 373 K, we can use the ideal-gas law to estimate the pressure inside the test tube when the water is completely boiled away.

Using the ideal-gas law, relate the pressure inside the test tube to its volume and the temperature:

$$PV = \frac{m}{M}RT \Rightarrow P = \frac{m}{M}\frac{RT}{V}$$

Substitute numerical values and evaluate P:

$$P = \frac{0.00100\,\text{kg}}{18.0\times10^{-3}\,\frac{\text{kg}}{\text{m}^3}} \frac{\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(373\,\text{K})}{1.00\times10^{-5}\,\text{m}^3} = 1.723\times10^{7}\,\frac{\text{N}}{\text{m}^2}$$

$$= 1.723\times10^{7}\,\text{Pa}\times\frac{1\,\text{atm}}{101.3\,\text{kPa}} = \boxed{170\,\text{atm}}$$

25 •• **[SSM]** In Chapter 11, we found that the escape speed at the surface of a planet of radius R is $v_e = \sqrt{2gR}$, where g is the acceleration due to gravity at the surface of the planet. If the rms speed of a gas is greater than about 15 to 20 percent of the escape speed of a planet, virtually all of the molecules of that gas will escape the atmosphere of the planet.

(*a*) At what temperature is v_{rms} for O_2 equal to 15 percent of the escape speed for Earth?
(*b*) At what temperature is v_{rms} for H_2 equal to 15 percent of the escape speed for Earth?
(*c*) Temperatures in the upper atmosphere reach 1000 K. How does this help account for the low abundance of hydrogen in Earth's atmosphere?
(*d*) Compute the temperatures for which the rms speeds of O_2 and H_2 are equal to 15 percent of the escape speed at the surface of the moon, where g is about one-sixth of its value on Earth and R = 1738 km. How does this account for the absence of an atmosphere on the moon?

Picture the Problem We can find the escape temperatures for Earth and the moon by equating, in turn, $0.15v_e$ and v_{rms} of O_2 and H_2. We can compare these temperatures to explain the absence from Earth's upper atmosphere and from the surface of the moon. See Appendix C for the molar masses of O_2 and H_2.

(*a*) Express v_{rms} for O_2:

$$v_{rms,\,O_2} = \sqrt{\frac{3RT}{M_{O_2}}}$$

where R is the gas constant, T is the absolute temperature, and M_{O_2} is the molar mass of oxygen.

Equate $0.15v_e$ and v_{rms,O_2}:

$$0.15\sqrt{2gR_{earth}} = \sqrt{\frac{3RT}{M}}$$

Solve for T to obtain:

$$T = \frac{0.045gR_{earth}M}{3R} \quad (1)$$

Substitute numerical values and evaluate T for O_2:

$$T = \frac{0.045(9.81\,m/s^2)(6.37\times10^6\,m)(32.0\times10^{-3}\,kg/mol)}{3(8.314\,J/mol\cdot K)} = \boxed{3.60\times10^3\,K}$$

(*b*) Substitute numerical values and evaluate T for H_2:

$$T = \frac{0.045(9.81\,m/s^2)(6.37\times10^6\,m)(2.02\times10^{-3}\,kg/mol)}{3(8.314\,J/mol\cdot K)} = \boxed{230\,K}$$

(*c*) Because hydrogen is lighter than air it rises to the top of the atmosphere. Because the temperature is high there, a greater fraction of the molecules reach escape speed.

(*d*) Express equation (1) at the surface of the moon:

$$T = \frac{0.045g_{moon}R_{moon}M}{3R} = \frac{0.045(\frac{1}{6}g_{earth})R_{moon}M}{3R} = \frac{0.0025g_{earth}R_{moon}M}{R}$$

Substitute numerical values and evaluate T for O_2:

$$T = \frac{0.0025(9.81\,m/s^2)(1.738\times10^6\,m)(32.0\times10^{-3}\,kg/mol)}{8.314\,J/mol\cdot K} = \boxed{160\,K}$$

Substitute numerical values and evaluate T for H_2:

$$T = \frac{0.0025(9.81\,m/s^2)(1.738\times10^6\,m)(2.02\times10^{-3}\,kg/mol)}{8.314\,J/mol\cdot K} = \boxed{10\,K}$$

Because g is less on the moon, the escape speed is lower. Thus, a larger percentage of the molecules are moving at escape speed.

26 •• The escape speed for gas molecules in the atmosphere of Mars is 5.0 km/s and the surface temperature of Mars is typically 0ºC. Calculate the rms speeds for (*a*) H_2, (*b*) O_2, and (*c*) CO_2 at this temperature. (*d*) Are H_2, O_2, and CO_2 likely to be found in the atmosphere of Mars?

Picture the Problem We can use $v_{rms} = \sqrt{3RT/M}$ to calculate the rms speeds of H_2, O_2, and CO_2 at 273 K and then compare these speeds to 20% of the escape velocity on Mars to decide the likelihood of finding these gases in the atmosphere of Mars. See Appendix C for molar masses.

Express the rms speed of an atom as a function of the temperature:

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

(*a*) Substitute numerical values and evaluate v_{rms} for H_2:

$$v_{rms,H_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(273\,\text{K})}{2.02\times10^{-3}\,\text{kg/mol}}} = \boxed{1.84\,\text{km/s}}$$

(*b*) Evaluate v_{rms} for O_2:

$$v_{rms,O_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(273\,\text{K})}{32.0\times10^{-3}\,\text{kg/mol}}} = \boxed{461\,\text{m/s}}$$

(*c*) Evaluate v_{rms} for CO_2:

$$v_{rms,CO_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(273\,\text{K})}{44.0\times10^{-3}\,\text{kg/mol}}} = \boxed{393\,\text{m/s}}$$

(*d*) Calculate 20% of v_e for Mars:

$$0.20v_e = \tfrac{1}{5}(5.0\,\text{km/s}) = 1.0\,\text{km/s}$$

Because v is greater than v_{rms} for CO_2 but less than v_{rms} for H_2, O_2 and CO_2, but not H_2 should be present.

27 •• **[SSM]** The escape speed for gas molecules in the atmosphere of Jupiter is 60 km/s and the surface temperature of Jupiter is typically –150ºC. Calculate the rms speeds for (*a*) H_2, (*b*) O_2, and (*c*) CO_2 at this temperature. (*d*) Are H_2, O_2, and CO_2 likely to be found in the atmosphere of Jupiter?

Picture the Problem We can use $v_{rms} = \sqrt{3RT/M}$ to calculate the rms speeds of H_2, O_2, and CO_2 at 123 K and then compare these speeds to 20% of the escape speed on Jupiter to decide the likelihood of finding these gases in the atmosphere of Jupiter. See Appendix C for the molar masses of H_2, O_2, and CO_2.

Express the rms speed of an atom as a function of the temperature:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

(*a*) Substitute numerical values and evaluate v_{rms} for H_2:

$$v_{\text{rms},H_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(123\,\text{K})}{2.02\times10^{-3}\,\text{kg/mol}}} = \boxed{1.23\,\text{km/s}}$$

(*b*) Evaluate v_{rms} for O_2:

$$v_{\text{rms},O_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(123\,\text{K})}{32.0\times10^{-3}\,\text{kg/mol}}} = \boxed{310\,\text{m/s}}$$

(*c*) Evaluate v_{rms} for CO_2:

$$v_{\text{rms},CO_2} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(123\,\text{K})}{44.0\times10^{-3}\,\text{kg/mol}}} = \boxed{264\,\text{m/s}}$$

(*d*) Calculate 20% of v_e for Jupiter:

$$0.20v_e = \tfrac{1}{5}(60\,\text{km/s}) = 12\,\text{km/s}$$

Because v_e is greater than v_{rms} for O_2, CO_2, and H_2, all three gasses should be found on Jupiter.

28 •• Estimate the average pressure on the front wall of a racquetball court, due to the collisions of the ball with the wall during a game. Use any reasonable numbers for the mass of the ball, its typical speed, and the dimensions of the court. Is the average pressure from the ball significant compared to that from the air?

Picture the Problem The average pressure exerted by the ball on the wall is the ratio of the average force it exerts on the wall to the area of the wall. The average force, in turn, is the rate at which the momentum of the ball changes during each collision with the wall. Assume that the mass of a racquetball is 100 g, that the court measures 5 m by 5 m, and that the speed of the racquetball is 10 m/s. We'll also assume that the interval Δt between collisions with the wall is 1 s.

The average pressure exerted on the wall by the racquetball is given by:

$$P_{\text{av}} = \frac{F_{\text{av}}}{A} = \frac{F_{\text{av}}}{wh}$$

where w and h are the width and height of the wall, respectively.

Assuming head-on collisions with the wall, the change in momentum of the ball during each collision is $2mv$ and the average force exerted by the ball on the wall is:

$$F_{av} = \frac{\Delta p}{\Delta t} = \frac{2mv}{\Delta t}$$

where Δt is the elapsed time between collisions of the ball with the wall.

Substituting for F_{av} yields:

$$P_{av} = \frac{2mv}{wh\Delta t}$$

Substitute numerical values and evaluate P_{av}:

$$P_{av} = \frac{2(.1\text{ kg})(10\text{ m/s})}{(5\text{ m})(5\text{ m})(1\text{ s})} = 0.08\text{ N/m}^2$$
$$\approx \boxed{0.1\text{ Pa}}$$

Express the ratio of P_{av} and P_{atm} to obtain:

$$\frac{P_{av}}{P_{atm}} \approx \frac{0.1\text{ Pa}}{10^5\text{ Pa}} = 10^{-6}$$

or

$$P_{av} \approx \boxed{10^{-6}P_{atm}}$$

and the average pressure from the ball is not significant compared to atmospheric pressure.

29 •• To a first approximation, the Sun consists of a gas of equal numbers of protons and electrons. (The masses of these particles can be found in Appendix B.) The temperature at the center of the Sun is about 1×10^7 K, and the density of the Sun is about 1×10^5 kg/m^3. Because the temperature is so high, the protons and electrons are separate particles (rather than being joined together to form hydrogen atoms). (*a*) Estimate the pressure at the center of the Sun. (*b*) Estimate the rms speeds of the protons and the electrons at the center of the Sun.

Picture the Problem We'll assume that we can model the plasma as an ideal gas, at least to a first approximation. Then we can apply the ideal gas law to an arbitrary volume of the material, say one cubic meter.

(*a*) To the extent that the plasma acts like an ideal gas, the pressure at the center of the Sun is given by:

$$P = \frac{nRT}{V}$$

The mass of our 1 m^3 volume of matter is 10^5 kg and this mass consists mostly of protons. One mole of protons has a mass of 1 g, so in 10^5 kg, the number of moles of protons would be:

$$n_{protons} = \frac{m_{protons}}{M_{protons}} = \frac{10^5\text{ kg}}{10^{-3}\text{ kg}} = 10^8\text{ mol}$$

or, counting electrons,

$$n = 2n_{protons} = 2\times10^8\text{ mol}$$

Substitute numerical values and evaluate P:

$$P = \frac{\left(2\times10^{8}\ \text{mol}\right)\left(8.314\ \text{J/mol}\cdot\text{K}\right)\left(10^{7}\ \text{K}\right)}{1\ \text{m}^{3}} \approx 2\times10^{16}\ \text{Pa}\times\frac{1\ \text{atm}}{101.325\ \text{kPa}} \approx \boxed{2\times10^{11}\ \text{atm}}$$

(*b*) To one significant figure, the average speed of the protons is the same as the rms speed:

$$v_{\text{rms, protons}} = \sqrt{\frac{3RT}{M_{\text{protons}}}}$$

Substitute numerical values and evaluate $v_{\text{rms, protons}}$:

$$v_{\text{rms, protons}} = \sqrt{\frac{3\left(8.314\ \text{J/mol}\cdot\text{K}\right)\left(10^{7}\ \text{K}\right)}{0.001\ \text{kg}}} = \boxed{5\times10^{5}\ \text{m/s}}$$

The rms speed of an electron at the center of the Sun is given by:

$$v_{\text{rms, electrons}} = \sqrt{\frac{3RT}{M_{\text{electrons}}}} = \sqrt{\frac{3RT}{\frac{1}{2000}M_{\text{protons}}}}$$

Substitute numerical values and evaluate $v_{\text{rms, electrons}}$:

$$v_{\text{rms, electrons}} = \sqrt{\frac{3\left(8.314\ \text{J/mol}\cdot\text{K}\right)\left(10^{7}\ \text{K}\right)}{\frac{1}{2000}\left(0.001\ \text{kg}\right)}} = \boxed{2\times10^{7}\ \text{m/s}}$$

Remarks: The huge pressure we calculated in (*a*) is *required* to support the tremendous weight of the rest of the sun. The very high speed we calculated for the plasma electrons is still an order-of-magnitude smaller than the speed of light.

30 •• You are designing a vacuum chamber for fabricating reflective coatings. Inside this chamber, a small sample of metal will be vaporized so that its atoms travel in straight lines (the effects of gravity are negligible during the brief time of flight) to a surface where they land to form a very thin film. The sample of metal is 30 cm from the surface to which the metal atoms will adhere. How low must the pressure in the chamber be so that the metal atoms only rarely collide with air molecules before they land on the surface?

Picture the Problem We can use the expression for the mean free path of a molecule to eliminate the number of molecules per unit volume n_V from the ideal-gas law. Assume that the air in the chamber is at room temperature (300 K) and that the average diameter of an air molecule is 4×10^{-10} m.

Apply the ideal gas law to express the pressure in terms of the number of molecules per unit volume and the temperature:

$$P = \frac{NkT}{V} = n_V kT$$

The mean free path of an air molecule is given by:

$$\lambda = \frac{1}{\sqrt{2}n_V \pi d^2} \Rightarrow n_V = \frac{1}{\sqrt{2}\lambda\pi d^2}$$

Substitute for n_V to obtain:

$$P = \frac{kT}{\sqrt{2}\lambda\pi d^2}$$

Substitute numerical values and evaluate P:

$$P = \frac{\left(1.381\times10^{-23}\ \text{J/K}\right)\left(300\ \text{K}\right)}{\sqrt{2}\left(30\ \text{cm}\right)\pi\left(4\times10^{-10}\ \text{m}\right)^2} \approx \boxed{20\ \text{mPa}}$$

31 ••• [SSM] In normal breathing conditions, approximately 5 percent of each exhaled breath is carbon dioxide. Given this information and neglecting any difference in water-vapor content, estimate the typical difference in mass between an inhaled breath and an exhaled breath.

Picture the Problem One breath (one's lung capacity) is about half a liter. The only thing that occurs in breathing is that oxygen is exchanged for carbon dioxide. Let's estimate that of the 20% of the air that is breathed in as oxygen, ¼ is exchanged for carbon dioxide. Then the mass difference between breaths will be 5% of a breath multiplied by the molar mass difference between oxygen and carbon dioxide and by the number of moles in a breath. Because this is an estimation problem, we'll use 32 g/mol as an approximation for the molar mass of oxygen and 44 g/mol as an approximation for the molar mass of carbon dioxide.

Express the difference in mass between an inhaled breath and an exhaled breath:

$$\Delta m = m_{O_2} - m_{CO_2} = f_{CO_2}\left(M_{CO_2} - M_{O_2}\right)n_{\text{breath}}$$

where f_{CO_2} is the fraction of the air breathed in that is exchanged for carbon dioxide.

The number of moles per breath is given by:

$$n_{\text{breath}} = \frac{V_{\text{breath}}}{22.4\ \text{L/mol}}$$

Substituting for n_{breath} yields:

$$\Delta m = f_{CO_2}\left(M_{CO_2} - M_{O_2}\right)\left(\frac{V_{\text{breath}}}{22.4\ \text{L/mol}}\right)$$

Substitute numerical values and evaluate Δm:

$$\Delta m = (0.05)(44\text{ g/mol} - 32\text{ g/mol})\frac{(0.5\text{ L})}{22.4\text{ L/mol}} \approx \boxed{0.01\text{ g}}$$

Temperature Scales

32 • A certain ski wax is rated for use between –12 and –7.0ºC. What is this temperature range on the Fahrenheit scale?

Picture the Problem We can use the temperature conversion equation $t_F = \frac{9}{5}t_C + 32$ to convert the Celsius temperatures to Fahrenheit temperatures and then use these results to find the temperature range on the Fahrenheit scale.

Convert –12ºC to the equivalent Fahrenheit temperature:

$$t_F = \tfrac{9}{5}(-12) + 32 = 10°\text{F}$$

Convert –7.0ºC to the equivalent Fahrenheit temperature:

$$t_F = \tfrac{9}{5}(-7.0) + 32 = 19°\text{F}$$

The temperature range on the Fahrenheit scale is approximately $\boxed{10°\text{F to } 19°\text{F}}$.

33 • **[SSM]** The melting point of gold is 1945.4ºF. Express this temperature in degrees Celsius.

Picture the Problem We can use the Fahrenheit-Celsius conversion equation to find this temperature on the Celsius scale.

Convert 1945.4°F to the equivalent Celsius temperature:

$$t_C = \tfrac{5}{9}(t_F - 32°) = \tfrac{5}{9}(1945.4° - 32°)$$
$$= \boxed{1063°\text{C}}$$

34 • A weather report indicates that the temperature is expected to drop by 15.0°C over the next four hours. By how many degrees on the Fahrenheit scale will the temperature drop?

Picture the Problem We can use the fact that 5 C° = 9 F° to set up a proportion that allows us to make easy interval conversions from either the Celsius or Fahrenheit scale to the other.

The proportion relating a temperature range on the Fahrenheit scale of a temperature range on the Celsius scale is:

$$\frac{\Delta t_F}{\Delta t_C} = \frac{9\ \text{F}°}{5\ \text{C}°} \Rightarrow \Delta t_F = \left(\frac{9\ \text{F}°}{5\ \text{C}°}\right)\Delta t_C$$

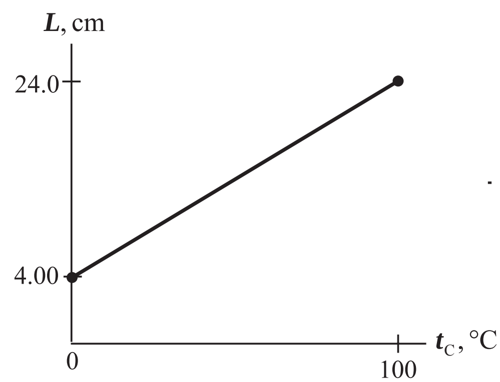
Substitute numerical values and evaluate Δt_F:

$$\Delta t_F = \left(\frac{9\ \text{F}°}{5\ \text{C}°}\right)(15.0\ \text{C}°) = \boxed{27.0\ \text{F}°}$$

35 • The length of the column of mercury in a thermometer is 4.00 cm when the thermometer is immersed in ice water at 1 atm of pressure, and 24.0 cm when the thermometer is immersed in boiling water at 1 atm of pressure. Assume that the length of the mercury column varies linearly with temperature. (*a*) Sketch a graph of the length of the mercury column versus temperature (in degrees Celsius). (*b*) What is the length of the column at room temperature (22.0ºC)? (*c*) If the mercury column is 25.4 cm long when the thermometer is immersed in a chemical solution, what is the temperature of the solution?

Picture the Problem We can use the equation of the graph plotted in (*a*) to (*b*) find the length of the mercury column at room temperature and (*c*) the temperature of the solution when the height of the mercury column is 25.4 cm.

(*a*) A graph of the length of the mercury column versus temperature (in degrees Celsius) is shown to the right. The equation of the line is:



$$L = \left(0.200\ \frac{\text{cm}}{°\text{C}}\right)t_C + 4.00\ \text{cm} \quad (1)$$

(*b*) Evaluate $L(22.0°\text{C})$

$$L = \left(0.200\ \frac{\text{cm}}{°\text{C}}\right)(22.0°\text{C}) + 4.00\ \text{cm}$$

$$= \boxed{8.40\ \text{cm}}$$

(*c*) Solve equation (1) for t_C to obtain:

$$t_C = \frac{L - 4.00\ \text{cm}}{0.200\ \frac{\text{cm}}{°\text{C}}}$$

Substitute numerical values and evaluate $t_C(25.4\,\text{cm})$:

$$t_C(25.4\,\text{cm}) = \frac{25.4\,\text{cm} - 4.00\,\text{cm}}{0.200\,\dfrac{\text{cm}}{°\text{C}}} = \boxed{107°\text{C}}$$

36 • The temperature of the interior of the Sun is about 1.0×10^7 K. What is this temperature in (*a*) Celsius degrees, and (*b*) Fahrenheit degrees?

Picture the Problem We can use the temperature conversion equations $t_F = \frac{9}{5}t_C + 32°$ and $t_C = T - 273.15\,\text{K}$ to convert 10^7 K to the Fahrenheit and Celsius temperatures.

Express the kelvin temperature in terms of the Celsius temperature:

$$T = t_C + 273.15\,\text{K}$$

(*a*) Solve for and evaluate t_C:

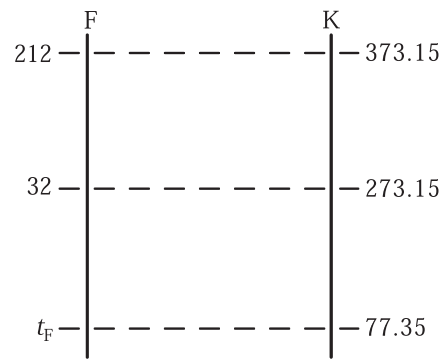
$$t_C = T - 273.15\,\text{K} = 1.0\times10^7\,\text{K} - 273.15\,\text{K} \approx \boxed{1.0\times10^7\,°\text{C}}$$

(*b*) Use the Celsius to Fahrenheit conversion equation to evaluate t_F:

$$t_F = \tfrac{9}{5}\left(1.0\times10^7\,°\text{C}\right) + 32° \approx \boxed{1.8\times10^7\,°\text{F}}$$

37 • The boiling point of nitrogen, N_2, is 77.35 K. Express this temperature in degrees Fahrenheit.

Picture the Problem While we could convert 77.35 K to a Celsius temperature and then convert the Celsius temperature to a Fahrenheit temperature, an alternative solution is to use the diagram to the right to set up a proportion for the direct conversion of the kelvin temperature to its Fahrenheit equivalent.



Use the diagram to set up the proportion:

$$\frac{32°\text{F} - t_F}{212°\text{F} - 32°\text{F}} = \frac{273.15\,\text{K} - 77.35\,\text{K}}{373.15\,\text{K} - 273.15\,\text{K}}$$

or

$$\frac{32°\text{F} - t_F}{180\,\text{F}°} = \frac{195.8}{100}$$

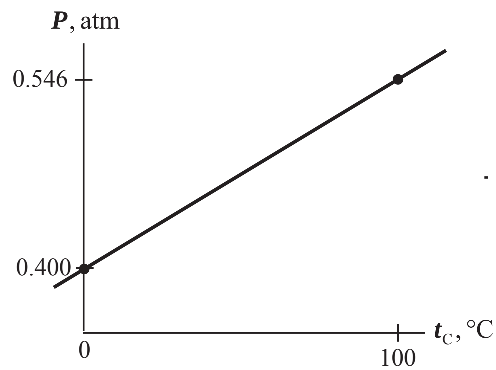
Solving for t_F yields:

$$t_F = 32°F - \frac{195.8}{100} \times 180\,F° = \boxed{-320°F}$$

38 • The pressure of a constant-volume gas thermometer is 0.400 atm at the ice point and 0.546 atm at the steam point. (a) Sketch a graph of pressure versus Celsius temperature for this thermometer. (*b*) When the pressure is 0.100 atm, what is the temperature? (*c*) What is the pressure at 444.6 ºC (the boiling point of sulfur)?

Picture the Problem We can use the equation of the graph plotted in (*a*) to (*b*) find the temperature when the pressure is 0.100 atm and (*c*) the pressure when the temperature is 444.6ºC.

(*a*) A graph of pressure (in atm) versus temperature (in degrees Celsius) for this thermometer is shown to the right. The equation of this graph is:



$$P = \left(1.46\times10^{-3}\,\frac{\text{atm}}{°\text{C}}\right)t_C + 0.400\,\text{atm}$$

(*b*) Solving for t_C yields:

$$t_C = \frac{P - 0.400\,\text{atm}}{1.46\times10^{-3}\,\frac{\text{atm}}{°\text{C}}}$$

Substitute numerical values and evaluate $t_C(0.100\,\text{atm})$:

$$t_C(0.100\,\text{atm}) = \frac{0.100\,\text{atm} - 0.400\,\text{atm}}{1.46\times10^{-3}\,\frac{\text{atm}}{°\text{C}}} = \boxed{-205°\text{C}}$$

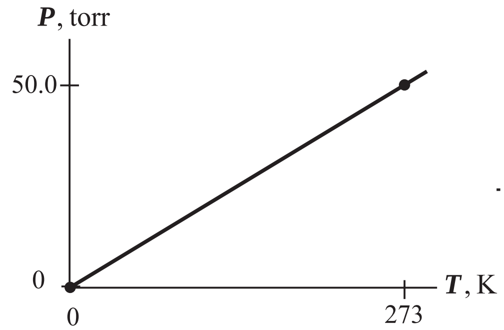
Substitute numerical values and evaluate $P(444.6°\text{C})$:

$$P(444.6°\text{C}) = \left(1.46\times10^{-3}\,\frac{\text{atm}}{°\text{C}}\right)(444.6°\text{C}) + 0.400\,\text{atm} = \boxed{1.05\,\text{atm}}$$

39 • **[SSM]** A constant-volume gas thermometer reads 50.0 torr at the triple point of water. (*a*) Sketch a graph of pressure vs. absolute temperature for this thermometer. (*b*) What will be the pressure when the thermometer measures a temperature of 300 K? (*c*) What ideal-gas temperature corresponds to a pressure of 678 torr?

Picture the Problem We can use the equation of the graph plotted in (*a*) to (*b*) find the pressure when the temperature is 300 K and (*c*) the ideal-gas temperature when the pressure is 678 torr.

(*a*) A graph of pressure (in torr) versus temperature (in kelvins) for this thermometer is shown to the right. The equation of this graph is:



$$P = \left(\frac{50.0\ \text{torr}}{273\ \text{K}}\right)T \qquad (1)$$

(*b*) Evaluate P when $T = 300$ K:

$$P(300\ \text{K}) = \left(\frac{50.0\ \text{torr}}{273\ \text{K}}\right)(300\ \text{K}) = \boxed{54.9\ \text{torr}}$$

(*c*) Solve equation (1) for T to obtain:

$$T = \left(\frac{273\ \text{K}}{50.0\ \text{torr}}\right)P$$

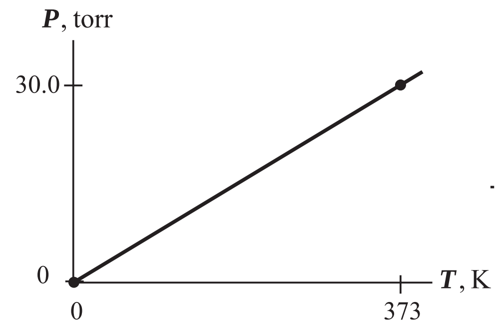
Evaluate $T(678\ \text{torr})$:

$$T(678\ \text{torr}) = \left(\frac{273\ \text{K}}{50.0\ \text{torr}}\right)(678\ \text{torr}) = \boxed{3.70\times10^{3}\ \text{K}}$$

40 • A constant-volume gas thermometer has a pressure of 30.0 torr when it reads a temperature of 373 K. (*a*) Sketch a graph of pressure vs. absolute temperature for this thermometer. (*b*) What is its triple-point pressure P_3? (*c*) What temperature corresponds to a pressure of 0.175 torr?

Picture the Problem We can use the equation of the graph plotted in (*a*) to (*b*) find the triple-point pressure P_3 and (*c*) the temperature when the pressure is 0.175 torr.

(*a*) A graph of pressure versus absolute temperature for this thermometer is shown to the right. The equation of this graph is:



$$P = \left(\frac{30.0\ \text{torr}}{373\ \text{K}}\right)T \quad (1)$$

(*b*) Solve for and evaluate the thermometer's triple-point (273.16 K) pressure:

$$P(273.16\ \text{K}) = \left(\frac{30.0\ \text{torr}}{373\ \text{K}}\right)(273.16\ \text{K}) = \boxed{22.0\ \text{torr}}$$

(*c*) Solve equation (1) for *T* to obtain:

$$T = \left(\frac{373\ \text{K}}{30.0\ \text{torr}}\right)P$$

Evaluate $T(0.175\ \text{torr})$:

$$T(0.175\ \text{torr}) = \left(\frac{373\ \text{K}}{30.0\ \text{torr}}\right)(0.175\ \text{torr}) = \boxed{2.18\ \text{K}}$$

41 • At what temperature do the Fahrenheit and Celsius temperature scales give the same reading?

Picture the Problem We can find the temperature at which the Fahrenheit and Celsius scales give the same reading by setting $t_F = t_C$ in the temperature-conversion equation.

Set $t_F = t_C$ in $t_C = \frac{5}{9}(t_F - 32°)$:

$$t_F = \tfrac{5}{9}(t_F - 32°)$$

Solve for and evaluate t_F:

$$t_C = t_F = \boxed{-40.0°\text{C}} = \boxed{-40.0°\text{F}}$$

Remarks: If you've not already thought of doing so, you might use your graphing calculator to plot t_C versus t_F and $t_F = t_C$ (a straight line at 45°) on the same graph. Their intersection is at (–40, –40).

42 • Sodium melts at 371 K. What is the melting point of sodium on the Celsius and Fahrenheit temperature scales?

Picture the Problem We can use the Celsius-to-absolute conversion equation to find 371 K on the Celsius scale and the Celsius-to-Fahrenheit conversion equation to find the Fahrenheit temperature corresponding to 371 K.

Express the absolute temperature as a function of the Celsius temperature: $T = t_C + 273.15\,\text{K}$

Solve for and evaluate t_C:

$$t_C = T - 273.15\,\text{K}$$
$$= 371\,\text{K} - 273.15\,\text{K} = \boxed{98°\text{C}}$$

Use the Celsius-to-Fahrenheit conversion equation to find t_F:

$$t_F = \tfrac{9}{5}t_C + 32° = \tfrac{9}{5}(97.9°) + 32°$$
$$= \boxed{208°\text{F}}$$

43 • The boiling point of oxygen at 1.00 atm is 90.2 K. What is the boiling point of oxygen at 1.00 atm on the Celsius and Fahrenheit scales?

Picture the Problem We can use the Celsius-to-absolute conversion equation to find 90.2 K on the Celsius scale and the Celsius-to-Fahrenheit conversion equation to find the Fahrenheit temperature corresponding to 90.2 K.

Express the absolute temperature as a function of the Celsius temperature: $T = t_C + 273.15\,\text{K}$

Solve for and evaluate t_C:

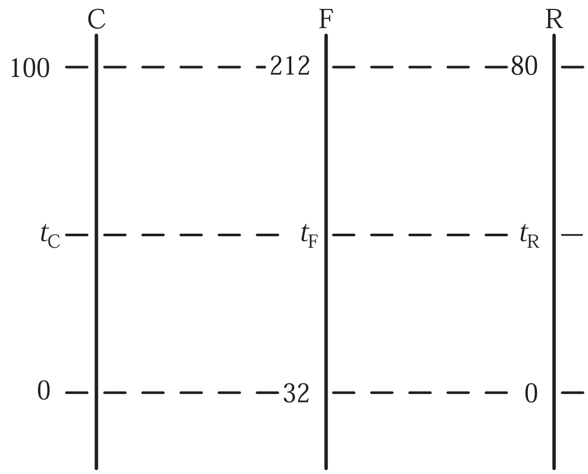
$$t_C = T - 273.15\,\text{K}$$
$$= 90.2\,\text{K} - 273.15\,\text{K} = \boxed{-183°\text{C}}$$

Use the Celsius-to-Fahrenheit conversion equation to find t_F:

$$t_F = \tfrac{9}{5}t_C + 32° = \tfrac{9}{5}(-183°) + 32°$$
$$= \boxed{-297°\text{F}}$$

44 •• On the Réaumur temperature scale, the melting point of ice is 0ºR and the boiling point of water is 80ºR. Derive expressions for converting temperatures on the Réaumur scale to the Celsius and Fahrenheit scales.

Picture the Problem We can use the following diagram to set up proportions that will allow us to convert temperatures on the Réaumur scale to Celsius and Fahrenheit temperatures.



Referring to the diagram, set up a proportion to convert temperatures on the Réaumur scale to Celsius temperatures:

$$\frac{t_C - 0°C}{100°C - 0°C} = \frac{t_R - 0°R}{80°R - 0°R}$$

Simplify to obtain:

$$\frac{t_C}{100} = \frac{t_R}{80} \Rightarrow t_C = \boxed{\tfrac{5}{4}t_R}$$

Referring to the diagram, set up a proportion to convert temperatures on the Réaumur scale to Fahrenheit temperatures:

$$\frac{t_F - 32°F}{212°F - 32°F} = \frac{t_R - 0°R}{80°R - 0°R}$$

Simplify to obtain:

$$\frac{t_F - 32}{180} = \frac{t_R}{80} \Rightarrow t_F = \boxed{\tfrac{9}{4}t_R + 32}$$

45 ••• [SSM] A thermistor is a solid-state device widely used in a variety of engineering applications. Its primary characteristic is that its electrical resistance varies greatly with temperature. Its temperature dependence is given approximately by $R = R_0e^{B/T}$, where R is in ohms (Ω), T is in kelvins, and R_0 and B are constants that can be determined by measuring R at calibration points such as the ice point and the steam point. (*a*) If R = 7360 Ω at the ice point and 153 Ω at the steam point, find R_0 and B. (*b*) What is the resistance of the thermistor at t = 98.6ºF? (*c*) What is the rate of change of the resistance with temperature (dR/dT) at the ice point and the steam point? (*d*) At which temperature is the thermistor most sensitive?

Picture the Problem We can use the temperature dependence of the resistance of the thermistor and the given data to determine R_0 and B. Once we know these quantities, we can use the temperature-dependence equation to find the resistance at any temperature in the calibration range. Differentiation of R with respect to T

will allow us to express the rate of change of resistance with temperature at both the ice point and the steam point temperatures.

(*a*) Express the resistance at the ice point as a function of temperature of the ice point:

$$7360\,\Omega = R_0 e^{B/273\,\text{K}} \quad (1)$$

Express the resistance at the steam point as a function of temperature of the steam point:

$$153\,\Omega = R_0 e^{B/373\,\text{K}} \quad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{7360\,\Omega}{153\,\Omega} = 48.10 = e^{B/273\,\text{K} - B/373\,\text{K}}$$

Solve for B by taking the logarithm of both sides of the equation:

$$\ln 48.1 = B\left(\frac{1}{273} - \frac{1}{373}\right)\text{K}^{-1}$$

and

$$B = \frac{\ln 48.1}{\left(\frac{1}{273} - \frac{1}{373}\right)\text{K}^{-1}} = 3.944 \times 10^3\ \text{K}$$

$$= \boxed{3.94 \times 10^3\ \text{K}}$$

Solve equation (1) for R_0 and substitute for B:

$$R_0 = \frac{7360\,\Omega}{e^{B/273\,\text{K}}} = (7360\,\Omega)e^{-B/273\,\text{K}}$$

$$= (7360\,\Omega)e^{-3.944\times10^3\,\text{K}/273\,\text{K}}$$

$$= 3.913 \times 10^{-3}\ \Omega$$

$$= \boxed{3.91 \times 10^{-3}\ \Omega}$$

(*b*) From (*a*) we have:

$$R = \left(3.913 \times 10^{-3}\ \Omega\right)e^{3.944\times10^3\,\text{K}/T}$$

Convert 98.6°F to kelvins to obtain:

$$T = 310\,\text{K}$$

Substitute for T to obtain:

$$R(310\,\text{K}) = \left(3.913 \times 10^{-3}\ \Omega\right)e^{3.944\times10^3\,\text{K}/310\,\text{K}}$$

$$= \boxed{1.31\,\text{k}\Omega}$$

(*c*) Differentiate R with respect to T to obtain:

$$\frac{dR}{dT} = \frac{d}{dT}\left(R_0 e^{B/T}\right) = R_0 e^{B/T}\frac{d}{dT}\left(\frac{B}{T}\right)$$

$$= \frac{-B}{T^2}R_0 e^{B/T} = -\frac{RB}{T^2}$$

Evaluate dR/dT at the ice point:

$$\left(\frac{dR}{dT}\right)_{\text{ice point}} = -\frac{(7360\,\Omega)(3.943\times10^{3}\text{ K})}{(273.16\text{K})^{2}} = \boxed{-389\,\Omega/\text{K}}$$

Evaluate dR/dT at the steam point:

$$\left(\frac{dR}{dT}\right)_{\text{steam point}} = -\frac{(153\,\Omega)(3.943\times10^{3}\text{ K})}{(373.16\text{K})^{2}} = \boxed{-4.33\,\Omega/\text{K}}$$

(*d*) The thermistor is more sensitive (has greater sensitivity) at lower temperatures.

The Ideal-Gas Law

46 • An ideal gas in a cylinder fitted with a piston (Figure 17-20) is held at fixed pressure. If the temperature of the gas increases from 50° to 100°C, by what factor does the volume change?

Picture the Problem Let the subscript 50 refer to the gas at 50°C and the subscript 100 to the gas at 100°C. We can apply the ideal-gas law for a fixed amount of gas to find the ratio of the final and initial volumes.

Apply the ideal-gas law for a fixed amount of gas:

$$\frac{P_{100}V_{100}}{T_{100}} = \frac{P_{50}V_{50}}{T_{50}}$$

or, because $P_{100} = P_{50}$,

$$\frac{V_{100}}{V_{50}} = \frac{T_{100}}{T_{50}}$$

Substitute numerical values and evaluate V_{100}/V_{50}:

$$\frac{V_{100}}{V_{50}} = \frac{(273.15+100)\text{K}}{(273.15+50)\text{K}} = \boxed{1.15}$$

or a 15% increase in volume.

47 • **[SSM]** A 10.0-L vessel contains gas at a temperature of 0.00ºC and a pressure of 4.00 atm. How many moles of gas are in the vessel? How many molecules?

Picture the Problem We can use the ideal-gas law to find the number of moles of gas in the vessel and the definition of Avogadro's number to find the number of molecules.

Apply the ideal-gas law to the gas:

$$PV = nRT \Rightarrow n = \frac{PV}{RT}$$

Substitute numerical values and evaluate n:

$$n = \frac{(4.00\,\text{atm})(10.0\,\text{L})}{(8.206\times10^{-2}\ \text{L}\cdot\text{atm/mol}\cdot\text{K})(273\,\text{K})} = 1.786\ \text{mol} = \boxed{1.79\,\text{mol}}$$

Relate the number of molecules N in the gas in terms of the number of moles n:

$$N = nN_{\text{A}}$$

Substitute numerical values and evaluate N:

$$N = (1.786\,\text{mol})(6.022\times10^{23}\ \text{molecules/mol}) = \boxed{1.08\times10^{24}\ \text{molecules}}$$

48 •• A pressure as low as 1.00×10^{-8} torr can be achieved using an oil diffusion pump. How many molecules are there in $1.00\ \text{cm}^3$ of a gas at this pressure if its temperature is 300 K?

Picture the Problem We can use the ideal-gas law to relate the number of molecules in the gas to its pressure, volume, and temperature.

Solve the ideal-gas law for the number of molecules in a gas as a function of its pressure, volume, and temperature:

$$N = \frac{PV}{kT}$$

Substitute numerical values and evaluate N:

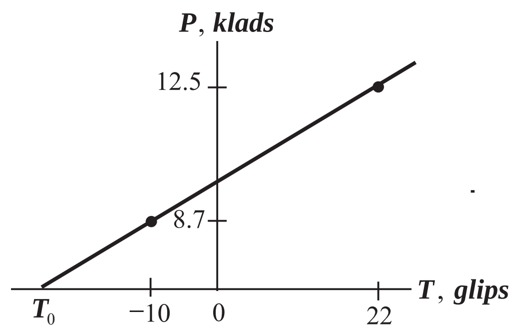
$$N = \frac{(1.00\times10^{-8}\ \text{torr})(133.32\,\text{Pa/torr})(1.00\times10^{-6}\ \text{m}^3)}{(1.381\times10^{-23}\ \text{J/K})(300\,\text{K})} = \boxed{3.22\times10^{8}}$$

49 •• You copy the following paragraph from a Martian physics textbook: "1 *snorf* of an ideal gas occupies a volume of 1.35 *zaks*. At a temperature of 22 *glips*, the gas has a pressure of 12.5 *klads*. At a temperature of –10 *glips*, the same gas now has a pressure of 8.7 *klads*." Determine the temperature of absolute zero in *glips*.

Picture the Problem Because the gas is ideal, its pressure is directly proportional to its temperature. Hence, a graph of P versus T will be linear and the linear equation relating P and T can be solved for the temperature corresponding to zero pressure. We'll assume that the data was taken at constant volume.

A graph of P, in *klads*, as function of T, in *glips*, is shown to the right. The equation of this graph is:

$$P = \left(\frac{3.8\ klads}{32\ glips}\right)T + 9.9\ klads$$



When $P = 0$:

$$0 = \left(\frac{3.8\ klads}{32\ glips}\right)T_0 + 9.9\ klads$$

Solve for T_0 to obtain:

$$T_0 = \boxed{-83\,glips}$$

50 •• A motorist inflates the tires of her car to a gauge pressure of 180 kPa on a day when the temperature is –8.0ºC. When she arrives at her destination, the tire pressure has increased to 245 kPa. What is the temperature of the tires if we assume that (*a*) the tires do not expand or (*b*) that the tires expand so the volume of the enclosed air increases by 7 percent?

Picture the Problem Let the subscript 1 refer to the tires when their gauge pressure is 180 kPa and the subscript 2 to conditions when their gauge pressure is 245 kPa. Assume that the air in the tires behaves as an ideal gas. Then, we can apply the ideal-gas law for a fixed amount of gas to relate the temperatures to the pressures and volumes of the tires.

(*a*) Apply the ideal-gas law for a fixed amount of gas to the air in the tires:

$$\frac{P_2V_2}{T_2} = \frac{P_1V_1}{T_1} \quad (1)$$

where the temperatures and pressures are absolute.

Solve for T_2:

$$T_2 = T_1\frac{P_2}{P_1}$$ because $V_1 = V_2$.

Substitute numerical values to obtain:

$$T_2 = (265\,\text{K})\frac{245\,\text{kPa} + 101\,\text{kPa}}{180\,\text{kPa} + 101\,\text{kPa}} = 326.3\,\text{K} = \boxed{53°\text{C}}$$

(*b*) Use equation (1) with $V_2 = 1.07\ V_1$. Solving for T_2 gives:

$$T_2 = \frac{P_2V_2}{P_1V_1}T_1 = (1.07)\left(\frac{P_2}{P_1}T_1\right)$$

Substitute numerical values and evaluate T_2:

$$T_2 = (1.07)(326.3\,\text{K}) = 349.1\,\text{K}$$
$$= \boxed{76°\text{C}}$$

51 •• A room is 6.0 m by 5.0 m by 3.0 m. (*a*) If the air pressure in the room is 1.0 atm and the temperature is 300 K, find the number of moles of air in the room. (*b*) If the temperature increases by 5.0 K and the pressure remains constant, how many moles of air leave the room?

Picture the Problem We can apply the ideal-gas law to find the number of moles of air in the room as a function of the temperature.

(*a*) Use the ideal-gas law to relate the number of moles of air in the room to the pressure, volume, and temperature of the air:

$$n = \frac{PV}{RT} \qquad (1)$$

Substitute numerical values and evaluate *n*:

$$n = \frac{(101.325\,\text{kPa})(6.0\,\text{m})(5.0\,\text{m})(3.0\,\text{m})}{(8.314\,\text{J/mol}\cdot\text{K})(300\,\text{K})}$$
$$= 3.66\times10^3\ \text{mol} = \boxed{3.7\times10^3\ \text{mol}}$$

(*b*) Letting n' represent the number of moles in the room when the temperature rises by 5 K, express the number of moles of air that leave the room:

$$\Delta n = n - n'$$

Apply the ideal-gas law to obtain:

$$n' = \frac{PV}{RT'} \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{n'}{n} = \frac{T}{T'} \text{ and } n' = n\frac{T}{T'}$$

Substitute for n' to obtain:

$$\Delta n = n - n\frac{T}{T'} = n\left(1 - \frac{T}{T'}\right)$$

Substitute numerical values and evaluate Δn:

$$\Delta n = (3.66\times10^{3}\text{ mol})\left(1-\frac{300\text{ K}}{305\text{ K}}\right) = \boxed{60\text{ mol}}$$

52 •• Image that 10.0 g of liquid helium, initially at 4.20 K, evaporate into an empty balloon that is kept at 1.00 atm pressure. What is the volume of the balloon at (*a*) 25.0 K and (*b*) 293 K?

Picture the Problem Let the subscript 1 refer to helium gas at 4.2 K and the subscript 2 to the gas at 293 K. We can apply the ideal-gas law to find the volume of the gas at 4.2 K and a fixed amount of gas to find its volume at 293 K. See Appendix C for the molar mass of helium.

(*a*) Apply the ideal-gas law to the helium gas to express its volume:

$$V_1 = \frac{nRT_1}{P_1} = \frac{\frac{m}{M}RT_1}{P_1} = \frac{mRT_1}{MP_1}$$

Substitute numerical values and evaluate V_1:

$$V_1 = \frac{(10.0\text{ g})(0.08206\text{ L}\cdot\text{atm/mol}\cdot\text{K})(25.0\text{ K})}{(4.003\text{ g/mol})(1.00\text{ atm})} = 5.125\text{ L} = \boxed{5.13\text{ L}}$$

(*b*) Apply the ideal-gas law for a fixed amount of gas and solve for the volume of the helium gas at 293 K:

$$\frac{P_2V_2}{T_2} = \frac{P_1V_1}{T_1}$$

and, because $P_1 = P_2$,

$$V_2 = \frac{T_2}{T_1}V_1$$

Substitute numerical values and evaluate V_2:

$$V_2 = \frac{293\text{ K}}{25.0\text{ K}}(5.125\text{ L}) = \boxed{60.1\text{ L}}$$

53 •• A closed container with a volume of 6.00 L holds 10.0 g of liquid helium at 25.0 K and enough air to fill the rest of its volume at a pressure of 1.00 atm. The helium then evaporates and the container warms to room temperature (293 K). What is the final pressure inside the container?

Picture the Problem We can apply the law of partial pressures to find the final pressure inside the container. See Appendix C for the molar mass of helium and of air.

The final pressure inside the container is the sum of the partial pressures of helium gas and air:

$$P_{\text{final}} = P_{\text{He gas}} + P_{\text{air}} \quad (1)$$

The pressure exerted by the air molecules at room temperature is given by the ideal-gas law:

$$P_{\text{air}} = \frac{n_{\text{air}}RT}{V} = \frac{m_{\text{air}}RT}{M_{\text{air}}V} = \frac{\rho_{\text{air}}VRT}{M_{\text{air}}V} = \frac{\rho_{\text{air}}RT}{M_{\text{air}}}$$

The pressure exerted by the helium molecules at room temperature is also given by the ideal-gas law:

$$P_{\text{He gas}} = \frac{n_{\text{He}}RT}{V} = \frac{m_{\text{He}}RT}{M_{\text{He}}V}$$

Substituting in equation (1) yields:

$$P_{\text{final}} = \frac{m_{\text{He}}RT}{M_{\text{He}}V} + \frac{\rho_{\text{air}}RT}{M_{\text{air}}} = \left(\frac{m_{\text{He}}}{M_{\text{He}}V} + \frac{\rho_{\text{air}}}{M_{\text{air}}}\right)RT$$

Substitute numerical values and evaluate P_2:

$$P_{\text{final}} = \left(\frac{10.0\text{ g}}{\left(4.003\,\frac{\text{g}}{\text{mol}}\right)\left(6.00\text{ L}\times\frac{10^{-3}\text{ m}^3}{\text{L}}\right)} + \frac{1.293\,\frac{\text{kg}}{\text{m}^3}}{28.81\,\frac{\text{g}}{\text{mol}}}\right)\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(293\text{ K})$$

$$= 1.124\times10^{6}\text{ Pa}\times\frac{1\text{ atm}}{101.325\text{ kPa}} = \boxed{11.1\text{ atm}}$$

54 •• An automobile tire is filled to a gauge pressure of 200 kPa when its temperature is 20ºC. (Gauge pressure is the difference between the actual pressure and atmospheric pressure.) After the car has been driven at high speeds, the tire temperature increases to 50ºC. (*a*) Assuming that the volume of the tire does not change and that air behaves as an ideal gas, find the gauge pressure of the air in the tire. (*b*) Calculate the gauge pressure if the tire expands so the volume of the enclosed air increases by 10 percent.

Picture the Problem Let the subscript 1 refer to the tire when its temperature is 20°C and the subscript 2 to conditions when its temperature is 50°C. We can apply the ideal-gas law for a fixed amount of gas to relate the temperatures to the pressures of the air in the tire.

(*a*) Apply the ideal-gas law for a fixed amount of gas and solve for pressure at the higher temperature:

$$\frac{P_2V_2}{T_2}=\frac{P_1V_1}{T_1} \qquad (1)$$

and

$$P_2=\frac{T_2}{T_1}P_1$$

because $V_1 = V_2$

Substitute numerical values to obtain:

$$P_2=\frac{323\,\text{K}}{293\,\text{K}}\left(200\,\text{kPa}+101\,\text{kPa}\right)=332\,\text{kPa}$$

and

$$P_{2,\text{gauge}}=332\,\text{kPa}-101\,\text{kPa}=\boxed{231\,\text{kPa}}$$

(*b*) Solve equation (1) for P_2:

$$P_2=\frac{V_1T_2}{V_2T_1}P_1$$

Because $V_2 = 1.10\ V_1$:

$$P_2=\frac{V_1T_2}{1.10V_1T_1}P_1=\frac{T_2}{1.10T_1}P_1$$

Substitute numerical values and evaluate P_2 and $P_{2,\text{gauge}}$:

$$P_2=\frac{323\,\text{K}}{(1.10)(293\,\text{K})}\left(200\,\text{kPa}+101\,\text{kPa}\right)=302\,\text{kPa}$$

and

$$P_{2,\text{gauge}}=302\,\text{kPa}-101\,\text{kPa}=\boxed{201\,\text{kPa}}$$

55 •• [SSM] After nitrogen (N_2) and oxygen (O_2), the most abundant molecule in Earth's atmosphere is water, H_2O. However, the fraction of H_2O molecules in a given volume of air varies dramatically, from practically zero percent under the driest conditions to as high as 4 percent where it is very humid. (*a*) At a given temperature and pressure, would air be denser when its water vapor content is large or small? (*b*) What is the difference in mass, at room temperature and atmospheric pressure, between a cubic meter of air with no water vapor molecules, and a cubic meter of air in which 4 percent of the molecules are water vapor molecules?

Picture the Problem (*a*) At a given temperature and pressure, the ideal-gas law tells us that the total number of molecules per unit volume, *N*/*V*, is constant. The denser gas will, therefore, be the one in which the average mass *per molecule* is greater. (*b*) We can apply the ideal-gas law and use the relationship between the masses of the dry and humid air and their molar masses to find the difference in

mass, at room temperature and atmospheric pressure, between a cubic meter of air containing no water vapor, and a cubic meter of air containing 4% water vapor. See Appendix C for the molar masses of N_2, O_2, and H_2O.

(*a*) The molecular mass of N_2 is 28.014 u (see Appendix C), that of O_2 is 31.999 u, and that of H_2O is 18.0152 u. Because H_2O molecules are lighter than the predominant molecules in air, a given volume of air will be less dense when its water vapor content is higher.

(*b*) Express the difference in mass between a cubic meter of air containing no water vapor, and a cubic meter of air containing 4% water vapor:

$$\Delta m = m_{\text{dry}} - m_{\text{humid}}$$

The masses of the dry air and humid air are related to their molar masses according to:

$$m_{\text{dry}} = 0.04nM_{\text{dry}}$$

and

$$m_{\text{humid}} = 0.04nM_{\text{humid}}$$

Substitute for m_{dry} and m_{humid} and simplify to obtain:

$$\begin{aligned}\Delta m &= 0.04nM_{\text{dry}} - 0.04nM_{\text{humid}}\\ &= 0.04n\left(M_{\text{dry}} - M_{\text{humid}}\right)\end{aligned}$$

From the ideal-gas law we have:

$$n = \frac{P_{\text{atm}}V}{RT}$$

and so Δm can be written as

$$\Delta m = \frac{0.04P_{\text{atm}}V}{RT}\left(M_{\text{dry}} - M_{\text{humid}}\right)$$

Substitute numerical values (28.811 g/mol is an 80% nitrogen and 20% oxygen weighted average) and evaluate Δm:

$$\Delta m = \frac{(0.04)(101.325\ \text{kPa})\left(1.00\ \text{m}^3\right)}{(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K})}(28.811\ \text{g/mol} - 18.0152\ \text{g/mol}) \approx \boxed{18\ \text{g}}$$

Remarks: With 4% H_2O, the remaining 96% of the air is mostly a mixture of O_2 and N_2. With dry air, the 4% that was H_2O would be the same mix as the other 96%. Therefore, we need only consider the makeup of the 4%.

56 •• A scuba diver is 40 m below the surface of a lake, where the temperature is 5.0ºC. He releases an air bubble that has a volume of 15 cm^3. The bubble rises to the surface, where the temperature is 25ºC. Assume that the air in the bubble is always in thermal equilibrium with the surrounding water, and

assume that there is no exchange of molecules between the bubble and the surrounding water. What is the volume of the bubble right before it breaks the surface? *Hint: Remember that the pressure also changes*.

Picture the Problem Let the subscript 1 refer to the conditions at the bottom of the lake and the subscript 2 to the surface of the lake and apply the ideal-gas law for a fixed amount of gas.

Apply the ideal-gas law for a fixed amount of gas:

$$\frac{P_2V_2}{T_2}=\frac{P_1V_1}{T_1}\Rightarrow V_2=\frac{V_1T_2P_1}{T_1P_2}$$

The pressure at the bottom of the lake is the sum of the pressure at its surface (atmospheric) and the pressure due to the depth of the lake:

$$P_1=P_{\text{atm}}+\rho gh$$

Substituting for P_1 yields:

$$V_2=\frac{V_1T_2\left(P_{\text{atm}}+\rho gh\right)}{T_1P_2}$$

Substitute numerical values and evaluate V_2:

$$V_2=\frac{\left(15\text{ cm}^3\right)\left(298\text{ K}\right)\left[101.325\text{ kPa}+\left(1.00\times10^3\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)\left(40\text{ m}\right)\right]}{\left(278\text{ K}\right)\left(101.325\text{ kPa}\right)}$$
$$=\boxed{78\text{ cm}^3}$$

57 •• [SSM] A hot-air balloon is open at the bottom. The balloon has a volume of 446 m^3 is filled with air with an average temperature of 100°C. The air outside the balloon has a temperature of 20.0°C and a pressure of 1.00 atm. How large a payload (including the envelope of the balloon itself) can the balloon lift? Use 29.0 g/mol for the molar mass of air. (Neglect the volume of both the payload and the envelope of the balloon.)

Picture the Problem Assume that the volume of the balloon is not changing. Then the air inside and outside the balloon must be at the same pressure of about 1.00 atm. The contents of the balloon are the air molecules inside it. We can use Archimedes principle to express the buoyant force on the balloon and we can find the weight of the air molecules inside the balloon. You'll need to determine the molar mass of air. See Appendix C for the molar masses of oxygen and nitrogen.

Express the net force on the balloon and its contents:

$$F_{\text{net}}=B-w_{\text{air inside the balloon}} \qquad (1)$$

Using Archimedes principle, express the buoyant force on the balloon:

$B = w_{\text{displaced fluid}} = m_{\text{displaced fluid}}g$

or

$B = \rho_{\text{o}} V_{\text{balloon}} g$

where ρ_{o} is the density of the air outside the balloon.

Express the weight of the air inside the balloon:

$w_{\substack{\text{air inside}\\ \text{the balloon}}} = \rho_{\text{i}} V_{\text{balloon}} g$

where ρ_{i} is the density of the air inside the balloon.

Substitute in equation (1) for B and $w_{\substack{\text{air inside}\\ \text{the balloon}}}$ to obtain:

$$F_{\text{net}} = \rho_{\text{o}} V_{\text{balloon}} g - \rho_{\text{i}} V_{\text{balloon}} g = (\rho_{\text{o}} - \rho_{\text{i}}) V_{\text{balloon}} g \qquad (2)$$

Express the densities of the air molecules in terms of their number densities, molecular mass, and Avogadro's number:

$$\rho = \frac{M}{N_{\text{A}}}\left(\frac{N}{V}\right)$$

Using the ideal-gas law, relate the number density of air N/V to its temperature and pressure:

$$PV = NkT \text{ and } \frac{N}{V} = \frac{P}{kT}$$

Substitute for $\frac{N}{V}$ to obtain:

$$\rho = \frac{M}{N_{\text{A}}}\left(\frac{P}{kT}\right)$$

Substitute in equation (2) and simplify to obtain:

$$F_{\text{net}} = \frac{MP}{N_{\text{A}}k}\left(\frac{1}{T_{\text{o}}} - \frac{1}{T_{\text{i}}}\right) V_{\text{balloon}} g$$

Assuming that the average molar mass of air is 29.0 g/mol, substitute numerical values and evaluate F_{net}:

$$F_{\text{net}} = \frac{(29.0\text{ g/mol})(101.325\text{ kPa})\left(\frac{1}{293\text{ K}} - \frac{1}{373\text{ K}}\right)(446\text{ m}^3)(9.81\text{ m/s}^2)}{(6.022\times10^{23}\text{ particles/mol})(1.381\times10^{-23}\text{ J/K})} = \boxed{1.1\text{ kN}}$$

58 ••• A helium balloon is used to lift a load of 110 N. The weight of the envelope of the balloon is 50.0 N and the volume of the helium when the balloon is fully inflated is 32.0 m^3. The temperature of the air is 0ºC and the atmospheric pressure is 1.00 atm. The balloon is inflated with a sufficient amount of helium gas that the net upward force on the balloon and its load is 30.0 N. Neglect any

effects due to the changes of temperature as the altitude changes. (*a*) How many moles of helium gas are contained in the balloon? (*b*) At what altitude will the balloon be fully inflated? (*c*) Does the balloon ever reach the altitude at which it is fully inflated? (*d*) If the answer to (*c*) is ″Yes,″ what is the maximum altitude attained by the balloon?

Picture the Problem (*a*) We can find the number of moles of helium gas in the balloon by applying the ideal-gas law to relate *n* to the pressure, volume, and temperature of the helium and Archimedes principle to find the volume of the helium. In Part (*b*), we can apply the result of Problem 13-91 to relate atmospheric pressure to altitude and use the ideal-gas law to determine the pressure of the gas when the balloon is fully inflated. In Part (*c*), we'll find the net force acting on the balloon at the altitude at which it is fully inflated in order to decide whether it can rise to that altitude.

(*a*) Apply the ideal-gas law to the helium in the balloon and solve for *n*:

$$n = \frac{PV}{RT} \qquad (1)$$

Relate the net force on the balloon to its weight:

$$F_B - w_{skin} - w_{load} - w_{He} = 30\,\text{N}$$

Use Archimedes principle to express the buoyant force on the balloon in terms of the volume of the balloon:

$$F_B = w_{displaced\,air} = \rho_{air} V g$$

Substitute to obtain:

$$\rho_{air} V g - w_{skin} - w_{load} - \rho_{He} V g = 30.0\,\text{N}$$

Solve for the volume of the helium:

$$V = \frac{30.0\,\text{N} + w_{skin} + w_{load}}{(\rho_{air} - \rho_{He})g}$$

Substituting for *V* in equation (1) yields:

$$n = \frac{P(30.0\,\text{N} + w_{skin} + w_{load})}{RT(\rho_{air} - \rho_{He})g}$$

Substitute numerical values and evaluate n:

$$n = \frac{(1.00\,\text{atm})(30.0\,\text{N}+50.0\,\text{N}+110\,\text{N})\left(\dfrac{1\,\text{L}}{10^{-3}\,\text{m}^3}\right)}{(8.206\times10^{-2}\,\text{L}\cdot\text{atm/mol}\cdot\text{K})(273\,\text{K})(1.293\,\text{kg/m}^3-0.179\,\text{kg/m}^3)(9.81\,\text{m/s}^2)}$$
$$= \boxed{776\,\text{mol}}$$

(b) Using the result of Problem 13-91, express the variation in atmospheric pressure with altitude:

$$P(h) = P_0 e^{-Ch}$$
where $C = 0.13\,\text{km}^{-1}$.

Solving for h yields:

$$h = \frac{1}{C}\ln\left[\frac{P_0}{P(h)}\right] \qquad (2)$$

Using the ideal-gas law, express P:

$$P = \frac{nRT}{V}$$

Substitute for P in equation (2) and simplify to obtain:

$$h = \frac{1}{C}\ln\left[\frac{P_0}{\dfrac{nRT}{V}}\right] = \frac{1}{C}\ln\left[\frac{P_0V}{nRT}\right]$$

Substitute numerical values and evaluate h:

$$h = \left(\frac{1}{0.13\,\text{km}^{-1}}\right)\ln\left[\frac{(1.00\,\text{atm})\left(32.0\,\text{m}^3\times\dfrac{1\,\text{L}}{10^{-3}\,\text{m}^3}\right)}{(776\,\text{mol})(8.206\times10^{-2}\,\text{L}\cdot\text{atm/mol}\cdot\text{K})(273\,\text{K})}\right]$$
$$= 4.69\,\text{km} = \boxed{4.7\,\text{km}}$$

(c) Express the condition that must be satisfied if the balloon is to reach its fully inflated altitude:

$$F_{\text{net}} = F_{\text{B}} - w_{\text{tot}} \geq 0 \qquad (3)$$

The total weight is the sum of the weights of the load, balloon skin, and helium:

$$w_{\text{tot}} = w_{\text{load}} + w_{\text{skin}} + w_{\text{He}}$$
or, because $w_{\text{He}} = \rho_{\text{He}}Vg$,
$$w_{\text{tot}} = w_{\text{load}} + w_{\text{skin}} + \rho_{\text{He}}Vg$$

Express the buoyant force on the balloon at $h = 4.84$ km:

$$F_B = \rho_{\text{air},h} Vg \qquad (4)$$

Express the dependence of the density of the air on atmospheric pressure:

$$\frac{P}{P_0} = \frac{\rho_{\text{air},h}}{\rho_{\text{air}}} \Rightarrow \rho_{\text{air},h} = \frac{P}{P_0}\rho_{\text{air}} \qquad (5)$$

Substitute for $\rho_{\text{air},h}$ in equation (4) to obtain:

$$F_B = \frac{P}{P_0}\rho_{\text{air}} Vg$$

Substitute for w_{tot} and F_B in equation (3) and simplify to obtain:

$$F_{\text{net}} = \frac{P}{P_0} Vg(\rho_{\text{air}} - \rho_{\text{He}}) - w_{\text{load}} - w_{\text{skin}}$$

or, because $P(h) = P_0 e^{-Ch}$,

$$F_{\text{net}} = e^{-Ch} Vg(\rho_{\text{air}} - \rho_{\text{He}}) - w_{\text{load}} - w_{\text{skin}}$$

Substitute numerical values and evaluate F_{net}:

$$F_{\text{net}} = e^{-(0.13\,\text{km}^{-1})(4.69\,\text{km})}(32.0\,\text{m}^3)(9.81\,\text{m/s}^2)(1.293\,\text{kg/m}^3 - 0.179\,\text{kg/m}^3) - 110\,\text{N} - 50\,\text{N} = 30\,\text{N}$$

Because $F_{\text{net}} = 30$ N > 0, the balloon will rise higher than the altitude at which it is fully inflated.

(*d*) The balloon will rise until the net force acting on it is zero. Because the buoyant force depends on the density of the air, the balloon will rise until the density of the air has decreased sufficiently for the buoyant force to just equal the total weight of the balloon.

Substitute equation (5) in equation (2) to obtain:

$$h = \frac{1}{C}\ln\frac{\rho_{\text{air}}}{\rho_{\text{air},h}}$$

Using equation (4), express the density of the air in terms of F_B:

$$\rho_{\text{air},h} = \frac{F_B}{Vg}$$

Substituting for $\rho_{\text{air},h}$ and simplifying yields:

$$h = \frac{1}{C}\ln\left[\frac{\rho_{\text{air}}}{\frac{F_B}{Vg}}\right] = \frac{1}{C}\ln\left[\frac{Vg\rho_{\text{air}}}{F_B}\right]$$

Substitute numerical values and evaluate h:

$$h = \left(\frac{1}{0.13\ \text{km}^{-1}}\right)\ln\left[\frac{(32.0\ \text{m}^3)(9.81\ \text{m/s}^2)(1.293\ \text{kg/m}^3)}{190.5\ \text{N}}\right] = \boxed{5.8\ \text{km}}$$

Kinetic Theory of Gases

59 • **[SSM]** (*a*) One mole of argon gas is confined to a 1.0-liter container at a pressure of 10 atm. What is the rms speed of the argon atoms? (*b*) Compare your answer to the rms speed for helium atoms under the same conditions.

Picture the Problem We can express the rms speeds of argon and helium atoms by combining $PV = nRT$ and $v_{\text{rms}} = \sqrt{3RT/M}$ to obtain an expression for v_{rms} in terms of *P*, *V*, and *M*. See Appendix C for the molar masses of argon and helium.

Express the rms speed of an atom as a function of the temperature and its molar mass:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

From the ideal-gas law we have:

$$RT = \frac{PV}{n}$$

Substitute for *RT* to obtain:

$$v_{\text{rms}} = \sqrt{\frac{3PV}{nM}}$$

(*a*) Substitute numerical values and evaluate v_{rms} for argon atoms:

$$\boldsymbol{v}_{\text{rms, Ar}} = \sqrt{\frac{3(10\ \text{atm})(101.325\ \text{kPa/atm})(1.0\times10^{-3}\ \text{m}^3)}{(1\ \text{mol})(39.948\times10^{-3}\ \text{kg/mol})}} = \boxed{0.28\ \text{km/s}}$$

(*b*) Substitute numerical values and evaluate v_{rms} for helium atoms:

$$v_{\text{rms, He}} = \sqrt{\frac{3(10\ \text{atm})(101.325\ \text{kPa/atm})(1.0\times10^{-3}\ \text{m}^3)}{(1\ \text{mol})(4.003\times10^{-3}\ \text{kg/mol})}} = \boxed{0.87\ \text{km/s}}$$

The rms speed of argon atoms is slightly less than one third the rms speed of helium atoms.

60 • Find the total translational kinetic energy of the molecules of 1.0 L of oxygen gas at a temperature of 0.0ºC and a pressure of 1.0 atm.

Picture the Problem We can express the total translational kinetic energy of the oxygen gas by combining $K = \frac{3}{2}nRT$ and the ideal-gas law to obtain an expression for K in terms of the pressure and volume of the gas.

Relate the total translational kinetic energy of translation to the temperature of the gas:

$$K = \tfrac{3}{2}nRT$$

Using the ideal-gas law, substitute for nRT to obtain:

$$K = \tfrac{3}{2}PV$$

Substitute numerical values and evaluate K:

$$K = \tfrac{3}{2}(101.325\,\text{kPa})\left(1.0\,\text{L}\times\frac{10^{-3}\,\text{m}^3}{\text{L}}\right) = \boxed{0.15\,\text{kJ}}$$

61 • Estimate the rms speed and the average kinetic energy of a hydrogen atom in a gas at a temperature of 1.0×10^7 K. (At this temperature, which is approximately the temperature in the interior of a star, hydrogen atoms are ionized and become protons.)

Picture the Problem Because we're given the temperature of the hydrogen atom and know its molar mass, we can find its rms speed using $v_{\text{rms}} = \sqrt{3RT/M}$ and its average kinetic energy from $K_{\text{av}} = \frac{3}{2}kT$. See Appendix C for the molar mass of hydrogen.

Relate the rms speed of a hydrogen atom to its temperature and molar mass:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M_{\text{H}}}}$$

Substitute numerical values and evaluate v_{rms}:

$$v_{\text{rms}} = \sqrt{\frac{3(8.314\,\text{J/mol}\cdot\text{K})(1.0\times10^7\,\text{K})}{1.0079\times10^{-3}\,\text{kg/mol}}} = \boxed{5.0\times10^5\,\text{m/s}}$$

Express the average kinetic energy of the hydrogen atom as a function of its temperature:

$$K_{\text{av}} = \tfrac{3}{2}kT$$

Substitute numerical values and evaluate K_{av}:

$$K_{\text{av}} = \tfrac{3}{2}(1.381\times10^{-23}\,\text{J/K})(1.0\times10^7\,\text{K}) = \boxed{2.1\times10^{-16}\,\text{J}}$$

62 • Liquid helium has a temperature of only 4.20 K and is in equilibrium with its vapor at atmospheric pressure. Calculate the rms speed of a helium atom in the vapor at this temperature, and comment on the result.

Picture the Problem The rms speed of helium atoms is given by $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M_{\text{He}}}}$.

See Appendix C for the molar mass of helium.

The rms speed of helium atoms is given by:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M_{\text{He}}}}$$

Substitute numerical values and evaluate v_{rms}:

$$v_{\text{rms}} = \sqrt{\frac{3(8.314\ \text{J/mol}\cdot\text{K})(4.20\ \text{K})}{4.003\ \text{g/mol}}} = \boxed{162\ \text{m/s}}$$

Thermal speeds, even at temperatures as low as 4.20 K, are very large compared to most of the speeds we experience directly.

63 • Show that the mean free path for a molecule in an ideal gas at temperature T and pressure P is given by $\lambda = kT/\sqrt{2}P\pi d^2$.

Picture the Problem We can combine $\lambda = \dfrac{1}{\sqrt{2}n_v \pi d^2}$ and $PV = NkT$ to express the mean free path for a molecule in an ideal gas in terms of the pressure and temperature.

Express the mean free path of a molecule in an ideal gas:

$$\lambda = \frac{1}{\sqrt{2}n_v \pi d^2} \qquad (1)$$

where

$$n_v = \frac{N}{V}$$

From Equation 17-7:

$$V = \frac{NkT}{P}$$

Substitute for V in the expression for n_v to obtain:

$$n_v = \frac{N}{\dfrac{NkT}{P}} = \frac{P}{kT}$$

Substitute for n_v in equation (1) to obtain:

$$\lambda = \boxed{\frac{kT}{\sqrt{2}P\pi d^2}}$$

64 •• State-of-the-art vacuum equipment can attain pressures as low as 7.0×10^{-11} Pa. Suppose that a chamber contains helium at this pressure and at room temperature (300 K). Estimate the mean free path and the collision time for helium in the chamber. Assume the diameter of a helium atom is 1.0×10^{-10} m.

Picture the Problem We can find the collision time from the mean free path and the average (rms) speed of the helium molecules. We can use the result of Problem 63 to find the mean free path of the molecules and $v_{\text{rms}} = \sqrt{3RT/M}$ to find the average speed of the molecules. See Appendix C for the molar mass of helium.

Express the collision time in terms of the mean free path for and the average speed of a helium molecule:

$$\tau = \frac{\lambda}{v_{\text{av}}} \approx \frac{\lambda}{v_{\text{rms}}} \qquad (1)$$

From Problem 63, the mean free path of the gas is given by:

$$\lambda = \frac{kT}{\sqrt{2}P\pi d^2}$$

Substitute numerical values and evaluate the mean free path λ:

$$\lambda = \frac{\left(1.381\times10^{-23}\ \text{J/K}\right)\left(300\ \text{K}\right)}{\sqrt{2}\left(7.0\times10^{-11}\ \text{Pa}\right)\pi\left(1.0\times10^{-10}\ \text{m}\right)^2} = 1.332\times10^{9}\ \text{m} = \boxed{1.3\times10^{9}\ \text{m}}$$

Express the rms speed of the helium molecules:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M_{\text{He}}}}$$

Substitute for v_{rms} in equation (1) and simplify to obtain:

$$\tau = \frac{\lambda}{\sqrt{\dfrac{3RT}{M_{\text{He}}}}} = \lambda\sqrt{\frac{M_{\text{He}}}{3RT}}$$

Substitute numerical values and evaluate τ:

$$\tau = \left(1.332\times10^{9}\ \text{m}\right)\sqrt{\frac{4.007\ \text{g/mol}}{3\left(8.314\ \text{J/mol}\cdot\text{K}\right)\left(300\ \text{K}\right)}} = \boxed{9.7\times10^{5}\ \text{s}}$$

65 •• **[SSM]** Oxygen (O_2) is confined to a cube-shaped container 15 cm on an edge at a temperature of 300 K. Compare the average kinetic energy of a molecule of the gas to the change in its gravitational potential energy if it falls 15 cm (the height of the container).

Picture the Problem We can use $K = \frac{3}{2}kT$ and $\Delta U = mgh = Mgh/N_A$ to express the ratio of the average kinetic energy of a molecule of the gas to the change in its gravitational potential energy if it falls from the top of the container to the bottom. See Appendix C for the molar mass of oxygen.

Express the average kinetic energy of a molecule of the gas as a function of its temperature:

$$K_{av} = \tfrac{3}{2}kT$$

Letting h represent the height of the container, express the change in the potential energy of a molecule as it falls from the top of the container to the bottom:

$$\Delta U = mgh = \frac{M_{O_2}gh}{N_A}$$

Express the ratio of K_{av} to ΔU and simplify to obtain:

$$\frac{K_{av}}{\Delta U} = \frac{\frac{3}{2}kT}{\frac{M_{O_2}gh}{N_A}} = \frac{3N_A kT}{2M_{O_2}gh}$$

Substitute numerical values and evaluate $K_{av}/\Delta U$:

$$\frac{K_{av}}{\Delta U} = \frac{3\left(6.022\times10^{23}\text{ particles/mol}\right)\left(1.381\times10^{-23}\text{ J/K}\right)\left(300\text{ K}\right)}{2\left(32.0\times10^{-3}\text{ kg/mol}\right)\left(9.81\text{ m/s}^2\right)\left(0.15\text{ m}\right)} = \boxed{7.9\times10^{4}}$$

*The Distribution of Molecular Speeds

66 •• Use calculus to show that $f(v)$, given by Equation 17-36, has its maximum value at a speed $v = \sqrt{2kT/m}$.

Picture the Problem Equation 17-36 gives the Maxwell-Boltzmann speed distribution. Setting its derivative with respect to v equal to zero will tell us where the function's extreme values lie.

Differentiate Equation 17-36 with respect to v:

$$\frac{df}{dv}=\frac{d}{dv}\left[\frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}v^2e^{-mv^2/2kT}\right]$$
$$=\frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\left(2v-\frac{mv^3}{kT}\right)e^{-mv^2/2kT}$$

Set $df/dv = 0$ for extrema and solve for v:

$$2v-\frac{mv^3}{kT}=0\Rightarrow\ v=\boxed{\sqrt{\frac{2kT}{m}}}$$

Examination of the graph of $f(v)$ makes it clear that this extreme value is, in fact, a maximum. See Figure 17-17 and note that it is concave downward at $v=\sqrt{2kT/m}$.

Remarks: An alternative to the examination of $f(v)$ in order to conclude that $v=\sqrt{2kT/m}$ maximizes the Maxwell-Boltzmann speed distribution function is to show that $d^2f/dv^2 < 0$ at $v=\sqrt{2kT/m}$.

67 •• [SSM] The fractional distribution function $f(v)$ is defined in Equation 17-36. Because $f(v)\,dv$ gives the fraction of molecules that have speeds in the range between v and $v + dv$, the integral of $f(v)\,dv$ over all the possible ranges of speeds must equal 1. Given that the integral $\int_0^\infty v^2e^{-av^2}dv=\frac{\sqrt{\pi}}{4}a^{-3/2}$, show that $\int_0^\infty f(v)dv=1$, where $f(v)$ is given by Equation 17-36.

Picture the Problem We can show that $f(v)$ is normalized by using the given integral to integrate it over all possible speeds.

Express the integral of Equation 17-36:

$$\int_0^\infty f(v)dv=\frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\int_0^\infty v^2e^{-mv^2/2kT}dv$$

Let $a=m/2kT$ to obtain:

$$\int_0^\infty f(v)dv=\frac{4}{\sqrt{\pi}}a^{3/2}\int_0^\infty v^2e^{-av^2}dv$$

Use the given integral to obtain:

$$\int_0^\infty f(v)dv=\frac{4}{\sqrt{\pi}}a^{3/2}\left(\frac{\sqrt{\pi}}{4}a^{-3/2}\right)=\boxed{1}$$

That is, $f(v)$ is normalized.

68 •• Given that the integral $\int_0^\infty v^3 e^{-av^2}dv = \frac{1}{2a^2}$, calculate the average speed v_{av} of molecules in a gas using the Maxwell–Boltzmann distribution function.

Picture the Problem In Problem 67 we showed that $f(v)$ is normalized. Hence we can evaluate v_{av} using $\int_0^\infty vf(v)dv$.

The average speed of the molecules in the gas is given by:

$$v_{av} = \int_0^\infty vf(v)dv$$
$$= \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\int_0^\infty v^3 e^{-mv^2/2kT}dv$$

Substitute $a = m/2kT$:

$$v_{av} = \frac{4}{\sqrt{\pi}}a^{3/2}\int_0^\infty v^3 e^{-av^2}dv$$

Use the given integral to obtain:

$$v_{av} = \frac{4}{\sqrt{\pi}}a^{3/2}\left(\frac{a^{-2}}{2}\right) = \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{a}}$$
$$= \boxed{\frac{2}{\sqrt{\pi}}\sqrt{\frac{2kT}{m}}}$$

69 •• The translational kinetic energies of the molecules of a gas are distributed according to the Maxwell-Boltzmann energy distribution, Equation 17-38. (*a*) Determine the most probable value of the translational kinetic energy (in terms of the temperature *T*) and compare this value to the average value. (*b*) Sketch a graph of the translational kinetic energy distribution [$f(E)$ versus E] and label the most probable energy and the average energy. (Do not worry about calibrating the vertical scale of the graph.) (*c*) Your teacher says, "Just looking at the graph $f(E)$ versus E allows you to see that the average translational kinetic energy is considerably greater than the most probable translational kinetic energy." What feature(s) of the graph support her claim?

Picture the Problem We can set the derivative of $f(E)$ with respect to E equal to zero in order to determine the most probable value of the kinetic energy of the gas molecules.

(*a*) The Maxwell-Boltzmann energy distribution function is:

$$f(E) = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{1}{kT}\right)^{3/2}\sqrt{E}e^{-E/kT}$$

Differentiate this expression with respect to E to obtain:

$$\frac{d}{dE}f(E)=\left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{1}{kT}\right)^{3/2}\left(\tfrac{1}{2}E^{-1/2}e^{-E/kT}+E^{1/2}\left(\frac{-1}{kT}\right)e^{-E/kt}\right)$$

Set this derivative equal to zero for extrema values and simplify to obtain:

$$\tfrac{1}{2}E_{\text{peak}}^{-1/2}e^{-E_{\text{peak}}/kT}+E_{\text{peak}}^{1/2}\left(\frac{-1}{kT}\right)e^{-E_{\text{peak}}/kt}=0$$

Solving for E_{peak} yields:

$$E_{\text{peak}}=\boxed{\tfrac{1}{2}kT}$$

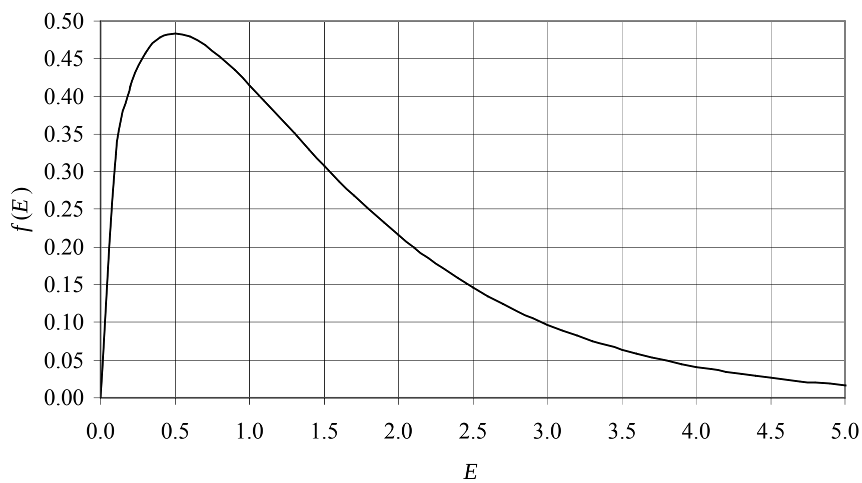
The average value of the energy of the gas molecules is given by:

$$E_{\text{av}}=\tfrac{3}{2}kT$$

Express the ratio of E_{peak} to E_{av}:

$$\frac{E_{\text{peak}}}{E_{\text{av}}}=\frac{\frac{1}{2}kT}{\frac{3}{2}kT}=\tfrac{1}{3}\Rightarrow E_{\text{peak}}=\boxed{\tfrac{1}{3}E_{\text{av}}}$$

(*b*) The following graph of the energy distribution was plotted using a spreadsheet program. Note that kT was set equal to 1 and that, as predicted by our result in (*a*), the peak value occurs for $E = 0.5$.



(*c*) The graph rises from zero to the peak much more rapidly than it falls off to the right of the peak. Because the distribution is so strongly skewed to the right of the peak, the outlying molecules with relatively high energies pull the average ($3kT/2$) far to the right of the most probable value ($kT/2$).

General Problems

70 • Find the temperature at which the rms speed of a molecule of hydrogen gas equals 343 m/s.

Picture the Problem We can use $v_{\text{rms}} = \sqrt{3RT/M}$ to relate the temperature of the H_2 molecules to their rms speed. See Appendix C for the molar mass of hydrogen.

Relate the rms speed of the hydrogen molecules to its temperature:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M_{H_2}}} \Rightarrow T = \frac{M_{H_2} v_{\text{rms}}^2}{3R}$$

Substitute numerical values and evaluate T:

$$T = \frac{(2.016\times10^{-3}\ \text{kg/mol})(343\ \text{m/s})^2}{3(8.314\ \text{J/mol}\cdot\text{K})} = \boxed{9.51\ \text{K}}$$

71 •• (*a*) If 1.0 mol of a gas in a cylindrical container occupies a volume of 10 L at a pressure of 1.0 atm, what is the temperature of the gas in kelvins? (*b*) The cylinder is fitted with a piston so that the volume of the gas (Figure 17-20) can vary. When the gas is heated at constant pressure, it expands to a volume of 20 L. What is the temperature of the gas in kelvins? (*c*) Next, the volume is fixed at 20 L, and the gas's temperature is increased to 350 K. What is the pressure of the gas now?

Picture the Problem We can use the ideal-gas law to find the initial temperature of the gas and the ideal-gas law for a fixed amount of gas to relate the volumes, pressures, and temperatures resulting from the given processes.

(*a*) Apply the ideal-gas law to express the temperature of the gas:

$$T = \frac{PV}{nR}$$

Substitute numerical values and evaluate T:

$$T = \frac{(1.0\ \text{atm})(10\ \text{L})}{(1.0\ \text{mol})\left(0.08206\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 121.9\ \text{K} = \boxed{1.2\times10^2\ \text{K}}$$

(*b*) Use the ideal-gas law for a fixed amount of gas to relate the temperatures and volumes:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

or, because $P_1 = P_2$,

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} \Rightarrow T_2 = \frac{V_2}{V_1}T$$

Substitute numerical values and evaluate T_2:

$$T_2 = 2(121.9\,\text{K}) = 243.7\,\text{K} = \boxed{2.4\times10^2\,\text{K}}$$

(*c*) Apply the ideal-gas law to express the pressure of the gas:

$$PV = nRT \Rightarrow P = \frac{nRT}{V}$$

Substitute numerical values and evaluate P:

$$P = \frac{(1.0\,\text{mol})\left(0.08206\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(350\,\text{K})}{20\,\text{L}} = \boxed{1.4\,\text{atm}}$$

72 •• (*a*) The volume per molecule of a gas is the reciprocal of the number density (the number of molecules per unit volume). (*a*) Find the average volume per molecule for dry air at room temperature and atmospheric pressure. (*b*) Take the cube root of your answer to Part (*a*) to obtain a rough estimate of the average distance *d* between air molecules. (*c*) Find or estimate the average diameter *D* of an air molecule, and compare it to your answer to Part (*b*). (*d*) Sketch the molecules in a cube-shaped volume of air, with the edge length of the cube equal to 3*d*. Make your figure to scale and place the molecules in what you think is a typical configuration. (*e*) Use your picture to explain why the mean free path of an air molecule is much greater than the average distance between molecules.

Picture the Problem (*a*) We can use the ideal-gas law to find the average volume per molecule. (*b*) If one were to divide a container of air into little cubes, with one cube per molecule, then this distance would be the width of each cube or, equivalently, the distance from the center of one cube to the centers of the neighboring cubes. On average, we can imagine that the molecules are at the centers of their respective cubes, so this distance is also the average distance between neighboring molecules.

(*a*) The ideal-gas law relates the number of molecules *N* in a gas to the volume *V* they occupy:

$$PV = NkT \Rightarrow \frac{V}{N} = \frac{kT}{P}$$

Substitute numerical values and evaluate *V*/*N*:

$$\frac{V}{N} = \frac{(1.381\times10^{-23}\,\text{J/K})(293\,\text{K})}{101.325\,\text{kPa}} = \boxed{3.99\times10^{-26}\,\text{m}^3}$$

(*b*) Taking the cube root of *V*/*N* yields:

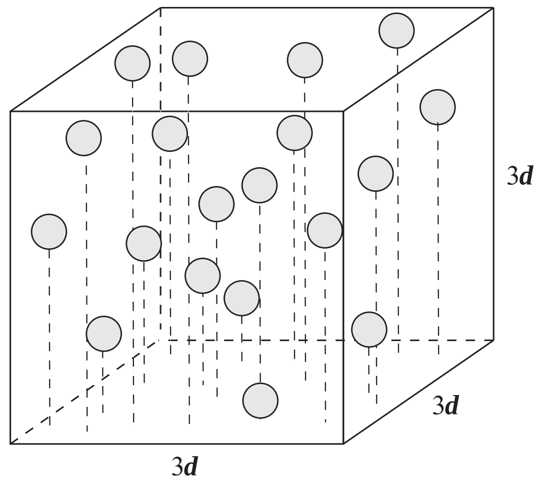
$$d = \sqrt[3]{3.99\times10^{-26}\,\text{m}^3} \approx \boxed{3.42\,\text{nm}}$$

(*c*) Example 17-10 gives the average diameter of an air molecule as:

$$D = 3.75\times10^{-10}\ \text{m} = \boxed{0.375\ \text{nm}}$$

or about 1/10 the average distance between molecules.

(*d*) A sketch of the molecules in a cube-shaped volume of air, with the edge length of the cube equal to 3*d*, follows. The random distribution of the molecules is a typical configuration.



(*e*) If a particular molecule in the diagram is moving in a random direction, its chance of colliding with a neighbor is very small because it can miss in either of the two directions perpendicular to its motion. So the mean free path, or average distance between collisions, should be many times larger than the distance to the nearest neighbor.

73 •• [SSM] The Maxwell-Boltzmann distribution applies not just to gases, but also to the molecular motions within liquids. The fact that not all molecules have the same speed helps us understand the process of evaporation. (*a*) Explain in terms of molecular motion why a drop of water becomes cooler as molecules evaporate from the drop's surface. (Evaporative cooling is an important mechanism for regulating our body temperatures, and is also used to cool buildings in hot, dry locations.) (*b*) Use the Maxwell-Boltzmann distribution to explain why even a slight increase in temperature can greatly increase the rate at which a drop of water evaporates.

Determine the Concept

(*a*) To escape from the surface of a droplet of water, molecules must have enough translational kinetic energy to overcome the attractive forces from their neighbors. Therefore the molecules that escape will be those that are moving faster, leaving the slower molecules behind. The slower molecules have less kinetic energy, so the temperature of the droplet, which is proportional to the average kinetic energy per molecule, decreases.

(*b*) As long as the temperature is not too high, the molecules that evaporate from a surface will be only those with the most extreme speeds, at the high-energy "tail" of the Maxwell-Boltzmann distribution. Within this part of the distribution, increasing the temperature only slightly can greatly increase the percentage of molecules with speeds above a certain threshold. For example, suppose that we set an initial threshold at $E = 5kT_1$, then imagine increasing the temperature by 10% so $T_2 = 1.1T_1$. At the threshold, the ratio of the new energy distribution to the old one is

$$\frac{F(T_2)}{F(T_1)} = \left(\frac{T_1}{T_2}\right)^{\frac{3}{2}} e^{\frac{-E}{kT_2}} e^{\frac{+E}{kT_1}} = (1.1)^{\frac{-3}{2}} e^{\frac{-5}{1.1}} e^5 = 1.366$$

an increase of almost 37%.

74 •• A cubic metal box that has 20-cm–long edges contains air at a pressure of 1.0 atm and a temperature of 300 K. The box is sealed so that the enclosed volume remains constant, and it is heated to a temperature of 400 K. Find the force due to the internal air pressure on each wall of the box.

Picture the Problem We can use the definition of pressure to express the net force on each wall of the box in terms of its area and the pressure differential between the inside and the outside of the box. We can apply the ideal-gas law for a fixed amount of gas to find the pressure inside the box after it has been heated.

Using the definition of pressure, express the net force on each wall of the box:

$$F = AP_{\text{inside}} \qquad (1)$$

Use the ideal-gas law for a fixed amount of gas to relate the initial and final pressures of the gas:

$$\frac{P_{\text{atm}}V_{\text{initial}}}{T_{\text{initial}}} = \frac{P_{\text{inside}}V_{\text{final}}}{T_{\text{final}}}$$

or, because $V_{\text{initial}} = V_{\text{final}}$,

$$\frac{P_{\text{atm}}}{T_{\text{initial}}} = \frac{P_{\text{inside}}}{T_{\text{final}}} \Rightarrow P_{\text{inside}} = \frac{T_{\text{final}}}{T_{\text{initial}}} P_{\text{atm}}$$

Substituting in equation (1) and simplifying yields:

$$F = \frac{T_{\text{final}}}{T_{\text{initial}}} AP_{\text{atm}}$$

Substitute numerical values and evaluate F:

$$F = (0.20\,\text{m})^2 \left(\frac{400\,\text{K}}{300\,\text{K}}\right)(101.325\,\text{kPa})$$

$$= \boxed{5.4\,\text{kN}}$$

75 •• [SSM] In attempting to create liquid hydrogen for fuel, one of the proposals is to convert plain old water (H_2O) into H_2 and O_2 gases by *electrolysis*. How many moles of each of these gases result from the electrolysis of 2.0 L of water?

Picture the Problem We can use the molar mass of water to find the number of moles in 2.0 L of water. Because there are two hydrogen atoms in each molecule of water, there must be as many hydrogen molecules in the gas formed by electrolysis as there were molecules of water and, because there is one oxygen atom in each molecule of water, there must be half as many oxygen molecules in the gas formed by electrolysis as there were molecules of water. See Appendix C for the molar masses of hydrogen and oxygen.

Express the electrolysis of water into H_2 and O_2:

$$n_{H_2O} \rightarrow n_{H_2} + \tfrac{1}{2} n_{O_2}$$

Express the number of moles in 2.0 L of water:

$$n_{H_2O} = \frac{2.0\,\text{L} \times 1000\,\frac{\text{g}}{\text{L}}}{18.02\,\text{g/mol}} = 110\,\text{mol}$$

Because there are two hydrogen atoms for each water molecule:

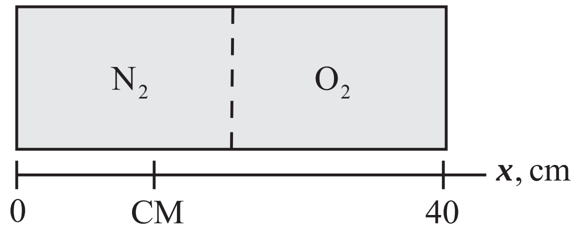
$$n_{H_2} = \boxed{110\,\text{mol}}$$

Because there is one oxygen atom for each water molecule:

$$n_{O_2} = \tfrac{1}{2} n_{H_2O} = \tfrac{1}{2}(110\,\text{mol}) = \boxed{55\,\text{mol}}$$

76 •• A hollow 40-cm-long cylinder of negligible mass rests on its side on a horizontal frictionless table. The cylinder is divided into two equal sections by a vertical non-porous membrane. One section contains nitrogen and the other contains oxygen. The pressure of the nitrogen is twice that of the oxygen. How far will the cylinder move if the membrane breaks?

Picture the Problem The diagram shows the cylinder before removal of the membrane. The distance the cylinder will move if the membrane breaks is the distance between the locations of the center of mass initially and after the membrane breaks. Because both gasses will uniformly fill the cylinder after the membrane breaks, the location of the center of mass after this break will be at the midpoint of the cylinder; that is, at 20 cm from either end of the cylinder. See Appendix C for the molar masses of oxygen and nitrogen.



Express the distance the cylinder will move in terms of the movement of the center of mass when the membrane is removed:

$$\Delta x = 20\text{ cm} - x_{\text{cm,before}} \qquad (1)$$

Apply the ideal-gas law to both collections of molecules to obtain:

$$P_{N_2}V_{N_2} = n_{N_2}kT$$

and

$$P_{O_2}V_{O_2} = n_{O_2}kT$$

Divide the first of these equations by the second to obtain:

$$\frac{P_{N_2}}{P_{O_2}} = \frac{n_{N_2}}{n_{O_2}}$$

or, because $P_{N_2} = 2P_{O_2}$,

$$\frac{2P_{O_2}}{P_{O_2}} = \frac{n_{N_2}}{n_{O_2}} \Rightarrow n_{N_2} = 2n_{O_2}$$

Express the mass of O_2 in terms of its molar mass and the number of moles of oxygen:

$$m_{O_2} = n_{O_2}M_{O_2}$$

Express the mass of N_2 in terms of its molar mass and the number of moles of nitrogen:

$$m_{N_2} = n_{O_2}M_{N_2}$$

Using the definition of center of mass, express the center of mass before the membrane is removed:

$$x_{\text{cm,before}} = \frac{\sum_i x_i m_i}{\sum_i m_i} = \frac{n_{\text{N}_2} M_{\text{N}_2} x_{\text{cm,N}_2} + n_{\text{O}_2} M_{\text{O}_2} x_{\text{cm,O}_2}}{n_{\text{N}_2} M_{\text{N}_2} + n_{\text{O}_2} M_{\text{O}_2}}$$
$$= \frac{2n_{\text{O}_2} M_{\text{N}_2} x_{\text{cm,N}_2} + n_{\text{O}_2} M_{\text{O}_2} x_{\text{cm,O}_2}}{2n_{\text{O}_2} M_{\text{N}_2} + n_{\text{O}_2} M_{\text{O}_2}}$$
$$= \frac{2M_{\text{N}_2} x_{\text{cm,N}_2} + M_{\text{O}_2} x_{\text{cm,O}_2}}{2M_{\text{N}_2} + M_{\text{O}_2}}$$

Substitute numerical values and evaluate $x_{\text{cm,before}}$:

$$x_{\text{cm, before}} = \frac{2(10\,\text{cm})(28\,\text{g}) + (30\,\text{cm})(32\,\text{g})}{2(28\,\text{g}) + 32\,\text{g}} = 17\,\text{cm}$$

Substitute numerical values in equation (1) to obtain:

$$\Delta x = 20\,\text{cm} - 17\,\text{cm} = \boxed{3\,\text{cm}}$$

Because momentum must be conserved during this process and the center of mass of the system consisting of the oxygen and nitrogen must not move. Therefore, the entire cylinder must move 3 cm to the left.

77 •• A cylinder of fixed volume contains a mixture of helium gas (He) and hydrogen gas (H_2) at a temperature T_1 and pressure P_1. If the temperature is doubled to $T_2 = 2T_1$, the pressure would also double, except for the fact that at this temperature the H_2 is essentially 100 percent dissociated into H_1. In reality, at pressure $P_2 = 2P_1$ the temperature is $T_2 = \frac{3}{2}T_1$. If the mass of the hydrogen in the cylinder is m, what is the mass of the helium in the cylinder?

Picture the Problem We can apply the ideal-gas law to the two processes to find the number of moles of hydrogen in terms of the number of moles of helium in the gas. Using the definition of molar mass, we can relate the mass of each gas to the number of moles of each gas and their molar masses. See Appendix C for the molar masses of helium gas and hydrogen gas.

Apply the ideal-gas law to the mixture when the pressure is P_1 and the temperature is T_1:

$$P_1V = \left[N_{\text{He}} + N_{\text{H}_2}\right]kT_1$$
$$= \left[n_{\text{He}}N_{\text{A}} + n_{\text{H}_2}N_{\text{A}}\right]kT_1$$
$$= \left[n_{\text{He}} + n_{\text{H}_2}\right]N_{\text{A}}kT_1$$
$$= \left[n_{\text{He}} + n_{\text{H}_2}\right]RT_1$$

Noting that when the pressure doubles, the temperature increases by a factor of 1.5 and H_2 dissociates into H_1, apply the ideal-gas law to the mixture to obtain:

$$\begin{aligned}(2P_1)V &= \left[N_{He} + 2N_{H_2}\right]k\left(\tfrac{3}{2}T_1\right)\\ &= \left[n_{He}N_A + 2n_{H_2}N_A\right]k\left(\tfrac{3}{2}T_1\right)\\ &= \left[n_{He} + 2n_{H_2}\right]N_A k\left(\tfrac{3}{2}T_1\right)\\ &= \left[n_{He} + 2n_{H_2}\right]R\left(\tfrac{3}{2}T_1\right)\end{aligned}$$

Dividing the second of these equations by the first and simplifying gives:

$$2n_{H_2} = n_{He} \qquad (1)$$

Relate m_{He} to n_{He}:

$$m_{He} = n_{He}M_{He} = n_{He}(4.003\,\text{g/mol})$$

and

$$n_{He} = \frac{m_{He}}{4.003\,\text{g/mol}}$$

Relate m to n_{H_2}:

$$m = n_{H_2}M_{H_2} = n_{H_2}(2.016\,\text{g/mol})$$

and

$$n_{H_2} = \frac{m}{2.016\,\text{g/mol}}$$

Substituting for n_{H_2} and n_{He} in equation (1) and solving for m_{He} gives:

$$\frac{2m}{2.016\,\text{g/mol}} = \frac{m_{He}}{4.003\,\text{g/mol}}$$

and

$$m_{He} \approx \boxed{4m}$$

78 •• The mean free path for O_2 molecules at a temperature of 300 K and at 1.00 atm pressure is 7.10×10^{-8} m. Use this data to estimate the size of an O_2 molecule.

Picture the Problem Because the O_2 molecule resembles 2 spheres stuck together, which in cross section look something like two circles, we can estimate the radius of the molecule from the formula for the area of a circle. We can express the area, and hence the radius, of the circle in terms of the mean free path and the number density of the molecules and use the ideal-gas law to express the number density.

Express the area of two circles of diameter d that touch each other:

$$A = 2\left(\frac{\pi d^2}{4}\right) = \frac{\pi d^2}{2} \Rightarrow d = \sqrt{\frac{2A}{\pi}} \qquad (1)$$

Relate the mean free path of the molecules to their number density and cross-sectional area:

$$\lambda = \frac{1}{n_v A} \Rightarrow A = \frac{1}{n_v \lambda}$$

Substitute for A in equation (1) to obtain:

$$d = \sqrt{\frac{2}{\pi n_v \lambda}}$$

Use the ideal-gas law to relate the number density of the O_2 molecules to their temperature and pressure:

$$PV = NkT \text{ or } n_v = \frac{N}{V} = \frac{P}{kT}$$

Substituting for n_v yields:

$$d = \sqrt{\frac{2kT}{\pi P \lambda}}$$

Substitute numerical values and evaluate d:

$$d = \sqrt{\frac{2(1.381\times10^{-23}\text{ J/K})(300\text{ K})}{\pi(101.325\text{ kPa})(7.10\times10^{-8}\text{ m})}} = \boxed{0.605\text{ nm}}$$

79 •• [SSM] Current experiments in atomic trapping and cooling can create low-density gases of rubidium and other atoms with temperatures in the nanokelvin (10^{-9} K) range. These atoms are trapped and cooled using magnetic fields and lasers in ultrahigh vacuum chambers. One method that is used to measure the temperature of a trapped gas is to turn the trap off and measure the time it takes for molecules of the gas to fall a given distance. Consider a gas of rubidium atoms at a temperature of 120 nK. Calculate how long it would take an atom traveling at the rms speed of the gas to fall a distance of 10.0 cm if (*a*) it were initially moving directly downward and (*b*) if it were initially moving directly upward. Assume that the atom doesn't collide with any others along its trajectory.

Picture the Problem Choose a coordinate system in which downward is the positive direction. We can use a constant-acceleration equation to relate the fall distance to the initial velocity of the molecule, the acceleration due to gravity, the fall time, and $v_{\text{rms}} = \sqrt{3kT/m}$ to find the initial velocity of the rubidium molecules.

(*a*) Using a constant-acceleration equation, relate the fall distance to the initial velocity of a molecule, the acceleration due to gravity, and the fall time:

$$y = v_0 t + \tfrac{1}{2} g t^2 \qquad (1)$$

Relate the rms speed of rubidium atoms to their temperature and mass:

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m_{\text{Rb}}}}$$

Substitute numerical values and evaluate v_{rms}:

$$v_{\text{rms}} = \sqrt{\frac{3(1.381\times10^{-23}\ \text{J/K})(120\,\text{nK})}{(85.47\,\text{u})(1.6606\times10^{-27}\ \text{kg/u})}} = 5.918\times10^{-3}\ \text{m/s}$$

Letting $v_{\text{rms}} = v_0$, substitute in equation (1) to obtain:

$$0.100\,\text{m} = \left(5.918\times10^{-3}\ \text{m/s}\right)t + \tfrac{1}{2}\left(9.81\,\text{m/s}^2\right)t^2$$

Use the quadratic formula or your graphing calculator to solve this equation for its positive root:

$$t = 0.14218\,\text{s} = \boxed{142\ \text{ms}}$$

(*b*) If the atom is initially moving upward:

$$v_{\text{rms}} = v_0 = -5.918\times10^{-3}\ \text{m/s}$$

Substitute in equation (1) to obtain:

$$0.100\,\text{m} = \left(-5.918\times10^{-3}\ \text{m/s}\right)t + \tfrac{1}{2}\left(9.81\,\text{m/s}^2\right)t^2$$

Use the quadratic formula or your graphing calculator to solve this equation for its positive root:

$$t = \boxed{143\ \text{ms}}$$

80 ••• A cylinder is filled with 0.10 mol of an ideal gas at standard temperature and pressure, and a 1.4-kg piston seals the gas in the cylinder (Figure 17-21) with a frictionless seal. The trapped column of gas is 2.4-m high. The piston and cylinder are surrounded by air, also at standard temperature and pressure. The piston is released from rest and starts to fall. The motion of the piston ceases after the oscillations stop with the piston and the trapped air in thermal equilibrium with the surrounding air. (*a*) Find the height of the gas column. (*b*) Suppose that the piston is pushed down below its equilibrium position by a small amount and then released. Assuming that the temperature of the gas remains constant, find the frequency of vibration of the piston.

Picture the Problem (*a*) Let *A* be the cross-sectional area of the cylinder. We can use the ideal-gas law to find the height of the piston under equilibrium conditions. In (*b*), we can apply Newton's second law and the ideal-gas law for a fixed amount of gas to show that, for small displacements from its equilibrium position, the piston executes simple harmonic motion.

(*a*) Express the pressure inside the cylinder:

$$P_{\text{in}} = P_{\text{atm}} + \frac{mg}{A}$$

Apply the ideal-gas law to obtain a second expression for the pressure of the gas in the cylinder:

$$P_{\text{in}} = \frac{nRT}{V} = \frac{nRT}{hA} \qquad (1)$$

Equating these two expressions yields:

$$P_{\text{atm}} + \frac{mg}{A} = \frac{nRT}{hA}$$

Solve for *h* to obtain:

$$h = \frac{nRT}{AP_{\text{atm}} + mg} = \frac{(2.4\text{ m})AP_{\text{atm}}}{AP_{\text{atm}} + mg} = \frac{2.4\text{ m}}{1 + \dfrac{mg}{AP_{\text{atm}}}}$$

At STP, 0.10 mol of gas occupies 2.24 L. Therefore:

$$(2.4\text{ m})A = 2.24\times10^{-3}\text{ m}^3$$

and

$$A = 9.333\times10^{-4}\text{ m}^2$$

Substitute numerical values and evaluate *h*:

$$h = \frac{2.4\text{ m}}{1 + \dfrac{(1.4\text{ kg})(9.81\text{ m/s}^2)}{(9.333\times10^{-4}\text{ m}^2)(101.325\text{ kPa})}} = 2.096\text{ m} = \boxed{2.1\text{ m}}$$

(*b*) Relate the frequency of vibration of the piston to its mass and a "stiffness" constant:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (2)$$

where *m* is the mass of the piston and *k* is a constant of proportionality.

Letting *y* be the displacement from equilibrium, apply $\sum F_y = ma_y$ to the piston in its equilibrium position:

$$P_{\text{in}}A - mg - P_{\text{atm}}A = 0$$

For a small displacement y above equilibrium:

$$P'_{\text{in}}A - mg - P_{\text{atm}}A = ma_y$$

or

$$P'_{\text{in}}A - P_{\text{in}}A = ma_y \qquad (3)$$

Using the ideal-gas law for a fixed amount of gas and constant temperature, relate P_{in}' to P_{in}:

$$P'_{\text{in}}V' = P_{\text{in}}V$$

or

$$P'_{\text{in}}(V + Ay) = P_{\text{in}}V \Rightarrow P'_{\text{in}} = P_{\text{in}}\frac{V}{V + Ay}$$

Substituting for V and simplifying yields:

$$P'_{\text{in}}A = P_{\text{in}}A\frac{Ah}{Ah + Ay} = P_{\text{in}}A\frac{1}{1 + \frac{y}{h}}$$

Substitute in equation (3) to obtain:

$$P_{\text{in}}A\left(1 + \frac{y}{h}\right)^{-1} - P_{\text{in}}A = ma_y$$

or, for $y << h$,

$$P_{\text{in}}A\left(1 - \frac{y}{h}\right) - P_{\text{in}}A \approx ma_y \qquad (4)$$

Simplify equation (4) to obtain:

$$-P_{\text{in}}A\frac{y}{h} \approx ma_y$$

Substitute in equation (1) to obtain:

$$-\left(\frac{nRT}{Ah}\right)A\frac{y}{h} \approx ma_y \Rightarrow -\left(\frac{nRT}{h^2}\right)y \approx ma_y$$

Solving for a_y yields:

$$a_y = -\frac{nRT}{mh^2}y$$

or

$a_y = -\frac{k}{m}y$, the condition for SHM

where $\frac{k}{m} = \frac{nRT}{mh^2}$

Substitute in equation (2) to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{nRT}{mh^2}}$$

Substitute numerical values and evaluate f:

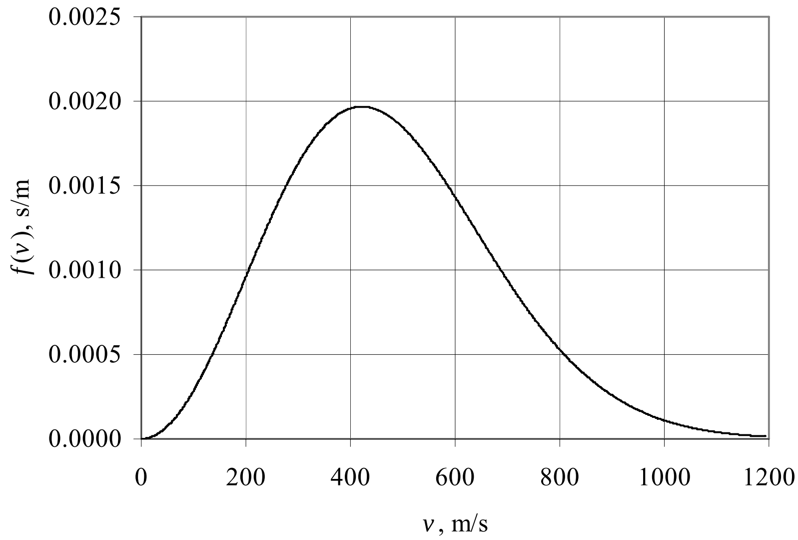
$$f = \frac{1}{2\pi}\sqrt{\frac{(0.10\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})(300\,\text{K})}{(1.4\,\text{kg})(2.096\,\text{m})^2}} = \boxed{1.0\,\text{Hz}}$$

81 ••• To solve this problem, you will use a **spreadsheet** to study the distribution of molecular speeds in a gas. Figure 17-22 should help you get started. (*a*) Enter the values for constants *R*, *M*, and *T* as shown. Then in column A, enter values of speed ranging from 0 to 1200 m/s, in increments of 1 m/s. (This spreadsheet will be long.) In cell B7, enter the formula for the Maxwell-Boltzmann fractional speed distribution. This formula contains parameters *v, R, M* and *T.* Substitute A7 for *v,* B$1 for *R,* B$2 for *M* and B$3 for *T.* Then use the FILL DOWN command to enter the formula in the cells below B7. Create a graph of *f*(*v*) versus *v* using the data in columns A and B. (*b*) Explore how the graph changes as you increase and decrease the temperature, and describe the results. (*c*) Add a third column in which each cell contains the cumulative sum of all *f*(*v*) values, multiplied by the interval size *dv* (which equals 1), in the rows above and including the row in question. What is the physical interpretation of the numbers in this column? (*d*) For nitrogen gas at 300 K, what percentage of the molecules has speeds less than 200 m/s? (*e*) For nitrogen gas at 300 K, what percentage of the molecules has speeds greater than 700 m/s?

(*a*) The first few rows of the spreadsheet are shown below. Note that the column for Part (*c*) is included.

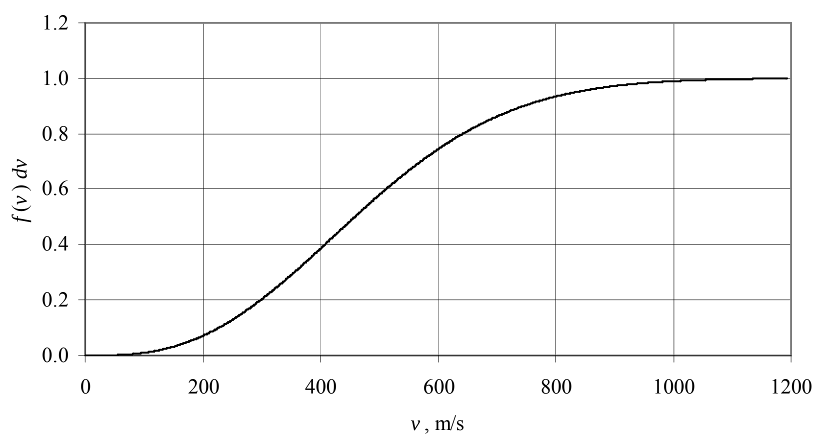
	A	B	C
1	*R*=	8.31	J/mol-K
2	*M*=	0.028	kg/mol
3	*T*=	300	K
4			
5	*v*	*f*(v)	sum *f*(*v*)*dv*
6	(m/s)	(s/m)	(unitless)
7	0	0.00E+00	0.00E+00
8	1	3.00E−08	3.00E−08
9	2	1.20E−07	1.50E−07
10	3	2.70E−07	4.20E−07
11	4	4.80E−07	9.01E−07
12	5	7.51E−07	1.65E−06

A graph of $f(v)$ for nitrogen at 300 K follows:



(*b*) As the temperature is increased the horizontal position of the peak moves to the right in proportion to the square root of the temperature, while the height of the peak drops by the same factor, preserving the total area under the graph (which must be 1.0).

(*c*) Each number in column C of the spreadsheet (shown in (*a*)) is approximately equal to the integral of $f(v)$ from zero up to the corresponding v value. This integral represents the probability of a molecule having a speed less than or equal to this value of v.



(*d*) Looking in cell C207, we find that the probability (at 300 K) of a nitrogen molecule having a speed less than 200 m/s is $\boxed{0.0706}$, or about 7%. Note that this value is consistent with the graph of $f(v)$ shown immediately above.

(*e*) Looking in cell C707, we find that the probability of a nitrogen molecule having a speed less than 700 m/s is approximately 0.862, so the probability of it having a speed greater than this would be 1 – 0.862 or $\boxed{0.138}$, or a little under 14%.

Chapter 18
Heat and the First Law of Thermodynamics

Conceptual Problems

1 • Object A has a mass that is twice the mass of object B and object A has a specific heat that is twice the specific heat of object B. If equal amounts of heat are transferred to these objects, how do the subsequent changes in their temperatures compare? (*a*) $\Delta T_A = 4\Delta T_B$, (*b*) $\Delta T_A = 2\Delta T_B$, (*c*) $\Delta T_A = \Delta T_B$, (*d*) $\Delta T_A = \frac{1}{2}\Delta T_B$, (*e*) $\Delta T_A = \frac{1}{4}\Delta T_B$

Picture the Problem We can use the relationship $Q = mc\Delta T$ to relate the temperature changes of objects A and B to their masses, specific heats, and the amount of heat supplied to each.

Express the change in temperature of object A in terms of its mass, specific heat, and the amount of heat supplied to it:

$$\Delta T_A = \frac{Q}{m_A c_A}$$

Express the change in temperature of object B in terms of its mass, specific heat, and the amount of heat supplied to it:

$$\Delta T_B = \frac{Q}{m_B c_B}$$

Divide the second of these equations by the first to obtain:

$$\frac{\Delta T_B}{\Delta T_A} = \frac{m_A c_A}{m_B c_B}$$

Substitute and simplify to obtain:

$$\frac{\Delta T_B}{\Delta T_A} = \frac{(2m_B)(2c_B)}{m_B c_B} = 4 \Rightarrow \Delta T_A = \tfrac{1}{4}\Delta T_B$$

and $\boxed{(e)}$ is correct.

2 • Object A has a mass that is twice the mass of object B. The temperature change of object A is equal to the temperature change of object B when the objects absorb equal amounts of heat. It follows that their specific heats are related by (*a*) $c_A = 2c_B$, (*b*) $2c_A = c_B$, (*c*) $c_A = c_B$, (*d*) none of the above

Picture the Problem We can use the relationship $Q = mc\Delta T$ to relate the temperature changes of objects A and B to their masses, specific heats, and the amount of heat supplied to each.

Relate the temperature change of object A to its specific heat and mass:

$$\Delta T_{\text{A}} = \frac{Q}{m_{\text{A}}c_{\text{A}}}$$

Relate the temperature change of object B to its specific heat and mass:

$$\Delta T_{\text{B}} = \frac{Q}{m_{\text{B}}c_{\text{B}}}$$

Equate the temperature changes and simplify to obtain:

$$\frac{1}{m_{\text{B}}c_{\text{B}}} = \frac{1}{m_{\text{A}}c_{\text{A}}}$$

Solve for c_{A}:

$$c_{\text{A}} = \frac{m_{\text{B}}}{m_{\text{A}}}c_{\text{B}} = \frac{m_{\text{B}}}{2m_{\text{B}}}c_{\text{B}} = \tfrac{1}{2}c_{\text{B}}$$

and $\boxed{(b)}$ is correct.

3 • [SSM] The specific heat of aluminum is more than twice the specific heat of copper. A block of copper and a block of aluminum have the same mass and temperature (20ºC). The blocks are simultaneously dropped into a single calorimeter containing water at 40ºC. Which statement is true when thermal equilibrium is reached? (*a*) The aluminum block is at a higher temperature than the copper block. (*b*) The aluminum block has absorbed less energy than the copper block. (*c*) The aluminum block has absorbed more energy than the copper block. (*d*) Both (*a*) and (*c*) are correct statements.

Picture the Problem We can use the relationship $Q = mc\Delta T$ to relate the amount of energy absorbed by the aluminum and copper blocks to their masses, specific heats, and temperature changes.

Express the energy absorbed by the aluminum block:

$$Q_{\text{Al}} = m_{\text{Al}}c_{\text{Al}}\Delta T$$

Express the energy absorbed by the copper block:

$$Q_{\text{Cu}} = m_{\text{Cu}}c_{\text{Cu}}\Delta T$$

Divide the second of these equations by the first to obtain:

$$\frac{Q_{\text{Cu}}}{Q_{\text{Al}}} = \frac{m_{\text{Cu}}c_{\text{Cu}}\Delta T}{m_{\text{Al}}c_{\text{Al}}\Delta T}$$

Because the block's masses are the same and they experience the same change in temperature:

$$\frac{Q_{\text{Cu}}}{Q_{\text{Al}}} = \frac{c_{\text{Cu}}}{c_{\text{Al}}} < 1$$

or

$Q_{\text{Cu}} < Q_{\text{Al}}$ and $\boxed{(c)}$ is correct.

4 • A block of copper is in a pot of boiling water and has a temperature of 100°C. The block is removed from the boiling water and immediately placed in an insulated container filled with a quantity of water that has a temperature of 20°C and has the same mass as the block of copper. (The heat capacity of the insulated container is negligible.) The final temperature will be closest to (*a*) 40°C, (*b*) 60°C, (*c*) 80°C.

Determine the Concept We can use the relationship $Q = mc\Delta T$ to relate the temperature changes of the block of copper and the boiling water to their masses, specific heats, and the amount of heat supplied to or absorbed by each.

Relate the heat supplied by the block of copper to its specific heat, mass, and temperature change as it cools to its equilibrium temperature:

$$\Delta Q_{Cu} = m_{Cu}c_{Cu}\Delta T_{Cu} = m_{Cu}c_{Cu}\left(T_{equil} - 100°C\right)$$

Relate the heat absorbed by the water to its specific heat, mass, and temperature change as it warms to its equilibrium temperature:

$$\Delta Q_{H_2O} = m_{H_2O}c_{H_2O}\Delta T_{H_2O} = m_{H_2O}c_{H_2O}\left(T_{equil} - 20°C\right)$$

Because thermal energy is conserved in this mixing process:

$$\Delta Q = \Delta Q_{Cu} + \Delta Q_{H_2O} = 0$$

or

$$m_{Cu}c_{Cu}\left(T_{equil} - 100°C\right) + m_{H_2O}c_{H_2O}\left(T_{equil} - 20°C\right) = 0$$

Because the mass of the water is equal to the mass of the block of copper:

$$c_{Cu}\left(T_{equil} - 100°C\right) + c_{H_2O}\left(T_{equil} - 20°C\right) = 0$$

Solving for T_{equil} yields:

$$T_{equil} = \frac{(100°C)c_{Cu} + (20°C)c_{H_2O}}{c_{Cu} + c_{H_2O}}$$

Substitute numerical values (See Table 18-1 for the specific heats of copper and water.)and evaluate T_{equil}:

$$T_{equil} = \frac{(100°C)\left(0.386\,\frac{kJ}{kg\cdot C°}\right) + (20°C)\left(4.18\,\frac{kJ}{kg\cdot C°}\right)}{0.386\,\frac{kJ}{kg\cdot C°} + 4.18\,\frac{kJ}{kg\cdot C°}} \approx 27°C \Rightarrow \boxed{(a)} \text{ is correct.}$$

5 • You pour both a certain amount of water at 100°C and an equal amount of water at 20°C into an insulated container. The final temperature of the mixture will be (a) 60°C, (b) less than 60°C, (c) greater than 60°C.

Determine the Concept We can use the relationship $Q = mc\Delta T$ to relate the temperature changes of the room-temperature water and the boiling water to their masses, specific heats, and the amount of heat supplied to or absorbed by each.

Because thermal energy is conserved in this mixing process:

$$\Delta Q = \Delta Q_{\text{hot H}_2\text{O}} + \Delta Q_{\text{H}_2\text{O}} = 0$$

or

$$m_{\text{hot H}_2\text{O}} c_{\text{hot H}_2\text{O}} \left(T_{\text{equil}} - 100°\text{C}\right) + m_{\text{H}_2\text{O}} c_{\text{H}_2\text{O}} \left(T_{\text{equil}} - 20°\text{C}\right) = 0$$

Because the mass of the boiling water is equal to the mass of the water that is initially at 20°C and the specific heat of water is independent of its temperature:

$$T_{\text{equil}} - 100°\text{C} + T_{\text{equil}} - 20°\text{C} = 0$$

Solving for T_{equil} yields: $T_{\text{equil}} = 60°\text{C}$

and $\boxed{(a)}$ is correct.

6 • You pour some water at 100°C and some ice cubes at 0°C into an insulated container. The final temperature of the mixture will be (*a*) 50°C, (*b*) less than 50°C but larger than 0°C, (*c*) 0°C, (*d*) You cannot tell the final temperature from the data given.

Determine the Concept We would need to know the mass of the boiling water and the mass of the melting ice in order to determine the final temperature of the mixture. $\boxed{(d)}$ is correct.

7 • You pour water at 100°C and some ice cubes at 0°C into an insulated container. When thermal equilibrium is reached, you notice some ice remains and floats in liquid water. The final temperature of the mixture is (*a*) above 0°C, (*b*) less than 0°C, (*c*) 0°C, (*d*) You cannot tell the final temperature from the data given

Determine the Concept Because some ice remains when thermal equilibrium is reached, the equilibrium temperature must be the temperature of the ice cubes at their melting point. $\boxed{(c)}$ is correct.

8 • Joule's experiment establishing the mechanical equivalence of heat involved the conversion of mechanical energy into internal energy. Give some everyday examples in which some of the internal energy of a system is converted into mechanical energy.

Determine the Concept One example from everyday life is that of a hot gas that expands and does work–as in the piston of an engine. Another example is the gas escaping from a can of spray paint. As it escapes, it moves air molecules and, in the process, does work against atmospheric pressure.

9 • Can a gas absorb heat while its internal energy does not change? If so, give an example. If not, explain why not.

Determine the Concept Yes. $\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$. If the gas does work at the same rate that it absorbs heat, its internal energy will remain constant.

10 • The equation $\Delta E_{\text{int}} = Q + W$ is the formal statement of the first law of thermodynamics. In this equation, the quantities Q and W, respectively, represent (*a*) the heat absorbed by the system and the work done by the system, (*b*) the heat absorbed by the system and the work done on the system, (*c*) the heat released by the system and the work done by the system, (*d*) the heat released by the system and the work done on the system.

Determine the Concept According to the first law of thermodynamics, the change in the internal energy of the system is equal to the heat that enters the system plus the work done on the system. $\boxed{(b)}$ is correct.

11 • **[SSM]** A real gas cools during a free expansion, while an ideal gas does not cool during a free expansion. Explain the reason for this difference.

Determine the Concept Particles that attract each other have more potential energy the farther apart they are. In a real gas the molecules exert weak attractive forces on each other. These forces increase the internal potential energy during an expansion. An increase in potential energy means a decrease in kinetic energy, and a decrease in kinetic energy means a decrease in translational kinetic energy. Thus, there is a decrease in temperature.

12 • An ideal gas that has a pressure of 1.0 atm and a temperature of 300 K is confined to half of an insulated container by a thin partition. The other half of the container is a vacuum. The partition is punctured and equilibrium is quickly reestablished. Which of the following is correct? (*a*) The gas pressure is 0.50 atm and the temperature of the gas is 150 K. (*b*) The gas pressure is 1.0 atm and the temperature of the gas is 150 K. (*c*) The gas pressure is 0.50 atm and the temperature of the gas is 300 K. (*d*) None of the above.

Determine the Concept Because the container is insulated, no energy is exchanged with the surroundings during the expansion of the gas. Neither is any work done on or by the gas during this process. Hence, the internal energy of the gas does not change and we can conclude that the equilibrium temperature will be the same as the initial temperature. Applying the ideal-gas law for a fixed amount of gas we see that the pressure at equilibrium must be half an atmosphere. $\boxed{(c)}$ is correct.

13 • A gas consists of ions that repel each other. The gas undergoes a free expansion in which no heat is absorbed or released and no work is done. Does the temperature of the gas increase, decrease, or remain the same? Explain your answer.

Determine the Concept Particles that repel each other have more potential energy the closer together they are. The repulsive forces decrease the internal potential energy during an expansion. A decrease in potential energy means an increase in kinetic energy, and an increase in kinetic energy means an increase in translational kinetic energy. Thus, there is an increase in temperature.

14 • Two gas-filled rubber balloons that have equal volumes are located at the bottom of a dark, cold lake. The temperature of the water decreases with increasing depth. One balloon rises rapidly and expands adiabatically as it rises. The other balloon rises more slowly and expands isothermally. The pressure in each balloon remains equal to the pressure in the water just next to the balloon. Which balloon has the larger volume when it reaches the surface of the lake? Explain your answer.

Determine the Concept The balloon that expands isothermally is larger when it reaches the surface. The balloon that expands adiabatically will be at a lower temperature than the one that expands isothermally. Because each balloon has the same number of gas molecules and are at the same pressure, the one with the higher temperature will be bigger. An analytical argument that leads to the same conclusion is shown below.

Letting the subscript ″a″ denote the adiabatic process and the subscript ″i″ denote the isothermal process, express the equation of state for the adiabatic balloon:

$$P_0V_0^\gamma = P_\mathrm{f}V_\mathrm{f,a}^\gamma \Rightarrow V_\mathrm{f,a} = V_0\left(\frac{P_0}{P_\mathrm{f}}\right)^{1/\gamma}$$

For the isothermal balloon:

$$P_0V_0 = P_\mathrm{f}V_\mathrm{f,i} \Rightarrow V_\mathrm{f,i} = V_0\left(\frac{P_0}{P_\mathrm{f}}\right)$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{V_{\text{f,i}}}{V_{\text{f,a}}}=\frac{V_0\left(\dfrac{P_0}{P_\text{f}}\right)}{V_0\left(\dfrac{P_0}{P_\text{f}}\right)^{1/\gamma}}=\left(\frac{P_0}{P_\text{f}}\right)^{1-1/\lambda}$$

Because $P_0/P_\text{f} > 1$ and $\gamma > 1$:

$$\boxed{V_{\text{f,i}} > V_{\text{f,a}}}$$

15 • A gas changes its state quasi-statically from A to C along the paths shown in Figure 18-21. The work done by the gas is (*a*) greatest for path A→B→C, (*b*) least for path A→C, (*c*) greatest for path A→D→C, (*d*) The same for all three paths.

Determine the Concept The work done along each of these paths equals the area under its curve. The area is greatest for the path A→B→C and least for the path A→D→C. $\boxed{(a)}$ is correct.

16 • When an ideal gas undergoes an adiabatic process, (*a*) no work is done by the system, (*b*) no heat is transferred to the system, (*c*) the internal energy of the system remains constant, (*d*) the amount of heat transferred into the system equals the amount of work done by the system.

Determine the Concept An adiabatic process is, by definition, one for which no heat enters or leaves the system. $\boxed{(b)}$ is correct.

17 • True or false:

(*a*) When a system can go from state 1 to state 2 by several different processes, the amount of heat absorbed by the system will be the same for all processes.
(*b*) When a system can go from state 1 to state 2 by several different processes, the work done on the system will be the same for all processes.
(*c*) When a system can go from state 1 to state 2 by several different processes, the change in the internal energy of the system will be the same for all processes.
(*d*) The internal energy of a given amount of an ideal gas depends only on its absolute temperature.
(*e*) A quasi-static process is one in which the system is never far from being in equilibrium.
(*f*) For any substance that expands when heated, its C_p is greater than its C_v.

(*a*) False. The amount the internal energy of the system changes along any path depends only on the temperatures at state 1 and state 2, but because $\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$, Q_{in} depends on W_{on}, which, in turn, depends on the path taken from state 1 to state 2.

(*b*) False. The work done on the system depends on the path taken from state 1 to state 2.

(*c*) True. The amount the internal energy of the system changes along any path depends only on the temperatures at state 1 and state 2.

(*d*) True. For an ideal gas, $E_{\text{int}} = \frac{3}{2}nRT$. For a ″given amount″ of an ideal gas, n is constant.

(*e*) True. This is the definition of a quasi-static process.

(*f*) True. All materials have values for C_{p} and C_{v} for which it is true that C_{p} is greater than C_{v}. For liquids and solids we generally ignore the very small difference between C_{p} and C_{v} .

18 • The volume of a sample of gas remains constant while its pressure increases. (*a*) The internal energy of the system is unchanged. (*b*) The system does work. (*c*) The system absorbs no heat. (*d*) The change in internal energy must equal the heat absorbed by the system. (e) None of the above.

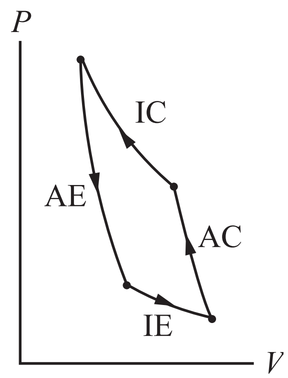
Determine the Concept For a constant-volume process, no work is done on or by the gas. Applying the first law of thermodynamics, we obtain $Q_{\text{in}} = \Delta E_{\text{int}}$. Because the temperature must change during such a process, we can conclude that $\Delta E_{\text{int}} \neq 0$ and hence $Q_{\text{in}} \neq 0$. $\boxed{(d)}$ is correct.

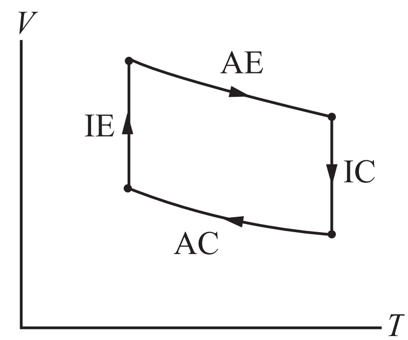
19 • When an ideal gas undergoes an isothermal process, (*a*) no work is done by the system, (*b*) no heat is absorbed by the system, (*c*) the heat absorbed by the system equals the change in the system's internal energy, (*d*) the heat absorbed by the system equals the work done by the system.

Determine the Concept Because the temperature does not change during an isothermal process, the change in the internal energy of the gas is zero. Applying the first law of thermodynamics, we obtain $Q_{\text{in}} = -W_{\text{on}} = W_{\text{by the system}}$. Hence $\boxed{(d)}$ is correct.

20 •• Consider the following series of sequential quasi-static processes that a system undergoes: (1) an adiabatic expansion, (2) an isothermal expansion, (3) an adiabatic compression and (4) an isothermal compression which brings the system back to its original state. Sketch the series of processes on a *PV* diagram, and then sketch the series of processes on a *VT* diagram (in which volume is plotted as function of temperature).

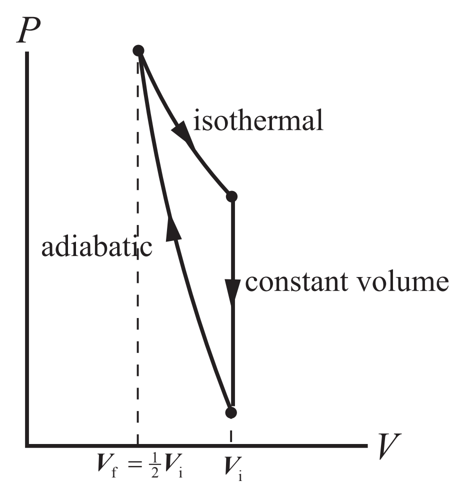
Determine the Concept In the graphs shown below, ″I″ denotes an isothermal process, ″A″ denotes an adiabatic process, ″E″ denotes an expansion, and ″C″ denotes a compression.





21 •• **[SSM]** An ideal gas in a cylinder is at pressure *P* and volume *V*. During a quasi-static adiabatic process, the gas is compressed until its volume has decreased to *V*/2. Then, in a quasi-static isothermal process, the gas is allowed to expand until its volume again has a value of *V*. What kind of process will return the system to its original state? Sketch the cycle on a graph.

Determine the Concept During a reversible adiabatic process, PV^{γ} is constant, where $\gamma < 1$ and during an isothermal processes PV is constant. Thus, the pressure rise during the compression is greater than the pressure drop during the expansion. The final process could be a constant volume process during which heat is absorbed from the system. A constant-volume cooling will decrease the pressure and return the gas to its original state.



22 •• Metal A is denser than metal B. Which would you expect to have a higher heat capacity per unit mass, metal A or metal B? Why?

Determine the Concept We can use the definition of heat capacity, the Dulong-Petit law, and the relationship between the mass of a substance and its molar mass to predict whether metal A or metal B has a higher heat capacity per unit mass.

The specific heat of metal A is given by:

$$c_A = \frac{c'}{M_A}$$

From the Dulong-Petit law, the molar heat capacity of most metals is approximately:

$$c' = 3R$$

Substituting for c' yields:

$$c_A = \frac{3R}{M_A}$$

Similarly, the specific heat of metal B is given by:

$$c_B = \frac{3R}{M_B}$$

Dividing c_A by c_B and simplifying yields:

$$\frac{c_A}{c_B} = \frac{\frac{3R}{M_A}}{\frac{3R}{M_B}} = \frac{M_B}{M_A} \Rightarrow c_A = \frac{M_B}{M_A} c_B$$

Because metal A is denser than metal B:

$$\boxed{c_A > c_B}$$

23 •• An ideal gas undergoes a process during which $P\sqrt{V} = \text{constant}$ and the volume of the gas decreases. Does its temperature increase, decrease, or remain the same during this process? Explain.

Picture the Problem We can use the given dependence of the pressure on the volume and the ideal-gas law to show that if the volume decreases, so does the temperature.

We're given that:

$$P\sqrt{V} = \text{constant}$$

Because the gas is an ideal gas:

$$PV = \left(P\sqrt{V}\right)\sqrt{V} = \text{constant} \times \sqrt{V}$$
$$= nRT$$

Solving for T yields:

$$T = \frac{(\text{constant})\sqrt{V}}{nR} \propto \sqrt{V}$$

Because T varies with the square root of V, if the volume decreases, the temperature decreases.

Estimation and Approximation

24 • During the early stages of designing a modern electric generating plant, you are in charge of the team of environmental engineers. The new plant is to be located on the ocean and will use ocean water for cooling. The plant will produce electrical power at the rate of 1.00 GW. Because the plant will have an efficiency of one-third (typical of most modern plants), heat will be released to the cooling water at the rate of 2.00 GW. If environmental codes require that only water with a temperature increase of 15°F or less can be returned to the ocean, estimate the flow rate (in kg/s) of cooling water through the plant.

Picture the Problem The rate at which thermal energy is delivered to the ocean by the power plant is given by $P = \Delta Q/\Delta t$ where $Q = mc\Delta T$

Express the rate at which thermal energy is delivered to the ocean:

$$P = \frac{\Delta Q}{\Delta t} = \frac{\Delta}{\Delta t}(mc\Delta T) = \frac{\Delta m}{\Delta t}c\Delta T$$

Our interest is in the flow rate of the cooling water. Solving for $\Delta m/\Delta t$ yields:

$$\frac{\Delta m}{\Delta t} = \frac{P}{c\Delta T}$$

Substitute numerical values and evaluate $\Delta m/\Delta t$:

$$\frac{\Delta m}{\Delta t} = \frac{2.00\times10^{9}\ \text{J/s}}{\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{C°}}\right)\left(15\ \text{F°}\times\frac{5\ \text{C°}}{9\ \text{F°}}\right)} = \boxed{5.7\times10^{4}\ \text{kg/s}}$$

25 •• **[SSM]** A ″typical″ microwave oven has a power consumption of about 1200 W. Estimate how long it should take to boil a cup of water in the microwave assuming that 50% of the electrical power consumption goes into heating the water. How does this estimate correspond to everyday experience?

Picture the Problem Assume that the water is initially at 30°C and that the cup contains 200 g of water. We can use the definition of power to express the required time to bring the water to a boil in terms of its mass, heat capacity, change in temperature, and the rate at which energy is supplied to the water by the microwave oven.

Use the definition of power to relate the energy needed to warm the water to the elapsed time:

$$P = \frac{\Delta W}{\Delta t} = \frac{mc\Delta T}{\Delta t} \Rightarrow \Delta t = \frac{mc\Delta T}{P}$$

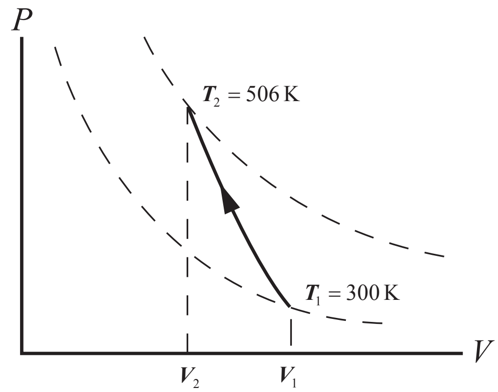
Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{(0.200\,\text{kg})\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(100°\text{C} - 30°\text{C})}{600\,\text{W}} = 97.63\,\text{s} \approx \boxed{1.6\,\text{min}},$$

an elapsed time that seems to be consistent with experience.

26 •• A demonstration of the heating of a gas under adiabatic compression involves putting a small strip of paper into a large glass test tube, which is then sealed off with a piston. If the piston compresses the trapped air very rapidly, the paper will catch fire. Assuming that the burning point of paper is 451ºF, estimate the factor by which the volume of the air trapped by the piston must be reduced for this demonstration to work.

Picture the Problem The adiabatic compression from an initial volume V_1 to a final volume V_2 between the isotherms at temperatures T_1 and T_2 is shown below. We'll assume a room temperature of 300 K and apply the equation for a quasi-static adiabatic compression to 451ºF (506 K) with $\gamma_{\text{air}} = 1.4$ to solve for the ratio of the initial to the final volume of the air.



Express $TV^{\gamma-1} = \text{constant}$ in terms of the initial and final values of T and V:

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow \frac{V_1}{V_2} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}$$

Substitute numerical values and evaluate V_1/V_2:

$$\frac{V_1}{V_2} = \left(\frac{506\,\text{K}}{300\,\text{K}}\right)^{\frac{1}{1.4-1}} = \boxed{3.69}$$

27 •• A small change in the volume of a liquid occurs when heating the liquid at constant pressure. Use the following data to estimate the fractional contribution this change makes to the heat capacity of water between 4.00ºC and 100ºC. The density of water at 4.00ºC and 1.00 atm pressure is 1.000 g/cm^3. The density of liquid water at 100ºC and 1.00 atm pressure is 0.9584 g/cm^3.

Picture the Problem The heat capacity C_p of a sample of water is equal to the mass m of the sample times the specific heat capacity c_p of water, where c_p = 4186 J/kg·K. When the sample is heated, its temperature and its volume increases. The increase in temperature reflects the increase in the internal energy of the sample, and the increase in volume reflects the amount of work W_{on} done by the sample on its surroundings. That is, $W_{on} = P\Delta V$, where P is the pressure and ΔV is the change in volume of the sample. We are looking for the ratio of the work done by the sample to the heat absorbed by the sample.

The heat that enters the water during the constant pressure process is given by:

$$Q_{in} = mc_p\Delta T$$

The work that is done by the sample during this process is given by:

$$W_{by} = P\Delta V = P(V_{100°C} - V_{4.00°C})$$

The volumes of the sample at these temperatures is related to its densities at these temperatures:

$$W_{by} = P\left(\frac{m}{\rho_{100°C}} - \frac{m}{\rho_{4.00°C}}\right) = Pm\left(\frac{1}{\rho_{100°C}} - \frac{1}{\rho_{4.00°C}}\right)$$

where m is the mass of the sample.

Taking the ratio of W_{by} to Q_{in} yields:

$$\frac{W_{by}}{Q_{in}} = \frac{Pm}{mc_p\Delta T}\left(\frac{1}{\rho_{100°C}} - \frac{1}{\rho_{4.00°C}}\right) = \frac{P}{c_p\Delta T}\left(\frac{1}{\rho_{100°C}} - \frac{1}{\rho_{4.00°C}}\right)$$

Substitute numerical values and evaluate W_{by}/Q_{in}:

$$\frac{W_{by}}{Q_{in}} = \frac{101.325\times 10^3\ \text{Pa}}{\left(4186\ \frac{\text{J}}{\text{kg}\cdot\text{K}}\right)(96\ \text{K})}\left(\frac{1}{954.3\ \frac{\text{kg}}{\text{m}^3}} - \frac{1}{1000\frac{\text{kg}}{\text{m}^3}}\right) = \boxed{1.2\times 10^{-3}\%}$$

Heat Capacity, Specific Heat, Latent Heat

28 • You designed a solar home that contains 1.00×10^5 kg of concrete (specific heat = 1.00 kJ/kg·K). How much heat is released by the concrete at night when it cools from 25.0ºC to 20.0ºC?

Picture the Problem We can use the relationship $Q = mc\Delta T$ to calculate the amount of heat given off by the concrete as it cools from 25.0°C to 20.0°C.

Relate the heat given off by the concrete to its mass, specific heat, and change in temperature:

$$Q = mc\Delta T$$

Substitute numerical values and evaluate Q:

$$Q = \left(1.00\times10^{5}\text{ kg}\right)\left(1.00\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(25.0°\text{C} - 20.0°\text{C}\right) = \boxed{500\,\text{kJ}}$$

29 • **[SSM]** How much heat must be absorbed by 60.0 g of ice at –10.0ºC to transform it into 60.0 g of water at 40.0ºC?

Picture the Problem We can find the amount of heat that must be absorbed by adding the heat required to warm the ice from −10.0°C to 0°C, the heat required to melt the ice, and the heat required to warm the water formed from the ice to 40.0°C.

Express the total heat required:

$$Q = Q_{\text{warm ice}} + Q_{\text{melt ice}} + Q_{\text{warm water}}$$

Substitute for each term to obtain:

$$\begin{aligned} Q &= mc_{\text{ice}}\Delta T_{\text{ice}} + mL_{\text{f}} + mc_{\text{water}}\Delta T_{\text{water}} \\ &= m\left(c_{\text{ice}}\Delta T_{\text{ice}} + L_{\text{f}} + c_{\text{water}}\Delta T_{\text{water}}\right) \end{aligned}$$

Substitute numerical values (See Tables 18-1 and 18-2) and evaluate Q:

$$\begin{aligned} Q &= \left(0.0600\,\text{kg}\right)\left[\left(2.05\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(0°\text{C} - \left(-10.0°\text{C}\right)\right) + 333.5\frac{\text{kJ}}{\text{kg}}\right. \\ &\qquad \left. + \left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(40.0°\text{C} - 0°\text{C}\right)\right] \\ &= \boxed{31.3\,\text{kJ}} \end{aligned}$$

30 •• How much heat must be released by 0.100 kg of steam at 150ºC to transform it into 0.100 kg of ice at 0ºC?

Picture the Problem We can find the amount of heat that must be removed by adding the heat that must be removed to cool the steam from 150°C to 100°C, the heat that must be removed to condense the steam to water, the heat that must be removed to cool the water from 100°C to 0°C, and the heat that must be removed to freeze the water.

Express the total heat that must be removed:

$$Q = Q_{\text{cool steam}} + Q_{\text{condense steam}} + Q_{\text{cool water}} + Q_{\text{freeze water}}$$

Substitute for each term and simplify to obtain:

$$\begin{aligned} Q &= mc_{\text{steam}}\Delta T_{\text{steam}} + mL_{\text{v}} + mc_{\text{water}}\Delta T_{\text{water}} + mL_{\text{f}} \\ &= m\left(c_{\text{steam}}\Delta T_{\text{steam}} + L_{\text{v}} + c_{\text{water}}\Delta T_{\text{water}} + L_{\text{f}}\right) \end{aligned}$$

Substitute numerical values (See Tables 18-1 and 18-2) and evaluate Q:

$$\begin{aligned} Q = (0.100\,\text{kg})\Bigg[&\left(2.02\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(150°\text{C} - 100°\text{C}) + 2.26\frac{\text{MJ}}{\text{kg}} \\ &+ \left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(100°\text{C} - 0°\text{C}) + 333.5\frac{\text{kJ}}{\text{kg}}\Bigg] \\ = {}&\boxed{311\,\text{kJ}} \end{aligned}$$

31 •• A 50.0-g piece of aluminum at 20ºC is cooled to –196ºC by placing it in a large container of liquid nitrogen at that temperature. How much nitrogen is vaporized? (Assume that the specific heat of aluminum is constant over this temperature range.)

Picture the Problem We can find the amount of nitrogen vaporized by equating the heat gained by the liquid nitrogen and the heat lost by the piece of aluminum.

Express the heat gained by the liquid nitrogen as it cools the piece of aluminum:

$$Q_{\text{N}} = m_{\text{N}}L_{\text{v,N}}$$

Express the heat lost by the piece of aluminum as it cools:

$$Q_{\text{Al}} = m_{\text{Al}}c_{\text{Al}}\Delta T_{\text{Al}}$$

Equate these two expressions and solve for m_{N}:

$$m_{\text{N}}L_{\text{v,N}} = m_{\text{Al}}c_{\text{Al}}\Delta T_{\text{Al}}$$

and

$$m_{\text{N}} = \frac{m_{\text{Al}}c_{\text{Al}}\Delta T_{\text{Al}}}{L_{\text{v,N}}}$$

Substitute numerical values (see Table 18-1 for the specific heat of aluminum) and evaluate m_N:

$$m_N = \frac{(0.0500\,\text{kg})\left(0.900\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(20°\text{C} - (-196°\text{C}))}{199\,\frac{\text{kJ}}{\text{kg}}} = \boxed{48.8\,\text{g}}$$

32 •• You are supervising the creation of some lead castings for use in the construction industry. Each casting involves one of your workers pouring 0.500 kg of molten lead that has a temperature of 327ºC into a cavity in a large block of ice at 0ºC. How much liquid water should you plan on draining per hour if there are 100 workers who are able to each average one casting every 10.0 min?

Picture the Problem Because the heat lost by the lead as it cools is gained by the block of ice (we're assuming no heat is lost to the surroundings), we can apply the conservation of energy to determine how much ice melts.

Apply the conservation of energy to this process:

$$\Delta Q = \Delta Q_{\text{Pb}} + \Delta Q_{\text{W}} = 0$$

or

$$-m_{\text{Pb}}\left(L_{\text{f,Pb}} + c_{\text{Pb}}\Delta T_{\text{Pb}}\right) + m_{\text{w, 1 casting}}L_{\text{f,w}} = 0$$

Solving for m_w yields:

$$m_{\text{W, 1 casting}} = \frac{m_{\text{Pb}}\left(L_{\text{f,Pb}} + c_{\text{Pb}}\Delta T_{\text{Pb}}\right)}{L_{\text{f,W}}}$$

Because there are N workers each turning out n castings per hour:

$$m = nNm_{\text{W.1 casting}} = \frac{nNm_{\text{Pb}}\left(L_{\text{f,Pb}} + c_{\text{Pb}}\Delta T_{\text{Pb}}\right)}{L_{\text{f,W}}}$$

Substitute numerical values and evaluate m_w:

$$m = \frac{(100)(6)(0.500\,\text{kg})\left(24.7\,\frac{\text{kJ}}{\text{kg}} + \left(0.128\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(327°\text{C} - 0°\text{C})\right)}{333.5\,\frac{\text{kJ}}{\text{kg}}} \approx \boxed{60\,\text{kg}}$$

Calorimetry

33 • **[SSM]** While spending the summer on your uncle's horse farm, you spend a week apprenticing with his farrier (a person who makes and fits

horseshoes). You observe the way he cools a shoe after pounding the hot, pliable shoe into the correct size and shape. Suppose a 750-g iron horseshoe is taken from the farrier's fire, shaped, and at a temperature of 650°C, dropped into a 25.0-L bucket of water at 10.0°C. What is the final temperature of the water after the horseshoe and water arrive at equilibrium? Neglect any heating of the bucket and assume the specific heat of iron is $460\ \text{J/(kg}\cdot\text{K)}$.

Picture the Problem During this process the water will gain energy at the expense of the horseshoe. We can use conservation of energy to find the equilibrium temperature. See Table 18-1 for the specific heat of water.

Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\text{warm the water}} + Q_{\text{cool the horseshoe}} = 0$$

or

$$m_{\text{water}}c_{\text{water}}\left(t_{\text{f}} - 10.0°\text{C}\right) + m_{\text{Fe}}c_{\text{Fe}}\left(t_{\text{f}} - 650°\text{C}\right) = 0$$

Solve for t_{f} to obtain:

$$t_{\text{f}} = \frac{m_{\text{water}}c_{\text{water}}\left(10.0°\text{C}\right) + m_{\text{Fe}}c_{\text{Fe}}\left(650°\text{C}\right)}{m_{\text{water}}c_{\text{water}} + m_{\text{Fe}}c_{\text{Fe}}}$$

Substitute numerical values and evaluate t_{f}:

$$t_{\text{f}} = \frac{\left(25.0\ \text{kg}\right)\left(4.184\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(10.0°\text{C}\right) + \left(0.750\ \text{kg}\right)\left(0.460\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(650°\text{C}\right)}{\left(25.0\ \text{kg}\right)\left(4.184\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) + \left(0.750\ \text{kg}\right)\left(0.460\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)}$$

$$= \boxed{12.1°\text{C}}$$

34 • The specific heat of a certain metal can be determined by measuring the temperature change that occurs when a piece of the metal is heated and then placed in an insulated container that is made of the same material and contains water. Suppose the piece of metal has a mass of 100 g and is initially at 100ºC. The container has a mass of 200 g and contains 500 g of water at an initial temperature of 20.0ºC. The final temperature is 21.4ºC. What is the specific heat of the metal?

Picture the Problem During this process the water and the container will gain thermal energy at the expense of the piece of metal. Applying conservation of energy will lead to an expression you can solve for the specific heat of the metal.

Apply conservation of energy to obtain:

$$\sum_i Q_i = Q_{\text{warm the water}} + Q_{\text{warm the container}} + Q_{\text{cool the metal sample}} = 0$$

or

$$m_w c_w \Delta T_w + m_{\text{container}} c_{\text{metal}} \Delta T_w + m_{\text{metal}} c_{\text{metal}} \Delta T_{\text{metal}} = 0$$

Solving for c_{metal} yields:

$$c_{\text{metal}} = \frac{m_w c_w \Delta T_w}{m_{\text{metal}} \Delta T_{\text{metal}} - m_{\text{container}} \Delta T_w}$$

Substitute numerical values and evaluate c_{metal} :

$$c_{\text{metal}} = \frac{(0.500\,\text{kg})\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(21.4°\text{C} - 20.0°\text{C})}{(0.100\,\text{kg})(100°\text{C} - 21.4°\text{C}) - (0.200\,\text{kg})(21.4°\text{C} - 20.0°\text{C})}$$
$$= \boxed{0.39\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}}$$

Remarks: Consulting Table 18-1, we note that the metal is copper.

35 •• During his many appearances at the Tour de France, champion bicyclist Lance Armstrong typically expended an average power of 400 W, 5.0 hours a day for 20 days. What quantity of water, initially at 24ºC, could be brought to a boil if you could harness all of that energy?

Picture the Problem We can use $Q = mc\Delta T$ to express the mass m of water that can be heated through a temperature interval ΔT by an amount of heat energy Q. We can then find the amount of heat energy expended by Armstrong from the definition of power.

Express the amount of heat energy Q required to raise the temperature of a mass m of water by ΔT:

$$Q = mc\Delta T \Rightarrow m = \frac{Q}{c\Delta T}$$

Use the definition of power to relate the heat energy expended by Armstrong to the rate at which he expended the energy:

$$P = \frac{Q}{\Delta t} \Rightarrow Q = P\Delta t$$

Substitute for Q to obtain:

$$m = \frac{P\Delta t}{c\Delta T}$$

Substitute numerical values and evaluate m:

$$m = \frac{\left(400\,\frac{\text{J}}{\text{s}} \times \frac{3600\,\text{s}}{\text{h}}\right)\left(5.0\,\frac{\text{h}}{\text{d}} \times 20\,\text{d}\right)}{\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(100°\text{C} - 24°\text{C})} = \boxed{4.5\times10^{2}\,\text{kg}}$$

36 •• A 25.0-g glass tumbler contains 200 mL of water at 24.0ºC. If two 15.0-g ice cubes, each at a temperature of –3.00ºC, are dropped into the tumbler, what is the final temperature of the drink? Neglect any heat transfer between the tumbler and the room.

Picture the Problem First you need to convince yourself that there is enough energy in the water to melt all the ice. Once you've done that you can use conservation of energy to find the final equilibrium temperature. See Tables 18-1 and 18-2 for specific heats and the latent heat of fusion of water.

First, determine the energy required to warm and melt all the ice:

$$\begin{aligned}
Q_{\text{needed}} &= Q_{\text{warm ice}} + Q_{\text{melt ice}} = m_{\text{ice}}c_{\text{ice}}\Delta T_{\text{warm ice}} + m_{\text{ice}}L_{\text{f}} \\
&= (0.0300\,\text{kg})\left(2.05\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(3.00\,\text{C}°) + (0.0300\,\text{kg})(333.5\,\text{kJ/kg}) \\
&= 0.1845\,\text{kJ} + 10.01\,\text{kJ} = 10.19\,\text{kJ}
\end{aligned}$$

The maximum amount of energy available from the water and the tumbler is:

$$\begin{aligned}
Q_{\text{available, max}} &= m_{\text{tumbler}}c_{\text{glass}}\Delta T_{\text{max, tumbler}} + m_{\text{water}}c_{\text{water}}\Delta T_{\text{max, water}} \\
&= m_{\text{tumbler}}c_{\text{glass}}\Delta T_{\text{max, tumbler}} + \rho_{\text{water}}V_{\text{water}}c_{\text{water}}\Delta T_{\text{max, water}} \\
&= (0.0250\,\text{kg})\left(0.840\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C} - 24.0°\text{C}) \\
&\quad + \left(1.00\times10^{3}\,\frac{\text{kg}}{\text{m}^3}\right)\left(200\times10^{-3}\,\text{L}\times\frac{10^{-3}\,\text{m}^3}{\text{L}}\right)\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) \\
&\quad \times(0°\text{C} - 24.0°\text{C}) \\
&= -0.5040\,\text{kJ} - 20.08\,\text{kJ} = -20.59\,\text{kJ}
\end{aligned}$$

Because $\left|Q_{\text{available, max}}\right| > Q_{\text{needed}}$, all the ice will melt and the final temperature will be greater than 0°C. Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\text{cool the tumbler}} + Q_{\text{cool the warm water}} + Q_{\text{warm the ice}} + Q_{\text{melt all the ice}} + Q_{\text{warm the ice water}} = 0$$

or

$$m_{\text{tumbler}} c_{\text{glass}} \Delta T_{\text{tumbler}} + m_{\text{water}} c_{\text{water}} \Delta T_{\text{water}} + m_{\text{ice}} c_{\text{ice}} \Delta T_{\text{warn ice}} + m_{\text{ice}} L_{\text{f, ice}} + m_{\text{ice}} c_{\text{water}} \Delta T_{\text{ice water}} = 0$$

Substituting numerical values yields:

$$(0.0250\,\text{kg})\left(0.840\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 24.0°\text{C}) + (0.200\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 24.0°\text{C}) + 0.1845\,\text{kJ} + 10.01\,\text{kJ} + (0.0300\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 0°\text{C}) = 0$$

Solving for t_{f} yields: $t_{\text{f}} = \boxed{10.6°\text{C}}$

37 •• [SSM] A 200-g piece of ice at 0ºC is placed in 500 g of water at 20ºC. This system is in a container of negligible heat capacity and is insulated from its surroundings. (*a*) What is the final equilibrium temperature of the system? (*b*) How much of the ice melts?

Picture the Problem Because we can not tell, without performing a couple of calculations, whether there is enough heat available in the 500 g of water to melt all of the ice, we'll need to resolve this question first. See Tables 18-1 and 18-2 for specific heats and the latent heat of fusion of water.

(*a*) Determine the energy required to melt 200 g of ice:

$$Q_{\text{melt ice}} = m_{\text{ice}} L_{\text{f}} = (0.200\,\text{kg})\left(333.5\frac{\text{kJ}}{\text{kg}}\right) = 66.70\,\text{kJ}$$

The energy available from 500 g of water at 20ºC is:

$$Q_{\text{available, max}} = m_{\text{water}} c_{\text{water}} \Delta T_{\text{water}} = (0.500\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C} - 20°\text{C}) = -41.84\,\text{kJ}$$

Because $\left|Q_{\text{available, max}}\right| < Q_{\text{melt ice}}$:

$$\boxed{\text{The final temperature is } 0°\text{C.}}$$

(*b*) Equate the energy available from the water $\left|Q_{\text{available, max}}\right|$ to $m_{\text{ice}}L_{\text{f}}$ and solve for m_{ice} to obtain:

$$m_{\text{ice}} = \frac{\left|Q_{\text{available, max}}\right|}{L_{\text{f}}}$$

Substitute numerical values and evaluate m_{ice}:

$$m_{\text{ice}} = \frac{41.84\,\text{kJ}}{333.5\,\frac{\text{kJ}}{\text{kg}}} = \boxed{125\,\text{g}}$$

38 •• A 3.5-kg block of copper at a temperature of 80ºC is dropped into a bucket containing a mixture of ice and water whose total mass is 1.2 kg. When thermal equilibrium is reached, the temperature of the water is 8.0ºC. How much ice was in the bucket before the copper block was placed in it? (Assume that the heat capacity of the bucket is negligible.)

Picture the Problem Because the bucket contains a mixture of ice and water initially, we know that its temperature must be 0°C. We can apply conservation of energy to find the amount of ice initially in the bucket. See Tables 18-1 and 18-2 for specific heats and the heats of fusion and vaporization of water.

Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\substack{\text{melt all}\\ \text{the ice}}} + Q_{\substack{\text{warm the}\\ \text{ice water}}} + Q_{\substack{\text{cool the}\\ \text{copper block}}} = 0$$

Substitute for $Q_{\substack{\text{melt all}\\ \text{the ice}}}$, $Q_{\substack{\text{warm the}\\ \text{ice water}}}$, and $Q_{\substack{\text{cool the}\\ \text{copper block}}}$ to obtain:

$$m_{\text{ice}}L_{\text{f}} + m_{\text{ice water}}c_{\text{water}}\Delta T_{\text{ice water}} + m_{\text{Cu}}c_{\text{Cu}}\Delta T_{\text{Cu}} = 0$$

Solving for m_{ice} yields:

$$m_{\text{ice}} = \frac{m_{\text{Cu}}c_{\text{Cu}}\Delta T_{\text{Cu}} - m_{\text{ice water}}c_{\text{water}}\Delta T_{\text{ice water}}}{L_{\text{f}}}$$

Substitute numerical values and evaluate m_{ice}:

$$m_{ice} = \frac{(3.5\,\text{kg})\left(0.386\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(80°\text{C} - 8.0°\text{C})}{333.5\,\frac{\text{kJ}}{\text{kg}}} - \frac{(1.2\,\text{kg})\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(8.0°\text{C} - 0°\text{C})}{333.5\,\frac{\text{kJ}}{\text{kg}}}$$

$$= \boxed{0.17\,\text{kg}}$$

39 •• A well-insulated bucket of negligible heat capacity contains 150 g of ice at 0ºC. (*a*) If 20 g of steam at 100ºC is injected into the bucket, what is the final equilibrium temperature of the system? (*b*) Is any ice left after the system reaches equilibrium?

Picture the Problem First you need to convince yourself that there is enough energy in the steam to melt all the ice. Once you've done that you can use conservation of energy to find the final equilibrium temperature. See Tables 18-1 and 18-2 for specific heats and the heats of fusion and vaporization of water.

(*a*) First, determine the energy required to melt all the ice:

$$\begin{aligned} Q_{\text{melt ice}} &= m_{\text{ice}} L_f \\ &= (0.150\,\text{kg})(333.5\,\text{kJ/kg}) \\ &= 50.03\,\text{kJ} \end{aligned}$$

Find the maximum amount of energy available from the steam is:

$$\begin{aligned} Q_{\text{steam, max}} &= m_{\text{steam}} L_v + m_{\text{steam water}} c_{\text{water}} \Delta T_{\text{max}} \\ &= (0.020\,\text{kg})\left(2257\,\frac{\text{kJ}}{\text{kg}}\right) + (0.020\,\text{kg})\left(4.184\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C} - 100°\text{C}) \\ &= -45.14\,\text{kJ} - 8.37\,\text{kJ} = -53.51\,\text{kJ} \end{aligned}$$

Because $\left|Q_{\text{steam, max}}\right| > Q_{\text{melt ice}}$, all the ice will melt and the final temperature will be greater than 0°C. Apply conservation of energy to obtain:

$$\sum_i Q_i = Q_{\substack{\text{condense}\\ \text{steam}}} + Q_{\substack{\text{cool the}\\ \text{hot water}}} + Q_{\substack{\text{melt all}\\ \text{the ice}}} + Q_{\substack{\text{warm the}\\ \text{ice water}}} = 0$$

or

$$-45.14\,\text{kJ} + m_{\text{hot water}}c_{\text{water}}\left(t_\text{f} - 100°\text{C}\right) + 50.03\,\text{kJ} + m_{\text{ice}}c_{\text{water}}\left(t_\text{f} - 0°\text{C}\right) = 0$$

Substituting numerical values yields:

$$-45.14\,\text{kJ} + (0.020\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(t_\text{f} - 100°\text{C}\right) + 50.03\,\text{kJ} + (0.150\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\left(t_\text{f} - 0°\text{C}\right) = 0$$

Solving for t_f yields: $$t_\text{f} = \boxed{4.9°\text{C}}$$

(*b*) Because the final temperature is greater than 0°C, no ice is left.

40 •• A calorimeter of negligible heat capacity contains 1.00 kg of water at 303 K and 50.0 g of ice at 273 K. (*a*) Find the final temperature *T*. (*b*) Find the final temperature *T* if the mass of ice is 500 g.

Picture the Problem First you need to convince yourself that there is enough energy in the water to melt all the ice. Once you've done that you can use conservation of energy to find the final equilibrium temperature. See Tables 18-1 and 18-2 for specific heats and the heat of fusion of water.

(*a*) Find the heat available to melt the ice:

$$Q_{\text{available, max}} = m_{\text{water}}c_{\text{water}}\Delta T_{\text{water}} = (1.00\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(273\,\text{K} - 303\,\text{K})$$
$$= -125.5\,\text{kJ}$$

Find the heat required to melt all of the ice:

$$Q_{\text{melt ice}} = m_{\text{ice}}L_\text{f} = (0.0500\,\text{kg})\left(333.5\frac{\text{kJ}}{\text{kg}}\right) = 16.68\,\text{kJ}$$

Because $\left|Q_{\text{available, max}}\right| > Q_{\text{melt ice}}$, we know that the final temperature will be greater than 273 K. Apply conservation of energy to obtain:

$$\sum_\text{i} Q_\text{i} = Q_{\substack{\text{cool the}\\\text{warm water}}} + Q_{\substack{\text{melt all}\\\text{the ice}}} + Q_{\substack{\text{warm the}\\\text{ice water}}} = 0$$

or

$$(1.00\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(T-303\,\text{K})+16.68\,\text{kJ}$$
$$+(0.0500\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(T-273\,\text{K})=0$$

Solving for T_f yields: $T_f = \boxed{298\,\text{K}}$

(*b*) Find the heat required to melt 500 g of ice:

$$Q_{\text{melt ice}} = m_{\text{ice}}L_f = (0.500\,\text{kg})(333.5\,\text{kJ/kg}) \approx 167\,\text{kJ}$$

Because the heat required to melt 500 g of ice is greater than the heat available, the final temperature will be 0°C.

41 •• A 200-g aluminum calorimeter contains 600 g of water at 20.0ºC. A 100-g piece of ice cooled to –20.0ºC is placed in the calorimeter. (*a*) Find the final temperature of the system, assuming no heat is transferred to or from the system. (*b*) A 200-g piece of ice at –20.0ºC is added. How much ice remains in the system after the system reaches equilibrium? (*c*) Would the answer for Part (*b*) change if both pieces of ice were added at the same time?

Picture the Problem First you need to convince yourself that there is enough energy in the water and the calorimeter to melt all the ice. Once you've done that you can use conservation of energy to find the final equilibrium temperature in (*a*). In Part (*b*) you can find the energy required to raise the temperature of the 200 g of ice to 0°C and then, noting that there are now 700 g of water in the calorimeter, find the energy available from cooling the calorimeter and water from their equilibrium temperature to 0°C. Finally, you can find the amount of ice that will melt in terms of the difference between the energy available and the energy required to warm the ice. See Tables 18-1 and 18-2 for specific heats and the latent heat of fusion of water.

Find the energy available to melt the ice:

$$Q_{\text{available, max}} = \left|m_{\text{water}}c_{\text{water}}\Delta T_{\text{water}} + m_{\text{cal}}c_{\text{cal}}\Delta T_{\text{water}}\right| = \left|(m_{\text{water}}c_{\text{water}} + m_{\text{cal}}c_{\text{cal}})\Delta T_{\text{water}}\right|$$
$$= \left|\left[(0.600\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)+(0.200\,\text{kg})\left(0.900\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\right](0°\text{C}-20.0°\text{C})\right|$$
$$= 53.81\,\text{kJ}$$

Find the energy required to warm and melt all of the ice:

$$\begin{aligned}
Q_{\substack{\text{warm and}\\ \text{melt the ice}}} &= m_{\text{ice}}c_{\text{ice}}\Delta T_{\text{ice}} + m_{\text{ice}}L_{\text{f}} \\
&= (0.100\,\text{kg})\left(2.05\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C}-(-20.0°\text{C})) + (0.100\,\text{kg})\left(333.5\frac{\text{kJ}}{\text{kg}}\right) \\
&= 37.45\,\text{kJ}
\end{aligned}$$

(*a*) Because $Q_{\text{available, max}} > Q_{\text{melt ice}}$, we know that the final temperature will be greater than 0°C. Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\substack{\text{cool the}\\ \text{calorimeter}}} + Q_{\substack{\text{cool the}\\ \text{warm water}}} + Q_{\substack{\text{warm and}\\ \text{melt the ice}}} + Q_{\substack{\text{warm the}\\ \text{ice water}}} = 0$$

or, using the results from our preliminary calculations,

$$\begin{aligned}
m_{\text{calorimeter}}c_{\text{Cu}}(t_{\text{f}} - 20.0°\text{C}) + m_{\text{water}}c_{\text{water}}(t_{\text{f}} - 20.0°\text{C}) + 37.45\,\text{kJ} \\
+ m_{\text{ice}}c_{\text{water}}(t_{\text{f}} - 0°\text{C}) = 0
\end{aligned}$$

Substituting numerical values yields:

$$\begin{aligned}
(0.200\,\text{kg})\left(0.900\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 20.0°\text{C}) + (0.600\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 20.0°\text{C}) \\
+ 37.45\,\text{kJ} + (0.100\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)t_{\text{f}} = 0
\end{aligned}$$

Solve for t_{f} to obtain: $t_{\text{f}} = 5.262°\text{C} = \boxed{5.26°\text{C}}$

(*b*) The mass of ice remaining in the system when equilibrium is reached is the difference between the initial mass of ice and the mass of the ice that has melted:

$$m_{\substack{\text{ice}\\ \text{remaining}}} = m_{\substack{\text{ice}\\ \text{initially}}} - m_{\substack{\text{ice}\\ \text{melted}}} = m_{\substack{\text{ice}\\ \text{initially}}} - \frac{Q_{\text{avail}} - Q_{\substack{\text{warm ice}\\ \text{to 0°C}}}}{L_{\text{f}}} \quad (1)$$

Find the energy required to raise the temperature of the 200 g of ice to 0°C:

$$Q_{\substack{\text{warm ice}\\ \text{to 0°C}}} = m_{\text{ice}}c_{\text{ice}}\Delta T_{\text{ice}} = (0.200\,\text{kg})\left(2.02\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C}-(-20.0°\text{C})) = 8.080\,\text{kJ}$$

Noting that there are now 700 g of water in the calorimeter, find the energy available from cooling the calorimeter and water from 5.262°C to 0°C:

$$\begin{aligned} Q_{\text{avail}} &= m_{\text{water}}c_{\text{water}}\Delta T_{\text{water}} + m_{\text{cal}}c_{\text{cal}}\Delta T_{\text{water}} = \left(m_{\text{water}}c_{\text{water}} + m_{\text{cal}}c_{\text{cal}}\right)\Delta T_{\text{water}} \\ &= \left[(0.700\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) + (0.200\,\text{kg})\left(0.900\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\right](5.262\text{°C} - 0\text{°C}) \\ &= 16.359\,\text{kJ} \end{aligned}$$

Substitute numerical values in equation (1) and evaluate $m_{\substack{\text{ice}\\\text{remaining}}}$:

$$m_{\substack{\text{ice}\\\text{remaining}}} = 200\text{ g} - \frac{16.359\,\text{kJ} - 8.080\,\text{kJ}}{333.5\,\text{kJ/kg}} = \boxed{200\text{ g}}$$

(*c*) No. Because the initial and final conditions are the same, the answer would be the same.

42 •• The specific heat of a 100-g block of a substance is to be determined. The block is placed in a 25-g copper calorimeter holding 60 g of water initially at 20ºC. Then, 120 mL of water at 80ºC are added to the calorimeter. When thermal equilibrium is reached, the temperature of the system is 54ºC. Determine the specific heat of the block.

Picture the Problem Let the subscript B denote the block, w_1 the water initially in the calorimeter, and w_2 the 120 mL of water that is added to the calorimeter vessel. We can use conservation of energy to find the specific heat of the block. See Table 18-1 for specific heats.

Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\substack{\text{warm}\\\text{the block}}} + Q_{\substack{\text{warm the}\\\text{calorimeter}}} + Q_{\text{warm } w_1} + Q_{\text{cool } w_2} = 0$$

or

$$m_{\text{B}}c_{\text{B}}\Delta T + m_{\text{Cu}}c_{\text{Cu}}\Delta T + m_{w_1}c_{w_1}\Delta T + m_{w_2}c_{w_2}\Delta T_{w_2} = 0$$

where ΔT is the common temperature change of the calorimeter, block, and water initially in the calorimeter.

Substitute numerical values to obtain:

$$(0.100\,\text{kg})\boldsymbol{c}_{\text{B}}(54°\text{C}-20°\text{C})+(0.025\,\text{kg})\left(0.386\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(54°\text{C}-20°\text{C})$$
$$+(0.060\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(54°\text{C}-20°\text{C})$$
$$+(0.120\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(54°\text{C}-80°\text{C})=0$$

Solving for c_{B} yields: $$c_{\text{B}}=\boxed{1.2\,\text{kJ/kg}\cdot\text{K}}$$

43 •• [SSM] A 100-g piece of copper is heated in a furnace to a temperature t_{C}. The copper is then inserted into a 150-g copper calorimeter containing 200 g of water. The initial temperature of the water and calorimeter is 16.0ºC, and the temperature after equilibrium is established is 38.0ºC. When the calorimeter and its contents are weighed, 1.20 g of water are found to have evaporated. What was the temperature t_{C}?

Picture the Problem We can find the temperature t by applying conservation of energy to this calorimetry problem. See Tables 18-1 and 18-2 for specific heats and the heat of vaporization of water.

Use conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\substack{\text{vaporize}\\ \text{water}}} + Q_{\substack{\text{warm}\\ \text{the water}}} + Q_{\substack{\text{warm the}\\ \text{calorimeter}}} + Q_{\substack{\text{cool the}\\ \text{Cu sample}}} = 0$$

or

$$m_{\text{H}_2\text{O, vaporized}}L_{\text{f, w}} + m_{\text{H}_2\text{O}}c_{\text{H}_2\text{O}}\Delta T_{\text{H}_2\text{O}} + m_{\text{cal}}c_{\text{cal}}\Delta T_{\text{w}} + m_{\text{Cu}}c_{\text{Cu}}\Delta T_{\text{Cu}} = 0$$

Substituting numerical values yields:

$$(1.20\,\text{g})\left(2257\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)+(200\,\text{g})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(38.0°\text{C}-16.0°\text{C})$$
$$+(150\,\text{g})\left(0.386\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(38.0°\text{C}-16.0°\text{C})$$
$$+(100\,\text{g})\left(0.386\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(38.0°\text{C}-t_{\text{C}})=0$$

Solving for t_{C} yields: $$t_{\text{C}}=\boxed{618°\text{C}}$$

44 •• A 200-g aluminum calorimeter contains 500 g of water at 20.0ºC. Aluminum shot with a mass equal to 300 g is heated to 100.0ºC and is then placed in the calorimeter. Find the final temperature of the system, assuming that there is no heat is transfer to the surroundings.

Picture the Problem We can find the final temperature by applying conservation of energy to the calorimeter, the water in the calorimeter, and to the cooling aluminum shot. See Table 18-1 for the specific heats of aluminum and water.

Use conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = Q_{\substack{\text{warm the}\\ \text{water}}} + Q_{\substack{\text{warm the}\\ \text{calorimeter}}} + Q_{\substack{\text{cool the}\\ \text{Al shot}}} = 0$$

or

$$\left[m_{\text{w}}c_{\text{w}} + m_{\text{cal}}c_{\text{Al}}\right]\Delta T + m_{\text{shot}}c_{\text{Al}}\Delta T_{\text{Al}} = 0$$

where ΔT is the common temperature change of the water and calorimeter.

Substituting numerical values yields:

$$\left[(0.500\,\text{kg})\left(4.184\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) + (0.200\,\text{kg})\left(0.900\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\right](t_{\text{f}} - 20.0°\text{C})$$
$$+ (0.300\,\text{kg})\left(0.900\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(t_{\text{f}} - 100°\text{C}) = 0$$

Solving for t_{f} gives:

$$t_{\text{f}} = \boxed{28.5°\text{C}}$$

First Law of Thermodynamics

45 • A diatomic gas does 300 J of work and also absorbs 2.50 kJ of heat. What is the change in internal energy of the gas?

Picture the Problem We can apply the first law of thermodynamics to find the change in internal energy of the gas during this process.

Apply the first law of thermodynamics to express the change in internal energy of the gas in terms of the heat added to the system and the work done on the gas:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$$

The work done by the gas equals the negative of the work done on the gas. Substitute numerical values and evaluate ΔE_{int}:

$$\Delta \boldsymbol{E}_{int} = 2.50\ \text{kJ} - 300\ \text{J} = \boxed{2.20\ \text{kJ}}$$

46 • If a gas absorbs 1.67 MJ of heat while doing 800 kJ of work, what is the change in the internal energy of the gas?

Picture the Problem We can apply the first law of thermodynamics to find the change in internal energy of the gas during this process.

Apply the first law of thermodynamics to express the change in internal energy of the gas in terms of the heat added to the system and the work done on the gas:

$$\Delta E_{int} = Q_{in} + W_{on}$$

The work done by the gas is the negative of the work done on the gas. Substitute numerical values and evaluate ΔE_{int}:

$$\Delta E_{int} = 1.67\ \text{MJ} - 800\ \text{kJ} = \boxed{0.87\ \text{MJ}}$$

47 • If a gas absorbs 84 J while doing 30 J of work, what is the change in the internal energy of the gas?

Picture the Problem We can apply the first law of thermodynamics to find the change in internal energy of the gas during this process.

Apply the first law of thermodynamics to express the change in internal energy of the gas in terms of the heat added to the system and the work done on the gas:

$$\Delta E_{int} = Q_{in} + W_{on}$$

The work done by the gas is the negative of the work done on the gas. Substitute numerical values and evaluate ΔE_{int}:

$$\Delta E_{int} = 84\ \text{J} - 30\ \text{J} = \boxed{54\ \text{J}}$$

48 •• A lead bullet initially at 30ºC just melts upon striking a target. Assuming that all of the initial kinetic energy of the bullet goes into the internal energy of the bullet, calculate the impact speed of the bullet.

Picture the Problem We can use the definition of kinetic energy to express the speed of the bullet upon impact in terms of its kinetic energy. The heat absorbed by the bullet is the sum of the heat required to warm the bullet from 303 K to its melting temperature of 600 K and the heat required to melt it. We can use the first law of thermodynamics to relate the impact speed of the bullet to the change in its internal energy. See Table 18-1 for the specific heat and melting temperature of lead.

Using the first law of thermodynamics, relate the change in the internal energy of the bullet to the work done on it by the target:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$$

or, because $Q_{\text{in}} = 0$,

$$\Delta E_{\text{int}} = W_{\text{on}} = \Delta K = -(K_{\text{f}} - K_i)$$

Substitute for ΔE_{int}, K_{f}, and K_{i} to obtain:

$$mc_{\text{Pb}}\Delta T_{\text{Pb}} + mL_{\text{f,Pb}} = -\left(0 - \tfrac{1}{2}mv^2\right) = \tfrac{1}{2}mv^2$$

or

$$mc_{\text{Pb}}(T_{\text{MP}} - T_{\text{i}}) + mL_{\text{f,Pb}} = \tfrac{1}{2}mv^2$$

Solving for v yields:

$$v = \sqrt{2\left[c_{\text{Pb}}(T_{\text{MP}} - T_{\text{i}}) + L_{\text{f,Pb}}\right]}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{2\left\{\left(0.128\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(600\,\text{K} - 303\,\text{K}) + 24.7\frac{\text{kJ}}{\text{kg}}\right\}} = \boxed{354\,\text{m/s}}$$

49 •• During a cold day, you can warm your hands by rubbing them together. Assume the coefficient of kinetic friction between your hands is 0.500, the normal force between your hands is 35.0 N, and that you rub them together at an average relative speed of 35.0 cm/s. (*a*) What is the rate at which mechanical energy is dissipated? (*b*) Assume further that the mass of each of your hands is 350 g, the specific heat of your hands is 4.00 kJ/kg·K, and that all the dissipated mechanical energy goes into increasing the temperature of your hands. How long must you rub your hands together to produce a 5.00°C increase in their temperature?

Picture the Problem We can find the rate at which heat is generated when you rub your hands together using the definition of power and the rubbing time to produce a 5.00°C increase in temperature from $\Delta Q = (dQ/dt)\Delta t$ and $Q = mc\Delta T$.

(*a*) The rate at which heat is generated as a function of the friction force and the average relative speed of your hands is given by:

$$\frac{dQ}{dt} = P = f_k v = \mu F_n v$$

Substitute numerical values and evaluate dQ/dt:

$$\frac{dQ}{dt} = (0.500)(35.0\,\text{N})(0.350\,\text{m/s})$$
$$= 6.125\ \text{W} = \boxed{6.13\,\text{W}}$$

(*b*) Relate the heat required to raise the temperature of your hands $5.00°\text{C}$ to the rate at which it is being generated:

$$\Delta Q = \frac{dQ}{dt}\Delta t = mc\Delta T \Rightarrow \Delta t = \frac{mc\Delta T}{dQ/dt}$$

Substitute numerical values and evaluate Δt:

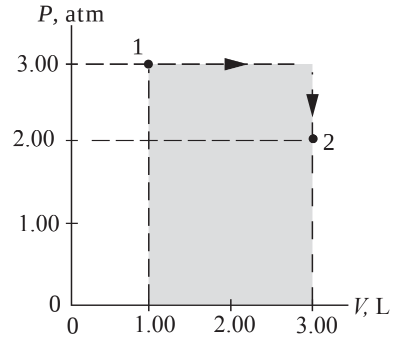
$$\Delta t = \frac{2(0.350\,\text{kg})\left(4.00\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(5.00\,\text{C}°)}{6.125\,\text{W}}$$
$$= 2286\,\text{s} \times \frac{1\,\text{min}}{60\,\text{s}} = \boxed{38.1\,\text{min}}$$

Work and the *PV* Diagram for a Gas

50 • The gas is allowed to expand at constant pressure until it reaches its final volume. It is then cooled at constant volume until it reaches its final pressure. (*a*) Illustrate this process on a *PV* diagram and calculate the work done by the gas. (*b*) Find the heat absorbed by the gas during this process.

Picture the Problem We can find the work done by the gas during this process from the area under the curve. Because no work is done along the constant volume (vertical) part of the path, the work done by the gas is done during its isobaric expansion. We can then use the first law of thermodynamics to find the heat added to the system during this process.

(*a*) The path from the initial state (1) to the final state (2) is shown on the *PV* diagram.



The work done by the gas equals the area under the shaded curve:

$$W_{\text{by gas}} = P\Delta V = (3.00\,\text{atm})(2.00\,\text{L}) = \left(3.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}\right)\left(2.00\,\text{L}\times\frac{10^{-3}\,\text{m}^3}{\text{L}}\right)$$
$$= \boxed{608\,\text{J}}$$

(*b*) The work done by the gas is the negative of the work done on the gas. Apply the first law of thermodynamics to the system to obtain:

$$\begin{aligned} Q_{\text{in}} &= \Delta E_{\text{int}} - W_{\text{on}} \\ &= \left(E_{\text{int},2} - E_{\text{int},1}\right) - \left(-W_{\text{by gas}}\right) \\ &= \left(E_{\text{int},2} - E_{\text{int},1}\right) + W_{\text{by gas}} \end{aligned}$$

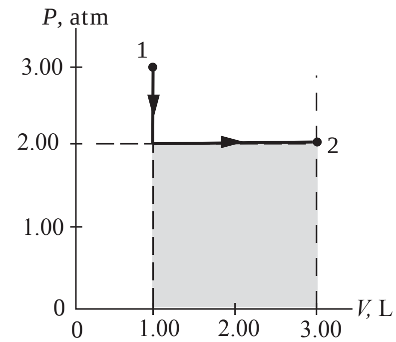
Substitute numerical values and evaluate Q_{in}:

$$\begin{aligned} Q_{\text{in}} &= (912\,\text{J} - 456\,\text{J}) + 608\,\text{J} \\ &= \boxed{1.06\,\text{kJ}} \end{aligned}$$

51 • [SSM] The gas is first cooled at constant volume until it reaches its final pressure. It is then allowed to expand at constant pressure until it reaches its final volume. (*a*) Illustrate this process on a *PV* diagram and calculate the work done by the gas. (*b*) Find the heat absorbed by the gas during this process.

Picture the Problem We can find the work done by the gas during this process from the area under the curve. Because no work is done along the constant volume (vertical) part of the path, the work done by the gas is done during its isobaric expansion. We can then use the first law of thermodynamics to find the heat absorbed by the gas during this process

(*a*) The path from the initial state (1) to the final state (2) is shown on the *PV* diagram.



The work done by the gas equals the area under the curve:

$$W_{\text{by gas}} = P\Delta V = (2.00\,\text{atm})(2.00\,\text{L}) = \left(2.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}\right)\left(2.00\,\text{L}\times\frac{10^{-3}\,\text{m}^3}{\text{L}}\right)$$
$$= \boxed{405\,\text{J}}$$

(*b*) The work done by the gas is the negative of the work done on the gas. Apply the first law of thermodynamics to the system to obtain:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}}$$
$$= \left(E_{\text{int,2}} - E_{\text{int,1}}\right) - \left(-W_{\text{by gas}}\right)$$
$$= \left(E_{\text{int,2}} - E_{\text{int,1}}\right) + W_{\text{by gas}}$$

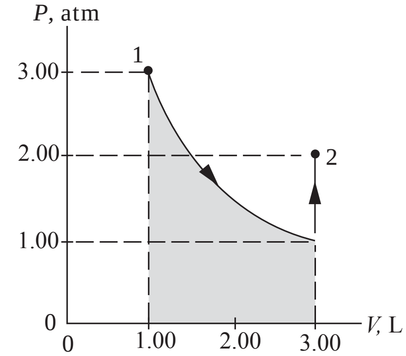
Substitute numerical values and evaluate Q_{in}:

$$Q_{\text{in}} = (912\,\text{J} - 456\,\text{J}) + 405\,\text{J} = \boxed{861\,\text{J}}$$

52 •• The gas is allowed to expand isothermally until it reaches its final volume and its pressure is 1.00 atm. It is then heated at constant volume until it reaches its final pressure. (*a*) Illustrate this process on a *PV* diagram and calculate the work done by the gas. (*b*) Find the heat absorbed by the gas during this process.

Picture the Problem We can find the work done by the gas during this process from the area under the curve. Because no work is done along the constant volume (vertical) part of the path, the work done by the gas is done during its isothermal expansion. We can then use the first law of thermodynamics to find the heat absorbed by the gas during this process.

(*a*) The path from the initial state (1) to the final state (2) is shown on the *PV* diagram.



The work done by the gas equals the area under the curve:

$$W_{\text{by gas}} = \int_{V_1}^{V_2} PdV = nRT_1 \int_{1\,\text{L}}^{3\,\text{L}} \frac{dV}{V}$$

$$= P_1V_1 \int_{1.00\,\text{L}}^{3.00\,\text{L}} \frac{dV}{V} = P_1V_1\left[\ln V\right]_{1.00\,\text{L}}^{3.00\,\text{L}}$$

$$= P_1V_1 \ln 3$$

Substitute numerical values and evaluate $W_{\text{by gas}}$:

$$W_{\text{by gas}} = \left(3.00\,\text{atm} \times \frac{101.325\,\text{kPa}}{\text{atm}}\right)\left(1.00\,\text{L} \times \frac{10^{-3}\,\text{m}^3}{\text{L}}\right)\ln 3 = \boxed{334\,\text{J}}$$

(*b*) The work done by the gas is the negative of the work done on the gas. Apply the first law of thermodynamics to the system to obtain:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}}$$
$$= \left(E_{\text{int},2} - E_{\text{int},1}\right) - \left(-W_{\text{by gas}}\right)$$
$$= \left(E_{\text{int},2} - E_{\text{int},1}\right) + W_{\text{by gas}}$$

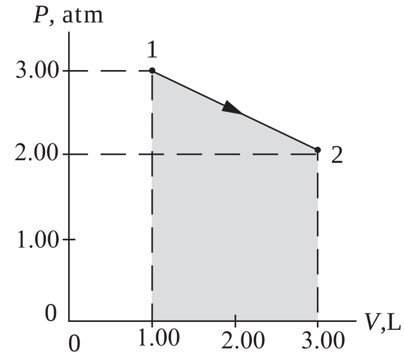
Substitute numerical values and evaluate Q_{in}:

$$Q_{\text{in}} = \left(912\,\text{J} - 456\,\text{J}\right) + 334\,\text{J} = \boxed{790\,\text{J}}$$

53 •• The gas is heated and is allowed to expand such that it follows a single straight-line path on a *PV* diagram from its initial state to its final state. (*a*) Illustrate this process on a *PV* diagram and calculate the work done by the gas. (*b*) Find the heat absorbed by the gas during this process.

Picture the Problem We can find the work done by the gas during this process from the area under the curve. We can then use the first law of thermodynamics to find the heat absorbed by the gas during this process.

(*a*) The path from the initial state (1) to the final state (2) is shown on the *PV* diagram:



The work done by the gas equals the area under the curve:

$$\begin{aligned} W_{\text{by gas}} &= A_{\text{trapezoid}} \\ &= \tfrac{1}{2}(3.00\,\text{atm} + 2.00\,\text{atm})(2.00\,\text{L}) \\ &= 5.00\,\text{atm}\cdot\text{L} \times \frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}} \\ &= \boxed{507\,\text{J}} \end{aligned}$$

(*b*) The work done by the gas is the negative of the work done on the gas. Apply the first law of thermodynamics to the system to obtain:

$$\begin{aligned} Q_{\text{in}} &= \Delta E_{\text{int}} - W_{\text{on}} \\ &= (E_{\text{int},2} - E_{\text{int},1}) - (-W_{\text{by gas}}) \\ &= (E_{\text{int},2} - E_{\text{int},1}) + W_{\text{by gas}} \end{aligned}$$

Substitute numerical values and evaluate Q_{in}:

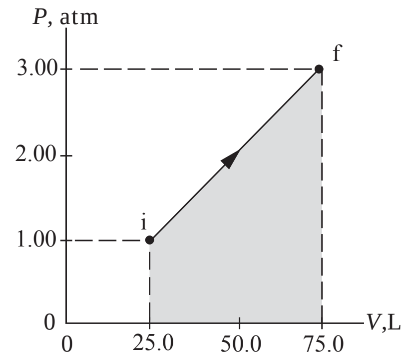
$$Q_{\text{in}} = (912\,\text{J} - 456\,\text{J}) + 507\,\text{J} = \boxed{963\,\text{J}}$$

Remarks: You could use the linearity of the path connecting the initial and final states and the coordinates of the endpoints to express *P* as a function of *V*. You could then integrate this function between 1.00 and 3.00 L to find the work done by the gas as it goes from its initial to its final state.

54 •• In this problem, 1.00 mol of a dilute gas initially has a pressure equal to 1.00 atm, a volume equal to 25.0 L and an internal energy equal to 456 J. As the gas is slowly heated, the plot of its state on a *PV* diagram moves in a straight line to the final state. The gas now has a pressure equal to 3.00 atm, a volume equal to 75.0 L and an internal energy equal to 912 J. Find the work done and the heat absorbed by the gas.

Picture the Problem We can find the work done by the gas during this process from the area under the curve and the heat absorbed by the gas from the 1$^{\text{st}}$ law of thermodynamics.

The path from the initial state i to the final state f is shown on the PV diagram:



The work done by the gas equals the area under the curve:

$$\begin{aligned}W_{\text{by gas}} &= A_{\text{trapezoid}} \\ &= \tfrac{1}{2}(1.00\,\text{atm} + 3.00\,\text{atm}) \\ &\qquad \times (75.0\,\text{L} - 25.0\,\text{L}) \\ &= 100\,\text{atm}\cdot\text{L} \times \frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}} \\ &= \boxed{10.1\,\text{kJ}}\end{aligned}$$

The work done by the gas is the negative of the work done on the gas. Apply the first law of thermodynamics to the system to obtain:

$$\begin{aligned}Q_{\text{in}} &= \Delta E_{\text{int}} - W_{\text{on}} \\ &= (E_{\text{int},2} - E_{\text{int},1}) - (-W_{\text{by gas}}) \\ &= (E_{\text{int},2} - E_{\text{int},1}) + W_{\text{by gas}}\end{aligned}$$

Substitute numerical values and evaluate Q_{in}:

$$Q_{\text{in}} = (912\,\text{J} - 456\,\text{J}) + 10.1\,\text{kJ} = \boxed{10.6\,\text{kJ}}$$

Remarks: You could use the linearity of the path connecting the initial and final states and the coordinates of the endpoints to express *P* as a function of *V*. You could then integrate this function between 25.0 and 75.0 L to find the work done by the gas as it goes from its initial to its final state.

55 •• In this problem, 1.00 mol of the ideal gas is heated while its volume changes, so that $T = AP^2$, where A is a constant. The temperature changes from T_0 to $4T_0$. Find the work done by the gas.

Picture the Problem We can find the work done by the gas from the area under the PV curve provided we can find the pressure and volume coordinates of the initial and final states. We can find these coordinates by using the ideal gas law and the condition $T = AP^2$.

Apply the ideal-gas law with $n =$ 1.00 mol and $T = AP^2$ to obtain:

$$PV = RAP^2 \Rightarrow V = RAP \qquad (1)$$

This result tells us that the volume varies linearly with the pressure.

Solve $T = AP^2$ for the pressure of the gas:

$$P_0 = \sqrt{\frac{T_0}{A}}$$

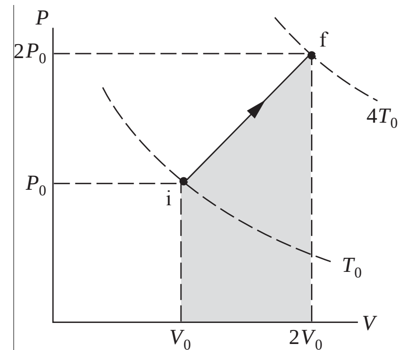
Find the pressure when the temperature is $4T_0$:

$$P = \sqrt{\frac{4T_0}{A}} = 2\sqrt{\frac{T_0}{A}} = 2P_0$$

Using equation (1), express the coordinates of the final state:

$$(2V_0, 2P_0)$$

The PV diagram for the process is shown to the right:



The work done by the gas equals the area under the curve:

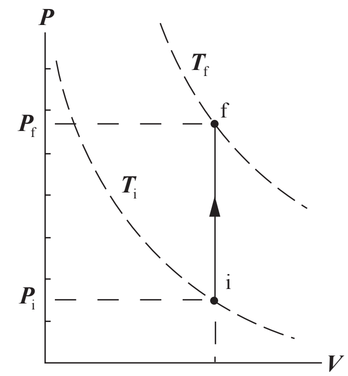
$$W_{\text{by gas}} = A_{\text{trapezoid}} = \tfrac{1}{2}(P_0 + 2P_0)(2V_0 - V_0)$$
$$= \boxed{\tfrac{3}{2}P_0V_0}$$

56 •• A sealed, almost-empty spray paint can still contains a residual amount of the propellant: 0.020 mol of nitrogen gas. The can's warning label clearly states: "Do Not Dispose by Incineration." (*a*) Explain this warning and draw the *PV* diagram for the gas if, in fact, the can is subject to a high temperature. (*b*) You are in charge of testing the can. The manufacturer claims it can withstand an internal gas pressure of 6.00 atm before it breaks. The can is initially at room-temperature and standard pressure in your testing laboratory. You begin to heat it uniformly using a heating element that has a power output of 200 W. The can and element are in an insulating oven, and you can assume 1.0% of the heat released by the heating element is absorbed by the gas in the can. How long should you expect the heating element to remain on before the can bursts?

Picture the Problem This is a constant-volume process and so no work is done on the system. The heat that is added to the gas by the heating element increases the internal energy (and, hence, the temperature) of the gas. We can use the definition of power to relate the minimum time-to-bursting to the internal energy required to raise the temperature of the gas in the can and the rate at which energy is supplied by the heating element. The energy required to burst the can is related

to the increase in the temperature of the can through the definition of the heat capacity at constant volume C_v.

(*a*) Because the volume of the can is fixed (constant), the heat absorbed by the gas in the can all goes into increasing the internal energy of the gas and, thus, its temperature. At constant volume, the pressure rise is the maximum possible, threatening the structural integrity of the can and possibly leading to a grenade-like shrapnel explosion.



(*b*) The heating time required to burst the container is related to the rate at which the heating element delivers energy to it:

$$\Delta t = \frac{\Delta Q}{P}$$

The heat that the heating element must supply to the gas is given by:

$$\Delta Q = C_v \Delta T = \tfrac{5}{2} nR\Delta T$$

Substituting for ΔQ yields:

$$\Delta t = \frac{\frac{5}{2} nR\Delta T}{P} = \frac{5nR\Delta T}{2P} \qquad (1)$$

The temperature difference that corresponds to the bursting threshold is given by:

$$\Delta T = T_f - T_i \qquad (2)$$

Apply the ideal-gas law to the gas in the can to obtain:

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f}$$

or, because $V_i = V_f$,

$$\frac{P_i}{T_i} = \frac{P_f}{T_f}$$

Solving for the ratio of the final temperature to the initial temperature yields:

$$\frac{T_f}{T_i} = \frac{P_f}{P_i} = 6 \Rightarrow T_f = 6T_i$$

Substitute for T_f in equation (2) and evaluate ΔT:

$$\Delta T = 6T_i - T_i = 5T_i$$

Substituting for ΔT in equation (1) yields:

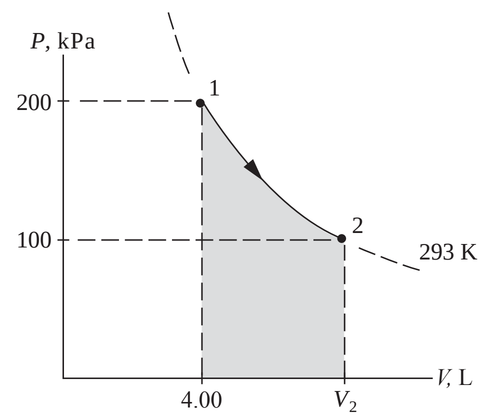
$$\Delta t = \frac{25nRT_i}{2P}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{25(0.020\ \text{mol})\left(8.314\ \frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(293\ \text{K})}{2(0.010)(200\ \text{W})} = 305\ \text{s}\cdot\frac{1\ \text{min}}{60\ \text{s}} = \boxed{5.1\ \text{min}}$$

57 •• [SSM] An ideal gas initially at 20ºC and 200 kPa has a volume of 4.00 L. It undergoes a quasi-static, isothermal expansion until its pressure is reduced to 100 kPa. Find (*a*) the work done by the gas, and (*b*) the heat absorbed by the gas during the expansion.

Picture the Problem The *PV* diagram shows the isothermal expansion of the ideal gas from its initial state 1 to its final state 2. We can use the ideal-gas law for a fixed amount of gas to find V_2 and then evaluate $\int PdV$ for an isothermal process to find the work done by the gas. In Part (*b*) of the problem we can apply the first law of thermodynamics to find the heat added to the gas during the expansion.



(*a*) Express the work done by a gas during an isothermal process:

$$W_{\text{by gas}} = \int_{V_1}^{V_2} PdV = nRT\int_{V_1}^{V_2}\frac{dV}{V} = P_1V_1\int_{V_1}^{V_2}\frac{dV}{V}$$

Apply the ideal-gas law for a fixed amount of gas undergoing an isothermal process:

$$P_1V_1 = P_2V_2 \Rightarrow V_2 = \frac{P_1}{P_2}V_1$$

Substitute numerical values and evaluate V_2:

$$V_2 = \frac{200\ \text{kPa}}{100\ \text{kPa}}(4.00\ \text{L}) = 8.00\ \text{L}$$

Substitute numerical values and evaluate W:

$$W_{\text{by gas}} = (200\ \text{kPa})(4.00\ \text{L})\int_{4.00\ \text{L}}^{8.00\ \text{L}} \frac{dV}{V} = (800\ \text{kPa}\cdot\text{L})[\ln V]_{4.00\ \text{L}}^{8.00\ \text{L}}$$

$$= (800\ \text{kPa}\cdot\text{L})\ln\left(\frac{8.00\ \text{L}}{4.00\ \text{L}}\right) = 554.5\ \text{kPa}\cdot\text{L}\times\frac{10^{-3}\ \text{m}^3}{\text{L}}$$

$$= \boxed{555\ \text{J}}$$

(*b*) Apply the first law of thermodynamics to the system to obtain:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}}$$

or, because $\Delta E_{\text{int}} = 0$ for an isothermal process,

$$Q_{\text{in}} = -W_{\text{on}}$$

Because the work done by the gas is the negative of the work done on the gas:

$$Q_{\text{in}} = -(-W_{\text{by gas}}) = W_{\text{by gas}} = \boxed{555\ \text{J}}$$

Remarks: in an isothermal expansion the heat added to the gas is always equal to the work done by the gas ($\Delta E_{\text{int}} = 0$).

Heat Capacities of Gases and the Equipartition Theorem

58 • The heat capacity at constant volume of a certain amount of a monatomic gas is 49.8 J/K. (*a*) Find the number of moles of the gas. (*b*) What is the internal energy of the gas at $T = 300$ K? (*c*) What is the heat capacity at constant pressure of the gas?

Picture the Problem We can find the number of moles of the gas from its heat capacity at constant volume using $C_{\text{V}} = \frac{3}{2}nR$. We can find the internal energy of the gas from $E_{\text{int}} = C_{\text{V}}T$ and the heat capacity at constant pressure using $C_{\text{P}} = C_{\text{V}} + nR$.

(*a*) Express C_{V} in terms of the number of moles in the monatomic gas:

$$C_{\text{V}} = \tfrac{3}{2}nR \Rightarrow n = \frac{2C_{\text{V}}}{3R}$$

Substitute numerical values and evaluate n:

$$n = \frac{2\left(49.8\,\frac{\text{J}}{\text{K}}\right)}{3\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = 3.993\ \text{mol} = \boxed{3.99\ \text{mol}}$$

(b) The internal energy of the gas is related to its temperature:

$$E_{\text{int}} = C_{\text{V}}T$$

Substitute numerical values and evaluate E_{int}:

$$E_{\text{int}} = \left(49.8\,\frac{\text{J}}{\text{K}}\right)(300\ \text{K}) = \boxed{14.9\ \text{kJ}}$$

(c) Relate the heat capacity at constant pressure to the heat capacity at constant volume:

$$C_{\text{P}} = C_{\text{V}} + nR = \tfrac{3}{2}nR + nR = \tfrac{5}{2}nR$$

Substitute numerical values and evaluate C_{P}:

$$C_{\text{P}} = \tfrac{5}{2}(3.993\ \text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right) = \boxed{83.0\ \text{J/K}}$$

59 •• [SSM] The heat capacity at constant pressure of a certain amount of a diatomic gas is 14.4 J/K. (a) Find the number of moles of the gas. (b) What is the internal energy of the gas at $T = 300$ K? (c) What is the molar heat capacity of this gas at constant volume? (d) What is the heat capacity of this gas at constant volume?

Picture the Problem (a) The number of moles of the gas is related to its heat capacity at constant pressure and its molar heat capacity at constant pressure according to $C_{\text{P}} = nc'_{\text{P}}$. For a diatomic gas, the molar heat capacity at constant pressure is $c'_{\text{P}} = \tfrac{7}{2}R$. ($b$) The internal energy of a gas depends on its number of degrees of freedom and, for a diatomic gas, is given by $E_{\text{int}} = \tfrac{5}{2}nRT$. ($c$) The molar heat capacity of this gas at constant volume is related to its molar heat capacity at constant pressure according to $c'_{\text{V}} = c'_{\text{P}} - R$. ($d$) The heat capacity of this gas at constant volume is the product of the number of moles in the gas and its molar heat capacity at constant volume.

(a) The number of moles of the gas is the ratio of its heat capacity at constant pressure to its molar heat capacity at constant pressure:

$$n = \frac{C_{\text{P}}}{c'_{\text{P}}}$$

For a diatomic gas, the molar heat capacity is given by:

$$c'_P = \tfrac{7}{2}R = 29.1\,\frac{\text{J}}{\text{mol}\cdot\text{K}}$$

Substitute numerical values and evaluate n:

$$n = \frac{14.4\,\frac{\text{J}}{\text{K}}}{29.1\,\frac{\text{J}}{\text{mol}\cdot\text{K}}} = 0.4948\ \text{mol} = \boxed{0.495\ \text{mol}}$$

(*b*) With 5 degrees of freedom at this temperature:

$$E_{\text{int}} = \tfrac{5}{2}nRT$$

Substitute numerical values and evaluate E_{int}:

$$E_{\text{int}} = \tfrac{5}{2}(0.4948\ \text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\ \text{K}) = \boxed{3.09\ \text{kJ}}$$

(*c*) The molar heat capacity of this gas at constant volume is the difference between the molar heat capacity at constant pressure and the gas constant R:

$$c'_V = c'_P - R$$

Because $c'_P = \tfrac{7}{2}R$ for a diatomic gas:

$$c'_V = \tfrac{7}{2}R - R = \tfrac{5}{2}R$$

Substitute the numerical value of R to obtain:

$$c'_V = \tfrac{5}{2}\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right) = 20.79\,\frac{\text{J}}{\text{mol}\cdot\text{K}} = \boxed{20.8\,\frac{\text{J}}{\text{mol}\cdot\text{K}}}$$

(*d*) The heat capacity of this gas at constant volume is given by:

$$C'_V = nc'_V$$

Substitute numerical values and evaluate $\boldsymbol{C'_V}$:

$$C'_V = (0.4948\ \text{mol})\left(20.79\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right) = \boxed{10.3\,\frac{\text{J}}{\text{K}}}$$

60 •• (*a*) Calculate the heat capacity per unit mass of air at constant volume and the heat capacity per unit mass of air at constant pressure. Assume that air has a temperature of 300 K and a pressure of $1.00 \times 10^5\ \text{N/m}^2$. Also assume that air is

composed of 74.0% N_2 molecules (molecular weight 28.0 g/mol) and 26.0% O_2 molecules (molar mass of 32.0 g/mol) and that both components are ideal gases. (*b*) Compare your answer for the specific heat at constant pressure to the value listed in the *Handbook of Chemistry and Physics* of 1.032 kJ/kg·K.

Picture the Problem The specific heats of air at constant volume and constant pressure are given by $c_V = C_V/m$ and $c_P = C_P/m$ and the heat capacities at constant volume and constant pressure are given by $C_V = \frac{5}{2}nR$ and $C_P = \frac{7}{2}nR$, respectively.

(*a*) Express the heat capacity per unit mass of air at constant volume and constant pressure:

$$c_V = \frac{C_V}{m} \quad (1)$$

and

$$c_P = \frac{C_P}{m} \quad (2)$$

Express the heat capacities of a diatomic gas in terms of the gas constant R, the number of moles n, and the number of degrees of freedom:

$$C_V = \tfrac{5}{2}nR$$

and

$$C_P = \tfrac{7}{2}nR$$

The mass of 1.00 mol of air is given by:

$$m = 0.74M_{N_2} + 0.26M_{O_2}$$

Substitute for C_V and m in equation (1) to obtain:

$$c_V = \frac{5nR}{2\left(0.74M_{N_2} + 0.26M_{O_2}\right)}$$

Substitute numerical values and evaluate c_V:

$$c_V = \frac{5(1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)}{2\left[(0.740)\left(28.0\times10^{-3}\,\text{kg}\right)+(0.260)\left(32.0\times10^{-3}\,\text{kg}\right)\right]} = \boxed{716\,\frac{\text{J}}{\text{kg}\cdot\text{K}}}$$

Substitute for C_P and m in equation (2) to obtain:

$$c_P = \frac{7nR}{2\left(0.74M_{N_2} + 0.26M_{O_2}\right)}$$

Substitute numerical values and evaluate c_P:

$$c_P = \frac{7(1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)}{2\left[(0.740)(28.0\times10^{-3}\,\text{kg})+(0.260)(32.0\times10^{-3}\,\text{kg})\right]} = \boxed{1.00\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}}$$

(*b*) The percent difference between the value from the *Handbook of Chemistry and Physics* and the calculated value is:

$$\frac{1.032\,\text{kJ/kg}\cdot\text{K} - 1.00\,\text{kJ/kg}\cdot\text{K}}{1.032\,\text{J/kg}\cdot\text{K}} \approx \boxed{3\%}$$

61 •• In this problem, 1.00 mol of an ideal diatomic gas is heated at constant volume from 300 to 600 K. (*a*) Find the increase in the internal energy of the gas , the work done by the gas, and the heat absorbed by the gas. (*b*) Find the same quantities if this gas is heated from 300 to 600 K at constant pressure. Use the first law of thermodynamics and your results for (*a*) to calculate the work done by the gas. (*c*) Again calculate the work done in Part (*b*). This time calculate it by integrating the equation $dW = P\,dV$.

Picture the Problem (*a*) We know that, during a constant-volume process, no work is done and that we can calculate the heat added during this expansion from the heat capacity at constant volume and the change in the absolute temperature. We can then use the first law of thermodynamics to find the change in the internal energy of the gas. In Part (*b*), we can proceed similarly; using the heat capacity at constant pressure rather than constant volume.

(*a*) The increase in the internal energy of a fixed amount of an ideal diatomic gas depends only on its change in temperature as it goes from one state to another:

$$\Delta E_{\text{int}} = \tfrac{5}{2} nR\Delta T$$

Substitute numerical values and evaluate ΔE_{int}:

$$\Delta E_{\text{int}} = \tfrac{5}{2}(1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K}) = 6.236\,\text{kJ} = \boxed{6.24\,\text{kJ}}$$

For a constant-volume process:

$$W_{\text{on}} = \boxed{0}$$

From the 1st law of thermodynamics we have:

$$Q_{in} = \Delta E_{int} - W_{on} = 6.24\,\text{kJ} - 0 = \boxed{6.24\,\text{kJ}}$$

(*b*) Because ΔE_{int} depends only on the temperature difference:

$$\Delta E_{int} = \boxed{6.24\,\text{kJ}}$$

Relate the heat added to the gas to its heat capacity at constant pressure and the change in its temperature:

$$Q_{in} = C_P \Delta T = \left(\tfrac{5}{2}nR + nR\right)\Delta T = \tfrac{7}{2}nR\Delta T$$

Substitute numerical values and evaluate Q_{in}:

$$Q_{in} = \tfrac{7}{2}(1.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K}) = 8.730\,\text{kJ} = \boxed{8.73\,\text{kJ}}$$

Apply the first law of thermodynamics to find W:

$$W_{on} = \Delta E_{int} - Q_{in} = 8.730\,\text{kJ} - 6.236\,\text{kJ} = \boxed{2.49\,\text{kJ}}$$

(*c*) Integrate $dW_{on} = P\,dV$ to obtain:

$$W_{on} = \int_{V_i}^{V_f} PdV = P(V_f - V_i) = nR(T_f - T_i)$$

Substitute numerical values and evaluate W_{on}:

$$W_{on} = (1.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K}) = \boxed{2.49\,\text{kJ}}$$

62 •• A diatomic gas is confined to a closed container of constant volume V_0 and at a pressure P_0. The gas is heated until its pressure triples. What amount of heat had to be absorbed by the gas in order to triple the pressure?

Picture the Problem Because this is a constant-volume process, we can use $Q = C_V\Delta T$ to express Q in terms of the temperature change and the ideal-gas law for a fixed amount of gas to find ΔT.

Express the amount of heat Q that must be absorbed by the gas if its pressure is to triple:

$$Q = C_V\Delta T = \tfrac{5}{2}nR(T_f - T_0)$$

Using the ideal-gas law for a fixed amount of gas, relate the initial and final temperatures, pressures and volumes:

$$\frac{P_0V_0}{T_0}=\frac{3P_0V_0}{T_f}\Rightarrow T_f=3T_0$$

Substitute for T_f and simplify to obtain:

$$Q=\tfrac{5}{2}nR(3T_0-T_0)=5(nRT_0)=\boxed{5P_0V_0}$$

63 •• In this problem, 1.00 mol of air is confined in a cylinder with a piston. The confined air is maintained at a constant pressure of 1.00 atm. The air is initially at 0ºC and has volume V_0. Find the volume after 13,200 J of heat are absorbed by the trapped air.

Picture the Problem Let the subscripts i and f refer to the initial and final states of the gas, respectively. We can use the ideal-gas law for a fixed amount of gas to express V' in terms of V and the change in temperature of the gas when 13,200 J of heat are transferred to it. We can find this change in temperature using $Q=C_p\Delta T$.

Using the ideal-gas law for a fixed amount of gas, relate the initial and final temperatures, volumes, and pressures:

$$\frac{P_iV}{T_i}=\frac{P_fV'}{T_f}$$

Noting that the process is isobaric, solve for V' to obtain:

$$V'=V\frac{T_f}{T_i}=V\frac{T_i+\Delta T}{T_i}=V\left(1+\frac{\Delta T}{T_i}\right)$$

Relate the heat transferred to the gas to the change in its temperature:

$$Q=C_p\Delta T=\tfrac{7}{2}nR\Delta T\Rightarrow\Delta T=\frac{2Q}{7nR}$$

Substitute for ΔT to obtain:

$$V'=V\left(1+\frac{2Q}{7nRT_i}\right)$$

One mol of gas at standard temperature and pressure occupies 22.4 L. Substitute numerical values and evaluate V':

$$V'=(22.4\,\text{L})\left(1+\frac{2(13.2\,\text{kJ})}{7(1.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})}\right)=\boxed{59.6\,\text{L}}$$

64 •• The heat capacity at constant pressure of a sample of gas is greater than the heat capacity at constant volume by 29.1 J/K. (*a*) How many moles of the gas are present? (*b*) If the gas is monatomic, what are C_v and C_p? (*c*) What are the values of C_v and C_p at normal room temperatures?

Picture the Problem We can use the relationship between C_P and C_V ($C_P = C_V + nR$) to find the number of moles of this particular gas. In Parts (*b*) and (*c*) we can use the number of degrees of freedom associated with monatomic and diatomic gases, respectively, to find C_P and C_V.

(*a*) Express the heat capacity of the gas at constant pressure to its heat capacity at constant volume:

$$C_P = C_V + nR \Rightarrow n = \frac{C_P - C_V}{R}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{29.1\,\frac{\text{J}}{\text{K}}}{8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}} = 3.500\,\text{mol}$$

$$= \boxed{3.50\,\text{mol}}$$

(*b*) C_V for a monatomic gas is given by:

$$C_V = \tfrac{3}{2}nR$$

Substitute numerical values and evaluate C_V:

$$C_V = \tfrac{3}{2}(3.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)$$

$$= 43.65\ \text{J/K} = \boxed{43.6\,\text{J/K}}$$

Express C_P for a monatomic gas:

$$C_P = \tfrac{5}{2}nR$$

Substitute numerical values and evaluate C_P:

$$C_P = \tfrac{5}{2}(3.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)$$

$$= \boxed{72.7\,\text{J/K}}$$

(*c*) At normal room temperature, if the diatomic molecules rotate but do not vibrate they have 5 degrees of freedom. Hence:

$$C_V = \tfrac{5}{2}nR$$

and

$$C_P = C_V + R = \tfrac{7}{2}nR$$

Substitute numerical values and evaluate C_V and C_P:

$$C_V = \tfrac{5}{2}(3.500\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right) = \boxed{72.7\,\text{J/K}}$$

and

$$C_P = \tfrac{7}{2}(3.500\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K}) = \boxed{102\,\text{J/K}}$$

65 •• **[SSM]** Carbon dioxide (CO_2) at a pressure of 1.00 atm and a temperature of –78.5ºC sublimates directly from a solid to a gaseous state without going through a liquid phase. What is the change in the heat capacity at constant pressure per mole of CO_2 when it undergoes sublimation? (Assume that the gas molecules can rotate but do not vibrate.) Is the change in the heat capacity positive or negative during sublimation? The CO_2 molecule is pictured in Figure 18-22.

Picture the Problem We can find the change in the heat capacity at constant pressure as CO_2 undergoes sublimation from the energy per molecule of CO_2 in the solid and gaseous states.

Express the change in the heat capacity (at constant pressure) per mole as the CO_2 undergoes sublimation:

$$\Delta C_P = C_{P,\text{gas}} - C_{P,\text{solid}} \qquad (1)$$

Express $C_{p,\text{gas}}$ in terms of the number of degrees of freedom per molecule:

$$C_{P,\text{gas}} = f\left(\tfrac{1}{2}Nk\right) = \tfrac{5}{2}Nk$$

because each molecule has three translational and two rotational degrees of freedom in the gaseous state.

We know, from the Dulong-Petit Law, that the molar specific heat of most solids is $3R = 3Nk$. This result is essentially a per-atom result as it was obtained for a monatomic solid with six degrees of freedom. Use this result and the fact CO_2 is triatomic to express $C_{P,\text{solid}}$:

$$C_{P,\text{solid}} = \frac{3Nk}{\text{atom}} \times 3\,\text{atoms} = 9Nk$$

Substitute in equation (1) to obtain:

$$\Delta C_P = \tfrac{5}{2}Nk - \tfrac{18}{2}Nk = \boxed{-\tfrac{13}{2}Nk}$$

66 •• In this problem, 1.00 mol of a monatomic ideal gas is initially at 273 K and 1.00 atm. (*a*) What is the initial internal energy of the gas? (*b*) Find the work done by the gas when 500 J of heat are absorbed by the gas at constant pressure.

What is the final internal energy of the gas? (*c*) Find the work done by the gas when 500 J of heat are absorbed by the gas at constant volume. What is the final internal energy of the gas?

Picture the Problem We can find the initial internal energy of the gas from $U_\mathrm{i} = \frac{3}{2}nRT$ and the final internal energy from the change in internal energy resulting from the addition of 500 J of heat. The work done during a constant-volume process is zero and the work done during the constant-pressure process can be found from the first law of thermodynamics.

(*a*) Express the initial internal energy of the gas in terms of its temperature:

$$E_\mathrm{int,i} = \tfrac{3}{2}nRT$$

Substitute numerical values and evaluate $E_\mathrm{int,i}$:

$$E_\mathrm{int,i} = \tfrac{3}{2}(1.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(273\,\mathrm{K}) = 3.405\,\mathrm{kJ} = \boxed{3.40\,\mathrm{kJ}}$$

(*b*) Relate the final internal energy of the gas to its initial internal energy:

$$E_\mathrm{int,f} = E_\mathrm{int,i} + \Delta E_\mathrm{int} = E_\mathrm{int,i} + C_\mathrm{V}\Delta T$$

Express the change in temperature of the gas resulting from the addition of heat:

$$\Delta T = \frac{Q_\mathrm{in}}{C_\mathrm{P}}$$

Substitute to obtain:

$$E_\mathrm{int,f} = E_\mathrm{int,i} + \frac{C_\mathrm{V}}{C_\mathrm{P}}Q_\mathrm{in}$$

Substitute numerical values and evaluate $E_\mathrm{int,f}$:

$$E_\mathrm{int,f} = 3.405\,\mathrm{kJ} + \left(\frac{\frac{3}{2}nR}{\frac{5}{2}nR}\right)(500\,\mathrm{J}) = 3.705\,\mathrm{kJ} = \boxed{3.71\,\mathrm{kJ}}$$

The work done by the gas is the negative of the work done on the gas and is related to the change in internal energy and the heat added to the gas through the first law of thermodynamics:

$$\begin{aligned} W_\text{by gas} &= -W_\mathrm{on} = -\Delta E_\mathrm{int} + Q_\mathrm{in} \\ &= -\left(E_\mathrm{int,f} - E_\mathrm{int,i}\right) + Q_\mathrm{in} \\ &= -E_\mathrm{int,f} + E_\mathrm{int,i} + Q_\mathrm{in} \end{aligned}$$

Substitute numerical values and evaluate $W_{\text{by gas}}$:

$$W_{\text{by gas}} = -3.71\,\text{kJ} + 3.40\,\text{kJ} + 500\,\text{J} = \boxed{0.19\,\text{kJ}}$$

(*c*) Relate the final internal energy of the gas to its initial internal energy:

$$E_{\text{int,f}} = E_{\text{int,i}} + \Delta E_{\text{int}}$$

Apply the first law of thermodynamics to the constant-volume process:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$$

or, because $W_{\text{on}} = 0$,

$$\Delta E_{\text{int}} = Q_{\text{in}} = 500\,\text{J}$$

Substitute numerical values and evaluate $E_{\text{int,f}}$:

$$E_{\text{int,f}} = 3.405\,\text{kJ} + 500\,\text{J} = \boxed{3.91\,\text{kJ}}$$

and, because $W_{\text{on}} = 0 = -W_{\text{by gas}}$,

$$W_{\text{by gas}} = \boxed{0}$$

67 •• List all of the degrees of freedom possible for a water molecule and estimate the heat capacity of water at a temperature very far above its boiling point. (Ignore the fact the molecule might dissociate at high temperatures.) Think carefully about all of the different ways in which a water molecule can vibrate.

Picture the Problem We can use $C_{\text{V,water}} = f\left(\frac{1}{2}Nk\right)$ to express $C_{\text{V,water}}$ and then count the number of degrees of freedom associated with a water molecule to determine *f*.

Express $C_{\text{V,water}}$ in terms of the number of degrees of freedom per molecule:

$$C_{\text{V,water}} = f\left(\tfrac{1}{2}Nk\right)$$

where *f* is the number of degrees of freedom associated with a water molecule.

There are 3 translational degrees of freedom and three rotational degrees of freedom. In addition, each of the hydrogen atoms can vibrate against the oxygen atom, resulting in an additional 4 degrees of freedom (2 per atom).

Substitute for *f* to obtain:

$$C_{\text{V,water}} = 10\left(\tfrac{1}{2}Nk\right) = \boxed{5Nk} = \boxed{5nR}$$

Heat Capacities of Solids and the Dulong-Petit Law

68 • The Dulong–Petit law was originally used to determine the molar mass of a substance from its measured heat capacity. The specific heat of a certain solid substance is measured to be 0.447 kJ/kg·K. (*a*) Find the molar mass of the substance. (*b*) What element has this specific heat value?

Picture the Problem The Dulong-Petit law gives the molar specific heat of a solid, c'. The specific heat is defined as $c = c'/M$ where M is the molar mass. Hence we can use this definition to find M and a periodic table to identify the element.

(*a*) Apply the Dulong-Petit law:

$$c' = 3R \Rightarrow c = \frac{3R}{M} \Rightarrow M = \frac{3R}{c}$$

Substitute numerical values and evaluate M:

$$M = \frac{24.9\,\frac{\text{J}}{\text{mol}\cdot\text{K}}}{0.447\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}} = \boxed{55.7\,\text{g/mol}}$$

(*b*) Consulting the periodic table of elements we see that the element is most likely $\boxed{\text{iron}}$.

Quasi-Static Adiabatic Expansion of a Gas

69 •• [SSM] A 0.500-mol sample of an ideal monatomic gas at 400 kPa and 300 K, expands quasi-statically until the pressure decreases to 160 kPa. Find the final temperature and volume of the gas, the work done by the gas, and the heat absorbed by the gas if the expansion is (*a*) isothermal and (*b*) adiabatic.

Picture the Problem We can use the ideal-gas law to find the initial volume of the gas. In Part (*a*) we can apply the ideal-gas law for a fixed amount of gas to find the final volume and the expression for the work done in an isothermal process. Application of the first law of thermodynamics will allow us to find the heat absorbed by the gas during this process. In Part (*b*) we can use the relationship between the pressures and volumes for a quasi-static adiabatic process to find the final volume of the gas. We can apply the ideal-gas law to find the final temperature and, as in (*a*), apply the first law of thermodynamics, this time to find the work done by the gas.

Use the ideal-gas law to express the initial volume of the gas:

$$V_i = \frac{nRT_i}{P_i}$$

Substitute numerical values and evaluate V_i:

$$V_i = \frac{(0.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K})}{400\,\text{kPa}} = 3.118\times10^{-3}\ \text{m}^3$$

(*a*) Because the process is isothermal:

$$T_f = T_i = \boxed{300\,\text{K}}$$

Use the ideal-gas law for a fixed amount of gas with T constant to express V_f:

$$P_iV = P_fV_f \Rightarrow V_f = V_i\frac{P_i}{P_f}$$

Substitute numerical values and evaluate V_f:

$$V_f = (3.118\,\text{L})\left(\frac{400\,\text{kPa}}{160\,\text{kPa}}\right) = 7.795\,\text{L}$$
$$= \boxed{7.80\,\text{L}}$$

Express the work done by the gas during the isothermal expansion:

$$W_{\text{by gas}} = nRT\ln\frac{V_f}{V_i}$$

Substitute numerical values and evaluate $W_{\text{by gas}}$:

$$W_{\text{by gas}} = (0.500\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)\times(300\,\text{K})\ln\left(\frac{7.795\,\text{L}}{3.118\,\text{L}}\right)$$
$$= \boxed{1.14\,\text{kJ}}$$

Noting that the work done by the gas during the process equals the negative of the work done on the gas, apply the first law of thermodynamics to find the heat absorbed by the gas:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}} = 0 - (-1.14\,\text{kJ})$$
$$= \boxed{1.14\,\text{kJ}}$$

(*b*) Using $\gamma = 5/3$ and the relationship between the pressures and volumes for a quasi-static adiabatic process, express V_f:

$$P_iV_i^\gamma = P_fV_f^\gamma \Rightarrow V_f = V_i\left(\frac{P_i}{P_f}\right)^{1/\gamma}$$

Substitute numerical values and evaluate V_f:

$$V_f = (3.118\,\text{L})\left(\frac{400\,\text{kPa}}{160\,\text{kPa}}\right)^{3/5} = 5.403\,\text{L}$$
$$= \boxed{5.40\,\text{L}}$$

Apply the ideal-gas law to find the final temperature of the gas:

$$T_f = \frac{P_fV_f}{nR}$$

Substitute numerical values and evaluate T_f:

$$T_f = \frac{(160\,\text{kPa})(5.403\times10^{-3}\,\text{m}^3)}{(0.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = \boxed{208\,\text{K}}$$

For an adiabatic process:

$$Q_{\text{in}} = \boxed{0}$$

Apply the first law of thermodynamics to express the work done on the gas during the adiabatic process:

$$W_{\text{on}} = \Delta E_{\text{int}} - Q_{\text{in}} = C_V\Delta T - 0 = \tfrac{3}{2}nR\Delta T$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = \tfrac{3}{2}(0.500\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})\times(208\,\text{K}-300\,\text{K}) = -574\,\text{J}$$

Because the work done by the gas equals the negative of the work done on the gas:

$$W_{\text{by gas}} = -(-574\,\text{J}) = \boxed{574\,\text{J}}$$

70 •• A 0.500-mol sample of an ideal diatomic gas at 400 kPa and 300 K expands until the pressure decreases to 160 kPa. Find the final temperature and volume of the gas, the work done by the gas, and the heat absorbed by the gas if the expansion is (*a*) isothermal and (*b*) adiabatic.

Picture the Problem We can use the ideal-gas law to find the initial volume of the gas. In Part (*a*) we can apply the ideal-gas law for a fixed amount of gas to find the final volume and the expression for the work done in an isothermal process. Application of the first law of thermodynamics will allow us to find the heat absorbed by the gas during this process. In Part (*b*) we can use the relationship between the pressures and volumes for a quasi-static adiabatic process to find the final volume of the gas. We can apply the ideal-gas law to find the final temperature and, as in (*a*), apply the first law of thermodynamics, this time to find the work done by the gas.

Use the ideal-gas law to express the initial volume of the gas:

$$V_i = \frac{nRT_i}{P_i}$$

Substitute numerical values and evaluate V_i:

$$V_i = \frac{(0.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K})}{400\,\text{kPa}} = 3.118\,\text{L}$$

(*a*) Because the process is isothermal:

$$T_f = T_i = \boxed{300\,\text{K}}$$

Use the ideal-gas law for a fixed amount of gas with T = constant to express V_f:

$$P_iV_i = P_fV_f \Rightarrow V_f = V_i\frac{P_i}{P_f}$$

Substitute numerical values and evaluate T_f:

$$V_f = (3.118\,\text{L})\left(\frac{400\,\text{kPa}}{160\,\text{kPa}}\right) = 7.795\,\text{L} = \boxed{7.80\,\text{L}}$$

Express the work done by the gas during the isothermal expansion:

$$W_{\text{by gas}} = nRT\ln\frac{V_f}{V_i}$$

Substitute numerical values and evaluate $W_{\text{by gas}}$:

$$W_{\text{by gas}} = (0.500\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)\times(300\,\text{K})\ln\left(\frac{7.795\,\text{L}}{3.118\,\text{L}}\right) = \boxed{1.14\,\text{kJ}}$$

Noting that the work done by the gas during the isothermal expansion equals the negative of the work done on the gas, apply the first law of thermodynamics to find the heat absorbed by the gas:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}} = 0 - (-1.14\,\text{kJ}) = \boxed{1.14\,\text{kJ}}$$

(*b*) Using γ = 1.4 and the relationship between the pressures and volumes for a quasi-static adiabatic process, express V_f:

$$P_iV_i^\gamma = P_fV_f^\gamma \Rightarrow V_f = V_i\left(\frac{P_i}{P_f}\right)^{1/\gamma}$$

Substitute numerical values and evaluate V_f:

$$V_f = (3.118\,\text{L})\left(\frac{400\,\text{kPa}}{160\,\text{kPa}}\right)^{1/1.4} = 6.000\,\text{L}$$
$$= \boxed{6.00\,\text{L}}$$

Apply the ideal-gas law to express the final temperature of the gas:

$$T_f = \frac{P_f V_f}{nR}$$

Substitute numerical values and evaluate T_f:

$$T_f = \frac{(160\,\text{kPa})(6.000\times10^{-3}\,\text{m}^3)}{(0.500\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})}$$
$$= \boxed{231\,\text{K}}$$

For an adiabatic process:

$$Q_{\text{in}} = \boxed{0}$$

Apply the first law of thermodynamics to express the work done on the gas during the adiabatic expansion:

$$W_{\text{on}} = \Delta E_{\text{int}} - Q_{\text{in}} = C_V \Delta T - 0 = \tfrac{5}{2} nR\Delta T$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = \tfrac{5}{2}(0.500\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K}) \times (231\,\text{K} - 300\,\text{K})$$
$$= -717\,\text{J}$$

Noting that the work done by the gas during the adiabatic expansion is the negative of the work done on the gas, we have:

$$W_{\text{by gas}} = -(-717\,\text{J}) = \boxed{717\,\text{J}}$$

71 •• A 0.500-mol sample of helium gas expands adiabatically and quasi-statically from an initial pressure of 5.00 atm and temperature of 500 K to a final pressure of 1.00 atm. Find (*a*) the final temperature of the gas, (*b*) the final volume of the gas, (*c*) the work done by the gas, and (*d*) the change in the internal energy of the gas.

Picture the Problem We can eliminate the volumes from the equations relating the temperatures and volumes and the pressures and volumes for a quasi-static adiabatic process to obtain a relationship between the temperatures and pressures. We can find the initial volume of the gas using the ideal-gas law and the final volume using the pressure-volume relationship. In Parts (*d*) and (*c*) we can find the change in the internal energy of the gas from the change in its temperature and use the first law of thermodynamics to find the work done by the gas during its expansion.

(*a*) Express the relationship between temperatures and volumes for a quasi-static adiabatic process:

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

Express the relationship between pressures and volumes for a quasi-static adiabatic process:

$$P_i V_i^{\gamma} = P_f V_f^{\gamma} \qquad (1)$$

Eliminate the volume between these two equations to obtain:

$$T_f = T_i\left(\frac{P_f}{P_i}\right)^{1-\frac{1}{\gamma}}$$

Substitute numerical values and evaluate T_f:

$$T_f = (500\,\text{K})\left(\frac{1.00\,\text{atm}}{5.00\,\text{atm}}\right)^{1-\frac{1}{5/3}} = \boxed{263\,\text{K}}$$

(*b*) Solve equation (1) for V_f:

$$V_f = V_i\left(\frac{P_i}{P_f}\right)^{\frac{1}{\gamma}}$$

Apply the ideal-gas law to express V_i:

$$V_i = \frac{nRT_i}{P_i}$$

Substitute numerical values and evaluate V_i:

$$V_i = \frac{(0.500\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(500\,\text{K})}{5.00\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}} = 4.103\,\text{L}$$

Substitute for V_i and evaluate V_f:

$$V_f = (4.103\,\text{L})\left(\frac{5.00\,\text{atm}}{1.00\,\text{atm}}\right)^{\frac{3}{5}} = \boxed{10.8\,\text{L}}$$

(*c*) The work done by the gas during an adiabatic expansion is the negative of the work done on the gas during the expansion:

$$W_{\text{by the gas}} = -W_{\text{on}} = -(\Delta E_{\text{int}} - Q_{\text{in}}) = -(\Delta E_{\text{int}} - 0) = -\Delta E_{\text{int}}$$

Because $\Delta E_{\text{int}} = C_V \Delta T = \frac{3}{2} nR\Delta T$:

$$W_{\text{by the gas}} = -\tfrac{3}{2} nR\Delta T$$

Substitute numerical values and evaluate $W_{\text{by the gas}}$:

$$W_{\text{by the gas}} = -\tfrac{3}{2}(0.500\text{ mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(263\text{ K} - 500\text{ K}) = \boxed{1.48\text{ kJ}}$$

(*d*) Apply the first law of thermodynamics for an adiabatic process to obtain:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}} = 0 + (-W_{\text{by the gas}}) = -W_{\text{by the gas}}$$

Substitute $W_{\text{by the gas}}$ from (*c*):

$$\Delta E_{\text{int}} = -(1.48\text{ kJ}) = \boxed{-1.48\text{ kJ}}$$

Cyclic Processes

72 •• A 1.00-mol sample of N_2 gas at 20.0ºC and 5.00 atm is allowed to expand adiabatically and quasi-statically until its pressure equals 1.00 atm. It is then heated at constant pressure until its temperature is again 20.0ºC. After it reaches a temperature of 20.0ºC, it is heated at constant volume until its pressure is again 5.00 atm. It is then compressed at constant pressure until it is back to its original state. (*a*) Construct a *PV* diagram showing each process in the cycle. (*b*) From your graph, determine the work done by the gas during the complete cycle. (*c*) How much heat is absorbed (or released) by the gas during the complete cycle?

Picture the Problem To construct the *PV* diagram we'll need to determine the volume occupied by the gas at the beginning and ending points for each process. Let these points be A, B, C, and D. We can apply the ideal-gas law to the starting point (A) to find V_A. To find the volume at point B, we can use the relationship between pressure and volume for a quasi-static adiabatic process. We can use the ideal-gas law to find the volume at point C and, because they are equal, the volume at point D. We can apply the first law of thermodynamics to find the amount of heat added to or subtracted from the gas during the complete cycle.

(*a*) Using the ideal-gas law, express the volume of the gas at the starting point A of the cycle:

$$V_A = \frac{nRT_A}{P_A}$$

Substitute numerical values and evaluate V_A:

$$V_A = \frac{(1.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(293\,\text{K})}{5.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}} = 4.808\times10^{-3}\,\text{m}^3 = 4.808\,\text{L}$$

Use the relationship between pressure and volume for a quasi-static adiabatic process to express the volume of the gas at point B; the end point of the adiabatic expansion:

$$V_B = V_A\left(\frac{P_A}{P_B}\right)^{\frac{1}{\gamma}}$$

Substitute numerical values and evaluate V_B:

$$V_B = (4.808\,\text{L})\left(\frac{5.00\,\text{atm}}{1.00\,\text{atm}}\right)^{\frac{1}{1.4}} = 15.18\,\text{L}$$

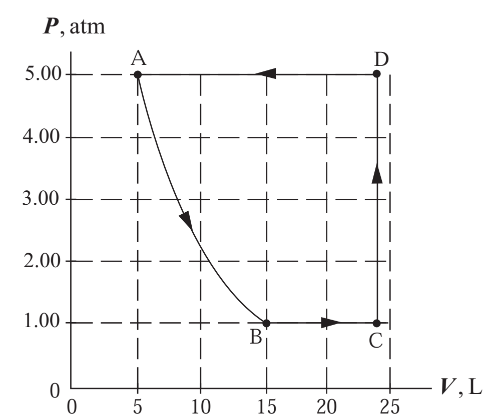
Using the ideal-gas law for a fixed amount of gas, express the volume occupied by the gas at points C and D:

$$V_C = V_D = \frac{nRT_C}{P_C}$$

Substitute numerical values and evaluate V_C:

$$V_C = \frac{(1.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(293\,\text{K})}{1.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}} = 24.04\times10^{-3}\,\text{m}^3 = 24.0\,\text{L}$$

The complete cycle is shown in the diagram.



(*b*) Note that for the paths A→B and B→C, $W_{\text{by gas}}$, the work done by the gas, is positive. For the path D→A, $W_{\text{by gas}}$ is negative, and greater in magnitude than $W_{A\to C}$. Therefore the total work done by the gas is negative. Find the area enclosed by the cycle by noting that each rectangle of dotted lines equals 5 atm·L and counting the rectangles:

$$W_{\text{by gas}} \approx -(13\,\text{rectangles})\left(5.00\frac{\text{atm}\cdot\text{L}}{\text{rectangle}}\right) = (-65\,\text{atm}\cdot\text{L})\left(\frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}}\right)$$
$$= -6.586\,\text{kJ} = \boxed{-6.6\,\text{kJ}}$$

(*c*) The work done on the gas equals the negative of the work done by the gas. Apply the first law of thermodynamics to find the amount of heat added to or subtracted from the gas during the complete cycle:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}} = 0 - (-6.59\,\text{kJ})$$
$$= \boxed{6.6\,\text{kJ}}$$

because $\Delta E_{\text{int}} = 0$ for the complete cycle.

73 •• [SSM] A 1.00-mol sample of an ideal diatomic gas is allowed to expand. This expansion is represented by the straight line from 1 to 2 in the *PV* diagram (Figure 18-23). The gas is then compressed isothermally. This compression is represented by the straight line from 2 to 1 in the *PV* diagram. Calculate the work per cycle done by the gas.

Picture the Problem The total work done as the gas is taken through this cycle is the area bounded by the two processes. Because the process from 1→2 is linear, we can use the formula for the area of a trapezoid to find the work done during this expansion. We can use $W_{\text{isothermal process}} = nRT\ln(V_f/V_i)$ to find the work done on the gas during the process 2→1. The net work done during this cycle is then the sum of these two terms.

Express the net work done per cycle:

$$W_{\text{net}} = W_{\text{by the gas}} + W_{\text{on the gas}} = W_{1\to2} + W_{2\to1} \quad (1)$$

Work is done by the gas during its expansion from 1 to 2 and hence is equal to the negative of the area of the trapezoid defined by this path and the vertical lines at $V_1 = 11.5$ L and $V_2 = 23$ L. Use the formula for the area of a trapezoid to express $W_{1\to2}$:

$$W_{1\to2} = -A_{\text{trap}}$$
$$= -\tfrac{1}{2}(23\,\text{L} - 11.5\,\text{L}) \times (2.0\,\text{atm} + 1.0\,\text{atm})$$
$$= -17.3\,\text{L}\cdot\text{atm}$$

Work is done on the gas during the isothermal compression from V_2 to V_1 and hence is equal to the area under the curve representing this process. Use the expression for the work done during an isothermal process to express $W_{2\to1}$:

$$W_{2\to1} = nRT\ln\left(\frac{V_\text{f}}{V_\text{i}}\right)$$

Apply the ideal-gas law at point 1 to find the temperature along the isotherm 2→1:

$$T = \frac{PV}{nR} = \frac{(2.0\,\text{atm})(11.5\,\text{L})}{(1.00\,\text{mol})(8.206\times10^{-2}\,\text{L}\cdot\text{atm/mol}\cdot\text{K})} = 280\,\text{K}$$

Substitute numerical values and evaluate $W_{2\to1}$:

$$W_{2\to1} = \left|(1.00\,\text{mol})(8.206\times10^{-2}\,\text{L}\cdot\text{atm/mol}\cdot\text{K})(280\,\text{K})\ln\left(\frac{11.5\,\text{L}}{23\,\text{L}}\right)\right| = 15.9\,\text{L}\cdot\text{atm}$$

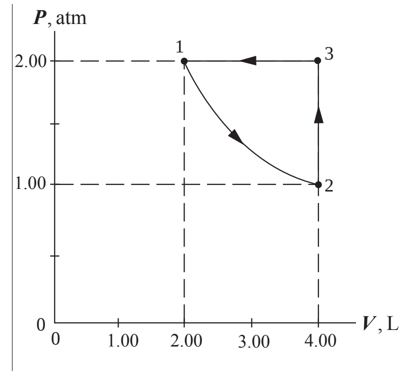
Substitute in equation (1) and evaluate W_net:

$$\begin{aligned} W_\text{net} &= -17.3\,\text{L}\cdot\text{atm} + 15.9\,\text{L}\cdot\text{atm} \\ &= -1.40\,\text{L}\cdot\text{atm}\times\frac{101.325\,\text{J}}{\text{L}\cdot\text{atm}} \\ &= \boxed{-0.14\,\text{kJ}} \end{aligned}$$

74 •• A 2.00-mol sample of an ideal monatomic gas has an initial pressure of 2.00 atm and an initial volume of 2.00 L. The gas is taken through the following quasi-static cycle: It is expanded isothermally until it has a volume of 4.00 L. It is next heated at constant volume until it has a pressure of 2.00 atm. It is then cooled at constant pressure until it is back to its initial state. (*a*) Show this cycle on a *PV* diagram. (*b*) Find the temperatures at the end of each part of the cycle. (*c*) Calculate the heat absorbed and the work done by the gas during each part of the cycle.

Picture the Problem Denote the initial and final points of each part of the cycle by the numerals 1, 2 and 3. We can apply the ideal-gas law to find the temperatures T_1, T_2, and T_3. We can use the appropriate work and heat equations to calculate the heat added and the work done by the gas for the isothermal process (1→2), the constant-volume process (2→3), and the isobaric process (3→1).

(*a*) The cycle is shown in the PV diagram to the right:



(*b*) Use the ideal-gas law to find T_1:

$$T_1 = \frac{P_1 V_1}{nR} = \frac{(2.00\,\text{atm})(2.00\,\text{L})}{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 24.37\ \text{K} = \boxed{24.4\,\text{K}}$$

Because the process 1→2 is isothermal:

$$T_2 = \boxed{24.4\,\text{K}}$$

Use the ideal-gas law to find T_3:

$$T_3 = \frac{P_3 V_3}{nR} = \frac{(2.00\,\text{atm})(4.00\,\text{L})}{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 48.74\ \text{K} = \boxed{48.7\,\text{K}}$$

(*c*) Because the process 1→2 is isothermal, $Q_{\text{in},1\to2} = W_{\text{by gas},1\to2}$:

$$Q_{\text{in},1\to2} = W_{\text{by gas},1\to2} = nRT\ln\left(\frac{V_2}{V_1}\right)$$

Substitute numerical values and evaluate $Q_{\text{in},1\to2}$:

$$Q_{\text{in},1\to2} = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(24.37\,\text{K})\ln\left(\frac{4.00\,\text{L}}{2.00\,\text{L}}\right) = \boxed{281\,\text{J}}$$

Because process 2→3 takes place at constant volume:

$$W_{2\to3} = \boxed{0}$$

Because process 2→3 takes place at constant volume, $W_{\text{on},2\to3} = 0$, and:

$$Q_{\text{in},2\to3} = \Delta E_{\text{int},2\to3} = C_\text{V}\Delta T = \tfrac{3}{2}nR(T_3 - T_2)$$

Substitute numerical values and evaluate $Q_{\text{in},2\to3}$:

$$Q_{\text{in},2\to3} = \tfrac{3}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(48.74\,\text{K} - 24.37\,\text{K}) = \boxed{608\,\text{J}}$$

Because process 3→1 is isobaric:

$$Q_{3\to1} = C_P\Delta T = \tfrac{5}{2}nR(T_1 - T_3)$$

Substitute numerical values and evaluate $Q_{3\to1}$:

$$Q_{3\to1} = \tfrac{5}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(24.37\,\text{K} - 48.74\,\text{K}) = \boxed{-1.01\,\text{kJ}}$$

The work done by the gas from 3 to 1 equals the negative of the work done on the gas:

$$W_{\text{by gas},3\to1} = -P_{1,3}\Delta V = P_{1,3}(V_1 - V_3)$$

Substitute numerical values and evaluate $W_{\text{by gas},3\to2}$:

$$W_{\text{by gas},3\to1} = -(2.00\,\text{atm})(2.00\,\text{L} - 4.00\,\text{L}) = -(-4.00\,\text{atm}\cdot\text{L})\left(\frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}}\right)$$

$$= \boxed{405\,\text{J}}$$

75 ••• [SSM] At point D in Figure 18-24 the pressure and temperature of 2.00 mol of an ideal monatomic gas are 2.00 atm and 360 K, respectively. The volume of the gas at point B on the *PV* diagram is three times that at point D and its pressure is twice that at point C. Paths AB and CD represent isothermal processes. The gas is carried through a complete cycle along the path DABCD. Determine the total amount of work done by the gas and the heat absorbed by the gas along each portion of the cycle.

Picture the Problem We can find the temperatures, pressures, and volumes at all points for this ideal monatomic gas (3 degrees of freedom) using the ideal-gas law and the work for each process by finding the areas under each curve. We can find the heat exchanged for each process from the heat capacities and the initial and final temperatures for each process.

Express the total work done by the gas per cycle:

$$W_{\text{by gas,tot}} = W_{\text{D}\to\text{A}} + W_{\text{A}\to\text{B}} + W_{\text{B}\to\text{C}} + W_{\text{C}\to\text{D}}$$

1. Use the ideal-gas law to find the volume of the gas at point D:

$$V_D = \frac{nRT_D}{P_D} = \frac{(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(360\,\text{K})}{(2.00\,\text{atm})\left(101.325\,\frac{\text{kPa}}{\text{atm}}\right)} = 29.54\,\text{L}$$

2. We're given that the volume of the gas at point B is three times that at point D:

$$V_B = V_C = 3V_D = 3(29.54\,\text{L}) = 88.62\,\text{L}$$

Use the ideal-gas law to find the pressure of the gas at point C:

$$P_C = \frac{nRT_C}{V_C} = \frac{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(360\,\text{K})}{88.62\,\text{L}} = 0.6667\,\text{atm}$$

We're given that the pressure at point B is twice that at point C:

$$P_B = 2P_C = 2(0.6667\,\text{atm}) = 1.333\,\text{atm}$$

3. Because path DC represents an isothermal process:

$$T_D = T_C = 360\,\text{K}$$

Use the ideal-gas law to find the temperatures at points B and A:

$$T_A = T_B = \frac{P_B V_B}{nR} = \frac{(1.333\,\text{atm})(88.62\,\text{L})}{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 719.8\,\text{K}$$

Because the temperature at point A is twice that at D and the volumes are the same, we can conclude that:

$$\boldsymbol{P}_A = 2\boldsymbol{P}_D = 4.00\,\text{atm}$$

The pressure, volume, and temperature at points A, B, C, and D are summarized in the table to the right.

Point	P	V	T
	(atm)	(L)	(K)
A	4.00	29.5	720
B	1.33	88.6	720
C	0.667	88.6	360
D	2.00	29.5	360

4. For the path D→A, $W_{D\to A} = 0$ and:

$$Q_{D\to A} = \Delta E_{\text{int, D}\to\text{A}} = \tfrac{3}{2}nR\Delta T_{D\to A} = \tfrac{3}{2}nR(T_A - T_D)$$

Substitute numerical values and evaluate $Q_{D\to A}$:

$$Q_{D\to A} = \tfrac{3}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(720\,\text{K} - 360\,\text{K}) = 8.979\,\text{kJ}$$

For the path A→B:

$$W_{A\to B} = Q_{A\to B} = nRT_{A,B}\ln\left(\frac{V_B}{V_A}\right)$$

Substitute numerical values and evaluate $W_{A\to B}$:

$$W_{A\to B} = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(720\,\text{K})\ln\left(\frac{88.62\,\text{L}}{29.54\,\text{L}}\right) = 13.15\,\text{kJ}$$

and, because process A→B is isothermal, $\Delta E_{\text{int, A}\to\text{B}} = 0$

For the path B→C, $W_{B\to C} = 0$, and:

$$Q_{B\to C} = \Delta U_{B\to C} = C_V\Delta T = \tfrac{3}{2}nR(T_C - T_B)$$

Substitute numerical values and evaluate $Q_{B\to C}$:

$$Q_{B\to C} = \tfrac{3}{2}(2.00\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})(360\,\text{K} - 720\,\text{K}) = -8.979\,\text{kJ}$$

For the path C→D:

$$W_{C\to D} = nRT_{C,D}\ln\left(\frac{V_D}{V_C}\right)$$

Substitute numerical values and evaluate $W_{C\to D}$:

$$W_{C\to D} = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(360\,\text{K})\ln\left(\frac{29.54\,\text{L}}{88.62\,\text{L}}\right) = -6.576\,\text{kJ}$$

Also, because process A→B is isothermal, $\Delta E_{\text{int,A}\to\text{B}} = 0$, and $Q_{C\to D} = W_{C\to D} = -6.58\,\text{kJ}$

Q_{in}, W_{on}, and ΔE_{int} are summarized for each of the processes in the table to the right.

Process	Q_{in}	W_{on}	ΔE_{int}
	(kJ)	(kJ)	(kJ)
D→A	$\boxed{8.98}$	0	8.98
A→B	$\boxed{13.2}$	−13.2	0
B→C	$\boxed{-8.98}$	0	−8.98
C→D	$\boxed{-6.58}$	6.58	0

Referring to the table, find the total work done by the gas per cycle:

$$\begin{aligned} W_{\text{by gas, tot}} &= W_{D\to A} + W_{A\to B} + W_{B\to C} + W_{C\to D} \\ &= 0 + 13.2\,\text{kJ} + 0 - 6.58\,\text{kJ} \\ &= \boxed{6.6\,\text{kJ}} \end{aligned}$$

Remarks: Note that, as it should be, ΔE_{int} is zero for the complete cycle.

76 ••• At point D in Figure 18-24 the pressure and temperature of 2.00 mol of an ideal diatomic gas are 2.00 atm and 360 K, respectively. The volume of the gas at point B on the *PV* diagram is three times that at point D and its pressure is twice that at point C. Paths AB and CD represent isothermal processes. The gas is carried through a complete cycle along the path DABCD. Determine the total amount of work done by the gas and the heat absorbed by the gas along each portion of the cycle.

Picture the Problem We can find the temperatures, pressures, and volumes at all points for this ideal diatomic gas (5 degrees of freedom) using the ideal-gas law and the work for each process by finding the areas under each curve. We can find the heat exchanged for each process from the heat capacities and the initial and final temperatures for each process.

Express the total work done by the gas per cycle:

$$W_{\text{by gas,tot}} = W_{\text{D}\to\text{A}} + W_{\text{A}\to\text{B}} + W_{\text{B}\to\text{C}} + W_{\text{C}\to\text{D}}$$

1. Use the ideal-gas law to find the volume of the gas at point D:

$$V_{\text{D}} = \frac{nRT_{\text{D}}}{P_{\text{D}}} = \frac{(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(360\,\text{K})}{(2.00\,\text{atm})\left(101.325\,\frac{\text{kPa}}{\text{atm}}\right)} = 29.54\,\text{L}$$

2. We're given that the volume of the gas at point B is three times that at point D:

$$V_{\text{B}} = V_{\text{C}} = 3V_{\text{D}} = 3(29.54\,\text{L}) = 88.62\,\text{L}$$

Use the ideal-gas law to find the pressure of the gas at point C:

$$P_{\text{C}} = \frac{nRT_{\text{C}}}{V_{\text{C}}} = \frac{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(360\,\text{K})}{88.62\,\text{L}} = 0.6667\,\text{atm}$$

We're given that the pressure at point B is twice that at point C:

$$P_{\text{B}} = 2P_{\text{C}} = 2(0.6667\,\text{atm}) = 1.333\,\text{atm}$$

3. Because path DC represents an isothermal process:

$$T_{\text{D}} = T_{\text{C}} = 360\,\text{K}$$

Use the ideal-gas law to find the temperatures at points B and A:

$$T_{\text{A}} = T_{\text{B}} = \frac{P_{\text{B}}V_{\text{B}}}{nR} = \frac{(1.333\,\text{atm})(88.62\,\text{L})}{(2.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 719.8\,\text{K}$$

Because the temperature at point A is twice that at D and the volumes are the same, we can conclude that:

$$P_{\text{A}} = 2P_{\text{D}} = 4.00\,\text{atm}$$

The pressure, volume, and temperature at points A, B, C, and D are summarized in the table to the right.

Point	P	V	T
	(atm)	(L)	(K)
A	4.00	29.5	720
B	1.33	88.6	720
C	0.667	88.6	360
D	2.00	29.5	360

4. For the path D→A, $W_{\mathrm{D}\to\mathrm{A}} = 0$ and:

$$\begin{aligned} Q_{\mathrm{D}\to\mathrm{A}} &= \Delta U_{\mathrm{D}\to\mathrm{A}} = \tfrac{5}{2}nR\Delta T_{\mathrm{D}\to\mathrm{A}} = \tfrac{5}{2}nR(T_\mathrm{A} - T_\mathrm{D}) \\ &= \tfrac{5}{2}(2.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(720\,\mathrm{K} - 360\,\mathrm{K}) \\ &= 14.97\,\mathrm{kJ} \end{aligned}$$

For the path A→B:

$$W_{\mathrm{A}\to\mathrm{B}} = Q_{\mathrm{A}\to\mathrm{B}} = nRT_{\mathrm{A,B}}\ln\left(\frac{V_\mathrm{B}}{V_\mathrm{A}}\right) = (2.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(720\,\mathrm{K})\ln\left(\frac{88.62\,\mathrm{L}}{29.54\,\mathrm{L}}\right)$$
$$= 13.15\,\mathrm{kJ}$$

and, because process A→B is isothermal, $\Delta E_{\mathrm{int,A}\to\mathrm{B}} = 0$

For the path B→C, $W_{\mathrm{B}\to\mathrm{C}} = 0$ and:

$$\begin{aligned} Q_{\mathrm{B}\to\mathrm{C}} &= \Delta U_{\mathrm{B}\to\mathrm{C}} = C_\mathrm{V}\Delta T = \tfrac{5}{2}nR(T_\mathrm{C} - T_\mathrm{B}) \\ &= \tfrac{5}{2}(2.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(360\,\mathrm{K} - 720\,\mathrm{K}) \\ &= -14.97\,\mathrm{kJ} \end{aligned}$$

For the path C→D:

$$W_{\mathrm{C}\to\mathrm{D}} = nRT_{\mathrm{C,D}}\ln\left(\frac{V_\mathrm{D}}{V_\mathrm{C}}\right) = (2.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(360\,\mathrm{K})\ln\left(\frac{29.54\,\mathrm{L}}{88.62\,\mathrm{L}}\right) = -6.576\,\mathrm{kJ}$$

Also, because process A→B is isothermal, $\Delta E_{\mathrm{int,A}\to\mathrm{B}} = 0$ and
$Q_{\mathrm{C}\to\mathrm{D}} = W_{\mathrm{C}\to\mathrm{D}} = -6.576\,\mathrm{kJ}$

Q_{in}, W_{on}, and ΔE_{int} are summarized for each of the processes in the table to the right.

Process	Q_{in}	W_{on}	ΔE_{int}
	(kJ)	(kJ)	(kJ)
D→A	$\boxed{15.0}$	0	15.0
A→B	$\boxed{13.2}$	−13.2	0
B→C	$\boxed{-15.0}$	0	−15.0
C→D	$\boxed{-6.58}$	6.58	0

Referring to the table and noting that the work done by the gas equals the negative of the work done on the gas, find the total work done by the gas per cycle:

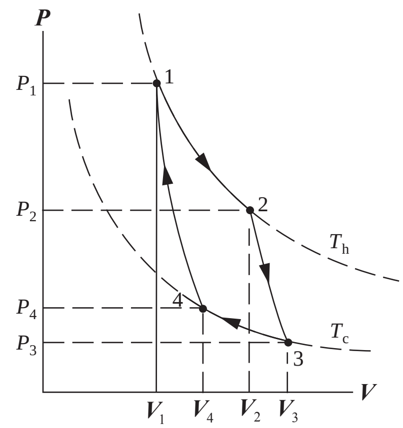
$$W_{\text{by gas,tot}} = W_{\text{D}\to\text{A}} + W_{\text{A}\to\text{B}} + W_{\text{B}\to\text{C}} + W_{\text{C}\to\text{D}} = 0 + 13.2\,\text{kJ} + 0 - 6.58\,\text{kJ} = \boxed{6.6\,\text{kJ}}$$

Remarks: Note that ΔE_{int} for the complete cycle is zero and that the total work done is the same for the diatomic gas of this problem and the monatomic gas of problem 75.

77 ••• A sample consisting of n moles of an ideal gas is initially at pressure P_1, volume V_1, and temperature T_h. It expands isothermally until its pressure and volume are P_2 and V_2. It then expands adiabatically until its temperature is T_c and its pressure and volume are P_3 and V_3. It is then compressed isothermally until it is at a pressure P_4 and a volume V_4, which is related to its initial volume V_1 by $T_c V_4^{\gamma-1} = T_h V_1^{\gamma-1}$. The gas is then compressed adiabatically until it is back in its original state. (*a*) Assuming that each process is quasi-static, plot this cycle on a PV diagram. (This cycle is known as the Carnot cycle for an ideal gas.) (*b*) Show that the heat Q_h absorbed during the isothermal expansion at T_h is $Q_h = nRT_h \ln(V_2/V_1)$. (*c*) Show that the heat Q_c released by the gas during the isothermal compression at T_c is $Q_c = nRT_c \ln(V_3/V_4)$. (*d*) Using the result that $TV^{\gamma-1}$ is constant for a quasi-static adiabatic expansion, show that $V_2/V_1 = V_3/V_4$. (*e*) The efficiency of a Carnot cycle is defined as the net work done by the gas, divided by the heat absorbed Q_h by the gas. Using the first law of thermodynamics, show that the efficiency is $1 - Q_c/Q_h$. (*f*) Using your results from the previous parts of this problem, show that $Q_c/Q_h = T_c/T_h$.

Picture the Problem We can use the equations of state for adiabatic and isothermal processes to express the work done on or by the system, the heat entering or leaving the system, and the change in internal energy for each of the four processes making up the Carnot cycle. We can use the first law of thermodynamics and the definition of the efficiency of a Carnot cycle to show that the efficiency is $1 - Q_c / Q_h$.

(*a*) The cycle is shown on the *PV* diagram to the right:



(*b*) Because the process 1→2 is isothermal:

$$\Delta E_{\text{int},1\to2} = 0$$

Apply the first law of thermodynamics to obtain:

$$Q_h = Q_{1\to2} = W_{1\to2} = \boxed{nRT_h \ln\left(\frac{V_2}{V_1}\right)}$$

(*c*) Because the process 3→4 is isothermal:

$$\Delta U_{3\to4} = 0$$

Apply the first law of thermodynamics to obtain:

$$Q_c = Q_{3\to4} = W_{3\to4} = nRT_c \ln\left(\frac{V_4}{V_3}\right)$$

$$= \boxed{-nRT_c \ln\left(\frac{V_3}{V_4}\right)}$$

where the minus sign tells us that heat is given off by the gas during this process.

(*d*) Apply the equation for a quasi-static adiabatic process at points 4 and 1 to obtain:

$$T_c V_4^{\gamma-1} = T_h V_1^{\gamma-1} \Rightarrow \frac{V_1}{V_4} = \left(\frac{T_c}{T_h}\right)^{\frac{1}{\gamma-1}} \quad (1)$$

Apply the equation for a quasi-static adiabatic process at points 2 and 3 to obtain:

$$T_h V_2^{\gamma-1} = T_c V_3^{\gamma-1} \Rightarrow \frac{V_2}{V_3} = \left(\frac{T_c}{T_h}\right)^{\frac{1}{\gamma-1}} \quad (2)$$

Equate equations (1) and (2) and rearrange to obtain:

$$\frac{V_3}{V_4} = \boxed{\frac{V_2}{V_1}}$$

(*e*) Express the efficiency of the Carnot cycle:

$$\varepsilon = \frac{W}{Q_h}$$

Apply the first law of thermodynamics to obtain:

$$W_{on} = \Delta E_{int,\,cycle} - Q_{in}$$
$$= 0 - (Q_h - Q_c) = -(Q_h - Q_c)$$

because E_{int} is a state function and $\Delta E_{int,\,cycle} = 0$.

Substitute to obtain:

$$\varepsilon = \frac{W_{\text{by the gas}}}{Q_h} = \frac{-W_{on}}{Q_h} = \frac{Q_h - Q_c}{Q_h}$$
$$= \boxed{1 - \frac{Q_c}{Q_h}}$$

(*f*) In Part (*b*) we established that:

$$Q_h = nRT_h \ln\left(\frac{V_2}{V_1}\right)$$

In Part (*c*) we established that the heat leaving the system along the path 3→4 is given by:

$$Q_c = nRT_c \ln\left(\frac{V_3}{V_4}\right)$$

Divide the second of these equations by the first to obtain:

$$\frac{Q_c}{Q_h} = \frac{nRT_c \ln\left(\frac{V_3}{V_4}\right)}{nRT_h \ln\left(\frac{V_2}{V_1}\right)} = \boxed{\frac{T_c}{T_h}}$$

because $\frac{V_3}{V_4} = \frac{V_2}{V_1}$.

Remarks: This last result establishes that the efficiency of a Carnot cycle is also given by $\varepsilon_C = 1 - \frac{T_c}{T_h}$.

General Problems

78 • During the process of quasi-statically compressing an ideal diatomic gas to one-fifth of its initial volume, 180 kJ of work are done on the gas. (*a*) If this compression is accomplished isothermally at room temperature (293 K), how much heat is released by the gas? (*b*) How many moles of gas are in this sample?

Picture the Problem (*a*) We can use the first law of thermodynamics to relate the heat removed from the gas to the work done on the gas. (*b*) We can find the number of moles of the gas from the expression for the work done on or by a gas during an isothermal process.

(*a*) Apply the first law of thermodynamics to this process:

$$Q_{\text{in}} = \Delta E_{\text{int}} - W_{\text{on}} = -W_{\text{on}}$$

because $\Delta E_{\text{int}} = 0$ for an isothermal process.

Substitute numerical values to obtain:

$$Q_{\text{in}} = -180\,\text{kJ}$$

Because $Q_{\text{removed}} = -Q_{\text{in}}$:

$$Q_{\text{removed}} = \boxed{180\,\text{kJ}}$$

(*b*) The work done on the gas during the isothermal process is given by:

$$W = nRT\ln\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right) \Rightarrow n = \frac{W}{RT\ln\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{-180\,\text{kJ}}{\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(293\,\text{K})\ln\left(\frac{1}{5}\right)} = \boxed{45.9\,\text{mol}}$$

79 • **[SSM]** The *PV* diagram in Figure 18-25 represents 3.00 mol of an ideal monatomic gas. The gas is initially at point A. The paths AD and BC represent isothermal changes. If the system is brought to point C along the path AEC, find (*a*) the initial and final temperatures of the gas, (*b*) the work done by the gas, and (*c*) the heat absorbed by the gas.

Picture the Problem We can use the ideal-gas law to find the temperatures T_{A} and T_{C}. Because the process EDC is isobaric, we can find the area under this line geometrically and then use the 1st law of thermodynamics to find Q_{AEC}.

(*a*) Using the ideal-gas law, find the temperature at point A:

$$T_A = \frac{P_A V_A}{nR} = \frac{(4.0\,\text{atm})(4.01\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 65.2\,\text{K} = \boxed{65\,\text{K}}$$

Using the ideal-gas law, find the temperature at point C:

$$T_C = \frac{P_C V_C}{nR} = \frac{(1.0\,\text{atm})(20.0\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 81.2\,\text{K} = \boxed{81\,\text{K}}$$

(*b*) Express the work done by the gas along the path AEC:

$$\begin{aligned} W_{AEC} &= W_{AE} + W_{EC} = 0 + P_{EC}\Delta V_{EC} \\ &= (1.0\,\text{atm})(20.0\,\text{L} - 4.01\,\text{L}) \\ &= 15.99\,\text{L}\cdot\text{atm}\times\frac{101.325\,\text{J}}{\text{L}\cdot\text{atm}} \\ &= 1.62\,\text{kJ} = \boxed{1.6\,\text{kJ}} \end{aligned}$$

(*c*) Apply the first law of thermodynamics to express Q_{AEC}:

$$\begin{aligned} Q_{AEC} &= W_{AEC} + \Delta E_{int} = W_{AEC} + C_V\Delta T \\ &= W_{AEC} + \tfrac{3}{2}nR\Delta T \\ &= W_{AEC} + \tfrac{3}{2}nR(T_C - T_A) \end{aligned}$$

Substitute numerical values and evaluate Q_{AEC}:

$$Q_{AEC} = 1.62\,\text{kJ} + \tfrac{3}{2}(3.00\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})(81.2\,\text{K} - 65.2\,\text{K}) = \boxed{2.2\,\text{kJ}}$$

Remarks The difference between W_{AEC} and Q_{AEC} is the change in the internal energy $\Delta E_{int,AEC}$ during this process.

80 •• The *PV* diagram in Figure 18-25 represents 3.00 mol of an ideal monatomic gas. The gas is initially at point A. The paths AD and BC represent isothermal changes. If the system is brought to point C along the path ABC, find (*a*) the initial and final temperatures of the gas, (*b*) the work done by the gas, and (*c*) the heat absorbed by the gas.

Picture the Problem We can use the ideal-gas law to find the temperatures T_A and T_C. Because the process AB is isobaric, we can find the area under this line

geometrically. We can use the expression for the work done during an isothermal expansion to find the work done between B and C and the first law of thermodynamics to find Q_{ABC}.

(*a*) Using the ideal-gas law, find the temperature at point A:

$$T_A = \frac{P_A V_A}{nR} = \frac{(4.0\,\text{atm})(4.01\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 65.2\,\text{K} = \boxed{65\,\text{K}}$$

Use the ideal-gas law to find the temperature at point C:

$$T_C = \frac{P_C V_C}{nR} = \frac{(1.0\,\text{atm})(20.0\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 81.2\,\text{K} = \boxed{81\,\text{K}}$$

(*b*) Express the work done by the gas along the path ABC:

$$W_{ABC} = W_{AB} + W_{BC} = P_{AB}\Delta V_{AB} + nRT_B \ln\left(\frac{V_C}{V_B}\right)$$

Use the ideal-gas law to find the volume of the gas at point B:

$$V_B = \frac{nRT_B}{P_B} = \frac{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(81.2\,\text{K})}{4.0\,\text{atm}} = 5.00\,\text{L}$$

Substitute numerical values and evaluate W_{ABC}:

$$W_{ABC} = (4.0\,\text{atm})(5.00\,\text{L} - 4.01\,\text{L}) + (3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right) \times (81.2\,\text{K})\ln\left(\frac{20.0\,\text{L}}{5.00\,\text{L}}\right)$$

$$= 31.7\,\text{L}\cdot\text{atm}\times\frac{101.325\,\text{J}}{\text{atm}} = 3.21\,\text{kJ} = \boxed{3.2\,\text{kJ}}$$

(*c*) Apply the first law of thermodynamics to obtain:

$$\begin{aligned} Q_{\text{ABC}} &= W_{\text{ABC}} + \Delta E_{\text{int}} = W_{\text{ABC}} + C_{\text{V}}\Delta T \\ &= W_{\text{ABC}} + \tfrac{3}{2}nR\Delta T \\ &= W_{\text{ABC}} + \tfrac{3}{2}nR\left(T_{\text{C}} - T_{A}\right) \end{aligned}$$

Substitute numerical values and evaluate Q_{ABC}:

$$Q_{\text{ABC}} = 3.21\,\text{kJ} + \tfrac{3}{2}(3.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(81.2\,\text{K} - 65.2\,\text{K}) = \boxed{3.8\,\text{kJ}}$$

Remarks: The difference between W_{ABC} and Q_{ABC} is the change in the internal energy $\Delta E_{\text{int,ABC}}$ during this process.

81 •• The *PV* diagram in Figure 18-25 represents 3.00 mol of an ideal monatomic gas. The gas is initially at point A. The paths AD and BC represent isothermal changes. If the system is brought to point C along the path ADC, find (*a*) the initial and final temperatures of the gas, (*b*) the work done by the gas, and (*c*) the heat absorbed by the gas.

Picture the Problem We can use the ideal-gas law to find the temperatures T_{A} and T_{C}. Because the process DC is isobaric, we can find the area under this line geometrically. We can use the expression for the work done during an isothermal expansion to find the work done between A and D and the first law of thermodynamics to find Q_{ADC}.

(*a*) Using the ideal-gas law, find the temperature at point A:

$$\begin{aligned} T_{\text{A}} &= \frac{P_{\text{A}}V_{\text{A}}}{nR} \\ &= \frac{(4.0\,\text{atm})(4.01\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} \\ &= 65.2\,\text{K} = \boxed{65\,\text{K}} \end{aligned}$$

Use the ideal-gas law to find the temperature at point C:

$$\begin{aligned} T_{\text{C}} &= \frac{P_{\text{C}}V_{\text{C}}}{nR} \\ &= \frac{(1.0\,\text{atm})(20\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} \\ &= \boxed{81\,\text{K}} \end{aligned}$$

(*b*) Express the work done by the gas along the path ADC:

$$W_{ADC} = W_{AD} + W_{DC} = nRT_A \ln\left(\frac{V_D}{V_A}\right) + P_{DC}\Delta V_{DC}$$

Use the ideal-gas law to find the volume of the gas at point D:

$$V_D = \frac{nRT_D}{P_D} = \frac{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(65.2\,\text{K})}{1.0\,\text{atm}} = 16.051\,\text{L}$$

Substitute numerical values and evaluate W_{ADC}:

$$\begin{aligned} W_{ADC} &= (3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)(65.2\,\text{K})\ln\left(\frac{16.1\,\text{L}}{4.01\,\text{L}}\right) \\ &\quad + (1.0\,\text{atm})(20.0\,\text{L} - 16.1\,\text{L}) \\ &= 26.2\,\text{L}\cdot\text{atm}\times\frac{101.325\,\text{J}}{\text{L}\cdot\text{atm}} = 2.65\,\text{kJ} = \boxed{2.7\,\text{kJ}} \end{aligned}$$

(*c*) Apply the first law of thermodynamics to obtain:

$$\begin{aligned} Q_{ADC} &= W_{ADC} + \Delta E_{int} = W_{ADC} + C_V\Delta T \\ &= W_{ADC} + \tfrac{3}{2}nR\Delta T \\ &= W_{ADC} + \tfrac{3}{2}nR(T_C - T_A) \end{aligned}$$

Substitute numerical values and evaluate Q_{ADC}:

$$Q_{ADC} = 2.65\,\text{kJ} + \tfrac{3}{2}(3.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(81.2\,\text{K} - 65.2\,\text{K}) = \boxed{3.3\,\text{kJ}}$$

82 •• Suppose that the paths AD and BC in Figure 18-25 represent adiabatic processes. What are the work done by the gas and the heat absorbed by the gas in following the path ABC?

Picture the Problem We can use the ideal-gas law to find the temperatures T_A and T_C. Because the process AB is isobaric, we can find the area under this line geometrically. We can find the work done during the adiabatic expansion between B and C using $W_{BC} = -C_V\Delta T_{BC}$ and the first law of thermodynamics to find Q_{ABC}.

The work done by the gas along path ABC is given by:

$$W_{ABC} = W_{AB} + W_{BC}$$
$$= P_{AB}\Delta V_{AB} - C_V\Delta T_{BC}$$
$$= P_{AB}\Delta V_{AB} - \tfrac{3}{2}nR\Delta T_{BC}$$

because, with $Q_{in} = 0$, $W_{BC} = -\Delta E_{int,BC}$.

Use the ideal-gas law to find T_A:

$$T_A = \frac{P_A V_A}{nR} = \frac{(4.0\,\text{atm})(4.01\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 65.2\,\text{K}$$

Use the ideal-gas law to express T_B:

$$T_B = \frac{P_B V_B}{nR}$$

Apply the pressure-volume relationship for a quasi-static adiabatic process to the gas at points B and C to find the volume of the gas at point B:

$$P_B V_B^\gamma = P_C V_C^\gamma$$

and

$$V_B = \left(\frac{P_C}{P_B}\right)^{\frac{1}{\gamma}} V_C = \left(\frac{1.0\,\text{atm}}{4.0\,\text{atm}}\right)^{\frac{3}{5}}(20.0\,\text{L}) = 8.71\,\text{L}$$

Substitute numerical values and evaluate T_B:

$$T_B = \frac{(4.0\,\text{atm})(8.71\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 142\,\text{K}$$

Use the ideal-gas law to express T_C:

$$T_C = \frac{P_C V_C}{nR}$$

Substitute numerical values and evaluate T_C:

$$T_C = \frac{(1.0\,\text{atm})(20.0\,\text{L})}{(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 81.2\,\text{K}$$

Substitute numerical values and evaluate W_{ABC}:

$$\begin{aligned} W_{\text{ABC}} &= (4.0\,\text{atm})(8.71\,\text{L} - 4.01\,\text{L}) - \tfrac{3}{2}(3.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right) \\ &\quad \times(81.2\,\text{K} - 142\,\text{K}) \\ &= 41.3\,\text{L}\cdot\text{atm}\times\frac{101.325\,\text{J}}{\text{atm}} = 4.18\,\text{kJ} = \boxed{4.2\,\text{kJ}} \end{aligned}$$

Apply the 1st law of thermodynamics to obtain:

$$\begin{aligned} Q_{\text{ABC}} &= W_{\text{ABC}} + \Delta E_{\text{int}} = W_{\text{ABC}} + C_{\text{V}}\Delta T \\ &= W_{\text{ABC}} + \tfrac{3}{2}nR\Delta T \\ &= W_{\text{ABC}} + \tfrac{3}{2}nR(T_{\text{C}} - T_{A}) \end{aligned}$$

Substitute numerical values and evaluate Q_{ABC}:

$$Q_{\text{ABC}} = 4.18\,\text{kJ} + \tfrac{3}{2}(3.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(81.2\,\text{K} - 65.2\,\text{K}) = \boxed{4.8\,\text{kJ}}$$

83 •• [SSM] As part of a laboratory experiment, you test the calorie content of various foods. Assume that when you eat these foods, 100% of the energy released by the foods is absorbed by your body. Suppose you burn a 2.50-g potato chip, and the resulting flame warms a small aluminum can of water. After burning the potato chip, you measure its mass to be 2.20 g. The mass of the can is 25.0 g, and the volume of water contained in the can is 15.0 ml. If the temperature increase in the water is 12.5°C, how many kilocalories (1 kcal = 1 dietary calorie) per 150-g serving of these potato chips would you estimate there are? Assume the can of water captures 50.0 percent of the heat released during the burning of the potato chip. *Note: Although the joule is the SI unit of choice in most thermodynamic situations, the food industry in the United States currently expresses the energy released during metabolism in terms of the "dietary calorie," which is our kilocalorie.*

Picture the Problem The ratio of the energy in a 150-g serving to the energy in 0.30 g of potato chip is the same as the ratio of the masses of the serving and the amount of the chip burned while heating the aluminum can and the water in it.

The ratio of the energy in a 150-g serving to the energy in 0.30 g of potato chip is the same as the ratio of the masses of the serving and the amount of the chip burned while heating the aluminum can and the water in it:

$$\frac{Q_{150\text{-g serving}}}{Q_{0.30\,\text{g}}}=\frac{150\text{ g}}{0.30\text{ g}}=500$$

or

$$Q_{150\text{ g serving}}=500Q_{0.30\,\text{g}}$$

Letting f represent the fraction of the heat captured by the can of water, express the energy transferred to the aluminum can and the water in it during the burning of the potato chip:

$$\begin{aligned}fQ_{0.30\,\text{g}}&=Q_{\text{Al}}+Q_{\text{H}_2\text{O}}\\&=m_{\text{Al}}c_{\text{Al}}\Delta T+m_{\text{H}_2\text{O}}c_{\text{H}_2\text{O}}\Delta T\\&=\left(m_{\text{Al}}c_{\text{Al}}+m_{\text{H}_2\text{O}}c_{\text{H}_2\text{O}}\right)\Delta T\end{aligned}$$

where ΔT is the common temperature change of the aluminum cup and the water it contains.

Substituting for $Q_{0.30\,\text{g}}$ yields and solving for $Q_{150\text{-g serving}}$ yields:

$$Q_{150\text{-g serving}}=\frac{500\left(m_{\text{Al}}c_{\text{Al}}+m_{\text{H}_2\text{O}}c_{\text{H}_2\text{O}}\right)\Delta T}{f}$$

Substitute numerical values and evaluate $Q_{150\text{-g serving}}$:

$$\begin{aligned}Q_{150\text{-g serving}}&=\frac{500\left[(0.0250\text{ kg})\left(0.900\dfrac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)+(0.0150\text{ kg})\left(4.184\dfrac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\right](12.5\text{ C}^\circ)}{0.500}\\&=1.07\times10^{6}\text{ J}\times\frac{1\text{ cal}}{4.184\text{ J}}=256\times10^{3}\text{ cal}\approx\boxed{256\text{ kcal}}\end{aligned}$$

84 •• Diesel engines operate without spark plugs, unlike gasoline engines. The cycle that diesel engines undergo involves adiabatically compressing the air in a cylinder, and then fuel is injected. When the fuel is injected, if the air temperature inside the cylinder is above the fuel's flashpoint, the fuel-air mixture will ignite. Most diesel engines have compression ratios in the range from 14:1 to 25:1. For this range of compression ratios (which are the ratio of maximum to minimum volume), what is the range of maximum temperatures of the air in the cylinder, assuming the air is taken into the cylinder at 35°C? Most modern gasoline engines typically have compression ratios on the order of 8:1. Explain why you expect the diesel engine to require a better (more efficient) cooling system than its gasoline counterpart.

Picture the Problem You can use the equation-of-state for an adiabatic process to find the range of maximum temperatures of the air in the cylinder.

For an adiabatic process from state i to state f:

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

where, for a diatomic gas, $\gamma = 1.4$.

Solving for T_f yields:

$$T_f = T_i \frac{V_i^{\gamma-1}}{V_f^{\gamma-1}} = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1}$$

Substitute numerical values and evaluate T_f for an engine with a compression ratio of 14:1:

$$T_f = (273\ \text{K} + 35\ \text{C°})\left(\frac{14}{1}\right)^{1.4-1}$$
$$= 885\ \text{K} = 612°\text{C}$$

Substitute numerical values and evaluate T_f for an engine with a compression ratio of 25:1:

$$T_f = (273\ \text{K} + 35\ \text{C°})\left(\frac{25}{1}\right)^{1.4-1}$$
$$= 1116\ \text{K} = 843°\text{C}$$

The range of maximum temperatures of the air in the cylinder, assuming it is taken into the cylinder at 35°C, is $\boxed{612°\text{C} \rightarrow 843°\text{C}}$.

Gasoline engines, with much lower compression ratios, would clearly have lower maximum operating temperatures and thus be subject to overall, average, lower temperatures and not require as extensive cooling systems.

85 •• At very low temperatures, the specific heat of a metal is given by $c = aT + bT^3$. For copper, $a = 0.0108$ J/kg·K^2 and $b = 7.62 \times 10^{-4}$ J/kg·K^4. (*a*) What is the specific heat of copper at 4.00 K? (*b*) How much heat is required to heat copper from 1.00 to 3.00 K?

Picture the Problem We can find c at $T = 4.00$ K by direct substitution. Because c is a function of T, we can integrate dQ over the given temperature interval in order to find the heat required to heat copper from 1.00 to 3.00 K.

(*a*) Substitute for *a* and *b* to obtain:

$$c = \left(0.0108 \frac{\text{J}}{\text{kg}\cdot\text{K}^2}\right)T + \left(7.62\times10^{-4}\frac{\text{J}}{\text{kg}\cdot\text{K}^4}\right)T^3$$

Evaluate c at $T = 4.00$ K:

$$c(4.00\,\text{K}) = \left(0.0108\frac{\text{J}}{\text{kg}\cdot\text{K}^2}\right)(4.00\,\text{K}) + \left(7.62\times10^{-4}\frac{\text{J}}{\text{kg}\cdot\text{K}^4}\right)(4.00\,\text{K})^3$$
$$= \boxed{9.20\times10^{-2}\frac{\text{J}}{\text{kg}\cdot\text{K}}}$$

(*b*) The heat required to heat copper from 1.00 to 3.00 K is given by:

$$Q = \int_{1.00\,\text{K}}^{3.00\,\text{K}} c(T)\,dT$$

Substituting for $c(T)$ yields:

$$Q = \left(0.0108\frac{\text{J}}{\text{kg}\cdot\text{K}^2}\right)\int_{1.00\,\text{K}}^{3.00\,\text{K}} T\,dT + \left(7.62\times10^{-4}\frac{\text{J}}{\text{kg}\cdot\text{K}^4}\right)\int_{1.00\,\text{K}}^{3.00\,\text{K}} T^3\,dT$$

Evaluate this integral to obtain:

$$Q = \left(0.0108\frac{\text{J}}{\text{kg}\cdot\text{K}^2}\right)\left[\frac{T^2}{2}\right]_{1.00\,\text{K}}^{3.00\,\text{K}} + \left(7.62\times10^{-4}\frac{\text{J}}{\text{kg}\cdot\text{K}^4}\right)\left[\frac{T^4}{4}\right]_{1.00\,\text{K}}^{3.00\,\text{K}} = \boxed{0.0584\,\text{J/kg}}$$

86 •• How much work must be done on 30.0 g of carbon monoxide (CO) at standard temperature and pressure to compress it to one-fifth of its initial volume if the process is (*a*) isothermal, (*b*) adiabatic?

Picture the Problem (*a*) Let the subscripts 1 and 2 refer to the initial and final state respectively. Because the gas is initially at standard temperature and pressure, we know that V_1 = 22.4 L/mol, P_1 = 1 atm, and T_1 = 273 K. We can use $W = -nRT\ln(V_2/V_1)$ to find the work done on the gas during an isothermal compression. (*b*) We can use $W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2)$ to find the work done on the gas during the adiabatic compression. Note that γ = 1.4 for diatomic gasses and see Appendix C for the molar mass of CO.

(*a*) Express the work done on the gas in compressing it isothermally:

$$W_{\text{on}} = -nRT\ln\left(\frac{V_2}{V_1}\right)$$

Because the number of moles of CO is the mass m of CO divided by its molar mass:

$$W_{\text{on}} = -\frac{m}{M}RT\ln\left(\frac{V_2}{V_1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -\left(\frac{30.0\,\text{g}}{28.01\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\ln\left(\frac{1}{5}\right) = \boxed{3.91\,\text{kJ}}$$

(*b*) Express the work done on the gas in compressing it adiabatically:

$$W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2) = -\frac{m}{M}C_{\text{V}}(T_1 - T_2)$$

Using the equation for a quasi-static adiabatic process, relate the initial and final temperatures and volumes:

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Substituting for T_2 and V_2 and simplifying gives:

$$W_{\text{on}} = -\frac{m}{M}C_{\text{V}}\left(T_1 - T_1\left(\frac{V_1}{\frac{1}{5}V_1}\right)^{\gamma-1}\right)$$

$$= -\frac{m}{M}C_{\text{V}}T_1\left(1-(5)^{\gamma-1}\right)$$

Because $C_{\text{V}} = 2.49R$ for an ideal diatomic gas (see Table 18-3):

$$W_{\text{on}} = -2.49\frac{m}{M}RT_1\left(1-(5)^{\gamma-1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -2.49\left(\frac{30.0\,\text{g}}{28.01\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\left(1-(5)^{1.4-1}\right) = \boxed{5.49\,\text{kJ}}$$

87 •• How much work must be done on 30.0 g of carbon dioxide (CO_2) at standard temperature and pressure to compress it to one-fifth of its initial volume if the process is (*a*) isothermal, (*b*) adiabatic?

Picture the Problem (*a*) Let the subscripts 1 and 2 refer to the initial and final state respectively. Because the gas is initially at standard temperature and pressure, we know that V_1 = 22.4 L/mol, P_1 = 1 atm, and T_1 = 273 K. We can use $W = -nRT\ln(V_2/V_1)$ to find the work done on the gas during an isothermal compression. (*b*) We can use $W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2)$ to find the work done on the gas during the adiabatic compression. Note that γ = 1.3 for triatomic gasses and see Appendix C for the molar mass of CO_2.

(*a*) Express the work done on the gas in compressing it isothermally:

$$W_{\text{on}} = -nRT\ln\left(\frac{V_2}{V_1}\right)$$

Because the number of moles of CO_2 is the mass m of CO_2 divided by its molar mass:

$$W_{\text{on}} = -\frac{m}{M}RT\ln\left(\frac{V_2}{V_1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -\left(\frac{30.0\,\text{g}}{44.01\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\ln\left(\frac{1}{5}\right) = \boxed{2.49\,\text{kJ}}$$

(*b*) Express the work done on the gas in compressing it adiabatically:

$$W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2) = -\frac{m}{M}C_{\text{V}}(T_1 - T_2)$$

Using the equation for a quasi-static adiabatic process, relate the initial and final temperatures and volumes:

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Substituting for T_2 and V_2 and simplifying gives:

$$W_{\text{on}} = -\frac{m}{M}C_{\text{V}}\left(T_1 - T_1\left(\frac{V_1}{\frac{1}{5}V_1}\right)^{\gamma-1}\right)$$

$$= -\frac{m}{M}C_{\text{V}}T_1\left(1-(5)^{\gamma-1}\right)$$

Because $C_{\text{V}} = 3.39R$ for CO_2 (see Table 18-3):

$$W_{\text{on}} = -3.39\frac{m}{M}RT_1\left(1-(5)^{\gamma-1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -3.39\left(\frac{30.0\,\text{g}}{44.01\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\left(1-(5)^{1.3-1}\right) = \boxed{3.26\,\text{kJ}}$$

88 •• How much work must be done on 30.0 g of argon (Ar) at standard temperature and pressure to compress it to one-fifth of its initial volume if the process is (*a*) isothermal, (*b*) adiabatic?

Picture the Problem (*a*) Let the subscripts 1 and 2 refer to the initial and final state respectively. Because the gas is initially at standard temperature and pressure, we know that V_1 = 22.4 L/mol, P_1 = 1 atm, and T_1 = 273 K. We can use $W = -nRT\ln(V_2/V_1)$ to find the work done on the gas during an isothermal

compression. (*b*) We can use $W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2)$ to find the work done on the gas during the adiabatic compression. Note that γ = 1.67 for monatomic gasses and see Appendix C for the molar mass of Ar.

(*a*) Express the work done on the gas in compressing it isothermally:

$$W_{\text{on}} = -nRT\ln\left(\frac{V_2}{V_1}\right)$$

Because the number of moles of Ar is the mass *m* of Ar divided by its molar mass:

$$W_{\text{on}} = -\frac{m}{M}RT\ln\left(\frac{V_2}{V_1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -\left(\frac{30.0\,\text{g}}{39.948\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\ln\left(\frac{1}{5}\right) = \boxed{2.74\,\text{kJ}}$$

(*b*) Express the work done on the gas in compressing it adiabatically:

$$W_{\text{on}} = -nC_{\text{V}}(T_1 - T_2) = -\frac{m}{M}C_{\text{V}}(T_1 - T_2)$$

Using the equation for a quasi-static adiabatic process, relate the initial and final temperatures and volumes:

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Substituting for T_2 and V_2 and simplifying gives:

$$W_{\text{on}} = -\frac{m}{M}C_{\text{V}}\left(T_1 - T_1\left(\frac{V_1}{\frac{1}{5}V_1}\right)^{\gamma-1}\right)$$

$$= -\frac{m}{M}C_{\text{V}}T_1\left(1 - (5)^{\gamma-1}\right)$$

Because $C_{\text{V}} = 1.50R$ for Ar (see Table 18-3):

$$W_{\text{on}} = -1.50\frac{m}{M}RT_1\left(1 - (5)^{\gamma-1}\right)$$

Substitute numerical values and evaluate W_{on}:

$$W_{\text{on}} = -1.50\left(\frac{30.0\,\text{g}}{39.95\,\text{g/mol}}\right)\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K})\left(1 - (5)^{1.67-1}\right) = \boxed{4.92\,\text{kJ}}$$

89 •• **[SSM]** A thermally insulated system consists of 1.00 mol of a diatomic gas at 100 K and 2.00 mol of a solid at 200 K that are separated by a rigid insulating wall. Find the equilibrium temperature of the system after the

insulating wall is removed, assuming that the gas obeys the ideal-gas law and that the solid obeys the Dulong–Petit law.

Picture the Problem We can use conservation of energy to relate the equilibrium temperature to the heat capacities of the gas and the solid. We can apply the Dulong-Petit law to find the heat capacity of the solid at constant volume and use the fact that the gas is diatomic to find its heat capacity at constant volume.

Apply conservation of energy to this process:

$$\Delta Q = Q_{\text{gas}} + Q_{\text{solid}} = 0$$

Use $Q = C_{\text{V}}\Delta T$ to substitute for Q_{gas} and Q_{solid}:

$$C_{\text{V,gas}}\left(T_{\text{equil}} - 100\,\text{K}\right) + C_{\text{V,solid}}\left(T_{\text{equil}} - 200\,\text{K}\right) = 0$$

Solving for T_{equil} yields:

$$T_{\text{equil}} = \frac{(100\,\text{K})\left(C_{\text{V,gas}}\right) + (200\,\text{K})\left(C_{\text{V,solid}}\right)}{C_{\text{V,gas}} + C_{\text{V,solid}}}$$

Using the Dulong-Petit law, determine the heat capacity of the solid at constant volume:

$$C_{\text{V,solid}} = 3n_{\text{solid}}R$$

The heat capacity of the gas at constant volume is given by:

$$C_{\text{V,gas}} = \tfrac{5}{2}n_{\text{gas}}R$$

Substitute for $C_{\text{V,solid}}$ and $C_{\text{V,gas}}$ and simplify to obtain:

$$T_{\text{equil}} = \frac{(100\,\text{K})\left(\frac{5}{2}n_{\text{gas}}R\right) + (200\,\text{K})\left(3n_{\text{solid}}R\right)}{\frac{5}{2}n_{\text{gas}}R + 3n_{\text{solid}}R} = \frac{(100\,\text{K})\left(\frac{5}{2}n_{\text{gas}}\right) + (200\,\text{K})\left(3n_{\text{solid}}\right)}{\frac{5}{2}n_{\text{gas}} + 3n_{\text{solid}}}$$

Substitute numerical values for n_{gas} and n_{solid} and evaluate T_{equil}:

$$T_{\text{equil}} = \frac{(100\,\text{K})\left(\frac{5}{2}\right)(1.00\,\text{mol}) + (200\,\text{K})(3)(2.00\,\text{mol})}{\frac{5}{2}(1.00\,\text{mol}) + 3(2.00\,\text{mol})} = \boxed{171\,\text{K}}$$

90 •• When an ideal gas undergoes a temperature change at constant volume, its internal energy change is given by the formula $\Delta E_{\text{int}} = C_{\text{v}}\Delta T$. However, this formula correctly gives the change in internal energy whether the volume remains constant or not. (*a*) Explain why this formula gives correct results for an ideal gas even when the volume changes. (*b*) Using this formula, along with the first law of thermodynamics, show that for an ideal gas $C_{\text{p}} = C_{\text{v}} + nR$.

Picture the Problem

(*a*) $\Delta E_{\text{int}} = C_{\text{V}}\Delta T$ correctly gives the change in internal energy when the temperature changes and the volume is not constant because ΔE_{int} is the same for *all* gas processes that have the same ΔT. For an ideal gas, the internal energy is the sum of the kinetic energies of the gas molecules and this sum is proportional to kT. Any two processes that change the thermal energy of the gas by ΔE_{int} will cause the same temperature change ΔT. Consequently, ΔE_{int} is independent of the process that takes the gas from one state to another.

(*b*) Use the first law of thermodynamics to relate the work done on the gas, the heat entering the gas, and the change in the internal energy of the gas:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$$

Substituting for ΔE_{int} yields:

$$C_{\text{V}}\Delta T = Q_{\text{in}} + W_{\text{on}}$$

The work done on the gas during a constant-pressure process is given by:

$$W_{\text{on}} = -P\Delta V = -P(V_{\text{f}} - V_{\text{i}})$$

Use the ideal gas law to substitute for V_{f} and V_{i} and then simplify to obtain:

$$W_{\text{on}} = -P\left(\frac{nRT_{\text{f}}}{P} - \frac{nRT_{\text{i}}}{P}\right) = -nR(T_{\text{f}} - T_{\text{i}})$$
$$= -nR\Delta T$$

Substituting for W_{on} gives:

$$C_{\text{V}}\Delta T = Q_{\text{in}} - nR\Delta T$$

Solve for Q_{in} and simplify to obtain:

$$Q_{\text{in}} = C_{\text{V}}\Delta T + nR\Delta T = (C_{\text{V}} + nR)\Delta T$$
$$= C_{\text{P}}\Delta T$$

where $\boxed{C_{\text{P}} = C_{\text{V}} + nR}$

91 •• An insulated cylinder is fitted with an insulated movable piston to maintain constant pressure. The cylinder initially contains 100 g of ice at –10ºC. Heat is transferred to the ice at a constant rate by a 100-W heater. Make a graph showing the temperature of the ice/water/ steam as a function of time starting at t_i when the temperature is –10ºC and ending at t_{f} when the temperature is 110ºC.

Picture the Problem Knowing the rate at which energy is supplied, we can obtain the data we need to plot this graph by finding the time required to warm the ice to 0°C, melt the ice, warm the water formed from the ice to 100°C, vaporize the water, and warm the water to 110°C.

Find the time required to warm the ice to 0°C:

$$\Delta t_1 = \frac{mc_{\text{ice}}\Delta T}{P} = \frac{(0.100\,\text{kg})\left(2.0\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(0°\text{C}-(-10°\text{C}))}{100\,\frac{\text{J}}{\text{s}}} = 20.0\,\text{s} \approx 0.3\,\text{min}$$

Find the time required to melt the ice:

$$\Delta t_2 = \frac{mL_{\text{f}}}{P} = \frac{(0.100\,\text{kg})\left(333.5\,\frac{\text{kJ}}{\text{kg}}\right)}{100\,\frac{\text{J}}{\text{s}}}$$

$$= 333.5\,\text{s} \approx 5.6\,\text{min}$$

Find the time required to heat the water to 100°C:

$$\Delta t_3 = \frac{mc_{\text{w}}\Delta T}{P}$$

$$= \frac{(0.100\,\text{kg})\left(4.18\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(100°\text{C})}{100\,\frac{\text{J}}{\text{s}}}$$

$$= 418\,\text{s} \approx 7.0\,\text{min}$$

Find the time required to vaporize the water:

$$\Delta t_4 = \frac{mL_{\text{V}}}{P} = \frac{(0.100\,\text{kg})\left(2257\,\frac{\text{kJ}}{\text{kg}}\right)}{100\,\frac{\text{J}}{\text{s}}}$$

$$= 2257\,\text{s} \approx 37.6\,\text{min}$$

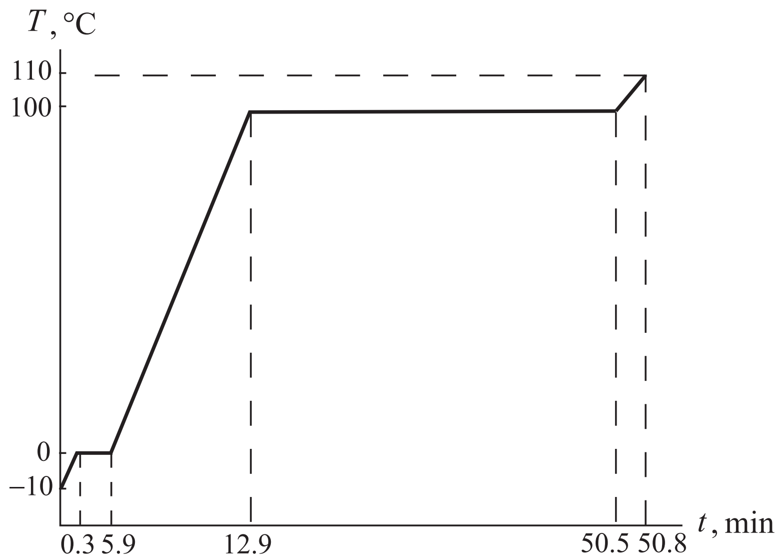
Find the time required to heat the vapor to 110°C:

$$\Delta t_5 = \frac{mc_{\text{steam}}\Delta T}{P}$$

$$= \frac{(0.100\,\text{kg})\left(2.0\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(10°\text{C})}{100\,\frac{\text{J}}{\text{s}}}$$

$$= 20.0\,\text{s} \approx 0.3\,\text{min}$$

A graph of T as a function of time follows:



92 •• (*a*) In this problem, 2.00 mol of a diatomic ideal gas expands adiabatically and quasi-statically. The initial temperature of the gas is 300 K. The work done by the gas during the expansion is 3.50 kJ. What is the final temperature of the gas? (*b*) Compare your result to the result you would get if the gas were monatomic.

Picture the Problem We know that, for an adiabatic process, $Q_{in} = 0$. Hence the work done by the expanding gas equals the change in its internal energy. Because we're given the work done by the gas during the expansion, we can express the change in the temperature of the gas in terms of this work and C_V.

(*a*) Express the final temperature of the gas as a result of its expansion:

$$T_f = T_i + \Delta T$$

Apply the equation for adiabatic work and solve for ΔT:

$$W_{adiabatic} = -C_V \Delta T$$

and

$$\Delta T = -\frac{W_{adiabatic}}{C_V} = -\frac{W_{adiabatic}}{\frac{5}{2}nR}$$

Substitute for ΔT to obtain:

$$T_f = T_i - \frac{W_{adiabatic}}{\frac{5}{2}nR}$$

Substitute numerical values and evaluate T_f for a diatomic gas:

$$T_f = 300\,\text{K} - \frac{3.50\,\text{kJ}}{\frac{5}{2}(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = \boxed{216\,\text{K}}$$

(*b*) Because a monatomic gas has only 3 degrees of freedom:

$$C_V = \tfrac{3}{2}nR$$

and

$$T_f = T_i - \frac{W_{\text{adiabatic}}}{\frac{3}{2}nR}$$

Substitute numerical values and evaluate T_f for a monatomic gas:

$$T_f = 300\,\text{K} - \frac{3.50\,\text{kJ}}{\frac{3}{2}(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = \boxed{160\,\text{K}}$$

93 •• A vertical insulated cylinder is divided into two parts by a movable piston of mass *m*. The piston is initially held at rest. The top part is evacuated and the bottom part is filled with 1.00 mol of diatomic ideal gas at temperature 300 K. After the piston is released and the system comes to equilibrium, the volume occupied by gas is halved. Find the final temperature of the gas.

Picture the Problem Let the subscripts 1 and 2 refer to the initial and final states in this adiabatic expansion. We can use an equation describing a quasi-static adiabatic process to express the final temperature as a function of the initial temperature and the initial and final volumes.

Using the equation for a quasi-static adiabatic process, relate the initial and final volumes and temperatures:

$$T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1}$$

Solve for T_2 and simplify to obtain:

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = T_1\left(\frac{V_1}{\frac{1}{2}V_1}\right)^{\gamma-1} = T_1(2)^{\gamma-1}$$

Substitute the numerical value for T_1 and evaluate T_2:

$$T_2 = (300\,\text{K})(2)^{1.4-1} = \boxed{396\,\text{K}}$$

94 ••• According to the Einstein model of a crystalline solid, the internal energy per mole is given by $U=\dfrac{3N_{\text{A}}kT_{\text{E}}}{e^{T_{\text{E}}/T}-1}$ where T_{E} is a characteristic temperature called the *Einstein temperature*, and T is the temperature of the solid in kelvins. Use this expression to show that a crystalline solid's molar heat capacity at constant volume is given by $c'_{\text{V}}=3R\left(\dfrac{T_E}{T}\right)^2\dfrac{e^{T_{\text{E}}/T}}{\left(e^{T_{\text{E}}/T}-1\right)^2}$.

Picture the Problem The molar heat capacity at constant volume is related to the internal energy per mole according to $c'_v=\dfrac{1}{n}\dfrac{dU}{dT}$. We can differentiate U with respect to temperature and use $nR = Nk$ or $R = N_{\text{A}}k$ to establish the result given in the problem statement.

From the Einstein model of a crystalline solid, the internal energy per mole is given by:

$$U=\frac{3N_{\text{A}}kT_{\text{E}}}{e^{T_{\text{E}}/T}-1}$$

Relate the molar heat capacity at constant volume to the internal energy per mol:

$$c'_{\text{V}}=\frac{1}{n}\frac{dU}{dT}$$

Use $c'_{\text{V}}=\dfrac{1}{n}\dfrac{dU}{dT}$ to express c'_{V}:

$$c'_{\text{V}}=\frac{1}{n}\frac{d}{dT}\left[\frac{3N_{\text{A}}kT_{\text{E}}}{e^{T_{\text{E}}/T}-1}\right]=3RT_{\text{E}}\frac{d}{dT}\left[\frac{1}{e^{T_{\text{E}}/T}-1}\right]=3RT_{\text{E}}\left[\frac{-1}{\left(e^{T_{\text{E}}/T}-1\right)^2}\right]\frac{d}{dT}\left(e^{T_{\text{E}}/T}-1\right)$$

$$=3RT_{\text{E}}\left[\frac{-1}{\left(e^{T_{\text{E}}/T}-1\right)^2}\right]\left[e^{T_{\text{E}}/T}\left(-\frac{T_E}{T^2}\right)\right]=\boxed{3R\left(\frac{T_E}{T}\right)^2\frac{e^{T_{\text{E}}/T}}{\left(e^{T_{\text{E}}/T}-1\right)^2}}$$

95 ••• [SSM] (*a*) Use the results of Problem 94 to show that in the limit that $T >> T_{\text{E}}$, the Einstein model gives the same expression for specific heat that the Dulong–Petit law does. (*b*) For diamond, T_{E} is approximately 1060 K. Integrate numerically $dE_{\text{int}}=c'_{\text{v}}\,dT$ to find the increase in the internal energy if 1.00 mol of diamond is heated from 300 to 600 K.

Picture the Problem (*a*) We can rewrite our expression for c'_{V} by dividing its numerator and denominator by $e^{T_{\mathrm{E}}/T}$ and then using the power series for e^x to show that, for $T > T_{\mathrm{E}}$, $c'_{\mathrm{V}} \approx 3R$. In Part (*b*), we can use the result of Problem 94 to obtain values for c'_{V} every 100 K between 300 K and 600 K and use this data to find ΔU numerically.

(*a*) From Problem 94 we have:

$$c'_{\mathrm{V}} = 3R\left(\frac{T_{\mathrm{E}}}{T}\right)^2 \frac{e^{T_{\mathrm{E}}/T}}{\left(e^{T_{\mathrm{E}}/T}-1\right)^2}$$

Divide the numerator and denominator by $e^{T_{\mathrm{E}}/T}$ to obtain:

$$c'_{\mathrm{V}} = 3R\left(\frac{T_{\mathrm{E}}}{T}\right)^2 \frac{1}{\dfrac{e^{2T_{\mathrm{E}}/T}-2e^{T_{\mathrm{E}}/T}+1}{e^{T_{\mathrm{E}}/T}}}$$

$$= 3R\left(\frac{T_{\mathrm{E}}}{T}\right)^2 \frac{1}{e^{T_{\mathrm{E}}/T}-2+e^{-T_{\mathrm{E}}/T}}$$

Express the exponential terms in their power series to obtain:

$$e^{T_{\mathrm{E}}/T}-2+e^{-T_{\mathrm{E}}/T} = 1+\frac{T_{\mathrm{E}}}{T}+\frac{1}{2}\left(\frac{T_{\mathrm{E}}}{T}\right)^2+\ldots-2+1-\frac{T_{\mathrm{E}}}{T}+\frac{1}{2}\left(\frac{T_{\mathrm{E}}}{T}\right)^2+\ldots$$

$$\approx \left(\frac{T_{\mathrm{E}}}{T}\right)^2 \text{ for } T >> T_{\mathrm{E}}$$

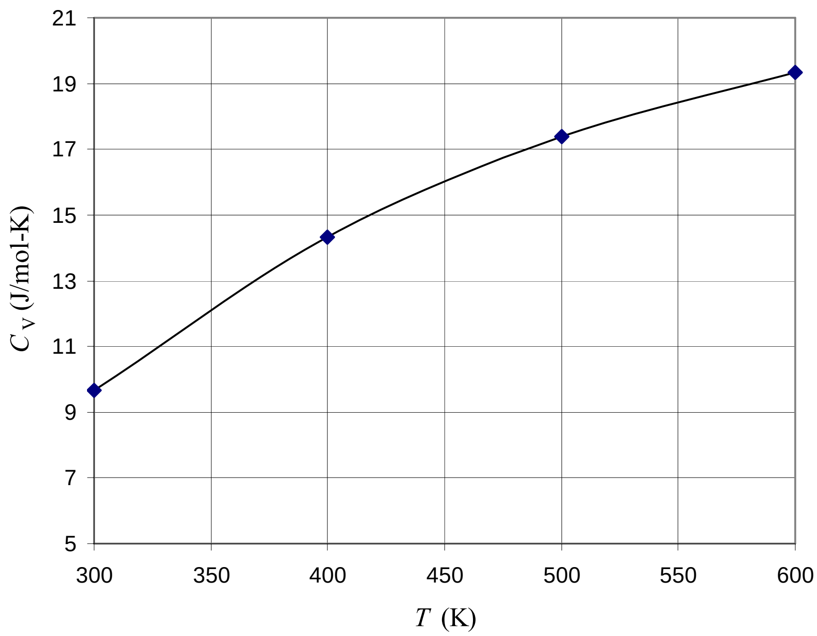
Substitute for $e^{T_{\mathrm{E}}/T}-2+e^{-T_{\mathrm{E}}/T}$ to obtain:

$$c'_{\mathrm{V}} \approx 3R\left(\frac{T_{\mathrm{E}}}{T}\right)^2 \frac{1}{\left(\dfrac{T_{\mathrm{E}}}{T}\right)^2} = \boxed{3R}$$

(*b*) Use the result of Problem 94 to verify the following table:

T	(K)	300	400	500	600
c_{V}	(J/mol·K)	9.65	14.33	17.38	19.35

The following graph of specific heat as a function of temperature was plotted using a spreadsheet program:



Integrate numerically, using the formula for the area of a trapezoid, to obtain:

$$\begin{aligned}\Delta U &= \tfrac{1}{2}(1.00\text{ mol})(100\text{ K})(9.65+14.33)\frac{\text{J}}{\text{mol}\cdot\text{K}}\\ &\quad+\tfrac{1}{2}(1.00\text{ mol})(100\text{ K})(14.33+17.38)\frac{\text{J}}{\text{mol}\cdot\text{K}}\\ &\quad+\tfrac{1}{2}(1.00\text{ mol})(100\text{ K})(17.38+19.35)\frac{\text{J}}{\text{mol}\cdot\text{K}}\\ &= \boxed{4.62\text{ kJ}}\end{aligned}$$

96 ••• Use the results of the Einstein model in Problem 94 to determine the molar internal energy of diamond (T_E = 1060 K) at 300 K and 600 K, and thereby the increase in internal energy as diamond is heated from 300 K to 600 K. Compare your result to that of Problem 95.

Picture the Problem We can simplify our calculations by relating Avogadro's number N_A, Boltzmann's constant k, the number of moles n, and the number of molecules N in the gas and solving for N_Ak. We can then calculate $U_{300\text{ K}}$, $U_{600\text{ K}}$, and their difference.

Express the increase in internal energy per mole resulting from the heating of diamond:

$$\Delta U = U_{600\,\text{K}} - U_{300\,\text{K}} \qquad (1)$$

Express the relationship between Avogadro's number N_A, Boltzmann's constant k, the number of moles n, and the number of molecules N in the gas:

$$nR = Nk \Rightarrow R = \frac{N}{n}k = N_A k$$

Substitute in the given equation to obtain:

$$U = \frac{3RT_E}{e^{T_E/T} - 1}$$

Determine $U_{300\,\text{K}}$:

$$U_{300\,\text{K}} = \frac{3\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(1060\,\text{K})}{e^{1060\,\text{K}/300\,\text{K}} - 1} = 795.4\,\text{J} = \boxed{795\,\text{J}}$$

Determine $U_{600\,\text{K}}$:

$$U_{600\,\text{K}} = \frac{3\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(1060\,\text{K})}{e^{1060\,\text{K}/600\,\text{K}} - 1} = 5.4498\,\text{kJ} = \boxed{5.45\,\text{kJ}}$$

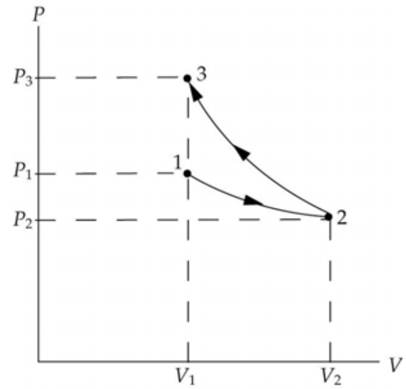
Substituting in equation (1) yields:

$$\Delta U = U_{600\,\text{K}} - U_{300\,\text{K}} = 5.4498\,\text{kJ} - 795.4\,\text{J} = \boxed{4.65\,\text{kJ}}$$

This result agrees with the result of Problem 95 to within 1%.

97 ••• During an isothermal expansion, an ideal gas at an initial pressure P_0 expands until its volume is twice its initial volume V_0. (*a*) Find its pressure after the expansion. (*b*) The gas is then compressed adiabatically and quasi-statically until its volume is V_0 and its pressure is $1.32P_0$. Is the gas monatomic, diatomic, or polyatomic? (*c*) How does the translational kinetic energy of the gas change in each stage of this process?

Picture the Problem The isothermal expansion followed by an adiabatic compression is shown on the PV diagram. The path 1→2 is isothermal and the path 2→3 is adiabatic. We can apply the ideal-gas law for a fixed amount of gas and an isothermal process to find the pressure at point 2 and the pressure-volume relationship for a quasi-static adiabatic process to determine γ.



(*a*) Relate the initial and final pressures and volumes for the isothermal expansion and solve for and evaluate the final pressure:

$$P_1V_1 = P_2V_2$$

and

$$P_2 = P_1\frac{V_1}{V_2} = P_0\frac{V_1}{2V_1} = \boxed{\tfrac{1}{2}P_0}$$

(*b*) Relate the initial and final pressures and volumes for the adiabatic compression:

$$P_2V_2^{\gamma} = P_3V_3^{\gamma}$$

or

$$\tfrac{1}{2}P_0(2V_0)^{\gamma} = 1.32P_0V_0^{\gamma}$$

which simplifies to $2^{\gamma} = 2.64$

Take the natural logarithm of both sides of this equation and solve for and evaluate γ:

$$\gamma\ln 2 = \ln 2.64 \Rightarrow \gamma = \frac{\ln 2.64}{\ln 2} = 1.40$$

and the gas is $\boxed{\text{diatomic}}$.

(*c*) During the isothermal process, *T* is constant and the translational kinetic energy is unchanged. During the adiabatic process, the translational kinetic energy increases by a factor of 1.32.

98 ••• If a hole is punctured in a tire, the gas inside will gradually leak out. Assume the following: the area of the hole is *A*; the tire volume is *V*; and the time, τ, it takes for most of the air to leak out of the tire can be expressed in terms of the ratio A/V, the temperature *T*, the Boltzmann constant *k*, and the initial mass *m* of the gas inside the tire. (*a*) Based on these assumptions, use dimensional analysis to find an estimate for τ. (*b*) Use the result of Part (*a*) to estimate the time it takes for a car tire with a nail hole punched in it to go flat.

Picture the Problem In (*a*) we'll assume that $\tau = f(A/V, T, k, m)$ with the factors dependent on constants *a, b, c,* and *d* that we'll find using dimensional analysis. In (*b*) we'll use our result from (*a*) and assume that the diameter of the puncture is about 2 mm, that the tire volume is 0.1 m^3, and that the air temperature is 20°C.

(*a*) Express $\tau = f(A/V, T, k, m)$:

$$\tau = \left(\frac{A}{V}\right)^a (T)^b (k)^c (m)^d \qquad (1)$$

Rewrite this equation in terms of the dimensions of the physical quantities to obtain:

$$\mathrm{T}^1 = (\mathrm{L})^{-a}(\mathrm{K})^b\left(\frac{\mathrm{ML}^2}{\mathrm{T}^2\mathrm{K}}\right)^c(\mathrm{M})^d$$

where K represents the dimension of temperature.

Simplify this dimensional equation to obtain:

$$\mathrm{T}^1 = \mathrm{L}^{-a}\mathrm{K}^b\mathrm{M}^c\mathrm{L}^{2c}\mathrm{K}^{-c}\mathrm{T}^{-2c}\mathrm{M}^d$$

or

$$\mathrm{T}^1 = \mathrm{L}^{2c-a}\mathrm{K}^{b-c}\mathrm{M}^{c+d}\mathrm{T}^{-2c}$$

Equate exponents to obtain:

$\mathrm{T}:\ -2c = 1$,
$\mathrm{L}:\ 2c - a = 0$,
$\mathrm{K}:\ b - c = 0$,
and
$\mathrm{M}:\ c + d = 0$

Solve these equations simultaneously to obtain:

$a = -1$, $b = -\frac{1}{2}$, $c = -\frac{1}{2}$, and $d = \frac{1}{2}$

Substituting in equation (1) yields:

$$\tau = \left(\frac{A}{V}\right)^{-1}(T)^{-\frac{1}{2}}(k)^{-\frac{1}{2}}(m)^{\frac{1}{2}} = \boxed{\frac{V}{A}\sqrt{\frac{m}{kT}}}$$

(*b*) Substitute numerical values and evaluate τ:

$$\tau = \frac{0.1\,\mathrm{m}^3}{\frac{\pi}{4}\left(2\times10^{-3}\,\mathrm{m}\right)^2}\sqrt{\frac{\left(1.293\,\mathrm{kg/m}^3\right)\left(0.1\,\mathrm{m}^3\right)}{\left(1.381\,\mathrm{J/K}\right)\left(293\,\mathrm{K}\right)}} = 569\,\mathrm{s} \approx \boxed{9\,\mathrm{min}}$$

Chapter 19
The Second Law of Thermodynamics

Conceptual Problems

1 • Modern automobile gasoline engines have efficiencies of about 25%. About what percentage of the heat of combustion is not used for work but released as heat? (*a*) 25%, (*b*) 50%, (*c*) 75%, (*d*) 100%, (*e*) You cannot tell from the data given.

Determine the Concept The efficiency of a heat engine is the ratio of the work done per cycle W to the heat absorbed from the high-temperature reservoir Q_h. The percentage of the heat of combustion (heat absorbed from the high-temperature reservoir) is the ratio of Q_c to Q_h. We can use the relationship between W, Q_h, and Q_c ($W = Q_h - Q_c$) to find Q_c/Q_h.

Use the definition of efficiency and the relationship between W, Q_h, and Q_c to obtain:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$

Solving for Q_c/Q_h yields:

$$\frac{Q_c}{Q_h} = 1 - \varepsilon$$

Substitute for ε to obtain:

$$\frac{Q_c}{Q_h} = 1 - 0.25 = 0.75$$

and $\boxed{(c)}$ is correct.

2 • If a heat engine does 100 kJ of work per cycle while releasing 400 kJ of heat, what is its efficiency? (*a*) 20%, (*b*) 25%, (*c*) 80%, (*d*) 400%, (*e*) You cannot tell from the data given.

Determine the Concept The efficiency of a heat engine is the ratio of the work done per cycle W to the heat absorbed from the high-temperature reservoir Q_h. We can use the relationship between W, Q_h, and Q_c ($W = Q_h - Q_c$) to express the efficiency of the heat engine in terms of Q_c and W.

Use the definition of efficiency and the relationship between W, Q_h, and Q_c to obtain:

$$\varepsilon = \frac{W}{Q_h} = \frac{W}{W + Q_c} = \frac{1}{1 + \frac{Q_c}{W}}$$

Substitute for Q_c and W to obtain:

$$\varepsilon = \frac{1}{1 + \frac{400\text{ kJ}}{100\text{ kJ}}} = 0.2$$

and $\boxed{(a)}$ is correct.

3 • If the heat absorbed by a heat engine is 600 kJ per cycle, and it releases 480 kJ of heat each cycle, what is its efficiency? (*a*) 20%, (*b*) 80%, (*c*) 100%, (*d*) You cannot tell from the data given.

Determine the Concept The efficiency of a heat engine is the ratio of the work done per cycle W to the heat absorbed from the high-temperature reservoir Q_h. We can use the relationship between W, Q_h, and Q_c ($W = Q_h - Q_c$) to express the efficiency of the heat engine in terms of Q_c and Q_h.

Use the definition of efficiency and the relationship between W, Q_h, and Q_c to obtain:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$

Substitute for Q_c and Q_h to obtain:

$$\varepsilon = 1 - \frac{480\text{ kJ}}{600\text{ kJ}} = 0.2$$

and $\boxed{(a)}$ is correct.

4 • Explain what distinguishes a refrigerator from a ″heat pump.″

Determine the Concept The job of a refrigerator is to move heat from its cold interior to the warmer kitchen environment. This process moves heat in a direction that is opposite its ″natural″ direction of flow, analogous to the use of a water pump to pump water out of a boat. The term heat pumps is used to describe devices, such as air conditioners, that are used to cool living and working spaces in the summer and warm them in the winter.

5 • **[SSM]** An air conditioner's COP is mathematically identical to that of a refrigerator, that is, $\text{COP}_{\text{AC}} = \text{COP}_{\text{ref}} = \frac{Q_c}{W}$. However a heat pump's COP is defined differently, as $\text{COP}_{\text{hp}} = \frac{Q_h}{W}$. Explain clearly *why* the two COPs are defined differently. *Hint*: *Think of the end use of the three different devices.*

Determine the Concept The COP is defined so as to be a measure of the

effectiveness of the device. For a refrigerator or air conditioner, the important quantity is the heat drawn from the already colder interior, Q_c. For a heat pump, the ideas is to focus on the heat drawn into the warm interior of the house, Q_h.

6 • Explain why you cannot cool your kitchen by leaving your refrigerator door open on a hot day. (Why does turning on a room air conditioner cool down the room, but opening a refrigerator door does not?)

Determine the Concept As described by the second law of thermodynamics, more heat must be transmitted to the outside world than is removed by a refrigerator or air conditioner. The heating coils on a refrigerator are inside the room, so the refrigerator actually heats the room in which it is located. The heating coils on an air conditioner are outside one's living space, so the waste heat is vented to the outside.

7 • Why do steam-power-plant designers try to increase the temperature of the steam as much as possible?

Determine the Concept Increasing the temperature of the steam increases its energy content. In addition, it increases the Carnot efficiency, and generally increases the efficiency of any heat engine.

8 • To increase the efficiency of a Carnot engine, you should (*a*) decrease the temperature of the hot reservoir, (*b*) increase the temperature of the cold reservoir, (*c*) increase the temperature of the hot reservoir, (*d*) change the ratio of maximum volume to minimum volume.

Determine the Concept Because the efficiency of a Carnot cycle engine is given by $\varepsilon_C = 1 - \frac{T_c}{T_h}$, you should increase the temperature of the hot reservoir. $\boxed{(c)}$ is correct.

9 •• **[SSM]** Explain why the following statement is true: To increase the efficiency of a Carnot engine, you should make the difference between the two operating temperatures as large as possible; but to increase the efficiency of a Carnot cycle *refrigerator*, you should make the difference between the two operating temperatures as small as possible.

Determine the Concept A Carnot-cycle refrigerator is more efficient when the temperatures are close together because it requires less work to extract heat from an already cold interior if the temperature of the exterior is close to the temperature of the interior of the refrigerator. A Carnot-cycle heat engine is more efficient when the temperature difference is large because then more work is done by the engine for each unit of heat absorbed from the hot reservoir.

10 •• A Carnot engine operates between a cold temperature reservoir of

27°C and a high temperature reservoir of 127°C. Its efficiency is (*a*) 21%, (*b*) 25%, (*c*) 75%, (*d*) 79%.

Determine the Concept The efficiency of a Carnot cycle engine is given by $\varepsilon_{\mathrm{C}} = 1 - T_{\mathrm{c}}/T_{\mathrm{h}}$ where T_c and T_h (in kelvins) are the temperatures of the cold and hot reservoirs, respectively.

Substituting numerical values for T_c and T_h yields:

$$\varepsilon_{\mathrm{C}} = 1 - \frac{300\,\mathrm{K}}{400\,\mathrm{K}} = 0.25$$

$\boxed{(b)}$ is correct.

11 •• The Carnot engine in Problem 10 is run in reverse as a refrigerator. Its COP is (*a*) 0.33, (*b*) 1.3, (*c*) 3.0 (*d*) 4.7.

Determine the Concept The coefficient of performance of a Carnot cycle engine run in reverse as refrigerator is given by $\mathrm{COP}_{\mathrm{ref}} = \dfrac{Q_{\mathrm{c}}}{W}$. We can use the relationship between W, Q_c, and Q_h to eliminate W from this expression and then use the relationship, applicable only to a device operating in a Carnot cycle, $\dfrac{Q_{\mathrm{c}}}{Q_{\mathrm{h}}} = \dfrac{T_{\mathrm{c}}}{T_{\mathrm{h}}}$ to express the refrigerator's COP in terms of T_c and T_h.

The coefficient of performance of a refrigerator is given by:

$$\mathrm{COP}_{\mathrm{ref}} = \frac{Q_{\mathrm{c}}}{W}$$

or, because $W = Q_{\mathrm{h}} - Q_{\mathrm{c}}$,

$$\mathrm{COP}_{\mathrm{ref}} = \frac{Q_{\mathrm{c}}}{Q_{\mathrm{h}} - Q_{\mathrm{c}}}$$

Dividing the numerator and denominator of this fraction by Q_c yields:

$$\mathrm{COP}_{\mathrm{ref}} = \frac{1}{\dfrac{Q_{\mathrm{h}}}{Q_{\mathrm{c}}} - 1}$$

For a device operating in a Carnot cycle:

$$\frac{Q_{\mathrm{c}}}{Q_{\mathrm{h}}} = \frac{T_{\mathrm{c}}}{T_{\mathrm{h}}}$$

Substitute in the expression for $\mathrm{COP}_{\mathrm{ref}}$ to obtain:

$$\mathrm{COP}_{\mathrm{ref,C}} = \frac{1}{\dfrac{T_{\mathrm{h}}}{T_{\mathrm{c}}} - 1}$$

Substitute numerical values and evaluate $COP_{ref,\,C}$:

$$COP_{ref,\,C} = \frac{1}{\dfrac{400\text{ K}}{300\text{ K}} - 1} = 3.0$$

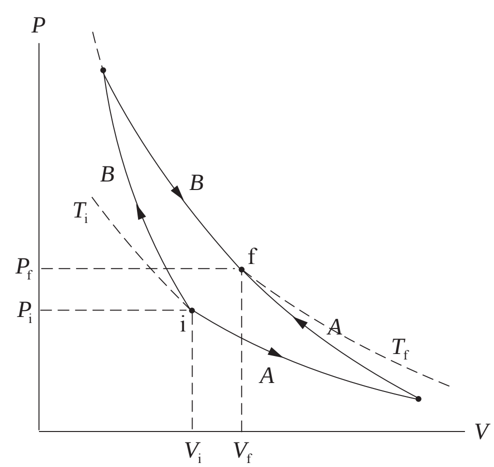
$\boxed{(c)}$ is correct.

12 •• On a humid day, water vapor condenses on a cold surface. During condensation, the entropy of the water (*a*) increases, (*b*) remains constant, (*c*) decreases, (*d*) may decrease or remain unchanged. Explain your answer.

Determine the Concept When water vapor condenses, its entropy decreases (the liquid state is a more ordered state than is the vapor state). $\boxed{(c)}$ is correct.

13 •• An ideal gas is taken reversibly from an initial state P_i, V_i, T_i to the final state P_f, V_f, T_f. Two possible paths are (A) an isothermal expansion followed by an adiabatic compression and (B) an adiabatic compression followed by an isothermal expansion. For these two paths, (*a*) $\Delta E_{int\,A} > \Delta E_{int\,B}$, (*b*) $\Delta S_A > \Delta S_B$, (*c*) $\Delta S_A < \Delta S_B$, (*d*) None of the above.

Determine the Concept The two paths are shown on the *PV* diagram to the right. We can use the concept of a state function to choose from among the alternatives given as possible answers to the problem.



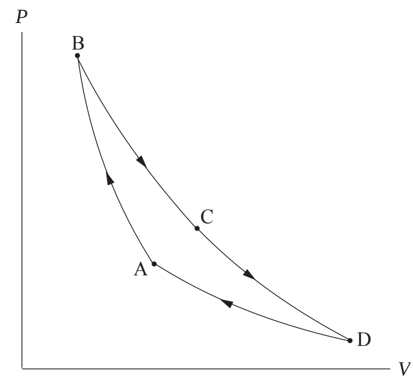
(*a*) Because E_{int} is a state function and the initial and final states are the same for the two paths, $\Delta E_{int,\,A} = \Delta E_{int,\,B}$.

(*b*) and (*c*) *S*, like E_{int}, is a state function and its change when the system moves from one state to another depends only on the system's initial and final states. It is not dependent on the process by which the change occurs. Thus $\Delta S_A = \Delta S_B$.

(*d*) $\boxed{(d)}$ is correct.

14 •• Figure 19-12 shows a thermodynamic cycle for an ideal gas on an *ST* diagram. Identify this cycle and sketch it on a *PV* diagram.

Determine the Concept The processes A→B and C→D are adiabatic and the processes B→C and D→A are isothermal. Therefore, the cycle is the Carnot cycle shown in the adjacent *PV* diagram.



15 •• Figure 19-13 shows a thermodynamic cycle for an ideal gas on an *SV* diagram. Identify the type of engine represented by this diagram.

Determine the Concept Note that A→B is an adiabatic expansion, B→C is a constant-volume process in which the entropy decreases, C→D is an adiabatic compression and D→A is a constant-volume process that returns the gas to its original state. The cycle is that of the Otto engine (see Figure 19-3). The points A, B, C, and D in Figure 19-13 correspond to points *c, d, a,* and *b,* respectively, in Figure 19-3.

16 •• Sketch an *ST* diagram of the Otto cycle. (The Otto cycle is discussed in Section 19-1.)

Determine the Concept The Otto cycle consists of four quasi-static steps. Refer to Figure 19-3. There $a \to b$ is an adiabatic compression, $b \to c$ is a constant volume heating, $c \to d$ is an adiabatic expansion and $d \to a$ is a constant-volume cooling. So, from a to b, S is constant and T increases, from b to c, heat is added to the system and both S and T increase, from $c \to d$ S is constant while T decreases, and from d to a both S and T decrease.

To determine how S depends on T along $b \to c$ and $d \to a$, consider the entropy change of the gas from point b to an arbitrary point on the path $b \to c$ where the entropy and temperature of the gas are S and T, respectively:

$$\Delta S = \frac{Q}{T}$$

where, because heat is entering the system, Q is positive.

Because $W_{\text{on}} = 0$ for this constant-volume process:

$$\Delta E_{\text{int}} = Q_{\text{in}} = Q = C_{\text{V}} \Delta T = C_{\text{V}}(T - T_b)$$

Substituting for Q yields:

$$\Delta S = \frac{C_{\text{V}}(T - T_b)}{T} = C_{\text{V}}\left(1 - \frac{T_b}{T}\right)$$

On path $b \rightarrow c$ the entropy is given by:

$$S = S_b + \Delta S = S_b + C_V\left(1 - \frac{T_b}{T}\right)$$

The first and second derivatives, dS/dT and d^2S/dT^2, give the slope and concavity of the path. Calculate these derivatives assuming C_V is constant. (For an ideal gas C_V is a positive constant.):

$$\frac{dS}{dT} = C_V \frac{T_b}{T^2}$$

$$\frac{d^2S}{dT^2} = -2C_V \frac{T_b}{T^3}$$

These results tell us that, along path $b \rightarrow c$, the slope of the path is positive and the slope decreases as T increases. The concavity of the path is negative for all T.

Following the same procedure on path $d \rightarrow a$ gives:

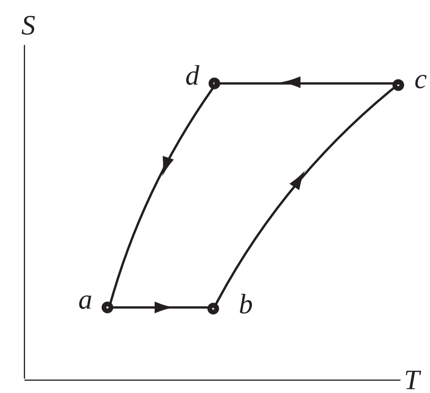
$$S = S_d + C_V\left(1 - \frac{T_d}{T}\right)$$

$$\frac{dS}{dT} = C_V \frac{T_d}{T^2}$$

$$\frac{d^2S}{dT^2} = -2C_V \frac{T_d}{T^3}$$

These results tell us that, along path $d \rightarrow a$, the slope of the path is positive and the slope decreases as T increases. The concavity of the path is negative for all T.

An ST diagram for the Otto cycle is shown to the right.



17 •• **[SSM]** Sketch an SV diagram of the Carnot cycle for an ideal gas.

Determine the Concept Referring to Figure 19-8, process $1 \rightarrow 2$ is an isothermal expansion. In this process heat is added to the system and the entropy and volume increase. Process $2 \rightarrow 3$ is adiabatic, so S is constant as V increases. Process $3 \rightarrow 4$ is an isothermal compression in which S decreases and V also decreases. Finally, process $4 \rightarrow 1$ is adiabatic, that is, isentropic, and S is constant while V decreases.

During the isothermal expansion (from point 1 to point 2) the work done by the gas equals the heat added to the gas. The change in entropy of the gas from point 1 (where the temperature is T_1) to an arbitrary point on the curve is given by:

$$\Delta S = \frac{Q}{T_1}$$

For an isothermal expansion, the work done by the gas, and thus the heat added to the gas, are given by:

$$Q = W = nRT_1 \ln\left(\frac{V}{V_1}\right)$$

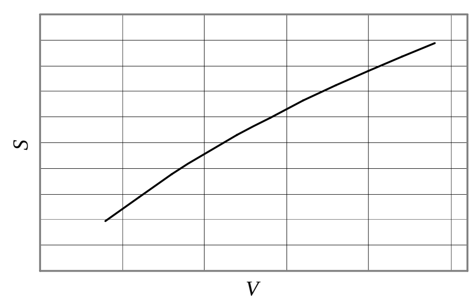
Substituting for Q yields:

$$\Delta S = nR \ln\left(\frac{V}{V_1}\right)$$

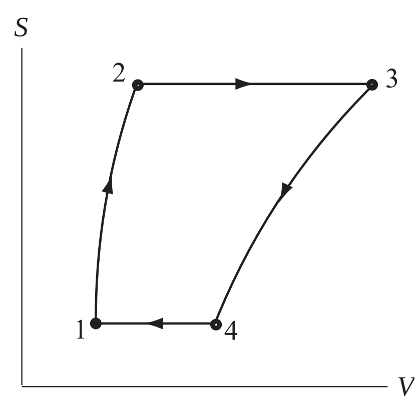
Since $S = S_1 + \Delta S$, we have:

$$S = S_1 + nR \ln\left(\frac{V}{V_1}\right)$$

The graph of S as a function of V for an isothermal expansion shown to the right was plotted using a spreadsheet program. This graph establishes the curvature of the 1→2 and 3→4 paths for the SV graph.

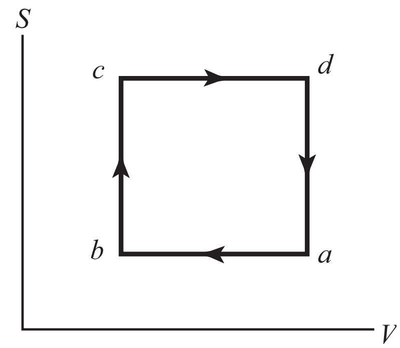


An SV graph for the Carnot cycle (see Figure 19-8) is shown to the right.



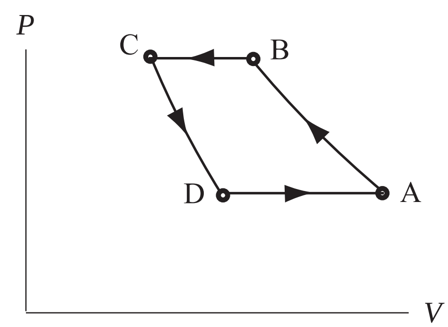
18 •• Sketch an SV diagram of the Otto cycle. (The Otto cycle is discussed in Section 19-1.)

Determine the Concept The Otto cycle is shown in Figure 19-3. Process $a\to b$ takes place adiabatically and so both $Q = 0$ and $\Delta S = 0$ along this path. Process $b\to c$ takes place at constant volume. Q_{in}, however, is positive and so, while $\Delta V = 0$ along this path, $Q > 0$ and, therefore $\Delta S > 0$. Process $c\to d$ also takes place adiabatically and so, again, both $Q = 0$ and $\Delta S = 0$ along this path. Finally, process $d\to a$ is a constant-volume process, this time with heat leaving the system and $\Delta S < 0$. A sketch of the *SV* diagram for the Otto cycle follows:



19 •• Figure 19-14 shows a thermodynamic cycle for an ideal gas on an *SP* diagram. Make a sketch of this cycle on a *PV* diagram.

Determine the Concept Process A→B is at constant entropy; that is, it is an adiabatic process in which the pressure increases. Process B→C is one in which *P* is constant and *S* decreases; heat is exhausted from the system and the volume decreases. Process C→D is an adiabatic compression. Process D→A returns the system to its original state at constant pressure. The cycle is shown in the adjacent *PV* diagram.



20 •• One afternoon, the mother of one of your friends walks into his room and finds a mess. She asks your friend how the room came to be in such a state, and your friend replies, ″Well, it is the natural destiny of any closed system to degenerate toward greater and greater levels of entropy. That’s all, Mom.″ Her reply is a sharp ″Nevertheless, you’d better clean your room!″ Your friend retorts, ″But that can’t happen. It would violate the second law of thermodynamics.″ Critique your friend’s response. Is his mother correct to ground him for not cleaning his room, or is cleaning the room really impossible?

Determine the Concept The son is out of line, here, but besides that, he’s also wrong. While it is true that systems tend to degenerate to greater levels of disorder, it is not true that order cannot be brought forth from disorder. What is

required is an agent doing work – for example, your friend – on the system in order to reduce the level of chaos and bring about order. His cleanup efforts will be rewarded with an orderly system after a sufficient time for him to complete the task. It is true that order will not come about from the disordered chaos of his room – unless he applies some elbow grease.

Estimation and Approximation

21 • Estimate the change in COP of your electric food freezer when it is removed from your kitchen to its new location in your basement, which is 8°C cooler than your kitchen.

Picture the Problem We can use the definition of the coefficient of performance to express the ratio of the coefficient of performance in your basement to the coefficient of performance in the kitchen. If we further assume that the freezer operates in a Carnot cycle, then we can use the proportion $Q_h/Q_c = T_h/T_c$ to express the ratio of the coefficients of performance in terms of the temperatures in the kitchen, basement, and freezer.

The ratio of the coefficients of performance in the basement and kitchen is given by:

$$\frac{\mathrm{COP_{basement}}}{\mathrm{COP_{kit}}} = \frac{\dfrac{Q_{c,basement}}{W_{c,basement}}}{\dfrac{Q_{c,kit}}{W_{c,kit}}}$$

Because $W = Q_h - Q_c$ for a heat engine or refrigerator:

$$\frac{\mathrm{COP_{basement}}}{\mathrm{COP_{kit}}} = \frac{\dfrac{Q_{c,basement}}{Q_{h,basement} - Q_{c,basement}}}{\dfrac{Q_{c,kit}}{Q_{h,kit} - Q_{c,kit}}}$$

Divide the numerators and denominators by $Q_{c,basement}$ and $Q_{c,kit}$ and simplify to obtain:

$$\frac{\mathrm{COP_{basement}}}{\mathrm{COP_{kit}}} = \frac{\dfrac{1}{\dfrac{Q_{h,basement}}{Q_{c,basement}} - 1}}{\dfrac{1}{\dfrac{Q_{h,kit}}{Q_{c,kit}} - 1}} = \frac{\dfrac{Q_{h,kit}}{Q_{c,kit}} - 1}{\dfrac{Q_{h,basement}}{Q_{c,basement}} - 1}$$

If we assume that the freezer unit operates in a Carnot cycle, then $\frac{Q_h}{Q_c}=\frac{T_h}{T_c}$ and our expression for the ratio of the COPs becomes:

$$\frac{\text{COP}_{\text{basement}}}{\text{COP}_{\text{kit}}}=\frac{\dfrac{T_{\text{h,kit}}}{T_{\text{c,kit}}}-1}{\dfrac{T_{\text{h,basement}}}{T_{\text{c,basement}}}-1}$$

Assuming that the temperature in your kitchen is 20°C and that the temperature of the interior of your freezer is −5°C, substitute numerical values and evaluate the ratio of the coefficients of performance:

$$\frac{\text{COP}_{\text{basement}}}{\text{COP}_{\text{kit}}}=\frac{\dfrac{293\ \text{K}}{268\ \text{K}}-1}{\dfrac{285\ \text{K}}{268\ \text{K}}-1}=1.47$$

or an increase of $\boxed{47\%}$ in the performance of the freezer!

22 •• Estimate the probability that all the molecules in your bedroom are located in the (open) closet which accounts for about 10% of the total volume of the room.

Picture the Problem The probability that all the molecules in your bedroom are located in the (open) closet is given by $p=\left(\frac{V_2}{V_1}\right)^N$ where N is the number of air molecules in your bedroom and V_1 and V_2 are the volumes of your bedroom and closet, respectively. We can use the ideal-gas law to find the number of molecules N. We'll assume that the volume of your room is about 50 m^3 and that the temperature of the air is 20°C.

If the original volume of the air in your bedroom is V_1, the probability p of finding the N molecules, normally in your bedroom, confined to your closet whose volume is V_2 is given by:

$$p=\left(\frac{V_2}{V_1}\right)^N$$

or, because $V_2=\frac{1}{10}V_1$,

$$p=\left(\frac{1}{10}\right)^N \qquad (1)$$

Use the ideal-gas law to express N:

$$N=\frac{PV}{kT}$$

Substitute numerical values and evaluate N:

$$N=\frac{(101.325\ \text{kPa})(50\ \text{m}^3)}{(1.381\times10^{-23}\ \text{J/K})(293\ \text{K})}=1.252\times10^{27}\ \text{molecules}$$

Substitute for N in equation (1) and evaluate p:

$$p = \left(\frac{1}{10}\right)^{1.252\times10^{27}} = \frac{1}{10^{1.252\times10^{27}}} = 10^{-1.252\times10^{27}} \approx \boxed{10^{-10^{27}}}$$

23 •• **[SSM]** Estimate the maximum efficiency of an automobile engine that has a compression ratio of 8.0:1.0. Assume the engine operates according to the Otto cycle and assume $\gamma = 1.4$. (The Otto cycle is discussed in Section 19-1.)

Picture the Problem The maximum efficiency of an automobile engine is given by the efficiency of a Carnot engine operating between the same two temperatures. We can use the expression for the Carnot efficiency and the equation relating V and T for a quasi-static adiabatic expansion to express the Carnot efficiency of the engine in terms of its compression ratio.

Express the Carnot efficiency of an engine operating between the temperatures T_c and T_h:

$$\varepsilon_C = 1 - \frac{T_c}{T_h}$$

Relate the temperatures T_c and T_h to the volumes V_c and V_h for a quasi-static adiabatic compression from V_c to V_h:

$$T_c V_c^{\gamma-1} = T_h V_h^{\gamma-1} \Rightarrow \frac{T_c}{T_h} = \frac{V_h^{\gamma-1}}{V_c^{\gamma-1}} = \left(\frac{V_h}{V_c}\right)^{\gamma-1}$$

Substitute for $\frac{T_c}{T_h}$ to obtain:

$$\varepsilon_C = 1 - \left(\frac{V_h}{V_c}\right)^{\gamma-1}$$

Express the compression ratio r:

$$r = \frac{V_c}{V_h}$$

Substituting for r yields:

$$\varepsilon_C = 1 - \frac{1}{r^{\gamma-1}}$$

Substitute numerical values for r and γ (1.4 for diatomic gases) and evaluate ε_C:

$$\varepsilon_C = 1 - \frac{1}{(8.0)^{1.4-1}} \approx \boxed{56\%}$$

24 •• You are working as an appliance salesperson during the summer. One day, your physics professor comes into your store to buy a new refrigerator. Wanting to buy the most efficient refrigerator possible, she asks you about the efficiencies of the available models. She decides to return the next day to buy the most efficient refrigerator. To make the sale, you need to provide her with the following estimates: (*a*) the highest COP possible for a household refrigerator, and (*b*) and the highest rate possible for the heat to be released by the refrigerator if the refrigerator uses 600 W of electrical power.

Picture the Problem If we assume that the temperature on the inside of the refrigerator is 0°C (273 K) and the room temperature to be 20°C (293 K), then the refrigerator must be able to maintain a temperature difference of 20 K. We can use the definition of the COP of a refrigerator and the relationship between the temperatures of the hot and cold reservoir and Q_h and Q_c to find an upper limit on the COP of a household refrigerator. In (*b*) we can solve the definition of COP for Q_c and differentiate the resulting equation with respect to time to estimate the rate at which heat is being drawn from the refrigerator compartment.

(*a*) Using its definition, express the COP of a household refrigerator:

$$\text{COP} = \frac{Q_c}{W} \tag{1}$$

Apply conservation of energy to the refrigerator to obtain:

$$W + Q_c = Q_h \Rightarrow W = Q_h - Q_c$$

Substitute for W and simplify to obtain:

$$\text{COP} = \frac{Q_c}{Q_h - Q_c} = \frac{1}{\dfrac{Q_h}{Q_c} - 1}$$

Assume, for the sake of finding the upper limit on the COP, that the refrigerator is a Carnot refrigerator and relate the temperatures of the hot and cold reservoirs to Q_h and Q_c:

$$\frac{Q_h}{Q_c} = \frac{T_h}{T_c}$$

Substitute for $\dfrac{Q_h}{Q_c}$ to obtain:

$$\text{COP}_{max} = \frac{1}{\dfrac{T_h}{T_c} - 1}$$

Substitute numerical values and evaluate COP_{max}:

$$\text{COP}_{max} = \frac{1}{\dfrac{293\,\text{K}}{273\,\text{K}} - 1} = 13.65 \approx \boxed{14}$$

(*b*) Solve equation (1) for Q_c:

$$Q_c = W(\text{COP}) \tag{2}$$

Differentiate equation (2) with respect to time to obtain:

$$\frac{dQ_c}{dt} = (\text{COP})\frac{dW}{dt}$$

Substitute numerical values and evaluate $\dfrac{dQ_c}{dt}$:

$$\frac{dQ_c}{dt} = (13.65)(600\,\text{J/s}) = \boxed{8.2\,\text{kW}}$$

25 •• [SSM] The average temperature of the surface of the Sun is about 5400 K, the average temperature of the surface of Earth is about 290 K. The solar constant (the intensity of sunlight reaching Earth's atmosphere) is about 1.37 kW/m^2. (*a*) Estimate the total power of the sunlight hitting Earth. (*b*) Estimate the net rate at which Earth's entropy is increasing due to this solar radiation.

Picture the Problem We can use the definition of intensity to find the total power of sunlight hitting Earth and the definition of the change in entropy to find the changes in the entropy of Earth and the Sun resulting from the radiation from the Sun.

(*a*) Using its definition, express the intensity of the Sun's radiation on Earth in terms of the power P delivered to Earth and Earth's cross sectional area A:

$$I = \frac{P}{A}$$

Solve for P and substitute for A to obtain:

$$P = IA = I\pi R^2$$

where R is the radius of Earth.

Substitute numerical values and evaluate P:

$$P = \pi\left(1.37\,\text{kW/m}^2\right)\left(6.37\times10^6\,\text{m}\right)^2 = 1.746\times10^{17}\,\text{W} = \boxed{1.75\times10^{17}\,\text{W}}$$

(*b*) Express the rate at which Earth's entropy S_{Earth} changes due to the flow of solar radiation:

$$\frac{dS_{\text{Earth}}}{dt} = \frac{P}{T_{\text{Earth}}}$$

Substitute numerical values and evaluate $\frac{dS_{\text{Earth}}}{dt}$:

$$\frac{dS_{\text{Earth}}}{dt} = \frac{1.746\times10^{17}\,\text{W}}{290\,\text{K}} = \boxed{6.02\times10^{14}\,\text{J/K}\cdot\text{s}}$$

26 •• A 1.0-L box contains N molecules of an ideal gas, and the positions of the molecules are observed 100 times per second. Calculate the average time it should take before we observe all N molecules in the left half of the box if N is equal to (*a*) 10, (*b*) 100, (*c*) 1000, and (*d*) 1.0 mole. (*e*) The best vacuums that have been created to date have pressures of about 10^{-12} torr. If a vacuum chamber has the same volume as the box, how long will a physicist have to wait before all of the gas molecules in the vacuum chamber occupy only the left half of it? Compare that to the expected lifetime of the universe, which is about 10^{10} years.

Picture the Problem If you had one molecule in a box, it would have a 50% chance of being on one side or the other. We don't care which side the molecules are on as long as they all are on one side, so with one molecule you have a 100%

chance of it being on one side or the other. With two molecules, there are four possible combinations (both on one side, both on the other, one on one side and one on the other, and the reverse), so there is a 25% (1 in 4) chance of them both being on a particular side, or a 50% chance of them both being on either side. Extending this logic, the probability of N molecules all being on one side of the box is $P = 2/2^N$, which means that, if the molecules shuffle 100 times a second, the time it would take them to cover all the combinations and all get on one side or the other is $t = \frac{2^N}{2(100)}$. In (e) we can apply the ideal gas law to find the number of molecules in 1.0 L of air at a pressure of 10^{-12} torr and an assumed temperature of 300 K.

(a) Evaluate t for $N = 10$ molecules:

$$t = \frac{2^{10}}{2(100\ \text{s}^{-1})} = 5.12\ \text{s} \approx \boxed{5\ \text{s}}$$

(b) Evaluate t for $N = 100$ molecules:

$$t = \frac{2^{100}}{2(100\ \text{s}^{-1})} = 6.34\times10^{27}\ \text{s}\times\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}} \approx \boxed{2\times10^{20}\ \text{y}}$$

(c) Evaluate t for $N = 1000$ molecules:

$$t = \frac{2^{1000}}{2(100\ \text{s}^{-1})}$$

To evaluate 2^{1000} let $10^x = 2^{1000}$ and take the logarithm of both sides of the equation to obtain:

$$(1000)\ln 2 = x\ln 10 \Rightarrow x = 301$$

Substitute to obtain:

$$t = \frac{10^{301}}{2(100\ \text{s}^{-1})} = 0.5\times10^{299}\ \text{s}\times\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}} \approx \boxed{2\times10^{291}\ \text{y}}$$

(d) Evaluate t for $N = 1.0\ \text{mol} = 6.022\times10^{23}$ molecules:

$$t = \frac{2^{6.022\times10^{23}}}{2(100\ \text{s}^{-1})}$$

To evaluate $2^{6.022\times10^{23}}$ let $10^x = 2^{6.022\times10^{23}}$ and take the logarithm of both sides of the equation to obtain:

$$(6.022\times10^{23})\ln 2 = x\ln 10 \Rightarrow x \approx 10^{23}$$

Substituting for x yields:

$$t \approx \frac{10^{10^{23}}}{2(100\ \text{s}^{-1})}\left(\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}}\right) \approx \boxed{10^{10^{23}}\ \text{y}}$$

(e) Solve the ideal gas law for the number of molecules N in the gas:

$$N = \frac{PV}{kT}$$

Assuming the gas to be at room temperature (300 K), substitute numerical values and evaluate N:

$$N = \frac{(10^{-12}\ \text{torr})(133.32\ \text{Pa/torr})(1.0\ \text{L})}{(1.381\times10^{-23}\ \text{J/K})(300\ \text{K})} = 3.22\times10^{7}\ \text{molecules}$$

Evaluate t for $N = 3.22\times10^{7}$ molecules:

$$t = \frac{2^{3.22\times10^{7}}}{2(100\ \text{s}^{-1})}$$

To evaluate $2^{3.22\times10^{7}}$ let $10^{x} = 2^{3.22\times10^{7}}$ and take the logarithm of both sides of the equation to obtain:

$$(3.22\times10^{7})\ln 2 = x\ln 10 \Rightarrow x \approx 10^{7}$$

Substituting for x yields:

$$t = \frac{10^{10^{7}}}{2(100\ \text{s}^{-1})}\times\frac{1\ \text{y}}{3.156\times10^{7}\ \text{s}} \approx \boxed{10^{10^{7}}\ \text{y}}$$

Express the ratio of this waiting time to the lifetime of the universe t_{universe}:

$$\frac{t}{t_{\text{universe}}} = \frac{10^{10^{7}}\ \text{y}}{10^{10}\ \text{y}} \approx 10^{10^{7}}$$

or

$$t \approx \boxed{10^{10^{7}}\, t_{\text{universe}}}$$

Heat Engines and Refrigerators

27 • **[SSM]** A heat engine with 20.0% efficiency does 0.100 kJ of work during each cycle. (a) How much heat is absorbed from the hot reservoir during each cycle? (b) How much heat is released to the cold reservoir during each cycle?

Picture the Problem (*a*) The efficiency of the engine is defined to be $\varepsilon = W/Q_h$ where W is the work done per cycle and Q_h is the heat absorbed from the hot reservoir during each cycle. (*b*) Because, from conservation of energy, $Q_h = W + Q_c$, we can express the efficiency of the engine in terms of the heat Q_c released to the cold reservoir during each cycle.

(*a*) Q_h absorbed from the hot reservoir during each cycle is given by:

$$Q_h = \frac{W}{\varepsilon} = \frac{100\ \text{J}}{0.200} = \boxed{500\ \text{J}}$$

(*b*) Use $Q_h = W + Q_c$ to obtain:

$$Q_c = Q_h - W = 500\ \text{J} - 100\ \text{J} = \boxed{400\ \text{J}}$$

28 • A heat engine absorbs 0.400 kJ of heat from the hot reservoir and does 0.120 kJ of work during each cycle. (*a*) What is its efficiency? (*b*) How much heat is released to the cold reservoir during each cycle?

Picture the Problem (*a*) The efficiency of the engine is defined to be $\varepsilon = W/Q_h$ where W is the work done per cycle and Q_h is the heat absorbed from the hot reservoir during each cycle. (*b*) We can apply conservation of energy to the engine to obtain $Q_h = W + Q_c$ and solve this equation for the heat Q_c released to the cold reservoir during each cycle.

(*a*) The efficiency of the heat engine is given by:

$$\varepsilon = \frac{W}{Q_h} = \frac{120\ \text{J}}{400\ \text{J}} = \boxed{30\%}$$

(*b*) Apply conservation of energy to the engine to obtain:

$$Q_h = W + Q_c \Rightarrow Q_c = Q_h - W$$

Substitute numerical values and evaluate Q_c:

$$Q_c = 400\ \text{J} - 120\ \text{J} = \boxed{280\ \text{J}}$$

29 • A heat engine absorbs 100 J of heat from the hot reservoir and releases 60 J of heat to the cold reservoir during each cycle. (*a*) What is its efficiency? (*b*) If each cycle takes 0.50 s, find the power output of this engine.

Picture the Problem We can use its definition to find the efficiency of the engine and the definition of power to find its power output.

(*a*) The efficiency of the heat engine is given by:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$

Substitute numerical values and evaluate ε:

$$\varepsilon = 1 - \frac{60\,\text{J}}{100\,\text{J}} = \boxed{40\%}$$

(*b*) The power output *P* of this engine is the rate at which it does work:

$$P = \frac{dW}{dt} = \frac{d}{dt}\varepsilon Q_{\text{h}} = \varepsilon \frac{dQ_{\text{h}}}{dt}$$

Substitute numerical values and evaluate *P*:

$$P = (0.40)\left(\frac{100\,\text{J}}{0.500\,\text{s}}\right) = \boxed{80\,\text{W}}$$

30 • A refrigerator absorbs 5.0 kJ of heat from a cold reservoir and releases 8.0 kJ to a hot reservoir. (*a*) Find the coefficient of performance of the refrigerator. (*b*) The refrigerator is reversible. If it is run backward as a heat engine between the same two reservoirs, what is its efficiency?

Picture the Problem We can apply their definitions to find the COP of the refrigerator and the efficiency of the heat engine.

(*a*) The COP of a refrigerator is defined to be:

$$\text{COP} = \frac{Q_{\text{c}}}{W}$$

Apply conservation of energy to relate the work done per cycle to Q_{h} and Q_{c}:

$$W = Q_{\text{h}} - Q_{\text{c}}$$

Substitute for *W* to obtain:

$$\text{COP} = \frac{Q_{\text{c}}}{Q_{\text{h}} - Q_{\text{c}}}$$

Substitute numerical values and evaluate COP:

$$\text{COP} = \frac{5.0\,\text{kJ}}{8.0\,\text{kJ} - 5.0\,\text{kJ}} = \boxed{1.7}$$

(*b*) The efficiency of a heat pump is defined to be:

$$\varepsilon = \frac{W}{Q_{\text{h}}}$$

Apply conservation of energy to the heat pump to obtain:

$$\varepsilon = \frac{Q_{\text{h}} - Q_{\text{c}}}{Q_{\text{h}}} = 1 - \frac{Q_{\text{c}}}{Q_{\text{h}}}$$

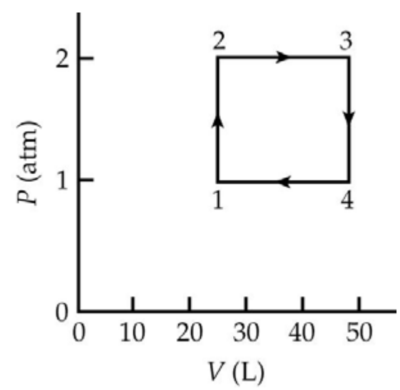
Substitute numerical values and evaluate ε:

$$\varepsilon = 1 - \frac{5.0\,\text{kJ}}{8.0\,\text{kJ}} = \boxed{38\%}$$

31 •• **[SSM]** The working substance of an engine is 1.00 mol of a monatomic ideal gas. The cycle begins at P_1 = 1.00 atm and V_1 = 24.6 L. The gas is heated at constant volume to P_2 = 2.00 atm. It then expands at constant pressure until its volume is 49.2 L. The gas is then cooled at constant volume until its pressure is again 1.00 atm. It is then compressed at constant pressure to its original state. All the steps are quasi-static and reversible. (*a*) Show this cycle on a *PV* diagram. For each step of the cycle, find the work done by the gas, the heat absorbed by the gas, and the change in the internal energy of the gas. (*b*) Find the efficiency of the cycle.

Picture the Problem To find the heat added during each step we need to find the temperatures in states 1, 2, 3, and 4. We can then find the work done on the gas during each process from the area under each straight-line segment and the heat that enters the system from $Q = C_V\Delta T$ and $Q = C_P\Delta T$. We can use the 1st law of thermodynamics to find the change in internal energy for each step of the cycle. Finally, we can find the efficiency of the cycle from the work done each cycle and the heat that *enters* the system each cycle.

(*a*) The cycle is shown to the right:



Apply the ideal-gas law to state 1 to find T_1:

$$T_1 = \frac{P_1V_1}{nR} = \frac{(1.00\,\text{atm})(24.6\,\text{L})}{(1.00\,\text{mol})\left(8.206\times10^{-2}\,\frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}\right)} = 300\,\text{K}$$

The pressure doubles while the volume remains constant between states 1 and 2. Hence:

$$T_2 = 2T_1 = 600\,K$$

The volume doubles while the pressure remains constant between states 2 and 3. Hence:

$$T_3 = 2T_2 = 1200\,K$$

The pressure is halved while the volume remains constant between states 3 and 4. Hence:

$$T_4 = \tfrac{1}{2}T_3 = 600\,K$$

For path 1→2:

$$W_{12} = P\Delta V_{12} = \boxed{0}$$

and

$$Q_{12} = C_V\Delta T_{12} = \tfrac{3}{2}R\Delta T_{12} = \tfrac{3}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K} - 300\,\text{K}) = \boxed{3.74\,\text{kJ}}$$

The change in the internal energy of the system as it goes from state 1 to state 2 is given by the 1st law of thermodynamics:

$$\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$$

Because $W_{12} = 0$:

$$\Delta E_{\text{int},12} = Q_{12} = \boxed{3.74\text{ kJ}}$$

For path 2→3:

$$W_{\text{on}} = -W_{23} = -P\Delta V_{23} = -(2.00\,\text{atm})(49.2\,\text{L} - 24.6\,\text{L})\left(\frac{101.325\,\text{J}}{\text{L}\cdot\text{atm}}\right) = \boxed{-4.99\,\text{kJ}}$$

$$Q_{23} = C_P\Delta T_{23} = \tfrac{5}{2}R\Delta T_{23} = \tfrac{5}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(1200\,\text{K} - 600\,\text{K}) = \boxed{12.5\,\text{kJ}}$$

Apply $\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$ to obtain:

$$\Delta E_{\text{int, 23}} = 12.5\text{ kJ} - 4.99\text{ kJ} = \boxed{7.5\text{ kJ}}$$

For path 3→4:

$$W_{34} = P\Delta V_{34} = \boxed{0}$$

and

$$Q_{34} = \Delta E_{\text{int},34} = C_V\Delta T_{34} = \tfrac{3}{2}R\Delta T_{34} = \tfrac{3}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K} - 1200\,\text{K}) = \boxed{-7.48\,\text{kJ}}$$

Apply $\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$ to obtain:

$$\Delta E_{\text{int, 34}} = -7.48\text{ kJ} + 0 = \boxed{-7.48\text{ kJ}}$$

For path 4→1:

$$W_{\text{on}} = -W_{41} = -P\Delta V_{41} = -(1.00\,\text{atm})(24.6\,\text{L} - 49.2\,\text{L})\left(\frac{101.325\,\text{J}}{\text{L}\cdot\text{atm}}\right) = \boxed{2.49\,\text{kJ}}$$

and

$$Q_{41} = C_{\text{P}}\Delta T_{41} = \tfrac{5}{2}R\Delta T_{41} = \tfrac{5}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(300\,\text{K} - 600\,\text{K}) = \boxed{-6.24\,\text{kJ}}$$

Apply $\Delta E_{\text{int}} = Q_{\text{in}} + W_{\text{on}}$ to obtain:

$$\Delta E_{\text{int, 41}} = -6.24\text{ kJ} + 2.49\text{ kJ} = \boxed{-3.75\text{ kJ}}$$

For easy reference, the results of the preceding calculations are summarized in the following table:

Process	W_{on}, kJ	Q_{in}, kJ	$\Delta E_{\text{int}}\left(= Q_{\text{in}} + W_{\text{on}}\right)$, kJ
1→2	0	3.74	3.74
2→3	−4.99	12.5	7.5
3→4	0	−7.48	−7.48
4→1	2.49	−6.24	−3.75

(*b*) The efficiency of the cycle is given by:

$$\varepsilon = \frac{W_{\text{by}}}{Q_{\text{in}}} = \frac{-W_{23} + (-W_{41})}{Q_{12} + Q_{23}}$$

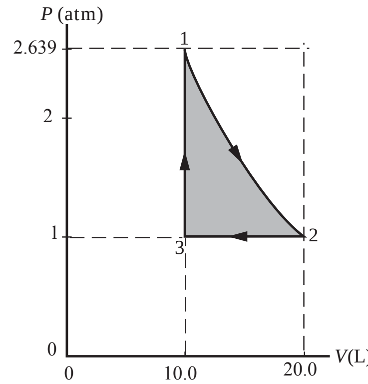
Substitute numerical values and evaluate ε:

$$\varepsilon = \frac{4.99\,\text{kJ} - 2.49\,\text{kJ}}{3.74\,\text{kJ} + 12.5\,\text{kJ}} \approx \boxed{15\%}$$

Remarks: Note that the work done per cycle is the area bounded by the rectangular path. Note also that, as expected because the system returns to its initial state, the sum of the changes in the internal energy for the cycle is zero.

32 •• The working substance of an engine is 1.00 mol of a diatomic ideal gas. The engine operates in a cycle consisting of three steps: (1) an adiabatic expansion from an initial volume of 10.0 L to a pressure of 1.00 atm and a volume of 20.0 L, (2) a compression at constant pressure to its original volume of 10.0 L, and (3) heating at constant volume to its original pressure. Find the efficiency of this cycle.

Picture the Problem The three steps in the process are shown on the *PV* diagram. We can find the efficiency of the cycle by finding the work done by the gas and the heat that enters the system per cycle.



The pressures and volumes at the end points of the adiabatic expansion are related according to:

$$P_1V_1^\gamma = P_2V_2^\gamma \Rightarrow P_1 = \left(\frac{V_2}{V_1}\right)^\gamma P_2$$

Substitute numerical values and evaluate P_1:

$$P_1 = \left(\frac{20.0\text{ L}}{10.0\text{ L}}\right)^{1.4}(1.00\text{ atm}) = 2.639\text{ atm}$$

Express the efficiency of the cycle:

$$\varepsilon = \frac{W}{Q_\text{h}} \qquad (1)$$

No heat enters or leaves the system during the adiabatic expansion:

$$Q_{12} = 0$$

Find the heat entering or leaving the system during the isobaric compression:

$$\begin{aligned} Q_{23} &= C_\text{P}\Delta T_{23} = \tfrac{7}{2}R\Delta T_{23} = \tfrac{7}{2}P\Delta V_{23} \\ &= \tfrac{7}{2}(1.00\,\text{atm})(10.0\,\text{L} - 20.0\,\text{L}) \\ &= -35.0\,\text{atm}\cdot\text{L} \end{aligned}$$

Find the heat entering or leaving the system during the constant-volume process:

$$\begin{aligned} Q_{31} &= C_\text{V}\Delta T_{31} = \tfrac{5}{2}R\Delta T_{31} = \tfrac{5}{2}\Delta PV_{31} \\ &= \tfrac{5}{2}(2.639\,\text{atm} - 1.00\,\text{atm})(10.0\,\text{L}) \\ &= 41.0\,\text{atm}\cdot\text{L} \end{aligned}$$

Apply the 1st law of thermodynamics to the cycle ($\Delta E_\text{int, cycle} = 0$) to obtain:

$$\begin{aligned} W_\text{on} &= \Delta E_\text{int} - Q_\text{in} = -Q_\text{in} \\ &= Q_{12} + Q_{23} + Q_{31} \\ &= 0 - 35.0\,\text{atm}\cdot\text{L} + 41.0\,\text{atm}\cdot\text{L} \\ &= 6.0\,\text{atm}\cdot\text{L} \end{aligned}$$

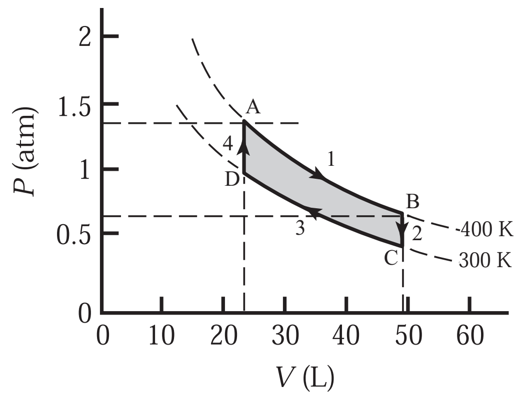
Substitute numerical values in equation (1) and evaluate ε:

$$\varepsilon = \frac{6.0\,\text{atm}\cdot\text{L}}{41\,\text{atm}\cdot\text{L}} = \boxed{15\%}$$

33 •• An engine using 1.00 mol of an ideal gas initially at a volume of 24.6 L and a temperature of 400 K performs a cycle consisting of four steps: (1) an isothermal expansion at 400 K to twice its initial volume, (2) cooling at constant volume to a temperature of 300 K (3) an isothermal compression to its original volume, and (4) heating at constant volume to its original temperature of 400 K. Assume that C_v = 21.0 J/K. Sketch the cycle on a *PV* diagram and find its efficiency.

Picture the Problem We can find the efficiency of the cycle by finding the work done by the gas and the heat that enters the system per cycle.

The *PV* diagram of the cycle is shown to the right. A, B, C, and D identify the four states of the gas and the numerals 1, 2, 3, and 4 represent the four steps through which the gas is taken.



Express the efficiency of the cycle:

$$\varepsilon = \frac{W}{Q_h} = \frac{W_1 + W_2 + W_3 + W_4}{Q_{h,1} + Q_{h,2} + Q_{h,3} + Q_{h,4}}$$

Because steps 2 and 4 are constant-volume processes, $W_2 = W_4 = 0$:

$$\varepsilon = \frac{W}{Q_h} = \frac{W_1 + 0 + W_3 + 0}{Q_{h,1} + Q_{h,2} + Q_{h,3} + Q_{h,4}}$$

Because the internal energy of the gas increases in step 4 while no work is done, and because the internal energy does not change during step 1 while work is done by the gas, heat enters the system only during these processes:

$$\varepsilon = \frac{W}{Q_h} = \frac{W_1 + W_3}{Q_{h,1} + Q_{h,4}} \qquad (1)$$

The work done during the isothermal expansion (1) is given by:

$$W_1 = nRT \ln\left(\frac{V_B}{V_A}\right)$$

The work done during the isothermal compression (3) is given by:

$$W_3 = nRT_c \ln\left(\frac{V_D}{V_C}\right)$$

Because there is no change in the internal energy of the system during step 1, the heat that enters the system during this isothermal expansion is given by:

$$Q_1 = W_1 = nRT_h \ln\left(\frac{V_B}{V_A}\right)$$

The heat that enters the system during the constant-volume step 4 is given by:

$$Q_4 = C_V \Delta T = C_V (T_h - T_c)$$

Substituting in equation (1) yields:

$$\varepsilon = \frac{nRT_h \ln\left(\frac{V_B}{V_A}\right) + nRT_c \ln\left(\frac{V_D}{V_C}\right)}{nRT_h \ln\left(\frac{V_B}{V_A}\right) + C_V (T_h - T_c)}$$

Noting the $\frac{V_B}{V_A} = 2$ and $\frac{V_D}{V_C} = \frac{1}{2}$, substitute and simplify to obtain:

$$\varepsilon = \frac{T_h \ln(2) + T_c \ln\left(\frac{1}{2}\right)}{T_h \ln(2) + \frac{C_V}{nR}(T_h - T_c)} = \frac{T_h \ln(2) - T_c \ln(2)}{T_h \ln(2) + \frac{C_V}{nR}(T_h - T_c)} = \frac{T_h - T_c}{T_h + \frac{C_V}{nR\ln(2)}(T_h - T_c)}$$

Substitute numerical values and evaluate ε:

$$\varepsilon = \frac{400\text{ K} - 300\text{ K}}{400\text{ K} + \frac{21.0\,\frac{\text{J}}{\text{K}}}{(1.00\text{ mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)\ln(2)}(400\text{ K} - 300\text{ K})} = \boxed{13.1\%}$$

34 •• Figure 19-15 shows the cycle followed by 1.00 mol of an ideal monatomic gas initially at a volume of 25.0 L. All the processes are quasi-static. Determine (*a*) the temperature of each numbered state of the cycle, (*b*) the heat transfer for each part of the cycle, and (*c*) the efficiency of the cycle.

Picture the Problem We can use the ideal-gas law to find the temperatures of each state of the gas and the heat capacities at constant volume and constant pressure to find the heat flow for the constant-volume and isobaric processes. Because the change in internal energy is zero for the isothermal process, we can use the expression for the work done on or by a gas during an isothermal process

to find the heat flow during such a process. Finally, we can find the efficiency of the cycle from its definition.

(*a*) Use the ideal-gas law to find the temperature at point 1:

$$T_1 = \frac{P_1V_1}{nR} = \frac{(100\,\text{kPa})(25.0\,\text{L})}{(1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = \boxed{301\,\text{K}}$$

Use the ideal-gas law to find the temperatures at points 2 and 3:

$$T_2 = T_3 = \frac{P_2V_2}{nR} = \frac{(200\,\text{kPa})(25.0\,\text{L})}{(1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)} = \boxed{601\,\text{K}}$$

(*b*) Find the heat entering the system for the constant-volume process from $1 \to 2$:

$$Q_{12} = C_V\Delta T_{12} = \tfrac{3}{2}R\Delta T_{12} = \tfrac{3}{2}\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(601\,\text{K} - 301\,\text{K}) = \boxed{3.74\,\text{kJ}}$$

Find the heat entering or leaving the system for the isothermal process from $2 \to 3$:

$$Q_{23} = nRT_2\ln\left(\frac{V_3}{V_2}\right) = (1.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(601\,\text{K})\ln\left(\frac{50.0\,\text{L}}{25.0\,\text{L}}\right) = \boxed{3.46\,\text{kJ}}$$

Find the heat leaving the system during the isobaric compression from $3 \to 1$:

$$Q_{31} = C_P\Delta T_{31} = \tfrac{5}{2}R\Delta T_{31} = \tfrac{5}{2}\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(301\,\text{K} - 601\,\text{K}) = \boxed{-6.24\,\text{kJ}}$$

(*c*) Express the efficiency of the cycle:

$$\varepsilon = \frac{W}{Q_{\text{in}}} = \frac{W}{Q_{12} + Q_{23}} \qquad (1)$$

Apply the 1st law of thermodynamics to the cycle:

$$W = \sum Q = Q_{12} + Q_{23} + Q_{31} = 3.74\,\text{kJ} + 3.46\,\text{kJ} - 6.24\,\text{kJ} = 0.96\,\text{kJ}$$

because, for the cycle, $\Delta E_{\text{int}} = 0$.

Substitute numerical values in equation (1) and evaluate ε:

$$\varepsilon = \frac{0.96\,\text{kJ}}{3.74\,\text{kJ} + 3.46\,\text{kJ}} = \boxed{13\%}$$

35 •• An ideal diatomic gas follows the cycle shown in Figure 19-16. The temperature of state 1 is 200 K. Determine (*a*) the temperatures of the other three numbered states of the cycle and (*b*) the efficiency of the cycle.

Picture the Problem We can use the ideal-gas law to find the temperatures of each state of the gas. We can find the efficiency of the cycle from its definition; using the area enclosed by the cycle to find the work done per cycle and the heat entering the system between states 1 and 2 and 2 and 3 to determine Q_{in}.

(*a*) Use the ideal-gas law for a fixed amount of gas to find the temperature in state 2 to the temperature in state 1:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \Rightarrow T_2 = T_1\frac{P_2V_2}{P_1V_1} = T_1\frac{P_2}{P_1}$$

Substitute numerical values and evaluate T_2:

$$T_2 = (200\,\text{K})\frac{(3.0\,\text{atm})}{(1.0\,\text{atm})} = \boxed{600\,\text{K}}$$

Apply the ideal-gas law for a fixed amount of gas to states 2 and 3 to obtain:

$$T_3 = T_2\frac{P_3V_3}{P_2V_2} = T_2\frac{V_3}{V_2}$$

Substitute numerical values and evaluate T_3:

$$T_3 = (600\,\text{K})\frac{(300\,\text{L})}{(100\,\text{L})} = \boxed{1800\,\text{K}}$$

Apply the ideal-gas law for a fixed amount of gas to states 3 and 4 to obtain:

$$T_4 = T_3\frac{P_4V_4}{P_3V_3} = T_3\frac{P_4}{P_3}$$

Substitute numerical values and evaluate T_4:

$$T_4 = (1800\,\text{K})\frac{(1.0\,\text{atm})}{(3.0\,\text{atm})} = \boxed{600\,\text{K}}$$

(*b*) The efficiency of the cycle is:

$$\varepsilon = \frac{W}{Q_{\text{in}}} \qquad (1)$$

Use the area of the rectangle to find the work done each cycle:

$$\begin{aligned} W &= \Delta P\Delta V \\ &= (300\,\text{L} - 100\,\text{L})(3.0\,\text{atm} - 1.0\,\text{atm}) \\ &= 400\,\text{atm}\cdot\text{L} \end{aligned}$$

Apply the ideal-gas law to state 1 to find the product of n and R:

$$nR = \frac{P_1V_1}{T_1} = \frac{(1.0\,\text{atm})(100\,\text{L})}{200\,\text{K}} = 0.50\,\text{L}\cdot\text{atm/K}$$

Noting that heat enters the system between states 1 and 2 and states 2 and 3, express Q_{in}:

$$Q_{\text{in}} = Q_{12} + Q_{23} = C_V\Delta T_{12} + C_P\Delta T_{23} = \tfrac{5}{2}nR\Delta T_{12} + \tfrac{7}{2}nR\Delta T_{23} = \left(\tfrac{5}{2}\Delta T_{12} + \tfrac{7}{2}\Delta T_{23}\right)nR$$

Substitute numerical values and evaluate Q_{in}:

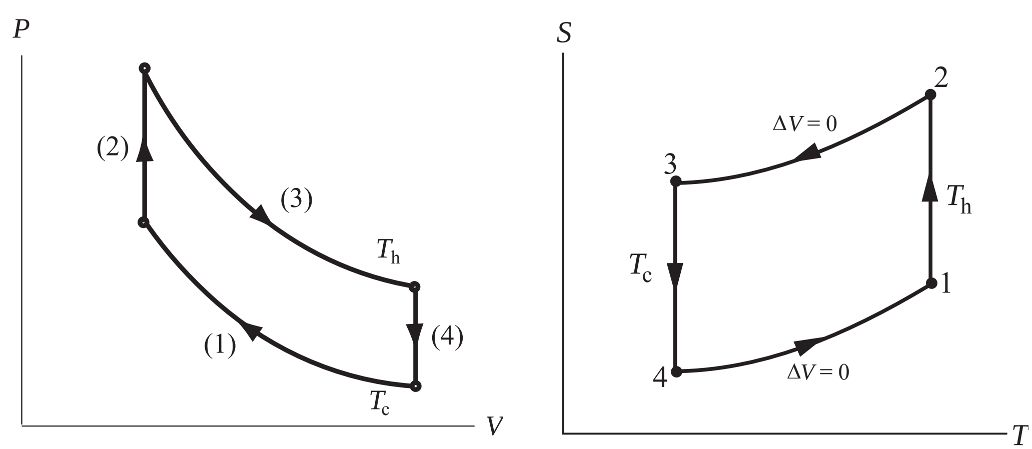
$$Q_{\text{in}} = \left[\tfrac{5}{2}(600\,\text{K} - 200\,\text{K}) + \tfrac{7}{2}(1800\,\text{K} - 600\,\text{K})\right]\left(0.50\frac{\text{L}\cdot\text{atm}}{\text{K}}\right) = 2600\,\text{atm}\cdot\text{L}$$

Substitute numerical values in equation (1) and evaluate ε:

$$\varepsilon = \frac{400\,\text{atm}\cdot\text{L}}{2600\,\text{atm}\cdot\text{L}} = \boxed{15\%}$$

36 ••• Recently, an old design for a heat engine, known as the *Stirling engine* has been promoted as a means of producing power from solar energy. The cycle of a Stirling engine is as follows: (1) isothermal compression of the working gas (2) heating of the gas at constant volume, (3) an isothermal expansion of the gas, and (4) cooling of the gas at constant volume. (*a*) Sketch *PV* and *ST* diagrams for the Stirling cycle. (*b*) Find the entropy change of the gas for each step of the cycle and show that the sum of these entropy changes is equal to zero.

Picture the Problem (*a*) The *PV* and *ST* cycles are shown below. (*b*) We can show that the entropy change during one Stirling cycle is zero by adding up the entropy changes for the four processes.



(*b*) The change in entropy for one Stirling cycle is the sum of the entropy changes during the cycle:

$$\Delta S_{\text{cycle}} = \Delta S_{12} + \Delta S_{23} + \Delta S_{34} + \Delta S_{41} \quad (1)$$

Express the entropy change for the isothermal process from state 1 to state 2:

$$\Delta S_{12} = nR\ln\left(\frac{V_2}{V_1}\right)$$

Similarly, the entropy change for the isothermal process from state 3 to state 4 is:

$$\Delta S_{34} = nR\ln\left(\frac{V_4}{V_3}\right)$$

or, because $V_2 = V_3$ and $V_1 = V_4$,

$$\Delta S_{34} = nR\ln\left(\frac{V_1}{V_2}\right) = -nR\ln\left(\frac{V_2}{V_1}\right)$$

The change in entropy for a constant-volume process is given by:

$$\Delta S_{\text{isochoric}} = \int\frac{dQ}{T} = \int_{T_\text{i}}^{T_\text{f}}\frac{nC_\text{V}dT}{T}$$

$$= nC_\text{V}\ln\left(\frac{T_\text{f}}{T_\text{i}}\right)$$

For the constant-volume process from state 2 to state 3:

$$\Delta S_{23} = C_\text{V}\ln\left(\frac{T_\text{c}}{T_\text{h}}\right)$$

For the constant-volume process from state 4 to state 1:

$$\Delta S_{41} = C_\text{V}\ln\left(\frac{T_\text{h}}{T_\text{c}}\right) = -C_\text{V}\ln\left(\frac{T_\text{c}}{T_\text{h}}\right)$$

Substituting in equation (1) yields:

$$\Delta S_{\text{cycle}} = nR\ln\left(\frac{V_2}{V_1}\right) + C_\text{V}\ln\left(\frac{T_\text{c}}{T_\text{h}}\right) - nR\ln\left(\frac{V_2}{V_1}\right) - C_\text{V}\ln\left(\frac{T_\text{c}}{T_\text{h}}\right) = \boxed{0}$$

37 •• ″As far as we know, Nature has never evolved a heat engine″—Steven Vogel, *Life's Devices*, Princeton University Press (1988). (*a*) Calculate the efficiency of a heat engine operating between body temperature (98.6ºF) and a typical outdoor temperature (70ºF), and compare this to the human body's efficiency for converting chemical energy into work (approximately 20%). Does this efficiency comparison contradict the second law of thermodynamics?
(*b*) From the result of Part (*a*), and a general knowledge of the conditions under which most warm-blooded organisms exist, give a reason why no warm-blooded organisms have evolved heat engines to increase their internal energies.

Picture the Problem We can use the efficiency of a Carnot engine operating between reservoirs at body temperature and typical outdoor temperatures to find an upper limit on the efficiency of an engine operating between these temperatures.

(*a*) Express the maximum efficiency of an engine operating between body temperature and 70°F:

$$\varepsilon_C = 1 - \frac{T_c}{T_h}$$

Use $T = \frac{5}{9}(t_F - 32) + 273$ to obtain:

$$T_{body} = 310\ \text{K} \text{ and } T_{room} = 294\ \text{K}$$

Substitute numerical values and evaluate ε_C:

$$\varepsilon_C = 1 - \frac{294\,\text{K}}{310\,\text{K}} = \boxed{5.16\%}$$

The fact that this efficiency is considerably less than the actual efficiency of a human body does not contradict the second law of thermodynamics. The application of the second law to chemical reactions such as the ones that supply the body with energy have not been discussed in the text but we can note that we don't get our energy from heat swapping between our body and the environment. Rather, we eat food to get the energy that we need.

(*b*) Most warm-blooded animals survive under roughly the same conditions as humans. To make a heat engine work with appreciable efficiency, internal body temperatures would have to be maintained at an unreasonably high level.

38 ••• The *diesel cycle* shown in Figure 19-17 approximates the behavior of a diesel engine. Process *ab* is an adiabatic compression, process *bc* is an expansion at constant pressure, process *cd* is an adiabatic expansion, and process *da* is cooling at constant volume. Find the efficiency of this cycle in terms of the volumes V_a, V_b and V_c.

Picture the Problem The working fluid will be modeled as an ideal gas and the process will be modeled as quasistatic. To find the efficiency of the diesel cycle we can find the heat that enters the system and the heat that leaves the system and use the expression that gives the efficiency in terms of these quantities. Note that no heat enters or leaves the system during the adiabatic processes *ab* and *cd.* Heat enters the system during the isobaric process *bc* and leaves the system during the isovolumetric process *da.*

Express the efficiency of the cycle in terms of Q_c and Q_h:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$

Express Q for the isobaric warming process *bc*:

$$Q_{bc} = Q_h = C_P(T_c - T_b)$$

Because C_V is independent of T, Q_{da} (the constant-volume cooling process) is given by:

$$Q_{da} = Q_c = C_V(T_d - T_a)$$

Substitute for Q_h and Q_c and simplify using $\gamma = C_P/C_V$ to obtain:

$$\varepsilon = 1 - \frac{C_V(T_d - T_a)}{C_P(T_c - T_b)} = 1 - \frac{(T_d - T_a)}{\gamma(T_c - T_b)}$$

Using an equation for a quasistatic adiabatic process, relate the temperatures T_a and T_b to the volumes V_a and V_b:

$$T_a V_a^{\gamma-1} = T_b V_b^{\gamma-1} \Rightarrow T_a = T_b \frac{V_b^{\gamma-1}}{V_a^{\gamma-1}} \quad (1)$$

Proceeding similarly, relate the temperatures T_c and T_d to the volumes V_c and V_d:

$$T_c V_c^{\gamma-1} = T_d V_d^{\gamma-1} \Rightarrow T_d = T_c \frac{V_c^{\gamma-1}}{V_d^{\gamma-1}} \quad (2)$$

Use equations (1) and (2) to eliminate T_a and T_d:

$$\varepsilon = 1 - \frac{\left(T_c \frac{V_c^{\gamma-1}}{V_d^{\gamma-1}} - T_b \frac{V_b^{\gamma-1}}{V_a^{\gamma-1}}\right)}{\gamma(T_c - T_b)}$$

Because $V_a = V_d$:

$$\varepsilon = 1 - \frac{\left(\left(\frac{V_c}{V_a}\right)^{\gamma-1} - \frac{T_b}{T_c}\left(\frac{V_b}{V_a}\right)^{\gamma-1}\right)}{\gamma\left(1 - \frac{T_b}{T_c}\right)}$$

Noting that $P_b = P_c$, apply the ideal-gas law to relate T_b and T_c:

$$\frac{T_b}{T_c} = \frac{V_b}{V_c}$$

Substitute for the ratio of T_b to T_c and simplify to obtain:

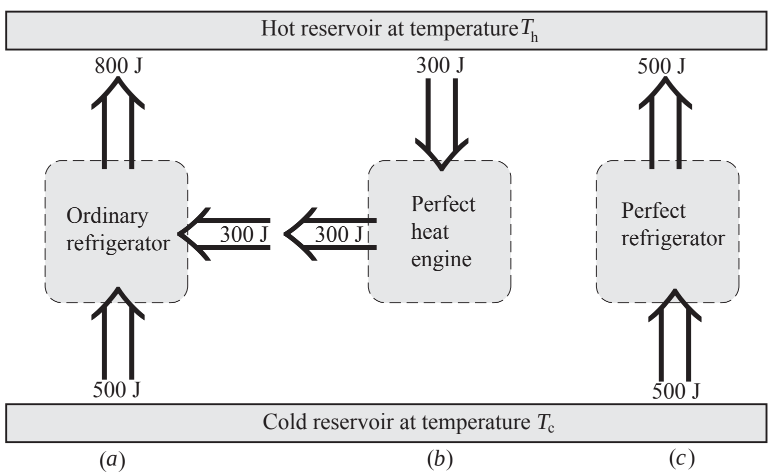
$$\varepsilon = 1 - \frac{\left(\frac{V_c}{V_a}\right)^{\gamma-1} - \frac{V_b}{V_c}\left(\frac{V_b}{V_a}\right)^{\gamma-1}}{\gamma\left(1 - \frac{V_b}{V_c}\right)} \cdot \frac{\frac{V_c}{V_a}}{\frac{V_c}{V_a}} = 1 - \frac{\left(\frac{V_c}{V_a}\right)^{\gamma-1} - \frac{V_b}{V_c}\left(\frac{V_b}{V_a}\right)^{\gamma-1}}{\gamma\left(\frac{V_c}{V_a} - \frac{V_b}{V_a}\right)}$$

$$= 1 - \frac{\left(\frac{V_c}{V_a}\right)^{\gamma} - \left(\frac{V_b}{V_a}\right)^{\gamma}}{\gamma\left(\frac{V_c}{V_a} - \frac{V_b}{V_a}\right)} = \boxed{1 - \frac{V_c^{\gamma} - V_b^{\gamma}}{\gamma V_a^{\gamma-1}(V_c - V_b)}}$$

Second Law of Thermodynamics

39 •• [SSM] A refrigerator absorbs 500 J of heat from a cold reservoir and releases 800 J to a hot reservoir. Assume that the heat-engine statement of the second law of thermodynamics is false, and show how a perfect engine working with this refrigerator can violate the refrigerator statement of the second law of thermodynamics.

Determine the Concept The following diagram shows an ordinary refrigerator that uses 300 J of work to remove 500 J of heat from a cold reservoir and releases 800 J of heat to a hot reservoir (see (*a*) in the diagram). Suppose the heat-engine statement of the second law is false. Then a ″perfect″ heat engine could remove energy from the hot reservoir and convert it completely into work with 100 percent efficiency. We could use this perfect heat engine to remove 300 J of energy from the hot reservoir and do 300 J of work on the ordinary refrigerator (see (*b*) in the diagram). Then, the combination of the perfect heat engine and the ordinary refrigerator would be a perfect refrigerator; transferring 500 J of heat from the cold reservoir to the hot reservoir without requiring any work (see (*c*) in the diagram).This violates the refrigerator statement of the second law.



40 •• If two curves that represent quasi-static adiabatic processes could intersect on a *PV* diagram, a cycle could be completed by an isothermal path between the two adiabatic curves shown in Figure 19-18. Show that such a cycle violates the second law of thermodynamics.

Determine the Concept The work done by the system is the area enclosed by the cycle, where we assume that we start with the isothermal expansion. It is only in this expansion that heat is extracted from a reservoir. There is no heat transfer in the adiabatic expansion or compression. Thus, we would completely convert heat to mechanical energy, without exhausting any heat to a cold reservoir, in violation of the second law of thermodynamics.

Carnot Cycles

41 • [SSM] A Carnot engine works between two heat reservoirs at temperatures T_h = 300 K and T_c = 200 K. (*a*) What is its efficiency? (*b*) If it absorbs 100 J of heat from the hot reservoir during each cycle, how much work does it do each cycle? (*c*) How much heat does it release during each cycle? (*d*) What is the COP of this engine when it works as a refrigerator between the same two reservoirs?

Picture the Problem We can find the efficiency of the Carnot engine using $\varepsilon = 1 - T_c / T_h$ and the work done per cycle from $\varepsilon = W / Q_h$. We can apply conservation of energy to find the heat rejected each cycle from the heat absorbed and the work done each cycle. We can find the COP of the engine working as a refrigerator from its definition.

(*a*) The efficiency of the Carnot engine depends on the temperatures of the hot and cold reservoirs:

$$\varepsilon_C = 1 - \frac{T_c}{T_h} = 1 - \frac{200\,\text{K}}{300\,\text{K}} = \boxed{33.3\%}$$

(*b*) Using the definition of efficiency, relate the work done each cycle to the heat absorbed from the hot reservoir:

$$W = \varepsilon_C Q_h = (0.333)(100\,\text{J}) = \boxed{33.3\,\text{J}}$$

(*c*) Apply conservation of energy to relate the heat given off each cycle to the heat absorbed and the work done:

$$Q_c = Q_h - W = 100\,\text{J} - 33.3\,\text{J} = 66.7\,\text{J}$$
$$= \boxed{67\,\text{J}}$$

(*d*) Using its definition, express and evaluate the refrigerator's coefficient of performance:

$$\text{COP} = \frac{Q_c}{W} = \frac{66.7\,\text{J}}{33.3\,\text{J}} = \boxed{2.0}$$

42 • An engine absorbs 250 J of heat per cycle from a reservoir at 300 K and releases 200 J of heat per cycle to a reservoir at 200 K. (*a*) What is its efficiency? (*b*) How much additional work per cycle could be done if the engine were reversible?

Picture the Problem We can find the efficiency of the engine from its definition and the additional work done if the engine were reversible from $W = \varepsilon_C Q_h$, where ε_C is the Carnot efficiency.

(*a*) Express the efficiency of the engine in terms of the heat absorbed from the high-temperature reservoir and the heat exhausted to the low-temperature reservoir:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$
$$= 1 - \frac{200\,\text{J}}{250\,\text{J}} = \boxed{20.0\%}$$

(*b*) Express the additional work done if the engine is reversible:

$$\Delta W = W_{\text{Carnot}} - W_{\text{Part}(a)} \qquad (1)$$

Relate the work done by a reversible engine to its Carnot efficiency:

$$W = \varepsilon_C Q_h = \left(1 - \frac{T_c}{T_h}\right) Q_h$$

Substitute numerical values and evaluate *W*:

$$W = \left(1 - \frac{200\,\text{K}}{300\,\text{K}}\right)(250\,\text{J}) = 83.3\,\text{J}$$

Substitute numerical values in equation (1) and evaluate Δ*W*:

$$\Delta W = 83.3\,\text{J} - 50\,\text{J} = \boxed{33\,\text{J}}$$

43 •• A reversible engine working between two reservoirs at temperatures T_h and T_c has an efficiency of 30%. Working as a heat engine, it releases 140 J per cycle of heat to the cold reservoir. A second engine working between the same two reservoirs also releases 140 J per cycle to the cold reservoir. Show that if the second engine has an efficiency greater than 30%, the two engines working together would violate the heat-engine statement of the second law.

Determine the Concept Let the first engine be run as a refrigerator. Then it will remove 140 J from the cold reservoir, deliver 200 J to the hot reservoir, and require 60 J of energy to operate. Now take the second engine and run it between the same reservoirs, and let it eject 140 J into the cold reservoir, thus replacing the heat removed by the refrigerator. If ε_2, the efficiency of this engine, is greater than 30%, then Q_{h2}, the heat removed from the hot reservoir by this engine, is 140 J/$(1 - \varepsilon_2)$ > 200 J, and the work done by this engine is $W = \varepsilon_2 Q_{h2} > 60$ J. The end result of all this is that the second engine can run the refrigerator, replacing the heat taken from the cold reservoir, and do additional mechanical work. The two systems working together then convert heat into mechanical energy without rejecting any heat to a cold reservoir, in violation of the second law.

44 •• A reversible engine working between two reservoirs at temperatures T_h and T_c has an efficiency of 20%. Working as a heat engine, it does 100 J of work per cycle. A second engine working between the same two reservoirs also does 100 J of work per cycle. Show that if the efficiency of the second engine is greater

than 20%, the two engines working together would violate the refrigerator statement of the second law.

Determine the Concept If the reversible engine is run as a refrigerator, it will require 100 J of mechanical energy to take 400 J of heat from the cold reservoir and deliver 500 J to the hot reservoir. Now let the second engine, with $\varepsilon_2 > 0.2$, operate between the same two heat reservoirs and use it to drive the refrigerator. Because $\varepsilon_2 > 0.2$, this engine will remove less than 500 J from the hot reservoir in the process of doing 100 J of work. The net result is then that no net work is done by the two systems working together, but a finite amount of heat is transferred from the cold reservoir to the hot reservoir, in violation of the refrigerator statement of the second law.

45 •• A Carnot engine works between two heat reservoirs as a refrigerator. During each cycle, 100 J of heat are absorbed and 150 J are released to the hot reservoir. (*a*) What is the efficiency of the Carnot engine when it works as a heat engine between the same two reservoirs? (*b*) Show that no other engine working as a refrigerator between the same two reservoirs can have a COP greater than 2.00.

Picture the Problem We can use the definition of efficiency to find the efficiency of the Carnot engine operating between the two reservoirs.

(*a*) The efficiency of the Carnot engine is given by:

$$\varepsilon_C = \frac{W}{Q_h} = \frac{50\,\text{J}}{150\,\text{J}} = \boxed{33\%}$$

(*b*) If the COP > 2, then 50 J of work will remove more than 100 J of heat from the cold reservoir and put more than 150 J of heat into the hot reservoir. So running the engine described in Part (*a*) to operate the refrigerator with a COP > 2 will result in the transfer of heat from the cold to the hot reservoir without doing any net mechanical work in violation of the second law.

46 •• A Carnot engine works between two heat reservoirs at temperatures $T_h = 300$ K and $T_c = 77.0$ K. (*a*) What is its efficiency? (*b*) If it absorbs 100 J of heat from the hot reservoir during each cycle, how much work does it do? (*c*) How much heat does it release to the low-temperature reservoir during each cycle? (*d*) What is the coefficient of performance of this engine when it works as a refrigerator between these two reservoirs?

Picture the Problem We can use the definitions of the efficiency of a Carnot engine and the coefficient of performance of a refrigerator to find these quantities. The work done each cycle by the Carnot engine is given by $W = \varepsilon_C Q_h$ and we can use the conservation of energy to find the heat rejected to the low-temperature reservoir.

(*a*) The efficiency of a Carnot engine depends on the temperatures of the hot and cold reservoirs:

$$\varepsilon_C = 1 - \frac{T_c}{T_h} = 1 - \frac{77.0\,\text{K}}{300\,\text{K}} = \boxed{74.3\%}$$

(*b*) Express the work done each cycle in terms of the efficiency of the engine and the heat absorbed from the high-temperature reservoir:

$$W = \varepsilon_C Q_h = (0.743)(100\,\text{J}) = \boxed{74.3\,\text{J}}$$

(*c*) Apply conservation of energy to obtain:

$$Q_c = Q_h - W = 100\,\text{J} - 74.3\,\text{J} = \boxed{26\,\text{J}}$$

(*d*) Using its definition, express and evaluate the refrigerator's coefficient of performance:

$$\text{COP} = \frac{Q_c}{W} = \frac{26\,\text{J}}{74.3\,\text{J}} = \boxed{0.35}$$

47 •• **[SSM]** In the cycle shown in Figure 19-19, 1.00 mol of an ideal diatomic gas is initially at a pressure of 1.00 atm and a temperature of 0.0ºC. The gas is heated at constant volume to T_2 = 150ºC and is then expanded adiabatically until its pressure is again 1.00 atm. It is then compressed at constant pressure back to its original state. Find (*a*) the temperature after the adiabatic expansion, (*b*) the heat absorbed or released by the system during each step, (*c*) the efficiency of this cycle, and (*d*) the efficiency of a Carnot cycle operating between the temperature extremes of this cycle.

Picture the Problem We can use the ideal-gas law for a fixed amount of gas and the equations of state for an adiabatic process to find the temperatures, volumes, and pressures at the end points of each process in the given cycle. We can use $Q = C_V \Delta T$ and $Q = C_P \Delta T$ to find the heat entering and leaving during the constant-volume and isobaric processes and the first law of thermodynamics to find the work done each cycle. Once we've calculated these quantities, we can use its definition to find the efficiency of the cycle and the definition of the Carnot efficiency to find the efficiency of a Carnot engine operating between the extreme temperatures.

(*a*) Apply the ideal-gas law for a fixed amount of gas to relate the temperature at point 3 to the temperature at point 1:

$$\frac{P_1V_1}{T_1}=\frac{P_3V_3}{T_3}$$

or, because $P_1 = P_3$,

$$T_3=T_1\frac{V_3}{V_1} \qquad (1)$$

Apply the ideal-gas law for a fixed amount of gas to relate the pressure at point 2 to the temperatures at points 1 and 2 and the pressure at 1:

$$\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}\Rightarrow P_2=\frac{P_1V_1T_2}{V_2T_1}$$

Because $V_1 = V_2$:

$$P_2=P_1\frac{T_2}{T_1}=(1.00\,\text{atm})\frac{423\,\text{K}}{273\,\text{K}}=1.55\,\text{atm}$$

Apply an equation for an adiabatic process to relate the pressures and volumes at points 2 and 3:

$$P_1V_1^{\gamma}=P_3V_3^{\gamma}\Rightarrow V_3=V_1\left(\frac{P_1}{P_3}\right)^{\frac{1}{\gamma}}$$

Noting that V_1 = 22.4 L, evaluate V_3:

$$V_3=(22.4\,\text{L})\left(\frac{1.55\,\text{atm}}{1\,\text{atm}}\right)^{\frac{1}{1.4}}=30.6\,\text{L}$$

Substitute numerical values in equation (1) and evaluate T_3 and t_3:

$$T_3=(273\,\text{K})\frac{30.6\,\text{L}}{22.4\,\text{L}}=373\,\text{K}$$

and

$$t_3=T_3-273=\boxed{100°\text{C}}$$

(*b*) Process 1→2 takes place at constant volume (note that γ = 1.4 corresponds to a diatomic gas and that $C_P - C_V = R$):

$$Q_{12}=C_V\Delta T_{12}=\tfrac{5}{2}R\Delta T_{12}$$
$$=\tfrac{5}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(423\,\text{K}-273\,\text{K})$$
$$=\boxed{3.12\,\text{kJ}}$$

Process 2→3 takes place adiabatically:

$$Q_{23}=\boxed{0}$$

Process 3→1 is isobaric (note that $C_P = C_V + R$):

$$Q_{31}=C_P\Delta T_{31}=\tfrac{7}{2}R\Delta T_{12}$$
$$=\tfrac{7}{2}\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(273\,\text{K}-373\,\text{K})$$
$$=\boxed{-2.91\,\text{kJ}}$$

(*c*) The efficiency of the cycle is given by:

$$\varepsilon = \frac{W}{Q_{in}} \quad (2)$$

Apply the first law of thermodynamics to the cycle:

$\Delta E_{int} = Q_{in} + W_{on}$
or, because $\Delta E_{int,\,cycle} = 0$ (the system begins and ends in the same state) and $W_{on} = -W_{by\ the\ gas} = Q_{in}$.

Evaluating *W* yields:

$$W = \sum Q = Q_{12} + Q_{23} + Q_{31} = 3.12\,kJ + 0 - 2.91\,kJ = 0.21\,kJ$$

Substitute numerical values in equation (2) and evaluate ε:

$$\varepsilon = \frac{0.21\,kJ}{3.12\,kJ} = \boxed{6.7\%}$$

(*d*) Express and evaluate the efficiency of a Carnot cycle operating between 423 K and 273 K:

$$\varepsilon_C = 1 - \frac{T_c}{T_h} = 1 - \frac{273\,K}{423\,K} = \boxed{35.5\%}$$

48 •• You are part of a team that is completing a mechanical-engineering project. Your team built a steam engine that takes in superheated steam at 270ºC and discharges condensed steam from its cylinder at 50.0ºC. Your team has measured its efficiency to be 30.0%. (*a*) How does this efficiency compare with the maximum possible efficiency for your engine? (*b*) If the useful power output of the engine is known to be 200 kW, how much heat does the engine release to its surroundings in 1.00 h?

Picture the Problem We can find the maximum efficiency of the steam engine by calculating the Carnot efficiency of an engine operating between the given temperatures. We can apply the definition of efficiency to find the heat discharged to the engine's surroundings in 1.00 h.

(*a*) The efficiency of the steam engine as a percentage of the maximum possible efficiency is given by:

$$\frac{\varepsilon_{steam\,engine}}{\varepsilon_{max}} = \frac{0.300}{\varepsilon_{max}}$$

The efficiency of a Carnot engine operating between temperatures F_c and T_h is:

$$\varepsilon_{max} = 1 - \frac{T_c}{T_h} = 1 - \frac{323\,K}{543\,K} = 40.52\%$$

Substituting for ε_{max} yields:

$$\frac{\varepsilon_{\text{steam engine}}}{\varepsilon_{max}} = \frac{0.300}{0.4052} = 74.05\%$$

or

$$\varepsilon_{\text{steam engine}} = \boxed{0.740\varepsilon_{max}}$$

(*b*) Relate the heat Q_c discharged to the engine's surroundings to Q_h and the efficiency of the engine:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} \Rightarrow Q_c = (1-\varepsilon)Q_h$$

Using its definition, relate the efficiency of the engine to the heat intake of the engine and the work it does each cycle:

$$Q_h = \frac{W}{\varepsilon} = \frac{P\Delta t}{\varepsilon}$$

Substitute for Q_h in the expression for $\boldsymbol{Q}_c$ and simplify to obtain:

$$Q_c = (1-\varepsilon)\frac{P\Delta t}{\varepsilon} = \left(\frac{1}{\varepsilon} - 1\right)P\Delta t$$

Substitute numerical values and evaluate $Q_c(1.00\text{ h})$:

$$Q_c(1.00\text{ h}) = \left(\frac{1}{0.300} - 1\right)\left(200\frac{\text{kJ}}{\text{s}}\right)(3600\text{ s}) = \boxed{1.68\text{ GJ}}$$

*Heat Pumps

49 • **[SSM]** As an engineer, you are designing a heat pump that is capable of delivering heat at the rate of 20 kW to a house. The house is located where, in January, the average outside temperature is –10ºC. The temperature of the air in the air handler inside the house is to be 40ºC. (*a*) What is maximum possible COP for a heat pump operating between these temperatures? (*b*) What must the minimum power of the electric motor driving the heat pump be? (*c*) In reality, the COP of the heat pump will be only 60 percent of the ideal value. What is the minimum power of the electric motor when the COP is 60 percent of the ideal value?

Picture the Problem We can use the definition of the COP_{HP} and the Carnot efficiency of an engine to express the maximum efficiency of the refrigerator in terms of the reservoir temperatures. We can apply the definition of power to find the minimum power needed to run the heat pump.

(*a*) Express the COP_{HP} in terms of T_h and T_c:

$$COP_{HP} = \frac{Q_h}{W} = \frac{Q_h}{Q_h - Q_c}$$
$$= \frac{1}{1-\frac{Q_c}{Q_h}} = \frac{1}{1-\frac{T_c}{T_h}} = \frac{T_h}{T_h - T_c}$$

Substitute numerical values and evaluate COP_{HP}:

$$COP_{HP} = \frac{313\,\text{K}}{313\,\text{K} - 263\,\text{K}} = 6.26$$
$$= \boxed{6.3}$$

(*b*) The COP_{HP} is also given by:

$$COP_{HP} = \frac{P_{out}}{P_{motor}} \Rightarrow P_{motor} = \frac{P_{out}}{COP_{HP}}$$

Substitute numerical values and evaluate P_{motor}:

$$P_{motor} = \frac{20\,\text{kW}}{6.26} = \boxed{3.2\,\text{kW}}$$

(*c*) The minimum power of the electric motor is given by:

$$P_{min} = \frac{\frac{dQ_c}{dt}}{\varepsilon_{HP}} = \frac{\frac{dQ_c}{dt}}{\varepsilon\left(COP_{HP,max}\right)}$$

where ε_{HP} is the efficiency of the heat pump.

Substitute numerical values and evaluate P_{min}:

$$P_{min} = \frac{20\,\text{kW}}{(0.60)(6.26)} = \boxed{5.3\,\text{kW}}$$

50 • A refrigerator is rated at 370 W. (*a*) What is the maximum amount of heat it can absorb from the food compartment in 1.00 min if the food-compartment temperature of the refrigerator is 0.0ºC and it releases heat into a room at 20.0ºC? (*b*) If the COP of the refrigerator is 70% of that of a reversible refrigerator, how much heat can it absorb from the food compartment in 1.00 min under these conditions?

Picture the Problem We can use the definition of the COP to relate the heat removed from the refrigerator to its power rating and operating time. By expressing the COP in terms of T_c and T_h we can write the amount of heat removed from the refrigerator as a function of T_c, T_h, P, and Δt.

(*a*) Express the amount of heat the refrigerator can remove in a given period of time as a function of its COP:

$$Q_c = (COP)W$$
$$= (COP)P\Delta t$$

Express the COP in terms of T_h and T_c and simplify to obtain:

$$\text{COP} = \frac{Q_c}{W} = \frac{Q_c}{\varepsilon Q_h} = \frac{Q_h - W}{\varepsilon Q_h} = \frac{1-\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} - 1 = \frac{1}{1-\frac{T_c}{T_h}} - 1 = \frac{T_c}{T_h - T_c}$$

Substituting for COP yields:

$$Q_c = \left(\frac{T_c}{T_h - T_c}\right)P\Delta t$$

Substitute numerical values and evaluate Q_c:

$$Q_c = \left(\frac{273\ \text{K}}{293\ \text{K} - 273\ \text{K}}\right)(370\ \text{W})\left(1.00\ \text{min} \times \frac{60\ \text{s}}{\text{min}}\right) = 303\ \text{kJ} = \boxed{0.30\ \text{MJ}}$$

(*b*) If the COP is 70% of the efficiency of an ideal pump:

$$Q_c^{'} = (0.70)(303\ \text{kJ}) = \boxed{0.21\ \text{MJ}}$$

51 • A refrigerator is rated at 370 W. (*a*) What is the maximum amount of heat it can absorb for the food compartment in 1.00 min if the temperature in the compartment is 0.0ºC and it releases heat into a room at 35ºC? (*b*) If the COP of the refrigerator is 70% of that of a reversible pump, how much heat can it absorb from the food compartment in 1.00 min? Is the COP for the refrigerator greater when the temperature of the room is 35ºC or 20ºC? Explain.

Picture the Problem We can use the definition of the COP to relate the heat removed from the refrigerator to its power rating and operating time. By expressing the COP in terms of T_c and T_h we can write the amount of heat removed from the refrigerator as a function of T_c, T_h, P, and Δt.

(*a*) Express the amount of heat the refrigerator can remove in a given period of time as a function of its COP:

$$Q_c = (\text{COP})W = (\text{COP})P\Delta t$$

Express the COP in terms of T_h and T_c and simplify to obtain:

$$\text{COP} = \frac{Q_c}{W} = \frac{Q_c}{\varepsilon Q_h} = \frac{Q_h - W}{\varepsilon Q_h} = \frac{1-\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} - 1 = \frac{1}{1-\dfrac{T_c}{T_h}} - 1 = \frac{T_c}{T_h - T_c}$$

Substituting for COP yields:

$$Q_c = \left(\frac{T_c}{T_h - T_c}\right) P\Delta t$$

Substitute numerical values and evaluate Q_c:

$$Q_c = \left(\frac{273\ \text{K}}{308\ \text{K} - 273\ \text{K}}\right)(370\ \text{W})\left(1.00\ \text{min} \times \frac{60\ \text{s}}{\text{min}}\right) = 173\ \text{kJ} = \boxed{0.17\ \text{MJ}}$$

(*b*) If the COP is 70% of the efficiency of an ideal pump:

$$Q_c' = (0.70)(173\ \text{kJ}) = \boxed{0.12\ \text{MJ}}$$

Because the temperature difference increases when the room is warmer, the COP decreases.

52 ••• You are installing a heat pump, whose COP is half the COP of a reversible heat pump. You will use the pump on chilly winter nights to increase the air temperature in your bedroom. Your bedroom's dimensions are 5.00 m × 3.50 m × 2.50 m. The air temperature should increase from 63°F to 68°F. The outside temperature is 35°F, and the temperature at the air handler in the room is 112°F. If the pump's electric power consumption is 750 W, how long will you have to wait in order for the room's air to warm (take the specific heat of air to be 1.005 kJ/(kg·°C)? Assume you have good window draperies and good wall insulation so that you can neglect the release of heat through windows, walls, ceilings and floors. Also assume that the heat capacity of the floor, ceiling, walls and furniture are negligible.

Picture the Problem We can use the definition of the coefficient of performance of a heat pump and the relationship between the work done per cycle and the pump's power consumption to find your waiting time.

The coefficient of performance of the heat pump is defined as:

$$\mathrm{COP_{HP}} = \frac{Q_h}{W} = \frac{Q_h}{P\Delta t} \Rightarrow \Delta t = \frac{Q_h}{(\mathrm{COP_{HP}})P}$$

where Q_h is the heat required to raise the temperature of your bedroom, P is the power consumption of the heat pump, and Δt is the time required to warm the bedroom.

We're given that the coefficient of performance of the heat pump is half the coefficient of performance of an ideal heat pump:

$$\mathrm{COP_{HP}} = \frac{Q_h}{W} = \tfrac{1}{2}\mathrm{COP_{max}} = \tfrac{1}{2}\left(\frac{T_h}{T_h - T_c}\right)$$

Substituting for $\mathrm{COP_{HP}}$ yields:

$$\Delta t = \frac{2Q_h}{\left(\frac{T_h}{T_h - T_c}\right)P}$$

The heat required to warm the room is related to the volume of the room, the density of air, and the desired increase in temperature:

$$Q_h = mc\Delta T = \rho Vc\Delta T$$

where ρ is the density of air and c is its specific heat capacity.

Substitute for Q_h to obtain:

$$\Delta t = \frac{2\rho Vc\Delta T}{\left(\frac{T_h}{T_h - T_c}\right)P}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2\left(1.293\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)(5.00\ \mathrm{m}\times 3.50\ \mathrm{m}\times 2.50\ \mathrm{m})\left(1005\,\frac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}}\right)\left(5\ \mathrm{F^\circ}\times\frac{5\ \mathrm{C^\circ}}{9\ \mathrm{F^\circ}}\right)}{\left(\frac{317\ \mathrm{K}}{317\ \mathrm{K} - 275\ \mathrm{K}}\right)(750\ \mathrm{W})} = \boxed{56\ \mathrm{s}}$$

Entropy Changes

53 • **[SSM]** You inadvertently leave a pan of water boiling away on the hot stove. You return just in time to see the last drop converted into steam. The pan originally held 1.00 L of boiling water. What is the change in entropy of the water associated with its change of state from liquid to gas?

Picture the Problem Because the water absorbed heat in the vaporization process its change in entropy is positive and given by $\Delta S_{H_2O} = \frac{Q_{\substack{\text{absorbed}\\ \text{by } H_2O}}}{T}$. See Table 18-2 for the latent heat of vaporization of water.

The change in entropy of the water is given by:

$$\Delta S_{H_2O} = \frac{Q_{\substack{\text{absorbed}\\ \text{by } H_2O}}}{T}$$

The heat absorbed by the water as it vaporizes is the product of its mass and latent heat of vaporization:

$$Q_{\substack{\text{absorbed}\\ \text{by } H_2O}} = mL_v = \rho V L_v$$

Substituting for $Q_{\substack{\text{absorbed}\\ \text{by } H_2O}}$ yields:

$$\Delta S_{H_2O} = \frac{\rho V L_v}{T}$$

Substitute numerical values and evaluate $\Delta \boldsymbol{S}_{H_2O}$:

$$\Delta S_{H_2O} = \frac{\left(1.00\,\frac{\text{kg}}{\text{L}}\right)(1.00\text{ L})\left(2257\,\frac{\text{kJ}}{\text{kg}}\right)}{373\text{ K}} = \boxed{6.05\,\frac{\text{kJ}}{\text{K}}}$$

54 • What is the change in entropy of 1.00 mol of liquid water at 0.0ºC that freezes to ice at 0.0°C?

Picture the Problem We can use the definition of entropy change to find the change in entropy of the liquid water as it freezes. Because heat is removed from liquid water when it freezes, the change in entropy of the liquid water is negative. See Appendix C for the molar mass of water and Table 18-2 for the latent heat of fusion of water.

The change in entropy of the water is given by::

$$\Delta S_{H_2O} = \frac{Q_{\substack{\text{removed}\\ \text{from } H_2O}}}{T}$$

The heat removed from the water as it freezes is the product of its mass and latent heat of fusion:

$$Q_{\substack{\text{removed}\\ \text{from } H_2O}} = -mL_f$$

or, because $m = nM_{H_2O}$,

$$Q_{\substack{\text{removed}\\ \text{from } H_2O}} = -nM_{H_2O}L_f$$

Substitute numerical values and evaluate ΔS_{H_2O}:

$$\Delta S_{H_2O} = \frac{-(1.00\,\text{mol})\left(18.015\,\frac{\text{g}}{\text{mol}}\right)\left(333.5\,\frac{\text{J}}{\text{g}}\right)}{273\,\text{K}} = \boxed{-22.0\,\frac{\text{J}}{\text{K}}}$$

55 •• Consider the freezing of 50.0 g of water once it is placed in the freezer compartment of a refrigerator. Assume the walls of the freezer are maintained at –10ºC. The water, initially liquid at 0.0ºC, is frozen into ice and cooled to –10ºC. Show that even though the entropy of the water decreases, the net entropy of the universe increases.

Picture the Problem The change in the entropy of the universe resulting from the freezing of this water and the cooling of the ice formed is the sum of the entropy changes of the water-ice and the freezer. Note that, while the entropy of the water decreases, the entropy of the freezer increases.

The change in entropy of the universe resulting from this freezing and cooling process is given by:

$$\Delta S_u = \Delta S_{\text{water}} + \Delta S_{\text{freezer}} \quad (1)$$

Express ΔS_{water}:

$$\Delta S_{\text{water}} = \Delta S_{\text{freezing}} + \Delta S_{\text{cooling}} \quad (2)$$

Express $\Delta S_{\text{freezing}}$:

$$\Delta S_{\text{freezing}} = \frac{-Q_{\text{freezing}}}{T_{\text{freezing}}} \quad (3)$$

where the minus sign is a consequence of the fact that energy is leaving the water as it freezes.

Relate Q_{freezing} to the latent heat of fusion and the mass of the water:

$$Q_{\text{freezing}} = mL_f$$

Substitute in equation (3) to obtain:

$$\Delta S_{\text{freezing}} = \frac{-mL_f}{T_{\text{freezing}}}$$

Express $\Delta S_{\text{cooling}}$:

$$\Delta S_{\text{cooling}} = mC_p \ln\left(\frac{T_f}{T_i}\right)$$

Substitute in equation (2) to obtain:

$$\Delta S_{\text{water}} = \frac{-mL_f}{T_{\text{freezing}}} + mC_p \ln\left(\frac{T_f}{T_i}\right)$$

Noting that the freezer gains heat (at 263 K) from the freezing water and cooling ice, express $\Delta S_{\text{freezer}}$:

$$\Delta S_{\text{freezer}} = \frac{\Delta Q_{\text{ice}}}{T_{\text{freezer}}} + \frac{\Delta Q_{\text{cooling ice}}}{T_{\text{freezer}}} = \frac{mL_{\text{f}}}{T_{\text{freezer}}} + \frac{mC_{\text{p}}\Delta T}{T_{\text{freezer}}}$$

Substitute for ΔS_{water} and $\Delta S_{\text{freezer}}$ in equation (1):

$$\Delta S_{\text{u}} = \frac{-mL_{\text{f}}}{T_{\text{freezing}}} + mC_{\text{p}}\ln\left(\frac{T_{\text{f}}}{T_{\text{i}}}\right) + \frac{mL_{\text{f}}}{T_{\text{freezer}}} + \frac{mC_{\text{p}}\Delta T}{T_{\text{freezer}}} = m\left[\frac{-L_{\text{f}}}{T_{\text{freezing}}} + C_{\text{p}}\ln\left(\frac{T_{\text{f}}}{T_{\text{i}}}\right) + \frac{L_{\text{f}} + C_{\text{p}}\Delta T}{T_{\text{freezer}}}\right]$$

Substitute numerical values and evaluate ΔS_{u}:

$$\Delta S_{\text{u}} = (0.0500\,\text{kg})\left[-\frac{333.5\times10^{3}\,\frac{\text{J}}{\text{kg}}}{273\,\text{K}} + \left(2100\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{263\,\text{K}}{273\,\text{K}}\right) + \frac{333.5\times10^{3}\,\frac{\text{J}}{\text{kg}} + \left(2100\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)(273\,\text{K} - 263\,\text{K})}{263\,\text{K}}\right] = \boxed{2.40\,\text{J/K}}$$

and, because $\Delta S_{\text{u}} > 0$, the entropy of the universe increases.

56 • In this problem, 2.00 mol of an ideal gas at 400 K expand quasi-statically and isothermally from an initial volume of 40.0 L to a final volume of 80.0 L. (*a*) What is the entropy change of the gas? (*b*) What is the entropy change of the universe for this process?

Picture the Problem We can use the definition of entropy change and the 1st law of thermodynamics to express ΔS for the ideal gas as a function of its initial and final volumes.

(*a*) The entropy change of the gas is given by:

$$\Delta S_{\text{gas}} = \frac{Q}{T} \qquad (1)$$

Apply the first law of thermodynamics to the isothermal process to express Q in terms of W_{on}:

$$Q = \Delta E_{int} - W_{on}$$

or, because $\Delta E_{int} = 0$ for an isothermal expansion of a gas,

$$Q = -W_{on}$$

The work done on the gas is given by:

$$W_{on} = nRT\ln\left(\frac{V_i}{V_f}\right) \Rightarrow Q = -nRT\ln\left(\frac{V_i}{V_f}\right)$$

Substitute for Q in equation (1) to obtain:

$$\Delta S_{gas} = -nR\ln\left(\frac{V_i}{V_f}\right)$$

Substitute numerical values and evaluate ΔS:

$$\Delta S_{gas} = -(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)\ln\left(\frac{40.0\,\text{L}}{80.0\,\text{L}}\right) = \boxed{11.5\frac{\text{J}}{\text{K}}}$$

(b) Because the process is reversible:

$$\Delta S_u = \boxed{0}$$

Remarks: The entropy change of the environment of the gas is −11.5 J/K.

57 •• [SSM] A system completes a cycle consisting of six quasi-static steps, during which the total work done by the system is 100 J. During step 1 the system absorbs 300 J of heat from a reservoir at 300 K, during step 3 the system absorbs 200 J of heat from a reservoir at 400 K, and during step 5 it absorbs heat from a reservoir at temperature T_3. (During steps 2, 4 and 6 the system undergoes adiabatic processes in which the temperature of the system changes from one reservoir's temperature to that of the next.) (a) What is the entropy change of the system for the complete cycle? (b) If the cycle is reversible, what is the temperature T_3?

Picture the Problem We can use the fact that the system *returns to its original state* to find the entropy change for the complete cycle. Because the entropy change for the complete cycle is the sum of the entropy changes for each process, we can find the temperature T_3 from the entropy changes during the 1st two processes and the heat released during the third.

(a) Because S is a state function of the system, and because the system's final state is identical to its initial state:

$$\Delta S_{\substack{\text{system}\\ \text{1 complete cycle}}} = \boxed{0}$$

(*b*) Relate the entropy changes for each of the three heat reservoirs and the system for one complete cycle of the system:

$$\Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_{\text{system}} = 0$$

or

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} + \frac{Q_3}{T_3} + 0 = 0$$

Substitute numerical values. Heat is rejected by the two high-temperature reservoirs and absorbed by the cold reservoir:

$$\frac{-300\,\text{J}}{300\,\text{K}} + \frac{-200\,\text{J}}{400\,\text{K}} + \frac{400\,\text{J}}{T_3} = 0$$

Solving for T_3 yields:

$$T_3 = \boxed{267\,\text{K}}$$

58 •• In this problem, 2.00 mol of an ideal gas initially has a temperature of 400 K and a volume of 40.0 L. The gas undergoes a free adiabatic expansion to twice its initial volume. What is (*a*) the entropy change of the gas and (*b*) the entropy change of the universe?

Picture the Problem The initial and final temperatures are the same for a free expansion of an ideal gas. Thus, the entropy change ΔS for a free expansion from V_i to V_f is the same as ΔS for an isothermal process from V_i to V_f. We can use the definition of entropy change and the 1st law of thermodynamics to express ΔS for the ideal gas as a function of its initial and final volumes.

(*a*) The entropy change of the gas is given by:

$$\Delta S_{\text{gas}} = \frac{Q}{T} \qquad (1)$$

Apply the first law of thermodynamics to the isothermal process to express Q:

$$Q = \Delta E_{\text{int}} - W_{\text{on}}$$

or, because $\Delta E_{\text{int}} = 0$ for a free expansion of a gas,

$$Q = -W_{\text{on}}$$

The work done on the gas is given by:

$$W_{\text{on}} = nRT\ln\left(\frac{V_\text{i}}{V_\text{f}}\right) \Rightarrow Q = -nRT\ln\left(\frac{V_\text{i}}{V_\text{f}}\right)$$

Substitute for Q in equation (1) to obtain:

$$\Delta S_{\text{gas}} = -nR\ln\left(\frac{V_\text{i}}{V_\text{f}}\right)$$

Substitute numerical values and evaluate ΔS:

$$\Delta S_{\text{gas}} = -(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)\ln\left(\frac{40.0\,\text{L}}{80.0\,\text{L}}\right) = \boxed{11.5\frac{\text{J}}{\text{K}}}$$

(*b*) The change in entropy of the universe is the sum of the entropy changes of the gas and the surroundings:

$$\Delta S_{\text{u}} = \Delta S_{\text{gas}} + \Delta S_{\text{surroundings}}$$

For the change in entropy of the surroundings we use the fact that, during the free expansion, the surroundings are unaffected:

$$\Delta S_{\text{surroundings}} = \frac{Q_{\text{rev}}}{T} = \frac{0}{T} = 0$$

The change in entropy of the universe is the change in entropy of the gas:

$$\Delta S_{\text{u}} = \boxed{11.5\,\frac{\text{J}}{\text{K}}}$$

59 •• A 200-kg block of ice at 0.0ºC is placed in a large lake. The temperature of the lake is just slightly higher than 0.0ºC, and the ice melts very slowly. (*a*) What is the entropy change of the ice? (*b*) What is the entropy change of the lake? (*c*) What is the entropy change of the universe (the ice plus the lake)?

Picture the Problem Because the ice gains heat as it melts, its entropy change is positive and can be calculated from its definition. Because the temperature of the lake is just slightly greater than 0°C and the mass of water is so much greater than that of the block of ice, the absolute value of the entropy change of the lake will be approximately equal to the entropy change of the ice as it melts.

(*a*) The entropy change of the ice is given by:

$$\Delta S_{\text{ice}} = \frac{mL_{\text{f}}}{T}$$

Substitute numerical values and evaluate ΔS_{ice}:

$$\Delta S_{\text{ice}} = \frac{(200\,\text{kg})\left(333.5\,\frac{\text{kJ}}{\text{kg}}\right)}{273\,\text{K}} = \boxed{244\,\frac{\text{kJ}}{\text{K}}}$$

(*b*) Relate the entropy change of the lake to the entropy change of the ice:

$$\Delta S_{\text{lake}} \approx -\Delta S_{\text{ice}} = \boxed{-244\,\frac{\text{kJ}}{\text{K}}}$$

(*c*) The entropy change of the universe due to this melting process is the sum of the entropy changes of the ice and the lake:

$$\Delta S_{\text{u}} = \Delta S_{\text{ice}} + \Delta S_{\text{lake}}$$

Because the temperature of the lake is slightly greater than that of the ice, the magnitude of the entropy change of the lake is less than 244 kJ/K and the entropy change of the universe is just slightly greater than zero. The melting of the ice is an irreversible process and $\boxed{\Delta S_u > 0}$.

60 •• A 100-g piece of ice at 0.0ºC is placed in an insulated calorimeter with negligible heat capacity containing 100 g of water at 100ºC. (*a*) What is the final temperature of the water once thermal equilibrium is established? (*b*) Find the entropy change of the universe for this process.

Picture the Problem We can use conservation of energy to find the equilibrium temperature of the water and apply the equations for the entropy change during a melting process and for constant-pressure processes to find the entropy change of the universe (the entropy change of the piece of ice plus the entropy change of the water in the insulated container).

(*a*) Apply conservation of energy to obtain:

$$\sum_i Q_i = 0$$

or

$$Q_{\substack{\text{melting}\\\text{ice}}} + Q_{\substack{\text{warming}\\\text{water}}} - Q_{\substack{\text{cooling}\\\text{water}}} = 0$$

Substitute to relate the masses of the ice and water to their temperatures, specific heats, and the final temperature of the water:

$$(100\,\text{g})\left(333.5\,\frac{\text{kJ}}{\text{kg}}\right) + (100\,\text{g})\left(4.18\,\frac{\text{kJ}}{\text{kg}\cdot\text{C}^\circ}\right)t - (100\,\text{g})\left(4.18\,\frac{\text{kJ}}{\text{kg}\cdot\text{C}^\circ}\right)(100^\circ\text{C} - t) = 0$$

Solving for t yields:

$$t = \boxed{10.1^\circ\text{C}}$$

(*b*) The entropy change of the universe is the sum of the entropy changes of the ice and the water:

$$\Delta S_u = \Delta S_{\text{ice}} + \Delta S_{\text{water}}$$

Using the expression for the entropy change for a constant-pressure process, express the entropy change of the melting ice and warming ice-water:

$$\Delta S_{\text{ice}} = \Delta S_{\text{melting ice}} + \Delta S_{\text{warming water}} = \frac{mL_f}{T_f} + mc_P \ln\left(\frac{T_f}{T_i}\right)$$

Substitute numerical values to obtain:

$$\Delta S_{\text{ice}} = \frac{(0.100\,\text{kg})\left(333.5\,\frac{\text{kJ}}{\text{kg}}\right)}{273\,\text{K}} + (0.100\,\text{kg})\left(4.18\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{283\,\text{K}}{273\,\text{K}}\right) = 137\,\frac{\text{J}}{\text{K}}$$

Find the entropy change of the cooling water:

$$\Delta S_{\text{water}} = (0.100\,\text{kg})\left(4.18\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{283\,\text{K}}{373\,\text{K}}\right) = -115\,\frac{\text{J}}{\text{K}}$$

Substitute for ΔS_{ice} and ΔS_{water} and evaluate the entropy change of the universe:

$$\Delta S_{\text{u}} = 137\,\frac{\text{J}}{\text{K}} - 115\,\frac{\text{J}}{\text{K}} = \boxed{22\,\frac{\text{J}}{\text{K}}}$$

Remarks: The result that $\Delta S_{\text{u}} > 0$ tells us that this process is irreversible.

61 •• [SSM] A 1.00-kg block of copper at 100ºC is placed in an insulated calorimeter of negligible heat capacity containing 4.00 L of liquid water at 0.0ºC. Find the entropy change of (*a*) the copper block, (*b*) the water, and (*c*) the universe.

Picture the Problem We can use conservation of energy to find the equilibrium temperature of the water and apply the equations for the entropy change during a constant pressure process to find the entropy changes of the copper block, the water, and the universe.

(*a*) Use the equation for the entropy change during a constant-pressure process to express the entropy change of the copper block:

$$\Delta S_{\text{Cu}} = m_{\text{Cu}} c_{\text{Cu}} \ln\left(\frac{T_{\text{f}}}{T_{\text{i}}}\right) \qquad (1)$$

Apply conservation of energy to obtain:

$$\sum_{\text{i}} Q_{\text{i}} = 0$$

or

$$Q_{\substack{\text{copper}\\\text{block}}} + Q_{\substack{\text{warming}\\\text{water}}} = 0$$

Substitute to relate the masses of the block and water to their temperatures, specific heats, and the final temperature T_f of the water:

$$(1.00\,\text{kg})\left(0.386\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(T_f - 373\,\text{K}) + (4.00\,\text{L})\left(1.00\frac{\text{kg}}{\text{L}}\right)\left(4.18\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(T_f - 273\,\text{K}) = 0$$

Solve for T_f to obtain: $T_f = 275.26\,\text{K}$

Substitute numerical values in equation (1) and evaluate ΔS_{Cu}:

$$\Delta S_{\text{Cu}} = (1.00\,\text{kg})\left(0.386\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{275.26\,\text{K}}{373\,\text{K}}\right) = \boxed{-117\frac{\text{J}}{\text{K}}}$$

(*b*) The entropy change of the water is given by:

$$\Delta S_{\text{water}} = m_{\text{water}}c_{\text{water}}\ln\left(\frac{T_f}{T_i}\right)$$

Substitute numerical values and evaluate ΔS_{water}:

$$\Delta S_{\text{water}} = (4.00\,\text{kg})\left(4.18\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{275.26\,\text{K}}{273\,\text{K}}\right) = \boxed{138\frac{\text{J}}{\text{K}}}$$

(*c*) Substitute for ΔS_{Cu} and ΔS_{water} and evaluate the entropy change of the universe:

$$\Delta S_u = \Delta S_{\text{Cu}} + \Delta S_{\text{water}} = -117\frac{\text{J}}{\text{K}} + 138\frac{\text{J}}{\text{K}} = \boxed{21\frac{\text{J}}{\text{K}}}$$

Remarks: The result that $\Delta S_u > 0$ tells us that this process is irreversible.

62 •• If a 2.00-kg piece of lead at 100ºC is dropped into a lake at 10ºC, find the entropy change of the universe.

Picture the Problem Because the mass of the water in the lake is so much greater than the mass of the piece of lead, the temperature of the lake will increase only slightly and we can reasonably assume that its final temperature is 10°C. We can apply the equation for the entropy change during a constant pressure process to find the entropy changes of the piece of lead, the water in the lake, and the universe.

Express the entropy change of the universe in terms of the entropy changes of the lead and the water in the lake:

$$\Delta S_u = \Delta S_{Pb} + \Delta S_w \quad (1)$$

Using the equation for the entropy change during a constant-pressure process, express and evaluate the entropy change of the lead:

$$\Delta S_{Pb} = m_{Pb}c_{Pb}\ln\left(\frac{T_f}{T_i}\right) = (2.00\,\text{kg})\left(0.128\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ln\left(\frac{283\,\text{K}}{373\,\text{K}}\right) = -70.69\frac{\text{J}}{\text{K}}$$

The entropy change of the water in the lake is given by:

$$\Delta S_w = \frac{Q_w}{T_w} = \frac{Q_{Pb}}{T_w} = \frac{m_{Pb}c_{Pb}\Delta T_{Pb}}{T_w}$$

Substitute numerical values and evaluate ΔS_w:

$$\Delta S_w = \frac{(2.00\,\text{kg})\left(0.128\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(90\,\text{K})}{283\,\text{K}} = 81.41\,\text{J/K}$$

Substitute numerical values in equation (1) and evaluate ΔS_u:

$$\Delta S_u = -70.69\frac{\text{J}}{\text{K}} + 81.41\frac{\text{J}}{\text{K}} = \boxed{11\frac{\text{J}}{\text{K}}}$$

Entropy and ″Lost″ Work

63 •• **[SSM]** A a reservoir at 300 K absorbs 500 J of heat from a second reservoir at 400 K. (*a*) What is the change in entropy of the universe, and (*b*) how much work is lost during the process?

Picture the Problem We can find the entropy change of the universe from the entropy changes of the high- and low-temperature reservoirs. The maximum amount of the 500 J of heat that could be converted into work can be found from the maximum efficiency of an engine operating between the two reservoirs.

(*a*) The entropy change of the universe is the sum of the entropy changes of the two reservoirs:

$$\Delta S_u = \Delta S_h + \Delta S_c = -\frac{Q}{T_h} + \frac{Q}{T_c} = -Q\left(\frac{1}{T_h} - \frac{1}{T_c}\right)$$

Substitute numerical values and evaluate ΔS_u:

$$\Delta S_u = (-500\,\text{J})\left(\frac{1}{400\,\text{K}} - \frac{1}{300\,\text{K}}\right) = \boxed{0.42\,\text{J/K}}$$

(*b*) Relate the heat that could have been converted into work to the maximum efficiency of an engine operating between the two reservoirs:

$$W = \varepsilon_{\text{max}} Q_h$$

The maximum efficiency of an engine operating between the two reservoir temperatures is the efficiency of a Carnot device operating between the reservoir temperatures:

$$\varepsilon_{\text{max}} = \varepsilon_C = 1 - \frac{T_c}{T_h}$$

Substitute for ε_{max} to obtain:

$$W = \left(1 - \frac{T_c}{T_h}\right) Q_h$$

Substitute numerical values and evaluate W:

$$W = \left(1 - \frac{300\,\text{K}}{400\,\text{K}}\right)(500\,\text{J}) = \boxed{125\,\text{J}}$$

64 •• In this problem, 1.00 mol of an ideal gas at 300 K undergoes a free adiabatic expansion from $V_1 = 12.3$ L to $V_2 = 24.6$ L. It is then compressed isothermally and reversibly back to its original state. (*a*) What is the entropy change of the universe for the complete cycle? (*b*) How much work is lost in this cycle? (*c*) Show that the work lost is $T\Delta S_u$.

Picture the Problem Although no energy is lost by the gas in the adiabatic free expansion, the process is irreversible and the entropy of the gas (and the universe) increases. In the isothermal reversible process that returns the gas to its original state, the gas releases energy to the surroundings. However, because the process is reversible, the entropy change of the universe is zero. Consequently, the net entropy change is the negative of that of the gas in the isothermal compression.

(*a*) Relate the entropy change of the universe to the entropy changes of the gas during 1 complete cycle:

$$\Delta S_{\mathrm{u}} = \Delta S_{\substack{\text{gas during}\\ \text{free expansion}}} + \Delta S_{\substack{\text{gas during}\\ \text{isothermal compresion}}}$$

or, because $\Delta S_{\substack{\text{gas during}\\ \text{isothermal compression}}} = 0$,

$$\Delta S_{\mathrm{u}} = \Delta S_{\substack{\text{gas during}\\ \text{free expansion}}} = \frac{Q}{T}$$

The work done by the gas during its isothermal compression is given by:

$$W_{\text{by}} = -W_{\text{on}} = -Q = -nRT\ln\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)$$

Substituting for Q in the expression for ΔS_{u} and simplifying yields:

$$\Delta S_{\mathrm{u}} = -nR\ln\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right) \qquad (1)$$

Substitute numerical values and evaluate ΔS_{u}:

$$\Delta S_{\mathrm{u}} = -(1.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)\ln\left(\frac{12.3\,\mathrm{L}}{24.6\,\mathrm{L}}\right) = 5.763\frac{\mathrm{J}}{\mathrm{K}} = \boxed{5.76\frac{\mathrm{J}}{\mathrm{K}}}$$

(*b*) Use Equation 19-22 to find the amount of energy that becomes unavailable for doing work during this process:

$$W_{\text{lost}} = T\Delta S_{\mathrm{u}} = (300\,\mathrm{K})\left(5.763\frac{\mathrm{J}}{\mathrm{K}}\right) = \boxed{1.73\,\mathrm{kJ}}$$

(*c*) No work is done in the free expansion. In the adiabatic compression, the work done on the gas is:

$$W_{\text{by gas, f}\to\text{i}} = -W_{\text{on gas, f}\to\text{i}} = -nRT\ln\left(\frac{V_{\mathrm{i}}}{V_{\mathrm{f}}}\right)$$

$$= nRT\ln\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right) = T\left(nR\ln\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)\right)$$

$$= \boxed{T\Delta S_{\mathrm{u}}}$$

General Problems

65 • A heat engine with an output of 200 W has an efficiency of 30%. It operates at 10.0 cycles/s. (*a*) How much work is done by the engine during each cycle? (*b*) How much heat is absorbed from the hot reservoir and how much is released to the cold reservoir during each cycle?

Picture the Problem We can use the definition of power to find the work done each cycle and the definition of efficiency to find the heat that is absorbed each cycle. Application of the first law of thermodynamics will yield the heat given off each cycle.

(*a*) Use the definition of power to relate the work done in each cycle to the frequency of each cycle:

$$W_{\text{cycle}} = P\Delta t = \frac{P}{f}$$

where f is the frequency of the engine.

Substitute numerical values and evaluate W_{cycle}:

$$W_{\text{cycle}} = \frac{200\text{ W}}{10.0\text{ s}^{-1}} = \boxed{20.0\text{ J}}$$

(*b*) Express the heat absorbed in each cycle in terms of the work done and the efficiency of the engine:

$$Q_{\text{h,cycle}} = \frac{W_{\text{cycle}}}{\varepsilon} = \frac{20.0\text{ J}}{0.30} = \boxed{67\text{ J}}$$

Apply the 1st law of thermodynamics to find the heat given off in each cycle:

$$Q_{\text{c,cycle}} = Q_{\text{h,cycle}} - W = 67\text{ J} - 20\text{ J} = \boxed{47\text{ J}}$$

66 • During each cycle, a heat engine operating between two heat reservoirs absorbs 150 J from the reservoir at 100ºC and releases 125 J to the reservoir at 20ºC. (*a*) What is the efficiency of this engine? (*b*) What is the ratio of its efficiency to that of a Carnot engine working between the same reservoirs? (This ratio is called the *second law efficiency*.)

Picture the Problem We can use their definitions to find the efficiency of the engine and that of a Carnot engine operating between the same reservoirs.

(*a*) The efficiency of the engine is given by:

$$\varepsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$$

Substitute numerical values and evaluate ε:

$$\varepsilon = 1 - \frac{125\text{ J}}{150\text{ J}} = 16.67\% = \boxed{16.7\%}$$

(*b*) Find the efficiency of a Carnot engine operating between the same reservoirs:

$$\varepsilon_C = 1 - \frac{T_c}{T_h} = 1 - \frac{293\text{ K}}{373\text{ K}} = 21.45\%$$

Express the ratio of the two efficiencies:

$$\frac{\varepsilon}{\varepsilon_C} = \frac{16.67\%}{21.45\%} = \boxed{0.777}$$

67 • **[SSM]** An engine absorbs 200 kJ of heat per cycle from a reservoir at 500 K and releases heat to a reservoir at 200 K. Its efficiency is 85 percent of that of a Carnot engine working between the same reservoirs. (*a*) What is the

efficiency of this engine? (*b*) How much work is done in each cycle? (*c*) How much heat is released to the low-temperature reservoir during each cycle?

Picture the Problem We can use the definition of efficiency to find the work done by the engine during each cycle and the first law of thermodynamics to find the heat released to the low-temperature reservoir during each cycle.

(*a*) Express the efficiency of the engine in terms of the efficiency of a Carnot engine working between the same reservoirs:

$$\varepsilon = 0.85\varepsilon_C = 0.85\left(1-\frac{T_c}{T_h}\right)$$

Substitute numerical values and evaluate ε:

$$\varepsilon = 0.85\left(1-\frac{200\,\text{K}}{500\,\text{K}}\right) = 0.510 = \boxed{51\%}$$

(*b*) Use the definition of efficiency to find the work done in each cycle:

$$W = \varepsilon Q_h = (0.510)(200\,\text{kJ}) = 102\text{ kJ} = \boxed{0.10\,\text{MJ}}$$

(*c*) Apply the first law of thermodynamics to the cycle to obtain:

$$Q_{c,\text{cycle}} = Q_{h,\text{cycle}} - W = 200\,\text{kJ} - 102\,\text{kJ} = \boxed{98\,\text{kJ}}$$

68 • Estimate the change in entropy of the universe associated with an Olympic diver diving into the water from the 10-m platform.

Picture the Problem Assume that the mass of the diver is 75 kg and that the temperature of the water in the pool is 25°C. The energy added to the water in the pool is the change in the gravitational potential energy of the diver during the dive.

The change in entropy of the universe associated with a dive is given by:

$$\Delta S_u = \Delta S_{\text{water}} = \frac{Q_{\text{added to water}}}{T_{\text{water}}}$$

where $Q_{\text{added to water}}$ is the energy entering the water as a result of the kinetic energy of the diver as he enters the water.

The energy added to the water is the change in the gravitational potential energy of the diver:

$$\Delta S_u = \frac{mgh}{T_{\text{water}}}$$

Substitute numerical values and evaluate ΔS_u:

$$\Delta S_u = \frac{(75\text{ kg})(9.81\text{ m/s}^2)(10\text{ m})}{(25+273)\text{K}} \approx \boxed{25\frac{\text{J}}{\text{K}}}$$

69 • To maintain the temperature inside a house at 20ºC, the electric power consumption of the electric baseboard heaters is 30.0 kW on a day when the outside temperature is –7ºC. At what rate does this house contribute to the increase in the entropy of the universe?

Picture the Problem The change in entropy of the universe is the change in entropy of the house plus the change in entropy of the environment. We can find the change in entropy of the house by exploiting the given information that the temperature inside the house is maintained at a constant temperature. We can find the change in entropy of the surrounding by dividing the heat added by the temperature.

Entropy is a state function, and the state of the house does not change. Therefore the entropy of the house does not change:

$$\Delta S_u = \Delta S_{\text{house}} + \Delta S_{\text{surroundings}}$$

or, because $\Delta S_{\text{house}} = 0$,

$$\Delta S_u = \Delta S_{\text{surroundings}}$$

Heat is absorbed by the surroundings at the same rate R that energy is delivered to the house:

$$\Delta S_{\text{surroundings}} = \frac{Q}{T_{\text{surroundings}}} = \frac{R\Delta t}{T_{\text{surroundings}}}$$

Substitute for $\Delta S_{\text{surroundings}}$ yields:

$$\Delta S_u = \frac{R\Delta t}{T_{\text{surroundings}}} \Rightarrow \frac{\Delta S_u}{\Delta t} = \frac{R}{T_{\text{surroundings}}}$$

Substitute numerical values and evaluate $\Delta S_u/\Delta t$:

$$\frac{\Delta S_u}{\Delta t} = \frac{30.0\text{ kW}}{266\text{ K}} = \boxed{113\frac{\text{W}}{\text{K}}}$$

70 •• Calvin Cliffs Nuclear Power Plant, located on the Hobbes River, generates 1.00 GW of power. In this plant, liquid sodium circulates between the reactor core and a heat exchanger located in the superheated steam that drives the turbine. Heat is absorbed by the liquid sodium in the core, and released by the liquid sodium (and into the superheated steam) in the heat exchanger. The temperature of the superheated steam is 500 K. Heat is released into the river, and the water in the river flows by at a temperature of 25ºC. (*a*) What is the highest efficiency that this plant can have? (*b*) How much heat is released into the river every second? (*c*) How much heat must be released by the core to supply

1.00 GW of electrical power? (*d*) Assume that new environmental laws have been passed to preserve the unique wildlife of the river. Because of these laws, the plant is not allowed to heat the river by more than 0.50ºC. What is the minimum flow rate that the water in the Hobbes River must have?

Picture the Problem We can use the expression for the Carnot efficiency of the plant to find the highest efficiency this plant can have. We can then use this efficiency to find the power that must be supplied to the plant to generate 1.00 GW of power and, from this value, the power that is wasted. The rate at which heat is being released to the river is related to the requisite flow rate of the river by $dQ/dt = c\Delta T\rho\, dV/dt$.

(*a*) The Carnot efficiency of a plant operating between temperatures T_c and T_h is given by:

$$\varepsilon_{max} = \varepsilon_C = 1 - \frac{T_c}{T_h}$$

Substitute numerical values and evaluate ε_C:

$$\varepsilon_{max} = 1 - \frac{298\,\text{K}}{500\,\text{K}} = \boxed{0.404}$$

(*c*) The power that must be supplied, at 40.4% efficiency, to produce an output of 1.00 GW is given by:

$$P_{supplied} = \frac{P_{output}}{\varepsilon_{max}} = \frac{1.00\,\text{GW}}{0.404} = \boxed{2.48\,\text{GW}}$$

(*b*) Relate the wasted power to the power generated and the power supplied:

$$P_{wasted} = P_{supplied} - P_{generated}$$

Substitute numerical values and evaluate P_{wasted}:

$$P_{wasted} = 2.48\,\text{GW} - 1.00\,\text{GW} = \boxed{1.48\,\text{GW}}$$

(*d*) Express the rate at which heat is being dumped into the river:

$$\frac{dQ}{dt} = c\Delta T\frac{dm}{dt} = c\Delta T\frac{d}{dt}(\rho V) = c\Delta T\rho\frac{dV}{dt}$$

Solve for the flow rate dV/dt of the river:

$$\frac{dV}{dt} = \frac{dQ/dt}{c\Delta T\rho}$$

Substitute numerical values (see Table 19-1 for the specific heat of water) and evaluate dV/dt:

$$\frac{dV}{dt} = \frac{1.48\times10^{9}\ \frac{\text{J}}{\text{s}}}{\left(4180\ \frac{\text{J}}{\text{kg}}\right)(0.50\ \text{K})\left(10^{3}\ \frac{\text{kg}}{\text{m}^{3}}\right)} = \boxed{7.1\times10^{5}\ \text{L/s}}$$

71 •• An inventor comes to you to explain his new invention. It is a novel heat engine using water vapor as the working substance. He claims that the water vapor absorbs heat at 100°C, does work at the rate of 125 W, and releases heat to the air at the rate of only 25.0 W, when the air temperature is 25°C. (*a*) Explain to him why he cannot be correct. (*b*) After careful analysis of the data in his prospectus folder, you decide he has made an error in the measurement of his exhausted-heat value. What is the minimum rate of exhausting heat that would make you consider believing him?

Picture the Problem We can use the inventor's data to calculate the thermal efficiency of his steam engine and then compare this value to the efficiency of a Carnot engine operating between the same temperatures.

(*a*) The Carnot efficiency of an engine operating between these temperatures is:

$$\varepsilon_{\text{C}} = 1 - \frac{T_{\text{c}}}{T_{\text{h}}} = 1 - \frac{298\ \text{K}}{373\ \text{K}} = 20.1\%$$

The thermal efficiency of the inventor's device, in terms of the rate at which it expels heat to the air and does work is:

$$\varepsilon = \frac{\frac{dW}{dt}}{\frac{dQ_{\text{h}}}{dt}} = \frac{\frac{dW}{dt}}{\frac{dW}{dt} + \frac{dQ_{\text{c}}}{dt}} = \frac{125\ \text{W}}{125\ \text{W} + 25.0\ \text{W}} = 83.3\%$$

You should explain to him that, because the efficiency he claims for his invention is greater than the efficiency of a Carnot engine operating between the same two temperatures, his data is not consistent with what is known about the thermodynamics of engines. He must have made a mistake in his analysis of his data–or he is a con man looking for suckers to swindle.

(*b*) The maximum efficiency of a steam engine that has ever been achieved is about 50% of the Carnot efficiency of an engine operating between the same temperatures.

Setting the efficiency of his steam engine equal to half the Carnot efficiency of the engine yields:

$$\tfrac{1}{2}\varepsilon_{\text{C}} = \frac{\frac{dW}{dt}}{\frac{dQ_{\text{h}}}{dt}} = \frac{\frac{dW}{dt}}{\frac{dW}{dt} + \frac{dQ_{\text{c}}}{dt}}$$

Solve for dQ_c/dt to obtain:

$$\frac{dQ_c}{dt} = \left(\frac{2}{\varepsilon_C} - 1\right)\frac{dW}{dt}$$

Assuming that the inventor has measured the work done per cycle by his invention correctly:

$$\frac{dQ_c}{dt} = \left(\frac{2}{0.201} - 1\right)(125\text{ W}) \approx 1100\text{ W}$$

a value totally inconsistent with the inventor's claims for his engine.

Ignoring his claim that 125.0 W of work are done per cycle, let's assume that his device does take in energy at the rate of 150 W each cycle and find how much work it would do with an efficiency half that of a Carnot engine:

$$\tfrac{1}{2}\varepsilon_C = \frac{\frac{dW}{dt}}{\frac{dQ_h}{dt}} \Rightarrow \frac{dW}{dt} = \tfrac{1}{2}\varepsilon_C \frac{dQ_h}{dt}$$

Substituting numerical values yields:

$$\frac{dW}{dt} = \tfrac{1}{2}(0.201)(150\text{ W}) \approx 15\text{ W}$$

Because $\frac{dQ_c}{dt} = \frac{dQ_h}{dt} - \frac{dW}{dt}$, a reasonable value for dQ_c/dt is:

$$\frac{dQ_c}{dt} = 150\text{ W} - 15\text{ W} = \boxed{135\text{ W}}$$

72 •• The cycle represented in Figure 19-12 (next to Problem 19-14) is for 1.00 mol of an ideal monatomic gas. The temperatures at points A and B are 300 and 750 K, respectively. What is the efficiency of the cyclic process ABCDA?

Picture the Problem Because the cycle represented in Figure 19-12 is a Carnot cycle, its efficiency is that of a Carnot engine operating between the temperatures of its isotherms.

The Carnot efficiency of the cycle is given by:

$$\varepsilon_C = 1 - \frac{T_c}{T_h}$$

Substitute numerical values and evaluate ε_C:

$$\varepsilon_C = 1 - \frac{300\text{ K}}{750\text{ K}} = \boxed{60.0\%}$$

73 •• **[SSM]** (*a*) Which of these two processes is more wasteful? (1) A block moving with 500 J of kinetic energy being slowed to rest by sliding (kinetic) friction when the temperature of the environment is 300 K, or (2) A reservoir at 400 K releasing 1.00 kJ of heat to a reservoir at 300 K? Explain your choice. *Hint: How much of the 1.00 kJ of heat could be converted into work by an ideal cyclic process?* (*b*) What is the change in entropy of the universe for each process?

Picture the Problem All 500 J of mechanical energy are lost, i.e., transformed into heat in process (1). For process (2), we can find the heat that would be converted to work by a Carnot engine operating between the given temperatures and subtract that amount of work from 1.00 kJ to find the energy that is lost. In Part (*b*) we can use its definition to find the change in entropy for each process.

(*a*) For process (2):

$$W_{2,\max} = W_{\text{recovered}} = \varepsilon_{\text{C}} Q_{\text{in}}$$

The efficiency of a Carnot engine operating between temperatures T_{h} and T_{c} is given by:

$$\varepsilon_{\text{C}} = 1 - \frac{T_{\text{c}}}{T_{\text{h}}}$$

and hence

$$W_{\text{recovered}} = \left(1 - \frac{T_{\text{c}}}{T_{\text{h}}}\right) Q_{\text{in}}$$

Substitute for ε_{C} to obtain:

$$W_{\text{recovered}} = \left(1 - \frac{300\text{ K}}{400\text{ K}}\right)(1.00\text{ kJ}) = 250\text{ J}$$

or

750 J are lost.

Process (1) produces more waste heat. Process (2) is more wasteful of ***available*** work.

(*b*) Find the change in entropy of the universe for process (1):

$$\Delta S_1 = \frac{\Delta Q}{T} = \frac{500\text{ J}}{300\text{ K}} = \boxed{1.67\text{ J/K}}$$

Express the change in entropy of the universe for process (2):

$$\Delta S_2 = \Delta S_{\text{h}} + \Delta S_{\text{c}} = -\frac{\Delta Q}{T_{\text{h}}} + \frac{\Delta Q}{T_{\text{c}}}$$

$$= \Delta Q\left(\frac{1}{T_{\text{c}}} - \frac{1}{T_{\text{h}}}\right)$$

Substitute numerical values and evaluate ΔS_2:

$$\Delta S_2 = (1.00\text{ kJ})\left(\frac{1}{300\text{ K}} - \frac{1}{400\text{ K}}\right)$$

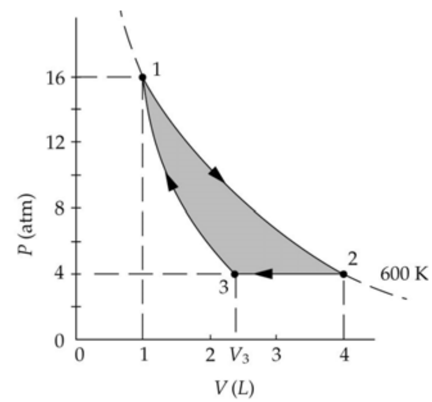
$$= \boxed{0.833\text{ J/K}}$$

74 •• Helium, a monatomic gas, is initially at a pressure of 16 atm, a volume of 1.0 L, and a temperature of 600 K. It is quasi-statically expanded at constant temperature until its volume is 4.0 L and is then quasi-statically compressed at constant pressure until its volume and temperature are such that a quasi-static adiabatic compression will return the gas to its original state. (*a*) Sketch this

cycle on a *PV* diagram. (*b*) Find the volume and temperature after the compression at constant pressure. (*c*) Find the work done during each step of the cycle. (*d*) Find the efficiency of the cycle.

Picture the Problem Denote the three states of the gas as 1, 2, and 3 with 1 being the initial state. We can use the ideal-gas law and the equation of state for an adiabatic process to find the temperatures, volumes, and pressures at points 1, 2, and 3. To find the work done during each cycle, we can use the equations for the work done during isothermal, isobaric, and adiabatic processes. Finally, we find the efficiency of the cycle from the work done each cycle and the heat that enters the system during the isothermal expansion.

(*a*) The *PV* diagram of the cycle is shown to the right.



(*b*) Apply the ideal-gas law to the isothermal expansion 1→2 to find P_2:

$$P_2 = P_1\frac{V_1}{V_2} = (16\,\text{atm})\left(\frac{1.0\,\text{L}}{4.0\,\text{L}}\right) = 4.0\,\text{atm}$$

Apply an equation for an adiabatic process to relate the pressures and volumes at 1 and 3:

$$P_1V_1^\gamma = P_3V_3^\gamma \Rightarrow V_3 = V_1\left(\frac{P_1}{P_3}\right)^{1/\gamma}$$

Substitute numerical values and evaluate V_3:

$$V_3 = (1.0\,\text{L})\left(\frac{16\,\text{atm}}{4.0\,\text{atm}}\right)^{1/1.67} = 2.294\,\text{L}$$

$$= \boxed{2.3\,\text{L}}$$

Apply an equation for an adiabatic process (γ=1.67) to relate the temperatures and volumes at 1 and 3:

$$T_3V_3^{\gamma-1} = T_1V_1^{\gamma-1} \Rightarrow T_3 = T_1\left(\frac{V_1}{V_3}\right)^{\gamma-1}$$

Substitute numerical values and evaluate T_3:

$$T_3 = (600\,\text{K})\left(\frac{1.0\,\text{L}}{2.294\,\text{L}}\right)^{1.67-1} = 344\,\text{K}$$
$$= \boxed{3.4\times10^2\ \text{K}}$$

(*c*) Express the work done each cycle:

$$W = W_{12} + W_{23} + W_{31} \qquad (1)$$

For the process 1→2:

$$W_{12} = nRT_1 \ln\left(\frac{V_2}{V_1}\right) = P_1V_1 \ln\left(\frac{V_2}{V_1}\right)$$
$$= (16\,\text{atm})(1.0\,\text{L})\ln\left(\frac{4.0\,\text{L}}{1.0\,\text{L}}\right)$$
$$= 22.18\,\text{atm}\cdot\text{L}$$

For the process 2→3:

$$W_{23} = P_2\Delta V_{23}$$
$$= (4.0\,\text{atm})(2.294\,\text{L} - 4.00\,\text{L})$$
$$= -6.824\,\text{atm}\cdot\text{L}$$

For the process 3→1:

$$W_{31} = -C_\text{V}\Delta T_{31} = -\tfrac{3}{2}nR(T_1 - T_3) = -\tfrac{3}{2}(P_1V_1 - P_3V_3)$$
$$= -\tfrac{3}{2}\left[(16\,\text{atm})(1.0\,\text{L}) - (4.0\,\text{atm})(2.294\,\text{L})\right]$$
$$= -10.24\,\text{atm}\cdot\text{L}$$

Substitute numerical values in equation (1) and evaluate W:

$$W = 22.18\,\text{atm}\cdot\text{L} - 6.824\,\text{atm}\cdot\text{L} - 10.24\,\text{atm}\cdot\text{L}$$
$$= 5.116\,\text{atm}\cdot\text{L} = \boxed{5\ \text{atm}\cdot\text{L}}$$

(*d*) Use its definition to express the efficiency of the cycle:

$$\varepsilon = \frac{W}{Q_\text{in}} = \frac{W}{Q_{12}} = \frac{W}{W_{12}}$$

Substitute numerical values and evaluate ε:

$$\varepsilon = \frac{5.116\,\text{atm}\cdot\text{L}}{22.18\,\text{atm}\cdot\text{L}} \approx \boxed{20\%}$$

75 •• **[SSM]** A heat engine that does the work of blowing up a balloon at a pressure of 1.00 atm absorbs 4.00 kJ from a reservoir at 120ºC. The volume of the balloon increases by 4.00 L, and heat is released to a reservoir at a temperature T_c, where $T_\text{c} < 120$ºC. If the efficiency of the heat engine is 50% of the efficiency of a Carnot engine working between the same two reservoirs, find the temperature T_c.

Picture the Problem We can express the temperature of the cold reservoir as a function of the Carnot efficiency of an ideal engine and, given that the efficiency of the heat engine is half that of a Carnot engine, relate T_c to the work done by and the heat input to the real heat engine.

Using its definition, relate the efficiency of a Carnot engine working between the same reservoirs to the temperature of the cold reservoir:

$$\varepsilon_C = 1 - \frac{T_c}{T_h} \Rightarrow T_c = T_h\left(1 - \varepsilon_C\right)$$

Relate the efficiency of the heat engine to that of a Carnot engine working between the same temperatures:

$$\varepsilon = \frac{W}{Q_{in}} = \tfrac{1}{2}\varepsilon_C \Rightarrow \varepsilon_C = \frac{2W}{Q_{in}}$$

Substitute for ε_C to obtain:

$$T_c = T_h\left(1 - \frac{2W}{Q_{in}}\right)$$

The work done by the gas in expanding the balloon is:

$$W = P\Delta V = (1.00\,\text{atm})(4.00\,\text{L})$$
$$= 4.00\,\text{atm}\cdot\text{L}$$

Substitute numerical values and evaluate T_c:

$$T_c = (393\,\text{K})\left(1 - \frac{2\left(4.00\,\text{atm}\cdot\text{L}\times\dfrac{101.325\,\text{J}}{\text{atm}\cdot\text{L}}\right)}{4.00\,\text{kJ}}\right) = \boxed{313\,\text{K}}$$

76 •• Show that the coefficient of performance of a Carnot engine run as a refrigerator is related to the efficiency of a Carnot engine operating between the same two temperatures by $\varepsilon_C \times \text{COP}_C = T_c/T_h$.

Picture the Problem We can use the definitions of the COP and ε_C to show that their relationship is $\varepsilon_C \times \text{COP}_C = T_C/T_h$.

Using the definition of the COP, relate the heat removed from the cold reservoir to the work done each cycle:

$$\text{COP} = \frac{Q_c}{W}$$

Apply energy conservation to relate Q_c, Q_h, and W:

$$Q_c = Q_h - W$$

Substitute for Q_c to obtain:

$$\text{COP} = \frac{Q_h - W}{W}$$

Divide the numerator and denominator by Q_h and simplify to obtain:

$$\text{COP} = \frac{Q_h - W}{W} = \frac{1 - \dfrac{W}{Q_h}}{\dfrac{W}{Q_h}}$$

Because $\varepsilon_C = \dfrac{W}{Q_h} = 1 - \dfrac{T_c}{T_h}$:

$$\text{COP}_c = \frac{1 - \varepsilon_C}{\varepsilon_C} = \frac{1 - \left(1 - \dfrac{T_c}{T_h}\right)}{\varepsilon_C} = \frac{\dfrac{T_c}{T_h}}{\varepsilon_C}$$

and

$$\boxed{\varepsilon_C \times \text{COP}_c = \frac{T_c}{T_h}}$$

77 •• A freezer has a temperature $T_c = -23$ºC. The air in the kitchen has a temperature $T_h = 27$ºC. The freezer is not perfectly insulated and some heat leaks through the walls of the freezer at a rate of 50 W. Find the power of the motor that is needed to maintain the temperature in the freezer.

Picture the Problem We can use the definition of the COP to express the work the motor must do to maintain the temperature of the freezer in terms of the rate at which heat flows into the freezer. Differentiation of this expression with respect to time will yield an expression for the power of the motor that is needed to maintain the temperature in the freezer.

Using the definition of the COP, relate the heat that must be removed from the freezer to the work done by the motor:

$$\text{COP} = \frac{Q_c}{W} \Rightarrow W = \frac{Q_c}{\text{COP}}$$

Differentiate this expression with respect to time to express the power of the motor:

$$P = \frac{dW}{dt} = \frac{dQ_c/dt}{\text{COP}}$$

Express the maximum COP of the motor:

$$\text{COP}_{\max} = \frac{T_c}{\Delta T}$$

Substitute for COP_{max} to obtain:

$$P=\frac{dQ_{\text{c}}}{dt}\frac{\Delta T}{T_{\text{c}}}$$

Substitute numerical values and evaluate P:

$$P=(50\,\text{W})\left(\frac{50\,\text{K}}{250\,\text{K}}\right)=\boxed{10\,\text{W}}$$

78 •• In a heat engine, 2.00 mol of a diatomic gas are taken through the cycle ABCA as shown in Figure 19-20. (The *PV* diagram is not drawn to scale.) At A the pressure and temperature are 5.00 atm and 600 K. The volume at B is twice the volume at A. The segment BC is an adiabatic expansion and the segment CA is an isothermal compression. (*a*) What is the volume of the gas at A? (*b*) What are the volume and temperature of the gas at B? (*c*) What is the temperature of the gas at C? (*d*) What is the volume of the gas at C? (*e*) How much work is done by the gas in each of the three segments of the cycle? (*f*) How much heat is absorbed or released by the gas in each segment of this cycle?

Picture the Problem We can use the ideal-gas law to find the unknown temperatures, pressures, and volumes at points A, B, and C and then find the work done by the gas and the efficiency of the cycle by using the expressions for the work done on or by the gas and the heat that enters the system for the constant-pressure, adiabatic, and isothermal processes of the cycle.

(*a*) Apply the ideal-gas law to find the volume of the gas at A:

$$V_{\text{A}}=\frac{nRT_{\text{A}}}{P_{\text{A}}}$$

$$=\frac{(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})}{5.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}}$$

$$=1.969\times10^{-2}\ \text{m}^3\times\frac{1\,\text{L}}{10^{-3}\ \text{m}^3}=19.69\,\text{L}$$

$$=\boxed{19.7\,\text{L}}$$

(*b*) We're given that $V_{\text{B}}=2V_{\text{A}}$. Hence:

$$V_{\text{B}}=2(19.69\,\text{L})=39.38\,\text{L}=\boxed{39.4\,\text{L}}$$

Apply the ideal-gas law to this constant-pressure process to obtain:

$$T_{\text{B}}=T_{\text{A}}\frac{V_{\text{B}}}{V_{\text{A}}}=(600\,\text{K})\frac{2V_{\text{A}}}{V_{\text{A}}}$$

$$=\boxed{1200\,\text{K}}$$

(*c*) Because the process C→A is isothermal:

$$T_C = T_A = \boxed{600\,\text{K}}$$

(*d*) Apply an equation for an adiabatic process (γ= 1.4) to find the volume of the gas at C:

$$T_B V_B^{\gamma-1} = T_C V_C^{\gamma-1} \Rightarrow V_C = V_B \left(\frac{T_B}{T_C}\right)^{\frac{1}{\gamma-1}}$$

Substitute numerical values and evaluate V_C:

$$V_C = (39.38\,\text{L})\left(\frac{1200\,\text{K}}{600\,\text{K}}\right)^{\frac{1}{1.4-1}} = 222.77\,\text{L}$$
$$= \boxed{223\,\text{L}}$$

(*e*) The work done by the gas during the constant-pressure process AB is given by:

$$W_{AB} = P_A(V_B - V_A) = P_A(2V_A - V_A)$$
$$= P_A V_A$$

Substitute numerical values and evaluate W_{AB}:

$$W_{AB} = (5.00\,\text{atm})(19.69\,\text{L})$$
$$= 98.45\,\text{atm}\cdot\text{L} \times \frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}}$$
$$= 9.9754 \times 10^3\ \text{J} = \boxed{9.98\,\text{kJ}}$$

Apply the first law of thermodynamics to express the work done on the gas during the adiabatic expansion BC:

$$W_{BC} = \Delta E_{\text{int,BC}} - Q_{\text{in, BC}} = \Delta E_{\text{int, BC}} - 0$$
$$= \Delta E_{\text{int, BC}} = -nc_V \Delta T_{BC}$$
$$= -\tfrac{5}{2} nR\Delta T_{BC}$$

Substitute numerical values and evaluate W_{BC}:

$$W_{BC} = -\tfrac{5}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K} - 1200\,\text{K}) = 2.494 \times 10^4\ \text{J}$$
$$= \boxed{24.9\,\text{kJ}}$$

The work done by the gas during the isothermal compression CA is:

$$W_{CA} = nRT_C \ln\left(\frac{V_A}{V_C}\right) = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})\ln\left(\frac{19.69\,\text{L}}{222.77\,\text{L}}\right)$$
$$= -24.20\,\text{kJ} = \boxed{-24.2\,\text{kJ}}$$

(*f*) The heat absorbed during the constant-pressure expansion AB is:

$$Q_{\mathrm{AB}} = nc_{\mathrm{P}}\Delta T_{\mathrm{A-B}} = \tfrac{7}{2}nR\Delta T_{\mathrm{A-B}} = \tfrac{7}{2}(2.00\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\cdot K}}\right)(1200\,\mathrm{K}-600\,\mathrm{K})$$
$$= 34.92\,\mathrm{kJ} = \boxed{34.9\,\mathrm{kJ}}$$

The heat absorbed during the adiabatic expansion BC is:

$$Q_{\mathrm{BC}} = \boxed{0}$$

Use the first law of thermodynamics to find the heat absorbed during the isothermal compression CA:

$$Q_{\mathrm{CA}} = W_{\mathrm{CA}} + \Delta E_{\mathrm{int,\,CA}} = W_{\mathrm{CA}}$$
$$= \boxed{-24.2\,\mathrm{kJ}}$$

because $\Delta E_{\mathrm{int,\,CA}} = 0$ for an isothermal process.

79 •• [SSM] In a heat engine, 2.00 mol of a diatomic gas are carried through the cycle ABCDA shown in Figure 19-21. (The *PV* diagram is not drawn to scale.) The segment AB represents an isothermal expansion, the segment BC an adiabatic expansion. The pressure and temperature at A are 5.00 atm and 600 K. The volume at B is twice the volume at A. The pressure at D is 1.00 atm. (*a*) What is the pressure at B? (*b*) What is the temperature at C? (*c*) Find the total work done by the gas in one cycle.

Picture the Problem We can use the ideal-gas law to find the unknown temperatures, pressures, and volumes at points B, C, and D. We can then find the work done by the gas and the efficiency of the cycle by using the expressions for the work done on or by the gas and the heat that enters the system for the various thermodynamic processes of the cycle.

(*a*) Apply the ideal-gas law for a fixed amount of gas to the isothermal process AB to find the pressure at B:

$$P_{\mathrm{B}} = P_{\mathrm{A}}\frac{V_{\mathrm{A}}}{V_{\mathrm{B}}} = (5.00\,\mathrm{atm})\frac{V_{\mathrm{A}}}{2V_{\mathrm{A}}}$$
$$= 2.50\,\mathrm{atm}\times\frac{101.325\,\mathrm{kPa}}{1\,\mathrm{atm}} = 253.3\,\mathrm{kPa}$$
$$= \boxed{253\,\mathrm{kPa}}$$

(*b*) Apply the ideal-gas law for a fixed amount of gas to the adiabatic process BC to express the temperature at C:

$$T_{\mathrm{C}} = T_{\mathrm{B}}\frac{P_{\mathrm{C}}V_{\mathrm{C}}}{P_{\mathrm{B}}V_{\mathrm{B}}} \qquad (1)$$

Use the ideal-gas law to find the volume of the gas at B:

$$V_B = \frac{nRT_B}{P_B} = \frac{(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})}{253.3\,\text{kPa}} = 39.39\,\text{L}$$

Use the equation of state for an adiabatic process and $\gamma = 1.4$ to find the volume occupied by the gas at C:

$$V_C = V_B\left(\frac{P_B}{P_C}\right)^{1/\gamma} = (39.39\,\text{L})\left(\frac{2.50\,\text{atm}}{1.00\,\text{atm}}\right)^{1/1.4} = 75.78\,\text{L}$$

Substitute numerical values in equation (1) and evaluate T_C:

$$T_C = (600\,\text{K})\frac{(1.00\,\text{atm})(75.78\,\text{L})}{(2.50\,\text{atm})(39.39\,\text{L})} = \boxed{462\,\text{K}}$$

(*c*) The work done by the gas in one cycle is given by:

$$W = W_{AB} + W_{BC} + W_{CD} + W_{DA}$$

The work done during the isothermal expansion AB is:

$$W_{AB} = nRT_A \ln\left(\frac{V_B}{V_A}\right) = (2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})\ln\left(\frac{2\text{V}_A}{\text{V}_A}\right) = 6.915\,\text{kJ}$$

The work done during the adiabatic expansion BC is:

$$W_{BC} = -C_V\Delta T_{BC} = -\tfrac{5}{2}nR\Delta T_{BC} = -\tfrac{5}{2}(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(462\,\text{K} - 600\,\text{K}) = 5.737\,\text{kJ}$$

The work done during the isobaric compression CD is:

$$W_{CD} = P_C(V_D - V_C) = (1.00\,\text{atm})(19.7\,\text{L} - 75.78\,\text{L}) = -56.09\,\text{atm}\cdot\text{L}\times\frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}} = -5.680\,\text{kJ}$$

Express and evaluate the work done during the constant-volume process DA:

$$W_{DA} = 0$$

Substitute numerical values and evaluate W:

$$W = 6.915\,\text{kJ} + 5.737\,\text{kJ} - 5.680\,\text{kJ} + 0$$
$$= 6.972\,\text{kJ} = \boxed{6.97\,\text{kJ}}$$

80 •• In a heat engine, 2.00 mol of a monatomic gas are taken through the cycle ABCA as shown in Figure 19-20. (The *PV* diagram is not drawn to scale.) At A the pressure and temperature are 5.00 atm and 600 K. The volume at B is twice the volume at A. The segment BC is an adiabatic expansion and the segment CA is an isothermal compression. (*a*) What is the volume of the gas at A? (*b*) What are the volume and temperature of the gas at B? (*c*) What is the temperature of the gas at C? (*d*) What is the volume of the gas at C? (*e*) How much work is done by the gas in each of the three segments of the cycle? (*f*) How much heat is absorbed by the gas in each segment of the cycle?

Picture the Problem We can use the ideal-gas law to find the unknown temperatures, pressures, and volumes at points A, B, and C and then find the work done by the gas and the efficiency of the cycle by using the expressions for the work done on or by the gas and the heat that enters the system for the isobaric, adiabatic, and isothermal processes of the cycle.

(*a*) Apply the ideal-gas law to find the volume of the gas at A:

$$V_{\text{A}} = \frac{nRT_{\text{A}}}{P_{\text{A}}}$$
$$= \frac{(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})}{5.00\,\text{atm}\times\frac{101.325\,\text{kPa}}{\text{atm}}}$$
$$= 19.69\,\text{L} = \boxed{19.7\,\text{L}}$$

(*b*) We're given that:

$$V_{\text{B}} = 2V_{\text{A}} = 2(19.69\,\text{L}) = 39.39\,\text{L}$$
$$= \boxed{39.4\,\text{L}}$$

Apply the ideal-gas law to this isobaric process to find the temperature at B:

$$T_{\text{B}} = T_{\text{A}}\frac{V_{\text{B}}}{V_{\text{A}}} = (600\,\text{K})\frac{2V_{\text{A}}}{V_{\text{A}}}$$
$$= \boxed{1200\,\text{K}}$$

(*c*) Because the process CA is isothermal:

$$T_{\text{C}} = T_{\text{A}} = \boxed{600\,\text{K}}$$

(*d*) Apply an equation for an adiabatic process ($\gamma = 5/3$) to express the volume of the gas at C:

$$T_{\text{B}}V_{\text{B}}^{\gamma-1} = T_{\text{C}}V_{\text{C}}^{\gamma-1} \Rightarrow V_{\text{C}} = V_{\text{B}}\left(\frac{T_{\text{B}}}{T_{\text{C}}}\right)^{\frac{1}{\gamma-1}}$$

Substitute numerical values and evaluate V_C:

$$V_C = (39.39\,\text{L})\left(\frac{1200\,\text{K}}{600\,\text{K}}\right)^{\frac{3}{2}} = 111.4\,\text{L} = \boxed{111\,\text{L}}$$

(*e*) The work done by the gas during the isobaric process AB is given by:

$$W_{AB} = P_A(V_B - V_A) = P_A(2V_A - V_A) = P_A V_A$$

Substitute numerical values and evaluate W_{AB}:

$$W_{AB} = (5.00\,\text{atm})(19.69\,\text{L}) = 98.45\,\text{atm}\cdot\text{L}\times\frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}} = 9.975\,\text{kJ} = \boxed{9.98\,\text{kJ}}$$

Apply the first law of thermodynamics to express the work done by the gas during the adiabatic expansion BC:

$$W_{BC} = \Delta E_{\text{int, BC}} - Q_{\text{in, BC}} = \Delta E_{\text{int, BC}} - 0 = \Delta E_{\text{int, BC}} = -(nc_V\Delta T_{BC}) = -\tfrac{3}{2}nR\Delta T_{BC}$$

Substitute numerical values and evaluate W_{BC}:

$$W_{BC} = -\tfrac{3}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K} - 1200\,\text{K}) = 14.97\,\text{kJ} = \boxed{15.0\,\text{kJ}}$$

The work done by the gas during the isothermal compression CA is:

$$W_{CA} = nRT_C\ln\left(\frac{V_A}{V_C}\right) = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})\ln\left(\frac{19.69\,\text{L}}{111.4\,\text{L}}\right) = -17.29\,\text{kJ}\ \boxed{-17.3\,\text{kJ}}$$

(*f*) The heat absorbed during the isobaric expansion AB is:

$$Q_{\text{in, AB}} = nc_P\Delta T_{AB} = \tfrac{5}{2}nR\Delta T_{AB} = \tfrac{5}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(1200\,\text{K} - 600\,\text{K}) = \boxed{24.9\,\text{kJ}}$$

Express and evaluate the heat absorbed during the adiabatic expansion BC:

$$Q_{BC} = \boxed{0}$$

Use the first law of thermodynamics to express and evaluate the heat absorbed during the isothermal compression CA:

$$Q_{CA} = W_{CA} + \Delta E_{int,\,CA} = W_{CA}$$
$$= \boxed{-17.3\,\text{kJ}}$$

because $\Delta E_{int} = 0$ for an isothermal process.

81 •• In a heat engine, 2.00 mol of a monatomic gas are carried through the cycle ABCDA shown in Figure 19-21. (The *PV* diagram is not drawn to scale.) The segment AB represents an isothermal expansion, the segment BC an adiabatic expansion. The pressure and temperature at A are 5.00 atm and 600 K. The volume at B is twice the volume at A. The pressure at D is 1.00 atm. (*a*) What is the pressure at B? (*b*) What is the temperature at C? (*c*) Find the total work done by the gas in one cycle.

Picture the Problem We can use the ideal-gas law to find the unknown temperatures, pressures, and volumes at points B, C, and D and then find the work done by the gas and the efficiency of the cycle by using the expressions for the work done on or by the gas and the heat that enters the system for the various thermodynamic processes of the cycle.

(*a*) Apply the ideal-gas law for a fixed amount of gas to the isothermal process AB:

$$P_B = P_A \frac{V_A}{V_B} = (5.00\,\text{atm})\frac{V_A}{2V_A}$$
$$= 2.50\,\text{atm} \times \frac{101.325\,\text{kPa}}{1\,\text{atm}}$$
$$= 253.3\,\text{kPa} = \boxed{253\,\text{kPa}}$$

(*b*) Apply the ideal-gas law for a fixed amount of gas to the adiabatic process BC:

$$T_C = T_B \frac{P_C V_C}{P_B V_B} \qquad (1)$$

Use the ideal-gas law to find the volume at B:

$$V_B = \frac{nRT_B}{P_B}$$
$$= \frac{(2.00\,\text{mol})\left(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})}{253.3\,\text{kPa}}$$
$$= 39.39\,\text{L}$$

Use the equation of state for an adiabatic process and $\gamma = 5/3$ to find the volume occupied by the gas at C:

$$V_C = V_B\left(\frac{P_B}{P_C}\right)^{1/\gamma} = (39.39\,\text{L})\left(\frac{2.50\,\text{atm}}{1.00\,\text{atm}}\right)^{3/5}$$
$$= 68.26\,\text{L}$$

Substitute numerical values in equation (1) and evaluate T_C:

$$T_C = (600\,\text{K})\frac{(1.00\,\text{atm})(68.26\,\text{L})}{(2.50\,\text{atm})(39.39\,\text{L})}$$
$$= 415.9\,\text{K} = \boxed{416\,\text{K}}$$

(*c*) The work done by the gas in one cycle is given by:

$$W = W_{AB} + W_{BC} + W_{CD} + W_{DA} \quad (2)$$

The work done during the isothermal expansion AB is:

$$W_{AB} = nRT_A \ln\left(\frac{V_B}{V_A}\right) = (2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(600\,\text{K})\ln\left(\frac{2V_A}{V_A}\right) = 6.915\,\text{kJ}$$

The work done during the adiabatic expansion BC is:

$$W_{BC} = -C_V\Delta T_{BC} = -\tfrac{3}{2}nR\Delta T_{BC}$$
$$= -\tfrac{3}{2}(2.00\,\text{mol})\left(8.314\frac{\text{J}}{\text{mol}\cdot\text{K}}\right)(415.9\,\text{K} - 600\,\text{K})$$
$$= 4.592\,\text{kJ}$$

The work done during the isobaric compression CD is:

$$W_{CD} = P_C(V_D - V_C) = (1.00\,\text{atm})(19.7\,\text{L} - 68.26\,\text{L}) = -48.56\,\text{atm}\cdot\text{L} \times \frac{101.325\,\text{J}}{\text{atm}\cdot\text{L}}$$
$$= -4.920\,\text{kJ}$$

The work done during the constant-volume process DA is:

$$W_{DA} = 0$$

Substitute numerical values in equation (2) to obtain:

$$W = 6.915\,\text{kJ} + 4.592\,\text{kJ} - 4.920\,\text{kJ} + 0$$
$$= \boxed{6.59\,\text{kJ}}$$

82 •• Compare the efficiency of the Otto cycle to the efficiency of the Carnot cycle operating between the same maximum and minimum temperatures. (The Otto cycle is discussed in Section 19-1.)

Picture the Problem We can express the efficiency of the Otto cycle using the result from Example 19-2. We can apply the relation $TV^{\gamma-1} = \text{constant}$ to the adiabatic processes of the Otto cycle to relate the end-point temperatures to the volumes occupied by the gas at these points and eliminate the temperatures at *c* and *d*. We can use the ideal-gas law to find the highest temperature of the gas during its cycle and use this temperature to express the efficiency of a Carnot engine. Finally, we can compare the efficiencies by examining their ratio.

The efficiency of the Otto engine is given in Example 19-2:

$$\varepsilon_{\text{Otto}} = 1 - \frac{T_d - T_a}{T_c - T_b} \qquad (1)$$

where the subscripts refer to the various points of the cycle as shown in Figure 19-3.

Apply the relation $TV^{\gamma-1}$ = constant to the adiabatic process $a \to b$ to obtain:

$$T_b = T_a\left(\frac{V_a}{V_b}\right)^{\gamma-1}$$

Apply the relation $TV^{\gamma-1}$ = constant to the adiabatic process $c \to d$ to obtain:

$$T_c = T_d\left(\frac{V_d}{V_c}\right)^{\gamma-1}$$

Subtract the first of these equations from the second to obtain:

$$T_c - T_b = T_d\left(\frac{V_d}{V_c}\right)^{\gamma-1} - T_a\left(\frac{V_a}{V_b}\right)^{\gamma-1}$$

In the Otto cycle, $V_a = V_d$ and $V_c = V_b$. Substitute to obtain:

$$T_c - T_b = T_d\left(\frac{V_a}{V_b}\right)^{\gamma-1} - T_a\left(\frac{V_a}{V_b}\right)^{\gamma-1} = (T_d - T_a)\left(\frac{V_a}{V_b}\right)^{\gamma-1}$$

Substitute in equation (1) and simplify to obtain:

$$\varepsilon_{\text{Otto}} = 1 - \frac{T_d - T_a}{(T_d - T_a)\left(\frac{V_a}{V_b}\right)^{\gamma-1}} = 1 - \left(\frac{V_b}{V_a}\right)^{\gamma-1} = 1 - \frac{T_a}{T_b}$$

Note that, while T_a is the lowest temperature of the cycle, T_b is not the highest temperature.

Apply the ideal-gas law to c and b to obtain an expression for the cycle's highest temperature T_c:

$$\frac{P_c}{T_c}=\frac{P_b}{T_b}\Rightarrow T_c=T_b\frac{P_c}{P_b}>T_b$$

The efficiency of a Carnot engine operating between the maximum and minimum temperatures of the Otto cycle is given by:

$$\varepsilon_{\text{Carnot}}=1-\frac{T_a}{T_c}$$

Express the ratio of the efficiency of a Carnot engine to the efficiency of an Otto engine operating between the same temperatures:

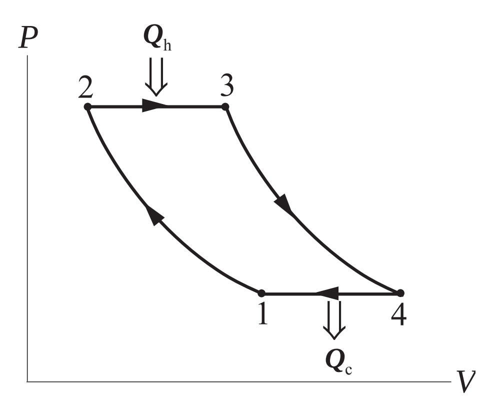
$$\frac{\varepsilon_{\text{Carnot}}}{\varepsilon_{\text{Otto}}}=\frac{1-\dfrac{T_a}{T_c}}{1-\dfrac{T_a}{T_b}}>1 \text{ because } T_c>T_b.$$

Hence, $\boxed{\varepsilon_{\text{Carnot}}>\varepsilon_{\text{Otto}}}$

83 ••• [SSM] A common practical cycle, often used in refrigeration, is the *Brayton cycle*, which involves (1) an adiabatic compression, (2) an isobaric (constant pressure) expansion,(3) an adiabatic expansion, and (4) an isobaric compression back to the original state. Assume the system begins the adiabatic compression at temperature T_1, and transitions to temperatures T_2, T_3 and T_4 after each leg of the cycle. (*a*) Sketch this cycle on a *PV* diagram. (*b*) Show that the efficiency of the overall cycle is given by $\varepsilon=1-\frac{(T_4-T_1)}{(T_3-T_2)}$. (*c*) Show that this efficiency, can be written as $\varepsilon=1-r^{(1-\gamma)/\gamma}$, where r is the pressure ratio $P_{\text{high}}/P_{\text{low}}$ of the maximum and minimum pressures in the cycle.

Picture the Problem The efficiency of the cycle is the ratio of the work done to the heat that flows into the engine. Because the adiabatic transitions in the cycle do not have heat flow associated with them, all we must do is consider the heat flow in and out of the engine during the isobaric transitions.

(*a*) The *Brayton heat engine cycle* is shown to the right. The paths 1→2 and 3→4 are adiabatic. Heat Q_h enters the gas during the isobaric transition from state 2 to state 3 and heat Q_c leaves the gas during the isobaric transition from state 4 to state 1.



(*b*) The efficiency of a heat engine is given by:

$$\varepsilon=\frac{W}{Q_{\text{in}}}=\frac{Q_{\text{h}}-Q_{\text{c}}}{Q_{\text{in}}} \qquad (1)$$

During the constant-pressure expansion from state 1 to state 2 heat enters the system:

$$Q_{23}=Q_{\text{h}}=nC_{\text{P}}\Delta T=nC_{\text{P}}\left(T_3-T_2\right)$$

During the constant-pressure compression from state 3 to state 4 heat enters the system:

$$Q_{41}=-Q_{\text{c}}=-nC_{\text{P}}\Delta T=-nC_{\text{P}}\left(T_1-T_4\right)$$

Substituting in equation (1) and simplifying yields:

$$\begin{aligned}\varepsilon&=\frac{nC_{\text{P}}\left(T_3-T_2\right)-\left(-nC_{\text{P}}\left(T_1-T_4\right)\right)}{nC_{\text{P}}\left(T_3-T_2\right)}\\&=\frac{\left(T_3-T_2\right)+\left(T_1-T_4\right)}{\left(T_3-T_2\right)}\\&=\boxed{1-\frac{\left(T_4-T_1\right)}{\left(T_3-T_2\right)}}\end{aligned}$$

(*c*) Given that, for an adiabatic transition, $TV^{\gamma-1}=\text{constant}$, use the ideal-gas law to eliminate V and obtain:

$$\frac{T^{\gamma}}{P^{\gamma-1}}=\text{constant}$$

Let the pressure for the transition from state 1 to state 2 be P_{low} and the pressure for the transition from state 3 to state 4 be P_{high}. Then for the adiabatic transition from state 1 to state 2:

$$\frac{T_1^{\gamma}}{P_{\text{low}}^{\gamma-1}}=\frac{T_2^{\gamma}}{P_{\text{high}}^{\gamma-1}}\Rightarrow T_1=\left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}T_2$$

Similarly, for the adiabatic transition from state 3 to state 4:

$$T_4=\left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}T_3$$

Subtract T_1 from T_4 and simplify to obtain:

$$\begin{aligned}T_4-T_1&=\left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}T_3-\left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}T_2\\&=\left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}\left(T_3-T_2\right)\end{aligned}$$

Dividing both sides of the equation by $T_3 - T_2$ yields:

$$\frac{T_4 - T_1}{T_3 - T_2} = \left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}}$$

Substitute in the result of Part (*b*) and simplify to obtain:

$$\varepsilon = 1 - \left(\frac{P_{\text{low}}}{P_{\text{high}}}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \left(\frac{P_{\text{high}}}{P_{\text{low}}}\right)^{\frac{1-\gamma}{\gamma}}$$

$$= \boxed{1 - (r)^{\frac{1-\gamma}{\gamma}}}$$

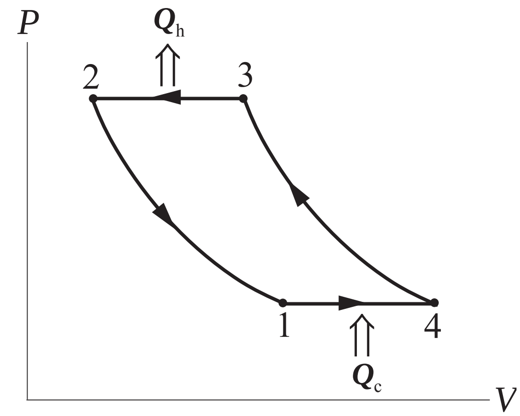
where $r = \dfrac{P_{\text{high}}}{P_{\text{low}}}$

84 ••• Suppose the Brayton cycle engine (see Problem 83) is run in reverse as a refrigerator in your kitchen. In this case, the cycle begins at temperature T_1 and expands at constant pressure until its temperature T_4. Then the gas is adiabatically compressed, until its temperature is T_3. And then it is compressed at constant pressure until its temperature T_2. Finally, it adiabatically expands until it returns to its initial state at temperature T_1. (*a*) Sketch this cycle on a *PV* diagram. (*b*) Show that the coefficient of performance is $\text{COP}_\text{B} = \dfrac{(T_4 - T_1)}{(T_3 - T_2 - T_4 + T_1)}$.

(*c*) Suppose your "Brayton cycle refrigerator" is run as follows. The cylinder containing the refrigerant (a monatomic gas) has an initial volume and pressure of 60 mL and 1.0 atm. After the expansion at constant pressure, the volume and temperature are 75 mL and –25°C. The pressure ratio $r = P_{\text{high}}/P_{\text{low}}$ for the cycle is 5.0. What is the coefficient of performance for your refrigerator? (*d*) To absorb heat from the food compartment at the rate of 120 W, what is the rate at which electrical energy must be supplied to the motor of this refrigerator? (*e*) Assuming the refrigerator motor is actually running for only 4.0 h each day, how much does it add to your monthly electric bill. Assume 15 cents per kWh of electric energy and thirty days in a month.

Picture the Problem The efficiency of the Brayton refrigerator cycle is the ratio of the heat that enters the system to the work done to operate the refrigerator. Because the adiabatic transitions in the cycle do not have heat flow associated with them, all we must do is consider the heat flow in and out of the refrigerator during the isobaric transitions.

(*a*) The *Brayton refrigerator cycle* is shown to the right. The paths 1→2 and 3→4 are adiabatic. Heat Q_c enters the gas during the constant-pressure transition from state 1 to state 4 and heat Q_h leaves the gas during the constant-pressure transition from state 3 to state 2.



(*b*) The coefficient of performance of the Brayton cycle refrigerator is given by:

$$\mathrm{COP_B} = \frac{Q_c}{W} \qquad (1)$$

where $W = Q_{32} - Q_{14}$

During the constant-pressure compression from state 3 to state 2 heat leaves the system:

$$Q_{32} = -Q_h = -nC_P\Delta T = -nC_P(T_2 - T_3)$$

During the constant-pressure expansion from state 1 to state 4 heat enters the system:

$$Q_{14} = Q_c = nC_P\Delta T = nC_P(T_4 - T_1)$$

Substituting in equation (1) and simplifying yields:

$$\begin{aligned}\mathrm{COP_B} &= \frac{nC_P(T_4 - T_1)}{-nC_P(T_2 - T_3) - nC_P(T_4 - T_1)} \\ &= \frac{(T_4 - T_1)}{-(T_4 - T_1) - (T_2 - T_3)} \\ &= \boxed{\frac{T_4 - T_1}{T_3 - T_2 - T_4 + T_1}}\end{aligned}$$

(*c*) The $\mathrm{COP_B}$ requires the temperatures corresponding to states 1, 2, 3, and 4. We're given that the temperature in state 4 is:

$$T_4 = -25°\mathrm{C} + 273\ \mathrm{K} = 248\ \mathrm{K}$$

For the constant-pressure transition from state 1 to state 4, the quotient T/V is constant:

$$\frac{T_1}{V_1} = \frac{T_4}{V_4} \Rightarrow T_1 = \left(\frac{V_1}{V_4}\right)T_4$$

Substitute numerical values and evaluate T_1:

$$T_1 = \left(\frac{60\ \mathrm{mL}}{75\ \mathrm{mL}}\right)(248\ \mathrm{K}) = 198\ \mathrm{K}$$

Given that, for an adiabatic transition, $TV^{\gamma-1} = \text{constant}$, use the ideal-gas law to eliminate V and obtain:

$$\frac{T^{\gamma}}{P^{\gamma-1}} = \text{constant}$$

For the adiabatic transition from state 4 to state 3:

$$\frac{T_3^{\gamma}}{P_3^{\gamma-1}} = \frac{T_4^{\gamma}}{P_4^{\gamma-1}} \Rightarrow T_3 = \left(\frac{P_3}{P_4}\right)^{\frac{\gamma-1}{\gamma}} T_4$$

Substitute numerical values and evaluate T_3:

$$T_3 = (5)^{\frac{1.67-1}{1.67}}(248\text{ K}) = 473\text{ K}$$

Similarly, for the adiabatic transition from state 2 to state 1:

$$T_2 = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} T_1 = (5)^{\frac{1.67-1}{1.67}}(198\text{ K})$$
$$= 378\text{ K}$$

Substitute numerical values in the expression derived in Part (a) and evaluate COP_B:

$$\text{COP}_\text{B} = \frac{248\text{ K} - 198\text{ K}}{473\text{ K} - 378\text{ K} - 248\text{ K} + 198\text{ K}}$$
$$= \boxed{1.1}$$

(d) From the definition of COP_B:

$$W = \frac{Q_c}{\text{COP}_\text{B}}$$

The rate at which energy must be supplied to this refrigerator is given by:

$$\frac{dW}{dt} = \frac{1}{\text{COP}_\text{B}}\frac{dQ_\text{c}}{dt}$$

or, if the frequency of the AC power input is f,

$$\frac{dW}{dt} = \frac{fQ_\text{c}}{\text{COP}_\text{B}}$$

Express the heat Q_c that is drawn from the cold reservoir:

$$Q_\text{c} = nC_\text{P}\Delta T = nC_\text{P}(T_4 - T_1)$$

Substituting for Q_c yields:

$$\frac{dW}{dt} = \frac{fnC_\text{P}(T_4 - T_1)}{\text{COP}_\text{B}}$$

Use the ideal-gas law to express the number of moles of the gas:

$$n = \frac{P_4V_4}{RT_4}$$

Because the gas is monatomic, $C_P = \frac{5}{2}R$. Substitute for n and C_P to obtain:

$$\frac{dW}{dt} = \frac{\frac{5}{2} f \frac{P_4V_4}{RT_4} R(T_4 - T_1)}{\text{COP}_\text{B}} = \frac{\frac{5}{2} fP_4V_4(T_4 - T_1)}{(\text{COP}_\text{B})T_4}$$

Substitute numerical values and evaluate dW/dt:

$$\frac{dW}{dt} = \frac{\frac{5}{2}(60\ \text{s}^{-1})(101.325\ \text{kPa})\left(75\ \text{mL} \times \frac{10^{-3}\ \text{m}^3}{\text{L}}\right)(248\ \text{K} - 198\ \text{K})}{(1.11)(248\ \text{K})} = 207\ \text{W}$$
$$= \boxed{0.21\ \text{kW}}$$

(*e*) The monthly cost of operation is given by

$$\text{Monthly Cost} = \text{Cost Per Unit of Power} \times \text{Power Consumption}$$
$$= \text{rate} \times \text{daily consumption} \times \text{number of days per month}$$

Substitute numerical values and evaluate the monthly cost of operation:

$$\text{Monthly Cost} = \frac{\$0.15}{\text{kWh}} \times 0.207\ \text{kW} \times \frac{4.0\ \text{h}}{\text{d}} \times 30\ \text{d} \approx \boxed{\$4}$$

85 ••• Using $\Delta S = C_V \ln(T_2/T_1) + nR\ln(V_2/V_1)$ (Equation 19-16) for the entropy change of an ideal gas, show explicitly that the entropy change is zero for a quasi-static adiabatic expansion from state (V_1, T_1) to state (V_2, T_2).

Picture the Problem We can use $nR = C_P - C_V$, $\gamma = C_P/C_V$, and $TV^{\gamma-1}$ = a constant to show that the entropy change for a quasi-static adiabatic expansion that proceeds from state (V_1,T_1) to state (V_2,T_2) is zero.

Express the entropy change for a general process that proceeds from state 1 to state 2:

$$\Delta S = C_V \ln\left(\frac{T_2}{T_1}\right) + nR\ln\left(\frac{V_2}{V_1}\right)$$

For an adiabatic process:

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Substitute for $\frac{T_2}{T_1}$ and simplify to obtain:

$$\Delta S = C_V \ln\left(\frac{V_1}{V_2}\right)^{\gamma-1} + nR\ln\left(\frac{V_2}{V_1}\right) = \ln\left(\frac{V_2}{V_1}\right)\left[nR + \frac{C_V \ln\left(\frac{V_1}{V_2}\right)^{\gamma-1}}{\ln\frac{V_2}{V_1}}\right]$$

$$= \ln\left(\frac{V_2}{V_1}\right)\left[nR + \frac{(\gamma-1)C_V \ln\left(\frac{V_1}{V_2}\right)}{-\ln\frac{V_1}{V_2}}\right] = \ln\left(\frac{V_2}{V_1}\right)\left[nR - (\gamma-1)C_V\right]$$

Use the relationship between C_P and C_V to obtain:

$$nR = C_P - C_V$$

Substituting for nR and γ and simplifying yields:

$$\Delta S = \ln\left(\frac{V_2}{V_1}\right)\left[C_P - C_V - \left(\frac{C_p}{C_V} - 1\right)C_V\right]$$
$$= \boxed{0}$$

86 ••• (a) Show that if the refrigerator statement of the second law of thermodynamics were not true, then the entropy of the universe could decrease. (b) Show that if the heat-engine statement of the second law were not true, then the entropy of the universe could decrease. (c) A third statement of the second law is that the entropy of the universe cannot decrease. Have you just proved that this statement is equivalent to the refrigerator and heat-engine statements?

Picture the Problem

(a) Suppose the refrigerator statement of the second law is violated in the sense that heat Q_c is taken from the cold reservoir and an equal amount of heat is transferred to the hot reservoir and $W = 0$. The entropy change of the universe is then $\Delta S_u = Q_c/T_h - Q_c/T_c$. Because $T_h > T_c$, $S_u < 0$, i.e., the entropy of the universe would decrease.

(b) In this case, heat Q_h is taken from the hot reservoir and no heat is rejected to the cold reservoir; that is, $Q_c = 0$, then the entropy change of the universe is $\Delta S_u = -Q_h/T_h + 0$, which is negative. Again, the entropy of the universe would decrease.

(*c*) The heat-engine and refrigerator statements of the second law only state that *some* heat must be rejected to a cold reservoir and *some* work must be done to transfer heat from the cold to the hot reservoir, but these statements do not specify the minimum amount of heat rejected or work that must be done. The statement $\Delta S_u \geq 0$ is more restrictive. The heat-engine and refrigerator statements in conjunction with the Carnot efficiency are equivalent to $\Delta S_u \geq 0$.

87 ••• Suppose that two heat engines are connected in series, such that the heat released by the first engine is used as the heat absorbed by the second engine as shown in Figure 19-22. The efficiencies of the engines are ε_1 and ε_2, respectively. Show that the net efficiency of the combination is given by $\varepsilon_{\text{net}} = \varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2$.

Picture the Problem We can express the net efficiency of the two engines in terms of W_1, W_2, and Q_h and then use $\varepsilon_1 = W_1/Q_h$ and $\varepsilon_2 = W_2/Q_m$ to eliminate W_1, W_2, Q_h, and Q_m.

Express the net efficiency of the two heat engines connected in series:

$$\varepsilon_{\text{net}} = \frac{W_1 + W_2}{Q_h}$$

Express the efficiencies of engines 1 and 2:

$$\varepsilon_1 = \frac{W_1}{Q_h} \text{ and } \varepsilon_2 = \frac{W_2}{Q_m}$$

Solve for W_1 and W_2 and substitute to obtain:

$$\varepsilon_{\text{net}} = \frac{\varepsilon_1 Q_h + \varepsilon_2 Q_m}{Q_h} = \varepsilon_1 + \frac{Q_m}{Q_h}\varepsilon_2$$

Express the efficiency of engine 1 in terms of Q_m and Q_h:

$$\varepsilon_1 = 1 - \frac{Q_m}{Q_h} \Rightarrow \frac{Q_m}{Q_h} = 1 - \varepsilon_1$$

Substitute for Q_m/Q_h and simplify to obtain:

$$\varepsilon_{\text{net}} = \varepsilon_1 + (1 - \varepsilon_1)\varepsilon_2 = \boxed{\varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2}$$

88 ••• Suppose that two heat engines are connected in series, such that the heat released by the first engine is used as the heat absorbed by the second engine, as shown in Figure 19-22. Suppose that each engine is an ideal reversible heat engine. Engine 1 operates between temperatures T_h and T_m and Engine 2 operates between T_m and T_c, where $T_h > T_m > T_c$. Show that the net efficiency of the combination is given by $\varepsilon_{\text{net}} = 1 - \frac{T_c}{T_h}$. (Note that this result means that two

reversible heat engines operating ″in series″ are equivalent to one reversible heat engine operating between the hottest and coldest reservoirs.)

Picture the Problem We can express the net efficiency of the two engines in terms of W_1, W_2, and Q_h and then use $\varepsilon_1 = W_1/Q_h$ and $\varepsilon_2 = W_2/Q_m$ to eliminate W_1, W_2, Q_h, and Q_m. Finally, we can substitute the expressions for the efficiencies of the ideal reversible engines to obtain $\varepsilon_{net} = 1 - T_c/T_h$.

Express the efficiencies of ideal reversible engines 1 and 2:

$$\varepsilon_1 = 1 - \frac{T_m}{T_h} \quad (1)$$

and

$$\varepsilon_2 = 1 - \frac{T_c}{T_m} \quad (2)$$

The net efficiency of the two engines connected in series is given by:

$$\varepsilon_{net} = \frac{W_1 + W_2}{Q_h} \quad (3)$$

Express the efficiencies of engines 1 and 2:

$$\varepsilon_1 = \frac{W_1}{Q_h} \text{ and } \varepsilon_2 = \frac{W_2}{Q_m}$$

Solve for W_1 and W_2 and substitute in equation (3) to obtain:

$$\varepsilon_{net} = \frac{\varepsilon_1 Q_h + \varepsilon_2 Q_m}{Q_h} = \varepsilon_1 + \frac{Q_m}{Q_h}\varepsilon_2$$

Express the efficiency of engine 1 in terms of Q_m and Q_h:

$$\varepsilon_1 = 1 - \frac{Q_m}{Q_h} \Rightarrow \frac{Q_m}{Q_h} = 1 - \varepsilon_1$$

Substitute for $\frac{Q_m}{Q_h}$ to obtain:

$$\varepsilon_{net} = \varepsilon_1 + (1 - \varepsilon_1)\varepsilon_2$$

Substitute for ε_1 and ε_2 and simplify to obtain:

$$\varepsilon_{net} = 1 - \frac{T_m}{T_h} + \left(\frac{T_m}{T_h}\right)\left(1 - \frac{T_c}{T_m}\right)$$

$$= 1 - \frac{T_m}{T_h} + \frac{T_m}{T_h} - \frac{T_c}{T_h} = \boxed{1 - \frac{T_c}{T_h}}$$

89 ••• [SSM] The English mathematician and philosopher Bertrand Russell (1872-1970) once said that if a million monkeys were given a million typewriters and typed away at random for a million years, they would produce all of Shakespeare’s works. Let us limit ourselves to the following fragment of Shakespeare (*Julius Caesar* III:ii):

Friends, Romans, countrymen! Lend me your ears.
I come to bury Caesar, not to praise him.
The evil that men do lives on after them,
The good is oft interred with the bones.
So let it be with Caesar.
The noble Brutus hath told you that Caesar was ambitious,
And, if so, it were a grievous fault,
And grievously hath Caesar answered it . . .

Even with this small fragment, it will take a lot longer than a million years! By what factor (roughly speaking) was Russell in error? Make any reasonable assumptions you want. (You can even assume that the monkeys are immortal.)

Picture the Problem There are 26 letters and four punctuation marks (space, comma, period, and exclamation point) used in the English language, disregarding capitalization, so we have a grand total of 30 characters to choose from. This fragment is 330 characters (including spaces) long; there are then 30^{330} different possible arrangements of the character set to form a fragment this long. We can use this number of possible arrangements to express the probability that one monkey will write out this passage and then an estimate of a monkey's typing speed to approximate the time required for one million monkeys to type the passage from Shakespeare.

Assuming the monkeys type at random, express the probability P that one monkey will write out this passage:

$$P = \frac{1}{30^{330}}$$

Use the approximation $30 \approx \sqrt{1000} = 10^{1.5}$ to obtain:

$$P = \frac{1}{10^{(1.5)(330)}} = \frac{1}{10^{495}} = 10^{-495}$$

Assuming the monkeys can type at a rate of 1 character per second, it would take about 330 s to write a passage of length equal to the quotation from Shakespeare. Find the time T required for a million monkeys to type this particular passage by accident:

$$T = \frac{(330\,\text{s})(10^{495})}{10^6} = (3.30\times10^{491}\,\text{s})\left(\frac{1\,\text{y}}{3.16\times10^{7}\,\text{s}}\right) \approx \boxed{10^{484}\,\text{y}}$$

Express the ratio of T to Russell's estimate:

$$\frac{T}{T_{\text{Russell}}} = \frac{10^{484}\,\text{y}}{10^{6}\,\text{y}} = 10^{478}$$

or

$$T \approx \boxed{10^{478}T_{\text{Russell}}}$$

Chapter 20
Thermal Properties and Processes

Conceptual Problems

1 • Why does the mercury level of a thermometer first decrease slightly when the thermometer is first placed in warm water?

Determine the Concept The glass bulb warms and expands first, before the mercury warms and expands.

2 • A large sheet of metal has a hole cut in the middle of it. When the sheet is heated, the area of the hole will (*a*) not change, (*b*) always increase, (*c*) always decrease, (*d*) increase if the hole is not in the exact center of the sheet, (*e*) decrease only if the hole is in the exact center of the sheet.

Determine the Concept The heating of the sheet causes the average separation of its molecules to increase. The consequence of this increased separation is that the area of the hole always increases. $\boxed{(b)}$ is correct.

3 • **[SSM]** Why is it a bad idea to place a tightly sealed glass bottle that is completely full of water, into your kitchen freezer in order to make ice?

Determine the Concept Water expands greatly as it freezes. If a sealed glass bottle full of water is placed in a freezer, as the water freezes there will be no room for the expansion to take place. The bottle will be broken.

4 • The windows of your physics laboratory are left open on a night when the temperature of the outside dropped well below freezing. A steel ruler and a wooden ruler were left on the window sill, and when you arrive in the morning they are both very cold. The coefficient of linear expansion of wood is about $5 \times 10^{-6}\ \text{K}^{-1}$. Which ruler should you use to make the most accurate length measurements? Explain your answer.

Determine the Concept You should use the wooden ruler. Because the coefficient of expansion for wood is about half that for metal, the metal ruler will have shrunk considerably more than will have the wooden ruler.

5 • Bimetallic strips are used both for thermostats and for electrical circuit breakers. A bimetallic strip consists of a pair of thin strips of metal that have different coefficients of linear expansion and are bonded together to form one doubly thick strip. Suppose a bimetallic strip is constructed out of one steel strip and one copper strip, and suppose the bimetallic strip is curled in the shape of a circular arc with the steel strip on the outside. If the temperature of the strip is decreased, will the strip straighten out or curl more tightly?

Determine the Concept The strip will curl more tightly. Because the coefficient of linear expansion for copper (17×10^{-6} K^{-1}) is greater than the coefficient of linear expansion for steel (11×10^{-6} K^{-1}), the length of the copper strip will decrease more than the length of the steel strip–resulting in a tighter curl.

6 • Metal A has a coefficient of linear expansion that is three times the coefficient of linear expansion of metal B. How do their coefficients of volume expansion β compare? (*a*) $\beta_A = \beta_B$, (*b*) $\beta_A = 3\beta_B$, (*c*) $\beta_A = 6\beta_B$, (*d*) $\beta_A = 9\beta_B$, (*e*) You cannot tell from the data given.

Determine the Concept We know that the coefficient of volume expansion is three times the coefficient of linear expansion and so can use this fact to express the ratio of β_A to β_B.

Express the coefficient of volume expansion of metal A in terms of its coefficient of linear expansion:

$$\beta_A = 3\alpha_A$$

Express the coefficient of volume expansion of metal B in terms of its coefficient of linear expansion:

$$\beta_B = 3\alpha_B$$

Dividing the first of these equations by the second yields:

$$\frac{\beta_A}{\beta_B} = \frac{3\alpha_A}{3\alpha_B} = \frac{\alpha_A}{\alpha_B}$$

Because $\boldsymbol{\alpha}_A = 3\boldsymbol{\alpha}_B$:

$$\frac{\beta_A}{\beta_B} = \frac{3\alpha_B}{\alpha_B} = 3 \Rightarrow \boxed{(b)} \text{ is correct.}$$

7 • The summit of Mount Rainier is 14 410 ft above sea level. Mountaineers say that you cannot hard boil an egg at the summit. This statement is true because at the summit of Mount Rainier (*a*) the air temperature is too low to boil water, (*b*) the air pressure is too low for alcohol fuel to burn, (*c*) the temperature of boiling water is not hot enough to hard boil the egg, (*d*) the oxygen content of the air is too low to support combustion, (*e*) eggs always break in climbers′ backpacks.

Determine the Concept Actually, an egg can be hard boiled, but it takes quite a bit longer than at sea level. $\boxed{(c)}$ is the best response.

8 • Which gases in Table 20-3 cannot be condensed by applying pressure at 20ºC? Explain your answer.

Determine the Concept Gases that cannot be liquefied by applying pressure at 20°C are those for which $T_c < 293$ K. These are He, Ar, Ne, H_2, O_2, NO.

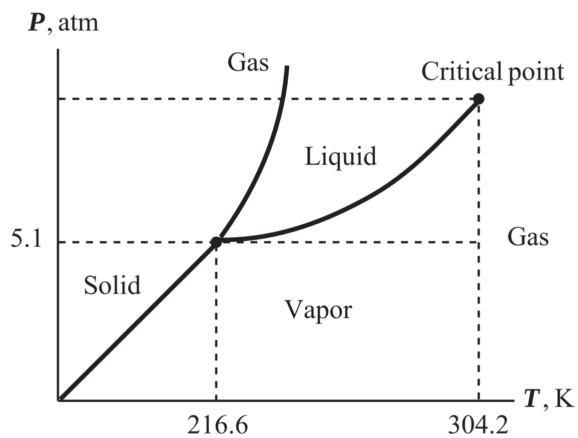
9 •• **[SSM]** The phase diagram in Figure 20-15 can be interpreted to yield information on how the boiling and melting points of water change with altitude. (*a*) Explain how this information can be obtained. (*b*) How might this information affect cooking procedures in the mountains?

Determine the Concept
(*a*) With increasing altitude, decreases; from curve OC, the temperature of the liquid-gas interface decreases as the pressure decreases, so the boiling temperature decreases. Likewise, from curve OB, the melting temperature increases with increasing altitude.

(*b*) Boiling at a lower temperature means that the cooking time will have to be increased.

10 •• Sketch a phase diagram for carbon dioxide using information from Section 20-3.

Determine the Concept The following phase diagram for carbon dioxide was constructed using information in Section 20-3.



11 •• Explain why the carbon dioxide on Mars is found in the solid state in the polar regions even though the atmospheric pressure at the surface of Mars is only about 1 percent of the atmospheric pressure at the surface of Earth.

Determine the Concept At very low pressures and temperatures, carbon dioxide can exist only as a solid or gas (or vapor above the gas). The atmosphere of Mars is 95 percent carbon dioxide. Mars, on average, is warm enough so that the atmosphere is mostly gaseous carbon dioxide. The polar regions are cold enough to enable solid carbon dioxide (dry ice) to exist, even at the low pressure.

12 •• Explain why decreasing the temperature of your house at night in winter can save money on heating costs. Why doesn't the cost of the fuel consumed to heat the house back to the daytime temperature in the morning equal

the savings realized by cooling it down in the evening and keeping it cool throughout the night?

Determine the Concept The amount of heat lost by the house is proportional to the difference between the temperature inside the house and that of the outside air. Hence, the rate at which the house loses heat (that must be replaced by the furnace) is greater at night when the temperature of the house is kept high than when it is allowed to cool down.

13 •• [SSM] Two solid cylinders made of materials A and B have the same lengths; their diameters are related by $d_A = 2d_B$. When the same temperature difference is maintained between the ends of the cylinders, they conduct heat at the same rate. Their thermal conductivities are therefore related by which of the following equations? (*a*) $k_A = k_B/4$, (*b*) $k_A = k_B/2$, (*c*) $k_A = k_B$, (*d*) $k_A = 2k_B$, (*e*) $k_A = 4k_B$

Picture the Problem The rate at which heat is conducted through a cylinder is given by $I = dQ/dt = kA\Delta T/\Delta x$ (see Equation 20-7) where A is the cross-sectional area of the cylinder.

The heat current in cylinder A is the same as the heat current in cylinder B:

$$I_A = I_B$$

Substituting for the heat currents yields:

$$k_A A_A \frac{\Delta T}{L} = k_B A_B \frac{\Delta T}{L} \Rightarrow k_A = k_B \frac{A_B}{A_A}$$

Because $d_A = 2d_B$:

$$k_A = k_B\left(\frac{A_B}{4A_B}\right) \Rightarrow k_A = \tfrac{1}{4}k_B$$

$\boxed{(a)}$ is correct.

14 •• Two solid cylinders made of materials A and B have the same diameter; their lengths are related by $L_A = 2L_B$. When the same temperature difference is maintained between the ends of the cylinders, they conduct heat at the same rate. Their thermal conductivities are therefore related by which of the following equations? (*a*) $k_A = k_B/4$, (*b*) $k_A = k_B/2$, (*c*) $k_A = k_B$, (*d*) $k_A = 2k_B$, (*e*) $k_A = 4k_B$.

Determine the Concept We can use the expression for the heat current in a conductor, Equation 20-7, to relate the heat current in each cylinder to its thermal conductivity, cross-sectional area, temperature difference, and length.

The heat current in cylinder A is the same as the heat current in cylinder B:

$$I_A = I_B$$

Substituting for the heat currents yields:

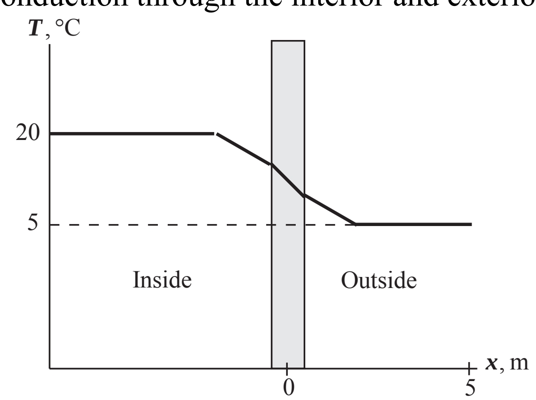
$$k_A A \frac{\Delta T}{L_A} = k_B A \frac{\Delta T}{L_B} \Rightarrow k_A = k_B \frac{L_A}{L_B}$$

Because $L_A = 2L_B$:

$$k_A = k_B \frac{2L_B}{L_B} = 2k_B \Rightarrow \boxed{(d)} \text{ is correct.}$$

15 •• If you feel the inside of a single pane window during a very cold day, it is cold, even though the room temperature can be quite comfortable. Assuming the room temperature is 20.0°C and the outside temperature is 5.0°C, Construct a plot of temperature versus position starting from a point 5.0 m in behind the window (inside the room) and ending at a point 5.0 m in front of the window. Explain the heat transfer mechanisms that occur along this path.

Determine the Concept The temperatures on both sides of the glass are almost the same. Because glass is an excellent conductor of heat, there need not be a huge temperature difference. Thus the temperature must drop quickly as you near the pane on the warm side and the same on the outside. This is sketched qualitatively in the following diagram. Convection and radiation are primarily responsible for heat transfer on the inside and outside, and it is mainly conduction through the glass. Conduction through the interior and exterior air is minimal.



16 •• During the thermal retrofitting of many older homes in California, it was found that the 3.5-in-deep spaces between the wallboards and the outer sheathing were filled with just air (no insulation). Filling the space with insulating material certainly reduces heating and cooling costs; although, the insulating material is a better conductor of heat than air is. Explain why adding the insulation is a good idea.

Determine the Concept The tradeoff is the reduction of convection cells between the walls by putting in the insulating material, versus a slight increase in conductivity. The net reduction in convection results in a higher R value.

Estimation and Approximation

17 •• [SSM] You are using a cooking pot to boil water for a pasta dish. The recipe calls for at least 4.0 L of water to be used. You fill the pot with 4.0 L of room temperature water and note that this amount of water filled the pot to the brim. Knowing some physics, you are counting on the volume expansion of the steel pot to keep all of the water in the pot while the water is heated to a boil. Is your assumption correct? Explain. If your assumption is not correct, how much water runs over the sides of the pot due to the thermal expansion of the water?

Determine the Concept The volume of water overflowing is the difference between the change in volume of the water and the change in volume of the pot. See Table 20-1 for the coefficient of volume expansion of water and the coefficient of linear expansion of steel.

Express the volume of water that overflows when the pot and the water are heated:

$$\begin{aligned} V_{\text{ovefrlow}} &= \Delta V_{H_2O} - \Delta V_{\text{pot}} \\ &= \beta_{H_2O} V_0 \Delta T - \beta_{\text{steel}} V_0 \Delta T \\ &= \left(\beta_{H_2O} - \beta_{\text{steel}}\right) V_0 \Delta T \end{aligned}$$

Because the coefficient of volume expansion of steel is three times its coefficient of linear expansion:

$$\beta_{\text{steel}} = 3\alpha_{\text{steel}}$$

Substituting for β_{H_2O} and β_{steel} yields:

$$V_{\text{overflow}} = \left(\alpha_{H_2O} - 3\alpha_{\text{steel}}\right) V_0 \Delta T$$

Substitute numerical values and evaluate V_{overflow} :

$$V_{\text{overflow}} = \left(0.207\times10^{-3}\ \text{K}^{-1} - 3\left(11\times10^{-6}\ \text{K}^{-1}\right)\right)(4.0\ \text{L})(100^\circ\text{C} - 20^\circ\text{C}) = \boxed{56\ \text{mL}}$$

Your assumption was not correct and 56 mL of water overflowed.

18 •• Liquid helium is stored in containers fitted with 7.00-cm-thick ″superinsulation″ consisting of numerous layers of very thin aluminized Mylar sheets. The rate of evaporation of liquid helium in a 200-L container is about 0.700 L per day when the container is stored at room temperature (20ºC). The density of liquid helium is 0.125 kg/L and the latent heat of vaporization is 21.0 kJ/kg. Estimate the thermal conductivity of the superinsulation.

Picture the Problem We can express the heat current through the insulation in terms of the rate of evaporation of the liquid helium and in terms of the temperature gradient across the superinsulation. Equating these equations will lead to an expression for the thermal conductivity *k* of the superinsulation. Note that the boiling temperature of liquid helium is 4.2 K.

Express the heat current in terms of the rate of evaporation of the liquid helium:

$$I = L_\text{v}\frac{dm}{dt}$$

Express the heat current in terms of the temperature gradient across the superinsulation and the conductivity of the superinsulation:

$$I = kA\frac{\Delta T}{\Delta x}$$

Equate these expressions and solve for k to obtain:

$$k = \frac{L_\text{v}\Delta x\dfrac{dm}{dt}}{A\Delta T}$$

Using the definition of density, express the rate of loss of liquid helium:

$$\frac{dm}{dt} = \rho\frac{dV}{dt}$$

Substitute for $\frac{dm}{dt}$ to obtain:

$$k = \frac{L_\text{v}\Delta x\rho\dfrac{dV}{dt}}{A\Delta T}$$

Express the ratio of the area of the spherical container to its volume:

$$\frac{A}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} \Rightarrow A = \sqrt[3]{36\pi V^2}$$

Substituting for A yields:

$$k = \frac{L_\text{v}\Delta x\rho\dfrac{dV}{dt}}{\sqrt[3]{36\pi V^2}\,\Delta T}$$

Substitute numerical values and evaluate k:

$$k = \frac{\left(21.0\dfrac{\text{kJ}}{\text{kg}}\right)\left(7.00\times10^{-2}\text{ m}\right)\left(125\dfrac{\text{kg}}{\text{m}^3}\right)\left(\dfrac{0.700\times10^{-3}\text{ m}^3}{86400\text{s}}\right)}{\sqrt[3]{36\pi\left(200\times10^{-3}\text{ m}^3\right)^2}\left(289\text{K}\right)} = \boxed{3.12\times10^{-6}\ \frac{\text{W}}{\text{m}\cdot\text{K}}}$$

19 •• [SSM] Estimate the thermal conductivity of human skin.

Picture the Problem We can use the thermal current equation for the thermal conductivity of the skin. If we model a human body as a rectangular parallelepiped that is 1.5 m high × 7 cm thick × 50 cm wide, then its surface area is about 1.8 m^2. We'll also assume that a typical human, while resting, produces energy at the rate of 120 W, that normal internal and external temperatures are 33°C and 37°C, respectively, and that an average skin thickness is 1.0 mm.

Use the thermal current equation to express the rate of conduction of thermal energy:

$$I = kA\frac{\Delta T}{\Delta x} \Rightarrow k = \frac{I}{A\frac{\Delta T}{\Delta x}}$$

Substitute numerical values and evaluate *k*:

$$k = \frac{120\,\text{W}}{\left(1.8\,\text{m}^2\right)\frac{37°\text{C} - 33°\text{C}}{1.0\times10^{-3}\,\text{m}}} = \boxed{17\frac{\text{mW}}{\text{m}\cdot\text{K}}}$$

20 •• You are visiting Finland with a college friend and have met some Finnish friends. They talk you into taking part in a traditional Finnish after-sauna exercise which consists of leaving the sauna, wearing only a bathing suit, and running out into the mid-winter Finnish air. Estimate the rate at which you initially lose energy to the cold air. Compare this rate of initial energy loss to the resting metabolic rate of a typical human under normal temperature conditions. Explain the difference.

Picture the Problem We can use the Stefan-Boltzmann law to estimate the rate at which you lose energy when you first step out of the sauna. If we model a human body as a rectangular parallelepiped that is 1.5 m high × 7 cm thick × 50 cm wide, then its surface area is about 1.8 m^2. Assume that your skin temperature is initially 37°C (310 K), that the mid-winter outside temperature is −10°C (263 K), and that the emissivity of your skin is 1.

Use the Stefan-Boltzmann law to express the net rate at which you radiate energy to the cold air:

$$P_{\text{net}} = \varepsilon\sigma A\left(T_{\text{skin}}^4 - T_{\text{air}}^4\right)$$

Substitute numerical values and evaluate P_{net}:

$$P_{\text{net}} = (1)\left(5.670\times10^{-8}\,\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right)\left(1.8\,\text{m}^2\right)\left((310\,\text{K})^4 - (263\,\text{K})^4\right) = \boxed{450\,\text{W}}$$

This result is almost four times greater than the basal metabolic rate of 120 W. We can understand the difference in terms of the temperature of your skin when you first step out of the sauna and the fact that radiation loses are dependent on the fourth power of the absolute temperature.

Remarks: The emissivity of your skin is, in fact, very close to 1.

21 •• Estimate the rate of heat conduction through a 2.0-in-thick wooden door during a cold winter day in Minnesota. Include the brass doorknob. What is the ratio of the heat that escapes through the doorknob to the heat that escapes through the whole door? What is the total overall *R*-factor for the door, including the knob? The thermal conductivity of brass is $85\,\text{W}/(\text{m}\cdot\text{K})$.

Picture the Problem We can use the thermal current equation (Equation 20-7) to estimate the rate of heat loss through the door and its knob. We'll assume that the door is made of oak with an area of 2.0 m^2, that the knob is made of brass, that the conducting path has a diameter of 3.0 cm, and that the inside temperature is 20°C. Take the outside temperature to be −20°C. The reciprocal of the equivalent R-factor is the sum of the reciprocals of the R-values of the door and the knob.

The total thermal current through the door is the sum of the thermal currents through the door and the knob:

$$I_{\text{tot}} = I_{\text{door}} + I_{\text{knob}} = \frac{k_{\text{door}}A_{\text{door}}\Delta T}{\Delta x_{\text{door}}} + \frac{k_{\text{knob}}A_{\text{knob}}\Delta T}{\Delta x_{\text{knob}}}$$

Assume that $\Delta \boldsymbol{x}_{\text{door}} = \Delta \boldsymbol{x}_{\text{knob}} = \Delta \boldsymbol{x}$ to obtain:

$$I_{\text{tot}} = \left(k_{\text{door}}A_{\text{door}} + k_{\text{knob}}A_{\text{knob}}\right)\frac{\Delta T}{\Delta x}$$

Substitute numerical values and evaluate I_{tot}:

$$I_{\text{tot}} = \left[\left(0.15\frac{\text{W}}{\text{m}\cdot\text{K}}\right)\left(2.0\ \text{m}^2\right) + \left(85\frac{\text{W}}{\text{m}\cdot\text{K}}\right)\frac{\pi}{4}\left(3.0\times10^{-2}\ \text{m}\right)^2\right]\frac{20°\text{C} - (-20°\text{C})}{(2.0\ \text{in})\left(\dfrac{1\ \text{m}}{39.37\ \text{in}}\right)}$$

$$= \boxed{0.28\ \text{kW}}$$

Express the ratio of the thermal currents through the knob and the door:

$$\frac{I_{\text{knob}}}{I_{\text{door}}} = \frac{k_{\text{door}}A_{\text{door}}\left(\dfrac{\Delta T}{\Delta x}\right)_{\text{door}}}{k_{\text{knob}}A_{\text{knob}}\left(\dfrac{\Delta T}{\Delta x}\right)_{\text{knob}}}$$

Because the temperature difference across the door and the knob are the same and we've assumed that the thickness of the door and length of the knob are the same:

$$\frac{I_{\text{knob}}}{I_{\text{door}}} = \frac{k_{\text{door}}A_{\text{door}}}{k_{\text{knob}}A_{\text{knob}}}$$

Substitute numerical values to obtain:

$$\frac{I_{\text{knob}}}{I_{\text{door}}} = \frac{\left(0.15\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)\left(2.0\ \text{m}^2\right)}{\left(85\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)\dfrac{\pi}{4}\left(3.0\times10^{-2}\ \text{m}\right)^2} = \boxed{5.0}$$

Relate the equivalent R-factor to the R-factors of the door and knob;

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_{\text{door}}} + \frac{1}{R_{\text{knob}}}$$

Substitute for R_{door} and R_{knob} to obtain:

$$\frac{1}{R_{eq}} = \frac{1}{\dfrac{\Delta x}{k_{door}A_{door}}} + \frac{1}{\dfrac{\Delta x}{k_{knob}A_{knob}}} = \frac{k_{door}A_{door} + k_{knob}A_{knob}}{\Delta x}$$

Solving for R_{eq} yields:

$$R_{eq} = \frac{\Delta x}{k_{door}A_{door} + k_{knob}A_{knob}}$$

Substitute numerical values and evaluate R_{eq}:

$$R_{eq} = \frac{(2.0\text{ in})\left(\dfrac{1\text{ m}}{39.37\text{ in}}\right)}{\left(0.15\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)(2.0\text{ m}^2) + \left(85\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)\dfrac{\pi}{4}\left(3.0\times10^{-2}\text{ m}\right)^2} = \boxed{0.14\frac{\text{K}}{\text{W}}}$$

22 •• Estimate the effective emissivity of Earth, given the following information. The solar constant, which is the intensity of radiation incident on Earth from the Sun, is about 1.37 kW/m^2. Seventy percent of this energy is absorbed by Earth, and Earth's average surface temperature is 288 K. (Assume that the effective area that is absorbing the light is πR^2, where R is Earth's radius, while the blackbody-emission area is $4\pi R^2$.)

Picture the Problem The amount of heat radiated by Earth must equal the solar flux from the Sun, or else the temperature on Earth would continually increase. The emissivity of Earth is related to the rate at which it radiates energy into space by the Stefan-Boltzmann law $P_r = e\sigma AT^4$.

Using the Stefan-Boltzmann law, express the rate at which Earth radiates energy as a function of its emissivity e and temperature T:

$$P_r = e\sigma A'T^4 \Rightarrow e = \frac{P_r}{\sigma A'T^4}$$

where A' is the surface area of Earth.

Use its definition to express the intensity of the radiation received by Earth:

$$I = \frac{P_{absorbed}}{A}$$

where A is the cross-sectional area of Earth.

For 70% absorption of the Sun's radiation incident on Earth:

$$I = \frac{(0.70)P_r}{A}$$

Substitute for P_r and A in the expression for e and simplify to

$$e = \frac{(0.70)AI}{\sigma A'T^4} = \frac{(0.70)\pi R^2 I}{4\pi R^2\sigma T^4} = \frac{(0.70)I}{4\sigma T^4}$$

obtain:

Substitute numerical values and evaluate e:

$$e = \frac{(0.70)(1.37\,\text{kW/m}^2)}{4(5.670\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(288\,\text{K})^4} = \boxed{0.61}$$

23 •• Black holes are highly condensed remnants of stars. Some black holes, together with a normal star, form binary systems. In such systems the black hole and the normal star orbit about the center of mass of the system. One way black holes can be detected from Earth is by observing the frictional heating of the atmospheric gases from the normal star that fall into the black hole. These gases can reach temperatures greater than 1.0×10^6 K. Assuming that the falling gas can be modeled as a blackbody radiator, estimate λ_{max} for use in an astronomical detection of a black hole. (Remark: This is in the X-ray region of the electromagnetic spectrum.)

Picture the Problem The wavelength at which maximum power is radiated by the gas falling into a black hole is related to its temperature by Wien's displacement law.

Wien's displacement law relates the wavelength at which maximum power is radiated by the gas to its temperature:

$$\lambda_{\text{max}} = \frac{2.898\,\text{mm}\cdot\text{K}}{T}$$

Substitute for T and evaluate λ_{max}:

$$\lambda_{\text{max}} = \frac{2.898\,\text{mm}\cdot\text{K}}{1.0\times10^6\ \text{K}} = \boxed{2.9\,\text{nm}}$$

24 ••• Your cabin in northern Michigan has walls that consist of pine logs that have average thicknesses of about 20 cm. You decide to finish the interior of the cabin to improve the look and to increase the insulation of the exterior walls. You choose to buy insulation with an R-factor of 31 to cover the walls. In addition, you cover the insulation with 1.0-in-thick gypsum wallboard. Assuming heat transfer is only due to conduction, estimate the ratio of thermal current through the walls during a cold winter night before the renovation to the thermal current through the walls following the renovation.

Picture the Problem We can use the thermal current equation to find the thermal current per square meter through the walls of the cabin both before and after the walls have been insulated. See Table 20-5 for the R-factor of gypsum wallboard and Table 20-4 for the thermal conductivity of white pine.

The rate at which heat is conducted through the walls is given by the thermal current equation:

$$I_{\text{before}} = k_{\text{before}} A \frac{\Delta T}{\Delta x} = \frac{\Delta T}{R_{\text{before}}}$$

With the insulation in place:

$$I_{\text{after}} = k_{\text{after}} A \frac{\Delta T}{\Delta x} = \frac{\Delta T}{R_{\text{after}}}$$

Divide the second of these equations by the first to obtain:

$$\frac{I_{\text{after}}}{I_{\text{before}}} = \frac{\dfrac{\Delta T}{R_{\text{after}}}}{\dfrac{\Delta T}{R_{\text{before}}}} = \frac{R_{\text{before}}}{R_{\text{after}}}$$

R_{before} is the R-factor for pine and R_{after} is the sum of the R-factors for pine, the insulating material, and the gypsum board:

$$R_{\text{before}} = R_{\text{pine}}$$

and

$$R_{\text{after}} = R_{\text{eq}} = R_{\text{pine}} + R_{31} + R_{\text{gypsum}}$$

Substitute for R_{before} and R_{after} to obtain:

$$\frac{I_{\text{after}}}{I_{\text{before}}} = \frac{R_{\text{pine}}}{R_{\text{pine}} + R_{31} + R_{\text{gypsum}}}$$

Because $R_{\text{pine}} = \dfrac{\Delta x}{k_{\text{pine}}}$:

$$\frac{I_{\text{after}}}{I_{\text{before}}} = \frac{\dfrac{\Delta x}{k_{\text{pine}}}}{\dfrac{\Delta x}{k_{\text{pine}}} + R_{31} + R_{\text{gypsum}}}$$

Substitute numerical values and evaluate the ratio $I_{\text{after}} / I_{\text{before}}$:

$$\frac{I_{\text{after}}}{I_{\text{before}}} = \frac{\dfrac{20\,\text{cm} \times \dfrac{1\,\text{in}}{2.54\,\text{cm}}}{0.78\,\dfrac{\text{Btu}\cdot\text{in}}{\text{h}\cdot\text{ft}^2\cdot\text{F}^\circ}}}{\dfrac{20\,\text{cm} \times \dfrac{1\,\text{in}}{2.54\,\text{cm}}}{0.78\,\dfrac{\text{Btu}\cdot\text{in}}{\text{h}\cdot\text{ft}^2\cdot\text{F}^\circ}} + 31\,\dfrac{\text{h}\cdot\text{ft}^2\cdot\text{F}^\circ}{\text{Btu}} + \dfrac{1\,\text{in}}{0.375\,\text{in}} \times 0.32\,\dfrac{\text{h}\cdot\text{ft}^2\cdot\text{F}^\circ}{\text{Btu}}} = \boxed{24\%}$$

25 ••• [SSM] You are in charge of transporting a liver from New York, New York to Los Angeles, California for a transplant surgery. The liver is kept cold in a Styrofoam ice chest initially filled with 1.0 kg of ice. It is crucial that the liver temperature is never warmer than 5.0°C. Assuming the trip from the hospital in New York to the hospital in Los Angeles takes 7.0 h, estimate the R-value the Styrofoam walls of the ice chest must have.

Picture the Problem The R factor is the thermal resistance per unit area of a slab of material. We can use the thermal current equation to express the thermal resistance of the styrofoam in terms of the maximum amount of heat that can enter the chest in 7.0 h without raising the temperature above 5.0°C. We'll

assume that the surface area of the ice chest is 1.0 m^2 and that the ambient temperature is 25°C

The R-factor needed for the styrofoam walls of the ice chest is the product of their thermal resistance and area:

$$R_f = RA \qquad (1)$$

Use the thermal current equation to express R:

$$R = \frac{\Delta T}{I} = \frac{\Delta T}{\frac{Q_{tot}}{\Delta t}} = \frac{\Delta T \Delta t}{Q_{tot}}$$

Substitute for R in equation (1) to obtain:

$$R_f = \frac{A\Delta T\Delta t}{Q_{tot}}$$

The total heat entering the chest in 7 h is given by:

$$Q_{tot} = Q_{\underset{\text{ice}}{\text{melt}}} + Q_{\underset{\text{ice water}}{\text{warm}}}$$
$$= m_{ice}L_f + m_{ice}c_{H_2O}\Delta T_{H_2O}$$

Substitute for Q_{tot} and simplify to obtain:

$$R_f = \frac{A\Delta T\Delta t}{m_{ice}\left(L_f + c_{H_2O}\Delta T_{H_2O}\right)}$$

Substitute numerical values and evaluate R_f:

$$R_f = \frac{\left(1.0\,\text{m}^2 \times \frac{1\,\text{ft}^2}{9.29\times10^{-2}\,\text{m}^2}\right)\left(20\,\text{C}^\circ \times \frac{9\,\text{F}^\circ}{5\,\text{C}^\circ}\right)(7\,\text{h})\left(\frac{1054.35\,\text{J}}{\text{Btu}}\right)}{(1\,\text{kg})\left(333.5\frac{\text{kJ}}{\text{kg}} + \left(4.18\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(5\,\text{C}^\circ)\right)} \approx \boxed{8\frac{\text{F}^\circ\cdot\text{h}\cdot\text{ft}^2}{\text{Btu}}}$$

Thermal Expansion

26 •• You have inherited your grandfather's grandfather clock that was calibrated when the temperature of the room was 20ºC. Assume that the pendulum consists of a thin brass rod of negligible mass with a compact heavy bob at its end. (*a*) During a hot day, the temperature is 30ºC, does the clock run fast or slow? Explain. (*b*) How much time does it gain or lose during this day?

Picture the Problem We can determine whether the clock runs fast or slow from the expression for the period of a simple pendulum and the dependence of its length on the temperature. We can use the expression for the period of a simple pendulum and the equation describing its length as a function of temperature to find the time gained or lost in a 24-h period.

(*a*) Express the period of the pendulum in terms of its length:

$$T_P = 2\pi\sqrt{\frac{L}{g}}$$

Because $T_P \propto \sqrt{L}$ and L is temperature dependent, the clock runs slow.

(*b*) The period of the pendulum when the temperature is 20°C is given by:

$$T_{20} = 2\pi\sqrt{\frac{L_{20}}{g}} \qquad (1)$$

When the temperature increases to 30°C, the period of the pendulum increases due to the increase in its length:

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{L_{20}(1+\alpha\Delta t_C)}{g}}$$
$$= T_{20}\sqrt{1+\alpha\Delta t_C}$$

The daily fractional gain or loss is given by :

$$\frac{\Delta\tau}{\tau} = \frac{\Delta T}{T_{20}} = \frac{T - T_{20}}{T_{20}}$$

Substituting for T and simplifying yields:

$$\frac{\Delta\tau}{\tau} = \frac{T_{20}\sqrt{1+\alpha\Delta t_C} - T_{20}}{T_{20}}$$
$$= \sqrt{1+\alpha\Delta t_C} - 1$$

Solve for $\Delta\tau$ to obtain:

$$\Delta\tau = \left(\sqrt{1+\alpha\Delta t_C} - 1\right)\tau$$

Substitute numerical values and $\Delta\tau$:

$$\Delta\tau = \left(\sqrt{1+\left(19\times10^{-6}\,\text{K}^{-1}\right)\left(30°\text{C} - 20°\text{C}\right)} - 1\right)\left(24\,\text{h}\times\frac{3600\,\text{s}}{\text{h}}\right) = \boxed{8.2\,\text{s}}$$

27 •• [SSM] You need to fit a copper collar tightly around a steel shaft that has a diameter of 6.0000 cm at 20ºC. The inside diameter of the collar at that temperature is 5.9800 cm. What temperature must the copper collar have so that it will just slip on the shaft, assuming the shaft itself remains at 20ºC?

Picture the Problem Because the temperature of the steel shaft does not change, we need consider just the expansion of the copper collar. We can express the required temperature in terms of the initial temperature and the change in temperature that will produce the necessary increase in the diameter D of the copper collar. This increase in the diameter is related to the diameter at 20°C and the increase in temperature through the definition of the coefficient of linear expansion.

Express the temperature to which the

$$T = T_i + \Delta T$$

copper collar must be raised in terms of its initial temperature and the increase in its temperature:

Apply the definition of the coefficient of linear expansion to express the change in temperature required for the collar to fit on the shaft:

$$\Delta T = \frac{\left(\dfrac{\Delta D}{D}\right)}{\alpha} = \frac{\Delta D}{\alpha D}$$

Substitute for ΔT to obtain:

$$T = T_{\text{i}} + \frac{\Delta D}{\alpha D}$$

Substitute numerical values and evaluate T:

$$T = 20°\text{C} + \frac{6.0000\,\text{cm} - 5.9800\,\text{cm}}{\left(17\times10^{-6}\,\text{K}^{-1}\right)\left(5.9800\,\text{cm}\right)}$$
$$= \boxed{220°\text{C}}$$

28 •• You have a copper collar and a steel shaft. At 20°C, the collar has an inside diameter of 5.9800 cm and the steel shaft has diameter of 6.0000 cm. The copper collar was heated. When its inside diameter exceeded 6.0000 cm is was slipped on the shaft. The collar fitted tightly on the shaft after they cooled to room temperature. Now, several years later, you need to remove the collar from the shaft. To do this you heat them both until you can just slip the collar off the shaft. What temperature must the collar have so that the collar will just slip off the shaft?

Picture the Problem Because the temperatures of both the steel shaft and the copper collar change together, we can find the temperature change required for the collar to fit the shaft by equating their diameters for a temperature increase ΔT. These diameters are related to their diameters at 20°C and the increase in temperature through the definition of the coefficient of linear expansion.

Express the temperature to which the collar and the shaft must be raised in terms of their initial temperature and the increase in their temperature:

$$T = T_{\text{i}} + \Delta T \qquad (1)$$

Express the diameter of the steel shaft when its temperature has been increased by ΔT:

$$D_{\text{steel}} = D_{\text{steel,20°C}}\left(1 + \alpha_{\text{steel}}\Delta T\right)$$

Express the diameter of the copper collar when its temperature has been

$$D_{\text{Cu}} = D_{\text{Cu,20°C}}\left(1 + \alpha_{\text{Cu}}\Delta T\right)$$

increased by ΔT:

If the collar is to fit over the shaft when the temperature of both has been increased by ΔT:

$$D_{\text{Cu,20°C}}(1+\alpha_{\text{Cu}}\Delta T) = D_{\text{steel,20°C}}(1+\alpha_{\text{steel}}\Delta T)$$

Solving for ΔT yields:

$$\Delta T = \frac{D_{\text{steel,20°C}} - D_{\text{Cu,20°C}}}{D_{\text{Cu,20°C}}\alpha_{\text{Cu}} - D_{\text{steel,20°C}}\alpha_{\text{steel}}}$$

Substitute in equation (1) to obtain:

$$T = T_{\text{i}} + \frac{D_{\text{steel,20°C}} - D_{\text{Cu,20°C}}}{D_{\text{Cu,20°C}}\alpha_{\text{Cu}} - D_{\text{steel,20°C}}\alpha_{\text{steel}}}$$

Substitute numerical values and evaluate T:

$$T = 20°\text{C} + \frac{6.0000\,\text{cm} - 5.9800\,\text{cm}}{(5.9800\,\text{cm})(17\times10^{-6}\,/\text{K}) - (6.0000\,\text{cm})(11\times10^{-6}\,/\text{K})} = \boxed{580°\text{C}}$$

29 •• A container is filled to the brim with 1.4 L of mercury at 20ºC. As the temperature of container and mercury is increased to 60ºC, a total of 7.5 mL of mercury spill over the brim of the container. Determine the linear expansion coefficient of the material that makes up the container.

Picture the Problem The linear expansion coefficient of the container is one-third its coefficient of volume expansion. We can relate the changes in volume of the mercury and the container to their initial volumes, temperature change, and coefficients of volume expansion, and, because we know the amount of spillage, obtain an equation that we can solve for β_{c}.

Relate the linear expansion coefficient of the container to its coefficient of volume expansion:

$$\alpha_{\text{c}} = \tfrac{1}{3}\beta_{\text{c}} \qquad (1)$$

Express the difference in the change in the volume of the mercury and the container in terms of the spillage:

$$\Delta V_{\text{Hg}} - \Delta V_{\text{c}} = 7.5\,\text{mL} \qquad (2)$$

Express ΔV_{Hg} using the definition of the coefficient of volume expansion:

$$\Delta V_{\text{Hg}} = \beta_{\text{Hg}} V_{\text{Hg}} \Delta T$$

Express ΔV_{c} using the definition of the coefficient of volume expansion:

$$\Delta V_{\text{c}} = \beta_{\text{c}} V_{\text{c}} \Delta T$$

Substitute for ΔV_{Hg} and ΔV_c in equation (2) to obtain:

$$\beta_{Hg}V_{Hg}\Delta T - \beta_c V_c \Delta T = 7.5\,\text{mL}$$

Solving for β_c yields:

$$\beta_c = \frac{\beta_{Hg}V_{Hg}\Delta T - 7.5\,\text{mL}}{V_c\Delta T}$$

or, because $V = V_{Hg} = V_c$,

$$\beta_c = \frac{\beta_{Hg}V\Delta T - 7.5\,\text{mL}}{V\Delta T} = \beta_{Hg} - \frac{7.5\,\text{mL}}{V\Delta T}$$

Substitute for β_c in equation (1) to obtain:

$$\alpha_c = \tfrac{1}{3}\beta_{Hg} - \frac{7.5\,\text{mL}}{3V\Delta T} = \alpha_{Hg} - \frac{7.5\,\text{mL}}{3V\Delta T}$$

Substitute numerical values and evaluate α_c:

$$\alpha_c = \tfrac{1}{3}\left(0.18\times10^{-3}\,\text{K}^{-1}\right) - \frac{7.5\,\text{mL}}{3(1.4\,\text{L})(40\,\text{K})} = \boxed{15\times10^{-6}\,\text{K}^{-1}}$$

30 •• A car has a 60.0-L steel gas tank filled to the brim with 60.0-L of gasoline when the temperature of the outside is 10ºC. The coefficient of volume expansion for gasoline at 20°C is 0.950 × 10^{-3} K⁻1. How much gasoline spills out of the tank when the outside temperature increases to 25ºC? Take the expansion of the steel tank into account.

Picture the Problem The amount of gas that spills is the difference between the change in the volume of the gasoline and the change in volume of the tank. We can find this difference by expressing the changes in volume of the gasoline and the tank in terms of their common volume at 10°C, their coefficients of volume expansion, and the change in the temperature.

Express the spill in terms of the change in volume of the gasoline and the change in volume of the tank:

$$V_{spill} = \Delta V_{gasoline} - \Delta V_{tank}$$

Relate $\Delta V_{gasoline}$ to the coefficient of volume expansion for gasoline:

$$\Delta V_{gas} = \beta_{gasoline}V\Delta T$$

Relate ΔV_{tank} to the coefficient of linear expansion for steel:

$$\Delta V_{\text{tank}} = \beta_{\text{tank}} V \Delta T$$

or, because $\beta_{\text{steel}} = 3\alpha_{\text{steel}}$,

$$\Delta V_{\text{tank}} = 3\alpha_{\text{steel}} V \Delta T$$

Substitute for $\Delta V_{\text{gasoline}}$ and ΔV_{tank} and simplify to obtain:

$$\begin{aligned} V_{\text{spill}} &= \beta_{\text{gas}} V \Delta T - 3\alpha_{\text{steel}} V \Delta T \\ &= V \Delta T \left(\beta_{\text{gasoline}} - 3\alpha_{\text{steel}}\right) \end{aligned}$$

Substitute numerical values and evaluate V_{spill}:

$$V_{\text{spill}} = (60.0\,\text{L})(25°\text{C} - 10°\text{C})\left[0.950\times10^{-3}\,\text{K}^{-1} - 3\left(11\times10^{-6}\,\text{K}^{-1}\right)\right] \approx \boxed{0.8\,\text{L}}$$

31 ••• What is the tensile stress in the copper collar of Problem 27 when its temperature returns to 20ºC?

Picture the Problem We can use the definition of Young's modulus to express the tensile stress in the copper in terms of the strain it undergoes as its temperature returns to 20°C. We can show that $\Delta L/L$ for the circumference of the collar is the same as $\Delta d/d$ for its diameter.

Using Young's modulus, relate the stress in the collar to its strain:

$$\text{Stress} = Y \times \text{Strain} = Y \frac{\Delta L}{L_{20°\text{C}}}$$

where $L_{20°\text{C}}$ is the circumference of the collar at 20°C.

Express the circumference of the collar at the temperature at which it fits over the shaft:

$$L_T = \pi d_T$$

Express the circumference of the collar at 20 °C:

$$L_{20°\text{C}} = \pi d_{20°\text{C}}$$

Substitute for L_T and $L_{20°\text{C}}$ and simplify to obtain:

$$\begin{aligned} \text{Stress} &= Y \frac{\pi d_T - \pi d_{20°\text{C}}}{\pi d_{20°\text{C}}} \\ &= Y \frac{d_T - d_{20°\text{C}}}{d_{20°\text{C}}} \end{aligned}$$

Substitute numerical values and evaluate the stress:

$$\text{Stress} = \left(11\times10^{-10}\ \text{N/m}^2\right)\frac{6.0000\,\text{cm} - 5.9800\,\text{cm}}{5.9800\,\text{cm}} = \boxed{3.7\times10^{-12}\ \text{N/m}^2}$$

The van der Waals Equation, Liquid-Vapor Isotherms, and Phase Diagrams

32 • (*a*) Calculate the volume of 1.00 mol of an ideal gas at a temperature of 100ºC and a pressure of 1.00 atm. (*b*) Calculate the temperature at which 1.00 mol of steam at a pressure of 1.00 atm has the volume calculated in Part (*a*). Use $a = 0.550\ \text{Pa}\cdot\text{m}^6/\text{mol}^2$ and $b = 30.0\ \text{cm}^3/\text{mol}$.

Picture the Problem We can apply the ideal-gas law to find the volume of 1.00 mol of steam at 100°C and a pressure of 1.00 atm and then use the van der Waals equation to find the temperature at which the steam will this volume.

(*a*) Solving the ideal-gas law for the volume gives:

$$V = \frac{nRT}{P}$$

Substitute numerical values and evaluate V:

$$V = \frac{(1.00\text{mol})(8.314\,\text{J/mol}\cdot\text{K})(373\,\text{K})}{1.00\,\text{atm}\times\dfrac{101.325\,\text{kPa}}{\text{atm}}} = 3.06\times10^{-2}\ \text{m}^3\times\frac{1\,\text{L}}{10^{-3}\ \text{m}^3} = \boxed{30.6\,\text{L}}$$

(*b*) Solve van der Waals equation for T to obtain:

$$T = \frac{\left(P + \dfrac{an^2}{V^2}\right)(V - bn)}{nR}$$

Substitute numerical values and evaluate T:

$$T = \left(101.325\,\text{kPa} + \frac{\left(0.550\,\text{Pa}\cdot\text{m}^6/\text{mol}^2\right)(1.00\,\text{mol})^2}{\left(3.06\times10^{-2}\ \text{m}^3\right)^2}\right) \times\frac{3.06\times10^{-2}\ \text{m}^3 - \left(30.0\times10^{-6}\ \text{m}^3/\text{mol}\right)(1.00\,\text{mol})}{(1.00\,\text{mol})\,(8.314\,\text{J/mol}\cdot\text{K})} = \boxed{375\,\text{K}}$$

33 •• **[SSM]** Using Figure 20-16, find the following quantities. (*a*) The temperature at which water boils on a mountain where the atmospheric pressure is 70.0 kPa, (*b*) the temperature at which water boils in a container where the pressure inside the container is 0.500 atm, and (*c*) the pressure at which water boils at 115ºC.

Picture the Problem Consulting Figure 20-16, we see that:

(*a*) At 70.0 kPa, water boils at approximately $\boxed{90°\text{C}}$.

(*b*) At 0.500 atm (about 51 kPa), water boils at approximately $\boxed{78°\text{C}}$.

(*c*) The pressure at which water boils at 115°C is approximately $\boxed{127\,\text{kPa}}$.

34 •• The van der Waals constants for helium are $a = 0.03412\ \text{L}^2\cdot\text{atm/mol}^2$ and $b = 0.0237$ L/mol. Use these data to find the volume in cubic centimeters occupied by one helium atom. Then, estimate the radius of the helium atom.

Picture the Problem Assume that a helium atom is spherical. Then we can find its volume from the van der Waals equation and its radius from $V = \frac{4}{3}\pi r^3$.

In the van der Waals equation, b is the volume of 1 mol of molecules. For He, 1 molecule = 1 atom. Use Avogadro's number to express b in cm^3/atom:

$$b = \frac{\left(0.0237\,\frac{\text{L}}{\text{mol}}\right)\left(10^3\,\frac{\text{cm}^3}{\text{L}}\right)}{6.022\times10^{23}\,\frac{\text{atoms}}{\text{mol}}} = \boxed{3.94\times10^{-23}\,\frac{\text{cm}^3}{\text{atom}}}$$

The volume of a spherical helium atom is given by:

$$V = \tfrac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3b}{4\pi}}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\frac{3\left(3.94\times10^{-23}\ \text{cm}^3\right)}{4\pi}} = \boxed{0.211\text{nm}}$$

Conduction

35 • **[SSM]** A 20-ft × 30-ft slab of insulation has an R factor of 11. At what rate is heat conducted through the slab if the temperature on one side is a constant 68ºF and the temperature of the other side is a constant 30ºF?

Picture the Problem We can use its definition to express the thermal current in the slab in terms of the temperature differential across it and its thermal resistance and use the definition of the R factor to express I as a function of ΔT, the cross-sectional area of the slab, and R_f.

Express the thermal current through the slab in terms of the temperature difference across it and its thermal resistance:

$$I = \frac{\Delta T}{R}$$

Substitute to express R in terms of the insulation's R factor:

$$I = \frac{\Delta T}{R_f / A} = \frac{A\Delta T}{R_f}$$

Substitute numerical values and evaluate I:

$$I = \frac{(20\,\text{ft})(30\,\text{ft})(68°\text{F} - 30°\text{F})}{11\dfrac{\text{h}\cdot\text{ft}^2\cdot\text{F}°}{\text{Btu}}} = \boxed{2.1\frac{\text{kBtu}}{\text{h}}}$$

36 •• A copper cube and an aluminum, cube each with 3.00-cm-long edges, are arranged as shown in Figure 20-17. Find (*a*) the thermal resistance of each cube, (*b*) the thermal resistance of the two-cube combination, (*c*) the thermal current I, and (*d*) the temperature at the interface of the two cubes.

Picture the Problem We can use $R = \Delta x/kA$ to find the thermal resistance of each cube and the fact that they are in series to find the thermal resistance of the two-cube system. We can use $I = \Delta T/R$ to find the thermal current through the cubes and the temperature at their interface. See Table 20-4 for the thermal conductivities of copper and aluminum.

(*a*) Using its definition, express the thermal resistance of each cube:

$$R = \frac{\Delta x}{kA}$$

Substitute numerical values and evaluate the thermal resistance of the copper cube:

$$R_{Cu} = \frac{3.00\,\text{cm}}{\left(401\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)(3.00\,\text{cm})^2} = 0.08313\,\text{K/W} = \boxed{0.0831\,\text{K/W}}$$

Substitute numerical values and evaluate the thermal resistance of the aluminum cube:

$$R_{\text{Al}} = \frac{3.00\,\text{cm}}{\left(237\,\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)(3.00\,\text{cm})^2} = 0.1406\,\text{K/W} = \boxed{0.141\,\text{K/W}}$$

(*b*) Because the cubes are in series, their thermal resistances are additive:

$$R = R_{\text{Cu}} + R_{\text{Al}} = 0.08313\,\text{K/W} + 0.1406\,\text{K/W} = 0.2237\,\text{K/W} = \boxed{0.224\,\text{K/W}}$$

(*c*) Using its definition, find the thermal current:

$$I = \frac{\Delta T}{R} = \frac{100°\text{C} - 20°\text{C}}{0.2237\,\text{K/W}} = 357.6\text{ W} = \boxed{0.36\,\text{kW}}$$

(*d*) Express the temperature at the interface between the two cubes:

$$T_{\text{interface}} = 100°\text{C} - \Delta T_{\text{Cu}}$$

Express the temperature differential across the copper cube:

$$\Delta T_{\text{Cu}} = I_{\text{Cu}} R_{\text{Cu}} = I R_{\text{Cu}}$$

Substitute for ΔT_{Cu} to obtain:

$$T_{\text{interface}} = 100°\text{C} - I R_{\text{Cu}}$$

Substitute numerical values and evaluate $T_{\text{interface}}$:

$$T_{\text{interface}} = 100°\text{C} - (357.6\,\text{W})(0.08313\,\text{K/W}) \approx \boxed{70°\text{C}}$$

37 •• Two metal cubes, one copper and one aluminum, with 3.00-cm-long edges, are arranged in parallel, as shown in Figure 20-18. Find (*a*) the thermal current in each cube, (*b*) the total thermal current, and (*c*) the thermal resistance of the two-cube combination.

Picture the Problem We can use Equation 20-9 to find the thermal current in each cube. Because the currents are additive, we can find the equivalent resistance of the two-cube system from $R_{\text{eq}} = \Delta T / I_{\text{total}}$.

(*a*) The thermal current in each cube is given by Equation 20-9:

$$I = \frac{\Delta T}{R} = kA\frac{\Delta T}{\Delta x}$$

Substitute numerical values and evaluate the thermal current in the copper cube:

$$I_{\text{Cu}} = \left(401\frac{\text{W}}{\text{m}\cdot\text{K}}\right)(3.00\,\text{cm})^2\left(\frac{100°\text{C}-20°\text{C}}{3.00\,\text{cm}}\right) = 962.4\text{ W} = \boxed{0.96\,\text{kW}}$$

Substitute numerical values and evaluate the thermal current in the aluminum cube:

$$I_{\text{Al}} = \left(237\frac{\text{W}}{\text{m}\cdot\text{K}}\right)(3.00\,\text{cm})^2\left(\frac{100°\text{C}-20°\text{C}}{3.00\,\text{cm}}\right) = 568.8\text{ W} = \boxed{0.57\,\text{kW}}$$

(*b*) Because the cubes are in parallel, their total thermal currents are additive:

$$I = I_{\text{Cu}} + I_{\text{Al}} = 962.4\,\text{W} + 568.8\,\text{W}$$
$$= 1.531\,\text{kW} = \boxed{1.5\,\text{kW}}$$

(*c*) Use the relationship between the total thermal current, temperature differential and thermal resistance to find R_{eq}:

$$R_{\text{eq}} = \frac{\Delta T}{I_{\text{total}}} = \frac{100°\text{C}-20°\text{C}}{1.531\,\text{kW}}$$
$$= \boxed{0.052\,\text{K/W}}$$

38 •• The cost of air conditioning a house is approximately proportional to the rate at which heat is absorbed by the house from its surroundings divided by the coefficient of performance (COP) of the air conditioner. Let us denote the temperature difference between the inside temperature and the outside temperature as ΔT. Assuming that the rate at which heat is absorbed by a house is proportional to ΔT and that the air conditioner is operating ideally, show that the cost of air conditioning is proportional to $(\Delta T)^2$ divided by the temperature inside the house.

Picture the Problem The cost of operating the air conditioner is proportional to the energy used in its operation. We can use the definition of the COP to relate the rate at which the air conditioner removes heat from the house to rate at which it must do work to maintain a constant temperature differential between the interior and the exterior of the house. To obtain an expression for the minimum rate at which the air conditioner must do work, we'll assume that it is operating with the maximum efficiency possible. Doing so will allow us to derive an expression for the rate at which energy is used by the air conditioner that we can integrate to obtain the energy (and hence the cost of operation) required.

Relate the cost C of air conditioning the energy W required to operate the air conditioner:

$$C = uW \qquad (1)$$

where u is the unit cost of the energy.

Express the rate dQ/dt at which heat flows into a house provided the house is maintained at a constant temperature:

$$P = \frac{dQ}{dt} = k\Delta T$$

where ΔT is the temperature difference between the interior and exterior of the house.

Use the definition of the COP to relate the rate at which the air conditioner must remove heat dW/dt to maintain a constant temperature:

$$\text{COP} = \frac{dQ/dt}{dW/dt} \Rightarrow \frac{dW}{dt} = \frac{1}{\text{COP}}\frac{dQ}{dt}$$

Express the maximum value of the COP:

$$\text{COP}_{\text{max}} = \frac{T_c}{\Delta T}$$

where $T_c = T_{\substack{\text{inside the}\\ \text{house}}}$ is the temperature of the cold reservoir.

Letting COP = COP_{max}, substitute to obtain an expression for the minimum rate at which the air conditioner must do work in order to maintain a constant temperature:

$$\frac{dW}{dt} = \frac{\frac{dQ}{dt}}{T_c}\Delta T$$

Substituting for dQ/dt gives:

$$\frac{dW}{dt} = \frac{k\Delta T}{T_c}\Delta T = \frac{k}{T_c}(\Delta T)^2$$

Separate variables and integrate this equation to obtain:

$$W = \frac{k}{T_c}(\Delta T)^2\int_0^{\Delta t} dt' = \frac{k}{T_c}(\Delta T)^2\Delta t$$

Substitute in equation (1) to obtain:

$$C = u\frac{k}{T_c}(\Delta T)^2\Delta t \Rightarrow C \propto \boxed{\frac{(\Delta T)^2}{T_c}}$$

39 •• A spherical shell of thermal conductivity k has inside radius r_1 and outside radius r_2 (Figure 20-19). The inside of the shell is held at a temperature T_1, and the outside of the shell is held at temperature T_2, with $T_1 < T_2$. In this problem, you are to show that the thermal current through the shell is given by

$$I = -\frac{4\pi k r_1 r_2}{r_2 - r_1}(T_2 - T_1)$$

where I is positive if heat is transferred in the $+r$ direction. Here is a suggested procedure for obtaining this result: (1) obtain an expression for the thermal current I through a thin spherical shell of radius r and thickness dr when there is a temperature difference dT across the thickness of the shell; (2) explain why the thermal current is the same through each such thin shell; (3) express the thermal current I through such a shell element in terms of the area $A = 4\pi r^2$, the thickness

dr, and the temperature difference *dT* across the element; and (4) separate variables (solve for *dT* in terms of *r* and *dr*) and integrate.

Picture the Problem We can follow the step-by-step instructions given in the problem statement to obtain the differential equation describing the variation of *T* with *r*. Integrating this equation will yield an equation that we can solve for the current *I*.

(1) The thermal current through a thin spherical shell surface of area *A* and thickness *dr* due to a temperature gradient $\frac{dT}{dr}$ is given by:

$$I = kA\frac{dT}{dr}$$

(2) Conservation of energy requires that the thermal current through each shell be the same.

(3) For a shell of area $A = 4\pi r^2$:

$$I = 4\pi kr^2 \frac{dT}{dr}$$

(4) Separating the variables yields:

$$dT = \frac{I}{4\pi k}\frac{dr}{r^2}$$

Integrate from $r = r_1$ to $r = r_2$ and simplify to obtain:

$$\int_{T_1}^{T_2} dT = \frac{I}{4\pi k}\int_{r_1}^{r_2}\frac{dr}{r^2}$$

and

$$T_2 - T_1 = -\frac{I}{4\pi k}\left[\frac{1}{r}\right]_{r_1}^{r_2} = -\frac{I}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Solving for *I* gives:

$$I = \boxed{-\frac{4\pi kr_1r_2}{r_2 - r_1}\left(T_2 - T_1\right)}$$

Radiation

40 • Calculate λ_{max} (the wavelength at which the emitted power is maximum) for a human skin. Assume the human skin is a blackbody emitter with a temperature of 33°C.

Picture the Problem We can apply Wein's displacement law to find the wavelength at which the power is a maximum.

Wein's law relates the maximum wavelength of the radiation to the temperature of its source:

$$\lambda_{max} = \frac{2.898\,\text{mm}\cdot\text{K}}{T}$$

Substitute numerical values and evaluate λ_{max}:

$$\lambda_{max} = \frac{2.898\,\text{mm}\cdot\text{K}}{273\,\text{K} + 33\,\text{K}} = \boxed{9.47\,\mu\text{m}}$$

41 • [SSM] The universe is filled with radiation that is believed to be remaining from the Big Bang. If the entire universe is considered to be a blackbody with a temperature equal to 2.3 K, what is the λ_{max} (the wavelength at which the power of the radiation is maximum) of this radiation?

Picture the Problem We can use Wein's law to find the peak wavelength of this radiation.

Wein's law relates the maximum wavelength of the background radiation to the temperature of the universe:

$$\lambda_{max} = \frac{2.898\,\text{mm}\cdot\text{K}}{T}$$

Substituting for T gives:

$$\lambda_{max} = \frac{2.898\,\text{mm}\cdot\text{K}}{2.3\,\text{K}} = \boxed{1.3\,\text{mm}}$$

42 • What is the range of temperatures for star surfaces for which λ_{max} (the wavelength at which the power of the emitted radiation is maximum) is in the visible range?

Picture the Problem The visible portion of the electromagnetic spectrum extends from approximately 400 nm to 700 nm. We can use Wein's law to find the range of temperatures corresponding to these wavelengths.

Wein's law relates the maximum wavelength of the radiation to the temperature of its source:

$$\lambda_{max} = \frac{2.898\,\text{mm}\cdot\text{K}}{T}$$

Solving for T yields:

$$T = \frac{2.898\,\text{mm}\cdot\text{K}}{\lambda_{max}}$$

For λ_{max} = 400 nm:

$$T = \frac{2.898\,\text{mm}\cdot\text{K}}{400\,\text{nm}} = 7250\,\text{K}$$

For λ_{max} = 700 nm:

$$T = \frac{2.898\,\text{mm}\cdot\text{K}}{700\,\text{nm}} = 4140\,\text{K}$$

The range of temperatures is $\boxed{4140\,\text{K} \le \boldsymbol{T} \le 7250\,\text{K}}$.

43 • The heating wires of a 1.00-kW electric heater are red hot at a temperature of 900ºC. Assuming that 100 percent of the heat released is due to radiation and that the wires act as blackbody emitters, what is the effective area of the radiating surface? (Assume a room temperature of 20ºC.)

Picture the Problem We can apply the Stefan-Boltzmann law to find the net power radiated by the wires of its heater to the room.

Relate the net power radiated to the surface area of the heating wires, their temperature, and the room temperature:

$$P_{\text{net}} = e\sigma A\left(T^4 - T_0^4\right) \Rightarrow A = \frac{P_{\text{net}}}{e\sigma\left(T^4 - T_0^4\right)}$$

Substitute numerical values and evaluate A:

$$A = \frac{1.00\,\text{kW}}{(1)\left(5.6703\times10^{-8}\,\dfrac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right)\left[(1173\,\text{K})^4 - (293\,\text{K})^4\right]} = \boxed{93.5\,\text{cm}^2}$$

44 •• A blackened, solid copper sphere that has a radius equal to 4.0 cm hangs in an evacuated enclosure whose walls have a temperature of 20ºC. If the sphere is initially at 0ºC, find the initial rate at which its temperature changes, assuming that heat is transferred by radiation only. (Assume the sphere is a blackbody emitter.)

Picture the Problem The rate at which the copper sphere absorbs radiant energy is given by $dQ/dt = mc\,dT/dt$ and, from the Stephan-Boltzmann law, $P_{\text{net}} = e\sigma A\left(T^4 - T_0^4\right)$ where A is the surface area of the sphere, T_0 is its temperature, and T is the temperature of the walls. We can solve the first equation for dT/dt and substitute P_{net} for dQ/dt in order to find the initial rate at which the temperature of the sphere changes.

Relate the rate at which the sphere absorbs radiant energy to the rate at which its temperature changes:

$$P_{\text{net}} = \frac{dQ}{dt} = mc\frac{dT}{dt}$$

Solving for $\dfrac{dT}{dt}$ and substituting for mc gives:

$$\frac{dT}{dt} = \frac{P_{\text{net}}}{mc} = \frac{P_{\text{net}}}{\rho Vc} = \frac{P_{\text{net}}}{\frac{4}{3}\pi r^3\rho c}$$

Apply the Stefan-Boltzmann law to relate the net power radiated to the sphere to the difference in temperature of the walls and the blackened copper sphere:

$$P_{net} = e\sigma A\left(T^4 - T_0^4\right) = 4\pi r^2 e\sigma\left(T^4 - T_0^4\right)$$

Substitute for P_{net} to obtain:

$$\frac{dT}{dt} = \frac{4\pi r^2 e\sigma\left(T^4 - T_0^4\right)}{\frac{4}{3}\pi r^3 \rho c} = \frac{3e\sigma\left(T^4 - T_0^4\right)}{r\rho c}$$

Substitute numerical values and evaluate dT/dt:

$$\frac{dT}{dt} = -\frac{3(1)\left(5.6703\times10^{-8}\,\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right)\left[(293\,\text{K})^4 - (273\,\text{K})^4\right]}{\left(4.0\times10^{-2}\,\text{m}\right)\left(8.93\times10^3\,\frac{\text{kg}}{\text{m}^3}\right)\left(0.386\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)} = \boxed{2.2\times10^{-3}\,\text{K/s}}$$

45 •• The surface temperature of the filament of an incandescent lamp is 1300ºC. If the electric power input is doubled, what will the new temperature be? *Hint: Show that you can neglect the temperature of the surroundings.*

Picture the Problem We can apply the Stephan-Boltzmann law to express the net power radiated by the incandescent lamp to its surroundings.

Express the rate at which energy is radiated to the surroundings:

$$P_{net} = e\sigma A\left(T^4 - T_0^4\right) = e\sigma A T^4\left(1 - \left(\frac{T_0}{T}\right)^4\right)$$

Evaluate $\left(\frac{T_0}{T}\right)^4$:

$$\left(\frac{T_0}{T}\right)^4 = \left(\frac{293\,\text{K}}{1573\,\text{K}}\right)^4 \approx 1\times10^{-3}$$

Because $\left(\frac{T_0}{T}\right)^4$ is so small, we can neglect the temperature of the surroundings. Hence:

$$P_{net} \approx e\sigma A T^4 \Rightarrow T = \left(\frac{P_{net}}{e\sigma A}\right)^{1/4}$$

Express the temperature T' when the electric power input is doubled:

$$T' = \left(\frac{2P_{net}}{e\sigma A}\right)^{1/4}$$

Dividing the second of these equations by the first and solving for T gives:

$$\frac{T'}{T} = (2)^{1/4} \Rightarrow T' = (2)^{1/4} T$$

Substitute numerical values and evaluate T'

$$T' = (2)^{1/4}(1573\,\text{K}) = 1871\,\text{K}$$
$$= \boxed{1598°\text{C}}$$

46 •• Liquid helium is stored at its boiling point (4.2 K) in a spherical can that is separated by an evacuated region of space from a surrounding shield that is maintained at the temperature of liquid nitrogen (77 K). If the can is 30 cm in diameter and is blackened on the outside so that it acts as a blackbody emitter, how much helium boils away per hour?

Picture the Problem We can differentiate $Q = mL$, where L is the latent heat of boiling for helium, with respect to time to obtain an expression for the rate at which the helium boils away.

Express the rate at which the helium boils away in terms of the rate at which its container absorbs radiant energy:

$$\frac{dm}{dt} = \frac{P_{\text{net}}}{L} = \frac{e\sigma A\left(T^4 - T_0^4\right)}{L}$$
$$= \frac{e\sigma\pi d^2\left(T^4 - T_0^4\right)}{L}$$
$$= \frac{e\sigma\pi d^2}{L} T^4\left(1 - \left(\frac{T_0}{T}\right)^4\right)$$

If $T_0 << T$, then:

$$\frac{dm}{dt} \approx \frac{e\sigma\pi d^2}{L} T^4$$

Substitute numerical values and evaluate dm/dt:

$$\frac{dm}{dt} \approx \frac{(1)\left(5.6703\times10^{-8}\,\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right)\pi(0.30\,\text{m})^2}{21\,\frac{\text{kJ}}{\text{kg}}}(77\,\text{K})^4$$
$$= 2.68\times10^{-5}\,\frac{\text{kg}}{\text{s}} \times \frac{3600\,\text{s}}{\text{h}} = \boxed{97\,\text{g/h}}$$

General Problems

47 • A steel tape is placed around Earth at the equator when the temperature is 0ºC. What will the clearance between the tape and the ground (assumed to be uniform) be if the temperature of the tape increases to 30ºC? Neglect the expansion of Earth.

Picture the Problem The distance by which the tape clears the ground equals the change in the radius of the circle formed by the tape placed around Earth at the equator.

Express the change in the radius of the circle defined by the steel tape:

$\Delta R = R\alpha\Delta T$
where R is the radius of Earth, α is the coefficient of linear expansion of steel, and ΔT is the increase in temperature.

Substitute numerical values and evaluate ΔR:

$$\Delta R = \left(6.37\times10^{6}\ \text{m}\right)\left(11\times10^{-6}\ \text{K}^{-1}\right)\left(30°\text{C}-0°\text{C}\right) = 2.10\times10^{3}\ \text{m} = \boxed{2.1\text{km}}$$

48 •• Show that change in the density ρ of an isotropic material due to an increase in temperature ΔT is given by $\Delta\rho = -\beta\rho\Delta T$.

Picture the Problem We can differentiate the definition of the density of an isotropic material with respect to T and use the definition of the coefficient of volume expansion to express the rate at which the density of the material changes with respect to temperature. Once we have an expression for $d\rho$ in terms of dT, we can apply a differential approximation to obtain $\Delta\rho$ in terms of ΔT.

Using its definition, relate the density of the material to its mass and volume:

$$\rho = \frac{m}{V}$$

Using its definition, relate the volume of the material to its coefficient of volume expansion:

$$\Delta V = \beta V\Delta T$$

Differentiate ρ with respect to T and simplify to obtain:

$$\frac{d\rho}{dT}=\frac{d\rho}{dV}\frac{dV}{dT}=-\frac{m}{V^2}\beta V$$
$$=-\frac{\rho V}{V^2}\beta V=-\rho\beta$$

or

$$d\rho=-\rho\beta dT$$

Using the differential approximations $d\rho \approx \Delta\rho$ and $dT \approx \Delta T$ yields:

$$\Delta\rho=\boxed{-\rho\beta\Delta T}$$

49 •• [SSM] The solar constant is the power received from the Sun per unit area perpendicular to the Sun's rays at the mean distance of Earth from the Sun. Its value at the upper atmosphere of Earth is about 1.37 kW/m^2. Calculate the effective temperature of the Sun if it radiates like a blackbody. (The radius of the Sun is 6.96×10^8 m.).

Picture the Problem We can apply the Stefan-Boltzmann law to express the effective temperature of the Sun in terms of the total power it radiates. We can, in turn, use the intensity of the Sun's radiation in the upper atmosphere of Earth to approximate the total power it radiates.

Apply the Stefan-Boltzmann law to relate the energy radiated by the Sun to its temperature:

$$P_{\rm r}=e\sigma AT^4 \Rightarrow T=\sqrt[4]{\frac{P_{\rm r}}{e\sigma A}}$$

Express the surface area of the sun:

$$A=4\pi R_{\rm S}^2$$

Relate the intensity of the Sun's radiation in the upper atmosphere to the total power radiated by the sun:

$$I=\frac{P_r}{4\pi R^2}\Rightarrow P_r=4\pi R^2 I$$

where R is the Earth-Sun distance.

Substitute for $P_{\rm r}$ and A in the expression for T and simplify to obtain:

$$T=\sqrt[4]{\frac{4\pi R^2 I}{e\sigma 4\pi R_{\rm S}^2}}=\sqrt[4]{\frac{R^2 I}{e\sigma R_{\rm S}^2}}$$

Substitute numerical values and evaluate *T*:

$$T=\sqrt[4]{\frac{\left(1.5\times10^{11}\text{ m}\right)^2\left(1.35\frac{\text{kW}}{\text{m}^2}\right)}{(1)\left(5.67\times10^{-8}\frac{\text{W}}{\text{m}^2\cdot\text{K}}\right)\left(6.96\times10^{8}\text{ m}\right)^2}}=\boxed{5800\text{ K}}$$

50 •• As part of your summer job as an engineering intern at an insulation manufacturer, you are asked to determine the *R* factor of insulating material. This particular material comes in $\frac{1}{2}$-in sheets. Using this material, you construct a hollow cube that has 12-in long edges. You place a thermometer and a 100-W heater inside the box. After thermal equilibrium has been attained, the temperature inside the box is 90ºC when the temperature outside the box is 20ºC. Determine the *R* factor of the material.

Picture the Problem We can solve the thermal-current equation for the *R* factor of the material.

Use the equation for the thermal current to express *I* in terms of the temperature gradient across the insulation:

$$I=kA\frac{\Delta T}{\Delta x}$$

Rewrite this expression in terms of the *R* factor of the material:

$$I=\frac{\Delta T}{\frac{\Delta x}{kA}}=\frac{\Delta T}{\frac{R_f}{A}}=\frac{A\Delta T}{R_f}$$

Solving for the *R* factor gives:

$$R_f=\frac{A\Delta T}{I}=\frac{6A_{\text{one side}}\Delta T}{I}$$

Substitute numerical values and evaluate *R*:

$$R_f=\frac{6\left(12\text{ in}\times\frac{2.54\times10^{-2}\text{ m}}{\text{in}}\right)^2}{100\text{ W}}\left(90°\text{C}-20°\text{C}\right)=\boxed{0.39\frac{\text{K}\cdot\text{m}^2}{\text{W}}}$$

$$=0.390\frac{\text{K}\cdot\text{m}^2}{\frac{\text{J}}{\text{s}}}\times\frac{9\text{ F°}}{5\text{ K}}\times\frac{10.76\text{ ft}^2}{\text{m}^2}\times\frac{1054\text{ J}}{\text{Btu}}\times\frac{1\text{ h}}{3600\text{ s}}=\boxed{2.2\frac{\text{F°}\cdot\text{ft}^2\cdot\text{h}}{\text{Btu}}}$$

51 • (*a*) From the definition of β, the coefficient of volume expansion (at constant pressure), show that $\beta = 1/T$ for an ideal gas. (*b*) The experimentally determined value of β for N_2 gas at 0ºC is 0.003673 K^{-1}. By what percent does this measured value of β differ from the value obtained by modeling N_2 as an ideal gas?

Picture the Problem We can use the definition of the coefficient of volume expansion with the ideal-gas law to show that $\beta = 1/T$.

(*a*) Use the definition of the coefficient of volume expansion to express β in terms of the rate of change of the volume with temperature:

$$\beta = \frac{1}{V}\frac{dV}{dT}$$

For an ideal gas:

$$V = \frac{nRT}{P} \text{ and } \frac{dV}{dT} = \frac{nR}{P}$$

Substitute for $\frac{dV}{dT}$ to obtain:

$$\beta = \frac{1}{V}\frac{nR}{P} = \boxed{\frac{1}{T}}$$

(*b*) Express the percent difference between the experimental value and the theoretical value:

$$\frac{\beta_{\text{exp}} - \beta_{\text{th}}}{\beta_{\text{th}}} = \frac{0.003673\,\text{K}^{-1} - \frac{1}{273}\text{K}^{-1}}{\frac{1}{273}\text{K}^{-1}} < \boxed{0.3\%}$$

52 •• A rod of length L_A, made from material A, is placed next to a rod of length L_B, made of material B. The rods remain in thermal equilibrium with each other. (*a*) Show that even though the lengths of each rod will change with changes in the ambient temperature, the difference between the two lengths will remain constant if the lengths L_A and L_B are chosen such that $L_A/L_B = \alpha_B/\alpha_A$, where α_A and α_B are the coefficients of linear expansion, respectively. (*b*) If material B is steel, material A is brass, and L_A = 250 cm at 0ºC, what is the value of L_B?

Picture the Problem Let ΔL be the difference between L_B and L_A and express L_B and L_A in terms of the coefficients of linear expansion of materials A and B. Requiring that ΔL be constant will lead us to the condition that $L_A/L_B = \alpha_B/\alpha_A$.

(*a*) Express the condition that ΔL does not change when the temperature of the materials changes:

$$\Delta L = L_B - L_A = \text{constant} \qquad (1)$$

Using the definition of the coefficient of linear expansion, substitute for L_B and L_A:

$$\begin{aligned}\Delta L &= (L_B + \alpha_B L_B \Delta T) - (L_A + \alpha_A L_A \Delta T) \\ &= (L_B - L_A) + (\alpha_B L_B - \alpha_A L_A)\Delta T \\ &= \Delta L + (\alpha_B L_B - \alpha_A L_A)\Delta T\end{aligned}$$

or

$$(\alpha_B L_B - \alpha_A L_A)\Delta T = 0$$

The condition that ΔL remain constant when the temperature changes by ΔT is:

$$\alpha_B L_B - \alpha_A L_A = 0$$

Solving for the ratio of L_A to L_B yields:

$$\frac{L_A}{L_B} = \boxed{\frac{\alpha_B}{\alpha_A}}$$

(*b*) From (*a*) we have:

$$L_B = L_{steel} = L_A \frac{\alpha_A}{\alpha_B} = L_{brass} \frac{\alpha_{brass}}{\alpha_{steel}}$$

Substitute numerical values and evaluate L_B:

$$L_B = (250\,\text{cm}) \frac{19 \times 10^{-6}\ \text{K}^{-1}}{11 \times 10^{-6}\ \text{K}^{-1}} = \boxed{430\,\text{cm}}$$

53 •• [SSM] On the average, the temperature of Earth's crust increases 1.0ºC for every increase in depth of 30 m. The average thermal conductivity of Earth's crust material is 0.74 J/m·s·K. What is the heat loss of Earth per second due to conduction from the core? How does this heat loss compare with the average power received from the Sun (which is about 1.37 kW/m^2)?

Picture the Problem We can apply the thermal-current equation to calculate the heat loss of Earth per second due to conduction from its core. We can also use the thermal-current equation to find the power per unit area radiated from Earth and compare this quantity to the solar constant.

Express the heat loss of Earth per unit time as a function of the thermal conductivity of Earth and its temperature gradient:

$$I = \frac{dQ}{dt} = kA\frac{\Delta T}{\Delta x} \qquad (1)$$

or

$$\frac{dQ}{dt} = 4\pi R_E^2 k \frac{\Delta T}{\Delta x}$$

Substitute numerical values and evaluate dQ/dt:

$$\frac{dQ}{dt}=4\pi\left(6.37\times10^{6}\text{ m}\right)^{2}\left(0.74\frac{\text{J}}{\text{m}\cdot\text{s}\cdot\text{K}}\right)\left(\frac{1.0\,\text{C}^\circ}{30\,\text{m}}\right)=\boxed{1.3\times10^{10}\text{ kW}}$$

Rewrite equation (1) to express the thermal current per unit area:

$$\frac{I}{A}=k\frac{\Delta T}{\Delta x}$$

Substitute numerical values and evaluate I/A:

$$\frac{I}{A}=\left(0.74\,\text{J/m}\cdot\text{s}\cdot\text{K}\right)\left(\frac{1.0\,\text{C}^\circ}{30\,\text{m}}\right)$$
$$=0.0247\,\text{W/m}^2$$

Express the ratio of I/A to the solar constant:

$$\frac{I/A}{\text{solar constant}}=\frac{0.0247\,\text{W/m}^2}{1.37\,\text{kW/m}^2}$$
$$<\boxed{0.002\%}$$

54 •• A copper-bottomed saucepan containing 0.800 L of boiling water boils dry in 10.0 min. Assuming that all the heat is transferred through the flat copper bottom, which has a diameter of 15.0 cm and a thickness of 3.00 mm, calculate the temperature of the outside of the copper bottom while some water is still in the pan.

Picture the Problem We can find the temperature of the outside of the copper bottom by finding the temperature difference between the outside of the saucepan and the boiling water. This temperature difference is related to the rate at which the water is evaporating through the thermal-current equation.

Express the temperature outside the pan in terms of the temperature inside the pan:

$$T_{\text{out}}=T_{\text{in}}+\Delta T=100°\text{C}+\Delta T \quad (1)$$

Relate the thermal current through the bottom of the saucepan to its thermal conductivity, area, and the temperature gradient between its surfaces:

$$\frac{\Delta Q}{\Delta t}=kA\frac{\Delta T}{\Delta x}\Rightarrow\Delta T=\frac{1}{kA}\frac{\Delta Q}{\Delta t}\Delta x$$

Because the water is boiling:

$$\Delta Q=mL_{\text{v}}$$

Substitute for ΔQ to obtain:

$$\Delta T=\frac{mL_\text{v}\Delta x}{kA\Delta t}$$

Substituting for ΔT in equation (1) yields:

$$T_\text{out}=100°\text{C}+\frac{mL_\text{v}\Delta x}{kA\Delta t}$$

Substitute numerical values and evaluate T_out:

$$T_\text{out}=100°\text{C}+\frac{(0.800\,\text{kg})\left(2.26\,\frac{\text{MJ}}{\text{kg}}\right)(3.00\times10^{-3}\,\text{m})}{\left(401\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)\left[\frac{\pi}{4}(0.150\,\text{m})^2\right](600\,\text{s})}=\boxed{101°\text{C}}$$

55 •• A cylindrical steel hot-water tank of cylindrical shape has an inside diameter of 0.550 m and inside height of 1.20 m. The tank is enclosed with a 5.00-cm-thick insulating layer of glass wool whose thermal conductivity is 0.0350 W/(m·K). The insulation is covered by a thin sheet-metal skin. The steel tank and the sheet-metal skin have thermal conductivities that are much greater than that of the glass wool. How much electrical power must be supplied to this tank in order to maintain the water temperature at 75.0ºC when the external temperature is 1.0ºC?

Picture the Problem Thermal energy is lost from this tank through its cylindrical side and its top and bottom. The power required to maintain the temperature of the water in the tank is equal to the rate at which thermal energy is conducted through the insulation.

Express the total thermal current as the sum of the thermal currents through the top and bottom and the side of the hot-water tank:

$$I=I_\text{top and bottom}+I_\text{side} \qquad (1)$$

Express I through the top and bottom surfaces:

$$I_{\substack{\text{top and}\\\text{bottom}}}=2\left(kA\frac{\Delta T}{\Delta x}\right)=\tfrac{1}{2}\pi d^2k\frac{\Delta T}{\Delta x}$$

Substitute numerical values and evaluate $I_{\substack{\text{top and}\\\text{bottom}}}$:

$$I_{\substack{\text{top and}\\\text{bottom}}}=\tfrac{1}{2}\pi(0.550\,\text{m})^2\frac{\left(0.0350\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)(75.0°\text{C}-1.0°\text{C})}{0.0500\,\text{m}}=24.6\,\text{W}$$

Letting r represent the inside radius of the tank, express the heat current through the cylindrical side:

$$I_{\text{side}} = -kA\frac{dT}{dr} = -2\pi\,kLr\frac{dT}{dr}$$

where the minus sign is a consequence of the heat current being opposite the temperature gradient.

Separating the variables yields:

$$dT = -\frac{I_{\text{side}}}{2\pi\,kL}\frac{dr}{r}$$

Integrate from $r = r_1$ to $r = r_2$ and $T = T_1$ to $T = T_2$ and simplify to obtain:

$$\int_{T_1}^{T_2} dT = -\frac{I_{\text{side}}}{2\pi\,kL}\int_{r_1}^{r_2}\frac{dr}{r}$$

and

$$T_2 - T_1 = -\frac{I_{\text{side}}}{2\pi\,kL}\ln r\Big]_{r_1}^{r_2} = -\frac{I_{\text{side}}}{2\pi\,kL}\ln\left(\frac{r_2}{r_1}\right) = \frac{I_{\text{side}}}{2\pi\,kL}\ln\left(\frac{r_1}{r_2}\right)$$

Solving for I_{side} gives:

$$I_{\text{side}} = \frac{2\pi\,kL}{\ln\left(\dfrac{r_1}{r_2}\right)}\left(T_2 - T_1\right)$$

Substitute numerical values and evaluate I_{side}:

$$I_{\text{side}} = \frac{2\pi\left(0.0350\,\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)(1.20\,\text{m})}{\ln\left(\dfrac{0.325\,\text{m}}{0.275\,\text{m}}\right)}\left(75.0°\text{C} - 1.0°\text{C}\right) = 117\,\text{W}$$

Substitute numerical values in equation (1) and evaluate I:

$$I = 24.6\,\text{W} + 117\,\text{W} = \boxed{142\,\text{W}}$$

56 ••• The diameter d of a tapered rod of length L is given by $d = d_0(1 + ax)$, where a is a constant and x is the distance from one end. If the thermal conductivity of the material is k what is the thermal resistance of the rod?

Picture the Problem We can use $R = \Delta T/I$ and $I = -kAdT/dt$ to express dT in terms of the linearly increasing diameter of the rod. Integrating this expression will allow us to find ΔT and, hence, R.

The thermal resistance of the rod is given by:

$$R = \frac{\Delta T}{I} \qquad (1)$$

where I the thermal current in the rod.

Relate the thermal current in the rod to its thermal conductivity k, cross-sectional area A, and temperature gradient:

$$I = -kA\frac{dT}{dx}$$

where the minus sign is a consequence of the heat current being opposite the temperature gradient.

Using the dependence of the diameter of the rod on x, express the area of the rod:

$$A = \frac{\pi d^2}{4} = \frac{\pi}{4}d_0^2(1+ax)^2$$

Substitute for A to obtain:

$$I = -k\left[\frac{\pi}{4}d_0^2(1+ax)^2\right]\frac{dT}{dx}$$

Separating variables yields:

$$dT = -\frac{Idx}{k\left[\frac{\pi}{4}d_0^2(1+ax)^2\right]} = -\frac{4I}{\pi k d_0^2}\frac{dx}{(1+ax)^2}$$

Integrating T from T_1 to T_2 and x from 0 to L gives:

$$\int_{T_1}^{T_2} dT = -\frac{4I}{\pi k d_0^2}\int_0^L \frac{dx}{(1+ax)^2}$$

and

$$T_2 - T_1 = \Delta T = \frac{4IL}{\pi k d_0^2(1+aL)}$$

Substitute for ΔT and I in equation (1) and simplify to obtain:

$$R = \frac{\frac{4IL}{\pi k d_0^2(1+aL)}}{I} = \boxed{\frac{4L}{\pi k d_0^2(1+aL)}}$$

57 ••• A solid disk of radius r and mass m is spinning about a frictionless axis through its center and perpendicular to the disk, with angular velocity ω_1 at temperature T_1. The temperature of the disk decreases to T_2. Express the angular velocity ω_2, rotational kinetic energy K_2, and angular momentum L_2 in terms of

their values at the temperature T_1 and the linear expansion coefficient α of the disk.

Picture the Problem Let $\Delta T = T_2 - T_1$. We can apply Newton's second law to establish the relationship between L_2 and L_1 and angular momentum conservation to relate ω_2 and ω_1. We can express both E_2 and E_1 in terms of their angular momenta and rotational inertias and take their ratio to establish their relationship.

Apply $\sum \tau = \frac{\Delta L}{\Delta t}$ to the spinning disk:

Because $\sum \tau = 0$, $\Delta L = 0$ and

$$\boxed{L_2 = L_1}$$

Apply conservation of angular momentum to relate the angular velocity of the disk at T_2 to the angular velocity at T_1:

$$I_2\omega_2 = I_1\omega_1 \Rightarrow \omega_2 = \frac{I_1}{I_2}\omega_1 \qquad (1)$$

Express I_2:

$$I_2 = mr_2^2 = mr_1^2(1+\alpha\Delta T)^2 = I_1\left(1 + 2\alpha\Delta T + (\alpha\Delta T)^2\right)$$

Because $(\alpha\Delta T)^2$ is small compared to $\alpha\Delta T$:

$$I_2 \approx I_1(1 + 2\alpha\Delta T)$$

Substitute for I_2 in equation (1) to obtain:

$$\omega_2 = \frac{I_1}{I_1(1+2\alpha\Delta T)}\omega_1 \qquad (2)$$

Expanding $(1+2\alpha\Delta T)^{-1}$ binomially yields:

$$(1+2\alpha\Delta T)^{-1} = 1 - 2\alpha\Delta T + \text{higher order terms}$$

Because the higher order terms are very small:

$$(1+2\alpha\Delta T)^{-1} \approx 1 - 2\alpha\Delta T$$

Substituting in equation (2) yields:

$$\omega_2 \approx \boxed{(1-2\alpha\Delta T)\omega_1}$$

Express E_2 in terms of L_2 and I_2:

$$E_2 = \frac{L_2^2}{2I_2} = \frac{L_1^2}{2I_2} \text{ because } L_2 = L_1.$$

Express E_1 in terms of L_1 and I_1:

$$E_1 = \frac{L_1^2}{2I_1}$$

Express the ratio of E_2 to E_1 and simplify to obtain:

$$\frac{E_2}{E_1}=\frac{\frac{L_1^2}{2I_2}}{\frac{L_1^2}{2I_1}}=\frac{I_1}{I_2}\Rightarrow E_2=E_1\frac{I_1}{I_2}$$

Substituting for the ratio of I_1 to I_2 yields:

$$E_2=E_1\frac{\omega_2}{\omega_1}=\boxed{E_1(1-2\alpha\Delta T)}$$

58 ••• Write a **spreadsheet** program to graph the average temperature of the surface of Earth as a function of emissivity, using the results of Problem 22. How much does the emissivity have to change in order for the average temperature to increase by 1 K? This result can be thought of as a model for the effect of increasing concentrations of greenhouse gases like methane and CO_2 in Earth's atmosphere.

Picture the Problem The amount of heat radiated by Earth must equal the solar flux from the Sun, or else the temperature on Earth would continually increase. The emissivity of Earth is related to the rate at which it radiates energy into space by the Stefan-Boltzmann law $P_r=e\sigma AT^4$.

Using the Stefan-Boltzmann law, express the rate at which Earth radiates energy as a function of its emissivity e and temperature T:

$$P_r=e\sigma A'T^4$$

where A' is the surface area of Earth.

Use its definition to express the intensity of the radiation P_a absorbed by Earth:

$$I=\frac{P_a}{A}\text{ or }P_a=AI$$

where A is the cross-sectional area of Earth.

For 70% absorption of the Sun's radiation incident on Earth:

$$P_a=(0.70)AI$$

Equate P_r and P_a and simplify:

$$(0.70)AI=e\sigma A'T^4$$

or

$$(0.70)\pi R^2I=e\left(4\pi R^2\sigma T^4\right)$$

Solve for T to obtain:

$$T=\sqrt[4]{\frac{(0.70)I}{4\sigma e}}=Ce^{-1/4} \tag{1}$$

Substitute numerical values for I and σ and simplify to obtain:

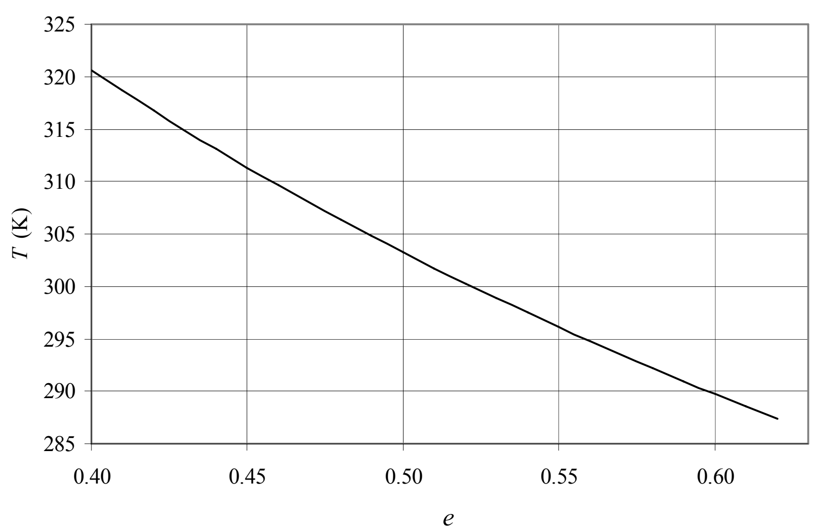
$$T=\sqrt[4]{\frac{(0.70)\left(1370\text{ W/m}^2\right)}{4\left(5.670\times10^{-8}\text{ W/m}^2\cdot\text{K}^4\right)e}}=(255\text{ K})e^{-1/4}$$

A spreadsheet program to evaluate T as a function of e is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	255	
B4	0.4	e
B5	B4+0.01	$e + 0.1$
C4	$B$1/(B4^0.25)	$(255\text{K})e^{-1/4}$

	A	B	C	D
1	$T=$	255	K	
2				
3		e	T	
4		0.40	321	
5		0.41	319	
6		0.42	317	
7		0.43	315	
23		0.59	291	
24		0.60	290	
25		0.61	289	
26		0.62	287	

A graph of T as a function of e follows.



Treating e as a variable, differentiate equation (1) to obtain:

$$\frac{dT}{de} = -\frac{1}{4}Ce^{-5/4}de \tag{2}$$

Divide equation (2) by equation (1) to obtain:

$$\frac{dT}{T}=\frac{-\frac{1}{4}Ce^{-5/4}de}{Ce^{-1/4}}=-\frac{1}{4}\frac{de}{e}$$

Use a differential approximation to obtain:

$$\frac{\Delta T}{T}=-\frac{1}{4}\frac{\Delta e}{e}\Rightarrow\frac{\Delta e}{e}=-4\frac{\Delta T}{T}$$

Substitute numerical values ($e\approx 0.615$ for $T_{\text{Earth}}=288$ K) and evaluate $\Delta e/e$:

$$\frac{\Delta e}{e}=-4\left(\frac{1\,\text{K}}{288\,\text{K}}\right)=\boxed{-0.014}$$

or about a 1.4% change.

59 ••• [SSM] A small pond has a layer of ice 1.00 cm thick floating on its surface. (*a*) If the air temperature is –10ºC on a day when there is a breeze, find the rate in centimeters per hour at which ice is added to the bottom of the layer. The density of ice is 0.917 g/cm^3. (*b*) How long do you and your friends have to wait for a 20.0-cm layer to be built up so you can play hockey?

Picture the Problem (*a*) We can differentiate the expression for the heat that must be removed from water in order to form ice to relate *dQ*/*dt* to the rate of ice build-up *dm*/*dt.* We can apply the thermal-current equation to express the rate at which heat is removed from the water to the temperature gradient and solve this equation for *dm*/*dt*. In Part (*b*) we can separate the variables in the differential equation relating *dm*/*dt* and Δ*T* and integrate to find out how long it takes for a 20.0-cm layer of ice to be built up.

(*a*) Relate the heat that must be removed from the water to freeze it to its mass and heat of fusion:

$$Q=mL_{\text{f}}\Rightarrow\frac{dQ}{dt}=L_{\text{f}}\frac{dm}{dt}$$

Using the definition of density, relate the mass of the ice added to the bottom of the layer to its density and volume:

$$m=\rho V=\rho Ax$$

Differentiate with respect to time to express the rate of build-up of the ice:

$$\frac{dm}{dt}=\rho A\frac{dx}{dt}$$

Substitute for $\frac{dm}{dt}$ to obtain:

$$\frac{dQ}{dt}=L_{\text{f}}\rho A\frac{dx}{dt}$$

The thermal-current equation is:

$$\frac{dQ}{dt}=kA\frac{\Delta T}{x}$$

Equate these expressions and solve for $\frac{dx}{dt}$:

$$L_f \rho A \frac{dx}{dt} = kA \frac{\Delta T}{x} \Rightarrow \frac{dx}{dt} = \frac{k}{L_f \rho} \frac{\Delta T}{x} \quad (1)$$

Substitute numerical values and evaluate $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{\left(0.592 \frac{\text{W}}{\text{m} \cdot \text{K}}\right)(0°\text{C} - (-10°\text{C}))}{\left(333.5 \frac{\text{kJ}}{\text{kg}}\right)(917\,\text{kg/m}^3)(0.0100\,\text{m})}$$
$$= 1.94 \frac{\mu\text{m}}{\text{s}} \times \frac{3600\,\text{s}}{\text{h}} = 0.697\,\text{cm/h}$$
$$= \boxed{0.70\,\text{cm/h}}$$

(*b*) Separating the variables in equation (1) gives:

$$x dx = \frac{k \Delta T}{L_f \rho} dt$$

Integrate x from x_i to x_f and t' from 0 to t:

$$\int_{x_i}^{x_f} x dx = \frac{k \Delta T}{L_f \rho} \int_0^t dt'$$

and

$$\tfrac{1}{2}\left(x_f^2 - x_i^2\right) = \frac{k \Delta T}{\rho L_f} t \Rightarrow t = \frac{\rho L_f \left(x_f^2 - x_i^2\right)}{2 k \Delta T}$$

Substitute numerical values and evaluate t:

$$t = \frac{\left(917 \frac{\text{kg}}{\text{m}^3}\right)\left(333.5 \frac{\text{kJ}}{\text{kg}}\right)}{2\left(0.592 \frac{\text{W}}{\text{m} \cdot \text{K}}\right)(0°\text{C} - (-10°\text{C}))}\left[(0.200\,\text{m})^2 - (0.010\,\text{m})^2\right]$$
$$= 1.03 \times 10^6\,\text{s} \times \frac{1\,\text{h}}{3600\,\text{s}} \times \frac{1\,\text{d}}{24\,\text{h}} = \boxed{12\,\text{d}}$$

60 ••• [SSM] A blackened copper cube that has 1.00-cm-long edges is heated to a temperature of 300ºC, and then is placed in a vacuum chamber whose walls are at a temperature of 0ºC. In the vacuum chamber, the cube cools radiatively. (*a*) Show that the (absolute) temperature T of the cube follows the differential equation: $\frac{dT}{dt} = -\frac{e\sigma A}{C}\left(T^4 - T_0^4\right)$ where C is the heat capacity of the cube, A is its surface area, e the emissivity, and T_0 the temperature of the vacuum chamber. (*b*) Using Euler's method (Section 5.4 of Chapter 5), numerically solve the differential equation to find $T(t)$, and graph it. Assume $e = 1.00$. How long does it take the cube to cool to a temperature of 15ºC?

Picture the Problem We can use the Stefan-Boltzmann equation and the definition of heat capacity to obtain the differential equation expressing the rate at which the temperature of the copper block decreases. We can then approximate the differential equation with a difference equation for the purpose of solving for the temperature of the block as a function of time using Euler's method.

(*a*) The rate at which heat is radiated away from the cube is given by:

$$\frac{dQ}{dt} = e\sigma A\left(T^4 - T_0^4\right)$$

Using the definition of heat capacity, relate the thermal current to the rate at which the temperature of the cube is changing:

$$\frac{dQ}{dt} = -C\frac{dT}{dt}$$

Equate these expressions and solve for $\frac{dT}{dt}$ to obtain:

$$\frac{dT}{dt} = \boxed{-\frac{e\sigma A}{C}\left(T^4 - T_0^4\right)}$$

Approximating the differential equation by the difference equation gives:

$$\frac{\Delta T}{\Delta t} = -\frac{e\sigma A}{C}\left(T^4 - T_0^4\right)$$

Solving for ΔT yields:

$$\Delta T = -\frac{e\sigma A}{C}\left(T^4 - T_0^4\right)\Delta t$$

or

$$T_{n+1} = T_n - \frac{e\sigma A}{C}\left(T_n^4 - T_0^4\right)\Delta t \qquad (1)$$

Use the definition of heat capacity to obtain:

$$C = mc = \rho Vc$$

Substitute numerical values (see Figure 13-1 for ρ_{Cu} and Table 19-1 for c_{Cu}) and evaluate C:

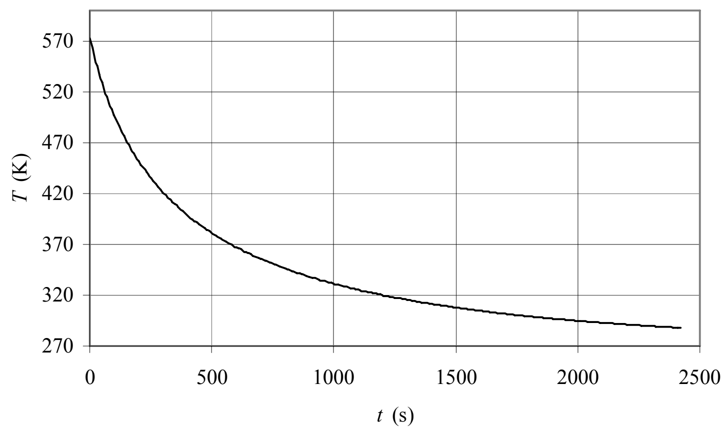
$$C = \left(8.93\times10^3\ \frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(1.00\times10^{-6}\ \mathrm{m}^3\right)\left(0.386\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right) = 3.45\frac{\mathrm{J}}{\mathrm{K}}$$

(b) A spreadsheet program to calculate T as a function of t using equation (1) is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	5.67E−08	σ
B2	6.00E−04	A
B3	3.45	C
B4	273	T_0
B5	10	Δt
A9	A8+$B$5	$t+\Delta t$
B9	B8-($B$1*$B$2/$B$3) *(B8^4−$B$4^4)*$B$5	$T_n - \frac{e\sigma A}{C}\left(T_n^4 - T_0^4\right)\Delta t$

	A	B	C
1	$\sigma=$	5.67×10^{-8}	$W/m^2 \cdot K^4$
2	$A=$	6.00×10^{-4}	m^2
3	$C=$	3.45	J/K
4	$T_0=$	273	K
5	$\Delta t=$	10	s
6			
7	t (s)	T (K)	
8	0	573.00	
9	10	562.92	
10	20	553.56	
11	30	544.85	
248	2400	288.22	
249	2410	288.08	
250	2420	287.95	
251	2430	287.82	

From the spreadsheet solution, the time to cool to 15°C (288 K) is about 2420 s or $\boxed{40.5\,\text{min}.}$ A graph of T as a function of t follows.



Chapter 21
The Electric Field 1: Discrete Charge Distributions

Conceptual Problems

1 • Objects are composed of atoms which are composed of charged particles (protons and electrons); however, we rarely observe the effects of the electrostatic force. Explain why we do not observe these effects.

Determine the Concept The net charge on large objects is always very close to zero. Hence the most obvious force is the gravitational force.

2 • A carbon atom can become a carbon *ion* if it has one or more of its electrons removed during a process called *ionization*. What is the net charge on a carbon atom that has had two of its electrons removed? (*a*) +*e*, (*b*) – *e*, (*c*) +2*e*, (*d*) –2*e*

Determine the Concept If two electrons are removed from a carbon atom, it will have a net positive charge of +2*e*. $\boxed{(c)}$ is correct.

3 •• You do a simple demonstration for your high school physics teacher in which you claim to disprove Coulomb's law. You first run a rubber comb through your dry hair, then use it to attract tiny neutral pieces of paper on the desk. You then say ″Coulomb's law states that for there to be electrostatic forces of attraction between two objects, both objects need to be charged. However, the paper was not charged. So according to Coulomb's law, there should be no electrostatic forces of attraction between them, yet there clearly was.″ You rest your case. (*a*) What is wrong with your assumptions? (*b*) Does attraction between the paper and the comb require that the net charge on the comb be negative? Explain.

Determine the Concept
(*a*) Coulomb's law is only valid for point particles. The paper bits cannot be modeled as point particles because the paper bits become polarized.

(*b*) No, the attraction does not depend on the sign of the charge on the comb. The induced charge on the paper that is closest to the comb is always opposite in sign to the charge on the comb, and thus the net force on the paper is always attractive.

4 •• You have a positively charged insulating rod and two metal spheres on insulating stands. Give step-by-step directions of how the rod, without actually touching either sphere, can be used to give one of the spheres (*a*) a negative charge, and (*b*) a positive charge.

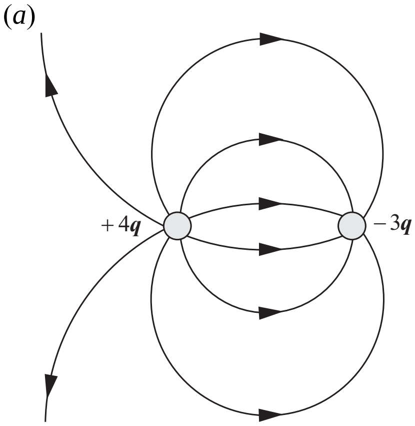
Determine the Concept
(*a*) Connect the metal sphere to ground; bring the insulating rod near the metal sphere and disconnect the sphere from ground; then remove the insulating rod. The sphere will be negatively charged.

(*b*) Bring the insulating rod in contact with the metal sphere; some of the positive charge on the rod will be transferred to the metal sphere.

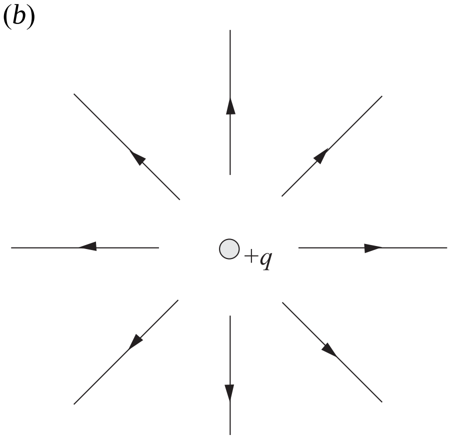
5 •• (*a*) Two point particles that have charges of $+4q$ and $-3q$ are separated by distance d. Use field lines to draw a visualization of the electric field in the neighborhood of this system. (*b*) Draw the field lines at distances much greater than d from the charges.

Determine the Concept (*a*) We can use the rules for drawing electric field lines to draw the electric field lines for this system. Two field lines have been assigned to each charge q. (*b*) At distances much greater than the separation distance between the two charges, the system of two charged bodies will ″look like″ a single charge of $+q$ and the field pattern will be that due to a point charge of $+q$. Eight field lines have been assigned to the single charge.

(*a*)



(*b*)



6 •• A metal sphere is positively charged. Is it possible for the sphere to electrically attract another positively charged ball? Explain your answer.

Comment: DAVID: The ISM solution to 10 needs to be fleshed out.

Determine the Concept Yes. Because a metal sphere is a conductor, the proximity of a positively charged ball (not necessarily a conductor), will induce a redistribution of charges on the metal sphere with the surface nearer the positively charged ball becoming negatively charged. Because the negative charges on the metal sphere are closer to the positively charged ball than are the positive charges on the metal sphere, the net force will be attractive.

7 •• A simple demonstration of electrostatic attraction can be done simply by dangling a small ball of crumpled aluminum foil on a string and bringing a charged rod near the ball. The ball initially will be attracted to the rod, but once they touch, the ball will be strongly repelled from it. Explain these observations.

Determine the Concept Assume that the rod has a negative charge. When the charged rod is brought near the aluminum foil, it induces a redistribution of charges with the side nearer the rod becoming positively charged, and so the ball of foil swings toward the rod. When it touches the rod, some of the negative charge is transferred to the foil, which, as a result, acquires a net negative charge and is now repelled by the rod.

8 •• Two positive point charges that are equal in magnitude are fixed in place, one at $x = 0.00$ m and the other at $x = 1.00$ m, on the x axis. A third positive point charge is placed at an equilibrium position. (*a*) Where is this equilibrium position? (*b*) Is the equilibrium position stable if the third particle is constrained to move parallel with the x axis? (*c*) What about if it is constrained to move parallel with the y axis? Explain.

Determine the Concept
(*a*) A third positive charge can be placed midway between the fixed positive charges. This is the only location.

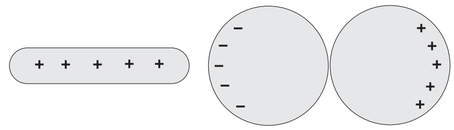
(*b*) Yes. The position identified in (*a*) is one of stable equilibrium. It is stable in the x direction because, regardless of whether you displace the third positive charge to the right or to the left, the net force acting on it is back toward the midpoint between the two fixed charges.

(*c*) If the third positive charge is displaced in the y direction, the net force acting on it will be away from its equilibrium position. Hence, the position midway between the fixed positive charges is one of unstable equilibrium in the y direction.

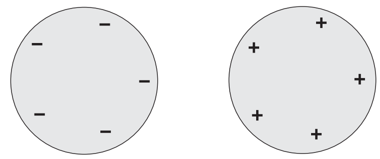
9 •• Two neutral conducting spheres are in contact and are supported on a large wooden table by insulated stands. A positively charged rod is brought up close to the surface of one of the spheres on the side opposite its point of contact with the other sphere. (*a*) Describe the induced charges on the two conducting spheres, and sketch the charge distributions on them. (*b*) The two spheres are separated and then the charged rod is removed. The spheres are then separated far apart. Sketch the charge distributions on the separated spheres.

Determine the Concept Because the spheres are conductors, there are free electrons on them that will reposition themselves when the positively charged rod is brought nearby.

(*a*) On the sphere near the positively charged rod, the induced charge is negative and near the rod. On the other sphere, the net charge is positive and on the side far from the rod. This is shown in the diagram.

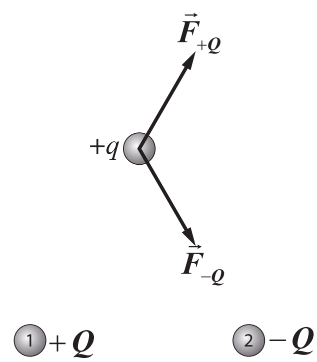


(*b*) When the spheres are separated and far apart and the rod has been removed, the induced charges are distributed uniformly over each sphere. The charge distributions are shown in the diagram.



10 •• Three point charges, $+q$, $+Q$, and $-Q$, are placed at the corners of an equilateral triangle as shown in Figure 21-33. No other charged objects are nearby. (*a*) What is the direction of the net force on charge $+q$ due to the other two charges? (*b*) What is the total electric force on the system of three charges? Explain.

Determine the Concept The forces acting on point charge $+q$ are shown in the diagram. The force acting on point charge $+q$ due to point charge $-Q$ is along the line joining them and directed toward $-Q$. The force acting on point charge $+q$ due to point charge $+Q$ is along the line joining them and directed away from point charge $+Q$.



(*a*) Because point charges $+Q$ and $-Q$ are equal in magnitude, the forces due to these charges are equal and their sum (the net force on charge $+q$) will be $\boxed{\text{to the right.}}$ Note that the vertical components of these forces add up to zero.

(*b*) Because no other charged objects are nearby, the forces acting on this system of three point charges are internal forces and the net force acting on the system is $\boxed{\text{zero}}$.

11 •• A positively charged particle is free to move in a region with an electric field $\vec{E}$. Which statements must be true?

(*a*) The particle is accelerating in the direction perpendicular to $\vec{E}$.

(*b*) The particle is accelerating in the direction of $\vec{E}$.
(*c*) The particle is moving in the direction of $\vec{E}$.
(*d*) The particle could be momentarily at rest.
(*e*) The force on the particle is opposite the direction of $\vec{E}$.
(*f*) The particle is moving opposite the direction of $\vec{E}$.

Determine the Concept
(*a*) False. The only force acting on the particle is in the direction of $\vec{E}$.

(*b*) True. The electrical force experienced by the particle is, by definition, in the direction of $\vec{E}$.

(*c*) False. We don't know whether the particle is moving or momentarily at rest. All we know is that the net force acting on it is in the direction of $\vec{E}$.

(*d*) Possibly, Whether the particle is ever at rest depends on how it was initially placed in the electric field. That is, it depends on whether its initial velocity was zero, in the direction of $\vec{E}$, or opposite $\vec{E}$. We do know that, if it is at any time at rest, it will not stay at rest.

(*e*) False. By definition, the electric force acting on a positively charged particle in an electric field is in the direction of the field.

(*f*) False. All we know for sure is that the electric force and, hence, the acceleration of the particle, is in the direction of $\vec{E}$. The particle could be moving in the direction of the field, in the direction opposite the field, or it could be momentarily at rest.

12 •• Four charges are fixed in place at the corners of a square as shown in Figure 21-34. No other charges are nearby. Which of the following statements is true?

(*a*) $\vec{E}$ is zero at the midpoints of all four sides of the square.
(*b*) $\vec{E}$ is zero at the center of the square.
(*c*) $\vec{E}$ is zero midway between the top two charges and midway between the bottom two charges.

Determine the Concept $\vec{E}$ is zero wherever the net force acting on a test charge is zero.

(*a*) False. A test charge placed at these locations will experience a net force.

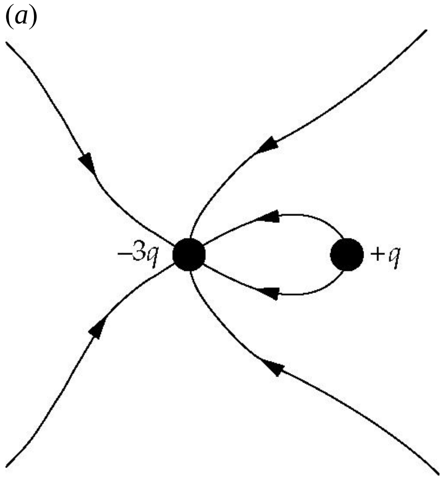
(*b*) True. At the center of the square the two positive charges alone produce a net electric field of zero, and the two negative charges alone also produce a net electric field of zero. Thus, the net force acting on a test charge at the midpoint of the square is zero.

(*c*) False. A test charge placed at either of these locations will experience a net force.

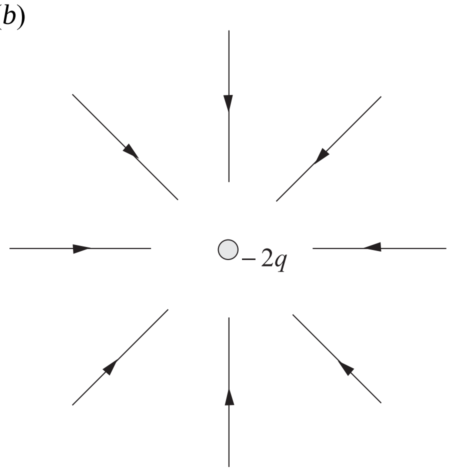
13 •• **[SSM]** Two point particles that have charges of $+q$ and $-3q$ are separated by distance d. (*a*) Use field lines to sketch the electric field in the neighborhood of this system. (*b*) Draw the field lines at distances much greater than d from the charges.

Determine the Concept (*a*) We can use the rules for drawing electric field lines to draw the electric field lines for this system. In the field-line sketch we've assigned 2 field lines to each charge q. (*b*) At distances much greater than the separation distance between the two charges, the system of two charged bodies will ″look like″ a single charge of $-2q$ and the field pattern will be that due to a point charge of $-2q$. Four field lines have been assigned to each charge $-q$.

(*a*)

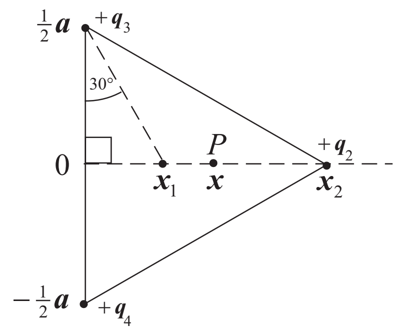


(*b*)



14 •• Three equal positive point charges (each charge $+q$) are fixed at the vertices of an equilateral triangle with sides of length a. The origin is at the midpoint of one side the triangle, the center of the triangle on the x axis at $x = x_1$ and the vertex opposite the origin is on the x axis at $x = x_2$. (*a*) Express x_1 and x_2 in terms of a. (*b*) Write an expression for the electric field on the x axis a distance x from the origin on the interval $0 \le x < x_2$. (*c*) Show that the expression you obtained in (*b*) gives the expected results for $x = 0$ and for $x = x_1$.

Picture the Problem (*a*) We can use the geometry of an equilateral triangle to express x_1 and x_2 in terms of the side a of the triangle. (*b*) The electric field on the x axis a distance x from the origin on the interval $0 \le x < x_2$ is the superposition of the electric fields due to the point charges q_2, q_3, and q_4 located at the vertices of the triangle.



(*a*) Apply the Pythagorean theorem to the triangle whose vertices are at $(0,0), (0,\frac{1}{2}a)$, and $(\boldsymbol{x}_2,0)$ to obtain:

$$x_2^2 + \left(\tfrac{1}{2}a\right)^2 = a^2 \Rightarrow x_2 = \boxed{\tfrac{\sqrt{3}}{2}a}$$

Referring to the triangle whose vertices are at $(0,0), (0,\frac{1}{2}a)$, and $(\boldsymbol{x}_1,0)$, note that:

$$\tan 30° = \frac{x_1}{\frac{1}{2}a} \Rightarrow x_1 = \tfrac{1}{2}a\tan 30°$$

Substitute for $\tan 30°$ and evaluate x_1:

$$x_1 = \boxed{\tfrac{\sqrt{3}}{6}a}$$

(*b*) The electric field at P is the superposition of the electric fields due to the charges q_2, q_3, and q_4:

$$\vec{\boldsymbol{E}}_P = \vec{\boldsymbol{E}}_2 + \vec{\boldsymbol{E}}_3 + \vec{\boldsymbol{E}}_4 \qquad (1)$$

Express the electric field at P due to q_2:

$$\vec{\boldsymbol{E}}_2 = \frac{kq_2}{r_{2,P}^2}\hat{\boldsymbol{r}}_{2,P} = \frac{kq_2}{r_{2,P}^3}\hat{\boldsymbol{r}}_{2,P}$$

where

$$\hat{\boldsymbol{r}}_{2,P} = (x - x_2)\hat{\boldsymbol{i}} = \left(x - \tfrac{\sqrt{3}}{2}a\right)\hat{\boldsymbol{i}}$$

and

$$r_{2,P} = x_2 - x$$

Substituting for q_2, $\vec{\boldsymbol{r}}_{2,P}$, and $r_{2,P}$ yields:

$$\vec{\boldsymbol{E}}_2 = \frac{kq}{\left(\frac{\sqrt{3}}{2}a - x\right)^3}\left(x - \tfrac{\sqrt{3}}{2}a\right)\hat{\boldsymbol{i}} = -\frac{kq}{\left(\frac{\sqrt{3}}{2}a - x\right)^2}\hat{\boldsymbol{i}}$$

Express the electric field at P due to q_3:

$$\vec{E}_3 = \frac{kq_3}{r_{3,P}^2}\vec{r}_{3,P} = \frac{kq_3}{r_{3,P}^3}\vec{r}_{3,P}$$

where

$$\vec{r}_{3,P} = (x-0)\hat{i} + \left(0-\tfrac{1}{2}a\right)\hat{j} = x\hat{i} - \tfrac{1}{2}a\hat{j}$$

and

$$r_{3,P} = \sqrt{(x-0)^2 + \left(0-\tfrac{1}{2}a\right)^2} = \sqrt{x^2 + \tfrac{1}{4}a^2}$$

Substituting for q_3, $\vec{r}_{3,P}$, and $r_{3,P}$ yields:

$$\vec{E}_3 = \frac{kq}{\left(x^2 + \frac{1}{4}a^2\right)^{3/2}}\left(x\hat{i} - \tfrac{1}{2}a\hat{j}\right)$$

Proceed similarly for $\vec{E}_4$ to obtain:

$$\vec{E}_4 = \frac{kq}{\left(x^2 + \frac{1}{4}a^2\right)^{3/2}}\left(x\hat{i} + \tfrac{1}{2}a\hat{j}\right)$$

Substituting for $\vec{E}_2, \vec{E}_3$, and $\vec{E}_4$ in equation (1) and simplifying yields:

$$\vec{E}_P = -\frac{kq}{\left(\frac{\sqrt{3}}{2}a - x\right)^2}\hat{i} + \frac{kq}{\left(x^2 + \frac{1}{4}a^2\right)^{3/2}}\left(x\hat{i} - \tfrac{1}{2}a\hat{j}\right) + \frac{kq}{\left(x^2 + \frac{1}{4}a^2\right)^{3/2}}\left(x\hat{i} + \tfrac{1}{2}a\hat{j}\right)$$

$$= \boxed{kq\left[-\frac{1}{\left(\frac{\sqrt{3}}{2}a - x\right)^2} + \frac{2x}{\left(x^2 + \frac{1}{4}a^2\right)^{3/2}}\right]\hat{i}}$$

(*c*) Evaluating $\vec{E}_P(0)$ yields:

$$\vec{E}_P(0) = kq\left[-\frac{1}{\left(\frac{\sqrt{3}}{2}a\right)^2} + \frac{2(0)}{\left(\frac{1}{4}a^2\right)^{3/2}}\right]\hat{i} = \boxed{-\frac{kq}{\left(\frac{\sqrt{3}}{2}a\right)^2}\hat{i}}$$

Note that, because the electric fields due to q_3 and q_4 cancel each other at the origin and the resultant field is that due to q_2, this is the expected result.

Evaluate $\vec{E}_P\left(\frac{\sqrt{3}}{6}a\right)$ to obtain:

$$\vec{E}_P\left(\tfrac{\sqrt{3}}{6}a\right) = kq\left[-\frac{1}{\left(\frac{\sqrt{3}}{2}a - \frac{\sqrt{3}}{6}a\right)^2} + \frac{2\left(\frac{\sqrt{3}}{6}a\right)}{\left(\left(\frac{\sqrt{3}}{6}a\right)^2 + \frac{1}{4}a^2\right)^{3/2}}\right]\hat{i} = kq\left[-\frac{3}{a^2} + \frac{3}{a^2}\right]\hat{i} = \boxed{0}$$

Note that, because the point at the center of the equilateral triangle is equidistant from the three vertices, the electric fields due to the charges at the vertices cancel and this is the expected result.

15 •• A molecule has a dipole moment given by $\vec{p}$. The molecule is momentarily at rest with $\vec{p}$ making an angle θ with a uniform electric field $\vec{E}$. Describe the subsequent motion of the dipole moment.

Comment: DAVID: The ISM solution to 23 is lacking. The first sentence in the solution is irrelevant. The second is close to being irrelevant. You need to show how at least one of these is done. I would avoid the field lines and just use Coulombs law qualitatively.

Determine the Concept The dipole moment rotates back and forth in oscillatory motion. The dipole moment gains angular speed as it rotates toward the direction of the electric field, and loses angular speed as it rotates away from the direction of the electric field.

16 •• True or false:

(*a*) The electric field of a point charge always points away from the charge.
(*b*) The electric force on a charged particle in an electric field is always in the same direction as the field.
(*c*) Electric field lines never intersect.
(*d*) All molecules have dipole moments in the presence of an external electric field.

(*a*) False. The direction of the field is toward a negative charge.

(*b*) False. The direction of the electric force acting on a point charge depends on the sign of the point charge.

(*c*) False. Electric field lines intersect any point in space occupied by a point charge.

(*d*) True. An electric field partially polarizes the molecules; resulting in the separation of their charges and the creation of electric dipole moments.

17 ••• **[SSM]** Two molecules have dipole moments of equal magnitude. The dipole moments are oriented in various configurations as shown in Figure 21-35. Determine the electric-field direction at each of the numbered locations. Explain your answers.

Comment:

Determine the Concept Figure 21-23 shows the electric field due to a single dipole, where the dipole moment is directed toward the right. The electric field due two a pair of dipoles can be obtained by superposing the two electric fields.

	1	2	3
(*a*)	down	up	up
(*b*)	up	right	left
(*c*)	down	up	up
(*d*)	down	up	up

Estimation and Approximation

18 •• Estimate the force required to bind the two protons in the He nucleus together. HINT: Model the protons as point charges. *You will need to have an estimate of the distance between them.*

Picture the Problem Because the nucleus is in equilibrium, the binding force must be equal to the electrostatic force of repulsion between the protons.

Apply $\sum \vec{F} = 0$ to a proton to obtain: $F_{\text{binding}} - F_{\text{electrostatic}} = 0$

Using Coulomb's law, substitute for $F_{\text{electrostatic}}$: $\frac{kq^2}{r^2} - F_{\text{electrostatic}} = 0 \Rightarrow F_{\text{binding}} = \frac{kq^2}{r^2}$

Assuming the diameter of the He nucleus to be approximately 10^{-15} m, substitute numerical values and evaluate $F_{\text{electrostatic}}$:

$$F_{\text{binding}} = \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(1.602\times10^{-19}\ \text{C})^2}{(10^{-15}\ \text{m})^2} \approx \boxed{0.2\ \text{kN}}$$

19 •• A popular classroom demonstration consists of rubbing a plastic rod with fur to give the rod charge, and then placing the rod near an empty soda can that is on its side (Figure 21-36). Explain why the can will roll toward the rod.

Comment: DAVID: I am sure the answer to 27 is correct. I think the can is grounded, so it becomes charged, not polarized.

Determine the Concept Because the can is grounded, the presence of the negatively charged plastic rod induces a positive charge on it. The positive charges induced on the can are attracted, via the Coulomb interaction, to the negative charges on the plastic rod. Unlike charges attract, so the can will roll toward the rod.

20 ••• Sparks in air occur when ions in the air are accelerated to a such a high speed by an electric field that when they impact on neutral gas molecules the neutral molecules become ions. If the electric field strength is large enough, the ionized collision products are themselves accelerated and produce more ions on impact, and so forth. This avalanche of ions is what we call a spark. (*a*) Assume that an ion moves, on average, exactly one mean free path through the air before hitting a molecule. If the ion needs to acquire approximately 1.0 eV of kinetic energy in order to ionize a molecule, estimate the minimum strength of the electric field required at standard room pressure and temperature. Assume that the cross-sectional area of an air molecule is about 0.10 nm^2. (*b*) How does the strength of the electric field in (*a*) depend on temperature? (*c*) How does the strength of the electric field in (*a*) depend on pressure?

Picture the Problem We can use the definition of electric field to express E in terms of the work done on the ionizing electrons and the distance they travel λ between collisions. We can use the ideal-gas law to relate the number density of molecules in the gas ρ and the scattering cross-section σ to the mean free path and, hence, to the electric field.

(*a*) Apply conservation of energy to relate the work done on the electrons by the electric field to the change in their kinetic energy:

$$W = \Delta K = F\Delta s$$

From the definition of electric field we have:

$$F = qE$$

Substitute for F and Δs to obtain:

$W = qE\lambda$, where the mean free path λ is the distance traveled by the electrons between ionizing collisions with nitrogen atoms.

The mean free path λ of a particle is related to the scattering cross-section σ and the number density for air molecules n:

$$\lambda = \frac{1}{\sigma n}$$

Substitute for λ to obtain:

$$W = \frac{qE}{\sigma n} \Rightarrow E = \frac{\sigma nW}{q}$$

Use the ideal-gas law to obtain:

$$n = \frac{N}{V} = \frac{P}{kT}$$

Substituting for n yields:

$$E = \frac{\sigma PW}{qkT} \qquad (1)$$

Substitute numerical values and evaluate E:

$$E = \frac{\left(0.10\,\text{nm}^2\right)\left(101.325\times10^3\,\frac{\text{N}}{\text{m}^2}\right)\left(1.0\,\text{eV}\right)\left(1.602\times10^{-19}\,\frac{\text{J}}{\text{eV}}\right)}{\left(1.602\times10^{-19}\,\text{C}\right)\left(1.381\times10^{-23}\,\frac{\text{J}}{\text{K}}\right)\left(300\,\text{K}\right)} \approx \boxed{2.4\times10^6\ \text{N/C}}$$

(*b*) From equation (1) we see that:

$$\boxed{E \propto T^{-1}}$$

(*c*) Also, from equation (1):

$$\boxed{E \propto P}$$

Charge

21 • A plastic rod is rubbed against a wool shirt, thereby acquiring a charge of –0.80 μC. How many electrons are transferred from the wool shirt to the plastic rod?

Picture the Problem The charge acquired by the plastic rod is an integral number of electronic charges, that is, $q = n_e(-e)$.

Relate the charge acquired by the plastic rod to the number of electrons transferred from the wool shirt:

$$q = n_e(-e) \Rightarrow n_e = \frac{q}{-e}$$

Substitute numerical values and evaluate n_e:

$$n_e = \frac{-0.80\ \mu\text{C}}{-1.602\times10^{-19}\ \dfrac{\text{C}}{\text{electron}}} = \boxed{5.0\times10^{12}\ \text{electrons}}$$

22 • A charge equal to the charge of Avogadro's number of protons ($N_A = 6.02 \times 10^{23}$) is called a *faraday*. Calculate the number of coulombs in a faraday.

Picture the Problem One faraday = N_Ae. We can use this definition to find the number of coulombs in a faraday.

Use the definition of a faraday to calculate the number of coulombs in a faraday:

$$1\ \text{faraday} = N_A e = (6.02\times10^{23}\ \text{electrons})(1.602\times10^{-19}\ \text{C/electron}) = \boxed{9.63\times10^{4}\ \text{C}}$$

23 • **[SSM]** What is the total charge of all of the protons in 1.00 kg of carbon?

Picture the Problem We can find the number of coulombs of positive charge there are in 1.00 kg of carbon from $Q = 6n_Ce$, where n_C is the number of atoms in 1.00 kg of carbon and the factor of 6 is present to account for the presence of 6 protons in each atom. We can find the number of atoms in 1.00 kg of carbon by setting up a proportion relating Avogadro's number, the mass of carbon, and the molecular mass of carbon to n_C. See Appendix C for the molar mass of carbon.

Express the positive charge in terms of the electronic charge, the number of protons per atom, and the number of atoms in 1.00 kg of carbon:

$$Q = 6n_{\text{C}}e$$

Using a proportion, relate the number of atoms in 1.00 kg of carbon n_{C}, to Avogadro's number and the molecular mass M of carbon:

$$\frac{n_{\text{C}}}{N_{\text{A}}} = \frac{m_{\text{C}}}{M} \Rightarrow n_{\text{C}} = \frac{N_{\text{A}}m_{\text{C}}}{M}$$

Substitute for n_{C} to obtain:

$$Q = \frac{6N_{\text{A}}m_{\text{C}}e}{M}$$

Substitute numerical values and evaluate Q:

$$Q = \frac{6\left(6.022\times10^{23}\,\frac{\text{atoms}}{\text{mol}}\right)(1.00\,\text{kg})\left(1.602\times10^{-19}\,\text{C}\right)}{0.01201\frac{\text{kg}}{\text{mol}}} = \boxed{4.82\times10^{7}\,\text{C}}$$

24 •• Suppose a cube of aluminum which is 1.00 cm on a side accumulates a net charge of +2.50 pC. (*a*)What percentage of the electrons originally in the cube was removed? (*b*) By what percentage has the mass of the cube decreased because of this removal?

Picture the Problem (*a*) The percentage of the electrons originally in the cube that was removed can be found from the ratio of the number of electrons removed to the number of electrons originally in the cube. (*b*) The percentage decrease in the mass of the cube can be found from the ratio of the mass of the electrons removed to the mass of the cube.

(*a*) Express the ratio of the electrons removed to the number of electrons originally in the cube:

$$\frac{N_{\text{rem}}}{N_{\text{ini}}} = \frac{\dfrac{Q_{\text{accumulated}}}{e}}{N_{\substack{\text{electrons}\\\text{per atom}}}N_{\text{atoms}}} \qquad (1)$$

The number of atoms in the cube is the ratio of the mass of the cube to the mass of an aluminum atom:

$$N_{\text{atoms}} = \frac{m_{\text{cube}}}{m_{\text{Al atom}}} = \frac{\rho_{\text{Al}}V_{\text{cube}}}{m_{\text{Al atom}}}$$

The mass of an aluminum atom is its molar mass divided by Avogadro's number:

$$m_{\text{Al atom}} = \frac{M_{\text{Al}}}{N_{\text{A}}}$$

Substituting and simplifying yields:

$$N_{\text{atoms}} = \frac{\rho_{\text{Al}} V_{\text{cube}}}{\frac{M_{\text{Al}}}{N_{\text{A}}}} = \frac{\rho_{\text{Al}} V_{\text{cube}} N_{\text{A}}}{M_{\text{Al}}}$$

Substitute for N_{atoms} in equation (1) and simplify to obtain:

$$\frac{N_{\text{rem}}}{N_{\text{ini}}} = \frac{\frac{Q_{\text{accumulated}}}{e}}{N_{\substack{\text{electrons}\\ \text{per atom}}} \left(\frac{\rho_{\text{Al}} V_{\text{cube}} N_{\text{A}}}{M_{\text{Al}}} \right)} = \frac{Q_{\text{accumulated}} M_{\text{Al}}}{N_{\substack{\text{electrons}\\ \text{per atom}}} \rho_{\text{Al}} V_{\text{cube}} e N_{\text{A}}}$$

Substitute numerical values and evaluate $\frac{N_{\text{rem}}}{N_{\text{ini}}}$:

$$\frac{N_{\text{rem}}}{N_{\text{ini}}} = \frac{(2.50\ \text{pC})\left(26.98\ \frac{\text{g}}{\text{mol}}\right)}{\left(13\frac{\text{electrons}}{\text{atom}}\right)\left(2.70\frac{\text{g}}{\text{cm}^3}\right)(1.00\ \text{cm})^3\left(1.602\times10^{-19}\ \text{C}\right)\left(6.022\times10^{23}\ \frac{\text{atoms}}{\text{mol}}\right)} \approx \boxed{1.99\times10^{-15}\ \%}$$

(*b*) Express the ratio of the mass of the electrons removed to the mass of the cube:

$$\frac{m_{\text{rem}}}{m_{\text{cube}}} = \frac{N_{\text{rem}} m_{\text{electron}}}{\rho_{\text{Al}} V_{\text{cube}}}$$

From (*a*), the number of electrons removed is given by:

$$N_{\text{rem}} = \frac{Q_{\text{accumulated}}}{e}$$

Substituting and simplifying yields:

$$\frac{m_{\text{rem}}}{m_{\text{cube}}} = \frac{\frac{Q_{\text{accumulated}}}{e} m_{\text{electron}}}{\rho_{\text{Al}} V_{\text{cube}}} = \frac{Q_{\text{accumulated}} m_{\text{electron}}}{e \rho_{\text{Al}} V_{\text{cube}}}$$

Substitute numerical values and evaluate $\frac{m_{\text{rem}}}{m_{\text{cube}}}$:

$$\frac{m_{\text{rem}}}{m_{\text{cube}}} = \frac{(2.50\ \text{pC})\left(9.109\times10^{-31}\ \text{kg}\right)}{\left(1.602\times10^{-19}\ \text{C}\right)\left(2.70\frac{\text{g}}{\text{cm}^3}\right)\left(1.00\ \text{cm}^3\right)} \approx \boxed{5.26\times10^{-19}\ \%}$$

25 •• During a process described by the *photoelectric effect*, ultra-violet light can be used to charge a piece of metal. (*a*) If such light is incident on a slab of conducting material and electrons are ejected with enough energy that they escape the surface of the metal, how long before the metal has a net charge of +1.50 nC if 1.00×10^6 electrons are ejected per second? (*b*) If 1.3 eV is needed to eject an electron from the surface, what is the power rating of the light beam? (Assume this process is 100% efficient.)

Picture the Problem (*a*) The required time is the ratio of the charge that accumulates to the rate at which it is delivered to the conductor. (*b*) We can use the definition of power to find the power rating of the light beam.

(*a*) The required time is the ratio of the charge that accumulates to the rate at which it is delivered:

$$\Delta t = \frac{\Delta q}{I} = \frac{\Delta q}{dq/dt}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1.50\,\text{nC}}{\left(1.00\times10^{6}\,\dfrac{\text{electrons}}{\text{s}}\right)\left(1.602\times10^{-19}\,\dfrac{\text{C}}{\text{electron}}\right)} = 9.363\times10^{3}\times\frac{1\,\text{h}}{3600\,\text{s}} = \boxed{2.60\,\text{h}}$$

(*b*) The power rating of the light beam is the rate at which it delivers energy:

$$P = \frac{\Delta E}{\Delta t}$$

The energy delivered by the beam is the product of the energy per electron, the electron current (that is, the number of electrons removed per unit time), and the elapsed time:

$$\Delta E = E_{\substack{\text{per}\\\text{electron}}} I_{\text{electron}} \Delta t$$

Substituting for ΔE in the expression for P and simplifying yields:

$$P = \frac{E_{\substack{\text{per}\\\text{electron}}} I_{\text{electron}} \Delta t}{\Delta t} = E_{\substack{\text{per}\\\text{electron}}} I_{\text{electron}}$$

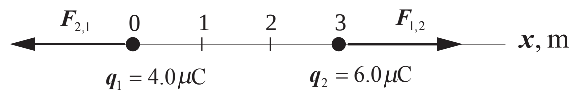
Substitute numerical values and evaluate *P*:

$$P = \left(1.3\,\frac{\text{eV}}{\text{electron}}\times\frac{1.602\times10^{-19}\,\text{J}}{\text{eV}}\right)\left(1.00\times10^{6}\,\frac{\text{electrons}}{\text{s}}\right) = \boxed{2.1\times10^{-13}\,\text{W}}$$

Coulomb's Law

26 • A point charge $q_1 = 4.0\ \mu C$ is at the origin and a point charge $q_2 = 6.0\ \mu C$ is on the x-axis at $x = 3.0$ m. (*a*) Find the electric force on charge q_2. (*b*) Find the electric force on q_1. (*c*) How would your answers for Parts (*a*) and (*b*) differ if q_2 were $-6.0\ \mu C$?

Picture the Problem We can find the electric forces the two charges exert on each by applying Coulomb's law and Newton's 3rd law. Note that $\hat{\boldsymbol{r}}_{1,2} = \hat{\boldsymbol{i}}$ because the vector pointing from q_1 to q_2 is in the positive x direction. The diagram shows the situation for Parts (*a*) and (*b*).



(*a*) Use Coulomb's law to express the force that q_1 exerts on q_2:

$$\vec{F}_{1,2} = \frac{kq_1q_2}{r_{1,2}^2}\hat{\boldsymbol{r}}_{1,2}$$

Substitute numerical values and evaluate $\vec{F}_{1,2}$:

$$\vec{F}_{1,2} = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)(4.0\,\mu\text{C})(6.0\,\mu\text{C})}{(3.0\,\text{m})^2}\hat{\boldsymbol{i}} = \boxed{(24\,\text{mN})\hat{\boldsymbol{i}}}$$

(*b*) Because these are action-and-reaction forces, we can apply Newton's 3rd law to obtain:

$$\vec{F}_{2,1} = -\vec{F}_{1,2} = \boxed{-(24\,\text{mN})\hat{\boldsymbol{i}}}$$

(*c*) If q_2 is $-6.0\ \mu C$, the force between q_1 and q_2 is attractive and both force vectors are reversed:

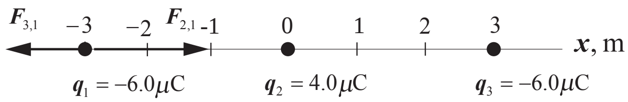
$$\vec{F}_{1,2} = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)(4.0\,\mu\text{C})(-6.0\,\mu\text{C})}{(3.0\,\text{m})^2}\hat{\boldsymbol{i}} = \boxed{-(24\,\text{mN})\hat{\boldsymbol{i}}}$$

and

$$\vec{F}_{2,1} = -\vec{F}_{1,2} = \boxed{(24\,\text{mN})\hat{\boldsymbol{i}}}$$

27 • **[SSM]** Three point charges are on the x-axis: $q_1 = -6.0\ \mu C$ is at $x = -3.0$ m, $q_2 = 4.0\ \mu C$ is at the origin, and $q_3 = -6.0\ \mu C$ is at $x = 3.0$ m. Find the electric force on q_1.

Picture the Problem q_2 exerts an attractive electric force $\vec{F}_{2,1}$ on point charge q_1 and q_3 exerts a repulsive electric force $\vec{F}_{3,1}$ on point charge q_1. We can find the net electric force on q_1 by adding these forces (that is, by using the superposition principle).



Express the net force acting on q_1:

$$\vec{F}_1 = \vec{F}_{2,1} + \vec{F}_{3,1}$$

Express the force that q_2 exerts on q_1:

$$\vec{F}_{2,1} = \frac{k|q_1||q_2|}{r_{2,1}^2}\hat{\boldsymbol{i}}$$

Express the force that q_3 exerts on q_1:

$$\vec{F}_{3,1} = \frac{k|q_1||q_3|}{r_{3,1}^2}\left(-\hat{\boldsymbol{i}}\right)$$

Substitute and simplify to obtain:

$$\vec{F}_1 = \frac{k|q_1||q_2|}{r_{2,1}^2}\hat{\boldsymbol{i}} - \frac{k|q_1||q_3|}{r_{3,1}^2}\hat{\boldsymbol{i}}$$
$$= k|q_1|\left(\frac{|q_2|}{r_{2,1}^2} - \frac{|q_3|}{r_{3,1}^2}\right)\hat{\boldsymbol{i}}$$

Substitute numerical values and evaluate $\vec{F}_1$:

$$\vec{F}_1 = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(6.0\,\mu\text{C})\left(\frac{4.0\,\mu\text{C}}{(3.0\,\text{m})^2} - \frac{6.0\,\mu\text{C}}{(6.0\,\text{m})^2}\right)\hat{\boldsymbol{i}} = \boxed{\left(1.5\times10^{-2}\ \text{N}\right)\hat{\boldsymbol{i}}}$$

28 •• A 2.0 μC point charge and a 4.0 μC point charge are a distance L apart. Where should a third point charge be placed so that the electric force on that third charge is zero?

Comment: DAVID: Note the changes in 36 nd 37.

Picture the Problem The third point charge should be placed at the location at which the forces on the third point charge due to each of the other two point charges cancel. There can be no such place except on the line between the two point charges. Denote the 2.0 μC and 4.0 μC point charges by the numerals 2 and 4, respectively, and the third point charge by the numeral 3. Assume that the 2.0 μC point charge is to the left of the 4.0 μC point charge, let the $+x$ direction be to the right. Then the 4.0 μC point charge is located at $x = L$.

Apply the condition for translational equilibrium to the third point charge:

$$\vec{F}_{4,3} + \vec{F}_{2,3} = 0$$

or

$$F_{4,3} = F_{2,3} \qquad (1)$$

Letting the distance from the third point charge to the 4.0 μC point charge be x, express the force that the 4.0 μC point charge exerts on the third point charge:

$$F_{4,3} = \frac{kq_3q_4}{(L-x)^2}$$

The force that the 2.0 μC point charge exerts on the third charge is given by:

$$F_{2,3} = \frac{kq_3q_2}{x^2}$$

Substitute in equation (1) to obtain:

$$\frac{kq_3q_4}{(L-x)^2} = \frac{kq_3q_2}{x^2}$$

or, simplifying,

$$\frac{q_4}{(L-x)^2} = \frac{q_2}{x^2}$$

Substituting $q_4 = 2q_2$ and rewriting this equation explicitly as a quadratic equations yields:

$$x^2 + 2Lx - L^2 = 0$$

Use the quadratic formula to obtain:

$$x = \frac{-2L \pm \sqrt{4L^2 + 4L^2}}{2} = -L \pm \sqrt{2}L$$

The root corresponding to the negative sign between the terms is extraneous because it corresponds to a position to the left of the 2.0 μC point charge and is, therefore, not a physically meaningful root. Hence the third point charge should be placed between the point charges and a distance equal to $0.41L$ away from the 2.0-μC charge.

29 •• A –2.0 μC point charge and a 4.0 μC point charge are a distance L apart. Where should a third point charge be placed so that the electric force on that third charge is zero?

Picture the Problem The third point charge should be placed at the location at which the forces on the third point charge due to each of the other two point charges cancel. There can be no such place between the two point charges. Beyond the 4.0 μC point charge, and on the line containing the two point charges, the force due to the 4.0 μC point charge overwhelms the force due to the –2.0 μC point charge. Beyond the –2.0 μC point charge, and on the line containing the two

point charges, however, we can find a place where these forces cancel because they are equal in magnitude and oppositely directed. Denote the –2.0 μC and 4.0 μC point charges by the numerals 2 and 4, respectively, and the third point charge by the numeral 3. Let the +x direction be to the right with the origin at the position of the –2.0 μC point charge and the 4.0 μC point charge be located at $x = L$.

Apply the condition for translational equilibrium to the third point charge:

$$\vec{F}_{4,3} + \vec{F}_{2,3} = 0$$

or

$$F_{4,3} = F_{2,3} \qquad (1)$$

Letting the distance from the third point charge to the 2.0 μC point charge be x, express the force that the 4.0 μC point charge exerts on the third point charge:

$$F_{4,3} = \frac{kq_3q_4}{(L+x)^2}$$

The force that the –2.0 μC point charge exerts on the third point charge is given by:

$$F_{2,3} = \frac{kq_3q_2}{x^2}$$

Substitute for $F_{4,3}$ and $F_{2,3}$ in equation (1) to obtain:

$$\frac{kq_3q_4}{(L+x)^2} = \frac{kq_3q_2}{x^2} \Rightarrow \frac{q_4}{(L+x)^2} = \frac{q_2}{x^2}$$

Substituting $q_4 = 2q_2$ and rewriting this equation explicitly as a quadratic equations yields:

$$x^2 - 2Lx - L^2 = 0$$

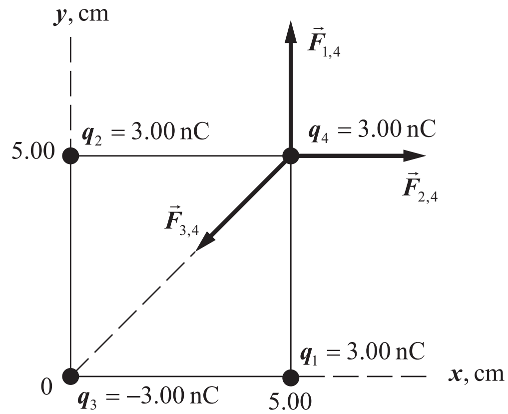
Use the quadratic formula to obtain:

$$x = \frac{2L \pm \sqrt{4L^2 + 4L^2}}{2} = L \pm \sqrt{2}L$$

The root corresponding to the positive sign between the terms is extraneous because it corresponds to a position to the right of the 2.0 μC point charge and is, therefore, not a physically meaningful root. Hence the third point charge should be placed a distance equal to 0.41L from the –2.0-μC charge on the side away from the 4.0-μC charge.

30 •• Three point charges, each of magnitude 3.00 nC, are at separate corners of a square of edge length 5.00 cm. The two point charges at opposite corners are positive, and the third point charge is negative. Find the electric force exerted by these point charges on a fourth point charge q_4 = +3.00 nC at the remaining corner.

Picture the Problem The configuration of the point charges and the forces on the fourth point charge are shown in the figure … as is a coordinate system. From the figure it is evident that the net force on the point charge q_4 is along the diagonal of the square and directed away from point charge q_3. We can apply Coulomb's law to express $\vec{F}_{1,4}$, $\vec{F}_{2,4}$ and $\vec{F}_{3,4}$ and then add them (that is, use the principle of superposition of forces) to find the net electric force on point charge q_4.



Express the net force acting on point charge q_4:

$$\vec{F}_4 = \vec{F}_{1,4} + \vec{F}_{2,4} + \vec{F}_{3,4} \quad (1)$$

Express the force that point charge q_1 exerts on point charge q_4:

$$\vec{F}_{1,4} = \frac{kq_1q_4}{r_{1,4}^2}\hat{j}$$

Substitute numerical values and evaluate $\vec{F}_{1,4}$:

$$\vec{F}_{1,4} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(3.00\,\text{nC})\left(\frac{3.00\,\text{nC}}{(0.0500\,\text{m})^2}\right)\hat{j} = \left(3.236\times10^{-5}\,\text{N}\right)\hat{j}$$

Express the force that point charge q_2 exerts on point charge q_4:

$$\vec{F}_{2,4} = \frac{kq_2q_4}{r_{2,4}^2}\hat{i}$$

Substitute numerical values and evaluate $\vec{F}_{2,4}$:

$$\vec{F}_{2,4} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(3.00\,\text{nC})\left(\frac{3.00\,\text{nC}}{(0.0500\,\text{m})^2}\right)\hat{i} = \left(3.236\times10^{-5}\,\text{N}\right)\hat{i}$$

Express the force that point charge q_3 exerts on point charge q_4:

$$\vec{F}_{3,4} = \frac{kq_3q_4}{r_{3,4}^2}\hat{r}_{3,4}$$, where $\hat{r}_{3,4}$ is a unit vector pointing from q_3 to q_4.

Express $\vec{r}_{3,4}$ in terms of $\vec{r}_{3,1}$ and $\vec{r}_{1,4}$:

$$\begin{aligned}\vec{r}_{3,4} &= \vec{r}_{3,1} + \vec{r}_{1,4}\\ &= (0.0500\,\text{m})\hat{i} + (0.0500\,\text{m})\hat{j}\end{aligned}$$

Convert $\vec{r}_{3,4}$ to $\hat{r}_{3,4}$:

$$\hat{r}_{3,4} = \frac{\vec{r}_{3,4}}{|\vec{r}_{3,4}|} = \frac{(0.0500\,\text{m})\hat{i} + (0.0500\,\text{m})\hat{j}}{\sqrt{(0.0500\,\text{m})^2 + (0.0500\,\text{m})^2}} = 0.707\,\hat{i} + 0.707\,\hat{j}$$

Substitute numerical values and evaluate $\vec{F}_{3,4}$:

$$\begin{aligned}\vec{F}_{3,4} &= \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(-3.00\,\text{nC})\left(\frac{3.00\,\text{nC}}{\left(0.0500\sqrt{2}\,\text{m}\right)^2}\right)\left(0.707\,\hat{i} + 0.707\,\hat{j}\right)\\ &= -\left(1.14\times10^{-5}\,\text{N}\right)\hat{i} - \left(1.14\times10^{-5}\,\text{N}\right)\hat{j}\end{aligned}$$

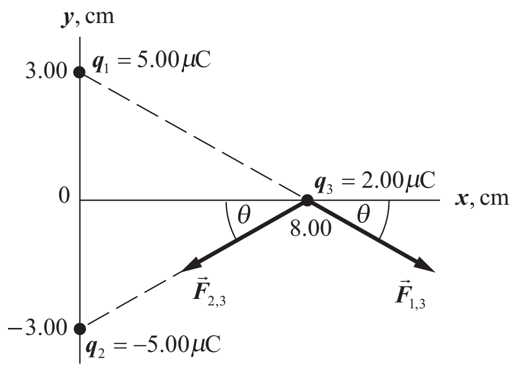
Substitute numerical values in equation (1) and simplify to find $\vec{F}_4$:

$$\begin{aligned}\vec{F}_4 &= \left(3.24\times10^{-5}\,\text{N}\right)\hat{j} + \left(3.24\times10^{-5}\,\text{N}\right)\hat{i} - \left(1.14\times10^{-5}\,\text{N}\right)\hat{i} - \left(1.14\times10^{-5}\,\text{N}\right)\hat{j}\\ &= \boxed{\left(2.10\times10^{-5}\,\text{N}\right)\hat{i} + \left(2.10\times10^{-5}\,\text{N}\right)\hat{j}}\end{aligned}$$

This result tells us that the net force is 2.97×10^{-5} N along the diagonal in the direction away from the −3.0 nC charge.

31 •• A point charge of 5.00 μC is on the y axis at y = 3.00 cm, and a second point charge of –5.00 μC is on the y axis at y = –3.00 cm. Find the electric force on a point charge of 2.00 μC on the x axis at x = 8.00 cm.

Picture the Problem The configuration of the point charges and the forces on point charge q_3 are shown in the figure … as is a coordinate system. From the geometry of the charge distribution it is evident that the net force on the 2.00 μC point charge is in the negative y direction. We can apply Coulomb's law to express $\vec{F}_{1,3}$ and $\vec{F}_{2,3}$ and then add them (that is, use the principle of superposition of forces) to find the net force on point charge q_3.



The net force acting on point charge q_3 is given by:

$$\vec{F}_3 = \vec{F}_{1,3} + \vec{F}_{2,3}$$

The force that point charge q_1 exerts on point charge q_3 is:

$$\vec{F}_{1,3} = F\cos\theta\,\hat{i} - F\sin\theta\,\hat{j}$$

F is given by:

$$F = \frac{kq_1q_3}{r^2} = \frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(5.00\ \mu\text{C})(2.00\ \mu\text{C})}{(0.0300\ \text{m})^2 + (0.0800\ \text{m})^2} = 12.32\ \text{N}$$

and

$$\theta = \tan^{-1}\left(\frac{3.00\ \text{cm}}{8.00\ \text{cm}}\right) = 20.56°$$

The force that point charge q_2 exerts on point charge q_3 is:

$$\vec{F}_{2,3} = -F\cos\theta\,\hat{i} - F\sin\theta\,\hat{j}$$

Substitute for $\vec{F}_{1,3}$ and $\vec{F}_{2,3}$ and simplify to obtain:

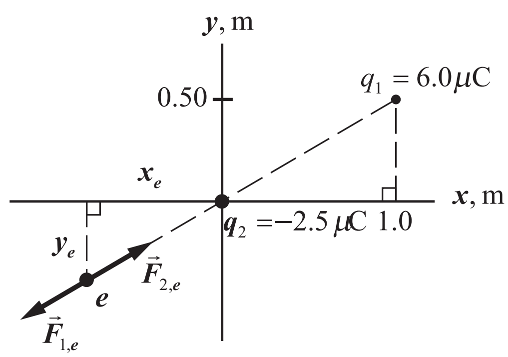
$$\begin{aligned}\vec{F}_3 &= F\cos\theta\,\hat{i} - F\sin\theta\,\hat{j} - F\cos\theta\,\hat{i} \\ &\quad - F\sin\theta\,\hat{j} \\ &= -2F\sin\theta\,\hat{j}\end{aligned}$$

Substitute numerical values and evaluate $\vec{F}_3$:

$$\begin{aligned}\vec{F}_3 &= -2(12.32\ \text{N})\sin 20.56°\,\hat{j} \\ &= \boxed{-(8.65\ \text{N})\hat{j}}\end{aligned}$$

32 •• A point particle that has a charge of –2.5 μC is located at the origin. A second point particle that has a charge of 6.0 μC is at x = 1.0 m, y = 0.50 m. A third point particle, and electron, is at a point with coordinates (x, y). Find the values of x and y such that the electron is in equilibrium.

Picture the Problem The positions of the point particles are shown in the diagram. It is apparent that the electron must be located along the line joining the two point particles. Moreover, because it is negatively charged, it must be closer to the particle with a charge of –2.5 μC than to the particle with a charge of 6.0 μC, as is indicated in the figure. We can find the x and y coordinates of the electron's position by equating the two electrostatic forces acting on it and solving for its distance from the origin. We can use similar triangles to express this radial distance in terms of the x and y coordinates of the electron.



Express the condition that must be satisfied if the electron is to be in equilibrium:

$$F_{1,e} = F_{2,e}$$

Letting r represent the distance from the origin to the electron, express the magnitude of the force that the particle whose charge is q_1 exerts on the electron:

$$F_{1,e} = \frac{kq_1e}{\left(r + \sqrt{1.25\,\text{m}}\right)^2}$$

Express the magnitude of the force that the particle whose charge is q_2 exerts on the electron:

$$F_{2,e} = \frac{k|q_2|e}{r^2}$$

Substitute and simplify to obtain:

$$\frac{q_1}{\left(r + \sqrt{1.25\,\text{m}}\right)^2} = \frac{|q_2|}{r^2}$$

Substitute for q_1 and q_2 and simplify to obtain:

$$\frac{6}{\left(r+\sqrt{1.25\,\text{m}}\right)^2}=\frac{2.5}{r^2}$$

Solving for r yields:

$r = 2.036\,\text{m}$

and

$r = -0.4386\,\text{m}$

Because $r < 0$ is unphysical, we'll consider only the positive root.

Use the similar triangles in the diagram to establish the proportion involving the y coordinate of the electron:

$$\frac{y_e}{0.50\,\text{m}}=\frac{2.036\,\text{m}}{1.12\,\text{m}} \Rightarrow y_e = 0.909\ \text{m}$$

Use the similar triangles in the diagram to establish the proportion involving the x coordinate of the electron:

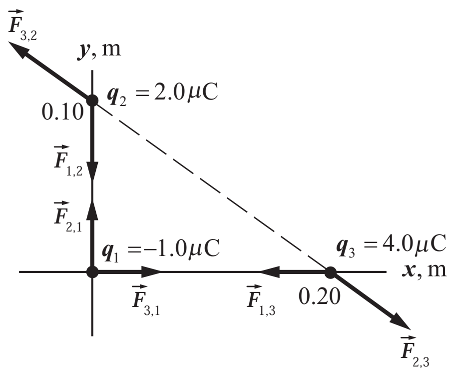
$$\frac{x_e}{1.0\ \text{m}}=\frac{2.036\,\text{m}}{1.12\,\text{m}} \Rightarrow x_e = 1.82\ \text{m}$$

The coordinates of the electron's position are:

$$(x_e, y_e) = \boxed{(-1.8\,\text{m}, -0.91\,\text{m})}$$

33 •• A point particle that has a charge of –1.0 μC is located at the origin; a second point particle that has a charge of 2.0 μC is located at $x = 0$, $y = 0.10$ m; and a third point particle that has a charge of 4.0 μC is located at $x = 0.20$ m, $y = 0$. Find the total electric force on each of the three point charges.

Picture the Problem Let q_1 represent the charge of the point particle at the origin, q_2 the charge of the point particle at (0, 0.10 m), and q_3 the charge of the point particle at (0.20 m, 0). The following diagram shows the forces acting on each of the point particles. Note the action-and-reaction pairs. We can apply Coulomb's law and the principle of superposition of forces to find the net force acting on each of the point particles.



Express the net force acting on the point particle whose charge is q_1:

$$\vec{F}_1 = \vec{F}_{2,1} + \vec{F}_{3,1}$$

Express the force that the point particle whose charge is q_2 exerts on the point particle whose charge is q_1:

$$\vec{F}_{2,1} = \frac{kq_2q_1}{r_{2,1}^2}\hat{r}_{2,1} = \frac{kq_2q_1}{r_{2,1}^2}\frac{\vec{r}_{2,1}}{r_{2,1}} = \frac{kq_2q_1}{r_{2,1}^3}\vec{r}_{2,1}$$

Substitute numerical values and evaluate $\vec{F}_{2,1}$:

$$\vec{F}_{2,1} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(2.0\,\mu\text{C})\frac{(-1.0\,\mu\text{C})}{(0.10\,\text{m})^3}(-0.10\,\text{m})\hat{j} = (1.80\,\text{N})\hat{j}$$

Express the force that the point particle whose charge is q_3 exerts on the point particle whose charge is q_1:

$$\vec{F}_{3,1} = \frac{kq_3q_1}{r_{3,1}^3}\vec{r}_{3,1}$$

Substitute numerical values and evaluate $\vec{F}_{3,1}$:

$$\vec{F}_{3,1} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(4.0\,\mu\text{C})\frac{(-1.0\,\mu\text{C})}{(0.20\,\text{m})^3}(-0.20\,\text{m})\hat{i} = (0.899\,\text{N})\hat{i}$$

Substitute to find $\vec{F}_1$:

$$\vec{F}_1 = \boxed{(0.90\,\text{N})\hat{i} + (1.8\,\text{N})\hat{j}}$$

Express the net force acting on the point particle whose charge is q_2:

$$\vec{F}_2 = \vec{F}_{3,2} + \vec{F}_{1,2} = \vec{F}_{3,2} - \vec{F}_{2,1} = \vec{F}_{3,2} - (1.80\,\text{N})\hat{j}$$

because $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$ are action-and-reaction forces.

Express the force that the point particle whose charge is q_3 exerts on the point particle whose charge is q_2:

$$\vec{F}_{3,2} = \frac{kq_3q_2}{r_{3,2}^3}\vec{r}_{3,2} = \frac{kq_3q_2}{r_{3,2}^3}\left[(-0.20\,\text{m})\hat{i} + (0.10\,\text{m})\hat{j}\right]$$

Substitute numerical values and evaluate $\vec{F}_{3,2}$:

$$\vec{F}_{3,2} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(4.0\,\mu\text{C})\frac{(2.0\,\mu\text{C})}{(0.224\,\text{m})^3}\left[(-0.20\,\text{m})\hat{i} + (0.10\,\text{m})\hat{j}\right]$$
$$= (-1.28\,\text{N})\hat{i} + (0.640\,\text{N})\hat{j}$$

Find the net force acting on the point particle whose charge is q_2:

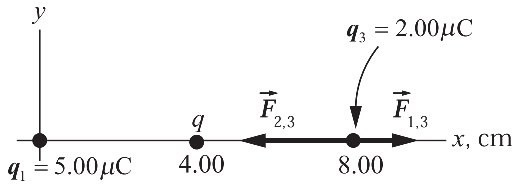
$$\vec{F}_2 = \vec{F}_{3,2} - (1.80\,\text{N})\hat{j} = (-1.28\,\text{N})\hat{i} + (0.640\,\text{N})\hat{j} - (1.80\,\text{N})\hat{j}$$
$$= \boxed{(-1.3\,\text{N})\hat{i} - (1.2\,\text{N})\hat{j}}$$

Noting that $\vec{F}_{1,3}$ and $\vec{F}_{3,1}$ are an action-and-reaction pair, as are $\vec{F}_{2,3}$ and $\vec{F}_{3,2}$, express the net force acting on the point particle whose charge is q_3:

$$\vec{F}_3 = \vec{F}_{1,3} + \vec{F}_{2,3} = -\vec{F}_{3,1} - \vec{F}_{3,2} = -(0.899\,\text{N})\hat{i} - \left[(-1.28\,\text{N})\hat{i} + (0.640\,\text{N})\hat{j}\right]$$
$$= \boxed{(0.4\,\text{N})\hat{i} - (0.64\,\text{N})\hat{j}}$$

34 •• A point particle that has a charge of 5.00 μC is located at $x = 0$, $y = 0$ and a point particle that has a charge q is located at $x = 4.00$ cm, $y = 0$. The electric force on a point particle that has a charge of 2.00 μC at $x = 8.00$ cm, $y = 0$ is $-(19.7\text{ N})\hat{i}$. When this 2.00-μC charge is repositioned at $x = 17.8$ cm, $y = 0$, the electric force on it is zero. Determine the charge q.

Picture the Problem Let q_1 represent the charge of the point particle at the origin and q_3 the charge of the point particle initially at (8.00 cm, 0) and later positioned at (17.8 cm, 0). The diagram shows the forces acting on the point particle whose charge is q_3 at (8.00 cm, 0). We can apply Coulomb's law and the principle of superposition of forces to find the net force acting on each of the point particles.



Express the net force on the point particle whose charge is q_2 when it is at (8.00 cm, 0):

$$\vec{F}_2(8.00\,\text{cm},0) = \vec{F}_{1,3} + \vec{F}_{2,3} = \frac{kq_1q_3}{r_{1,3}^3}\vec{r}_{1,3} + \frac{kqq_3}{r_{2,3}^3}\vec{r}_{2,3} = kq_3\left(\frac{q_1}{r_{1,3}^3}\vec{r}_{1,3} + \frac{q}{r_{2,3}^3}\vec{r}_{2,3}\right)$$

Substitute numerical values to obtain:

$$(-19.7\,\text{N})\hat{\boldsymbol{i}} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(2.00\,\mu\text{C}) \times\left[\frac{5.00\,\mu\text{C}}{(0.0800\,\text{m})^3}(0.0800\,\text{m})\hat{\boldsymbol{i}} + \frac{q}{(0.0400\,\text{m})^3}(0.0400\,\text{m})\hat{\boldsymbol{i}}\right]$$

Solving for q yields:

$$q = \boxed{-3.00\,\mu\text{C}}$$

Remarks: An alternative approach is to equate the electrostatic forces acting on q_2 when it is at (17.8 cm, 0).

35 ••• [SSM] Five identical point charges, each having charge Q, are equally spaced on a semicircle of radius R as shown in Figure 21-37. Find the force (in terms of k, Q, and R) on a charge q located equidistant from the five other charges.

Picture the Problem By considering the symmetry of the array of charged point particles, we can see that the y component of the force on q is zero. We can apply Coulomb's law and the principle of superposition of forces to find the net force acting on q.

Express the net force acting on the point charge q:

$$\vec{F}_q = \vec{F}_{Q\text{ on }x\text{ axis},q} + 2\vec{F}_{Q\text{ at }45°,q}$$

Express the force on point charge q due to the point charge Q on the x axis:

$$\vec{F}_{Q\text{ on }x\text{ axis},q} = \frac{kqQ}{R^2}\hat{\boldsymbol{i}}$$

Express the net force on point charge q due to the point charges at 45°:

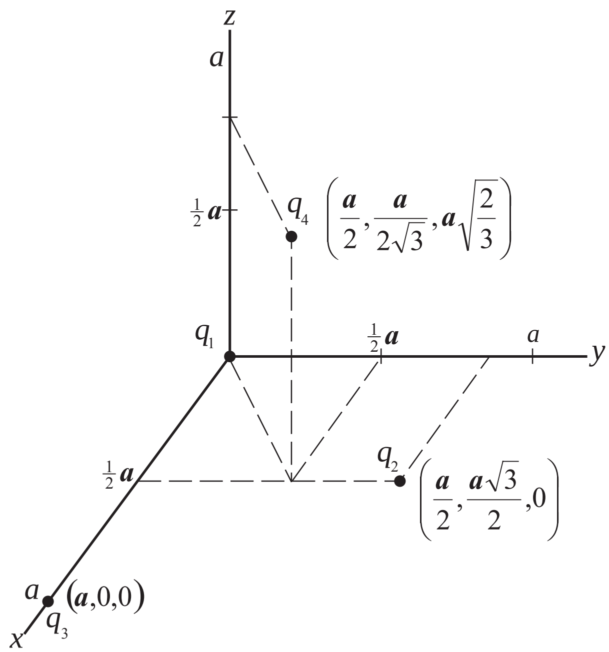
$$2\vec{F}_{Q\text{ at }45°,q} = 2\frac{kqQ}{R^2}\cos 45°\hat{\boldsymbol{i}} = \frac{2}{\sqrt{2}}\frac{kqQ}{R^2}\hat{\boldsymbol{i}}$$

Substitute for $\vec{F}_{Q\text{ on }x\text{ axis},q}$ and $2\vec{F}_{Q\text{ at }45°,q}$ to obtain:

$$\vec{F}_q = \frac{kqQ}{R^2}\hat{\boldsymbol{i}} + \frac{2}{\sqrt{2}}\frac{kqQ}{R^2}\hat{\boldsymbol{i}} = \boxed{\frac{kqQ}{R^2}\left(1+\sqrt{2}\right)\hat{\boldsymbol{i}}}$$

36 ••• The structure of the NH_3 molecule is approximately that of an equilateral tetrahedron, with three H^+ ions forming the base and an N^{3-} ion at the apex of the tetrahedron. The length of each side is 1.64×10^{-10} m. Calculate the electric force that acts on each ion.

Picture the Problem Let the H^+ ions be in the x-y plane with H_1 at (0, 0, 0), H_2 at (a, 0, 0), and H_3 at $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}, 0\right)$. The N^{-3} ion, with charge q_4 in our notation, is then at $\left(\frac{a}{2}, \frac{a}{2\sqrt{3}}, a\sqrt{\frac{2}{3}}\right)$ where a =1.64 × 10^{-10} m. To simplify our calculations we'll set $ke^2/a^2 = C = 8.56\times10^{-9}\,\text{N}$. We can apply Coulomb's law and the principle of superposition of forces to find the net force acting on each ion.



Express the net force acting on point charge q_1:

$$\vec{F}_1 = \vec{F}_{2,1} + \vec{F}_{3,1} + \vec{F}_{4,1}$$

Find $\vec{F}_{2,1}$:

$$\vec{F}_{2,1} = \frac{kq_1q_2}{r_{2,1}^2}\hat{r}_{2,1} = C\left(-\hat{i}\right) = -C\hat{i}$$

Find $\vec{F}_{3,1}$:

$$\vec{F}_{3,1} = \frac{kq_3q_1}{r_{3,1}^2}\hat{r}_{3,1}$$

$$= C\frac{\left(0-\frac{a}{2}\right)\hat{i} + \left(0-\frac{a\sqrt{3}}{2}\right)\hat{j}}{a}$$

$$= -C\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$$

Noting that the magnitude of point charge q_4 is three times that of the other point charges and that it is negative, express $\vec{F}_{4,1}$:

$$\vec{F}_{4,1} = 3C\hat{r}_{4,1} = -3C\frac{\left(0-\frac{a}{2}\right)\hat{i}+\left(0-\frac{a}{2\sqrt{3}}\right)\hat{j}+\left(0-\frac{a\sqrt{2}}{\sqrt{3}}\right)\hat{k}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2\sqrt{3}}\right)^2+\left(\frac{a\sqrt{2}}{\sqrt{3}}\right)^2}}$$

$$= 3C\frac{\left(\frac{a}{2}\right)\hat{i}+\left(\frac{a}{2\sqrt{3}}\right)\hat{j}+\left(\frac{a\sqrt{2}}{\sqrt{3}}\right)\hat{k}}{a} = 3C\left[\left(\frac{1}{2}\right)\hat{i}+\left(\frac{1}{2\sqrt{3}}\right)\hat{j}+\left(\sqrt{\frac{2}{3}}\right)\hat{k}\right]$$

Substitute in the expression for $\vec{F}_1$ to obtain:

$$\vec{F}_1 = -C\hat{i} - C\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right) + 3C\left[\left(\frac{1}{2}\right)\hat{i}+\left(\frac{1}{2\sqrt{3}}\right)\hat{j}+\left(\sqrt{\frac{2}{3}}\right)\hat{k}\right] = \boxed{C\sqrt{6}\hat{k}}$$

From symmetry considerations:

$$\vec{F}_2 = \vec{F}_3 = \vec{F}_1 = \boxed{C\sqrt{6}\hat{k}}$$

Express the condition that the molecule is in equilibrium:

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0$$

Solve for and evaluate $\vec{F}_4$:

$$\vec{F}_4 = -\left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) = -3\vec{F}_1 = \boxed{-3C\sqrt{6}\hat{k}}$$

The Electric Field

37 • **[SSM]** A point charge of 4.0 μC is at the origin. What are the magnitude and direction of the electric field on the x axis at (*a*) $x = 6.0$ m and (*b*) $x = -10$ m? (*c*) Sketch the function E_x versus x for both positive and negative values of x. (Remember that E_x is negative when $\vec{E}$ points in the $-x$ direction.)

Picture the Problem Let q represent the point charge at the origin and use Coulomb's law for $\vec{E}$ due to a point charge to find the electric field at $x = 6.0$ m and -10 m.

(*a*) Express the electric field at a point P located a distance *x* from a point charge *q*:

$$\vec{E}(x)=\frac{kq}{x^2}\hat{r}_{\mathrm{P},0}$$

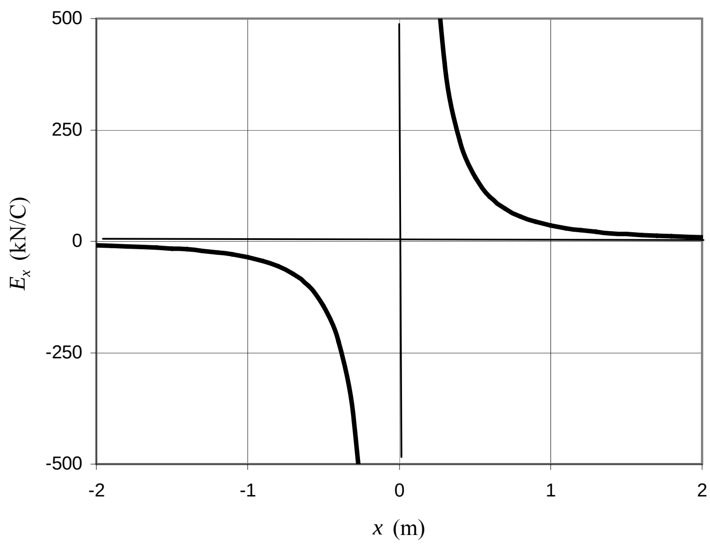
Evaluate this expression for $x = 6.0$ m:

$$\vec{E}(6.0\,\mathrm{m})=\frac{\left(8.988\times10^{9}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)(4.0\,\mu\mathrm{C})}{(6.0\,\mathrm{m})^2}\hat{i}=\boxed{(1.0\,\mathrm{kN/C})\hat{i}}$$

(*b*) Evaluate $\vec{E}$ at $x = -10$ m:

$$\vec{E}(-10\,\mathrm{m})=\frac{\left(8.988\times10^{9}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)(4.0\,\mu\mathrm{C})}{(10\,\mathrm{m})^2}\left(-\hat{i}\right)=\boxed{(-0.36\,\mathrm{kN/C})\hat{i}}$$

(*c*) The following graph was plotted using a spreadsheet program:



38 • Two point charges, each +4.0 μC, are on the *x* axis; one point charge is at the origin and the other is at $x = 8.0$ m. Find the electric field on the *x* axis at (*a*) $x = -2.0$ m, (*b*) $x = 2.0$ m, (*c*) $x = 6.0$ m, and (*d*) $x = 10$ m. (*e*) At what point on the *x* axis is the electric field zero? (*f*) Sketch E_x versus *x* for $-3.0\text{ m} < x < 11$ m.

Picture the Problem Let *q* represent the point charges of +4.0 μC and use Coulomb's law for $\vec{E}$ due to a point charge and the principle of superposition for fields to find the electric field at the locations specified.

Noting that $q_1 = q_2$, use Coulomb's law and the principle of superposition to express the electric field due to the given charges at point P a distance x from the origin:

$$\vec{E}(x) = \vec{E}_{q_1}(x) + \vec{E}_{q_2}(x) = \frac{kq_1}{x^2}\hat{r}_{q_1,P} + \frac{kq_2}{(8.0\,\mathrm{m} - x)^2}\hat{r}_{q_2,P}$$
$$= kq_1\left(\frac{1}{x^2}\hat{r}_{q_1,P} + \frac{1}{(8.0\,\mathrm{m} - x)^2}\hat{r}_{q_2,P}\right)$$
$$= (36\,\mathrm{kN\cdot m^2/C})\left(\frac{1}{x^2}\hat{r}_{q_1,P} + \frac{1}{(8.0\,\mathrm{m} - x)^2}\hat{r}_{q_2,P}\right)$$

(a) Apply this equation to the point at $x = -2.0$ m:

$$\vec{E}(-2.0\,\mathrm{m}) = (36\,\mathrm{kN\cdot m^2/C})\left[\frac{1}{(2.0\,\mathrm{m})^2}(-\hat{i}) + \frac{1}{(10\,\mathrm{m})^2}(-\hat{i})\right] = \boxed{(-9.4\,\mathrm{kN/C})\hat{i}}$$

(b) Evaluate $\vec{E}$ at $x = 2.0$ m:

$$\vec{E}(2.0\,\mathrm{m}) = (36\,\mathrm{kN\cdot m^2/C})\left[\frac{1}{(2.0\,\mathrm{m})^2}(\hat{i}) + \frac{1}{(6.0\,\mathrm{m})^2}(-\hat{i})\right] = \boxed{(8.0\,\mathrm{kN/C})\hat{i}}$$

(c) Evaluate $\vec{E}$ at $x = 6.0$ m:

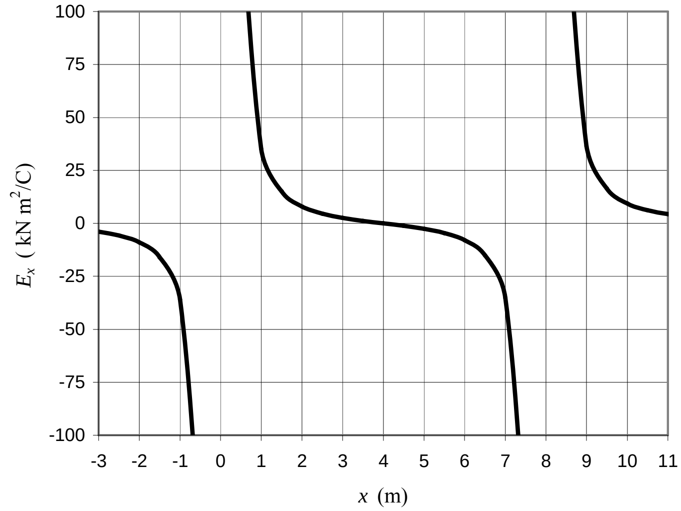
$$\vec{E}(6.0\,\mathrm{m}) = (36\,\mathrm{kN\cdot m^2/C})\left[\frac{1}{(6.0\,\mathrm{m})^2}(\hat{i}) + \frac{1}{(2.0\,\mathrm{m})^2}(-\hat{i})\right] = \boxed{(-8.0\,\mathrm{kN/C})\hat{i}}$$

(d) Evaluate $\vec{E}$ at $x = 10$ m:

$$\vec{E}(10\,\mathrm{m}) = (36\,\mathrm{kN\cdot m^2/C})\left[\frac{1}{(10\,\mathrm{m})^2}(\hat{i}) + \frac{1}{(2.0\,\mathrm{m})^2}(\hat{i})\right] = \boxed{(9.4\,\mathrm{kN/C})\hat{i}}$$

(e) From symmetry considerations: $\vec{E}(4.0\,\mathrm{m}) = \boxed{0}$

(*f*) The following graph was plotted using a spreadsheet program:



39 • When a 2.0-nC point charge is placed at the origin, it experiences an electric force of 8.0×10^{-4} N in the $+y$ direction. (*a*) What is the electric field at the origin? (*b*) What would be the electric force on a –4.0-nC point charge placed at the origin? (*c*) If this force is due to the electric field of a point charge on the *y* axis at *y* = 3.0 cm, what is the value of that charge?

Comment: Is the force related to q_0? 8.0×10^{-4} *N*?

Picture the Problem We can find the electric field at the origin from its definition and the electric force on a point charge placed there using $\vec{F} = q\vec{E}$. We can apply Coulomb's law to find the value of the point charge placed at y = 3 cm.

(*a*) Apply the definition of electric field to obtain:

$$\vec{E}(0,0) = \frac{\vec{F}(0,0)}{q_0} = \frac{\left(8.0\times10^{-4}\text{ N}\right)\hat{j}}{2.0\,\text{nC}}$$
$$= \boxed{\left(4.0\times10^{5}\text{ N/C}\right)\hat{j}}$$

(*b*) The force on a point charge in an electric field is given by:

$$\vec{F}(0,0) = q\vec{E}(0,0)$$
$$= (-4.0\,\text{nC})(400\,\text{kN/C})\hat{j}$$
$$= \boxed{(-1.6\,\text{mN})\hat{j}}$$

(*c*) Apply Coulomb's law to obtain:

$$\frac{kq(-4.0\,\text{nC})}{(0.030\,\text{m})^2}\left(-\hat{j}\right) = (-1.60\,\text{mN})\hat{j}$$

Solving for q yields:

$$q = -\frac{(1.60\,\text{mN})(0.030\,\text{m})^2}{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(4.0\,\text{nC})} = \boxed{-40\,\text{nC}}$$

40 • The electric field near the surface of Earth points downward and has a magnitude of 150 N/C. (*a*) Compare the magnitude of the upward electric force on an electron with the magnitude of the gravitational force on the electron. (*b*) What charge should be placed on a ping pong ball of mass 2.70 g so that the electric force balances the weight of the ball near Earth's surface?

Comment: DAVID: IN 48 the penny has been replaced by a ping pong ball

Picture the Problem We can compare the electric and gravitational forces acting on an electron by expressing their ratio. Because the ping pong ball is in equilibrium under the influence of the electric and gravitational forces acting on it, we can use the condition for translational equilibrium to find the charge that would have to be placed on it in order to balance Earth's gravitational force on it.

(*a*) Express the magnitude of the electric force acting on the electron:

$$F_e = eE$$

Express the magnitude of the gravitational force acting on the electron:

$$F_g = m_e g$$

The ratio of these forces is:

$$\frac{F_e}{F_g} = \frac{eE}{mg}$$

Substitute numerical values and evaluate F_e/F_g:

$$\frac{F_e}{F_g} = \frac{(1.602\times10^{-19}\,\text{C})(150\,\text{N/C})}{(9.109\times10^{-31}\,\text{kg})(9.81\,\text{m/s}^2)} = 2.69\times10^{12}$$

or

$$F_e = \boxed{(2.69\times10^{12})F_g}$$

Thus, the electric force is greater by a factor of 2.69×10^{12}.

(*b*) Letting the upward direction be positive, apply the condition for static equilibrium to the ping pong ball to obtain:

$$F_e - F_g = 0$$

or

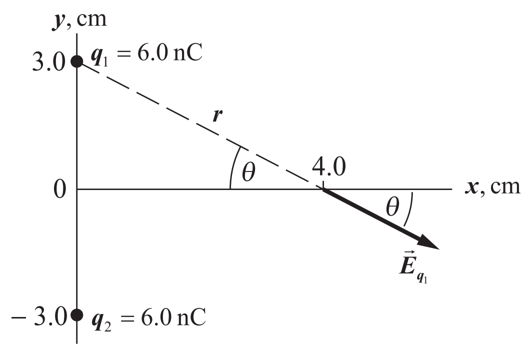
$$-qE - mg = 0 \Rightarrow q = -\frac{mg}{E}$$

Substitute numerical values and evaluate q:

$$q = -\frac{(2.70\times10^{-3}\text{ kg})(9.81\text{m/s}^2)}{150\text{ N/C}} = \boxed{-0.177\text{ mC}}$$

41 •• [SSM] Two point charges q_1 and q_2 both have a charge equal to +6.0 nC and are on the y axis at y_1 = +3.0 cm and y_2 = –3.0 cm respectively. (a) What is the magnitude and direction of the electric field on the x axis at x = 4.0 cm? (b) What is the force exerted on a third charge q_0 = 2.0 nC when it is placed on the x axis at x = 4.0 cm?

Picture the Problem The diagram shows the locations of the point charges q_1 and q_2 and the point on the x axis at which we are to find $\vec{E}$. From symmetry considerations we can conclude that the y component of $\vec{E}$ at any point on the x axis is zero. We can use Coulomb's law for the electric field due to point charges and the principle of superposition for fields to find the field at any point on the x axis and $\vec{F} = q\vec{E}$ to find the force on a point charge q_0 placed on the x axis at x = 4.0 cm.



(a) Letting $q = q_1 = q_2$, express the x component of the electric field due to one point charge as a function of the distance r from either point charge to the point of interest:

$$\vec{E}_x = \frac{kq}{r^2}\cos\theta\,\hat{\boldsymbol{i}}$$

Express $\vec{E}_x$ for both charges:

$$\vec{E}_x = 2\frac{kq}{r^2}\cos\theta\,\hat{\boldsymbol{i}}$$

Substitute for $\cos\theta$ and r, substitute numerical values, and evaluate to obtain:

$$\begin{aligned}\vec{E}(4.0\text{ cm})_x &= 2\frac{kq}{r^2}\frac{0.040\text{ m}}{r}\hat{\boldsymbol{i}} = \frac{2kq(0.040\text{ m})}{r^3}\hat{\boldsymbol{i}} \\ &= \frac{2\left(8.988\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2\right)(6.0\text{ nC})(0.040\text{ m})}{\left[(0.030\text{ m})^2+(0.040\text{ m})^2\right]^{3/2}}\hat{\boldsymbol{i}} \\ &= (34.5\text{ kN/C})\hat{i} = (35\text{ kN/C})\hat{\boldsymbol{i}}\end{aligned}$$

The magnitude and direction of the electric field at $x = 4.0$ cm is:

$$\boxed{35\text{ kN/C @ }0°}$$

(*b*) Apply $\vec{F} = q\vec{E}$ to find the force on a point charge q_0 placed on the x axis at $x = 4.0$ cm:

$$\begin{aligned}\vec{F} &= (2.0\text{ nC})(34.5\text{ kN/C})\hat{\boldsymbol{i}} \\ &= \boxed{(69\ \mu\text{N})\hat{\boldsymbol{i}}}\end{aligned}$$

42 •• A point charge of +5.0 μC is located on the x axis at $x = -3.0$ cm, and a second point charge of –8.0 μC is located on the x axi at $x = +4.0$ cm. Where should a third charge of +6.0 μC be placed so that the electric field at the origin is zero?

Picture the Problem If the electric field at $x = 0$ is zero, both its x and y components must be zero. The only way this condition can be satisfied with point charges of +5.0 μC and −8.0 μC on the x axis is if the point charge +6.0 μC is also on the x axis. Let the subscripts 5, −8, and 6 identify the point charges and their fields. We can use Coulomb's law for $\vec{E}$ due to a point charge and the principle of superposition for fields to determine where the +6.0 μC point charge should be located so that the electric field at $x = 0$ is zero.

Express the electric field at $x = 0$ in terms of the fields due to the point charges of +5.0 μC, −8.0 μC, and +6.0 μC:

$$\begin{aligned}\vec{E}(0) &= \vec{E}_{5\,\mu\text{C}} + \vec{E}_{-8\,\mu\text{C}} + \vec{E}_{6\,\mu\text{C}} \\ &= 0\end{aligned}$$

Substitute for each of the fields to obtain:

$$\frac{kq_5}{r_5^2}\hat{\boldsymbol{r}}_5 + \frac{kq_6}{r_6^2}\hat{\boldsymbol{r}}_6 + \frac{kq_{-8}}{r_{-8}^2}\hat{\boldsymbol{r}}_{-8} = 0$$

or

$$\frac{kq_5}{r_5^2}\hat{\boldsymbol{i}} + \frac{kq_6}{r_6^2}\left(-\hat{\boldsymbol{i}}\right) + \frac{kq_{-8}}{r_{-8}^2}\left(-\hat{\boldsymbol{i}}\right) = 0$$

Divide out the unit vector $\hat{\boldsymbol{i}}$ to obtain:

$$\frac{q_5}{r_5^2}-\frac{q_6}{r_6^2}-\frac{q_{-8}}{r_{-8}^2}=0$$

Substitute numerical values to obtain:

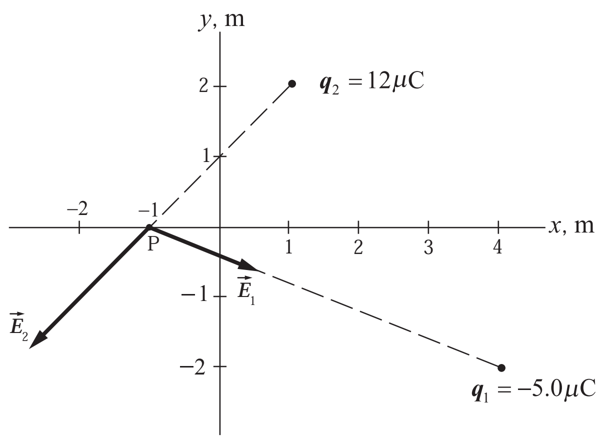
$$\frac{5}{(3.0\,\text{cm})^2}-\frac{6}{r_6^2}-\frac{-8}{(4.0\,\text{cm})^2}=0$$

Solving for r_6 yields:

$$r_6=\boxed{2.4\,\text{cm}}$$

43 •• A –5.0-μC point charge is located at x = 4.0 m, y = –2.0 m and a 12-μC point charge is located at x = 1.0 m, y = 2.0 m. (*a*) Find the magnitude and direction of the electric field at x = –1.0 m, y = 0. (*b*) Calculate the magnitude and direction of the electric force on an electron that is placed at x = –1.0 m, y = 0.

Picture the Problem The diagram shows the electric field vectors at the point of interest P due to the two point charges. We can use Coulomb's law for $\vec{E}$ due to point charges and the superposition principle for electric fields to find $\vec{E}_\text{P}$. We can apply $\vec{F}=q\vec{E}$ to find the force on an electron at (–1.0 m, 0).



(*a*) Express the electric field at (–1.0 m, 0) due to the point charges q_1 and q_2:

$$\vec{E}_\text{P}=\vec{E}_1+\vec{E}_2$$

Substitute numerical values and evaluate $\vec{E}_1$:

$$\vec{E}_1 = \frac{kq_1}{r_{1,\mathrm{P}}^2}\hat{r}_{1,\mathrm{P}} = \frac{(8.988\times10^9\ \mathrm{N\cdot m^2/C^2})(-5.0\,\mu\mathrm{C})}{(5.0\,\mathrm{m})^2+(2.0\,\mathrm{m})^2}\left(\frac{(-5.0\,\mathrm{m})\hat{i}+(2.0\,\mathrm{m})\hat{j}}{\sqrt{(5.0\,\mathrm{m})^2+(2.0\,\mathrm{m})^2}}\right)$$
$$= (-1.55\times10^3\ \mathrm{N/C})(-0.928\,\hat{i}+0.371\,\hat{j})$$
$$= (1.44\,\mathrm{kN/C})\hat{i} + (-0.575\,\mathrm{kN/C})\hat{j}$$

Substitute numerical values and evaluate $\vec{E}_2$:

$$\vec{E}_2 = \frac{kq_2}{r_{2,\mathrm{P}}^2}\hat{r}_{2,\mathrm{P}} = \frac{(8.988\times10^9\ \mathrm{N\cdot m^2/C^2})(12\,\mu\mathrm{C})}{(2.0\,\mathrm{m})^2+(2.0\,\mathrm{m})^2}\left(\frac{(-2.0\,\mathrm{m})\hat{i}+(-2.0\,\mathrm{m})\hat{j}}{\sqrt{(2.0\,\mathrm{m})^2+(2.0\,\mathrm{m})^2}}\right)$$
$$= (13.5\times10^3\ \mathrm{N/C})(-0.707\,\hat{i}-0.707\,\hat{j})$$
$$= (-9.54\,\mathrm{kN/C})\hat{i} + (-9.54\,\mathrm{kN/C})\hat{j}$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ and simplify to find $\vec{E}_\mathrm{P}$:

$$\vec{E}_\mathrm{P} = (1.44\,\mathrm{kN/C})\hat{i} + (-0.575\,\mathrm{kN/C})\hat{j} + (-9.54\,\mathrm{kN/C})\hat{i} + (-9.54\,\mathrm{kN/C})\hat{j}$$
$$= (-8.10\,\mathrm{kN/C})\hat{i} + (-10.1\,\mathrm{kN/C})\hat{j}$$

The magnitude of $\vec{E}_\mathrm{P}$ is:

$$E_\mathrm{P} = \sqrt{(-8.10\,\mathrm{kN/C})^2+(-10.1\,\mathrm{kN/C})^2} = \boxed{13\,\mathrm{kN/C}}$$

The direction of $\vec{E}_\mathrm{P}$ is:

$$\theta_E = \tan^{-1}\left(\frac{-10.1\,\mathrm{kN/C}}{-8.10\,\mathrm{kN/C}}\right) = \boxed{230°}$$

Note that the angle returned by your calculator for $\tan^{-1}\left(\frac{-10.1\,\mathrm{kN/C}}{-8.10\,\mathrm{kN/C}}\right)$ is the reference angle and must be increased by 180° to yield θ_E.

(*b*) Express and evaluate the force on an electron at point P:

$$\vec{F} = q\vec{E}_\mathrm{P} = (-1.602\times10^{-19}\,\mathrm{C})\left[(-8.10\,\mathrm{kN/C})\hat{i} + (-10.1\,\mathrm{kN/C})\hat{j}\right]$$
$$= (1.30\times10^{-15}\,\mathrm{N})\hat{i} + (1.62\times10^{-15}\,\mathrm{N})\hat{j}$$

Find the magnitude of $\vec{F}$:

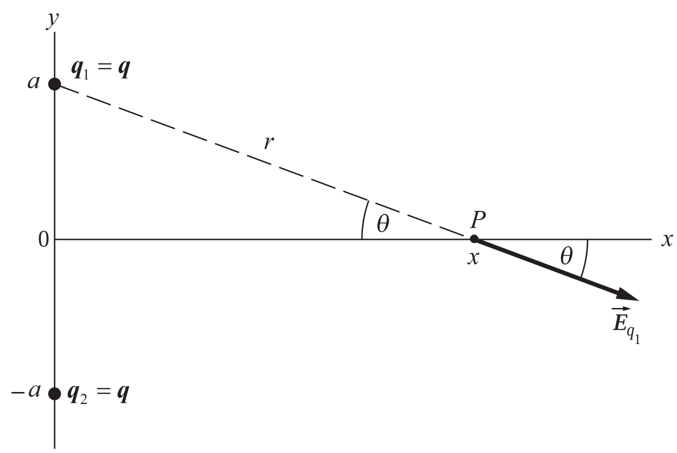
$$F = \sqrt{\left(1.30\times10^{-15}\ \text{N}\right)^2 + \left(1.62\times10^{-15}\ \text{N}\right)^2} = \boxed{2.1\times10^{-15}\ \text{N}}$$

Find the direction of $\vec{F}$:

$$\theta_F = \tan^{-1}\left(\frac{1.62\times10^{-15}\ \text{N}}{1.3\times10^{-15}\ \text{N}}\right) = \boxed{51°}$$

44 •• Two equal positive charges q are on the y axis; one point charge is at $y = +a$ and the other is at $y = -a$. (a) Show that on the x axis the x component of the electric field is given by $E_x = 2kqx/(x^2 + a^2)^{3/2}$. ($b$) Show that near the origin, where x is much smaller than a, $E_x \approx 2kqx/a^3$. (c) Show that for values of x much larger than a, $E_x \approx 2kq/x^2$. Explain why a person might expect this result even without deriving it by taking the appropriate limit.

Picture the Problem The diagram shows the locations of point charges q_1 and q_2 and the point on the x axis at which we are to find $\vec{E}$. From symmetry considerations we can conclude that the y component of $\vec{E}$ at any point on the x axis is zero. We can use Coulomb's law for the electric field due to point charges and the principle of superposition of fields to find the field at any point on the x axis. We can establish the results called for in Parts (b) and (c) by factoring the radicand and using the approximation $1 + \alpha \approx 1$ whenever $\alpha << 1$.



(a) Express the x-component of the electric field due to the point charges at $y = a$ and $y = -a$ as a function of the distance r from either charge to point P:

$$\vec{E}_x = 2\frac{kq}{r^2}\cos\theta\,\hat{\boldsymbol{i}}$$

Substitute for $\cos\theta$ and r to obtain:

$$\vec{E}_x = 2\frac{kq}{r^2}\frac{x}{r}\hat{i} = \frac{2kqx}{r^3}\hat{i} = \frac{2kqx}{\left(x^2+a^2\right)^{3/2}}\hat{i} = \frac{2kqx}{\left(x^2+a^2\right)^{3/2}}\hat{i}$$

The magnitude of $\vec{E}_x$ is:

$$E_x = \boxed{\frac{2kqx}{\left(x^2+a^2\right)^{3/2}}}$$

(*b*) For $|x| << a, x^2 + a^2 \approx a^2$, so:

$$E_x \approx \frac{2kqx}{\left(a^2\right)^{3/2}} = \boxed{\frac{2kqx}{a^3}}$$

(*c*) For $x >> a$, the charges separated by a would appear to be a single charge of magnitude $2q$. Its field would be given by $E_x = \dfrac{2kq}{x^2}$.

Factor the radicand to obtain:

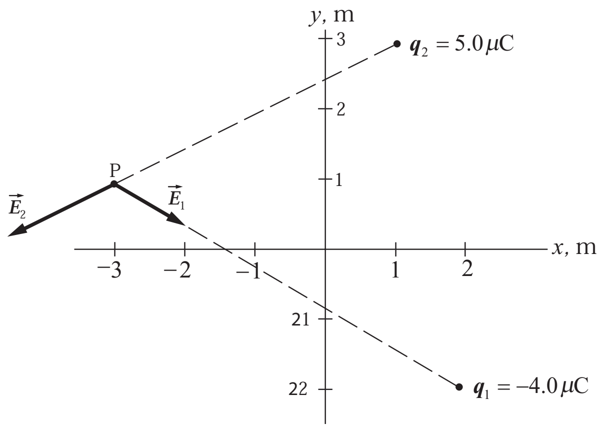
$$E_x = 2kqx\left[x^2\left(1+\frac{a^2}{x^2}\right)\right]^{-3/2}$$

For $a << x$, $1+\dfrac{a^2}{x^2} \approx 1$ and:

$$E_x = 2kqx\left[x^2\right]^{-3/2} = \boxed{\frac{2kq}{x^2}}$$

45 •• A 5.0-μC point charge is located at $x = 1.0$ m, $y = 3.0$ m and a –4.0-μC point charge is located at $x = 2.0$ m, $y = –2.0$ m. (*a*) Find the magnitude and direction of the electric field at $x = –3.0$ m, $y = 1.0$ m. (*b*) Find the magnitude and direction of the force on a proton placed at $x = –3.0$ m, $y = 1.0$ m.

Picture the Problem The diagram shows the electric field vectors at the point of interest P due to the two point charges. We can use Coulomb's law for $\vec{E}$ due to point charges and the superposition principle for electric fields to find $\vec{E}_P$. We can apply $\vec{F} = q\vec{E}$ to find the force on a proton at (–3.0 m, 1.0 m).



(*a*) Express the electric field at (−3.0 m, 1.0 m) due to the point charges q_1 and q_2:

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2$$

Evaluate $\vec{E}_1$:

$$\vec{E}_1 = \frac{kq_1}{r_{1,P}^2}\hat{r}_{1,P} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(-4.0\,\mu\text{C}\right)}{\left(5.0\,\text{m}\right)^2 + \left(3.0\,\text{m}\right)^2}\left(\frac{\left(-5.0\,\text{m}\right)\hat{i} + \left(3.0\,\text{m}\right)\hat{j}}{\sqrt{\left(5.0\,\text{m}\right)^2 + \left(3.0\,\text{m}\right)^2}}\right)$$
$$= \left(-1.06\,\text{kN/C}\right)\left(-0.857\,\hat{i} + 0.514\,\hat{j}\right) = \left(0.907\,\text{kN/C}\right)\hat{i} + \left(-0.544\,\text{kN/C}\right)\hat{j}$$

Evaluate $\vec{E}_2$:

$$\vec{E}_2 = \frac{kq_2}{r_{2,P}^2}\hat{r}_{2,P} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(5.0\,\mu\text{C}\right)}{\left(4.0\,\text{m}\right)^2 + \left(2.0\,\text{m}\right)^2}\left(\frac{\left(-4.0\,\text{m}\right)\hat{i} + \left(-2.0\,\text{m}\right)\hat{j}}{\sqrt{\left(4.0\,\text{m}\right)^2 + \left(2.0\,\text{m}\right)^2}}\right)$$
$$= \left(2.25\,\text{kN/C}\right)\left(-0.894\,\hat{i} - 0.447\,\hat{j}\right) = \left(-2.01\,\text{kN/C}\right)\hat{i} + \left(-1.01\,\text{kN/C}\right)\hat{j}$$

Substitute and simplify to find $\vec{E}_P$:

$$\vec{E}_P = \left(0.908\,\text{kN/C}\right)\hat{i} + \left(-0.544\,\text{kN/C}\right)\hat{j} + \left(-2.01\,\text{kN/C}\right)\hat{i} + \left(-1.01\,\text{kN/C}\right)\hat{j}$$
$$= \left(-1.10\,\text{kN/C}\right)\hat{i} + \left(-1.55\,\text{kN/C}\right)\hat{j}$$

The magnitude of $\vec{E}_P$ is:

$$E_P = \sqrt{\left(1.10\,\text{kN/C}\right)^2 + \left(1.55\,\text{kN/C}\right)^2} = \boxed{1.9\,\text{kN/C}}$$

The direction of $\vec{E}_P$ is:

$$\theta_E = \tan^{-1}\left(\frac{-1.55\,\text{kN/C}}{-1.10\,\text{kN/C}}\right) = \boxed{235°}$$

Note that the angle returned by your calculator for $\tan^{-1}\left(\frac{-1.55\,\text{kN/C}}{-1.10\,\text{kN/C}}\right)$ is the reference angle and must be increased by 180° to yield θ_E.

(*b*) Express and evaluate the force on a proton at point P:

$$\begin{aligned}\vec{F} = q\vec{E}_P &= \left(1.602\times10^{-19}\,\text{C}\right)\left[\left(-1.10\,\text{kN/C}\right)\hat{i} + \left(-1.55\,\text{kN/C}\right)\hat{j}\right]\\ &= \left(-1.76\times10^{-16}\,\text{N}\right)\hat{i} + \left(-2.48\times10^{-16}\,\text{N}\right)\hat{j}\end{aligned}$$

The magnitude of $\vec{F}$ is:

$$F = \sqrt{\left(-1.76\times10^{-16}\,\text{N}\right)^2 + \left(-2.48\times10^{-16}\,\text{N}\right)^2} = \boxed{3.0\times10^{-16}\,\text{N}}$$

The direction of $\vec{F}$ is:

$$\theta_F = \tan^{-1}\left(\frac{-2.48\times10^{-16}\,\text{N}}{-1.76\times10^{-16}\,\text{N}}\right) = \boxed{235°}$$

where, as noted above, the angle returned by your calculator for $\tan^{-1}\left(\frac{-2.48\times10^{-16}\,\text{N}}{-1.76\times10^{-16}\,\text{N}}\right)$ is the reference angle and must be increased by 180° to yield θ_E.

46 •• Two positive point charges, each having a charge Q, are on the y axis, one at $y = +a$ and the other at $y = -a$. (*a*) Show that the electric field strength on the x axis is greatest at the points $x = a/\sqrt{2}$ and $x = -a/\sqrt{2}$ by computing $\partial E_x/\partial x$ and setting the derivative equal to zero. (*b*) Sketch the function E_x versus x using your results for Part (*a*) of this problem and the fact that E_x is approximately $2kqx/a^3$ when x is much smaller than a and E_x is approximately $2kq/x^2$ when x is much larger than a,.

Picture the Problem In Problem 44 it is shown that the electric field on the x axis, due to equal positive charges located at (0, a) and (0,–a), is given by $E_x = 2kqx\left(x^2 + a^2\right)^{-3/2}$. We can identify the locations at which E_x has it greatest values by setting $\partial E_x/\partial x$ equal to zero.

(*a*) Evaluate $\frac{\partial E_x}{\partial x}$ to obtain:

$$\begin{aligned}\frac{\partial E_x}{\partial x} &= \frac{d}{dx}\left[2kqx\left(x^2+a^2\right)^{-3/2}\right] = 2kq\frac{d}{dx}\left[x\left(x^2+a^2\right)^{-3/2}\right]\\ &= 2kq\left[x\frac{d}{dx}\left(x^2+a^2\right)^{-3/2}+\left(x^2+a^2\right)^{-3/2}\right]\\ &= 2kq\left[x\left(-\frac{3}{2}\right)\left(x^2+a^2\right)^{-5/2}(2x)+\left(x^2+a^2\right)^{-3/2}\right]\\ &= 2kq\left[-3x^2\left(x^2+a^2\right)^{-5/2}+\left(x^2+a^2\right)^{-3/2}\right]\end{aligned}$$

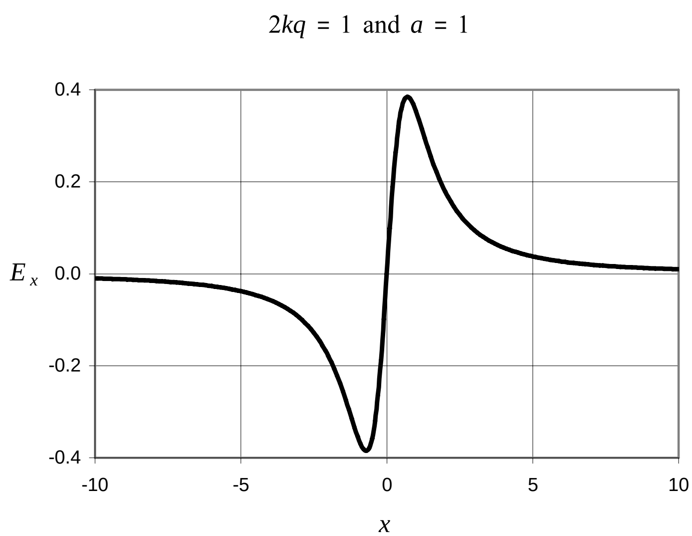
Set this derivative equal to zero for extreme values:

$$-3x^2\left(x^2+a^2\right)^{-5/2}+\left(x^2+a^2\right)^{-3/2}=0$$

Solving for x yields:

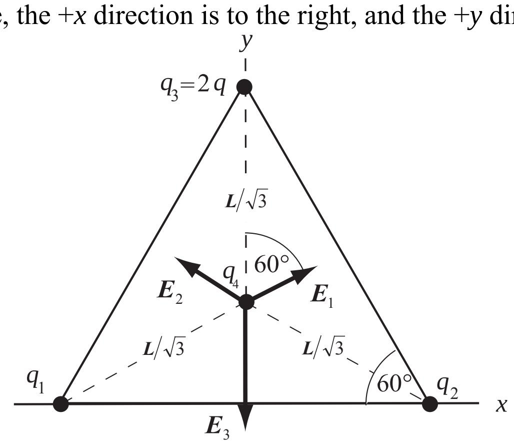
$$x=\boxed{\pm\frac{a}{\sqrt{2}}}$$

(*b*) The following graph was plotted using a spreadsheet program:



47 •• [SSM] Two point particles, each having a charge q, sit on the base of an equilateral triangle that has sides of length L as shown in Figure 21-38. A third point particle that has a charge equal to $2q$ sits at the apex of the triangle. Where must a fourth point particle that has a charge equal to q be placed in order that the electric field at the center of the triangle be zero? (The center is in the plane of the triangle and equidistant from the three vertices.)

Picture the Problem The electric field of the 4th charged point particle must cancel the sum of the electric fields due to the other three charged point particles. By symmetry, the position of the 4th charged point particle must lie on the vertical centerline of the triangle. Using trigonometry, one can show that the center of an equilateral triangle is a distance $L/\sqrt{3}$ from each vertex, where L is the length of the side of the triangle. Note that the x components of the fields due to the base charged particles cancel each other, so we only need concern ourselves with the y components of the fields due to the charged point particles at the vertices of the triangle. Choose a coordinate system in which the origin is at the midpoint of the base of the triangle, the $+x$ direction is to the right, and the $+y$ direction is upward.



Express the condition that must be satisfied if the electric field at the center of the triangle is to be zero:

$$\sum_{\text{i=1 to 4}} \vec{E}_{\text{i}} = 0$$

Substituting for $\vec{E}_1$, $\vec{E}_2$, $\vec{E}_3$, and $\vec{E}_4$ yields:

$$\frac{k(q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\cos 60°\,\hat{\boldsymbol{j}} + \frac{k(q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\cos 60°\,\hat{\boldsymbol{j}} - \frac{k(2q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\,\hat{j} + \frac{kq}{y^2}\,\hat{\boldsymbol{j}} = 0$$

Solving for y yields:

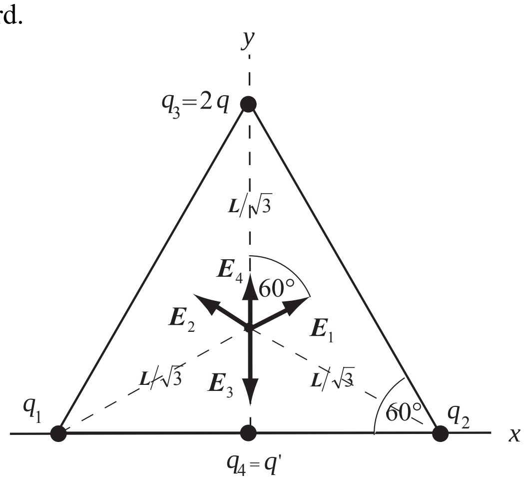
$$y = \pm\frac{L}{\sqrt{3}}$$

The positive solution corresponds to the 4th point particle being a distance $L/\sqrt{3}$ above the base of the triangle, where it produces the same strength and same direction electric field caused by the three charges at the corners of the triangle. So the charged point particle must be placed a distance $L/\sqrt{3}$ below the midpoint of the triangle.

48 •• Two point particles, each having a charge equal to q, sit on the base of an equilateral triangle that has sides of length L as shown in Figure 21-38. A third point particle that has a charge equal to $2q$ sits at the apex of the triangle. A fourth point particle that has charge q' is placed at the midpoint of the baseline making the electric field at the center of the triangle equal to zero. What is the value of q'? (The center is in the plane of the triangle and equidistant from all three vertices.)

Comment: DAVID: In 55 and 56 the solution should either show the center is a distance L/sqrt(3) from the verticies or state that using trig it can be shown?

Picture the Problem The electric field of 4th charge must cancel the sum of the electric fields due to the other three charges. Using trigonometry, one can show that the center of an equilateral triangle is a distance $L/\sqrt{3}$ from each vertex, where L is the length of the side of the triangle. The distance from the center point of the triangle to the midpoint of the base is half this distance. Note that the x components of the fields due to the base charges cancel each other, so we only need concern ourselves with the y components of the fields due to the charges at the vertices of the triangle. Choose a coordinate system in which the origin is at the midpoint of the base of the triangle, the $+x$ direction is to the right, and the $+y$ direction is upward.



Express the condition that must be satisfied if the electric field at the center of the triangle is to be zero:

$$\sum_{i=1\text{ to }4} \vec{E}_i = 0$$

Substituting for $\vec{E}_1$, $\vec{E}_2$, $\vec{E}_3$, and $\vec{E}_4$ yields:

$$\frac{k(q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\cos 60°\,\hat{j}+\frac{k(q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\cos 60°\,\hat{j}-\frac{k(2q)}{\left(\frac{L}{\sqrt{3}}\right)^2}\hat{j}+\frac{kq'}{\left(\frac{L}{2\sqrt{3}}\right)^2}\hat{j}=0$$

Solve for q' to obtain: $$q'=\boxed{\tfrac{1}{4}q}$$

49 ••• Two equal positive point charges $+q$ are on the y axis; one is at $y = +a$ and the other is at $y = –a$. The electric field at the origin is zero. A test charge q_0 placed at the origin will therefore be in equilibrium. (*a*) Discuss the stability of the equilibrium for a positive test charge by considering small displacements from equilibrium along the x axis and small displacements along the y axis. (*b*) Repeat Part (*a*) for a negative test charge. (*c*) Find the magnitude and sign of a charge q_0 that when placed at the origin results in a net force of zero on each of the three charges.

Picture the Problem We can determine the stability of the equilibrium in Part (*a*) and Part (*b*) by considering the forces the equal point charges q at $y = +a$ and $y = -a$ exert on the test charge when it is given a small displacement along either the x or y axis. (*c*) The application of Coulomb's law in Part (*c*) will lead to the magnitude and sign of the charge that must be placed at the origin in order that a net force of zero is experienced by each of the three point charges.

(*a*) Because E_x is in the x direction, a positive test charge that is displaced from (0, 0) in either the $+x$ direction or the $-x$ direction will experience a force pointing away from the origin and accelerate in the direction of the force. Consequently, the equilibrium at (0,0) is unstable for a small displacement along the x axis.

If the positive test charge is displaced in the direction of increasing y (the $+y$ direction), the charge at $y = +a$ will exert a greater force than the charge at $y = -a$, and the net force is then in the $-y$ direction; i.e., it is a restoring force. Similarly, if the positive test charge is displaced in the direction of decreasing y (the $-y$ direction), the charge at $y = -a$ will exert a greater force than the charge at $y = -a$, and the net force is then in the $-y$ direction; i.e., it is a restoring force. Consequently, the equilibrium at (0,0) is stable for a small displacement along the y axis.

(*b*) Following the same arguments as in Part (*a*), one finds that, for a negative test charge, the equilibrium is stable at (0,0) for displacements along the x axis and unstable for displacements along the y axis.

(*c*) Express the net force acting on the charge at $y = +a$:

$$\sum F_{q\text{ at }y=+a} = \frac{kqq_0}{a^2} + \frac{kq^2}{(2a)^2} = 0$$

Solve for q_0 to obtain:

$$q_0 = \boxed{-\tfrac{1}{4}q}$$

Remarks: In Part (*c*), we could just as well have expressed the net force acting on the charge at *y* = –*a*. Due to the symmetric distribution of the charges at *y* = –*a* and *y* = +*a*, summing the forces acting on q_0 at the origin does not lead to a relationship between q_0 and *q*.

50 ••• Two positive point charges $+q$ are on the y axis at $y = +a$ and $y = -a$. A bead of mass m and charge $-q$ slides without friction along a taut thread that runs along the x axis. Let x be the position of the bead. (*a*) Show that for $x << a$, the bead experiences a linear restoring force (a force that is proportional to x and directed toward the equilibrium position at $x = 0$) and therefore undergoes simple harmonic motion. (*b*) Find the period of the motion.

Picture the Problem In Problem 44 it is shown that the electric field on the x axis, due to equal positive point charges located at $(0, a)$ and $(0, -a)$, is given by $E_x = 2kqx\left(x^2 + a^2\right)^{-3/2}$. We can use $T = 2\pi\sqrt{m/k'}$ to express the period of the motion of the bead in terms of the restoring constant k'.

(*a*) Express the force acting on the bead when its displacement from the origin is x:

$$F_x = -qE_x = -\frac{2kq^2x}{\left(x^2 + a^2\right)^{3/2}}$$

Factor a^2 from the denominator to obtain:

$$F_x = -\frac{2kq^2x}{a^2\left(\dfrac{x^2}{a^2} + 1\right)^{3/2}}$$

For $x << a$:

$$F_x = \boxed{-\frac{2kq^2}{a^3}x}$$

That is, the bead experiences a linear restoring force.

(*b*) Express the period of a simple harmonic oscillator:

$$T = 2\pi\sqrt{\frac{m}{k'}}$$

Obtain k' from our result in Part (*a*):

$$k' = \frac{2kq^2}{a^3}$$

Substitute for k' and simplify to obtain:

$$T = 2\pi\sqrt{\frac{m}{\frac{2kq^2}{a^3}}} = \boxed{2\pi\sqrt{\frac{ma^3}{2kq^2}}}$$

Point Charges in Electric Fields

51 •• **[SSM]** The acceleration of a particle in an electric field depends on q/m (the charge-to-mass ratio of the particle). (*a*) Compute q/m for an electron. (*b*) What is the magnitude and direction of the acceleration of an electron in a uniform electric field that has a magnitude of 100 N/C? (*c*) Compute the time it takes for an electron placed at rest in a uniform electric field that has a magnitude of 100 N/C to reach a speed of $0.01c$. (When the speed of an electron approaches the speed of light c, relativistic kinematics must be used to calculate its motion, but at speeds of $0.01c$ or less, non-relativistic kinematics is sufficiently accurate for most purposes.) (*d*) How far does the electron travel in that time?

Picture the Problem We can use Newton's second law of motion to find the acceleration of the electron in the uniform electric field and constant-acceleration equations to find the time required for it to reach a speed of $0.01c$ and the distance it travels while acquiring this speed.

(*a*) Use data found at the back of your text to compute e/m for an electron:

$$\frac{e}{m_e} = \frac{1.602\times10^{-19}\text{ C}}{9.109\times10^{-31}\text{ kg}} = \boxed{1.76\times10^{11}\text{ C/kg}}$$

(*b*) Apply Newton's second law to relate the acceleration of the electron to the electric field:

$$a = \frac{F_{\text{net}}}{m_e} = \frac{eE}{m_e}$$

Substitute numerical values and evaluate a:

$$a = \frac{(1.602\times10^{-19}\text{ C})(100\text{ N/C})}{9.109\times10^{-31}\text{ kg}} = 1.759\times10^{13}\text{ m/s}^2 = \boxed{1.76\times10^{13}\text{ m/s}^2}$$

The direction of the acceleration of an electron is opposite the electric field.

(*c*) Using the definition of acceleration, relate the time required for an electron to reach $0.01c$ to its acceleration:

$$\Delta t = \frac{v}{a} = \frac{0.01c}{a}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{0.01\left(2.998\times10^{8}\ \text{m/s}\right)}{1.759\times10^{13}\ \text{m/s}^2} = 0.1704\ \mu\text{s}$$

$$= \boxed{0.2\ \mu\text{s}}$$

(*d*) Use a constant-acceleration equation to express the distance the electron travels in a given time interval:

$$\Delta x = v_i\Delta t + \tfrac{1}{2}a(\Delta t)^2$$

or, because $v_i = 0$,

$$\Delta x = \tfrac{1}{2}a(\Delta t)^2$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \tfrac{1}{2}\left(1.759\times10^{13}\ \text{m/s}^2\right)\left(0.1704\ \mu\text{s}\right)^2$$

$$= \boxed{0.3\ \text{m}}$$

52 • The acceleration of a particle in an electric field depends on the charge-to-mass ratio of the particle. (*a*) Compute q/m for a proton, and find its acceleration in a uniform electric field that has a magnitude of 100 N/C. (*b*) Find the time it takes for a proton initially at rest in such a field to reach a speed of $0.01c$ (where c is the speed of light). (When the speed of an electron approaches the speed of light c, relativistic kinematics must be used to calculate its motion, but at speeds of $0.01c$ or less, non-relativistic kinematics is sufficiently accurate for most purposes.)

Picture the Problem We can use Newton's second law of motion to find the acceleration of the proton in the uniform electric field and constant-acceleration equations to find the time required for it to reach a speed of $0.01c$ and the distance it travels while acquiring this speed.

(*a*) Use data found at the back of your text to compute e/m for an electron:

$$\frac{e}{m_p} = \frac{1.602\times10^{-19}\ \text{C}}{1.673\times10^{-27}\ \text{kg}}$$

$$= \boxed{9.58\times10^{7}\ \text{C/kg}}$$

Apply Newton's second law to relate the acceleration of the electron to the electric field:

$$a = \frac{F_{\text{net}}}{m_p} = \frac{eE}{m_p}$$

Substitute numerical values and evaluate a:

$$a = \frac{\left(1.602\times10^{-19}\ \text{C}\right)\left(100\ \text{N/C}\right)}{1.673\times10^{-27}\ \text{kg}}$$

$$= 9.576\times10^{9}\ \text{m/s}^2$$

$$= \boxed{9.58\times10^{9}\ \text{m/s}^2}$$

The direction of the acceleration of a proton is in the direction of the electric field.

(*b*) Using the definition of acceleration, relate the time required for an electron to reach $0.01c$ to its acceleration:

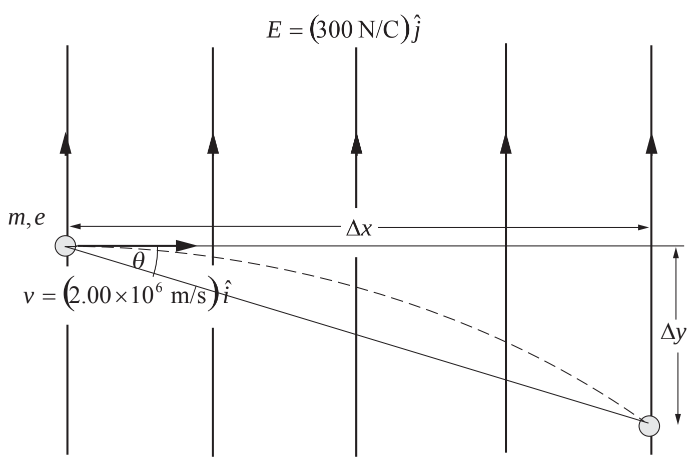
$$\Delta t = \frac{v}{a} = \frac{0.01c}{a}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{0.01\left(2.998\times10^{8}\ \text{m/s}\right)}{9.576\times10^{9}\ \text{m/s}^{2}} = \boxed{0.3\,\text{ms}}$$

53 • An electron has an initial velocity of 2.00×10^6 m/s in the $+x$ direction. It enters a region that has a uniform electric field $\vec{\boldsymbol{E}} = (300\ \text{N/C})\hat{\boldsymbol{j}}$. (*a*) Find the acceleration of the electron. (*b*) How long does it take for the electron to travel 10.0 cm in the *x* direction in the region that has the field? (*c*) Through what angle, and in what direction, is the electron deflected while traveling the 10.0 cm in the *x* direction?

Picture the Problem The electric force acting on the electron is opposite the direction of the electric field. We can apply Newton's second law to find the electron's acceleration and use constant acceleration equations to find how long it takes the electron to travel a given distance and its deflection during this interval of time. Finally, we can use the pictorial representation to obtain an expression for the angle through which the electron is deflected while traveling 10.0 cm in the *x* direction.



(*a*) Use Newton's second law to relate the acceleration of the electron first to the net force acting on it and then the electric field in which it finds itself:

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m_e} = \frac{-e\vec{E}}{m_e}$$

Substitute numerical values and evaluate $\vec{a}$:

$$\vec{a} = -\frac{1.602\times10^{-19}\ \text{C}}{9.109\times10^{-31}\ \text{kg}}(300\ \text{N/C})\hat{j}$$
$$= \left(-5.276\times10^{13}\ \text{m/s}^2\right)\hat{j}$$
$$= \boxed{\left(-5.28\times10^{13}\ \text{m/s}^2\right)\hat{j}}$$

(*b*) Relate the time to travel a given distance in the *x* direction to the electron's speed in the *x* direction:

$$\Delta t = \frac{\Delta x}{v_x} = \frac{0.100\ \text{m}}{2.00\times10^{6}\ \text{m/s}} = \boxed{50.0\ \text{ns}}$$

(*c*) The angle through which the electron is deflected as it travels a horizontal distance Δx is given by:

$$\theta = \tan^{-1}\left[\frac{\Delta y}{\Delta x}\right]$$

where Δy is its vertical deflection.

Using a constant-acceleration equation, relate the vertical deflection of the electron to its acceleration and the elapsed time:

$$\Delta y = \tfrac{1}{2}a_y(\Delta t)^2$$

Substituting for Δy and simplifying yields:

$$\theta = \tan^{-1}\left[\frac{\frac{1}{2}a_y(\Delta t)^2}{\Delta x}\right] = \tan^{-1}\left[\frac{a_y(\Delta t)^2}{2\Delta x}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[\frac{\left(5.28\times10^{13}\ \text{m/s}^2\right)(50.0\ \text{ns})^2}{2(10.0\ \text{cm})}\right]$$
$$= \boxed{33.4°}$$

Because the acceleration of the electron is downward and it was moving horizontally initially, it is deflected $\boxed{\text{downward}}$.

54 •• An electron is released from rest in a weak electric field $\vec{E} = \left(-1.50\times10^{-10}\ \text{N/C}\right)\hat{j}$. After the electron has traveled a vertical distance of 1.0 μm, what is its speed? (Do not neglect the gravitational force on the electron.)

Comment: DAVID: Note changes in 62.

Picture the Problem Because the electric field is in the $-y$ direction, the force it exerts on the electron is in the $+y$ direction. Applying Newton's second law to the electron will yield an expression for the acceleration of the electron in the y direction. We can then use a constant-acceleration equation to relate its speed to its acceleration and the distance it has traveled.

Apply $\sum F_y = ma_y$ to the electron to obtain:

$$F_E - F_g = ma_y$$

or, because $F_E = eE$ and $F_g = mg$,

$$eE - mg = ma_y$$

Solving for a_y yields:

$$a_y = \frac{eE}{m} - g$$

Use a constant-acceleration equation to relate the speed of the electron to its acceleration and the distance it travels:

$$v_y^2 = v_0^2 + 2a_y\Delta y$$

or, because the electron starts at rest,

$$v_y^2 = 2a_y\Delta y \Rightarrow v_y = \sqrt{2a_y\Delta y}$$

Substitute for a_y in the expression for v_y:

$$v_y = \sqrt{2\left(\frac{eE}{m} - g\right)\Delta y}$$

Substitute numerical values and evaluate v_y:

$$v_y = \sqrt{2\left[\frac{\left(1.602\times10^{-19}\text{ C}\right)\left(1.50\times10^{-10}\text{ N/C}\right)}{9.109\times10^{-31}\text{ kg}} - 9.81\text{ m/s}^2\right]\left(1.0\times10^{-6}\text{ m}\right)}$$

$$= \boxed{5.8\text{ mm/s}}$$

55 •• A 2.00-g charged particle is released from rest in a region that has a uniform electric field $\vec{E} = (300\text{ N/C})\hat{i}$. After traveling a distance of 0.500 m in this region, the particle has a kinetic energy of 0.120 J. Determine the charge of the particle.

Picture the Problem We can apply the work-kinetic energy theorem to relate the change in the object's kinetic energy to the net force acting on it. We can express the net force acting on the charged body in terms of its charge and the electric field.

Using the work-kinetic energy theorem, express the kinetic energy of the object in terms of the net force acting on it and its displacement:

$$W = \Delta K = F_{\text{net}}\Delta x$$

Relate the net force acting on the charged particle to the electric field:

$$F_{\text{net}} = qE$$

Substitute for F_{net} to obtain:

$$\Delta K = K_{\text{f}} - K_{\text{i}} = qE\Delta x$$

or, because Ki = 0,

$$K_{\text{f}} = qE\Delta x \Rightarrow q = \frac{K_{\text{f}}}{E\Delta x}$$

Substitute numerical values and evaluate q:

$$q = \frac{0.120\,\text{J}}{(300\,\text{N/C})(0.500\,\text{m})} = \boxed{800\,\mu\text{C}}$$

56 •• A charged particle leaves the origin with a speed of 3.00×10^6 m/s at an angle of 35º above the x axis. A uniform electric field given by $\vec{\boldsymbol{E}} = -E_0\hat{\boldsymbol{j}}$ exists throughout the region. Find E_0 such that the particle will cross the x axis at $x = 1.50$ cm if the particle is (a) an electron, and (b) a proton.

Picture the Problem We can use constant-acceleration equations to express the x and y coordinates of the particle in terms of the parameter t and Newton's second law to express the constant acceleration in terms of the electric field. Eliminating t will yield an equation for y as a function of x, q, and m that we can solve for E_y.

Express the x and y coordinates of the particle as functions of time:

$$x = (v\cos\theta)t$$

and

$$y = (v\sin\theta)t - \tfrac{1}{2}a_y t^2$$

Apply Newton's second law to relate the acceleration of the particle to the net force acting on it:

$$a_y = \frac{F_{\text{net,y}}}{m} = \frac{qE_0}{m}$$

Substitute in the y-coordinate equation to obtain:

$$y = (v\sin\theta)t - \frac{qE_0}{2m}t^2$$

Eliminate the parameter t between the two equations to obtain:

$$y = (\tan\theta)x - \frac{qE_0}{2mv^2\cos^2\theta}x^2$$

Set $y = 0$ and solve for E_0 to obtain:

$$E_0 = \frac{mv^2 \sin 2\theta}{qx}$$

Substitute the non-particle-specific data to obtain:

$$E_0 = \frac{m\left(3.00\times10^6\ \text{m/s}\right)^2 \sin 70°}{q(0.0150\,\text{m})} = \left(5.638\times10^{14}\ \text{m/s}^2\right)\frac{m}{q}$$

(*a*) Substitute for the mass and charge of an electron and evaluate E_0:

$$E_0 = \left(5.638\times10^{14}\ \text{m/s}^2\right)\frac{9.109\times10^{-31}\ \text{kg}}{-1.602\times10^{-19}\,\text{C}} = \boxed{-3.2\,\text{kN/C}}$$

(*b*) Substitute for the mass and charge of a proton and evaluate E_0:

$$E_0 = \left(5.64\times10^{14}\ \text{m/s}^2\right)\frac{1.673\times10^{-27}\ \text{kg}}{1.602\times10^{-19}\,\text{C}} = \boxed{5.9\,\text{MN/C}}$$

57 •• [SSM] An electron starts at the position shown in Figure 21-39 with an initial speed $v_0 = 5.00 \times 10^6$ m/s at 45º to the x axis. The electric field is in the $+y$ direction and has a magnitude of 3.50×10^3 N/C. The black lines in the figure are charged metal plates. On which plate and at what location will the electron strike?

Picture the Problem We can use constant-acceleration equations to express the x and y coordinates of the electron in terms of the parameter t and Newton's second law to express the constant acceleration in terms of the electric field. Eliminating t will yield an equation for y as a function of x, q, and m. We can decide whether the electron will strike the upper plate by finding the maximum value of its y coordinate. Should we find that it does not strike the upper plate, we can determine where it strikes the lower plate by setting $y(x) = 0$. Ignore any effects of gravitational forces.

Express the x and y coordinates of the electron as functions of time:

$$x = (v_0 \cos\theta)t$$

and

$$y = (v_0 \sin\theta)t - \tfrac{1}{2}a_y t^2$$

Apply Newton's second law to relate the acceleration of the electron to the net force acting on it:

$$a_y = \frac{F_{\text{net},y}}{m_e} = \frac{eE_y}{m_e}$$

Substitute in the y-coordinate equation to obtain:

$$y = (v_0 \sin\theta)t - \frac{eE_y}{2m_e}t^2$$

Eliminate the parameter t between the two equations to obtain:

$$y(x) = (\tan\theta)x - \frac{eE_y}{2m_e v_0^2 \cos^2\theta}x^2 \quad (1)$$

To find y_{max}, set $dy/dx = 0$ for extrema:

$$\frac{dy}{dx} = \tan\theta - \frac{eE_y}{m_e v_0^2 \cos^2\theta}x'$$
$$= 0 \text{ for extrema}$$

Solve for x' to obtain:

$$x' = \frac{m_e v_0^2 \sin 2\theta}{2eE_y} \text{ (See remark below.)}$$

Substitute x' in $y(x)$ and simplify to obtain y_{max}:

$$y_{\text{max}} = \frac{m_e v_0^2 \sin^2\theta}{2eE_y}$$

Substitute numerical values and evaluate y_{max}:

$$y_{\text{max}} = \frac{(9.109\times10^{-31}\text{ kg})(5.00\times10^{6}\text{ m/s})^2 \sin^2 45°}{2(1.602\times10^{-19}\text{ C})(3.50\times10^{3}\text{ N/C})} = 1.02\text{ cm}$$

Because the plates are separated by 2 cm, the electron does not strike the upper plate.

To determine where the electron will strike the lower plate, set $y = 0$ in equation (1) and solve for x to obtain:

$$x = \frac{m_e v_0^2 \sin 2\theta}{eE_y}$$

Substitute numerical values and evaluate x:

$$x = \frac{(9.109\times10^{-31}\text{ kg})(5.00\times10^{6}\text{ m/s})^2 \sin 90°}{(1.602\times10^{-19}\text{ C})(3.50\times10^{3}\text{ N/C})} = \boxed{4.1\text{ cm}}$$

Remarks: x' is an extremum, that is, either a maximum or a minimum. To show that it is a maximum we need to show that d^2y/dx^2, evaluated at x', is negative. A simple alternative is to use your graphing calculator to show that the graph of $y(x)$ is a maximum at x'. Yet another alternative is to recognize that, because equation (1) is quadratic and the coefficient of x^2 is negative, its graph is a parabola that opens downward.

58 •• An electron that has a kinetic energy equal to 2.00×10^{-16} J is moving to the right along the axis of a cathode-ray tube as shown in Figure 21-40. An electric field $\vec{E} = \left(2.00\times10^{4}\text{ N/C}\right)\hat{j}$ exists in the region between the deflection plates; and no electric field ($\vec{E} = 0$) exists outside this region. (*a*) How far is the electron from the axis of the tube when it exits the region between the plates? (*b*) At what angle is the electron moving, with respect to the axis, after exiting the region between the plates? (*c*) At what distance from the axis will the electron strike the fluorescent screen?

Picture the Problem The trajectory of the electron while it is in the electric field is parabolic (its acceleration is downward and constant) and its trajectory, once it is out of the electric field is, if we ignore the small gravitational force acting on it, linear. We can use constant-acceleration equations and Newton's second law to express the electron's x and y coordinates parametrically and then eliminate the parameter t to express $y(x)$. We can find the angle with the horizontal at which the electron leaves the electric field from the x and y components of its velocity and its total vertical deflection by summing its deflections over the first 4 cm and the final 12 cm of its flight.

(*a*) Using a constant-acceleration equation, express the x and y coordinates of the electron as functions of time:

$$x(t) = v_0 t$$
and
$$y(t) = v_{0,y}t + \tfrac{1}{2}a_y t^2$$

Because $v_{0,y} = 0$:

$$x(t) = v_0 t \qquad (1)$$
and
$$y(t) = \tfrac{1}{2}a_y t^2$$

Using Newton's second law, relate the acceleration of the electron to the electric field:

$$a_y = \frac{F_{\text{net}}}{m_e} = \frac{-eE_y}{m_e}$$

Substituting for a_y gives:

$$y(t) = -\frac{eE_y}{2m_e}t^2 \qquad (2)$$

Eliminate the parameter t between equations (1) and (2) to obtain:

$$y(x) = -\frac{eE_y}{2m_e v_0^2}x^2 = -\frac{eE_y}{4K}x^2$$

Substitute numerical values and evaluate y(4 cm):

$$y(0.04\,\text{m}) = -\frac{(1.602\times10^{-19}\,\text{C})(2.00\times10^{4}\,\text{N/C})(0.0400\,\text{m})^2}{4(2.00\times10^{-16}\,\text{J})} = \boxed{-6.40\,\text{mm}}$$

(b) The angle at which the electron is moving, with respect to the axis, after exiting the region between the plates is given by:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{v_y}{v_0}\right)$$

Using a constant-acceleration equation, express v_y as a function of the electron's acceleration and its time in the electric field:

$$v_y = v_{0,y} + a_y t$$

or, because $v_{0,y} = 0$

$$v_y = a_y t = \frac{F_{\text{net},y}}{m_e} t = -\frac{eE_y}{m_e}\frac{x}{v_0}$$

Substitute for v_y to obtain:

$$\theta = \tan^{-1}\left(-\frac{eE_y x}{m_e v_0^2}\right) = \tan^{-1}\left(-\frac{eE_y x}{2K}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \tan^{-1}\left[-\frac{(1.602\times10^{-19}\,\text{C})(2.00\times10^{4}\,\text{N/C})(0.0400\,\text{m})}{2(2.00\times10^{-16}\,\text{J})}\right] = \boxed{-17.8°}$$

(c) Express the total vertical displacement of the electron:

$$y_{\text{total}} = y_{4\,\text{cm}} + y_{12\,\text{cm}}$$

Relate the horizontal and vertical distances traveled to the screen after leaving the region between the plates to the horizontal and vertical components of its velocity:

$$x = v_0 \Delta t$$

and

$$y = v_y \Delta t$$

Eliminate Δt from these equations to obtain:

$$y = \left(\frac{v_y}{v_0}\right)x = (\tan\theta)x$$

Substitute numerical values and evaluate y:

$$y = [\tan(-17.7°)](0.120\,\text{m}) = -3.83\,\text{cm}$$

Substitute for $y_{4\text{ cm}}$ and $y_{12\text{ cm}}$ and evaluate y_{total}:

$$y_{\text{total}} = -0.640\,\text{cm} - 3.83\,\text{cm}$$
$$= \boxed{-4.47\,\text{cm}}$$

That is, the electron will strike the fluorescent screen 4.47 cm below the horizontal axis.

Dipoles

59 • Two point charges, $q_1 = 2.0$ pC and $q_2 = -2.0$ pC, are separated by 4.0 μm. (*a*) What is the magnitude of the dipole moment of this pair of charges? (*b*) Sketch the pair and show the direction of the dipole moment.

Picture the Problem We can use its definition to find the dipole moment of this pair of charges.

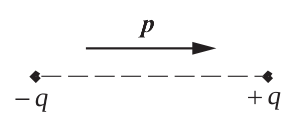
(*a*) Apply the definition of electric dipole moment to obtain:

$$\vec{\boldsymbol{p}} = q\vec{\boldsymbol{L}}$$

and

$$p = (2.0\,\text{pC})(4.0\,\mu\text{m})$$
$$= \boxed{8.0\times10^{-18}\,\text{C}\cdot\text{m}}$$

(*b*) If we assume that the dipole is oriented as shown to the right, then $\vec{\boldsymbol{p}}$ is to the right; pointing from the negative charge toward the positive charge.



60 • A dipole of moment 0.50 *e*·nm is placed in a uniform electric field that has a magnitude of 4.0×10^4 N/C. What is the magnitude of the torque on the dipole when (*a*) the dipole is aligned with the electric field, (*b*) the dipole is transverse to (perpendicular to) the electric field, and (*c*) the dipole makes an angle of 30º with the direction of the electric field? (*d*) Defining the potential energy to be zero when the dipole is transverse to the electric field, find the potential energy of the dipole for the orientations specified in Parts (*a*) and (*c*).

Picture the Problem The torque on an electric dipole in an electric field is given by $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}}\times\vec{\boldsymbol{E}}$ and the potential energy of the dipole by $U = -\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{E}}$.

Using its definition, express the torque on a dipole moment in a uniform electric field:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}}\times\vec{\boldsymbol{E}} \Rightarrow \tau = pE\sin\theta$$

where θ is the angle between the electric dipole moment and the electric field.

(*a*) Evaluate τ for θ = 0°:

$$\tau(0°) = pE\sin 0° = \boxed{0}$$

(*b*) Evaluate τ for θ = 90°:

$$\tau(90°) = (0.50\,e\cdot\text{nm})(4.0\times10^{4}\ \text{N/C})\sin 90° = \boxed{3.2\times10^{-24}\ \text{N}\cdot\text{m}}$$

(*c*) Evaluate τ for θ = 30°:

$$\tau(30°) = (0.50\,e\cdot\text{nm})(4.0\times10^{4}\ \text{N/C})\sin 30° = \boxed{1.6\times10^{-24}\ \text{N}\cdot\text{m}}$$

(*d*) Using its definition, express the potential energy of a dipole in an electric field:

$$U = -\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{E}} = -pE\cos\theta$$

Evaluate U for θ = 0°:

$$U(0°) = -(0.50\,e\cdot\text{nm})(4.0\times10^{4}\ \text{N/C})\cos 0° = \boxed{-3.2\times10^{-24}\ \text{J}}$$

Evaluate U for θ = 30°:

$$U(30°) = -(0.50\,e\cdot\text{nm})(4.0\times10^{4}\ \text{N/C})\cos 30° = \boxed{-2.8\times10^{-24}\ \text{J}}$$

General Problems

61 • [SSM] Show that it is only possible to place one isolated proton in an ordinary empty coffee cup by considering the following situation. Assume the first proton is fixed at the bottom of the cup. Determine the distance directly above this proton where a second proton would be in equilibrium. Compare this distance to the depth of an ordinary coffee cup to complete the argument.

Comment: DAVID: My coffee mug is more than 5 in high, so I changed mug to cup.

Comment: At close distances, protons will bind together by the strong nuclear force. This question confuses students.

Comment: To the best of my knowledge, Helium-2 does not exist.

Picture the Problem Equilibrium of the second proton requires that the sum of the electric and gravitational forces acting on it be zero. Let the upward direction be the +*y* direction and apply the condition for equilibrium to the second proton.

Apply $\sum F_y = 0$ to the second proton:

$$\vec{\boldsymbol{F}}_\text{e} + \vec{\boldsymbol{F}}_\text{g} = 0$$

or

$$\frac{kq_\text{p}^2}{h^2} - m_\text{p}g = 0 \Rightarrow h = \sqrt{\frac{kq_\text{p}^2}{m_\text{p}g}}$$

Substitute numerical values and evaluate h:

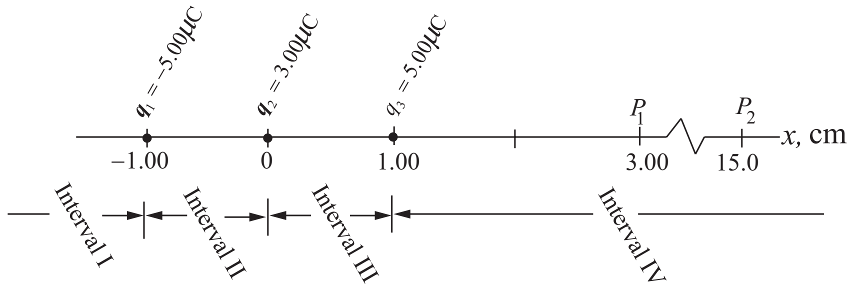
$$h = \sqrt{\frac{\left(8.988\times 10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times 10^{-19}\ \text{C}\right)^2}{\left(1.673\times 10^{-27}\ \text{kg}\right)\left(9.81\ \text{m/s}^2\right)}} \approx 12\ \text{cm} \approx 5\ \text{in}$$

This separation of about 5 in is greater than the height of a typical coffee cup. Thus the first proton will repel the second one out of the cup and the maximum number of protons in the cup is one.

62 •• Point charges of $-5.00\ \mu$C, $+3.00\mu$C, and $+5.00\ \mu$C are located on the x axis at $x = -1.00$ cm, $x = 0$, and $x = +1.00$ cm, respectively. Calculate the electric field on the x axis at $x = 3.00$ cm and at $x = 15.0$ cm. Are there any points on the x axis where the magnitude of the electric field is zero? If so, where are those points?

Comment: DAVID: In the second part of 76 the ISM solution is invalid. I get solutions at x = -6.95 cm and +4.17 cm

Picture the Problem The locations of the point charges q_1, q_2 and q_3 and the points P_1 and P_2 at which we are to calculate the electric field are shown in the diagram. From the diagram it is evident that $\vec{E}$ along the axis has no y component. (*a*) We can use Coulomb's law for $\vec{E}$ due to a point charge and the superposition principle for electric fields to find $\vec{E}$ at points P_1 and P_2. (*b*) To decide whether there are any points on the x axis where the magnitude of the electric field is zero we need to consider the intervals (identified on the following pictorial representation) I ($x < -1.00$ cm), II (-1.00 cm $< x < 0$), III ($0 < x < 1.00$ cm), and IV (1.00 cm $< x$). Throughout interval II, the three electric fields are in the same direction and so they cannot add up to zero. Throughout interval IV, the electric-field strength due to the positive charge at $x = 1.00$ cm is larger than the electric-field strength due to the negative charge at $x = -1.00$ cm. The fields due to the positive charges at $x = 0$ and $x = 1.00$ cm both point in the $+x$ direction, so the resultant field throughout this interval must also point in the $+x$ direction. Hence we can narrow our search for points on the x axis where the magnitude of the electric field is zero to intervals I and III. Setting the resultant electric-field strength on the x axis equal to zero throughout these intervals and solving the resulting quadratic equations will identify the desired points. Sketching the electric field vectors in intervals I and III will help you get the signs of the terms in the quadratic equations right.



Using Coulomb's law, and letting r_{11}, r_{21}, and r_{31} represent the distances from P_1 to point charges q_1, q_2, and q_3, respectively, express the electric field at P_1 due to the three charges:

$$\vec{E}_{P_1} = \vec{E}_{q_1} + \vec{E}_{q_2} + \vec{E}_{q_3} = \frac{kq_1}{r_{11}^2}\hat{i} + \frac{kq_2}{r_{21}^2}\hat{i} + \frac{kq_3}{r_{31}^2}\hat{i} = k\left[\frac{q_1}{r_{11}^2} + \frac{q_2}{r_{21}^2} + \frac{q_3}{r_{31}^2}\right]\hat{i}$$

Substitute numerical values and evaluate $\vec{E}_{P_1}$:

$$\vec{E}_{P_1} = \left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{-5.00\,\mu\text{C}}{(4.00\,\text{cm})^2} + \frac{3.00\,\mu\text{C}}{(3.00\,\text{cm})^2} + \frac{5.00\,\mu\text{C}}{(2.00\,\text{cm})^2}\right]\hat{i} = \boxed{\left(1.14\times10^8\ \text{N/C}\right)\hat{i}}$$

Using Coulomb's law, and letting r_{12}, r_{22}, and x_{32} represent the distances from P_2 to point charges q_1, q_2, and q_3, respectively, express the electric field at P_2 due to the three charges:

$$\vec{E}_{P_2} = \vec{E}_{q_1} + \vec{E}_{q_2} + \vec{E}_{q_3} = k\left[\frac{q_1}{r_{12}^2} + \frac{q_2}{r_{22}^2} + \frac{q_3}{r_{32}^2}\right]\hat{i}$$

Substitute numerical values and evaluate $\vec{E}_{P_2}$:

$$\vec{E}_{P_2} = \left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{-5.00\,\mu\text{C}}{(16.0\,\text{cm})^2} + \frac{3.00\,\mu\text{C}}{(15.0\,\text{cm})^2} + \frac{5.00\,\mu\text{C}}{(14.0\,\text{cm})^2}\right]\hat{i} = \boxed{\left(1.74\times10^6\ \text{N/C}\right)\hat{i}}$$

Applying Coulomb's law for electric fields and the superposition of fields in interval I yields:

$$\frac{5.00\,\mu\text{C}}{[x-(-1.00\,\text{cm})]^2}-\frac{3.00\,\mu\text{C}}{x^2}-\frac{5.00\,\mu\text{C}}{(x-1.00\,\text{cm})^2}=0$$

where k has been divided out and x is in the interval defined by $x < 0$. The root of this equation that is in the interval $x < -1.00$ cm is $\boldsymbol{x} = \boxed{-6.95\,\text{cm}}$.

Applying Coulomb's law for electric fields and the superposition of fields in interval III yields:

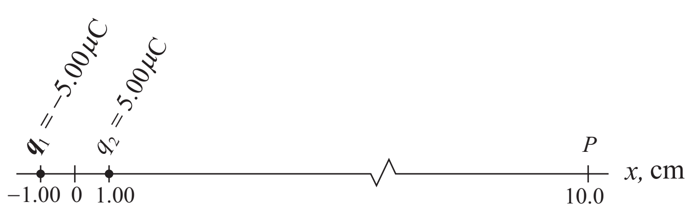
$$-\frac{5.00\,\mu\text{C}}{[x-(-1.00\,\text{cm})]^2}+\frac{3.00\,\mu\text{C}}{x^2}-\frac{5.00\,\mu\text{C}}{(1.00\,\text{cm}-x)^2}=0$$

The root of this equation that is in the interval $0 < x < 1.00$ cm is $x = \boxed{0.417\,\text{cm}}$

63 •• Point charges of $-5.00\ \mu$C and $+5.00\ \mu$C are located on the x axis at $x = -1.00$ cm and $x = +1.00$ cm, respectively. (*a*) Calculate the electric field strength at $x = 10.0$ cm. (*b*) Estimate the electric field strength at $x = 10.00$ cm by modeling the two charges as an electric dipole located at the origin and using $E = 2kp/|x|^3$ (Equation 21-10). Compare your result with the result obtained in Part (*a*), and explain the reason for the difference between the two results.

Comment: David: In 88 notice the many changes.

Picture the Problem Let the point of interest ($x = 10.0$ cm) be identified as point P. In Part (*a*) we can use Coulomb's law for $\vec{E}$ due to a point charge and the superposition principle to find the electric field strength at P. In Part (*b*) we can use Equation 21-10 with $p = 2aq$ to estimate the electric field strength at P.



(*a*) Express the electric field at P as the sum of the fields due to the point charges located at $x = -1.00$ cm and $x = 1.00$ cm:

$$\vec{E}_P = \vec{E}_{q_1} + \vec{E}_{q_2} = \frac{kq_1}{x^2_{-1.00\,\text{cm}\to P}}\hat{i} + \frac{kq_2}{x^2_{1.00\,\text{cm}\to P}}\hat{i} = k\left[\frac{q_1}{x^2_{-1.00\,\text{cm}\to P}} + \frac{q_2}{x^2_{1.00\,\text{cm}\to P}}\right]\hat{i}$$

Substitute numerical values and evaluate $\vec{E}_P$:

$$\vec{E}_P = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{-5.00\ \mu\text{C}}{(11.00\ \text{cm})^2} + \frac{5.00\ \mu\text{C}}{(9.00\ \text{cm})^2}\right]\hat{i} = \left(1.83\times10^6\ \text{N/C}\right)\hat{i}$$

and

$$E_P = \boxed{1.83\times10^6\ \text{N/C}}$$

(*b*) The electric field strength due to a dipole is given by Equation 21-10:

$$E = \frac{2kp}{|x|^3}$$

Because $p = 2qa$, where $2a$ is the separation of the charges that constitute the dipole, E is also given by:

$$E = \frac{4kqa}{|x|^3}$$

Substitute numerical values and evaluate E_P:

$$E_P = \frac{4\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(5.00\ \mu\text{C})(1.00\ \text{cm})}{|10.0\ \text{cm}|^3} = \boxed{1.80\times10^6\ \text{N/C}}$$

The exact and estimated values of E_P agree to within 2%. This difference is due to the fact that the separation of the two charges of the dipole is 20% of the distance from the center of the dipole to point P.

64 •• A fixed point charge of $+2q$ is connected by strings to point charges of $+q$ and $+4q$, as shown in Figure 21-41. Find the tensions T_1 and T_2.

Picture the Problem The electrostatic forces between the charges are responsible for the tensions in the strings. We'll assume that these are point charges and apply Coulomb's law and the principle of the superposition of forces to find the tension in each string.

Use Coulomb's law to express the net force on the charge $+q$:

$$T_1 = F_{2q} + F_{4q}$$

Substitute and simplify to obtain:

$$T_1 = \frac{kq(2q)}{d^2} + \frac{kq(4q)}{(2d)^2} = \boxed{\frac{3kq^2}{d^2}}$$

Use Coulomb's law to express the net force on the charge $+4q$:

$$T_2 = F_q + F_{2q}$$

Substitute and simplify to obtain:

$$T_2 = \frac{k(2q)(4q)}{d^2} + \frac{kq(4q)}{(2d)^2} = \boxed{\frac{9kq^2}{d^2}}$$

65 •• [SSM] A positive charge Q is to be divided into two positive point charges q_1 and q_2. Show that, for a given separation D, the force exerted by one charge on the other is greatest if $q_1 = q_2 = \frac{1}{2}Q$.

Picture the Problem We can use Coulomb's law to express the force exerted on one charge by the other and then set the derivative of this expression equal to zero to find the distribution of the charge that maximizes this force.

Using Coulomb's law, express the force that either charge exerts on the other:

$$F = \frac{kq_1q_2}{D^2}$$

Express q_2 in terms of Q and q_1:

$$q_2 = Q - q_1$$

Substitute for q_2 to obtain:

$$F = \frac{kq_1(Q - q_1)}{D^2}$$

Differentiate F with respect to q_1 and set this derivative equal to zero for extreme values:

$$\frac{dF}{dq_1} = \frac{k}{D^2}\frac{d}{dq_1}\left[q_1(Q - q_1)\right]$$
$$= \frac{k}{D^2}\left[q_1(-1) + Q - q_1\right]$$
$$= 0 \text{ for extrema}$$

Solve for q_1 to obtain:

$$q_1 = \tfrac{1}{2}Q \Rightarrow q_2 = Q - q_1 = \tfrac{1}{2}Q$$

To determine whether a maximum or a minimum exists at $q_1 = \frac{1}{2}Q$, differentiate F a second time and evaluate this derivative at $q_1 = \frac{1}{2}Q$:

$$\frac{d^2F}{dq_1^2} = \frac{k}{D^2}\frac{d}{dq_1}[Q - 2q_1]$$
$$= \frac{k}{D^2}(-2)$$
$$< 0 \text{ independently of } q_1.$$
$$\boxed{\therefore q_1 = q_2 = \tfrac{1}{2}Q \text{ maximizes } F.}$$

66 •• A point charge Q is located on the x axis at $x = 0$, and a point charge $4Q$ is located at $x = 12.0$ cm. The electric force on a point charge of $-2.00\ \mu\text{C}$ is zero if that charge is placed at $x = 4.00$ cm, and is 126 N in the $+x$ direction if placed at $x = 8.00$ cm. Determine the charge Q.

Picture the Problem We can apply Coulomb's law and the superposition of forces to relate the net force acting on the charge $q = -2.00\ \mu\text{C}$ to x. Because Q divides out of our equation when $F(x) = 0$, we'll substitute the data given for $x = 8.00$ cm.

Using Coulomb's law, express the net electric force on q as a function of x:

$$F(x) = -\frac{kqQ}{x^2} + \frac{kq(4Q)}{(12.0\,\text{cm} - x)^2}$$

Solving for Q yields:

$$Q = \frac{F(x)}{kq\left[-\dfrac{1}{x^2} + \dfrac{4}{(12.0\,\text{cm} - x)^2}\right]}$$

Evaluate Q for $x = 8.00$ cm:

$$Q = \frac{126\,\text{N}}{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(2.00\,\mu\text{C})\left[-\dfrac{1}{(8.00\,\text{cm})^2} + \dfrac{4}{(4.00\,\text{cm})^2}\right]} = \boxed{2.99\,\mu\text{C}}$$

67 •• Two point particles separated by 0.60 m carry a total charge of 200 μC. (*a*) If the two particles repel each other with a force of 80 N, what is the charge on each of the two particles? (*b*) If the two particles attract each other with a force of 80 N, what are the charges on the two particles?

Picture the Problem Knowing the total charge of the two point particles, we can use Coulomb's law to find the two combinations of charge that will satisfy the condition that both are positive and hence repel each other. If the particles attract each other, then there is just one distribution of charge that will satisfy the conditions that the force is attractive and the sum of the two charges is 200 μC.

(*a*) Use Coulomb's law to express the repulsive electric force each particle exerts on the other:

$$F = \frac{kq_1q_2}{r_{1,2}^2}$$

Express q_2 in terms of the total charge and q_1:

$$q_2 = Q - q_1$$

Substitute for q_2 to obtain:

$$F = \frac{kq_1(Q-q_1)}{r_{1,2}^2} = \frac{k(Qq_1 - q_1^2)}{r_{1,2}^2}$$

Writing this equation explicitly as a quadratic equation gives:

$$q_1^2 - Qq_1 + \frac{Fr_{1,2}^2}{k} = 0$$

Substitute numerical values to obtain:

$$q_1^2 - (200\,\mu\text{C})q_1 + \frac{(80\,\text{N})(0.60\,\text{m})^2}{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2} = 0$$

or

$$q_1^2 - \left(2.00\times10^{-4}\,\text{C}\right)q_1 + 3.20\times10^{-9}\,\text{C}^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$q_1 = \boxed{1.8\times10^{-5}\ \text{C}}$$

and

$$q_2 = Q - q_1 = \boxed{1.8\times10^{-4}\ \text{C}}$$

(*b*) Use Coulomb's law to express the attractive electric force each particle exerts on the other:

$$F = -\frac{kq_1q_2}{r_{1,2}^2}$$

Proceed as in (*a*) to obtain:

$$q_1^2 - \left(2.00\times10^{-4}\,\text{C}\right)q_1 - 3.20\times10^{-9}\,\text{C}^2 = 0$$

Solve this quadratic equation to obtain:

$$q_1 = \boxed{-1.5\times10^{-5}\ \text{C}}$$

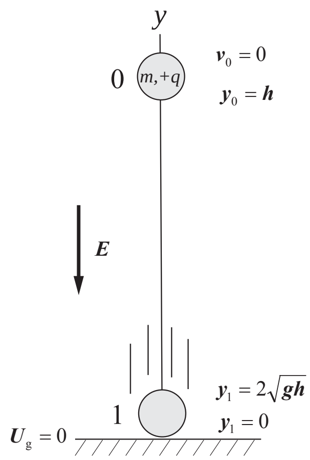
and

$$q_2 = Q - q_1 = \boxed{2.1\times10^{-4}\ \text{C}}$$

68 •• A point particle that has charge $+q$ and unknown mass m is released from rest in a region that has a uniform electric field $\vec{E}$ that is directed vertically

downward. The particle hits the ground at a speed $v = 2\sqrt{gh}$, where h is the initial height of the particle. Find m in terms of E, q, and g.

Picture the Problem Choose the coordinate system shown in the diagram and let $U_g = 0$ where $y = 0$. We'll let our system include the point particle and Earth. Then the work done on the point particle by the electric field will change the energy of the system. The diagram summarizes what we know about the motion of the point particle. We can apply the work-energy theorem to our system to relate the work done by the electric field to the change in its energy.



Using the work-energy theorem, relate the work done by the electric field to the change in the energy of the system:

$$W_{\substack{\text{electric}\\\text{field}}} = \Delta K + \Delta U_g$$

$$= K_1 - K_0 + U_{g,1} - U_{g,0}$$

or, because $K_1 = U_{g,1} = 0$,

$$W_{\substack{\text{electric}\\\text{field}}} = K_1 - U_{g,1}$$

Substitute for $W_{\substack{\text{electric}\\\text{field}}}$, K_1 and $U_{g,1}$ and simplify to obtain:

$$qEh = \tfrac{1}{2}mv_1^2 - mgh$$

$$= \tfrac{1}{2}m\left(2\sqrt{gh}\right)^2 - mgh = mgh$$

Solving for m yields:

$$m = \boxed{\frac{qE}{g}}$$

69 •• [SSM] A rigid 1.00-m-long rod is pivoted about its center (Figure 21-42). A charge $q_1 = 5.00 \times 10^{-7}$ C is placed on one end of the rod, and a charge $q_2 = -q_1$ is placed a distance $d = 10.0$ cm directly below it. (*a*) What is the force exerted by q_2 on q_1? (*b*) What is the torque (measured about the rotation axis) due to that force? (*c*) To counterbalance the attraction between the two charges, we hang a block 25.0 cm from the pivot as shown. What value should we choose for the mass of the block? (*d*) We now move the block and hang it a distance of 25.0 cm from the balance point, on the same side of the balance as the charge. Keeping q_1 the same, and d the same, what value should we choose for q_2 to keep this apparatus in balance?

Picture the Problem We can use Coulomb's law, the definition of torque, and the condition for rotational equilibrium to find the electrostatic force between the two charged bodies, the torque this force produces about an axis through the center of the rod, and the mass required to maintain equilibrium when it is located either 25.0 cm to the right or to the left of the mid-point of the rod.

(*a*) Using Coulomb's law, express the electric force between the two charges:

$$F = \frac{kq_1q_2}{d^2}$$

Substitute numerical values and evaluate F:

$$F = \frac{\left(8.988\times10^9\ \mathrm{N\cdot m^2/C^2}\right)\left(5.00\times10^{-7}\ \mathrm{C}\right)^2}{\left(0.100\ \mathrm{m}\right)^2} = 0.2247\ \mathrm{N} = \boxed{0.225\ \mathrm{N}}\text{, downward}$$

(*b*) The torque (measured about the rotation axis) due to the force F is:

$$\tau = F\ell$$

Substitute numerical values and evaluate τ:

$$\tau = \left(0.2247\ \mathrm{N}\right)\left(0.500\ \mathrm{m}\right) = 0.1124\ \mathrm{N\cdot m} = \boxed{0.112\ \mathrm{N\cdot m}},$$

counterclockwise.

(*c*) Apply $\sum\tau_{\substack{\text{center of}\\ \text{the rod}}} = 0$ to the rod:

$$\tau - mg\ell' = 0 \Rightarrow m = \frac{\tau}{g\ell'}$$

Substitute numerical values and evaluate m:

$$m = \frac{0.1124\ \mathrm{N\cdot m}}{\left(9.81\ \mathrm{m/s^2}\right)\left(0.250\ \mathrm{m}\right)} = 0.04583\ \mathrm{kg} = \boxed{45.8\ \mathrm{g}}$$

(*d*) Apply $\sum\tau_{\substack{\text{center of}\\ \text{the rod}}} = 0$ to the rod:

$$-\tau + mg\ell' = 0$$

Substitute for τ to obtain:

$$-F\ell + mg\ell' = 0$$

Substituting for F gives:

$$-\frac{kq_1q_2'}{d^2} + mg\ell' = 0 \Rightarrow q_2' = \frac{d^2mg\ell'}{kq_1\ell}$$

where q' is the required charge.

Substitute numerical values and evaluate q_2':

$$q_2' = \frac{\left(0.100\ \mathrm{m}\right)^2\left(0.04582\ \mathrm{kg}\right)\left(9.81\ \mathrm{m/s^2}\right)\left(0.250\ \mathrm{m}\right)}{\left(8.988\times10^9\ \mathrm{N\cdot m^2/C^2}\right)\left(5.00\times10^{-7}\ \mathrm{C}\right)\left(0.500\ \mathrm{m}\right)} = \boxed{5.00\times10^{-7}\ \mathrm{C}}$$

70 •• Two 3.0-μC point charges are located at $x = 0$, $y = 2.0$ m and at $x = 0$, $y = -2.0$ m. Two other point charges, each with charge equal to Q, are located at $x = 4.0$ m, $y = 2.0$ m and at $x = 4.0$ m, $y = -2.0$ m (Figure 21-43). The electric field at $x = 0$, $y = 0$ due to the presence of the four charges is $(4.0 \times 10^3\ \text{N/C})\hat{\boldsymbol{i}}$. Determine Q.

Picture the Problem The field at (0, 0) produced by the two 3.0 μC charges is zero. Thus the entire field at (0, 0) is due to the two point charges whose charges are Q. Let the numeral 1 refer to the point charge in the 1st quadrant and the numeral 2 to the point charge in the 4th quadrant. We can use Coulomb's law for the electric field due to a point charge and the superposition of forces to express the field at the origin and use this equation to solve for Q.

Express the electric field at the origin due to the point charges Q:

$$\begin{aligned}\vec{\boldsymbol{E}}(0,0) &= \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2 = \frac{kQ}{r_{1,0}^2}\hat{\boldsymbol{r}}_{1,0} + \frac{kQ}{r_{2,0}^2}\hat{\boldsymbol{r}}_{2,0} \\ &= \frac{kQ}{r^3}\left[(-4.0\,\text{m})\hat{\boldsymbol{i}} + (-2.0\,\text{m})\hat{\boldsymbol{j}}\right] + \frac{kQ}{r^3}\left[(-4.0\,\text{m})\hat{\boldsymbol{i}} + (2.0\,\text{m})\hat{\boldsymbol{j}}\right] = -\frac{(8.0\,\text{m})kQ}{r^3}\hat{\boldsymbol{i}} \\ &= E_x\hat{\boldsymbol{i}}\end{aligned}$$

where r is the distance from each charge to the origin and $E_x = -\dfrac{(8.0\,\text{m})kQ}{r^3}$.

Express r in terms of the coordinates (x, y) of the point charges:

$$r = \sqrt{x^2 + y^2}$$

Substitute for r to obtain:

$$E_x = -\frac{(8.0\,\text{m})kQ}{\left(x^2 + y^2\right)^{3/2}}$$

Solving for Q yields:

$$Q = -\frac{E_x\left(x^2 + y^2\right)^{3/2}}{k(8.0\,\text{m})}$$

Substitute numerical values and evaluate Q:

$$Q = -\frac{(4.0\,\text{kN/C})\left[(4.0\,\text{m})^2 + (2.0\,\text{m})^2\right]^{3/2}}{\left(8.988 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(8.0\,\text{m})} = \boxed{-5.0\,\mu\text{C}}$$

71 •• **[SSM]** Two point charges have a total charge of 200 μC and are separated by 0.600 m. (*a*) Find the charge of each particle if the particles repel each other with a force of 120 N. (*b*) Find the force on each particle if the charge on each particle is 100 μC.

Picture the Problem Let the numeral 1 denote one of the point charges and the numeral 2 the other. Knowing the total charge on the two spheres, we can use Coulomb's law to find the charge on each of them. A second application of Coulomb's law when the spheres carry the same charge and are 0.600 m apart will yield the force each exerts on the other.

(*a*) Use Coulomb's law to express the repulsive force each point charge exerts on the other:

$$F = \frac{kq_1q_2}{r_{1,2}^2}$$

Express q_2 in terms of the total charge and q_1:

$$q_2 = Q - q_1$$

Substitute for q_2 to obtain:

$$F = \frac{kq_1(Q - q_1)}{r_{1,2}^2}$$

Substitute numerical values to obtain:

$$120\,\text{N} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[(200\,\mu\text{C})q_1 - q_1^2\right]}{(0.600\,\text{m})^2}$$

Simplify to obtain the quadratic equation:

$$q_1^2 + (-200\,\mu\text{C})q_1 + 4806(\mu\text{C})^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$q_1 = 27.9\,\mu\text{C and } 172\,\mu\text{C}$$

Hence the charges on the particles are:

$$\boxed{27.9\,\mu\text{C}} \text{ and } \boxed{172\,\mu\text{C}}$$

(*b*) Use Coulomb's law to express the repulsive force each point charge exerts on the other when $q_1 = q_2 = 100\ \mu\text{C}$:

$$F = \frac{kq_1q_2}{r_{1,2}^2}$$

Substitute numerical values and evaluate F:

$$F = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\frac{(100\,\mu\text{C})^2}{(0.600\,\text{m})^2} = \boxed{250\,\text{N}}$$

72 •• Two point charges have a total charge of 200 μC and are separated by 0.600 m. (*a*) Find the charge of each particle if the particles attract each other with a force of 120 N. (*b*) Find the force on each particle if the charge on each particle is 100 μC.

Picture the Problem Let the numeral 1 denote one of the point charges and the numeral 2 the other. Knowing the total charge on the two point charges, we can use Coulomb's law to find the charge on each of them. A second application of Coulomb's law when the particles carry the same charge and are 0.600 m apart will yield the electric force each exerts on the other.

(*a*) Use Coulomb's law to express the attractive electric force each point charge exerts on the other:

$$F = -\frac{kq_1q_2}{r_{1,2}^2}$$

Express q_2 in terms of the total charge and q_1:

$$q_2 = Q - q_1$$

Substitute for q_2 to obtain:

$$F = -\frac{kq_1(Q - q_1)}{r_{1,2}^2}$$

Writing this explicitly as a quadratic equation gives:

$$q_1^2 - Qq_1 - \frac{r_{1,2}^2 F}{k} = 0$$

Substitute numerical values to obtain:

$$q_1^2 - (200\,\mu\text{C})q_1 - \frac{(0.600\,\text{m})^2(120\,\text{N})}{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2} = 0$$

or

$$q_1^2 + (-200\,\mu\text{C})q_1 - 4805(\mu\text{C})^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$q_1 = \boxed{-21.7\,\mu\text{C}}$$

and

$$q_2 = Q - q_1 = \boxed{222\,\mu\text{C}}$$

(*b*) Use Coulomb's law to express the repulsive electric force each point charge exerts on the other when $q_1 = q_2 = 100\ \mu$C:

$$F = \frac{kq_1q_2}{r_{1,2}^2}$$

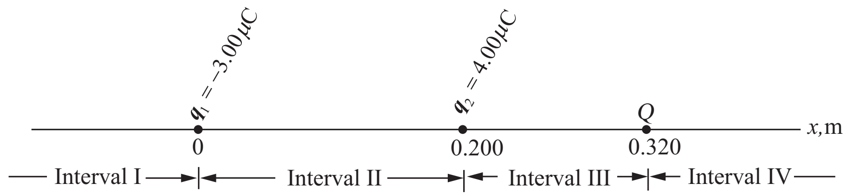
Substitute numerical values and evaluate F:

$$F = \left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\frac{\left(100\ \mu\mathrm{C}\right)^2}{\left(0.600\ \mathrm{m}\right)^2} = \boxed{250\ \mathrm{N}}$$

73 •• A point charge of –3.00 μC is located at the origin; a point charge of 4.00 μC is located on the x axis at $x = 0.200$ m; a third point charge Q is located on the x axis at $x = 0.320$ m. The electric force on the 4.00-μC charge is 240 N in the $+x$ direction. (a) Determine the charge Q. (b) With this configuration of three charges, at what location(s) is the electric field zero?

Comment: DAVID: In 87 the ism solution is cavilier about the region in which the electric field is zero. Just to say by inspection is little help to those who do not know how to go about the process.

Picture the Problem (a) We can use Coulomb's law for point charges and the superposition of forces to express the net electric force acting on q_2 and then solve this equation to determine the charge Q. (b) To identify the location(s) at which the electric field is zero, we'll need to systematically examine the resultant electric fields in the intervals I, II, III, and IV identified below on the pictorial representation. Drawing the electric field vectors in each of these intervals will help you get the signs of the terms in the field equations right.



(a) Use Coulomb's law to express the electric force on the particle whose charge is 4.00-μC:

$$\vec{F}_2 = \vec{F}_{1,2} + \vec{F}_{Q,2} = \frac{kq_1q_2}{r_{1,2}^2}\hat{i} + \frac{kQq_2}{r_{Q,2}^2}\left(-\hat{i}\right) = kq_2\left[\frac{q_1}{r_{1,2}^2} - \frac{Q}{r_{Q,2}^2}\right]\hat{i} = F_2\hat{i}$$

Solving for Q yields:

$$Q = r_{Q,2}^2\left[\frac{q_1}{r_{1,2}^2} - \frac{F_2}{kq_2}\right]$$

Substitute numerical values and evaluate Q:

$$Q = \left(0.120\ \mathrm{m}\right)^2\left[\frac{-3.00\ \mu\mathrm{C}}{\left(0.200\ \mathrm{m}\right)^2} - \frac{240\ \mathrm{N}}{\left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\left(4.00\ \mu\mathrm{C}\right)}\right] = \boxed{-97.2\ \mu\mathrm{C}}$$

(*b*) Applying Coulomb's law for electric fields and the superposition of fields in interval I yields:

$$\frac{3.00\ \mu\text{C}}{(x-0)^2}-\frac{4.00\ \mu\text{C}}{(x-0.200\,\text{m})^2}+\frac{97.2\ \mu\text{C}}{(x-0.320\,\text{m})^2}=0$$

where k has been divided out and x is in the interval defined by $x < 0$. The roots of this equation are at x = 0.170 m and x =0.220 m. Neither of these are in interval I.

Applying Coulomb's law for electric fields and the superposition of fields in interval II yields:

$$-\frac{3.00\ \mu\text{C}}{(x-0)^2}-\frac{4.00\ \mu\text{C}}{(x-0.200\,\text{m})^2}+\frac{97.2\ \mu\text{C}}{(x-0.320\,\text{m})^2}=0$$

The roots of this equation are at x = –0.0720 m, x = 0.0508 m, x = 0.169 m, and x = 0.220 m. The second and third of these are in interval II.

Applying Coulomb's law for electric fields and the superposition of fields in interval III yields:

$$-\frac{3.00\ \mu\text{C}}{(x-0)^2}+\frac{4.00\ \mu\text{C}}{(x-0.200\,\text{m})^2}+\frac{97.2\ \mu\text{C}}{(x-0.320\,\text{m})^2}=0$$

The roots of this equation are at x = –0.0649 m and x = 0.0454 m. Neither of these are in interval III.

Applying Coulomb's law for electric fields and the superposition of fields in interval IV yields:

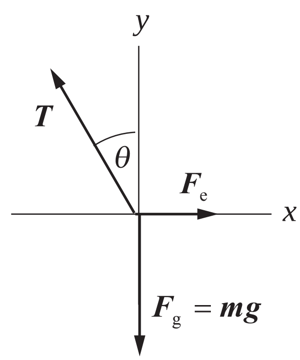
$$-\frac{3.00\ \mu\text{C}}{(x-0)^2}+\frac{4.00\ \mu\text{C}}{(x-0.200\,\text{m})^2}-\frac{97.2\ \mu\text{C}}{(x-0.320\,\text{m})^2}=0$$

The roots of this equation are at x = 0.170 m and x = 0.220. Neither of these are in interval IV.

Summarizing, the electric field is zero at two locations in interval II: $x=\boxed{0.0508\,\text{m}}$ and $x=\boxed{0.169\,\text{m}}$

74 •• Two point particles, each of mass m and charge q, are suspended from a common point by threads of length L. Each thread makes an angle θ with the vertical as shown in Figure 21-44. (*a*) Show that $q=2L\sin\theta\sqrt{(mg/k)\tan\theta}$ where k is the Coulomb constant. (*b*) Find q if m = 10.0 g, L = 50.0 cm, and θ = 10.0º.

Picture the Problem Each point particle is in static equilibrium under the influence of the tension $\vec{T}$, the gravitational force $\vec{F}_g$, and the electric force $\vec{F}_E$. We can use Coulomb's law to relate the electric force to the charge on each particle and their separation and the conditions for static equilibrium to relate these forces to the charge on each particle.



(*a*) Apply the conditions for static equilibrium to the point particle whose free-body diagram is shown above:

$$\sum F_x = F_E - T\sin\theta = \frac{kq^2}{r^2} - T\sin\theta = 0$$

and

$$\sum F_y = T\cos\theta - mg = 0$$

Eliminate T between these equations to obtain:

$$\tan\theta = \frac{kq^2}{mgr^2} \Rightarrow q = r\sqrt{\frac{mg\tan\theta}{k}}$$

Referring to the figure, relate the separation of the spheres r to the length of the pendulum L:

$$r = 2L\sin\theta$$

Substitute for r to obtain:

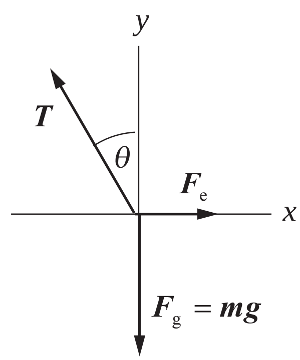
$$q = \boxed{2L\sin\theta\sqrt{\frac{mg\tan\theta}{k}}}$$

(*b*) Evaluate q for m = 10.0 g, L = 50.0 cm, and θ = 10°:

$$q = 2(0.500\,\text{m})\sin 10.0°\sqrt{\frac{(0.0100\,\text{kg})(9.81\,\text{m/s}^2)\tan 10.0°}{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2}} = \boxed{0.241\,\mu\text{C}}$$

75 •• Suppose that in Problem 74 L = 1.5 m and m = 0.010 kg. (*a*) What is the angle that each string makes with the vertical if q = 0.75 μC? (*b*) What is the angle that each string makes with the vertical if one particle has a charge of 0.50 μC, the other has a charge of 1.0 μC?

Picture the Problem Each sphere is in static equilibrium under the influence of the tension $\vec{T}$, the gravitational force $\vec{F}_g$, and the electric force $\vec{F}_e$. We can use Coulomb's law to relate the electric force to the charge on each sphere and their separation and the conditions for static equilibrium to relate these forces to the charge on each sphere.



(*a*)Apply the conditions for static equilibrium to the charged sphere:

$$\sum F_x = F_e - T\sin\theta = \frac{kq^2}{r^2} - T\sin\theta = 0$$

and

$$\sum F_y = T\cos\theta - mg = 0$$

Eliminate *T* between these equations to obtain:

$$\tan\theta = \frac{kq^2}{mgr^2}$$

Referring to the figure for Problem 74, relate the separation of the spheres *r* to the length of the pendulum *L*:

$$r = 2L\sin\theta$$

Substitute for *r* to obtain:

$$\tan\theta = \frac{kq^2}{4mgL^2\sin^2\theta}$$

or

$$\sin^2\theta\tan\theta = \frac{kq^2}{4mgL^2} \qquad (1)$$

Substitute numerical values and evaluate $\sin^2\theta\tan\theta$:

$$\sin^2\theta\tan\theta = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.75\,\mu\text{C}\right)^2}{4\left(0.010\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)\left(1.5\,\text{m}\right)^2} = 5.73\times10^{-3}$$

Because $\sin^2\theta\tan\theta << 1$:

$$\sin\theta \approx \tan\theta \approx \theta$$

and

$$\theta^3 \approx 5.73\times10^{-3}$$

Solving for θ gives: $\theta = 0.179\,\text{rad} = \boxed{10°}$

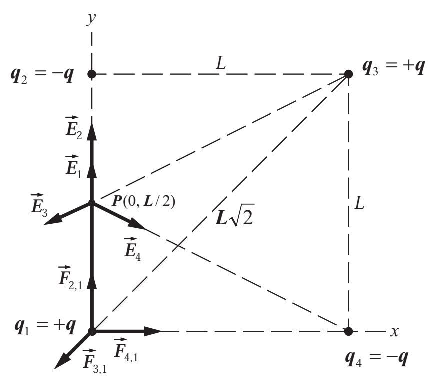
(*b*) Evaluate equation (1) with replacing q^2 with q_1q_2:

$$\sin^2\theta\tan\theta = \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(0.50\,\mu\text{C})(1.0\,\mu\text{C})}{4(0.010\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m})^2} = 5.09\times10^{-3} \approx \theta^3$$

Solve for θ to obtain: $\theta = 0.172\,\text{rad} = \boxed{9.9°}$

76 •• Four point charges of equal magnitude are arranged at the corners of a square of side L as shown in Figure 21-45. (*a*) Find the magnitude and direction of the force exerted on the charge in the lower left corner by the other three charges. (*b*) Show that the electric field at the midpoint of one of the sides of the square is directed along that side toward the negative charge and has a magnitude E given by $E = k\dfrac{8q}{L^2}\left(1-\dfrac{1}{5\sqrt{5}}\right)$.

Picture the Problem Let the origin be at the lower left-hand corner and designate the point charges as shown in the diagram. We can apply Coulomb's law for point charges to find the forces exerted on q_1 by q_2, q_3, and q_4 and superimpose these forces to find the net force exerted on q_1. In Part (*b*), we'll use Coulomb's law for the electric field due to a point charge and the superposition of fields to find the electric field at point $P(0, L/2)$.



(*a*) Using superposition of forces, express the net force exerted on q_1:

$$\vec{F}_1 = \vec{F}_{2,1} + \vec{F}_{3,1} + \vec{F}_{4,1}$$

Apply Coulomb's law to express $\vec{F}_{2,1}$:

$$\vec{F}_{2,1} = \frac{kq_2q_1}{r_{2,1}^2}\hat{r}_{2,1} = \frac{kq_2q_1}{r_{2,1}^3}\vec{r}_{2,1} = \frac{k(-q)q}{L^3}\left(-L\,\hat{j}\right) = \frac{kq^2}{L^2}\hat{j}$$

Apply Coulomb's law to express $\vec{F}_{4,1}$:

$$\vec{F}_{4,1}=\frac{kq_4q_1}{r_{4,1}^2}\hat{r}_{4,1}=\frac{kq_4q_1}{r_{4,1}^3}\vec{r}_{4,1}=\frac{k(-q)q}{L^3}\left(-L\,\hat{i}\right)=\frac{kq^2}{L^2}\hat{i}$$

Apply Coulomb's law to express $\vec{F}_{3,1}$:

$$\begin{aligned}\vec{F}_{3,1}&=\frac{kq_3q_1}{r_{3,1}^2}\hat{r}_{3,1}=\frac{kq_3q_1}{r_{3,1}^3}\vec{r}_{3,1}\\&=\frac{kq^2}{2^{3/2}L^3}\left(-L\,\hat{i}-L\,\hat{j}\right)\\&=-\frac{kq^2}{2^{3/2}L^2}\left(\hat{i}+\hat{j}\right)\end{aligned}$$

Substitute and simplify to obtain:

$$\begin{aligned}\vec{F}_1&=\frac{kq^2}{L^2}\hat{j}-\frac{kq^2}{2^{3/2}L^2}\left(\hat{i}+\hat{j}\right)+\frac{kq^2}{L^2}\hat{i}\\&=\frac{kq^2}{L^2}\left(\hat{i}+\hat{j}\right)-\frac{kq^2}{2^{3/2}L^2}\left(\hat{i}+\hat{j}\right)\\&=\boxed{\frac{kq^2}{L^2}\left(1-\frac{1}{2\sqrt{2}}\right)\left(\hat{i}+\hat{j}\right)}\end{aligned}$$

(*b*) Using superposition of fields, express the resultant field at point *P*:

$$\vec{E}_P=\vec{E}_1+\vec{E}_2+\vec{E}_3+\vec{E}_4 \qquad (1)$$

Use Coulomb's law to express $\vec{E}_1$:

$$\begin{aligned}\vec{E}_1&=\frac{kq_1}{r_{1,P}^2}\hat{r}_{1,P}=\frac{kq}{r_{1,P}^3}\left(\frac{L}{2}\hat{j}\right)\\&=\frac{kq}{\left(\frac{L}{2}\right)^3}\left(\frac{L}{2}\hat{j}\right)=\frac{4kq}{L^2}\hat{j}\end{aligned}$$

Use Coulomb's law to express $\vec{E}_2$:

$$\begin{aligned}\vec{E}_2&=\frac{kq_2}{r_{2,P}^2}\hat{r}_{2,P}=\frac{k(-q)}{r_{2,P}^3}\left(\frac{L}{2}\hat{j}\right)\\&=\frac{-kq}{\left(\frac{L}{2}\right)^3}\left(-\frac{L}{2}\hat{j}\right)=\frac{4kq}{L^2}\hat{j}\end{aligned}$$

Use Coulomb's law to express $\vec{E}_3$:

$$\vec{E}_3 = \frac{kq_3}{r_{3,P}^2}\hat{r}_{3,P} = \frac{kq}{r_{3,P}^3}\left(-L\hat{i} - \frac{L}{2}\hat{j}\right)$$
$$= \frac{8kq}{5^{3/2}L^2}\left(-\hat{i} - \frac{1}{2}\hat{j}\right)$$

Use Coulomb's law to express $\vec{E}_4$:

$$\vec{E}_4 = \frac{kq_4}{r_{4,P}^2}\hat{r}_{4,P} = \frac{k(-q)}{r_{4,P}^3}\left(-L\hat{i} + \frac{L}{2}\hat{j}\right)$$
$$= \frac{8kq}{5^{3/2}L^2}\left(\hat{i} - \frac{1}{2}\hat{j}\right)$$

Substitute in equation (1) and simplify to obtain:

$$\vec{E}_P = \frac{4kq}{L^2}\hat{j} + \frac{4kq}{L^2}\hat{j} + \frac{8kq}{5^{3/2}L^2}\left(-\hat{i} - \frac{1}{2}\hat{j}\right) + \frac{8kq}{5^{3/2}L^2}\left(\hat{i} - \frac{1}{2}\hat{j}\right) = \boxed{k\frac{8q}{L^2}\left(1 - \frac{1}{5\sqrt{5}}\right)\hat{j}}$$

77 •• [SSM] Figure 21-46 shows a dumbbell consisting of two identical small particles, each of mass m, attached to the ends of a thin (massless) rod of length a that is pivoted at its center. The particles carry charges of $+q$ and $-q$, and the dumbbell is located in a uniform electric field $\vec{E}$. Show that for small values of the angle θ between the direction of the dipole and the direction of the electric field, the system displays a rotational form of simple harmonic motion, and obtain an expression for the period of that motion.

Picture the Problem We can apply Newton's second law in rotational form to obtain the differential equation of motion of the dipole and then use the small angle approximation $\sin\theta \approx \theta$ to show that the dipole experiences a linear restoring torque and, hence, will experience simple harmonic motion.

Apply $\sum\tau = I\alpha$ to the dipole:

$$-pE\sin\theta = I\frac{d^2\theta}{dt^2}$$

where τ is negative because it acts in such a direction as to decrease θ.

For small values of θ, $\sin\theta \approx \theta$ and:

$$-pE\theta = I\frac{d^2\theta}{dt^2}$$

Express the moment of inertia of the dipole:

$$I = \tfrac{1}{2}ma^2$$

Relate the dipole moment of the dipole to its charge and the charge separation:

$$p = qa$$

Substitute for p and I to obtain:

$$\tfrac{1}{2}ma^2\frac{d^2\theta}{dt^2} = -qaE\theta$$

or

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{2qE}{ma}\theta}$$

the differential equation for a simple harmonic oscillator with angular frequency $\omega = \sqrt{2qE/ma}$.

Express the period of a simple harmonic oscillator:

$$T = \frac{2\pi}{\omega}$$

Substitute for ω and simplify to obtain:

$$T = \boxed{2\pi\sqrt{\frac{ma}{2qE}}}$$

78 •• For the dumbbell in Problem 77, let m = 0.0200 kg, a = 0.300 m, and $\vec{E} = (600\text{ N/C})\hat{i}$. The dumbbell is initially at rest and makes an angle of 60º with the x axis. The dumbbell is then released, and when it is momentarily aligned with the electric field, its kinetic energy is 5.00 × 10^{-3} J. Determine the magnitude of q.

Picture the Problem We can apply conservation of energy and the definition of the potential energy of a dipole in an electric field to relate q to the kinetic energy of the dumbbell when it is aligned with the field.

Using conservation of energy, relate the initial potential energy of the dumbbell to its kinetic energy when it is momentarily aligned with the electric field:

$$\Delta K + \Delta U = 0$$

or, because $K_i = 0$,

$$K + \Delta U = 0 \qquad (1)$$

where K is the kinetic energy when it is aligned with the field.

Express the change in the potential energy of the dumbbell as it aligns with the electric field in terms of its dipole moment, the electric field, and the angle through which it rotates:

$$\begin{aligned}\Delta U &= U_{\mathrm{f}} - U_{\mathrm{i}} \\ &= -pE\cos\theta_{\mathrm{f}} + pE\cos\theta_{\mathrm{i}} \\ &= qaE(\cos 60° - 1)\end{aligned}$$

Substitute for ΔU in equation (1) to obtain:

$$K + qaE(\cos 60° - 1) = 0$$

Solving for q yields:

$$q = \frac{K}{aE(1 - \cos 60°)}$$

Substitute numerical values and evaluate q:

$$q = \frac{5.00\times10^{-3}\ \mathrm{J}}{(0.300\,\mathrm{m})(600\,\mathrm{N/C})(1-\cos 60°)} = \boxed{56\,\mu\mathrm{C}}$$

79 •• [SSM] An electron (charge $-e$, mass m) and a positron (charge $+e$, mass m) revolve around their common center of mass under the influence of their attractive coulomb force. Find the speed v of each particle in terms of e, m, k, and their separation distance L.

Picture the Problem The forces the electron and the proton exert on each other constitute an action-and-reaction pair. Because the magnitudes of their charges are equal and their masses are the same, we find the speed of each particle by finding the speed of either one. We'll apply Coulomb's force law for point charges and Newton's second law to relate v to e, m, k, and their separation distance L.

Apply Newton's second law to the positron to obtain:

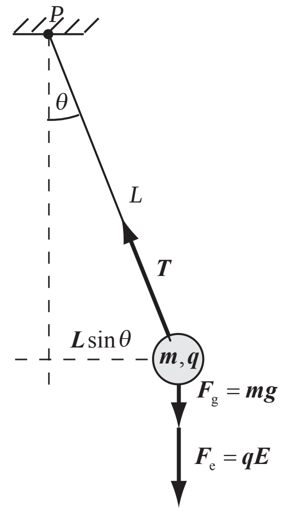
$$\frac{ke^2}{L^2} = m\frac{v^2}{\frac{1}{2}L} \Rightarrow \frac{ke^2}{L} = 2mv^2$$

Solving for v gives:

$$v = \boxed{\sqrt{\frac{ke^2}{2mL}}}$$

80 ••• A simple pendulum of length 1.0 m and mass 5.0×10^{-3} kg is placed in a uniform electric field $\vec{E}$ that is directed vertically upward. The bob has a charge of $-8.0\ \mu$C. The period of the pendulum is 1.2 s. What is the magnitude and direction of $\vec{E}$?

Picture the Problem Because the period of this pendulum is 60% of the period (2.006 s) of a simple pendulum with an uncharged bob, we know that the bob must be experiencing an additional downward force (in the direction of the gravitational force). Because the electric force acting on the negatively-charged bob is downward, the electric field must be upward. We can find the magnitude of the electric field by apply Newton's second law in rotational form to the simple pendulum. Doing so will lead us to the equation of motion for the pendulum and from this equation we can obtain an expression relating its period to the magnitude of E. Knowing the pendulum's period in the absence of the electric field will allow us to derive an expression for E from the ratio of the two periods.



Taking clockwise torques to be negative, apply $\sum \tau_P = I_P\alpha$ to the pendulum bob to obtain:

$$-(mg+qE)L\sin\theta = I_P\alpha$$

or, because $I_P = mL^2$ and $\alpha = \dfrac{d^2\theta}{dt^2}$,

$$-(mg+qE)L\sin\theta = mL^2\frac{d^2\theta}{dt^2}$$

For small displacements from equilibrium, $\sin\boldsymbol{\theta} \approx \boldsymbol{\theta}$:

$$mL^2\frac{d^2\theta}{dt^2} = -(mg+qE)L\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{mg+qE}{mL}\theta = 0$$

This is the equation of simple harmonic motion with:

$$\omega^2 = \frac{mg+qE}{mL}$$

The period of this motion is given by:

$$T' = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{mL}{mg+qE}}$$

In the absence of the electric field, the period of the simple pendulum would be:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Dividing the second of these equations by the first and simplifying yields:

$$\frac{T}{T'}=\frac{2\pi\sqrt{\frac{L}{g}}}{2\pi\sqrt{\frac{mL}{mg+qE}}}=\sqrt{\frac{mg+qE}{mg}}$$
$$=\sqrt{1+\frac{qE}{mg}}$$

Solving for E gives:

$$E=\frac{mg}{q}\left[\left(\frac{T}{T'}\right)-1\right]$$

Noting that the period of the simple pendulum in the absence of the electric field is 2.006 s, substitute numerical values and evaluate E:

$$E=\frac{(5.0\text{ mg})(9.81\text{ m/s}^2)}{8.0\ \mu\text{C}}\left[\left(\frac{2.006\text{ s}}{1.2\text{ s}}\right)^2-1\right]=\boxed{1.1\times10^4\text{ N/C}}\text{, upward}$$

81 ••• A point particle of mass m and charge q is constrained to move vertically inside a narrow, frictionless cylinder (Figure 21-47). At the bottom of the cylinder is a point charge Q having the same sign as q. (a) Show that the particle whose mass is m will be in equilibrium at a height $y_0 = (kqQ/mg)^{1/2}$. (b) Show that if the particle is displaced from its equilibrium position by a small amount and released, it will exhibit simple harmonic motion with angular frequency $\omega = (2g/y_0)^{1/2}$.

Picture the Problem We can use Coulomb's force law for point particles and the condition for translational equilibrium to express the equilibrium position as a function of k, q, Q, m, and g. In Part (b) we'll need to show that the displaced point charge experiences a linear restoring force and, hence, will exhibit simple harmonic motion.

(a) Apply the condition for translational equilibrium to the particle:

$$\frac{kqQ}{y_0^2}-mg=0\Rightarrow y_0=\boxed{\sqrt{\frac{kqQ}{mg}}}$$

(b) Express the restoring force that acts on the particle when it is displaced a distance Δy from its equilibrium position:

$$F=\frac{kqQ}{(y_0+\Delta y)^2}-\frac{kqQ}{y_0^2}$$

or, because $\Delta y \ll y_0$,

$$F\approx\frac{kqQ}{y_0^2+2y_0\Delta y}-\frac{kqQ}{y_0^2}$$

Simplify this expression further by writing it with a common denominator:

$$F=-\frac{2y_0\Delta ykqQ}{y_0^4+2y_0^3\Delta y}=-\frac{2y_0\Delta ykqQ}{y_0^4\left(1+2\dfrac{\Delta y}{y_0}\right)}\approx-\frac{2\Delta ykqQ}{y_0^3}$$

again, because $\Delta y << y_0$.

From the 1st step of our solution: $\dfrac{kqQ}{y_0^2}=mg$

Substitute for $\dfrac{kqQ}{y_0^2}$ and simplify to obtain: $F=-\dfrac{2mg}{y_0}\Delta y$

Apply Newton's second law to the displaced particle to obtain:

$$m\frac{d^2\Delta y}{dt^2}=-\frac{2mg}{y_0}\Delta y$$

or

$$\boxed{\frac{d^2\Delta y}{dt^2}+\frac{2g}{y_0}\Delta y=0}$$

the differential equation of simple harmonic motion with $\omega=\boxed{\sqrt{2g/y_0}}$.

82 ••• Two neutral molecules on the x axis attract each other. Each molecule has a dipole moment $\vec{p}$, and these dipole moments are on the $+x$ axis and are separated by a distance d. Derive an expression for the force of attraction in terms of p and d.

Picture the Problem We can relate the force of attraction that each molecule exerts on the other to the potential energy function of either molecule using $F=-dU/dx$. We can relate U to the electric field at either molecule due to the presence of the other through $U = -pE$. Finally, the electric field at either molecule is given by $E=2kp/x^3$.

Express the force of attraction between the dipoles in terms of the spatial derivative of the potential energy function of p_1:

$$F=-\frac{dU_1}{dx} \qquad (1)$$

Express the potential energy of the dipole p_1:

$U_1=-p_1E_1$

where E_1 is the field at p_1 due to p_2.

Express the electric field strength at p_1 due to p_2:

$$E_1 = \frac{2kp_2}{x^3}$$

where x is the separation of the dipoles.

Substitute for E_1 to obtain:

$$U_1 = -\frac{2kp_1p_2}{x^3}$$

Substitute in equation (1) and differentiate with respect to x:

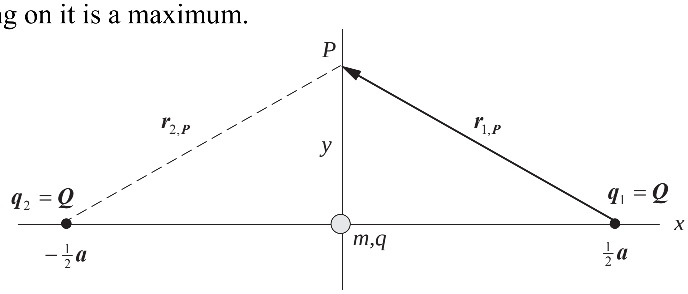
$$F = -\frac{d}{dx}\left[-\frac{2kp_1p_2}{x^3}\right] = \frac{6kp_1p_2}{x^4}$$

Evaluate F for $p_1 = p_2 = p$ and $x = d$ to obtain:

$$F = \boxed{\frac{6kp^2}{d^4}}$$

83 ••• Two equal positive point charges Q are on the x axis at $x = \frac{1}{2}a$ and $x = -\frac{1}{2}a$. (*a*) Obtain an expression for the electric field on the y axis as a function of y. (*b*) A bead of mass M, which has a charge q, moves along the y axis on a thin frictionless taut thread. Find the electric force that acts on the bead as a function of y and determine the sign of q such that this force always points away from the origin. (*c*) The bead is initially at rest at the origin. If it is given a slight nudge in the $+y$ direction, how fast will the bead be traveling the instant the net force on it is a maximum? (Assume any effects due to gravity are negligible.)

Picture the Problem (*a*) We can use Coulomb's law for the electric field due to a point charge and superposition of fields to find the electric field at any point on the y axis. (*b*) Using the definition of electric field will yield an expression for the electric force that acts on the bead. In Part (*c*) we can set $dF_y/dy = 0$ to find the value of y that maximizes F_y and then use the work-kinetic energy theorem (integrating F_y from 0 to this extreme value will yield an expression for the work done on the bead during this displacement) to find the speed of the bead when the force acting on it is a maximum.



(*a*) Use Coulomb's law for the electric field due to a point charge and superposition of fields, to express the field at point P on the y axis:

$$\vec{E}_y = \vec{E}_1 + \vec{E}_2 = \frac{kq_1}{r_{1,P}^2}\hat{r}_{1,P} + \frac{kq_2}{r_{2,P}^2}\hat{r}_{2,P}$$

The unit vectors $\hat{r}_{1,P}$ and $\hat{r}_{2,P}$ are given by:

$$\hat{r}_{1,P} = \frac{\hat{r}_{1,P}}{r_{1,P}} = \frac{-\frac{1}{2}a\hat{i} + y\hat{j}}{r_{1,P}}$$

and

$$\hat{r}_{2,P} = \frac{\hat{r}_{2,P}}{r_{2,P}} = \frac{\frac{1}{2}a\hat{i} + y\hat{j}}{r_{2,P}}$$

Substituting for q_1, q_2, $\hat{r}_{1,P}$, $\hat{r}_{2,P}$ and $r_{1,P} = r_{2,\mathrm{P}} = \left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{1/2}$ yields:

$$\vec{E}_y = \frac{kQ}{r_{1,P}^2}\left(\frac{-\frac{1}{2}a\hat{i} + y\hat{j}}{r_{1,P}}\right) + \frac{kQ}{r_{2,P}^2}\left(\frac{\frac{1}{2}a\hat{i} + y\hat{j}}{r_{2,P}}\right) = \frac{kQ}{r_{1,P}^3}\left(-\frac{1}{2}a\hat{i} + y\hat{j}\right) + \frac{kQ}{r_{2,P}^3}\left(\frac{1}{2}a\hat{i} + y\hat{j}\right)$$

$$= \frac{kQ}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}\left(y\hat{j}\right) + \frac{kQ}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}\left(y\hat{j}\right) = \boxed{\frac{2kQy}{\left[y^2 + \frac{1}{4}a^2\right]^{3/2}}\hat{j}}$$

(*b*) Relate the electric force on the bead to its charge and the electric field:

$$\vec{F} = q\vec{E}_y = \boxed{\frac{2kqQy}{\left[y^2 + \frac{1}{4}a^2\right]^{3/2}}\hat{j}}$$

where q must be positive if $\vec{F}$ always points away from the origin.

(*c*) Apply the work-kinetic energy theorem to the bead as it moves from the origin to the location at which F_y is a maximum to obtain:

$$W_{\mathrm{net}_y} = \Delta K = K_\mathrm{f} - K_\mathrm{i}$$

or, because $K_\mathrm{i} = 0$,

$$W_{\text{done by } F_y} = \frac{1}{2}Mv_\mathrm{f}^2$$

Solving for v_f yields:

$$v_\mathrm{f} = \sqrt{\frac{2W_{\text{done by } F_y}}{M}}$$

The work done on the bead by F_y as the bead moves from the origin to $y_{\max}$ (the y coordinate of the point at which F_y is a maximum) is given by:

$$W_{\text{done by } F_y} = \int_0^{y_{\max}} \frac{2kqQy}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}dy$$

Substituting for $W_{\text{done by } F_y}$ and simplifying yields:

$$v_f = \sqrt{\frac{4kqQ}{m}\int_0^{y_{max}} \frac{y}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}dy}$$

The condition that F_y is a maximum is:

$$\frac{dF_y}{dy} = \frac{d}{dy}\left[\frac{y}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}\right] = 0$$

Carrying out the details of the differentiation and solving for the critical value y_{max} yields:

$y_{max} = \dfrac{a}{2\sqrt{2}}$ where we've ignored the negative value because the bead is given a nudge in the $+y$ direction.

Substitute for y_{max} in the expression for v_f to obtain:

$$v_f = \sqrt{\frac{4kqQ}{M}\int_0^{\frac{a}{2\sqrt{2}}} \frac{y}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}dy}$$

Evaluating the integral yields:

$$\int_0^{\frac{a}{2\sqrt{2}}} \frac{y}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}dy \approx \frac{0.367}{a}$$

Substitute for $\int_0^{\frac{a}{2\sqrt{2}}} \frac{y}{\left[y^2 + \left(\frac{1}{2}a\right)^2\right]^{3/2}}dy$ and simplify to obtain:

$$v_f = \sqrt{\frac{4kqQ}{M}\left(\frac{0.367}{a}\right)} = \boxed{1.21\sqrt{\frac{kqQ}{aM}}}$$

84 ••• A gold nucleus is 100 fm (1 fm = 10^{-15} m) from a proton, which initially is at rest. When the proton is released, it speeds away because of the repulsion that it experiences due to the charge on the gold nucleus. What is the proton's speed a large distance (assume to be infinity) from the gold nucleus? (Assume the gold nucleus remains stationary.)

Picture the Problem The work done by the electric field of the gold nucleus changes the kinetic energy of the proton. We can apply the work-kinetic energy theorem to derive an expression for the speed of the proton as a function of its distance from the gold nucleus. Because the repulsive Coulomb force $\vec{F}_e$ varies with distance, we'll have to evaluate $\int \vec{F}_e \cdot d\vec{r}$ in order to find the work done on the proton by this force.

Apply the work-kinetic energy theorem to the proton to obtain:

$$W_{\text{net}} = \int_{r_0}^{\infty} \vec{\boldsymbol{F}}_{\text{e}} \cdot d\hat{\boldsymbol{r}} = \Delta K$$

or, because $\int_{r_0}^{\infty} \vec{\boldsymbol{F}}_{\text{e}} \cdot d\hat{\boldsymbol{r}} = \int_{r_0}^{\infty} \frac{ke(79e)}{r^2}dr$ and $K_{\text{i}} = 0$,

$$79ke^2 \int_{r_0}^{\infty} \frac{dr}{r^2} = \tfrac{1}{2} m_{\text{p}} v_{\text{f}}^2$$

Evaluating the integral yields:

$$-79ke^2\left[\frac{1}{r}\right]_{r_0}^{\infty} = \frac{79ke^2}{r_0} = \tfrac{1}{2} m_{\text{p}} v_{\text{f}}^2$$

Solve for v_{f} and simplify to obtain:

$$v_{\text{f}} = \sqrt{\frac{158ke^2}{m_{\text{p}} r_0}} = e\sqrt{\frac{158k}{m_{\text{p}} r_0}}$$

Substitute numerical values and evaluate v_{f}:

$$v_{\text{f}} = \left(1.602\times10^{-19}\,\text{C}\right)\sqrt{\frac{158\left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)}{\left(1.673\times10^{-27}\,\text{kg}\right)\left(1.00\times10^{-13}\,\text{m}\right)}} = \boxed{1.48\times10^{7}\,\text{m/s}}$$

85 ••• [SSM] During a famous experiment in 1919, Ernest Rutherford shot doubly ionized helium nuclei (also known as alpha particles) at a gold foil. He discovered that virtually all of the mass of an atom resides in an extremely compact nucleus. Suppose that during such an experiment, an alpha particle far from the foil has an initial kinetic energy of 5.0 MeV. If the alpha particle is aimed directly at the gold nucleus, and the only force acting on it is the electric force of repulsion exerted on it by the gold nucleus, how close will it approach the gold nucleus before turning back? That is, what is the minimum center-to-center separation of the alpha particle and the gold nucleus?

Comment: What????

Picture the Problem The work done by the electric field of the gold nucleus changes the kinetic energy of the alpha particle–eventually bringing it to rest. We can apply the work-kinetic energy theorem to derive an expression for the distance of closest approach. Because the repulsive Coulomb force $\vec{\boldsymbol{F}}_{\text{e}}$ varies with distance, we'll have to evaluate $\int \vec{\boldsymbol{F}}_{\text{e}} \cdot d\vec{\boldsymbol{r}}$ in order to find the work done on the alpha particles by this force.

Apply the work-kinetic energy theorem to the alpha particle to obtain:

$$W_{\text{net}} = \int_{\infty}^{r_{\min}} \vec{F}_{\text{e}} \cdot d\vec{r} = \Delta K$$

or, because

$$\int_{\infty}^{r_{\min}} \vec{F}_{\text{e}} \cdot d\vec{r} = -\int_{\infty}^{r_{\min}} \frac{k(2e)(79e)}{r^2}\,dr \text{ and}$$

$K_{\text{f}} = 0$,

$$-158ke^2 \int_{\infty}^{r_{\min}} \frac{dr}{r^2} = -K_{\text{i}}$$

Evaluating the integral yields:

$$-158ke^2\left[\frac{1}{r}\right]_{\infty}^{r_{\min}} = -\frac{158ke^2}{r_{\min}} = -K_{\text{i}}$$

Solve for $r_{\min}$ and simplify to obtain:

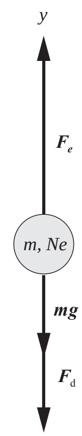
$$r_{\min} = \frac{158ke^2}{K_{\text{i}}}$$

Substitute numerical values and evaluate $r_{\min}$:

$$r_{\min} = \frac{158\left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\,\text{C}\right)^2}{5.0\,\text{Mev}\times\frac{1.602\times10^{-19}\,\text{J}}{\text{eV}}} = \boxed{4.6\times10^{-14}\ \text{m}}$$

86 ••• During the Millikan experiment used to determine the charge on the electron, a charged polystyrene microsphere is released in still air in a known vertical electric field. The charged microsphere will accelerate in the direction of the net force until it reaches terminal speed. The charge on the microsphere is determined by measuring the terminal speed. During one such experiment, the microsphere has radius of $r = 5.50 \times 10^{7}$ m, and the field has a magnitude $E = 6.00 \times 10^{4}$ N/C. The magnitude of the drag force on the sphere is given by $F_{\text{D}} = 6\pi\eta rv$, where v is the speed of the sphere and η is the viscosity of air ($\eta = 1.8 \times 10^{-5}$ N·s/m^2). Polystyrene has density 1.05×10^{3} kg/m^3. (*a*) If the electric field is pointing down and the polystyrene microsphere is rising with a terminal speed of 1.16×10^{-4} m/s, what is the charge on the sphere? (*b*) How many excess electrons are on the sphere? (*c*) If the direction of the electric field is reversed but its magnitude remains the same, what is the new terminal speed?

Picture the Problem The free body diagram shows the forces acting on the microsphere of mass m and having an excess charge of $q = Ne$ when the electric field is downward. Under terminal-speed conditions the sphere is in equilibrium under the influence of the electric force $\vec{F}_e$, its weight $m\vec{g}$, and the drag force $\vec{F}_d$. We can apply Newton's second law, under terminal-speed conditions, to relate the number of excess charges N on the sphere to its mass and, using Stokes' law, find its terminal speed.



(*a*) Apply Newton's second law to the microsphere to obtain:

$$F_e - mg - F_d = ma_y$$

or, because $a_y = 0$,

$$F_e - mg - F_{d,\text{terminal}} = 0$$

Substitute for F_e, m, and $F_{d,\text{terminal}}$ to obtain:

$$qE - \rho Vg - 6\pi\eta r v_t = 0$$

or, because $q = Ne$,

$$NeE - \tfrac{4}{3}\pi r^3\rho g - 6\pi\eta r v_t = 0$$

Solve for Ne to obtain:

$$Ne = \frac{\tfrac{4}{3}\pi r^3\rho g + 6\pi\eta r v_t}{E} \qquad (1)$$

Substitute numerical values and evaluate $\tfrac{4}{3}\pi r^3\rho g$:

$$\tfrac{4}{3}\pi r^3\rho g = \tfrac{4}{3}\pi\left(5.50\times10^{-7}\text{ m}\right)^3\left(1.05\times10^3\text{ kg/m}^3\right)\left(9.81\text{m/s}^2\right) = 7.18\times10^{-15}\text{ N}$$

Substitute numerical values and evaluate $6\pi\eta r v_t$:

$$6\pi\eta r v_t = 6\pi\left(1.8\times10^{-5}\text{ Pa}\cdot\text{s}\right)\left(5.50\times10^{-7}\text{ m}\right)\left(1.16\times10^{-4}\text{ m/s}\right) = 2.16\times10^{-14}\text{ N}$$

Substitute numerical values in equation (1) and evaluate Ne:

$$Ne = \frac{7.18\times10^{-15}\text{ N} + 2.16\times10^{-14}\text{ N}}{6.00\times10^4\text{ N/C}} = \boxed{4.8\times10^{-19}\text{ C}}$$

(*b*) Divide the result in (*a*) by e to obtain:

$$N = \frac{4.80\times10^{-19}\text{ C}}{1.602\times10^{-19}\text{ C}} = \boxed{3}$$

(*c*) With the field pointing upward, the electric force is downward and the application of $\sum F_y = ma_y$ to the bead yields:

$$F_{\text{d,terminal}} - F_{\text{e}} - mg = 0$$
or
$$6\pi\eta r v_{\text{t}} - NeE - \tfrac{4}{3}\pi r^3 \rho g = 0$$

Solve for v_{t} to obtain:

$$v_{\text{t}} = \frac{NeE + \tfrac{4}{3}\pi r^3 \rho g}{6\pi\eta r}$$

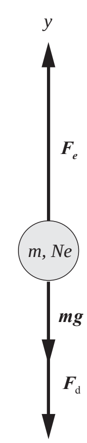
Substitute numerical values and evaluate v_{t}:

$$v_{\text{t}} = \frac{3\left(1.602\times10^{-19}\ \text{C}\right)\left(6.00\times10^{4}\ \dfrac{\text{N}}{\text{C}}\right) + \tfrac{4}{3}\pi\left(5.50\times10^{-7}\ \text{m}\right)^3\left(1.05\times10^{3}\ \dfrac{\text{kg}}{\text{m}^3}\right)\left(9.81\dfrac{\text{m}}{\text{s}^2}\right)}{6\pi\left(1.8\times10^{-5}\ \text{Pa}\cdot\text{s}\right)\left(5.50\times10^{-7}\ \text{m}\right)}$$

$$= \boxed{0.19\,\text{mm/s}}$$

87 ••• [SSM] In Problem 86, there is a description of the Millikan experiment used to determine the charge on the electron. During the experiment, a switch is used to reverse the direction of the electric field without changing its magnitude, so that one can measure the terminal speed of the microsphere both as it is moving upward and as it is moving downward. Let v_{u} represent the terminal speed when the particle is moving up, and v_{d} the terminal speed when moving down. (*a*) If we let $u = v_{\text{u}} + v_{\text{d}}$, show that $q = 3\pi\eta r u/E$, where q is the microsphere's net charge. For the purpose of determining q, what advantage does measuring both v_{u} and v_{d} have over measuring only one terminal speed?
(*b*) Because charge is quantized, u can only change by steps of magnitude N, where N is an integer. Using the data from Problem 86, calculate Δu.

Picture the Problem The free body diagram shows the forces acting on the microsphere of mass m and having an excess charge of $q = Ne$ when the electric field is downward. Under terminal-speed conditions the sphere is in equilibrium under the influence of the electric force $\vec{F}_{\text{e}}$, its weight $m\vec{g}$, and the drag force $\vec{F}_{\text{d}}$. We can apply Newton's second law, under terminal-speed conditions, to relate the number of excess charges N on the sphere to its mass and, using Stokes' law, to its terminal speed.



(*a*) Apply Newton's second law to the microsphere when the electric field is downward:

$$F_e - mg - F_d = ma_y$$

or, because $a_y = 0$,

$$F_e - mg - F_{d,\text{terminal}} = 0$$

Substitute for F_e and $F_{d,\text{terminal}}$ to obtain:

$$qE - mg - 6\pi\eta r v_u = 0$$

or, because $q = Ne$,

$$NeE - mg - 6\pi\eta r v_u = 0$$

Solve for v_u to obtain:

$$v_u = \frac{NeE - mg}{6\pi\eta r} \quad (1)$$

With the field pointing upward, the electric force is downward and the application of Newton's second law to the microsphere yields:

$$F_{d,\text{terminal}} - F_e - mg = 0$$

or

$$6\pi\eta r v_d - NeE - mg = 0$$

Solve for v_d to obtain:

$$v_d = \frac{NeE + mg}{6\pi\eta r} \quad (2)$$

Add equations (1) and (2) and simplify to obtain:

$$u = v_u + v_d = \frac{NeE - mg}{6\pi\eta r} + \frac{NeE + mg}{6\pi\eta r} = \frac{NeE}{3\pi\eta r} = \boxed{\frac{qE}{3\pi\eta r}}$$

Measuring both v_u and v_d has the advantage that you don't need to know the mass of the microsphere.

(*b*) Letting Δu represent the change in the terminal speed of the microsphere due to a gain (or loss) of one electron we have:

$$\Delta u = v_{N+1} - v_N$$

Noting that Δv will be the same whether the microsphere is moving upward or downward, express its terminal speed when it is moving upward with N electronic charges on it:

$$v_N = \frac{NeE - mg}{6\pi\eta r}$$

Express its terminal speed upward when it has $N + 1$ electronic charges:

$$v_{N+1} = \frac{(N+1)eE - mg}{6\pi\eta r}$$

Substitute and simplify to obtain:

$$\Delta u = \frac{(N+1)eE - mg}{6\pi\eta r} - \frac{NeE - mg}{6\pi\eta r} = \frac{eE}{6\pi\eta r}$$

Substitute numerical values and evaluate Δu:

$$\Delta u = \frac{(1.602\times10^{-19}\ \text{C})(6.00\times10^{4}\ \text{N/C})}{6\pi(1.8\times10^{-5}\ \text{Pa}\cdot\text{m})(5.50\times10^{-7}\ \text{m})} = \boxed{52\ \mu\text{m/s}}$$

Chapter 22
The Electric Field 2: Continuous Charge Distributions

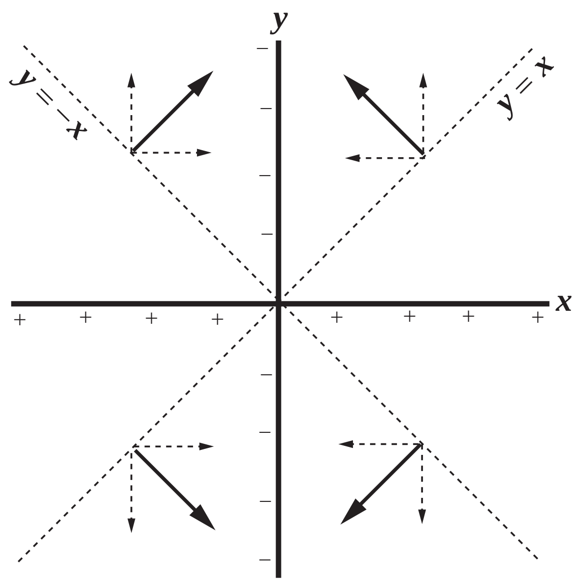
Conceptual Problems

1 • **[SSM]** Figure 22-37 shows an L-shaped object that has sides that are equal in length. Positive charge is distributed uniformly along the length of the object. What is the direction of the electric field along the dashed 45° line? Explain your answer.

Determine the Concept The resultant field is the superposition of the electric fields due to the charge distributions along the axes and is directed along the dashed line, pointing away from the intersection of the two sides of the L-shaped object. This can be seen by dividing each leg of the object into 10 (or more) equal segments and then drawing the electric field on the dashed line due to the charges on each pair of segments that are equidistant from the intersection of the legs.

2 • Positive charge is distributed uniformly along the entire length of the x axis, and negative charge is distributed uniformly along the entire length of the y axis. The charge per unit length on the two axes is identical, except for the sign. Determine the direction of the electric field at points on the lines defined by $y = x$ and $y = -x$. Explain your answer.

Determine the Concept The electric fields along the lines defined by $y = x$ and $y = -x$ are the superposition of the electric fields due to the charge distributions along the axes. The direction of the electric field is the direction of the force acting on a test charge at the point(s) of interest. Typical points are shown at two points on each of the two lines.



3 • True or false:

(*a*) The electric field due to a hollow uniformly charged thin spherical shell is zero at all points inside the shell.
(*b*) In electrostatic equilibrium, the electric field everywhere inside the material of a conductor must be zero.
(*c*) If the net charge on a conductor is zero, the charge density must be zero at every point on the surface of the conductor.

(*a*) True (assuming there are no charges inside the shell).

(*b*) True. The charges reside on the surface of conductor.

(*c*) False. Consider a spherical conducting shell. Such a surface will have equal charges on its inner and outer surfaces but, because their areas differ, so will their charge densities.

4 • If the electric flux through a closed surface is zero, must the electric field be zero everywhere on that surface? If not, give a specific example. From the given information can the net charge inside the surface be determined? If so, what is it?

Determine the Concept No, this is not necessarily true. The only conclusion that we can draw is that there is equal positive and negative flux. For example, the net flux through a Gaussian surface completely enclosing a dipole is zero. If the electric flux is zero through the closed surface, we can conclude that the net charge inside the surface is zero.

5 • True or false:

(*a*) Gauss's law holds only for symmetric charge distributions.
(*b*) The result that $E = 0$ everywhere inside the material of a conductor under electrostatic conditions can be derived from Gauss's law.

(*a*) False. Gauss's law states that the net flux through any surface is given by $\phi_{\text{net}} = \oint_S E_n\, dA = 4\pi k Q_{\text{inside}}$. While it is true that Gauss's law is easiest to apply to symmetric charge distributions, it holds for *any* surface.

(*b*) True. Because the charges on a conductor, under electrostatic conditions, reside on the surface of the conductor, the net flux inside the conductor is zero. Hence, by Gauss's law, the electric field inside the conductor must also be zero.

6 •• A single point charge q is located at the center of both an imaginary cube and an imaginary sphere. How does the electric flux through the surface of the cube compare to that through the surface of the sphere? Explain your answer.

Determine the Concept Because the net flux is proportional to the net charge enclosed, and this is the same for both surfaces, the electric flux through the surface of the cube is the same as the electric flux through the surface of the sphere.

7 •• **[SSM]** An electric dipole is completely inside a closed imaginary surface and there are no other charges. True or False:

(*a*) The electric field is zero everywhere on the surface.
(*b*) The electric field is normal to the surface everywhere on the surface.
(c) The electric flux through the surface is zero.
(*d*) The electric flux through the surface could be positive or negative.
(*e*) The electric flux through a portion of the surface might not be zero.

(*a*) False. Near the positive end of the dipole, the electric field, in accordance with Coulomb's law, will be directed outward and will be nonzero. Near the negative end of the dipole, the electric field, in accordance with Coulomb's law, will be directed inward and will be nonzero.

(*b*) False. The electric field is perpendicular to the Gaussian surface only at the intersections of the surface with a line defined by the axis of the dipole.

(*c*) True. Because the net charge enclosed by the Gaussian surface is zero, the net flux, given by $\phi_{\text{net}} = \oint_S E_n dA = 4\pi k Q_{\text{inside}}$, through this surface must be zero.

(*d*) False. The flux through the closed surface is zero.

(*e*) True. All Gauss's law tells us is that, because the net charge inside the surface is zero, the *net* flux through the surface must be zero.

8 •• Explain why the electric field strength increases linearly with r, rather than decreases inversely with r^2, between the center and the surface of a uniformly charged solid sphere.

Determine the Concept We can show that the charge inside a uniformly charged solid sphere of radius r is proportional to r^3 and that the area of a sphere is proportional to r^2. Using Gauss's law, it follows that the electric field must be proportional to $r^3/r^2 = r$.

Use Gauss's law to express the electric field inside a spherical charge distribution of constant volume charge density:

$$E = \frac{4\pi k Q_{\text{inside}}}{A}$$

where $A = 4\pi r^2$.

Express Q_{inside} as a function of ρ and r:

$$Q_{\text{inside}} = \rho V = \tfrac{4}{3}\pi\rho r^3$$

Substitute for Q_{inside} to obtain:

$$E = \frac{4\pi k \tfrac{4}{3}\pi\rho r^3}{4\pi r^2} = \frac{4k\pi\rho}{3}r$$

This result shows that the electric field *increases* linearly as you move out from the center of a spherical charge distribution.

9 •• **[SSM]** Suppose that the total charge on the conducting spherical shell in Figure 22-38 is zero. The negative point charge at the center has a magnitude given by Q. What is the direction of the electric field in the following regions? (*a*) $r < R_1$, (*b*) $R_2 > r > R_1$, (*c*) and $r > R_2$. Explain your answer.

Determine the Concept We can apply Gauss's law to determine the electric field for $r < R_1$, $R_2 > r > R_1$, and $r > R_2$. We also know that the direction of an electric field at any point is determined by the direction of the electric force acting on a positively charged object located at that point.

(*a*) From the application of Gauss's law we know that the electric field in this region is not zero. A positively charged object placed in the region for which $r < R_1$ will experience an attractive force from the charge $-Q$ located at the center of the shell. Hence the direction of the electric field is radially inward.

(*b*) Because the total charge on the conducting sphere is zero, the charge on its inner surface must be positive (the positive charges in the conducting sphere are drawn there by the negative charge at the center of the shell) and the charge on its outer surface must be negative. Hence the electric field in the region $R_2 > r > R_1$ is radially outward.

(*c*) Because the charge on the outer surface of the conducting shell is negative, the electric field in the region $r > R_2$ is radially inward.

10 •• The conducting shell in Figure 22-38 is grounded, and the negative point charge at the center has a magnitude given by Q. Which of the following statements is correct?

(*a*) The charge on the inner surface of the shell is $+Q$ and the charge on the outer surface is $-Q$.
(*b*) The charge on the inner surface of the shell is $+Q$ and the charge on the outer surface is zero.
(*c*) The charge on both surfaces of the shell is $+Q$.
(*d*) The charge on both surfaces of the shell is zero.

Determine the Concept We can decide what will happen when the conducting shell is grounded by thinking about the distribution of charge on the shell before it is grounded and the effect on this distribution of grounding the shell.

The negative point charge at the center of the conducting shell induces a positive charge on the inner surface of the shell and a negative charge on the outer surface. Grounding the shell attracts positive charge from ground; resulting in the outer surface becoming electrically neutral. $\boxed{(b)}$ is correct.

11 •• **T**he conducting shell in Figure 22-38 is grounded, and the negative point charge at the center has a magnitude given by Q. What is the direction of the electric field in the following regions? (*a*) $r < R_1$, (*b*) $R_2 > r > R_1$, (*c*) and $r > R_2$. Explain your answers.

Determine the Concept We can apply Gauss's law to determine the electric field for $r < R_1$, $R_2 > r > R_1$, and $r > R_2$. We also know that the direction of an electric field at any point is determined by the direction of the electric force acting on a positively charged object located at that point.

(*a*) From the application of Gauss's law we know that the electric field in this region is not zero. A positively charged object placed in the region for which $r < R_1$ will experience an attractive force from the charge $-Q$ located at the center of the shell. Hence the direction of the electric field is radially inward.

(*b*) Because the conducting shell is grounded, its inner surface is positively charged and its outer surface will have zero net charge. Hence the electric field in the region $R_2 > r > R_1$ is radially outward.

(*c*) Because the conducting shell is grounded, the net charge on the outer surface of the conducting shell is zero, and the electric field in the region $r > R_2$ is zero.

Estimation and Approximation

12 •• In the chapter, the expression for the electric field due to a uniformly charged disk (on its axis), was derived. At any location on the axis, the field

magnitude is $|E| = 2\pi k\sigma\left[1-\left(1+\frac{R^2}{z^2}\right)^{-1/2}\right]$. At large distances ($|z| >> R$), it was shown that this equation approaches $E \approx kQ/z^2$. Very near the disk ($|z| << R$), the field strength is approximately that of an infinite plane of charge or $|E| \approx 2\pi k\sigma$. Suppose you have a disk of radius 2.5 cm that has a uniform surface charge density of 3.6 μC/m^2. Use both the exact and appropriate expression from those given above to find the electric-field strength on the axis at distances of (*a*) 0.010 cm, (*b*) 0.040 cm, and (*c*) 5.0 m. Compare the two values in each case and comment on the how well the approximations work in their region of validity.

Picture the Problem For $z << R$, we can model the disk as an infinite plane. For $z >> R$, we can approximate the ring charge by a point charge.

(*a*) Evaluate the exact expression for z = 0.010 cm:

$$|E|_{z=0.010\,\text{cm}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\left(1+\frac{(2.5\ \text{cm})^2}{(0.010\ \text{cm})^2}\right)^{-1/2}\right)$$
$$= 2.025\times10^{5}\ \text{N/C} = \boxed{2.0\times10^{5}\ \text{N/C}}$$

For $z << R$, the electric field strength near an infinite plane of charge is given by:

$$|E| \approx 2\pi k\sigma$$

Evaluate the approximate expression for z = 0.010 cm:

$$|E|_{\text{approx}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right) = 2.033\times10^{5}\ \text{N/C}$$
$$= \boxed{2.0\times10^{5}\ \text{N/C}}$$

The approximate value agrees to within 0.40% with the exact value and is larger than the exact value.

(*b*) Evaluate the exact expression for $z = 0.040$ cm:

$$\left|E\right|_{z=0.040\,\text{cm}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\left(1+\frac{(2.5\ \text{cm})^2}{(0.040\ \text{cm})^2}\right)^{-1/2}\right)$$

$$= 2.001\times10^{5}\ \text{N/C} = \boxed{2.0\times10^{5}\ \text{N/C}}$$

The approximate value agrees to within 1.2% with the exact value and is smaller than the exact value.

(*c*) Evaluate the exact expression for $z = 5.0$ m:

$$\left|E\right|_{z=5.0\,\text{m}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\left(1+\frac{(2.5\ \text{cm})^2}{(5.0\ \text{m})^2}\right)^{-1/2}\right)$$

$$= 2.541\ \text{N/C} = \boxed{2.5\ \text{N/C}}$$

Because $z >> R$, we can use Coulomb's law for the electric field due to a point charge to obtain:

$$\left|E(z)\right| = \frac{kQ}{z^2} = \frac{k\pi r^2\sigma}{z^2}$$

Evaluate $\left|E(5.0\ \text{m})\right|$:

$$\left|E(5.0\,\text{m})\right|_{\text{approx}} = \frac{\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.5\,\text{cm})^2\left(3.6\,\mu\text{C/m}^2\right)}{(5.0\,\text{m})^2} = 2.541\,\text{N/C}$$

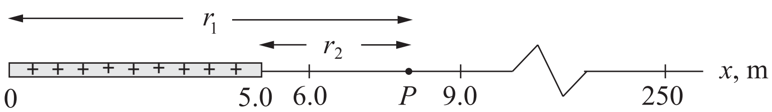
$$= \boxed{2.5\,\text{N/C}}$$

The approximate value agrees, to four significant figures, with the exact value.

Calculating $\vec{E}$ From Coulomb's Law

13 •• [SSM] A uniform line charge that has a linear charge density λ equal to 3.5 nC/m is on the x axis between $x = 0$ and $x = 5.0$ m. (*a*) What is its total charge? Find the electric field on the x axis at (*b*) $x = 6.0$ m, (*c*) $x = 9.0$ m, and (*d*) $x = 250$ m. (*e*) Estimate the electric field at $x = 250$ m, using the approximation that the charge is a point charge on the x axis at $x = 2.5$ m, and compare your result with the result calculated in Part (*d*). (To do this you will need to assume that the values given in this problem statement are valid to more than two significant figures.) Is your approximate result greater or smaller than the exact result? Explain your answer.

Picture the Problem (*a*) We can use the definition of λ to find the total charge of the line of charge. (*b*), (*c*) and (*d*) Equation 22-2*b* gives the electric field on the axis of a finite line of charge. In Part (*e*) we can apply Coulomb's law for the electric field due to a point charge to approximate the electric field at x = 250 m. In the following diagram, L = 5.0 m and P is a generic point on the x axis.



(*a*) Use the definition of linear charge density to express Q in terms of λ:

$$Q = \lambda L = (3.5\,\text{nC/m})(5.0\,\text{m}) = 17.5\,\text{nC} = \boxed{18\,\text{nC}}$$

The electric field on the axis of a finite line charge is given by Equation 22-2*b*:

$$E_x = k\lambda\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

(*b*) Substitute numerical values and evaluate $E_{x=6.0\,\text{m}}$:

$$E_{x=6.0\,\text{m}} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(3.5\times10^{-9}\,\frac{\text{C}}{\text{m}}\right)\left(\frac{1}{6.0\,\text{m}-5.0\,\text{m}} - \frac{1}{6.0\,\text{m}}\right) = \boxed{26\,\text{N/C}}$$

(*c*) Substitute numerical values and evaluate $E_{x=9.0\,\text{m}}$:

$$E_{x=9.0\,\text{m}} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(3.5\times10^{-9}\,\frac{\text{C}}{\text{m}}\right)\left(\frac{1}{9.0\,\text{m}-5.0\,\text{m}} - \frac{1}{9.0\,\text{m}}\right) = \boxed{4.4\,\text{N/C}}$$

(*d*) Substitute numerical values and evaluate E_x at x = 250 m:

$$E_{x=250\,\text{m}} = \left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(3.5\times10^{-9}\,\frac{\text{C}}{\text{m}}\right)\left(\frac{1}{250\,\text{m}-5.0\,\text{m}} - \frac{1}{250\,\text{m}}\right)$$
$$= 2.56800\,\text{mN/C} = \boxed{2.6\,\text{mN/C}}$$

(*e*) Using the approximation that the charge is a point charge on the x axis at x = 2.5 m, Coulomb's law gives:

$$E_x = \frac{kQ}{\left(r_1 - \frac{1}{2}L\right)^2}$$

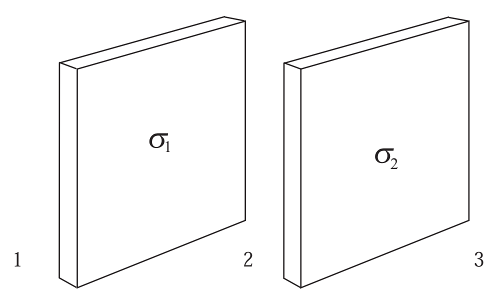
Substitute numerical values and evaluate $E_{x=250\text{ m}}$:

$$E_{x=250\text{ m}} = \frac{\left(8.988\times10^{9}\text{ N}\cdot\text{m}^2/\text{C}^2\right)\left(17.5\text{ nC}\right)}{\left(250\text{ m}-\frac{1}{2}\left(5.0\text{ m}\right)\right)^2} = 2.56774\text{ mN/C} = \boxed{2.6\text{ mN/C}}$$

This result is about 0.01% less than the exact value obtained in (*d*). This suggests that the line of charge is too long for its field at a distance of 250 m to be modeled exactly as that due to a point charge.

14 • Two infinite nonconducting sheets of charge are parallel to each other, with sheet A in the $x = -2.0$ m plane and sheet B in the $x = +2.0$ m plane. Find the electric field in the region $x < -2.0$ m, in the region $x > +2.0$ m, and between the sheets for the following situations. (*a*) When each sheet has a uniform surface charge density equal to $+3.0\ \mu\text{C/m}^2$ and (*b*) when sheet A has a uniform surface charge density equal to $+3.0\ \mu\text{C/m}^2$ and sheet B has a uniform surface charge density equal to $-3.0\ \mu\text{C/m}^2$. (*c*) Sketch the electric field-line pattern for each case.

Picture the Problem Choose a coordinate system in which the $+x$ direction is perpendicular to the surfaces and points to the right. Let the charge densities on the two plates be σ_1 and σ_2 and denote the three regions of interest as 1, 2, and 3. Choose a coordinate system in which the positive x direction is to the right. We can apply the equation for $\vec{E}$ near an infinite plane of charge and the superposition of fields to find the field in each of the three regions.



(*a*) Use the equation for $\vec{E}$ near an infinite plane of charge to express the field in region 1 when $\sigma_1 = \sigma_2 = +3.0\ \mu\text{C/m}^2$:

$$\begin{aligned}\vec{E}_1 &= \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} \\ &= -2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i} \\ &= -4\pi k\sigma\,\hat{i}\end{aligned}$$

Substitute numerical values and evaluate $\vec{E}_1$:

$$\vec{E}_1 = -4\pi\left(8.988\times10^{9}\text{ N}\cdot\text{m}^2/\text{C}^2\right)\left(3.0\ \mu\text{C/m}^2\right)\hat{i} = \boxed{-\left(3.4\times10^{5}\text{ N/C}\right)\hat{i}}$$

Proceed as above for region 2:

$$\vec{E}_2 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i}$$
$$= 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i}$$
$$= \boxed{0}$$

Proceed as above for region 3:

$$\vec{E}_3 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} + 2\pi k\sigma_2\hat{i} = 4\pi k\sigma\,\hat{i}$$
$$= 4\pi\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(3.0\ \mu\text{C/m}^2\right)\hat{i}$$
$$= \boxed{\left(3.4\times10^5\ \text{N/C}\right)\hat{i}}$$

(*b*) Use the equation for $\vec{E}$ near an infinite plane of charge to express and evaluate the field in region 1 when $\sigma_1 = +3.0\ \mu\text{C/m}^2$ and $\sigma_2 = -3.0\ \mu\text{C/m}^2$:

$$\vec{E}_1 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i}$$
$$= 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i}$$
$$= \boxed{0}$$

Proceed as above for region 2:

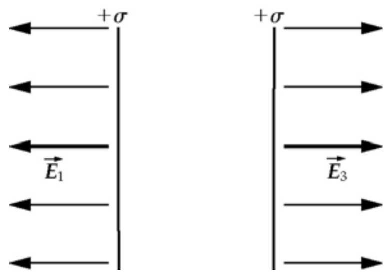
$$\vec{E}_2 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} + 2\pi k\sigma_2\hat{i} = 4\pi k\sigma\,\hat{i}$$
$$= 4\pi\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(3.0\ \mu\text{C/m}^2\right)\hat{i}$$
$$= \boxed{\left(3.4\times10^5\ \text{N/C}\right)\hat{i}}$$

Proceed as above for region 3:

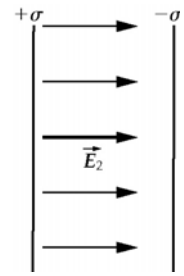
$$\vec{E}_3 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i}$$
$$= 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i}$$
$$= \boxed{0}$$

(c) The electric field lines for (a) and (b) are shown below:

(a)



(b)



15 • A charge of 2.75 μC is uniformly distributed on a ring of radius 8.5 cm. Find the electric field strength on the axis at distances of (a) 1.2 cm, (b) 3.6 cm, and (c) 4.0 m from the center of the ring. (d) Find the field strength at 4.0 m using the approximation that the ring is a point charge at the origin, and compare your results for Parts (c) and (d). Is your approximate result a good one? Explain your answer.

Picture the Problem The magnitude of the electric field at a distance z on the axis of a ring of charge Q and radius a is given by $E_z = \dfrac{kQz}{\left(z^2+a^2\right)^{3/2}}$.

Express $\vec{\boldsymbol{E}}$ on the axis of a ring charge:

$$E_z = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$$

(a) Substitute numerical values and evaluate E_z for z = 1.2 cm:

$$E_{z=1.2\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(1.2\,\text{cm})}{\left[(1.2\,\text{cm})^2+(8.5\,\text{cm})^2\right]^{3/2}} = \boxed{4.7\times10^5\ \text{N/C}}$$

(b) Proceed as in (a) with z = 3.6 cm:

$$E_{z=3.6\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(3.6\,\text{cm})}{\left[(3.6\,\text{cm})^2+(8.5\,\text{cm})^2\right]^{3/2}} = \boxed{1.1\times10^6\ \text{N/C}}$$

(c) Proceed as in (a) with z = 4.0 m:

$$E_{z=4.0\,\text{m}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(4.0\,\text{m})}{\left[(4.0\,\text{m})^2+(8.5\,\text{cm})^2\right]^{3/2}} = \boxed{1.5\times10^3\ \text{N/C}}$$

(*d*) Using Coulomb's law for the electric field due to a point charge, express E_z:

$$E_z = \frac{kQ}{z^2}$$

Substitute numerical values and evaluate $E_{z=4.0\,\text{m}}$:

$$E_{z=4.0\,\text{m}} = \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(2.75\,\mu\text{C})}{(4.0\,\text{m})^2} = \boxed{1.5\times10^3\ \text{N/C}}$$

While this result agrees exactly, to two significant figures, with the result obtained in Part (*c*), it should be slightly larger because the point charge is nearer $z = 4.0$ m than is the ring of charge.

16 • A non-conducting disk of radius R lies in the $z = 0$ plane with its center at the origin. The disk has a uniform surface charge density σ. Find the value of z for which $E_z = \sigma/\left(4\in_0\right)$. Note that at this distance, the magnitude of the electric-field strength is half the electric-field strength at points on the x axis that are very close to the disk.

Picture the Problem The magnitude of the electric field at an axial distance z from a disk that has a uniform charge density σ and whose radius is R is given by

$$E_z = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right).$$

We can equate this expression and $E_z = \frac{1}{2}\sigma/\left(2\in_0\right)$ and solve for z.

Express the electric field on the axis of a disk charge:

$$E_z = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right)$$

We're given that:

$$E_z = \tfrac{1}{2}\sigma/\left(2\in_0\right) = \frac{\sigma}{4\in_0}$$

Equating these expressions gives:

$$\frac{\sigma}{4\in_0} = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right)$$

Substituting for k yields:

$$\frac{\sigma}{4\in_0}=2\pi\left(\frac{1}{4\pi\in_0}\right)\sigma\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right)$$

Solve for z to obtain:

$$z=\boxed{\frac{R}{\sqrt{3}}}$$

17 • [SSM] A ring that has radius a lies in the $z = 0$ plane with its center at the origin. The ring is uniformly charged and has a total charge Q. Find E_z on the z axis at (a) $z = 0.2a$, (b) $z = 0.5a$, (c) $z = 0.7a$, (d) $z = a$, and (e) $z = 2a$. (f) Use your results to plot E_z versus z for both positive and negative values of z. (Assume that these distances are exact.)

Picture the Problem The electric field at a distance z from the center of a ring whose charge is Q and whose radius is a is given by $E_z=\frac{kQz}{\left(z^2+a^2\right)^{3/2}}$.

(a) Evaluating $E_{z=0.2a}$ gives:

$$E_{z=0.2a}=\frac{kQ(0.2a)}{\left[(0.2a)^2+a^2\right]^{3/2}}=\boxed{0.189\frac{kQ}{a^2}}$$

(b) Evaluating $E_{z=0.5a}$ gives:

$$E_{z=0.5a}=\frac{kQ(0.5a)}{\left[(0.5a)^2+a^2\right]^{3/2}}=\boxed{0.358\frac{kQ}{a^2}}$$

(c) Evaluating $E_{z=0.7a}$ gives:

$$E_{z=0.7a}=\frac{kQ(0.7a)}{\left[(0.7a)^2+a^2\right]^{3/2}}=\boxed{0.385\frac{kQ}{a^2}}$$

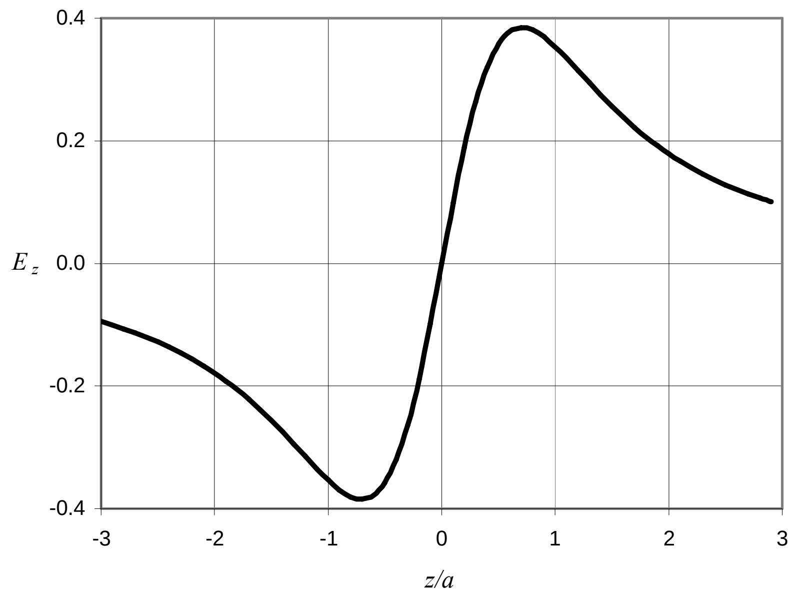
(d) Evaluating $E_{z=a}$ gives:

$$E_{z=a}=\frac{kQa}{\left[a^2+a^2\right]^{3/2}}=\boxed{0.354\frac{kQ}{a^2}}$$

(e) Evaluating $E_{z=2a}$ gives:

$$E_{z=2a}=\frac{2kQa}{\left[(2a)^2+a^2\right]^{3/2}}=\boxed{0.179\frac{kQ}{a^2}}$$

(f) The field along the z axis is plotted below. The z coordinates are in units of z/a and E is in units of kQ/a^2.



18 • A non-conducting disk of radius a lies in the $z = 0$ plane with its center at the origin. The disk is uniformly charged and has a total charge Q. Find E_z on the z axis at (a) $z = 0.2a$, (b) $z = 0.5a$, (c) $z = 0.7a$, (d) $z = a$, and (e) $z = 2a$. (f) Use your results to plot E_z versus z for both positive and negative values of z. (Assume that these distances are exact.)

Picture the Problem The magnitude of the electric field at a distance z from the center of a ring whose charge density is σ and whose radius is a is given by

$$E_z = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\frac{a^2}{z^2}}}\right).$$

The magnitude of the electric field on the axis of a charged disk of radius a is given by:

$$E_z = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\frac{a^2}{z^2}}}\right)$$

$$= \frac{Q}{2\pi \in_0 a^2}\left(1-\frac{1}{\sqrt{1+\frac{a^2}{z^2}}}\right)$$

(*a*) Evaluating $E_{z=0.2a}$ gives:

$$E_{z=0.2a} = \frac{Q}{2\pi \in_0 a^2}\left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{(0.2a)^2}}}\right)$$

$$= \boxed{0.128\frac{Q}{\in_0 a^2}}$$

(*b*) Evaluating $E_{z=0.5a}$ gives:

$$E_{z=0.5a} = \frac{Q}{2\pi \in_0 a^2}\left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{(0.5a)^2}}}\right)$$

$$= \boxed{0.088\frac{Q}{\in_0 a^2}}$$

(*c*) Evaluating $E_{z=0.7a}$ gives:

$$E_{z=0.7a} = \frac{Q}{2\pi \in_0 a^2}\left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{(0.7a)^2}}}\right)$$

$$= \boxed{0.068\frac{Q}{\in_0 a^2}}$$

(*d*) Evaluating $E_{z=a}$ gives:

$$E_{z=a} = \frac{Q}{2\pi \in_0 a^2}\left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{a^2}}}\right)$$

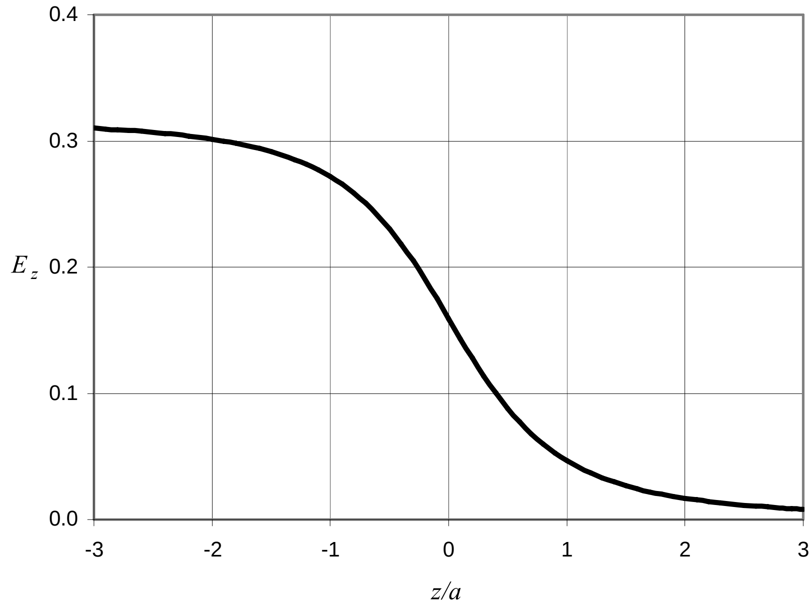
$$= \boxed{0.047\frac{Q}{\in_0 a^2}}$$

(*e*) Evaluating $E_{z=2a}$ gives:

$$E_{z=2a} = \frac{Q}{2\pi \in_0 a^2}\left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{(2a)^2}}}\right)$$

$$= \boxed{0.017\frac{Q}{\in_0 a^2}}$$

The field along the z axis is plotted below. The z coordinates are in units of z/a and E_z is in units of $\dfrac{Q}{\epsilon_0\, a^2}$.



19 •• (a) Using a **spreadsheet** program or graphing calculator, make a graph of the electric field strength on the axis of a disk that has a radius a = 30.0 cm and a surface charge density σ = 0.500 nC/m^2. (b) Compare your results to the results based on the approximation $E = 2\pi k\sigma$ (the formula for the electric-field strength of a uniformly charged infinite sheet). At what distance does the solution based on approximation differ from the exact solution by 10.0 percent?

Picture the Problem The magnitude of the electric field on the z axis of a disk of radius a carrying a surface charge density σ is given by

$$E_z = 2\pi k\sigma\left(1-\frac{1}{\sqrt{1+\dfrac{a^2}{z^2}}}\right) = 2\pi k\sigma\left(1-\frac{z}{\sqrt{z^2+a^2}}\right).$$

The electric field due to an infinite sheet of charge density σ is independent of the distance from the plane and is given by $E_{\text{sheet}} = 2\pi k\sigma$.

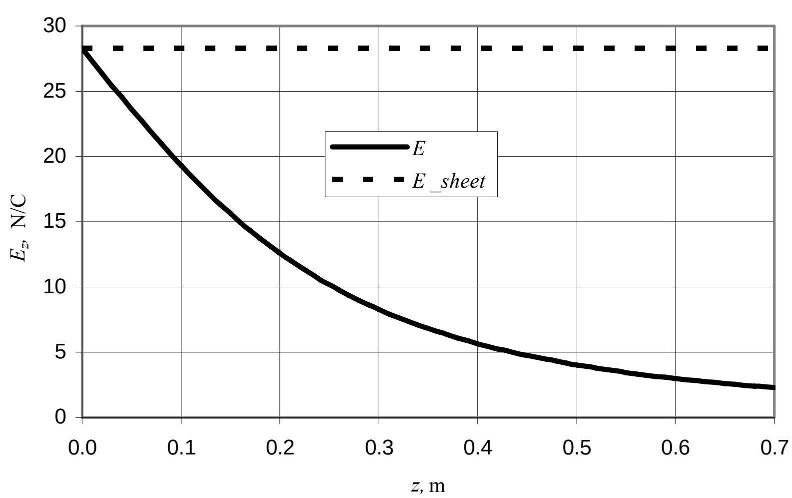
(a) A spreadsheet program to graph E_z is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
B3	9.00E+09	k
B4	5.00E−10	σ
B5	0.3	a

A8	0	z_0
A9	0.01	$x_0 + 0.01$
B8	2*PI()*$B$3*$B$4*(1−A8/(A8^2+$B$5^2)^2)^0.5)	$2\pi k\sigma\left(1-\dfrac{z}{\sqrt{z^2+a^2}}\right)$
C8	2*PI()*$B$3*$B$4	$2\pi k\sigma$

	A	B	C
1			
2			
3	k=	9.00E+09	N·m²/C²
4	σ=	5.00E-10	C/m²
5	a=	0.300	m
6			
7	z	E_z	E_{sheet}
8	0.00	28.27	28.3
9	0.01	27.33	28.3
77	0.69	2.34	28.3
78	0.70	2.29	28.3

(*b*) A graph of E_z follows. The electric field from an infinite sheet with the same charge density is shown for comparison. The magnitudes differ by more than 10.0 percent for $x \geq 0.0300$ m.



20 •• (*a*) Show that the electric-field strength E on the axis of a ring charge of radius a has maximum values at $z = \pm a/\sqrt{2}$. (*b*) Sketch the field strength E versus z for both positive and negative values of z. (*c*) Determine the maximum value of E.

Picture the Problem The electric field on the axis of a ring charge as a function of distance z along the axis from the center of the ring is given by $E_z = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$. We can show that it has its maximum and minimum values at $z = +a/\sqrt{2}$ and $z = -a/\sqrt{2}$ by setting its first derivative equal to zero and solving the resulting equation for z. The graph of E_z will confirm that the maximum and minimum occur at these coordinates.

(*a*) The variation of E_z with z on the axis of a ring charge is given by:

$$E_z = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$$

Differentiate this expression with respect to z to obtain:

$$\frac{dE_x}{dz} = kQ\frac{d}{dz}\left[\frac{x}{\left(z^2+a^2\right)^{3/2}}\right] = kQ\frac{\left(z^2+a^2\right)^{3/2} - z\frac{d}{dz}\left(z^2+a^2\right)^{3/2}}{\left(z^2+a^2\right)^{3}}$$

$$= kQ\frac{\left(z^2+a^2\right)^{3/2} - z\left(\tfrac{3}{2}\right)\left(z^2+a^2\right)^{1/2}(2z)}{\left(z^2+a^2\right)^{3}} = kQ\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^{3}}$$

Set this expression equal to zero for extrema and simplify:

$$\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^{3}} = 0,$$

$$\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2} = 0,$$

and

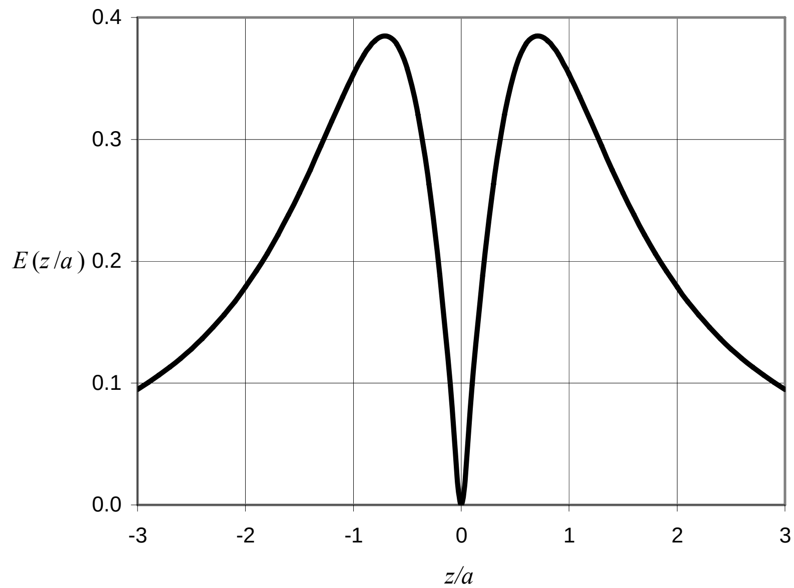
$$z^2 + a^2 - 3z^2 = 0$$

Solving for z yields:

$$z = \boxed{\pm\frac{a}{\sqrt{2}}}$$

as our candidates for maxima or minima.

(*b*) A graph of the magnitude of E_z, in units of kQ/a^2, versus z/a follows. This graph shows that the extrema at $z = \pm a/\sqrt{2}$ are, in fact, maxima.



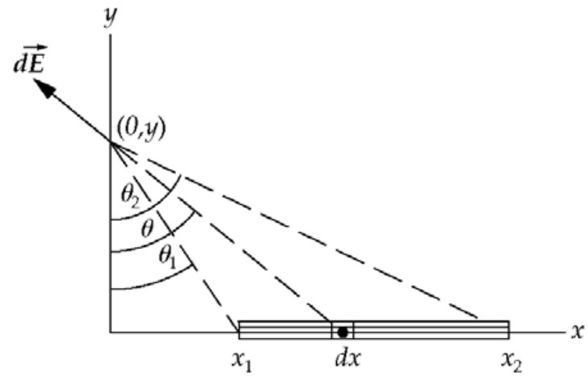
(*c*) Evaluate $E_z\left(\pm\frac{a}{\sqrt{2}}\right)$ and simplify to obtain the maximum value of the magnitude of E_z:

$$E_{z,\max} = E_{z=\pm\frac{a}{\sqrt{2}}} = \left|\frac{kQ\left(\pm\frac{a}{\sqrt{2}}\right)}{\left(\left(\pm\frac{a}{\sqrt{2}}\right)^2 + a^2\right)^{3/2}}\right| = \frac{kQ\frac{a}{\sqrt{2}}}{\left(\frac{1}{2}a^2 + a^2\right)^{3/2}} = \boxed{\frac{2\sqrt{3}}{9}\frac{kQ}{a^2}}$$

Remarks: Note that our result in Part (*c*) confirms the maxima obtained graphically in Part (*b*).

21 •• A line charge that has a uniform linear charge density λ lies along the x axis from $x = x_1$ to $x = x_2$ where $x_1 < x_2$. Show that the x component of the electric field at a point on the y-axis is given by $E_x = \frac{k\lambda}{y}\left(\cos\theta_2 - \cos\theta_1\right)$ where $\theta_1 = \tan^{-1}(x_1/y)$, $\theta_2 = \tan^{-1}(x_2/y)$ and $y \neq 0$.

Picture the Problem The line charge and point (0, y) are shown in the diagram. Also shown is a line element of length dx and the field $d\vec{E}$ its charge produces at (0, y). We can find dE_x from $d\vec{E}$ and then integrate from $x = x_1$ to $x = x_2$.



Express the x component of $d\vec{\mathbf{E}}$:

$$dE_x = -\frac{k\lambda}{x^2 + y^2}\sin\theta dx$$
$$= -\frac{k\lambda}{x^2 + y^2}\frac{x}{\sqrt{x^2 + y^2}}dx$$
$$= -\frac{k\lambda x}{\left(x^2 + y^2\right)^{3/2}}dx$$

Integrate from $x = x_1$ to x_2 and simplify to obtain:

$$E_x = -k\lambda\int_{x_1}^{x_2}\frac{x}{\left(x^2 + y^2\right)^{3/2}}dx$$
$$= -k\lambda\left[-\frac{1}{\sqrt{x^2 + y^2}}\right]_{x_1}^{x_2}$$
$$= -k\lambda\left[-\frac{1}{\sqrt{x_2^2 + y^2}} + \frac{1}{\sqrt{x_1^2 + y^2}}\right]$$
$$= -\frac{k\lambda}{y}\left[-\frac{y}{\sqrt{x_2^2 + y^2}} + \frac{y}{\sqrt{x_1^2 + y^2}}\right]$$

From the diagram we see that:

$$\cos\theta_2 = \frac{y}{\sqrt{x_2^2 + y^2}} \text{ or } \theta_2 = \tan^{-1}\left(\frac{x_2}{y}\right)$$

and

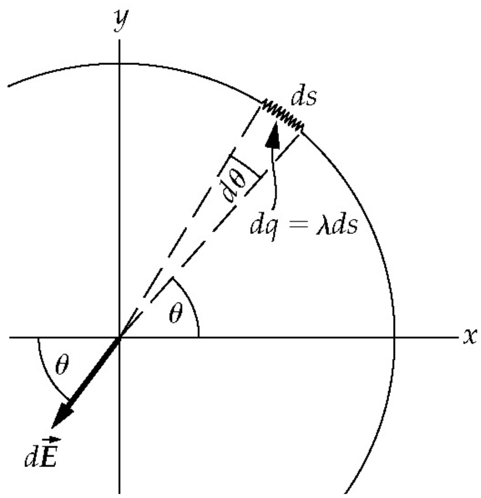
$$\cos\theta_1 = \frac{y}{\sqrt{x_1^2 + y^2}} \text{ or } \theta_1 = \tan^{-1}\left(\frac{x_1}{y}\right)$$

Substitute for $\frac{y}{\sqrt{x_2^2+y^2}}$ and $\frac{y}{\sqrt{x_1^2+y^2}}$ to obtain:

$$E_x = -\frac{k\lambda}{y}\left[-\cos\theta_2 + \cos\theta_1\right]$$
$$= \boxed{\frac{k\lambda}{y}\left[\cos\theta_2 - \cos\theta_1\right]}$$

22 •• A ring of radius a has a charge distribution on it that varies as $\lambda(\theta) = \lambda_0 \sin\theta$, as shown in Figure 22-39. (*a*) What is the direction of the electric field at the center of the ring? (*b*) What is the magnitude of the field at the center of the ring?

Picture the Problem The following diagram shows a segment of the ring of length ds that has a charge $dq = \lambda ds$. We can express the electric field $d\vec{E}$ at the center of the ring due to the charge dq and then integrate this expression from $\theta = 0$ to 2π to find the magnitude of the field in the center of the ring.



(*a*) and (*b*) The field $d\vec{E}$ at the center of the ring due to the charge dq is:

$$d\vec{E} = d\vec{E}_x + d\vec{E}_y = -dE\cos\theta\,\hat{\boldsymbol{i}} - dE\sin\theta\,\hat{\boldsymbol{j}} \quad (1)$$

The magnitude dE of the field at the center of the ring is:

$$dE = \frac{kdq}{r^2}$$

Because $dq = \lambda ds$:

$$dE = \frac{k\lambda ds}{r^2}$$

The linear charge density varies with θ according to $\lambda(\theta) = \lambda_0 \sin\theta$:

$$dE = \frac{k\lambda_0 \sin\theta\, ds}{r^2}$$

Substitute $rd\theta$ for ds:

$$dE = \frac{k\lambda_0 \sin\theta\, r d\theta}{r^2} = \frac{k\lambda_0 \sin\theta\, d\theta}{r}$$

Substitute for dE in equation (1) to obtain:

$$d\vec{E} = -\frac{k\lambda_0 \sin\theta \cos\theta\, d\theta}{r}\hat{i} - \frac{k\lambda_0 \sin^2\theta\, d\theta}{r}\hat{j}$$

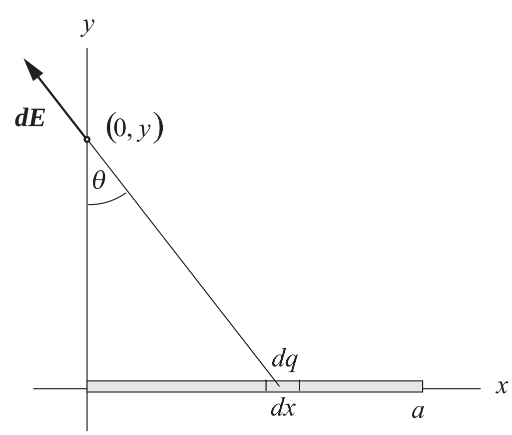
Integrate $d\vec{E}$ from $\theta = 0$ to 2π and simplify to obtain:

$$\vec{E} = -\frac{k\lambda_0}{2r}\int_0^{2\pi}\sin 2\theta\, d\theta\,\hat{i} - \frac{k\lambda_0}{r}\int_0^{2\pi}\sin^2\theta\, d\theta\,\hat{j} = 0 - \frac{\pi k\lambda_0}{r}\hat{j} = \boxed{-\frac{\pi k\lambda_0}{r}\hat{j}}$$

(*b*) The field at the origin is in the negative y direction and its magnitude is $\frac{\pi k\lambda_0}{r}$.

23 •• A line of charge that has uniform linear charge density λ lies on the x axis from $x = 0$ to $x = a$. Show that the y component of the electric field at a point on the y axis is given by $E_y = \frac{k\lambda}{y}\frac{a}{\sqrt{y^2 + a^2}}, \quad y \neq 0$.

Picture the Problem The line of charge and the point whose coordinates are (0, y) are shown in the diagram. Also shown is a segment of the line of length dx and charge dq. The field due to this charge at (0, y) is $d\vec{E}$. We can find dE_y from $d\vec{E}$ and then integrate from $x = 0$ to $x = a$ to find the y component of the electric field at a point on the y axis.



(*a*) Express the magnitude of the field $d\vec{E}$ due to charge dq of the element of length dx:

$$dE = \frac{kdq}{r^2}$$

where $r^2 = x^2 + y^2$

Because $dq = \lambda dx$:

$$dE = \frac{k\lambda dx}{x^2 + y^2}$$

Express the y component of dE:

$$dE_y = \frac{k\lambda}{x^2 + y^2}\cos\theta\, dx$$

Refer to the diagram to express $\cos\theta$ in terms of x and y:

$$\cos\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Substitute for $\cos\theta$ in the expression for dE_y to obtain:

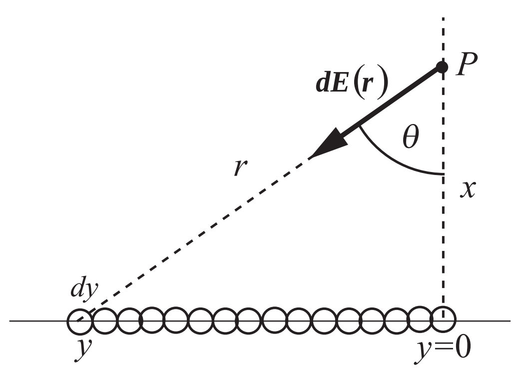
$$dE_y = \frac{k\lambda y}{\left(x^2 + y^2\right)^{3/2}}dx$$

Integrate from $x = 0$ to $x = a$ and simplify to obtain:

$$E_y = k\lambda y\int_0^a \frac{1}{\left(x^2 + y^2\right)^{3/2}}dx = k\lambda y\left[\frac{x}{y^2\sqrt{x^2 + y^2}}\right]_0^a = \boxed{\frac{k\lambda}{y}\frac{a}{\sqrt{a^2 + y^2}}}$$

24 ••• Calculate the electric field a distance z from a uniformly charged infinite flat non-conducting sheet by modeling the sheet as a continuum of infinite straight lines of charge.

Picture the Problem The field due to a line of charge is given by $E(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$ where r is the perpendicular distance to the line. The diagram shows a point P, at which we will calculate the electric field due a continuum of infinite straight non-conducting lines of charge, and a few of the lines of charge. P is a distance L from the plane and the origin of the coordinate system is directly below P. Note that the horizontal components of the field at P, by symmetry, add up to zero. Hence we need only find the sum of all the z components of the field.



Because the horizontal components of the electric field add up to zero, the resultant field is given by:

$$E = E_{\perp} = \int_{-\pi/2}^{\pi/2} dE(r)\cos\theta \qquad (1)$$

Express the field due to an infinite line of charge:

$$dE(r)=\frac{1}{2\pi\in_0}\frac{d\lambda}{r}$$, where r is the perpendicular distance to the line of charge.

The surface charge density σ of the plane and the linear charge density of the charged rings λ are related:

$$d\lambda=\sigma\,dy$$

Substitute for $d\lambda$ to obtain:

$$dE(r)=\frac{1}{2\pi\in_0}\frac{\sigma\,dy}{r}$$

Substituting for $dE(r)$ in equation (1) yields:

$$E=\frac{1}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}\frac{\sigma\,dy}{r}\cos\theta$$

Referring to the diagram, note that:

$$y=x\tan\theta\Rightarrow dy=x\sec^2\theta\,d\theta$$

Substitute for dy in the expression for E to obtain:

$$E=\frac{\sigma}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}\frac{x\sec^2\theta\cos\theta\,d\theta}{r}$$

Because $x=r\cos\theta$:

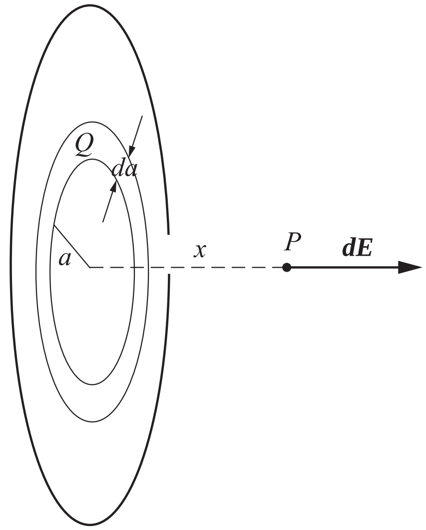
$$E_\perp=\frac{\sigma}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}\sec^2\theta\cos^2\theta\,d\theta$$
$$=\frac{\sigma}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}d\theta$$

Integrating this expression yields:

$$E=\frac{\sigma}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}d\theta=\frac{\sigma}{2\pi\in_0}\pi=\boxed{\frac{\sigma}{2\in_0}}$$

25 •• [SSM] Calculate the electric field a distance z from a uniformly charged infinite flat non-conducting sheet by modeling the sheet as a continuum of infinite circular rings of charge.

Picture the Problem The field at a point on the axis of a uniformly charged ring lies along the axis and is given by Equation 22-8. The diagram shows one ring of the continuum of circular rings of charge. The radius of the ring is a and the distance from its center to the field point P is x. The ring has a uniformly distributed charge Q. The resultant electric field at P is the sum of the fields due to the continuum of circular rings. Note that, by symmetry, the horizontal components of the electric field cancel.



Express the field of a single uniformly charged ring with charge Q and radius a on the axis of the ring at a distance x away from the plane of the ring:

$$\vec{\boldsymbol{E}} = E_x \hat{\boldsymbol{i}} \text{ , where } E_x = \frac{kQx}{\left(x^2 + a^2\right)^{3/2}}$$

Substitute dq for Q and dE_x for E_x to obtain:

$$dE_x = \frac{kxdq}{\left(x^2 + a^2\right)^{3/2}}$$

The resultant electric field at P is the sum of the fields due to all the circular rings. Integrate both sides to calculate the resultant field for the entire plane. The field point remains fixed, so x is constant:

$$E = \int \frac{kxdq}{\left(x^2 + a^2\right)^{3/2}} = kx\int \frac{dq}{\left(x^2 + a^2\right)^{3/2}}$$

To evaluate this integral we change integration variables from q to a. The charge $dq = \sigma\, dA$ where $dA = 2\pi\, a\, da$ is the area of a ring of radius a and width da:

$$dq = 2\pi\sigma\, a\, da$$

so

$$E = kx\int_0^{\infty} \frac{2\pi\, \sigma\, a\, da}{\left(x^2 + a^2\right)^{3/2}}$$

$$= 2\pi\, \sigma\, kx\int_0^{\infty} \frac{a\, da}{\left(x^2 + a^2\right)^{3/2}}$$

To integrate this expression, let $u = \sqrt{x^2 + a^2}$. Then:

$$du = \tfrac{1}{2}\frac{1}{\sqrt{x^2 + a^2}}\left(2ada\right) = \frac{a}{u}da$$

or

$$ada = udu$$

Noting that when $a = 0$, $u = x$, substitute and simplify to obtain:

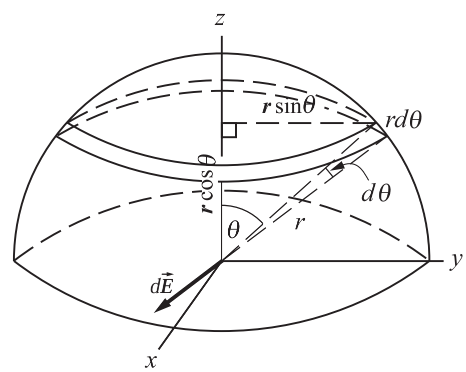
$$E = 2\pi\sigma kx\int_x^\infty \frac{u}{u^3}du = 2\pi\sigma kx\int_x^\infty u^{-2}du$$

Evaluating the integral yields:

$$E = 2\pi\sigma kx\left(-\frac{1}{u}\right)\Bigg|_x^\infty = 2\pi k\sigma = \boxed{\frac{\sigma}{2\in_0}}$$

26 ••• A thin hemispherical shell of radius R has a uniform surface charge σ. Find the electric field at the center of the base of the hemispherical shell.

Picture the Problem Consider the ring with its axis along the z direction shown in the diagram. Its radius is $z = r\cos\theta$ and its width is $rd\theta$. We can use the equation for the field on the axis of a ring charge and then integrate to express the field at the center of the hemispherical shell.



Express the field on the axis of the ring of charge:

$$dE = \frac{kzdq}{\left(r^2\sin^2\theta + r^2\cos^2\theta\right)^{3/2}} = \frac{kzdq}{r^3}$$

where $z = r\cos\theta$

Express the charge dq on the ring:

$$dq = \sigma dA = \sigma\left(2\pi r\sin\theta\right)rd\theta = 2\pi\sigma r^2\sin\theta d\theta$$

Substitute to obtain:

$$dE = \frac{k\left(r\cos\theta\right)2\pi\sigma r^2\sin\theta d\theta}{r^3} = 2\pi k\sigma\sin\theta\cos\theta d\theta$$

Integrating dE from $\theta = 0$ to $\pi/2$ yields:

$$E = 2\pi k\sigma\int_0^{\pi/2}\sin\theta\cos\theta d\theta = 2\pi k\sigma\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \boxed{\pi k\sigma}$$

Gauss's Law

27 • A square that has 10-cm-long edges is centered on the x axis in a region where there exists a uniform electric field given by $\vec{E} = (2.00\text{ kN/C})\hat{i}$. (a) What is the electric flux of this electric field through the surface of a square if the normal to the surface is in the $+x$ direction? (b) What is the electric flux through the same square surface if the normal to the surface makes a 60º angle with the y axis and an angle of 90° with the z axis?

Picture the Problem The definition of electric flux is $\phi = \oint_S \vec{E} \cdot \hat{n}\,dA$. We can apply this definition to find the electric flux through the square in its two orientations.

(a) Apply the definition of ϕ to find the flux of the field when the square is parallel to the yz plane:

$$\begin{aligned}\phi &= \oint_S (2.00\,\text{kN/C})\hat{i} \cdot \hat{i}\,dA \\ &= (2.00\,\text{kN/C})\oint_S dA \\ &= (2.00\,\text{kN/C})(0.100\,\text{m})^2 \\ &= \boxed{20.0\,\text{N}\cdot\text{m}^2/\text{C}}\end{aligned}$$

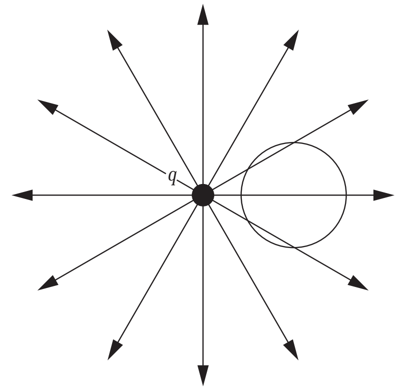
(b) Proceed as in (a) with $\hat{i} \cdot \hat{n} = \cos 30°$ to obtain:

$$\begin{aligned}\phi &= \oint_S (2.00\,\text{kN/C})\cos 30°\,dA \\ &= (2.00\,\text{kN/C})\cos 30° \oint_S dA \\ &= (2.00\,\text{kN/C})(0.100\,\text{m})^2 \cos 30° \\ &= \boxed{17\,\text{N}\cdot\text{m}^2/\text{C}}\end{aligned}$$

28 • A single point charge (q = +2.00 μC) is fixed at the origin. An imaginary spherical surface of radius 3.00 m is centered on the x axis at x = 5.00 m. (a) Sketch electric-field lines for this charge (in two dimensions) assuming twelve equally-spaced field lines in the xy plane leave the charge location, with one of the lines in the $+x$ direction. Do any lines enter the spherical surface? If so, how many? (b) Do any lines leave the spherical surface? If so, how many? (c) Counting the lines that enter as negative and the ones that leave as positive, what is the net number of field lines that penetrate the spherical surface? (d) What is the net electric flux through this spherical surface?

Determine the Concept We must show the twelve electric field lines originating at q and, in the absence of other charges, radially symmetric with respect to the location of q. While we're drawing twelve lines in this problem, the number of lines that we draw is always, by agreement, in proportion to the magnitude of q.

(*a*) The sketch of the field lines and of the spherical surface is shown in the diagram to the right.



Given the number of field lines drawn from *q,* 3 lines enter the spherical surface. Had we chosen to draw 24 field lines, 6 would have entered the spherical surface.

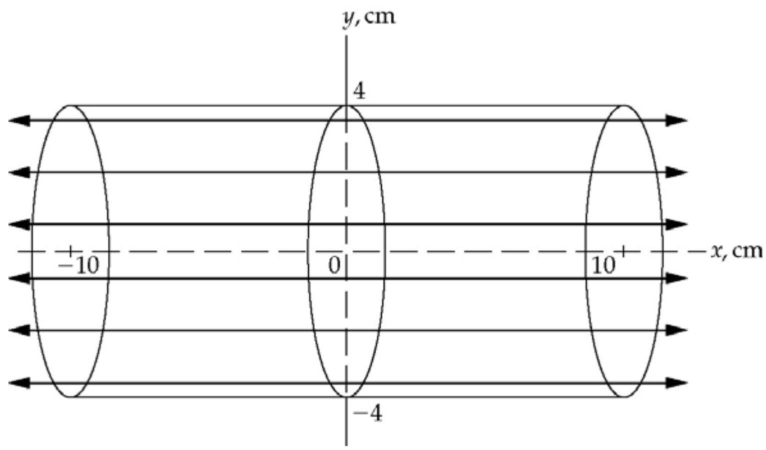
(*b*) Three lines leave the spherical surface.

(*c*) Because the three lines that enter the spherical surface also leave the spherical surface, the net number of field lines that pass through the surface is zero.

(*d*) Because as many field lines leave the spherical surface as enter it, the net flux is zero.

29 • [SSM] An electric field is given by $\vec{E} = \text{sign}(x)\cdot(300\text{ N/C})\hat{i}$, where sign($x$) equals –1 if $x < 0$, 0 if $x = 0$, and +1 if $x > 0$. A cylinder of length 20 cm and radius 4.0 cm has its center at the origin and its axis along the x axis such that one end is at $x = +10$ cm and the other is at $x = -10$ cm. (*a*) What is the electric flux through each end? (*b*) What is the electric flux through the curved surface of the cylinder? (*c*) What is the electric flux through the entire closed surface? (*d*) What is the net charge inside the cylinder?

Picture the Problem The field at both circular faces of the cylinder is parallel to the outward vector normal to the surface, so the flux is just *EA*. There is no flux through the curved surface because the normal to that surface is perpendicular to $\vec{E}$. The net flux through the closed surface is related to the net charge inside by Gauss's law.



(a) Use Gauss's law to calculate the flux through the right circular surface:

$$\phi_{\text{right}} = \vec{E}_{\text{right}} \cdot \hat{n}_{\text{right}} A = (300\,\text{N/C})\hat{i} \cdot \hat{i}(\pi)(0.040\,\text{m})^2 = \boxed{1.5\,\text{N}\cdot\text{m}^2/\text{C}}$$

Apply Gauss's law to the left circular surface to obtain:

$$\phi_{\text{left}} = \vec{E}_{\text{left}} \cdot \hat{n}_{\text{left}} A = (-300\,\text{N/C})\hat{i} \cdot (-\hat{i})(\pi)(0.040\,\text{m})^2 = \boxed{1.5\,\text{N}\cdot\text{m}^2/\text{C}}$$

(b) Because the field lines are parallel to the curved surface of the cylinder:

$$\phi_{\text{curved}} = \boxed{0}$$

(c) Express and evaluate the net flux through the entire cylindrical surface:

$$\phi_{\text{net}} = \phi_{\text{right}} + \phi_{\text{left}} + \phi_{\text{curved}} = 1.5\,\text{N}\cdot\text{m}^2/\text{C} + 1.5\,\text{N}\cdot\text{m}^2/\text{C} + 0 = \boxed{3.0\,\text{N}\cdot\text{m}^2/\text{C}}$$

(d) Apply Gauss's law to obtain:

$$\phi_{\text{net}} = 4\pi k Q_{\text{inside}} \Rightarrow Q_{\text{inside}} = \frac{\phi_{\text{net}}}{4\pi k}$$

Substitute numerical values and evaluate Q_{inside}:

$$Q_{\text{inside}} = \frac{3.0\,\text{N}\cdot\text{m}^2/\text{C}}{4\pi\left(8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\right)} = \boxed{2.7\times10^{-11}\,\text{C}}$$

30 • Careful measurement of the electric field at the surface of a black box indicates that the net outward electric flux through the surface of the box is 6.0 kN·m^2/C. (*a*) What is the net charge inside the box? (*b*) If the net outward electric flux through the surface of the box were zero, could you conclude that there were no charges inside the box? Explain your answer.

Picture the Problem We can use Gauss's law to find the net charge inside the box.

(*a*) Apply Gauss's law in terms of ϵ_0 to find the net charge inside the box:

$$\phi_{\text{net}} = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow Q_{\text{inside}} = \epsilon_0\, \phi_{\text{net}}$$

Substitute numerical values and evaluate Q_{inside}:

$$Q_{\text{inside}} = \left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(6.0\,\frac{\text{kN}\cdot\text{m}^2}{\text{C}}\right) = \boxed{5.3\times10^{-8}\ \text{C}}$$

(*b*) You can only conclude that the net charge is zero. There may be an equal number of positive and negative charges present inside the box.

31 • A point charge ($q = +2.00\ \mu$C) is at the center of an imaginary sphere that has a radius equal to 0.500 m. (*a*) Find the surface area of the sphere. (*b*) Find the magnitude of the electric field at all points on the surface of the sphere. (*c*) What is the flux of the electric field through the surface of the sphere? (*d*) Would your answer to Part (*c*) change if the point charge were moved so that it was inside the sphere but not at its center? (*e*) What is the flux of the electric field through the surface of an imaginary cube that has 1.00-m-long edges and encloses the sphere?

Picture the Problem We can apply Gauss's law to find the flux of the electric field through the surface of the sphere.

(*a*) Use the formula for the surface area of a sphere to obtain:

$$A = 4\pi r^2 = 4\pi(0.500\,\text{m})^2 = 3.142\,\text{m}^2 = \boxed{3.14\,\text{m}^2}$$

(*b*) Apply Coulomb's law to find *E*:

$$E = \frac{1}{4\pi\,\epsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\frac{2.00\,\mu\text{C}}{(0.500\,\text{m})^2} = 7.190\times10^4\ \text{N/C}$$
$$= \boxed{7.19\times10^4\ \text{N/C}}$$

(*c*) Apply Gauss's law to obtain:

$$\phi = \oint_S \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}dA = \oint_S EdA$$
$$= \left(7.190\times10^4\ \text{N/C}\right)\left(3.142\,\text{m}^2\right)$$
$$= \boxed{2.26\times10^5\ \text{N}\cdot\text{m}^2/\text{C}}$$

(*d*) No. The flux through the surface is independent of where the charge is located inside the sphere.

(*e*) Because the cube encloses the sphere, the flux through the surface of the sphere will also be the flux through the cube:

$$\phi_{\text{cube}} = \boxed{2.26\times10^5\ \text{N}\cdot\text{m}^2/\text{C}}$$

32 • What is the electric flux through one side of a cube that has a single point charge of –3.00 μC placed at its center? *HINT: You do not need to integrate any equations to get the answer.*

Picture the Problem The flux through the cube is given by $\phi_{\text{net}} = Q_{\text{inside}}/\in_0$, where Q_{inside} is the charge at the center of the cube. The flux through one side of the cube is one-sixth of the total flux through the cube.

The flux through one side of the cube is one-sixth of the total flux through the cube:

$$\phi_{1\text{ face}} = \tfrac{1}{6}\phi_{\text{tot}} = \frac{Q}{6\in_0}$$

Substitute numerical values and evaluate $\phi_{1\text{ face}}$:

$$\phi_{1\text{ face}} = \frac{-3.00\,\mu\text{C}}{6\left(8.854\times10^{-12}\ \dfrac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}$$
$$= \boxed{-5.65\times10^4\ \frac{\text{N}\cdot\text{m}^2}{\text{C}}}$$

33 • **[SSM]** A single point charge is placed at the center of an imaginary cube that has 20-cm-long edges. The electric flux out of one of the cube's sides is –1.50 kN·m^2/C. How much charge is at the center?

Picture the Problem The net flux through the cube is given by $\phi_{\text{net}} = Q_{\text{inside}}/\in_0$, where Q_{inside} is the charge at the center of the cube.

The flux through one side of the cube is one-sixth of the total flux through the cube:

$$\phi_{1\text{ faces}} = \tfrac{1}{6}\phi_{\text{net}} = \frac{Q_{\text{inside}}}{6\in_0}$$

Solving for Q_{inside} yields: $Q_{\text{inside}} = 6\in_0 \phi_{\text{2 faces}}$

Substitute numerical values and evaluate Q_{inside}:

$$Q_{\text{inside}} = 6\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(-1.50\frac{\text{kN}\cdot\text{m}^2}{\text{C}^2}\right) = \boxed{-79.7\ \text{nC}}$$

34 •• Because the formulas for Newton's law of gravity and for Coulomb's law have the same inverse-square dependence on distance, a formula analogous to the formula for Gauss's law can be found for gravity. The gravitational field $\vec{g}$ at a location is the force per unit mass on a test mass m_0 placed at that location. Then, for a point mass m at the origin, the gravitational field $\vec{g}$ at some position $\vec{r}$ is $\vec{g} = \left(Gm/r^2\right)\hat{r}$. Compute the flux of the gravitational field through a spherical surface of radius R centered at the origin, and verify that the gravitational analog of Gauss's law is $\phi_{\text{net}} = -4\pi Gm_{\text{inside}}$.

Picture the Problem We'll define the flux of the gravitational field in a manner that is analogous to the definition of the flux of the electric field and then substitute for the gravitational field and evaluate the integral over the closed spherical surface.

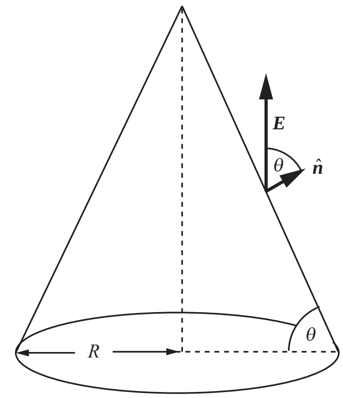
Define the gravitational flux as: $\phi_{\text{g}} = \oint_S \vec{g}\cdot\hat{n}dA$

Substitute for $\vec{g}$ and evaluate the integral to obtain:

$$\begin{aligned}\phi_{\text{net}} &= \oint_S\left(-\frac{Gm_{\text{inside}}}{r^2}\hat{r}\right)\cdot\hat{n}dA\\ &= -\frac{Gm_{\text{inside}}}{r^2}\oint_S dA\\ &= \left(-\frac{Gm_{\text{inside}}}{r^2}\right)\left(4\pi r^2\right)\\ &= \boxed{-4\pi Gm_{\text{inside}}}\end{aligned}$$

35 •• An imaginary right circular cone (Figure 22-40) that has a base angle θ and a base radius R is in a charge free region that has a uniform electric field $\vec{E}$ (field lines are vertical and parallel to the cone's axis). What is the ratio of the number of field lines per unit area penetrating the base to the number of field lines per unit area penetrating the conical surface of the cone? Use Gauss's law in your answer. (The field lines in the figure are only a representative sample.)

Picture the Problem Because the cone encloses no charge, we know, from Gauss's law, that the net flux of the electric field through the cone's surface is zero. Thus, the number of field lines penetrating the curved surface of the cone must equal the number of field lines penetrating the base and the entering flux must equal the exiting flux.



The flux penetrating the base of the cone is given by:

$$\phi_{\text{entering}} = EA_{\text{base}}$$

The flux penetrating the curved surface of the cone is given by:

$$\phi_{\text{exiting}} = \oint_S \vec{E}\cdot\hat{n}\,dA = \oint_S E\cos\theta\,dA$$

Equating the fluxes and simplifying yields:

$$A_{\text{base}} = \cos\theta\oint_S dA = (\cos\theta)A_{\substack{\text{curved}\\\text{surface}}}$$

The ratio of the density of field lines is:

$$\frac{A_{\text{base}}}{A_{\substack{\text{curved}\\\text{surface}}}} = \boxed{\cos\theta}$$

36 •• In the atmosphere and at an altitude of 250 m, you measure the electric field to be 150 N/C directed downward and you measure the electric field to be 170 N/C directed downward at an altitude of 400 m. Calculate the volume charge density of the atmosphere in the region between altitudes of 250 m and 400 m, assuming it to be uniform. (You may neglect the curvature of Earth. Why?)

Picture the Problem We'll model this portion of Earth's atmosphere as though it is a cylinder with cross-sectional area A and height h. Because the electric flux increases with altitude, we can conclude that there is charge inside the cylindrical region and use Gauss's law to find that charge and hence the charge density of the atmosphere in this region.

The definition of volume charge density is:

$$\rho = \frac{Q}{V}$$

Express the charge inside a cylinder of base area A and height h for a charge density ρ:

$$Q = \rho Ah$$

Taking upward to be the positive direction, apply Gauss's law to the charge in the cylinder:

$Q = -(E_h A - E_0 A)\epsilon_0 = (E_0 A - E_h A)\epsilon_0$

where we've taken our zero at 250 m above the surface of a flat Earth.

Substitute to obtain:

$$\rho = \frac{(E_0 A - E_h A)\epsilon_0}{Ah} = \frac{(E_0 - E_h)\epsilon_0}{h}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{(150\,\text{N/C} - 170\,\text{N/C})(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}{400\,\text{m} - 250\,\text{m}} = \boxed{-1.2\times10^{-12}\ \text{C/m}^3}$$

where we've been able to neglect the curvature of Earth because the maximum height of 400 m is approximately 0.006% of the radius of Earth and because the diameter of the cylindrical Gaussian surface can be arbitrarily small.

Gauss's Law Applications in Spherical Symmetry Situations

37 • A thin non-conducting spherical shell of radius R_1 has a total charge q_1 that is uniformly distributed on its surface. A second, larger thin non-conducting spherical shell of radius R_2 that is coaxial with the first has a charge q_2 that is uniformly distributed on its surface. (*a*) Use Gauss's law to obtain expressions for the electric field in each of the three regions: $r < R_1$, $R_1 < r < R_2$, and $r > R_2$. (*b*) What should the ratio of the charges q_1/q_2 and the relative signs for q_1 and q_2 be for the electric field to be zero throughout the region $r > R_2$? (*c*) Sketch the electric field lines for the situation in Part (*b*) when q_1 is positive.

Picture the Problem To find E_n in these three regions we can choose Gaussian surfaces of appropriate radii and apply Gauss's law. On each of these surfaces, E_r is constant and Gauss's law relates E_r to the total charge inside the surface.

(*a*) Use Gauss's law to find the electric field in the region $r < R_1$:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

and

$\vec{E}_{r<R_1} = \dfrac{Q_{\text{inside}}}{\epsilon_0 A}\hat{r}$ where $\hat{r}$ is a unit radial vector.

Because $Q_{\text{inside}} = 0$:

$$\vec{E}_{r<R_1} = \boxed{0}$$

Apply Gauss's law in the region $R_1 < r < R_2$:

$$\vec{E}_{R_1<r<R_1} = \frac{q_1}{\epsilon_0 (4\pi r^2)}\hat{r} = \boxed{\frac{kq_1}{r^2}\hat{r}}$$

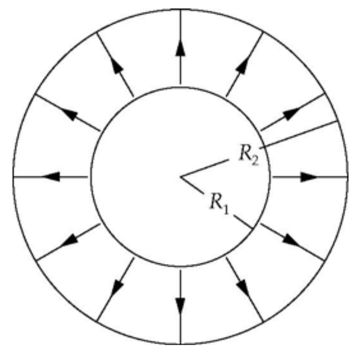
Using Gauss's law, find the electric field in the region $r > R_2$:

$$\vec{E}_{r>R_2} = \frac{q_1 + q_2}{\epsilon_0 \left(4\pi r^2\right)}\hat{r} = \boxed{\frac{k\left(q_1 + q_2\right)}{r^2}\hat{r}}$$

(*b*) Set $E_{r>R_2} = 0$ to obtain:

$$q_1 + q_2 = 0 \Rightarrow \frac{q_1}{q_2} = \boxed{-1}$$

(*c*) The electric field lines for the situation in (*b*) with q_1 positive is shown to the right.



38 • A spherical shell of radius 6.00 cm carries a uniform surface charge density of A non-conducting thin spherical shell of radius 6.00 cm has a uniform surface charge density of 9.00 nC/m^2. (*a*) What is the total charge on the shell? Find the electric field at the following distances from the sphere's center: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

Picture the Problem We can use the definition of surface charge density and the formula for the area of a sphere to find the total charge on the shell. Because the charge is distributed uniformly over a spherical shell, we can choose a spherical Gaussian surface and apply Gauss's law to find the electric field as a function of the distance from the center of the spherical shell.

(*a*) Using the definition of surface charge density, relate the charge on the sphere to its area:

$$Q = \sigma A = 4\pi\sigma r^2$$

Substitute numerical values and evaluate Q:

$$Q = 4\pi\left(9.00\,\text{nC/m}^2\right)\left(0.0600\,\text{m}\right)^2$$
$$= 0.4072\,\text{nC} = \boxed{0.407\,\text{nC}}$$

Apply Gauss's law to a spherical surface of radius r that is concentric the spherical shell to obtain:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Noting that, due to symmetry, $E_r = E_n$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{4\pi \in_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

(*b*) The charge inside a sphere whose radius is 2.00 cm is zero and hence:

$$E_{r=2.00\,\text{cm}} = \boxed{0}$$

(*c*) The charge inside a sphere whose radius is 5.90 cm is zero and hence:

$$E_{r=5.90\,\text{cm}} = \boxed{0}$$

(*d*) The charge inside a sphere whose radius is 6.10 cm is 0.4072 nC and hence:

$$E_{r=6.10\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0610\,\text{m}\right)^2} = \boxed{983\,\text{N/C}}$$

(*e*) The charge inside a sphere whose radius is 10.0 cm is 0.4072 nC and hence:

$$E_{r=10\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.100\,\text{m}\right)^2} = \boxed{366\,\text{N/C}}$$

39 •• [SSM] A non-conducting sphere of radius 6.00 cm has a uniform volume charge density of 450 nC/m^3. (*a*) What is the total charge on the sphere? Find the electric field at the following distances from the sphere's center: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

Picture the Problem We can use the definition of volume charge density and the formula for the volume of a sphere to find the total charge of the sphere. Because the charge is distributed uniformly throughout the sphere, we can choose a spherical Gaussian surface and apply Gauss's law to find the electric field as a function of the distance from the center of the sphere.

(*a*) Using the definition of volume charge density, relate the charge on the sphere to its volume:

$$Q = \rho V = \tfrac{4}{3}\pi\rho r^3$$

Substitute numerical values and evaluate Q:

$$Q = \tfrac{4}{3}\pi\left(450\,\text{nC/m}^3\right)\left(0.0600\,\text{m}\right)^3 = 0.4072\,\text{nC} = \boxed{0.407\,\text{nC}}$$

Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the spherical shell to obtain:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

Because the charge distribution is uniform, we can find the charge inside the Gaussian surface by using the definition of volume charge density to establish the proportion:

$$\frac{Q}{V} = \frac{Q_{\text{inside}}}{V'}$$

where V' is the volume of the Gaussian surface.

Solve for Q_{inside} to obtain:

$$Q_{\text{inside}} = Q\frac{V'}{V} = Q\frac{r^3}{R^3}$$

Substitute for Q_{inside} to obtain:

$$E_{r<R} = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ}{R^3} r$$

(b) Evaluate $E_{r = 2.00\text{ cm}}$:

$$E_{r=2.00\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0600\,\text{m}\right)^3}\left(0.0200\,\text{m}\right) = \boxed{339\,\text{N/C}}$$

(c) Evaluate $E_{r = 5.90\text{ cm}}$:

$$E_{r=5.90\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0600\,\text{m}\right)^3}\left(0.0590\,\text{m}\right) = \boxed{1.00\,\text{kN/C}}$$

Apply Gauss's law to the Gaussian surface with $r > R$:

$$4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow E_r = \frac{kQ_{\text{inside}}}{r^2} = \frac{kQ}{r^2}$$

(d) Evaluate $E_{r = 6.10\text{ cm}}$:

$$E_{r=6.10\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0610\,\text{m}\right)^2} = \boxed{983\,\text{N/C}}$$

(e) Evaluate $E_{r\,=\,10.0\text{ cm}}$:

$$E_{r=10.0\,\text{cm}} = \frac{\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.100\,\text{m}\right)^2} = \boxed{366\,\text{N/C}}$$

40 •• Consider the solid conducting sphere and the concentric conducting spherical shell in Figure 22-41. The spherical shell has a charge –7*Q*. The solid sphere has a charge +2*Q*. (*a*) How much charge is on the outer surface and how much charge is on the inner surface of the spherical shell? (*b*) Suppose a metal wire is now connected between the solid sphere and the shell. After electrostatic equilibrium is re-established, how much charge is on the solid sphere and on each surface of the spherical shell? Does the electric field at the surface of the solid sphere change when the wire is connected? If so, in what way? (*c*) Suppose we return to the conditions in Part (*a*), with +2*Q* on the solid sphere and –7*Q* on the spherical shell. We next connect the solid sphere to ground with a metal wire, and then disconnect it. Then how much total charge is on the solid sphere and on each surface of the spherical shell?

Determine the Concept The charges on a conducting sphere, in response to the repulsive Coulomb forces each experiences, will separate until electrostatic equilibrium conditions exit. The use of a wire to connect the two spheres or to ground the outer sphere will cause additional redistribution of charge.

(*a*) Because the outer sphere is conducting, the field in the thin shell must vanish. Therefore, $-2Q$, uniformly distributed, resides on the inner surface, and $-5Q$, uniformly distributed, resides on the outer surface.

(*b*) Now there is no charge on the inner surface and $-5Q$ on the outer surface of the spherical shell. The electric field just outside the surface of the inner sphere changes from a finite value to zero.

(*c*) In this case, the $-5Q$ is drained off, leaving no charge on the outer surface and $-2Q$ on the inner surface. The total charge on the outer sphere is then $-2Q$.

41 •• A non-conducting solid sphere of radius 10.0 cm has a uniform volume charge density. The magnitude of the electric field at 20.0 cm from the sphere's center is 1.88×10^3 N/C. (*a*) What is the sphere's volume charge density? (*b*) Find the magnitude of the electric field at a distance of 5.00 cm from the sphere's center.

Picture the Problem (*a*) We can use the definition of volume charge density, in conjunction with Equation 22-18a, to find the sphere's volume charge density.
(*b*) We can use Equation 22-18b, in conjunction with our result from Part (*a*), to find the electric field at a distance of 5.00 cm from the solid sphere's center.

(*a*) The solid sphere's volume charge density is the ratio of its charge to its volume:

$$\rho = \frac{Q_{\text{inside}}}{V} = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi R^3} \qquad (1)$$

For $r \geq R$, Equation 22-18*a* gives the electric field at a distance r from the center of the sphere:

$$E_{\text{r}} = \frac{1}{4\pi \in_0} \frac{Q_{\text{inside}}}{r^2} \qquad (2)$$

Solving for Q_{inside} yields:

$$Q_{\text{inside}} = 4\pi \in_0 E_{\text{r}} r^2$$

Substitute for Q_{inside} in equation (1) and simplify to obtain:

$$\rho = \frac{4\pi \in_0 E_{\text{r}} r^2}{\frac{4}{3}\pi R^3} = \frac{3 \in_0 E_{\text{r}} r^2}{R^3}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{3\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)\left(1.88\times10^3\ \text{N/C}\right)\left(20.0\ \text{cm}\right)^2}{\left(10.0\ \text{cm}\right)^3} = 1.997\,\mu\text{C/m}^3$$

$$= \boxed{2.00\,\mu\text{C/m}^3}$$

(*b*) For $r \leq R$, the electric field at a distance r from the center of the sphere is given by:

$$E_r = \frac{1}{4\pi \in_0} \frac{Q_{\text{inside}}}{R^3} r \qquad (3)$$

Express Q_{inside} for $r \leq R$:

$$Q_{\text{inside}} = \rho V_{\substack{\text{sphere whose}\\ \text{radius is } r}} = \tfrac{4}{3}\pi r^3 \rho$$

Substituting for Q_{inside} in equation (3) and simplifying yields:

$$E_r = \frac{1}{4\pi \in_0} \frac{\frac{4}{3}\pi r^3 \rho}{R^3} r = \frac{\rho r^4}{3 \in_0 R^3}$$

Substitute numerical values and evaluate $E_{r\,=\,5.00\ \text{cm}}$:

$$E_{r=5.00\,\text{cm}} = \frac{\left(1.997\,\mu\text{C/m}^3\right)\left(5.00\ \text{cm}\right)^4}{3\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)\left(10.0\ \text{cm}\right)^3} = \boxed{470\ \text{N/C}}$$

42 •• A non-conducting solid sphere of radius R has a volume charge density that is proportional to the distance from the center. That is, $\rho = Ar$ for $r \leq R$, where A is a constant. (*a*) Find the total charge on the sphere. (*b*) Find the expressions for the electric field inside the sphere ($r < R$) and outside the sphere ($r > R$). (*c*) Sketch the magnitude of the electric field as a function of the distance r from the sphere's center.

Picture the Problem We can find the total charge on the sphere by expressing the charge dq in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find E_r inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find E_r outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, E_r is constant. Gauss's law then relates E_r to the total charge inside the surface.

(*a*) Express the charge dq in a shell of thickness dr and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 (Ar)dr = 4\pi A r^3 dr$$

Integrate this expression from $r = 0$ to R to find the total charge on the sphere:

$$Q = 4\pi A\int_0^R r^3 dr = \left[\pi A r^4\right]_0^R = \boxed{\pi A R^4}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for E_r yields:

$$E_{r>R} = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2} = \frac{kA\pi R^4}{r^2} = \boxed{\frac{AR^4}{4\epsilon_0 r^2}}$$

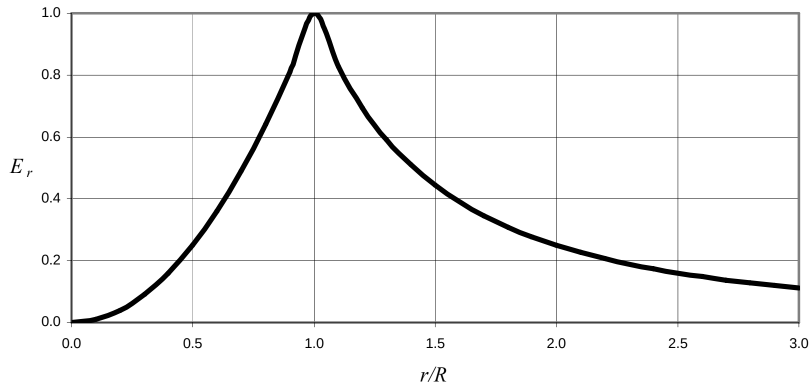
Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solve for E_r and simplify to obtain:

$$E_{r<R} = \frac{Q_{\text{inside}}}{4\pi r^2\epsilon_0} = \frac{\pi A r^4}{4\pi r^2\epsilon_0} = \boxed{\frac{Ar^2}{4\epsilon_0}}$$

(c) The following graph of E_r versus r/R, with E_r in units of $A/(4\epsilon_0)$, was plotted using a spreadsheet program.



Remarks: Note that the results for (*a*) and (*b*) agree at *r* = *R*.

43 •• [SSM] A sphere of radius R has volume charge density $\rho = B/r$ for $r < R$, where B is a constant and $\rho = 0$ for $r > R$. (a) Find the total charge on the sphere. (b) Find the expressions for the electric field inside and outside the charge distribution (c) Sketch the magnitude of the electric field as a function of the distance r from the sphere's center.

Picture the Problem We can find the total charge on the sphere by expressing the charge dq in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find E_r inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find E_r outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, E_r is constant. Gauss's law then relates E_r to the total charge inside the surface.

(a) Express the charge dq in a shell of thickness dr and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 \frac{B}{r} dr$$
$$= 4\pi B r dr$$

Integrate this expression from $r = 0$ to R to find the total charge on the sphere:

$$Q == 4\pi B \int_0^R r dr = \left[2\pi B r^2\right]_0^R$$
$$= \boxed{2\pi B R^2}$$

(b) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \text{ or } 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for E_r yields:

$$E_{r>R}=\frac{Q_{\text{inside}}}{4\pi\in_0}\frac{1}{r^2}=\frac{kQ_{\text{inside}}}{r^2}$$
$$=\frac{k2\pi BR^2}{r^2}=\boxed{\frac{BR^2}{2\in_0 r^2}}$$

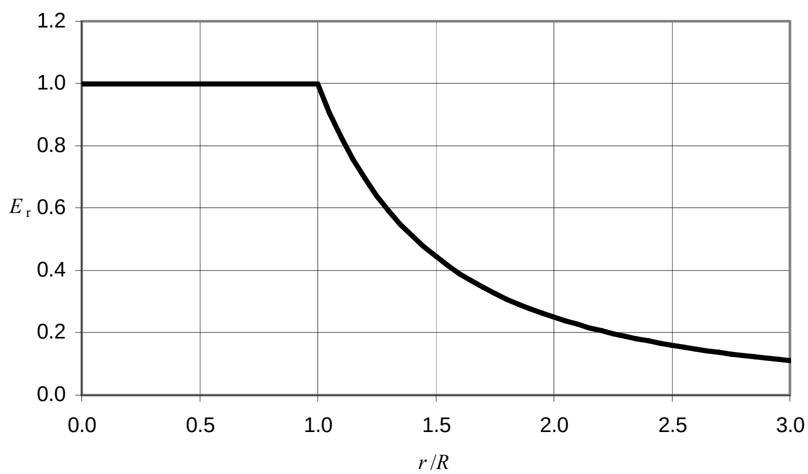
Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA=\frac{1}{\in_0}Q_{\text{inside}}\Rightarrow 4\pi r^2 E_r=\frac{Q_{\text{inside}}}{\in_0}$$

Solving for E_r yields:

$$E_{r<R}=\frac{Q_{\text{inside}}}{4\pi r^2\in_0}=\frac{2\pi Br^2}{4\pi r^2\in_0}=\boxed{\frac{B}{2\in_0}}$$

(*c*) The following graph of E_r versus r/R, with E_r in units of $B/(2\in_0)$, was plotted using a spreadsheet program.



Remarks: Note that our results for (*a*) and (*b*) agree at *r* = *R*.

44 •• A sphere of radius R has volume charge density $\rho = C/r^2$ for $r < R$, where C is a constant and $\rho = 0$ for $r > R$. (*a*) Find the total charge on the sphere. (*b*) Find the expressions for the electric field inside and outside the charge distribution (*c*) Sketch the magnitude of the electric field as a function of the distance r from the sphere's center.

Picture the Problem We can find the total charge on the sphere by expressing the charge dq in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find E_r inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find E_r outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$.

On each of these surfaces, E_r is constant. Gauss's law then relates E_r to the total charge inside the surface.

(*a*) Express the charge dq in a shell of thickness dr and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 \frac{C}{r^2} dr = 4\pi C dr$$

Integrate this expression from $r = 0$ to R to find the total charge on the sphere:

$$Q = 4\pi C \int_0^R dr = \left[4\pi Cr\right]_0^R = \boxed{4\pi CR}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for E_r yields:

$$E_{r>R} = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2} = \frac{k4\pi CR}{r^2} = \boxed{\frac{CR}{\epsilon_0 r^2}}$$

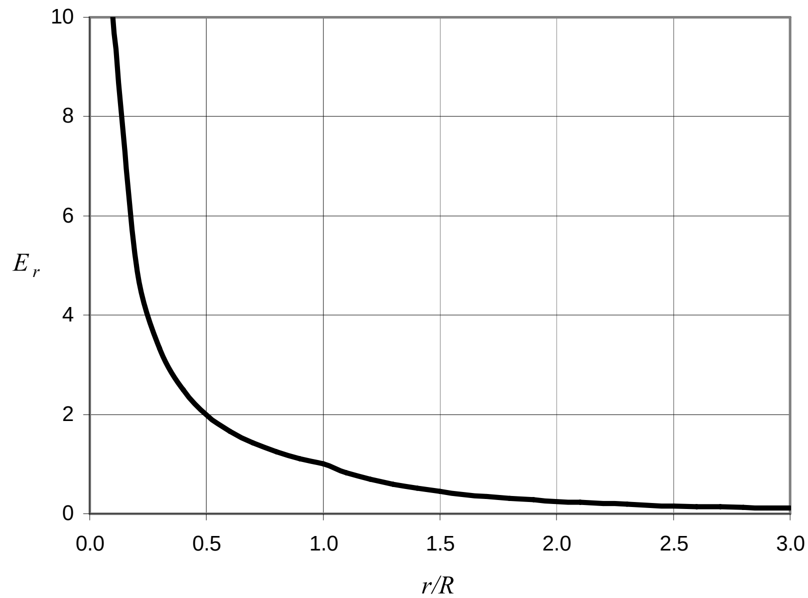
Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \text{ or } 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for E_r yields:

$$E_{r<R} = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} = \frac{4\pi Cr}{4\pi r^2 \epsilon_0} = \boxed{\frac{C}{\epsilon_0 r}}$$

(c) The following graph of E_r versus r/R, with E_r in units of $C/(\epsilon_0 R)$, was plotted using a spreadsheet program.



45 ••• A non-conducting spherical shell of inner radius R_1 and outer radius R_2 has a uniform volume charge density ρ. (a) Find the total charge on the shell. (b) Find expressions for the electric field everywhere.

Picture the Problem By symmetry, the electric fields resulting from this charge distribution must be radial. To find E_r for $r < R_1$ we choose a spherical Gaussian surface of radius $r < R_1$. To find E_r for $R_1 < r < R_2$ we choose a spherical Gaussian surface of radius $R_1 < r < R_2$. To find E_r for $r > R_2$ we choose a spherical Gaussian surface of radius $r > R_2$. On each of these surfaces, E_r is constant. Gauss's law then relates E_r to the total charge inside the surface.

(a) The charge in an infinitesimal spherical shell of radius r and thickness dr is:

$$dQ = \rho dV = 4\pi\rho r^2 dr$$

Integrate dQ from $r = R_1$ to R_2 to find the total charge in the spherical shell in the interval $R_1 < r < R_2$:

$$Q_{\text{total}} = 4\pi\rho\int_{R_1}^{R_2} r^2 dr = \left[\frac{4\pi C r^3}{3}\right]_{R_1}^{R_2}$$

$$= \boxed{\frac{4\pi\rho}{3}\left(R_2^3 - R_1^3\right)}$$

(*b*) Apply Gauss's law to a spherical surface of radius *r* that is concentric with the nonconducting spherical shell to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for E_r yields:

$$E_r = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

Evaluate $E_{r<R_1}$:

$$E_{r<R_1} = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2} = \boxed{0}$$

because $\rho_{r<R_1} = 0$ and, therefore, $Q_{\text{inside}} = 0$.

Evaluate $E_{R_1<r<R_2}$:

$$E_{R_1<r<R_2} = \frac{kQ_{\text{inside}}}{r^2} = \frac{4\pi k\rho}{3r^2}\left(r^3 - R_1^3\right)$$

$$= \boxed{\frac{\rho}{3\epsilon_0 r^2}\left(r^3 - R_1^3\right)}$$

For $r > R_2$:

$$Q_{\text{inside}} = \frac{4\pi\rho}{3}\left(R_2^3 - R_1^3\right)$$

and

$$E_{r>R_2} = \frac{4\pi k\rho}{3r^2}\left(R_2^3 - R_1^3\right)$$

$$= \boxed{\frac{\rho}{3\epsilon_0 r^2}\left(R_2^3 - R_1^3\right)}$$

Remarks: Note that *E* is continuous at *r* = R_2.

Gauss's Law Applications in Cylindrical Symmetry Situations

46 • For your senior project you are in charge of designing a Geiger tube for detecting radiation in the nuclear physics laboratory. This instrument will consist of a long metal cylindrical tube that has a long straight metal wire running down its central axis. The diameter of the wire is to be 0.500 mm and the inside diameter of the tube will be 4.00 cm. The tube is to be filled with a dilute gas in which electrical discharge (breakdown) occurs when the electric field reaches 5.50×10^6 N/C. Determine the maximum linear charge density on the wire if breakdown of the gas is not to happen. Assume that the tube and the wire are infinitely long.

Picture the Problem The electric field of a line charge of infinite length is given by $E_r = \dfrac{1}{2\pi \in_0} \dfrac{\lambda}{r}$, where r is the distance from the center of the line of charge and λ is the linear charge density of the wire.

The electric field of a line charge of infinite length is given by:

$$E_r = \frac{1}{2\pi \in_0} \frac{\lambda}{r}$$

Because E_r varies inversely with r, its maximum value occurs at the surface of the wire where $r = R$, the radius of the wire:

$$E_{\max} = \frac{1}{2\pi \in_0} \frac{\lambda}{R}$$

Solving for λ yields:

$$\lambda = 2\pi \in_0 RE_{\max}$$

Substitute numerical values and evaluate λ:

$$\lambda = 2\pi \left(8.854\times10^{-12}\ \frac{\mathrm{C^2}}{\mathrm{N\cdot m^2}}\right)(0.250\ \mathrm{mm})\left(5.50\times10^{6}\ \frac{\mathrm{N}}{\mathrm{C}}\right) = \boxed{76.5\ \mathrm{nC/m}}$$

47 ••• In Problem 54, suppose ionizing radiation produces an ion and an electron at a distance of 2.00 cm from the long axis of the central wire of the Geiger tube. Suppose that the central wire is positively charged and has a linear charge density equal to 76.5 pC/m. (*a*) In this case, what will be the electron's speed as it impacts the wire? (*b*) Qualitatively, how will the electron's speed compare to that of the ion's final speed when it impacts the outside cylinder? Explain your reasoning.

Picture the Problem Because the inward force on the electron increases as its distance from the wire decreases, we'll need to integrate the net electric force acting on the electron to obtain an expression for its speed as a function of its distance from the wire in the Geiger tube.

(*a*) The force the electron experiences is the radial component of the force on the electron and is the product of its charge and the radial component of the electric field due to the positively charged central wire:

$$F_{e,r} = eE_r$$

The radial electric field due to the charged wire is given by:

$$E_r = \frac{1}{2\pi \in_0} \frac{\lambda}{r}$$

Substituting for E_r yields:

$$F_{e,r}=-\left(\frac{e\lambda}{2\pi\in_0}\right)\frac{1}{r}$$ where the minus sign indicates that the force acting on the electron is radially inward.

Apply Newton's second law to the electron to obtain:

$$-\left(\frac{e\lambda}{2\pi\in_0}\right)\frac{1}{r}=m\frac{dv}{dt}=m\frac{dv}{dt}\frac{dr}{dr}$$
$$=m\frac{dv}{dr}\frac{dr}{dt}=mv\frac{dv}{dr}$$

Separating variables yields:

$$vdv=-\left(\frac{e\lambda}{2\pi m\in_0}\right)\frac{dr}{r}$$

Express the integral of this equation to obtain:

$$\int_0^{v_\mathrm{f}}vdv=-\left(\frac{e\lambda}{2\pi m\in_0}\right)\int_{r_1}^{r_2}\frac{dr}{r}$$ where the lower limit on the left-hand side is zero because the electron is initially at rest.

Integrating yields:

$$\tfrac{1}{2}v_\mathrm{f}^2=-\left(\frac{e\lambda}{2\pi m\in_0}\right)\ln\left(\frac{r_2}{r_1}\right)$$

Solve for v_f to obtain:

$$v_\mathrm{f}=\sqrt{\left(\frac{e\lambda}{\pi m\in_0}\right)\ln\left(\frac{r_1}{r_2}\right)}$$

Substitute numerical values and evaluate v_f:

$$v_\mathrm{f}=\sqrt{\left(\frac{\left(1.602\times10^{-19}\ \mathrm{C}\right)\left(76.5\ \frac{\mathrm{pC}}{\mathrm{m}}\right)}{\pi\left(9.109\times10^{-31}\ \mathrm{kg}\right)\left(8.854\times10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}\right)}\right)\ln\left(\frac{0.0200\ \mathrm{m}-0.0025\ \mathrm{m}}{0.250\ \mathrm{mm}}\right)}$$
$$=\boxed{1.43\times10^{6}\ \mathrm{m/s}}$$

(*b*) The positive ion is accelerated radially outward and will impact the tube instead of the wire. Because of its much larger mass, the impact speed of the ion will be much less than the impact speed of the electron.

48 •• Show that the electric field due to an infinitely long, uniformly charged thin cylindrical shell of radius a having a surface charge density σ is given by the following expressions: $E = 0$ for $0 \le R < a$ and $E_R = \sigma a/(\epsilon_0 R)$ for $R > a$.

Picture the Problem From symmetry, the field in the tangential direction must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shell.

Apply Gauss's law to the cylindrical surface of radius r and length L that is concentric with the infinitely long, uniformly charged cylindrical shell:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

or

$$2\pi r L E_R = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because no there is no flux through them and, due to symmetry, $E_n = E_R$.

Solve for E_R:

$$E_R = \frac{Q_{\text{inside}}}{2\pi r L \epsilon_0} = \frac{2kQ_{\text{inside}}}{Lr}$$

For $R < a$, $Q_{\text{inside}} = 0$ and:

$$E_{R<a} = \boxed{0}$$

For $R > a$, $Q_{\text{inside}} = \lambda L$ and:

$$E_{R>a} = \frac{2k\lambda L}{Lr} = \frac{2k\lambda}{r} = \frac{2k(2\pi a\sigma)}{r} = \boxed{\frac{a\sigma}{\epsilon_0 r}}$$

49 •• A thin cylindrical shell of length 200 m and radius 6.00 cm has a uniform surface charge density of 9.00 nC/m^2. (*a*) What is the total charge on the shell? Find the electric field at the following radial distances from the long axis of the cylinder. (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm. (Use the results of Problem 48.)

Picture the Problem We can use the definition of surface charge density to find the total charge on the shell. From symmetry, the electric field in the tangential direction must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the uniformly charged cylindrical shell.

(*a*) Using its definition, relate the surface charge density to the total charge on the shell:

$$Q = \sigma A = 2\pi RL\sigma$$

Substitute numerical values and evaluate Q:

$$Q = 2\pi(0.0600\,\text{m})(200\,\text{m})(9.00\,\text{nC/m}^2) = \boxed{679\,\text{nC}}$$

(*b*) From Problem 48 we have, for $r = 2.00$ cm:

$$E_{r=2.00\,\text{cm}} = \boxed{0}$$

(*c*) From Problem 48 we have, for $r = 5.90$ cm:

$$E_{r=5.90\,\text{cm}} = \boxed{0}$$

(*d*) From Problem 48 we have, for $r = 6.10$ cm:

$$E_r = \frac{\sigma R}{\epsilon_0\, r}$$

Substitute numerical values and evaluate $E_{r=6.10\,\text{cm}}$:

$$E_{r=6.10\,\text{cm}} = \frac{(9.00\,\text{nC/m}^2)(0.0600\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.0610\,\text{m})} = \boxed{1.00\,\text{kN/C}}$$

(*e*) From Problem 48 we have, for $r = 10.0$ cm:

$$E_{r=10.0\,\text{cm}} = \frac{(9.00\,\text{nC/m}^2)(0.0600\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.100\,\text{m})} = \boxed{610\,\text{N/C}}$$

50 •• An infinitely long non-conducting solid cylinder of radius a has a uniform volume charge density of ρ_0. Show that the electric field is given by the following expressions: $E_a = \rho_0 a/(2\epsilon_0)$ for $0 \le r < a$ and $E_a = \rho_0 a^2/(2\epsilon_0\, r)$ for $r > a$, where r is the distance from the long axis of the cylinder.

Picture the Problem From symmetry, the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

Apply Gauss's law to a cylindrical surface of radius r and length L that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

or

$$2\pi r L E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Due to symmetry, $E_n = E_r$. Solving for E_r yields:

$$E_r = \frac{Q_{\text{inside}}}{2\pi r L \epsilon_0} = \frac{2kQ_{\text{inside}}}{Lr}$$

Express Q_{inside} for $r < a$:

$$Q_{\text{inside}} = \rho(r)V = \rho_0(\pi r^2 L)$$

Substitute to obtain:

$$E_{r<R} = \frac{2k(\pi\rho_0 L r^2)}{Lr} = \boxed{\frac{\rho_0}{2\epsilon_0} r}$$

or, because $\lambda = \rho\pi a^2$,

$$E_{r<R} = \boxed{\frac{\lambda}{2\pi\epsilon_0 a^2} r}$$

Express Q_{inside} for $r > a$:

$$Q_{\text{inside}} = \rho(r)V = \rho_0(\pi a^2 L)$$

Substitute for Q_{inside} to obtain:

$$E_{r>a} = \frac{2k(\pi\rho_0 L a^2)}{Lr} = \boxed{\frac{\rho_0 a^2}{2\epsilon_0 r}}$$

or, because $\lambda = \rho\pi a^2$

$$E_{r>a} = \boxed{\frac{\lambda}{2\pi\epsilon_0 r}}$$

51 •• **[SSM]** A solid cylinder of length 200 m and radius 6.00 cm has a uniform volume charge density of 300 nC/m^3. (*a*) What is the total charge of the cylinder? Use the formulas given in Problem 50 to calculate the electric field at a point equidistant from the ends at the following radial distances from the cylindrical axis: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

Picture the Problem We can use the definition of volume charge density to find the total charge on the cylinder. From symmetry, the electric field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the uniformly charged cylinder.

(*a*) Use the definition of volume charge density to express the total charge of the cylinder:

$$Q_{\text{total}} = \rho V = \rho\left(\pi R^2 L\right)$$

Substitute numerical values to obtain:

$$Q_{\text{total}} = \pi\left(300\,\text{nC/m}^3\right)\left(0.0600\,\text{m}\right)^2\left(200\,\text{m}\right) = \boxed{679\,\text{nC}}$$

(*b*) From Problem 50, for $r < R$, we have:

$$E_{r<R} = \frac{\rho}{2\epsilon_0}r$$

For $r = 2.00$ cm:

$$E_{r=2.00\,\text{cm}} = \frac{\left(300\,\text{nC/m}^3\right)\left(0.0200\,\text{m}\right)}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)} = \boxed{339\,\text{N/C}}$$

(*c*) For $r = 5.90$ cm:

$$E_{r=5.90\,\text{cm}} = \frac{\left(300\,\text{nC/m}^3\right)\left(0.0590\,\text{m}\right)}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)} = \boxed{1.00\,\text{kN/C}}$$

From Problem 50, for $r > R$, we have:

$$E_{r>R} = \frac{\rho R^2}{2\epsilon_0 r}$$

(*d*) For $r = 6.10$ cm:

$$E_{r=6.10\,\text{cm}} = \frac{\left(300\,\text{nC/m}^3\right)\left(0.0600\,\text{m}\right)^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\left(0.0610\,\text{m}\right)} = \boxed{1.00\,\text{kN/C}}$$

(*e*) For $r = 10.0$ cm:

$$E_{r=10.0\,\text{cm}} = \frac{\left(300\,\text{nC/m}^3\right)\left(0.0600\,\text{m}\right)^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\left(0.100\,\text{m}\right)} = \boxed{610\,\text{N/C}}$$

52 •• Consider two infinitely long, coaxial thin cylindrical shells. The inner shell has a radius a_1 and has a uniform surface charge density of σ_1, and the outer shell has a radius a_2 and has a uniform surface charge density of σ_2. (*a*) Use Gauss's law to find expressions for the electric field in the three regions: $0 \le r < a_1$, $a_1 < r < a_2$, and $r > a_2$, where *r* is the distance from the axis. (*b*) What is the ratio of the surface charge densities σ_2/σ_1 and their relative signs if the electric field is to be zero everywhere outside the largest cylinder? (*c*) For the case in Part (*b*), what would be the electric field between the shells? (*d*) Sketch the electric field lines for the situation in Part (*b*) if σ_1 is positive.

Picture the Problem From symmetry; the field tangent to the surfaces of the shells must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shells.

(*a*) Apply Gauss's law to the cylindrical surface of radius *r* and length *L* that is concentric with the infinitely long, uniformly charged cylindrical shell:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 2\pi r L E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_n = E_r$, Solve for E_r to obtain:

$$E_r = \frac{2kQ_{\text{inside}}}{Lr} \qquad (1)$$

For $r < R_1$, $Q_{\text{inside}} = 0$ and:

$$E_{r<R_1} = \boxed{0}$$

Express Q_{inside} for $a_1 < r < a_2$:

$$Q_{\text{inside}} = \sigma_1 A_1 = 2\pi\sigma_1 a_1 L$$

Substitute in equation (1) to obtain:

$$E_{a_1<r<a_2} = \frac{2k(2\pi\sigma_1 a_1 L)}{Lr} = \boxed{\frac{\sigma_1 a_1}{\epsilon_0 r}}$$

Express Q_{inside} for $r > a_2$:

$$Q_{\text{inside}} = \sigma_1 A_1 + \sigma_2 A_2 = 2\pi\sigma_1 a_1 L + 2\pi\sigma_2 a_2 L$$

Substitute in equation (1) to obtain:

$$E_{r>a_2} = \frac{2k(2\pi\sigma_1 a_1 L + 2\pi\sigma_2 a_2 L)}{Lr} = \boxed{\frac{\sigma_1 a_1 + \sigma_2 a_2}{\epsilon_0 r}}$$

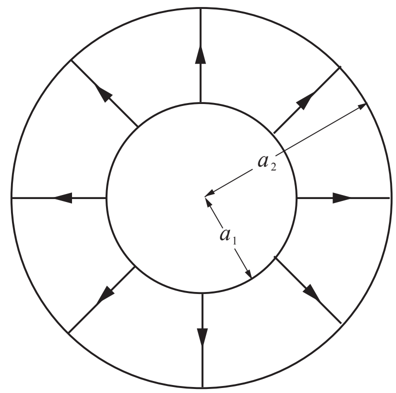
(*b*) Set $E = 0$ for $r > a_2$ to obtain:

$$\frac{\sigma_1 a_1 + \sigma_2 a_2}{\epsilon_0 r} = 0 \Rightarrow \frac{\sigma_2}{\sigma_1} = \boxed{-\frac{a_1}{a_2}}$$

(*c*) Because the electric field is determined by the charge inside the Gaussian surface, the field under these conditions would be as given above:

$$E_{a_1<r<a_2} = \boxed{\frac{\sigma_1 a_1}{\epsilon_0 r}}$$

(*d*) Because σ_1 is positive, the field lines are directed as shown to the right:



53 •• Figure 22-42 shows a portion of an infinitely long, concentric cable in cross section. The inner conductor has a charge of 6.00 nC/m and the outer conductor has no net charge. (*a*) Find the electric field for all values of *R*, where *R* is the perpendicular distance from the common axis of the cylindrical system. (*b*) What are the surface charge densities on the inside and the outside surfaces of the outer conductor?

Picture the Problem The electric field is directed radially outward. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shell.

(*a*) Apply Gauss's law to a cylindrical surface of radius *r* and length *L* that is concentric with the inner conductor:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 2\pi r L E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{2kQ_{\text{inside}}}{Lr} \qquad (1)$$

For $r < 1.50$ cm, $Q_{\text{inside}} = 0$ and:

$$E_{r<1.50\,\text{cm}} = \boxed{0}$$

Letting $R = 1.50$ cm, express Q_{inside} for 1.50 cm < r < 4.50 cm:

$$Q_{\text{inside}} = \lambda L = 2\pi\sigma RL$$

Substitute in equation (1) to obtain:

$$E_{1.50\,\text{cm}<r<4.50\,\text{cm}} = \frac{2k(\lambda L)}{Lr} = \frac{2k\lambda}{r}$$

Substitute numerical values and evaluate $E_{1.50\,\text{cm}<r<4.50\,\text{cm}}$:

$$E_{1.50\,\text{cm}<r<4.50\,\text{cm}} = 2\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\frac{(6.00\,\text{nC/m})}{r} = \boxed{\frac{(108\,\text{N}\cdot\text{m/C})}{r}}$$

Express Q_{inside} for 4.50 cm < r < 6.50 cm:

$$Q_{\text{inside}} = 0$$

and

$$E_{4.50\,\text{cm}<r<6.50\,\text{cm}} = \boxed{0}$$

Because the net charge on the outer conductor is zero:

$$Q_{\text{inside}} = \lambda L$$

Substitute in equation (1) to obtain:

$$E_r = \frac{2k\lambda L}{Lr} = \frac{\lambda}{2\pi \in_0 r}$$

Substitute numerical values and evaluate $E_{r>6.50\,\text{cm}}$:

$$E_{r>6.50\,\text{cm}} = \frac{6.00\times10^{-9}\ \text{C/m}}{2\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)r} = \boxed{\frac{108\,\text{N}\cdot\text{m/C}}{r}}$$

(*b*) The surface charge densities on the inside and the outside surfaces of the outer conductor are given by:

$$\sigma_{\text{inside}} = \frac{-\lambda}{2\pi R_{\text{inside}}}$$

and

$$\sigma_{\text{outside}} = \frac{\lambda}{2\pi R_{\text{outside}}}$$

Substitute numerical values and evaluate σ_{inside} and σ_{outside}:

$$\sigma_{\text{inside}} = \frac{-6.00\,\text{nC/m}}{2\pi(0.0450\,\text{m})} = -21.22\,\text{nC/m}^2 = \boxed{-21.2\,\text{nC/m}^2}$$

and

$$\sigma_{\text{outside}} = \frac{6.00\,\text{nC/m}}{2\pi(0.0650\,\text{m})} = \boxed{14.7\,\text{nC/m}^2}$$

54 •• An infinitely long non-conducting solid cylinder of radius a has a non-uniform volume charge density. This density varies linearly with R, the perpendicular distance from its axis, according to $\rho(R) = \beta R$, where β is a constant. (a) Show that the linear charge density of the cylinder is given by $\lambda = 2\pi\beta a^3/3$. ($b$) Find expressions for the electric field for $R < a$ and $R > a$.

Picture the Problem From symmetry considerations, we can conclude that the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

(a) The linear charge density of the cylinder is given by:

$$\lambda = \frac{q}{L} \qquad (1)$$

The total charge q in the long non-conducting solid cylinder is given by:

$$q = \int \rho(R)dV$$

Substituting for $\rho(R)$ and dV gives:

$$q = \int \beta R(2\pi RLdR)$$

Substitute for q in equation (1) and simplify to obtain:

$$\lambda = 2\pi\beta\int_0^a R^2 dR = \boxed{\frac{2\pi\beta a^3}{3}}$$

(b) Apply Gauss's law to a cylindrical surface of radius R and length L that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\epsilon_0}Q_{\text{inside}} \Rightarrow 2\pi RLE_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_n = E_R$, solve for E_R to obtain:

$$E_R = \frac{Q_{\text{inside}}}{2\pi RL\epsilon_0} \qquad (2)$$

Express dQ_{inside} for $\rho(R) = \beta R$:

$$dQ_{\text{inside}} = \rho(R)dV = \beta R(2\pi RL)dR = 2\pi\beta R^2 LdR$$

Integrate dQ_{inside} from $R = 0$ to r to obtain:

$$Q_{\text{inside}} = 2\pi\beta L\int_0^r R^2 dR = 2\pi aL\left[\frac{R^3}{3}\right]_0^r = \frac{2\pi\beta Lr^3}{3}$$

Substitute for Q_{inside} in equation (2) and simplify to obtain:

$$E_{R<a} = \frac{\frac{2\pi\beta Lr^3}{3}}{2\pi \in_0 Lr} = \boxed{\frac{\beta}{3\in_0}r^2}$$

For $R > a$:

$$Q_{\text{inside}} = \lambda L = \frac{2\pi\beta La^3}{3}$$

Substitute for Q_{inside} in equation (1) and simplify to obtain:

$$E_{R>a} = \frac{\frac{2\pi\beta La^3}{3}}{2\pi rL\in_0} = \boxed{\frac{\beta}{3r\in_0}a^3}$$

55 •• [SSM] An infinitely long non-conducting solid cylinder of radius a has a non-uniform volume charge density. This density varies with R, the perpendicular distance from its axis, according to $\rho(R) = bR^2$, where b is a constant. (a) Show that the linear charge density of the cylinder is given by $\lambda = \pi ba^4/2$. (b) Find expressions for the electric field for $R < a$ and $R > a$.

Picture the Problem From symmetry; the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

(a) Apply Gauss's law to a cylindrical surface of radius R and length L that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\in_0}Q_{\text{inside}} \Rightarrow 2\pi RLE_n = \frac{Q_{\text{inside}}}{\in_0}$$

where we've neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_n = E_R$, solve for E_R to obtain:

$$E_R = \frac{Q_{\text{inside}}}{2\pi RL\in_0} \qquad (1)$$

Express dQ_{inside} for $\rho(R) = bR^2$:

$$dQ_{\text{inside}} = \rho(R)dV = bR^2(2\pi RL)dR = 2\pi bR^3 LdR$$

Integrate dQ_{inside} from 0 to R to obtain:

$$Q_{\text{inside}} = 2\pi bL\int_0^R R^3 dr = 2\pi bL\left[\frac{R^4}{4}\right]_0^R = \frac{\pi bL}{2}R^4$$

Divide both sides of this equation by L to obtain an expression for the charge per unit length λ of the cylinder:

$$\lambda = \frac{Q_{\text{inside}}}{L} = \boxed{\frac{\pi bR^4}{2}}$$

(b) Substitute for Q_{inside} in equation (1) and simplify to obtain:

$$E_{R<a} = \frac{\frac{\pi bL}{2}r^4}{2\pi Lr\in_0} = \boxed{\frac{b}{4\in_0}r^3}$$

For $r > a$:

$$Q_{\text{inside}} = \frac{\pi bL}{2}a^4$$

Substitute for Q_{inside} in equation (1) and simplify to obtain:

$$E_{R>a} = \frac{\frac{\pi bL}{2}a^4}{2\pi rL\in_0} = \boxed{\frac{ba^4}{4\in_0 r}}$$

56 ••• An infinitely long, non-conducting cylindrical shell of inner radius a_1 and outer radius a_2 has a uniform volume charge density ρ. Find expressions for the electric field everywhere.

Picture the Problem From symmetry; the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylindrical shell.

Apply Gauss's law to a cylindrical surface of radius r and length L that is concentric with the infinitely long nonconducting cylindrical shell:

$$\oint_S E_n dA = \frac{1}{\in_0}Q_{\text{inside}} \Rightarrow 2\pi rLE_n = \frac{Q_{\text{inside}}}{\in_0}$$

where we've neglected the end areas because no flux crosses them.

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{2\pi rL \in_0}$$

For $r < a_1$, $Q_{\text{inside}} = 0$:

$$E_{r<a_1} = \boxed{0}$$

Express Q_{inside} for $a_1 < r < a_2$:

$$Q_{\text{inside}} = \rho V = \rho\pi r^2 L - \rho\pi a_1^2 L = \rho\pi L\left(r^2 - a_1^2\right)$$

Substitute for Q_{inside} and simplify to obtain:

$$E_{a_1<r<a_2} = \frac{\rho\pi L\left(r^2 - a_1^2\right)}{2\pi \in_0 Lr} = \boxed{\frac{\rho\left(r^2 - a_1^2\right)}{2 \in_0 r}}$$

Express Q_{inside} for $r > a_2$:

$$Q_{\text{inside}} = \rho V = \rho\pi a_2^2 L - \rho\pi a_1^2 L = \rho\pi L\left(a_2^2 - a_1^2\right)$$

Substitute for Q_{inside} and simplify to obtain:

$$E_{r>a_2} = \frac{\rho\pi L\left(a_2^2 - a_1^2\right)}{2\pi \in_0 rL} = \boxed{\frac{\rho\left(a_2^2 - a_1^2\right)}{2 \in_0 r}}$$

57 ••• [SSM] The inner cylinder of Figure 22-42 is made of non-conducting material and has a volume charge distribution given by $\rho(R) = C/R$, where $C = 200$ nC/m^2. The outer cylinder is metallic, and both cylinders are infinitely long. (*a*) Find the charge per unit length (that is, the linear charge density) on the inner cylinder. (*b*) Calculate the electric field for all values of R.

Picture the Problem We can integrate the density function over the radius of the inner cylinder to find the charge on it and then calculate the linear charge density from its definition. To find the electric field for all values of r we can construct a Gaussian surface in the shape of a cylinder of radius r and length L and apply Gauss's law to each region of the cable to find the electric field as a function of the distance from its centerline.

(*a*) Find the charge Q_{inner} on the inner cylinder:

$$Q_{\text{inner}} = \int_0^R \rho(r)dV = \int_0^R \frac{C}{r} 2\pi rLdr = 2\pi CL\int_0^R dr = 2\pi CLR$$

Relate this charge to the linear charge density:

$$\lambda = \frac{Q_{\text{inner}}}{L} = \frac{2\pi CLR}{L} = 2\pi CR$$

Substitute numerical values and evaluate λ:

$$\lambda = 2\pi(200\,\text{nC/m})(0.0150\,\text{m}) = \boxed{18.8\,\text{nC/m}}$$

(*b*) Apply Gauss's law to a cylindrical surface of radius *r* and length *L* that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 2\pi r L E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{2\pi r L \epsilon_0}$$

Substitute to obtain, for $r < 1.50$ cm:

$$E_{r<1.50\,\text{cm}} = \frac{2\pi C L r}{2\pi \epsilon_0 L r} = \frac{C}{\epsilon_0}$$

Substitute numerical values and evaluate $E_\text{n}(r < 1.50\text{ cm})$:

$$E_{r<1.50\,\text{cm}} = \frac{200\,\text{nC/m}^2}{8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{22.6\,\text{kN/C}}$$

Express Q_{inside} for 1.50 cm $< r <$ 4.50 cm:

$$Q_{\text{inside}} = 2\pi C L R$$

Substitute to obtain, for 1.50 cm $< r <$ 4.50 cm:

$$E_{1.50\,\text{cm}<r<4.50\,\text{cm}} = \frac{2C\pi R L}{2\pi \epsilon_0 r L} = \frac{CR}{\epsilon_0 r}$$

where $R = 1.50$ cm.

Substitute numerical values and evaluate $E_{1.50\,\text{cm}<r<4.50\,\text{cm}}$:

$$E_{1.50\,\text{cm}<r<4.50\,\text{cm}} = \frac{(200\,\text{nC/m}^2)(0.0150\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)r} = \boxed{\frac{339\,\text{N}\cdot\text{m/C}}{r}}$$

Because the outer cylindrical shell is a conductor:

$$E_{4.50\,\text{cm}<r<6.50\,\text{cm}} = \boxed{0}$$

For $r > 6.50$ cm, $Q_{\text{inside}} = 2\pi C L R$ and:

$$E_{r>6.50\,\text{cm}} = \boxed{\frac{339\,\text{N}\cdot\text{m/C}}{r}}$$

Electric Charge and Field at Conductor Surfaces

58 • An uncharged penny is in a region that has a uniform electric field of magnitude 1.60 kN/C directed perpendicular to its faces. (*a*) Find the charge density on each face of the penny, assuming the faces are planes. (*b*) If the radius of the penny is 1.00 cm, find the total charge on one face.

Picture the Problem Because the penny is in an external electric field, it will have charges of opposite signs induced on its faces. The induced charge σ is related to the electric field by $E = \sigma/\epsilon_0$. Once we know σ, we can use the definition of surface charge density to find the total charge on one face of the penny.

(*a*) Relate the electric field to the charge density on each face of the penny:

$$E = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = \epsilon_0 E$$

Substitute numerical values and evaluate σ:

$$\sigma = \left(8.854\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(1.60\,\mathrm{kN/C}\right) = 14.17\,\mathrm{nC/m^2} = \boxed{14.2\,\mathrm{nC/m^2}}$$

(*b*) Use the definition of surface charge density to obtain:

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi r^2} \Rightarrow Q = \sigma\pi r^2$$

Substitute numerical values and evaluate Q:

$$Q = \pi\left(14.17\,\mathrm{nC/m^2}\right)\left(0.0100\,\mathrm{m}\right)^2 = \boxed{4.45\,\mathrm{pC}}$$

59 • A thin metal slab has a net charge of zero and has square faces that have 12-cm-long sides. It is in a region that has a uniform electric field that is perpendicular to its faces. The total charge induced on one of the faces is 1.2 nC. What is the magnitude of the electric field?

Picture the Problem Because the metal slab is in an external electric field, it will have charges of opposite signs induced on its faces. The induced charge σ is related to the electric field by $E = \sigma/\epsilon_0$.

Relate the magnitude of the electric field to the charge density on the metal slab:

$$E = \frac{\sigma}{\epsilon_0}$$

Use its definition to express σ:

$$\sigma = \frac{Q}{A} = \frac{Q}{L^2}$$

Substitute for σ to obtain:

$$E = \frac{Q}{L^2 \in_0}$$

Substitute numerical values and evaluate E:

$$E = \frac{1.2\,\text{nC}}{(0.12\,\text{m})^2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)} = \boxed{9.4\,\text{kN/C}}$$

60 •• A charge of -6.00 nC is uniformly distributed on a thin square sheet of non-conducting material of edge length 20.0 cm. (*a*) What is the surface charge density of the sheet? (*b*) What are the magnitude and direction of the electric field next to the sheet and proximate to the center of the sheet?

Picture the Problem We can apply its definition to find the surface charge density of the nonconducting material and calculate the electric field at either of its surfaces from $\sigma/(2\in_0)$.

(*a*) Use its definition to find σ:

$$\sigma = \frac{Q}{A} = \frac{-6.00\,\text{nC}}{(0.200\,\text{m})^2} = \boxed{-150\,\text{nC/m}^2}$$

(*b*) The magnitude of the electric field just outside the surface of the sheet on the side that is charged is given by:

$$|E| = \frac{\sigma}{2\in_0} = \left|\frac{-150\,\text{nC/m}^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\right| = \boxed{8.47\,\text{kN/C}}$$

The direction of the field on the side of the sheet that is charged is the direction of the electric force acting on a test charge. Because the surface is negatively charged, this force and, hence, the electric field, is directed toward the surface.

Because the sheet is constructed from non-conducting material, no charge is induced on the second surface of the sheet and there is, therefore, no electric field just outside the sheet surface on this side.

61 • A conducting spherical shell that has zero net charge has an inner radius R_1 and an outer radius R_2. A positive point charge q is placed at the center of the shell. (*a*) Use Gauss's law and the properties of conductors in electrostatic equilibrium to find the electric field in the three regions: $0 \le r < R_1$, $R_1 < r < R_2$, and $r > R_2$, where r is the distance from the center. (*b*) Draw the electric field lines in all three regions. (*c*) Find the charge density on the inner surface ($r = R_1$) and on the outer surface ($r = R_2$) of the shell.

Picture the Problem We can construct a Gaussian surface in the shape of a sphere of radius r with the same center as the shell and apply Gauss's law to find

the electric field as a function of the distance from this point. The inner and outer surfaces of the shell will have charges induced on them by the charge q at the center of the shell.

(*a*) Apply Gauss's law to a spherical surface of radius r that is concentric with the point charge:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} \quad (1)$$

For $r < R_1$, $Q_{\text{inside}} = q$. Substitute in equation (1) and simplify to obtain:

$$E_{r<R_1} = \frac{q}{4\pi r^2 \epsilon_0} = \boxed{\frac{kq}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for $R_1 < r < R_2$:

$$Q_{\text{inside}} = 0$$

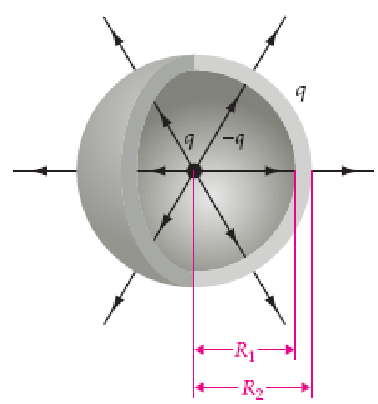
and

$$E_{R_1<r<R_2} = \boxed{0}$$

For $r > R_2$, $Q_{\text{inside}} = q$. Substitute in equation (1) and simplify to obtain:

$$E_{r>R_2} = \frac{q}{4\pi r^2 \epsilon_0} = \boxed{\frac{kq}{r^2}}$$

(*b*) The electric field lines are shown in the diagram to the right:



(*c*) A charge $-q$ is induced on the inner surface. Use the definition of surface charge density to obtain:

$$\sigma_{\text{inner}} = \boxed{-\frac{q}{4\pi R_1^2}}$$

A charge q is induced on the outer surface. Use the definition of surface charge density to obtain:

$$\sigma_{\text{outer}} = \boxed{\frac{q}{4\pi R_2^2}}$$

62 •• The electric field just above the surface of Earth has been measured to typically be 150 N/C pointing downward. (*a*) What is the sign of the net charge on Earth's surface under typical conditions? (*b*)What is the total charge on Earth's surface implied by this measurement?

Picture the Problem We can construct a spherical Gaussian surface at the surface of Earth (we'll assume Earth is a sphere) and apply Gauss's law to relate the electric field to its total charge.

(*a*) Because the direction of an electric field is the direction of the force acting on a positively charged object, the net charge on Earth's surface must be negative.

(*b*)Apply Gauss's law to a spherical surface of radius R_E that is concentric with Earth:

$$\oint_S E_n dA = \frac{1}{\epsilon_0}Q_{\text{inside}} \Rightarrow 4\pi R_E^2 E_E = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solve for $Q_{\text{inside}} = Q_{\text{Earth}}$ to obtain:

$$Q_{\text{Earth}} = 4\pi \epsilon_0 R_E^2 E_E = \frac{R_E^2 E_E}{k}$$

Substitute numerical values and evaluate Q_{Earth}:

$$Q_{\text{Earth}} = \frac{(6.37\times10^6\ \text{m})^2(150\ \text{N/C})}{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2} = \boxed{677\ \text{kC}}$$

63 •• **[SSM]** A positive point charge of 2.5 μC is at the center of a conducting spherical shell that has a net charge of zero, an inner radius equal to 60 cm, and an outer radius equal to 90 cm. (*a*) Find the charge densities on the inner and outer surfaces of the shell and the total charge on each surface. (*b*) Find the electric field everywhere. (*c*) Repeat Part (*a*) and Part (*b*) with a net charge of +3.5 μC placed on the shell.

Picture the Problem Let the inner and outer radii of the uncharged spherical conducting shell be R_1 and R_2 and q represent the positive point charge at the center of the shell. The positive point charge at the center will induce a negative charge on the inner surface of the shell and, because the shell is uncharged, an equal positive charge will be induced on its outer surface. To solve Part (*b*), we can construct a Gaussian surface in the shape of a sphere of radius r with the same center as the shell and apply Gauss's law to find the electric field as a function of the distance from this point. In Part (*c*) we can use a similar strategy with the additional charge placed on the shell.

(a) Express the charge density on the inner surface:

$$\sigma_{\text{inner}} = \frac{q_{\text{inner}}}{A}$$

Express the relationship between the positive point charge q and the charge induced on the inner surface q_{inner}:

$$q + q_{\text{inner}} = 0 \Rightarrow q_{\text{inner}} = -q$$

Substitute for q_{inner} and A to obtain:

$$\sigma_{\text{inner}} = \frac{-q}{4\pi R_1^2}$$

Substitute numerical values and evaluate σ_{inner}:

$$\sigma_{\text{inner}} = \frac{-2.5\,\mu\text{C}}{4\pi(0.60\,\text{m})^2} = \boxed{-0.55\,\mu\text{C/m}^2}$$

Express the charge density on the outer surface:

$$\sigma_{\text{outer}} = \frac{q_{\text{outer}}}{A}$$

Because the spherical shell is uncharged:

$$q_{\text{outer}} + q_{\text{inner}} = 0$$

Substitute for q_{outer} to obtain:

$$\sigma_{\text{outer}} = \frac{-q_{\text{inner}}}{4\pi R_2^2}$$

Substitute numerical values and evaluate σ_{outer}:

$$\sigma_{\text{outer}} = \frac{2.5\,\mu\text{C}}{4\pi(0.90\,\text{m})^2} = \boxed{0.25\,\mu\text{C/m}^2}$$

(b) Apply Gauss's law to a spherical surface of radius r that is concentric with the point charge:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Noting that, due to symmetry, $E_n = E_r$, solve for E_r to obtain:

$$E_r = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} \qquad (1)$$

For $r < R_1 = 60$ cm, $Q_{\text{inside}} = q$. Substitute in equation (1) to obtain:

$$E_{r<60\,\text{cm}} = \frac{q}{4\pi r^2 \epsilon_0} = \frac{kq}{r^2}$$

Substitute numerical values and evaluate $E_{r<60\,\text{cm}}$:

$$E_{r<60\,\text{cm}} = \frac{\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(2.5\ \mu\text{C}\right)}{r^2} = \boxed{\left(2.2\times10^{4}\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for 60 cm < r < 90 cm:

$Q_{\text{inside}} = 0$

and

$E_{60\,\text{cm}<r<90\,\text{cm}} = \boxed{0}$

For r > 90 cm, the net charge inside the Gaussian surface is q and:

$$E_{r>90\,\text{cm}} = \frac{kq}{r^2} = \boxed{\left(2.3\times10^{4}\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

(*c*) Because E = 0 in the conductor:

$q_{\text{inner}} = -2.5\ \mu\text{C}$

and

$\sigma_{\text{inner}} = \boxed{-0.55\ \mu\text{C/m}^2}$ as before.

Express the relationship between the charges on the inner and outer surfaces of the spherical shell:

$q_{\text{outer}} + q_{\text{inner}} = 3.5\ \mu\text{C}$

and

$q_{\text{outer}} = 3.5\ \mu\text{C} - q_{\text{inner}} = 6.0\ \mu\text{C}$

σ_{outer} is now given by:

$$\sigma_{\text{outer}} = \frac{6.0\ \mu\text{C}}{4\pi\left(0.90\ \text{m}\right)^2} = \boxed{0.59\ \mu\text{C/m}^2}$$

For $r < R_1$ = 60 cm, $Q_{\text{inside}} = q$ and $E_{r<60\,\text{cm}}$ is as it was in (*a*):

$$E_{r<60\,\text{cm}} = \boxed{\left(2.3\times10^{4}\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for 60 cm < r < 90 cm:

$Q_{\text{inside}} = 0$

and

$E_{60\,\text{cm}<r<90\,\text{cm}} = \boxed{0}$

For r > 0.90 m, the net charge inside the Gaussian surface is 6.0 μC and:

$$E_{r>90\,\text{cm}} = \frac{kq}{r^2} = \left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(6.0\ \mu\text{C}\right)\frac{1}{r^2} = \boxed{\left(5.4\times10^{4}\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

64 •• If the magnitude of an electric field in air is as great as 3.0×10^6 N/C, the air becomes ionized and begins to conduct electricity. This phenomenon is called *dielectric breakdown*. A charge of 18 μC is to be placed on a conducting sphere. What is the minimum radius of a sphere that can hold this charge without breakdown?

Picture the Problem From Gauss's law we know that the electric field at the surface of the charged sphere is given by $E = kQ/R^2$ where Q is the charge on the sphere and R is its radius. The minimum radius for dielectric breakdown corresponds to the maximum electric field at the surface of the sphere.

Use Gauss's law to express the electric field at the surface of the charged sphere:

$$E = \frac{kQ}{R^2}$$

Express the relationship between E and R for dielectric breakdown:

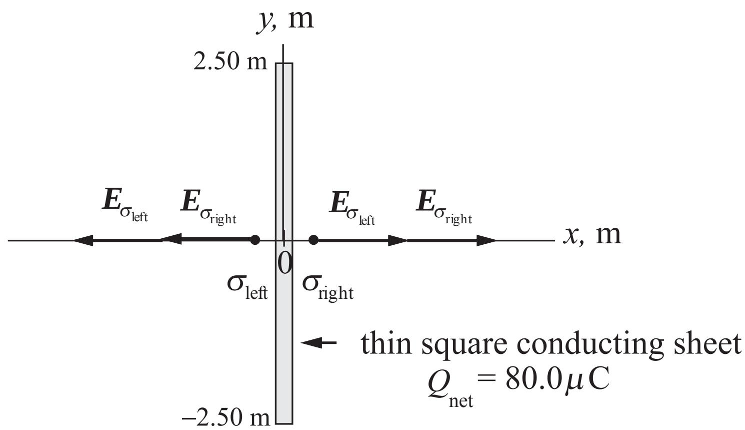
$$E_{\text{max}} = \frac{kQ}{R_{\text{min}}^2} \Rightarrow R_{\text{min}} = \sqrt{\frac{kQ}{E_{\text{max}}}}$$

Substitute numerical values and evaluate R_{min}:

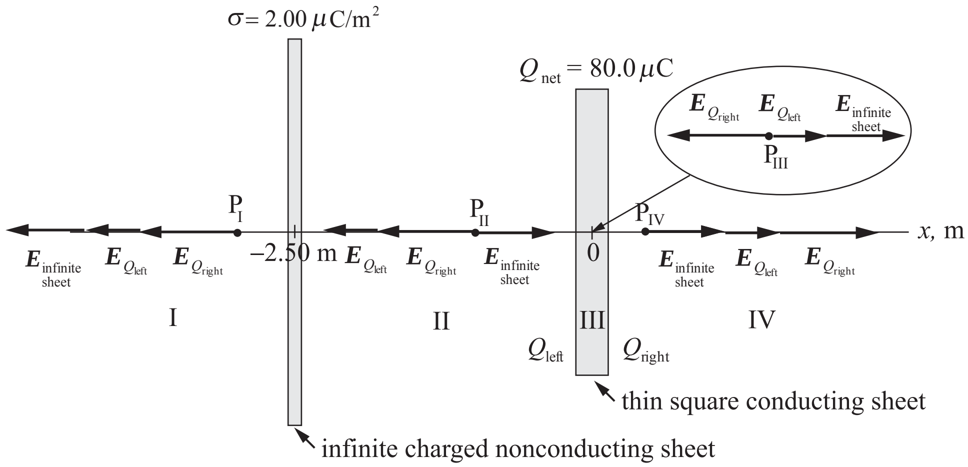
$$R_{\text{min}} = \sqrt{\frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(18\ \mu\text{C}\right)}{3.0\times10^6\ \text{N/C}}} = \boxed{23\,\text{cm}}$$

65 ••• **[SSM]** A thin square conducting sheet that has 5.00-m-long edges has a net charge of 80.0 μC. The square is in the $x = 0$ plane and is centered at the origin.(Assume the charge on each surface is uniformly distributed.) (*a*) Find the charge density on each side of the sheet and find the electric field on the x axis in the region $|x| << 5.00$ m. (*b*) A thin but infinite nonconducting sheet that has a uniform charge density of 2.00 $\mu\text{C/m}^2$ is now placed in the $x = -2.50$ m plane. Find the electric field on the x axis on each side of the square sheet in the region $|x| << 2.50$ m. Find the charge density on each surface of the square sheet.

Picture the Problem (*a*) One half of the total charge is on each side of the conducting sheet and the electric field inside the conducting sheet is zero. The electric field intensity just outside the surface of a conductor is given by $E = |\sigma|/\epsilon_0$. Typical field points to the left and right of the conducting sheet are shown in the following diagram.



(*b*) We can use the fact that the net charge on the conducting sheet is the sum of the charges Q_{left} and Q_{right} on its left and right surfaces to obtain an equation relating these charges. Because the resultant electric field is zero inside the sheet, we can obtain a second equation in Q_{left} and Q_{right} that we can solve simultaneously with the first equation to find Q_{left} and Q_{right}. The resultant electric field is the superposition of three fields–the field due to the charges on the infinite nonconducting sheet and the fields due to the charges on the surfaces of the thin square conducting sheet. The electric field intensity due to a uniformly charged nonconducting infinite sheet is given by $E = |\sigma|/2\in_0$. Typical field points for each of the four regions of interest are shown in the following diagram.



Note: The vectors in this figure are drawn consistent with the charges Q_{left} and Q_{right} both being positive. If either Q_{left} or Q_{right} are negative then the solution will produce a negative value for either Q_{left} or Q_{right}.

(*a*) Because the square sheet is a conductor, half the charge on each surface is half the net charge on the sheet:

$$\sigma_{\text{left}} = \sigma_{\text{right}} = \frac{\frac{1}{2}Q_{\text{net}}}{A}$$

Substitute numerical values and evaluate σ_{left} and σ_{right}:

$$\sigma_{\text{left}} = \sigma_{\text{right}} = \frac{\frac{1}{2}(80.0\ \mu\text{C})}{(5.00\ \text{m})^2} = \boxed{1.60\ \frac{\mu\text{C}}{\text{m}^2}}$$

For $|x| << 5.00$ m, the electric field is given by the expression for the field just outside a conductor:

$$E_{|x|<<5.00\ \text{m}} = \frac{\sigma}{\epsilon_0}$$

Substitute numerical values and evaluate $E_{|x|<<5.00\ \text{m}}$:

$$E_{|x|<<5.00\ \text{m}} = \frac{1.60\ \mu\text{C/m}^2}{8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = 180.7\ \text{kN/C} = \boxed{181\ \text{kN/C}}$$

For $x > 0$, $E_{|x|<<5.00\ \text{m}}$ is in the $+x$ direction and for $x < 0$, $E_{|x|<<5.00\ \text{m}}$ is in the $-x$ direction.

(*b*) The resultant electric field in Region II is the sum of the fields due to the infinite nonconducting sheet and the charge on the surfaces of the thin square conducting sheet:

$$\vec{E}_{\text{II}} = \vec{E}_{\substack{\text{infinite}\\ \text{sheet}}} + \vec{E}_{Q_{\text{left}}} + \vec{E}_{Q_{\text{right}}} = \frac{\sigma_{\substack{\text{infinite}\\ \text{sheet}}}}{2\epsilon_0}\hat{i} - \frac{\sigma_{\text{left}}}{2\epsilon_0}\hat{i} - \frac{\sigma_{\text{right}}}{2\epsilon_0}\hat{i} = \left(\frac{\sigma_{\substack{\text{infinite}\\ \text{sheet}}} - \sigma_{\text{left}} - \sigma_{\text{right}}}{2\epsilon_0}\right)\hat{i}$$

Due to the presence of the infinite nonconducting sheet, the charges on the thin square conducting sheet are redistributed on the left and right surfaces. The net charge on the thin square conducting sheet is the sum of the charges on its left- and right-hand surfaces:

$$Q_{\text{left}} + Q_{\text{right}} = 80.0\ \mu\text{C}$$

where we've assumed that Q_{left} and Q_{right} are both positive.

Writing this equation in terms of the surface charge densities yields:

$$\sigma_{\text{left}}+\sigma_{\text{right}}=\frac{Q_{\text{left}}}{A}+\frac{Q_{\text{right}}}{A}=\frac{Q_{\text{left}}+Q_{\text{right}}}{A}=\frac{80.0\ \mu\text{C}}{(5.00\ \text{m})^2}=3.20\ \mu\text{C/m}^2 \qquad (1)$$

where A is the area of one side of the thin square conducting sheet.

Because the electric field is zero inside the thin square conducting sheet:

$$\frac{\sigma_{\substack{\text{infinite}\\ \text{sheet}}}}{2\epsilon_0}+\frac{\sigma_{\text{left}}}{2\epsilon_0}-\frac{\sigma_{\text{right}}}{2\epsilon_0}=0$$

or

$$2.00\ \mu\text{C/m}^2+\sigma_{\text{left}}-\sigma_{\text{right}}=0 \qquad (2)$$

Solving equations (1) and (2) simultaneously yields:

$$\sigma_{\text{left}}=\boxed{0.60\ \mu\text{C/m}^2}$$

and

$$\sigma_{\text{right}}=\boxed{2.60\ \mu\text{C/m}^2}$$

Substitute numerical values and evaluate $\vec{E}_{\text{II}}$:

$$\vec{E}_{\text{II}}=\left(\frac{2.00\frac{\mu\text{C}}{\text{m}^2}-0.60\frac{\mu\text{C}}{\text{m}^2}-2.60\frac{\mu\text{C}}{\text{m}^2}}{2\left(8.854\times10^{-12}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\right)\hat{\boldsymbol{i}}=\boxed{\left(-67.8\frac{\text{kN}}{\text{C}}\right)\hat{\boldsymbol{i}}}$$

The resultant electric field in Region IV is the sum of the fields due to the charge on the infinite nonconducting sheet and the charges on the two surfaces of the thin square conducting sheet:

$$\vec{E}_{\text{IV}}=\vec{E}_{\substack{\text{infinite}\\ \text{sheet}}}+\vec{E}_{Q_{\text{left}}}+\vec{E}_{Q_{\text{right}}}=\frac{\sigma_{\substack{\text{infinite}\\ \text{sheet}}}}{2\epsilon_0}\hat{\boldsymbol{i}}+\frac{\sigma_{\text{left}}}{2\epsilon_0}\hat{\boldsymbol{i}}+\frac{\sigma_{\text{right}}}{2\epsilon_0}\hat{\boldsymbol{i}}=\left(\frac{\sigma_{\substack{\text{infinite}\\ \text{sheet}}}+\sigma_{\text{left}}+\sigma_{\text{right}}}{2\epsilon_0}\right)\hat{\boldsymbol{i}}$$

Substitute numerical values and evaluate $\vec{E}_{IV}$:

$$\vec{E}_{IV} = \left(\frac{2.00\frac{\mu C}{m^2} + 0.60\frac{\mu C}{m^2} + 2.60\frac{\mu C}{m^2}}{2\left(8.854\times10^{-12}\frac{C^2}{N\cdot m^2}\right)} \right)\hat{i} = \boxed{\left(294\frac{kN}{C}\right)\hat{i}}$$

General Problems

66 •• Consider the concentric metal sphere and spherical shells that are shown in Figure 22-43. The innermost is a solid sphere that has a radius R_1. A spherical shell surrounds the sphere and has an inner radius R_2 and an outer radius R_3. The sphere and the shell are both surrounded by a second spherical shell that has an inner radius R_4 and an outer radius R_5. None of these three objects initially have a net charge. Then, a negative charge $-Q_0$ is placed on the inner sphere and a positive charge $+Q_0$ is placed on the outermost shell. (*a*) After the charges have reached equilibrium, what will be the direction of the electric field between the inner sphere and the middle shell? (*b*) What will be the charge on the inner surface of the middle shell? (*c*) What will be the charge on the outer surface of the middle shell? (*d*) What will be the charge on the inner surface of the outermost shell? (*e*) What will be the charge on the outer surface of the outermost shell? (*f*) Plot E as a function of r for all values of r.

Determine the Concept We can determine the direction of the electric field between spheres I and II by imagining a test charge placed between the spheres and determining the direction of the force acting on it. We can determine the amount and sign of the charge on each sphere by realizing that the charge on a given surface induces a charge of the same magnitude but opposite sign on the next surface of larger radius.

(*a*) The charge placed on sphere III has no bearing on the electric field between spheres I and II. The field in this region will be in the direction of the force exerted on a test charge placed between the spheres. Because the charge at the center is negative, $\boxed{\text{the field will point toward the center.}}$

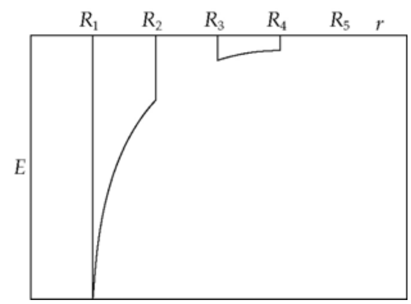
(*b*) The charge on sphere I ($-Q_0$) will induce a charge of the same magnitude but opposite sign on sphere II: $\boxed{+Q_0}$

(*c*) The induction of charge $+Q_0$ on the inner surface of sphere II will leave its outer surface with a charge of the same magnitude but opposite sign: $\boxed{-Q_0}$

(*d*) The presence of charge $-Q_0$ on the outer surface of sphere II will induce a charge of the same magnitude but opposite sign on the inner surface of sphere III: $\boxed{+Q_0}$

(*e*) The presence of charge $+Q_0$ on the inner surface of sphere III will leave the outer surface of sphere III neutral: $\boxed{0}$

(*f*) A graph of E as a function of r is shown to the right:



67 •• **[SSM]** A large, flat, nonconducting, non-uniformly charged surface lies in the $x = 0$ plane. At the origin, the surface charge density is $+3.10\ \mu\text{C/m}^2$. A small distance away from the surface on the positive x axis, the x component of the electric field is 4.65×10^5 N/C. What is E_x a small distance away from the surface on the negative x axis?

Picture the Problem Because the difference between the field just to the right of the surface $E_{x,\text{pos}}$ and the field just to the left of the surface $E_{x,\text{neg}}$ is the field due to the nonuniform surface charge, we can express $E_{x,\text{neg}}$ as the difference between $E_{x,\text{pos}}$ and σ/ϵ_0.

Express the electric field just to the left of the origin in terms of $E_{x,\text{pos}}$ and σ/ϵ_0:

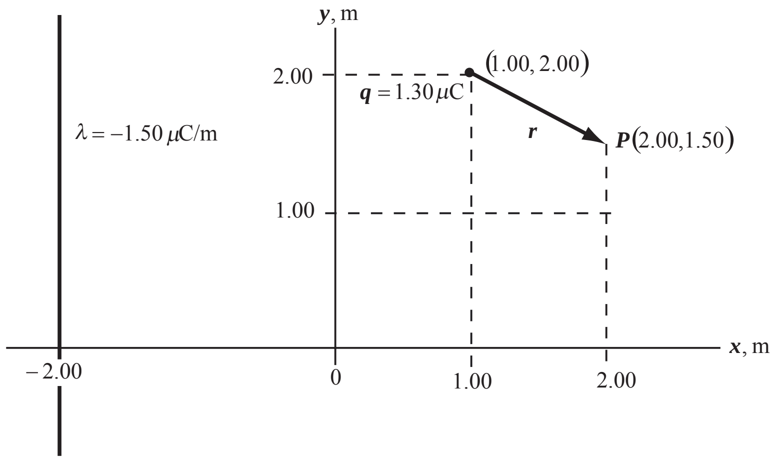
$$E_{x,\text{neg}} = E_{x,\text{pos}} - \frac{\sigma}{\epsilon_0}$$

Substitute numerical values and evaluate $E_{x,\text{neg}}$:

$$E_{x,\text{neg}} = 4.65\times10^5\ \text{N/C} - \frac{3.10\ \mu\text{C/m}^2}{8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{115\ \text{kN/C}}$$

68 •• An infinitely long line charge that has a uniform linear charge density equal to $-1.50\ \mu$C/m lies parallel to the y axis at $x = -2.00$ m. A positive point charge that has a magnitude equal to $1.30\ \mu$C is located at $x = 1.00$ m, $y = 2.00$ m. Find the electric field at $x = 2.00$ m, $y = 1.50$ m.

Picture the Problem Let P denote the point of interest at (2.00 m, 1.50 m). The electric field at P is the sum of the electric fields due to the infinite line charge and the point charge.



Express the resultant electric field at P: $\vec{E} = \vec{E}_\lambda + \vec{E}_q$ (1)

Find the field at P due the infinite line charge:

$$\vec{E}_\lambda = \frac{2k\lambda}{r}\hat{r} = \frac{2(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(-1.50\ \mu\text{C/m})}{4.00\,\text{m}}\hat{i} = (-6.741\,\text{kN/C})\hat{i}$$

Express the field at P due the point charge: $\vec{E}_q = \frac{kq}{r^2}\hat{r}$

Referring to the diagram above, determine r and $\hat{r}$: $r = 1.118\,\text{m}$ and $\hat{r} = 0.8944\hat{i} - 0.4472\hat{j}$

Substitute and evaluate $\vec{E}_q(2.00\,\text{m}, 1.50\,\text{m})$:

$$\begin{aligned}\vec{E}_q(2.00\,\text{m},1.50\,\text{m}) &= \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(1.30\ \mu\text{C})}{(1.118\,\text{m})^2}(0.8944\hat{i} - 0.4472\hat{j}) \\ &= (9.348\,\text{kN/C})(0.8944\hat{i} - 0.4472\hat{j}) \\ &= (8.361\,\text{kN/C})\hat{i} - (4.180\,\text{kN/C})\hat{j}\end{aligned}$$

Substitute for $\vec{E}_\lambda$ and $\vec{E}_q$ in equation (1) and simplify to obtain:

$$\begin{aligned}\vec{E}(2.00\,\text{m},1.50\,\text{m}) &= (-6.741\,\text{kN/C})\hat{i} + (8.361\,\text{kN/C})\hat{i} - (4.180\,\text{kN/C})\hat{j}\\ &= \boxed{(1.62\,\text{kN/C})\hat{i} - (4.18\,\text{kN/C})\hat{j}}\end{aligned}$$

69 •• [SSM] A thin, non-conducting, uniformly charged spherical shell of radius R (Figure 22-44*a*) has a total positive charge of Q. A small circular plug is removed from the surface. (*a*) What is the magnitude and direction of the electric field at the center of the hole? (*b*) The plug is now put back in the hole (Figure 22-44*b*). Using the result of Part (*a*), find the electric force acting on the plug. (*c*) Using the magnitude of the force, calculate the "electrostatic pressure" (force/unit area) that tends to expand the sphere.

Picture the Problem If the patch is small enough, the field at the center of the patch comes from two contributions. We can view the field in the hole as the sum of the field from a uniform spherical shell of charge Q plus the field due to a small patch with surface charge density equal but opposite to that of the patch cut out.

(*a*) Express the magnitude of the electric field at the center of the hole:

$$E = E_{\substack{\text{spherical}\\\text{shell}}} + E_{\text{hole}}$$

Apply Gauss's law to a spherical gaussian surface just outside the given sphere:

$$E_{\substack{\text{spherical}\\\text{shell}}}\left(4\pi r^2\right) = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for $E_{\substack{\text{spherical}\\\text{shell}}}$ to obtain:

$$E_{\substack{\text{spherical}\\\text{shell}}} = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field due to the small hole (small enough so that we can treat it as a plane surface) is:

$$E_{\text{hole}} = \frac{-\sigma}{2\epsilon_0}$$

Substitute for $E_{\substack{\text{spherical}\\\text{shell}}}$ and E_{hole} and simplify to obtain:

$$\begin{aligned}E &= \frac{Q}{4\pi\epsilon_0 r^2} + \frac{-\sigma}{2\epsilon_0}\\ &= \frac{Q}{4\pi\epsilon_0 r^2} - \frac{Q}{2\epsilon_0\left(4\pi r^2\right)}\\ &= \boxed{\frac{Q}{8\pi\epsilon_0 r^2}}\ \text{radially outward}\end{aligned}$$

(*b*) Express the force on the patch:

$$F = qE$$

where q is the charge on the patch.

Assuming that the patch has radius *a,* express the proportion between its charge and that of the spherical shell:

$$\frac{q}{\pi a^2}=\frac{Q}{4\pi r^2} \text{ or } q=\frac{a^2}{4r^2}Q$$

Substitute for q and E in the expression for F to obtain:

$$F=\left(\frac{a^2}{4r^2}Q\right)\left(\frac{Q}{8\pi \in_0 r^2}\right)$$

$$=\boxed{\frac{Q^2a^2}{32\pi \in_0 r^4}} \text{ radially outward}$$

(*c*) The pressure is the force exerted on the patch divided by the area of the patch:

$$P=\frac{\dfrac{Q^2a^2}{32\pi \in_0 r^4}}{\pi a^2}=\boxed{\frac{Q^2}{32\pi^2 \in_0 r^4}}$$

70 •• An infinite thin sheet in the $y = 0$ plane has a uniform surface charge density $\sigma_1 = +65\ \text{nC/m}^2$. A second infinite thin sheet has a uniform charge density $\sigma_2 = +45\ \text{nC/m}^2$ and intersects the $y = 0$ plane at the z axis and makes an angle of 30º with the xz plane, as shown in Figure 22-45. Find the electric field at (*a*) $x = 6.0$ m, $y = 2.0$ m and (*b*) $x = 6.0$ m, $y = 5.0$ m.

Picture the Problem Let the numeral 1 refer to the plane with charge density σ_1 and the numeral 2 to the plane with charge density σ_2. We can find the electric field at the two points of interest by adding the electric fields due to the charge distributions of the two infinite planes.

Express the electric field at any point in space due to the charge distributions on the two planes:

$$\vec{E}=\vec{E}_1+\vec{E}_2 \qquad (1)$$

(*a*) Express the electric field at (6.0 m, 2.0 m) due to plane 1:

$$\vec{E}_1(6.0\,\text{m},2.0\,\text{m})=\frac{\sigma_1}{2\in_0}\hat{\boldsymbol{j}}=\frac{+65\,\text{nC/m}^2}{2\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{\boldsymbol{j}}=(3.67\,\text{kN/C})\hat{\boldsymbol{j}}$$

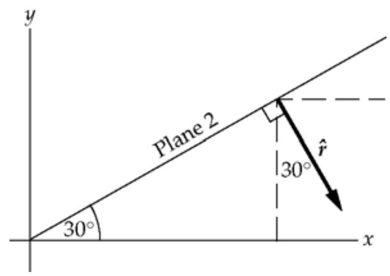
Express the electric field at (6.0 m, 2.0 m) due to plane 2:

$$\vec{E}_2(6.0\,\text{m},2.0\,\text{m})=\frac{\sigma_2}{2\in_0}\hat{\boldsymbol{r}}=\frac{+45\,\text{nC/m}^2}{2\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{\boldsymbol{r}}=(2.54\,\text{kN/C})\hat{\boldsymbol{r}}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing from plane 2 toward the point whose coordinates are (6.0 m, 2.0 m).

Refer to the following diagram to obtain:

$$\hat{\boldsymbol{r}} = \sin 30°\hat{\boldsymbol{i}} - \cos 30°\hat{\boldsymbol{j}}$$



Substitute to obtain:

$$\begin{aligned}\vec{\boldsymbol{E}}_2(6.0\,\text{m}, 2.0\,\text{m}) &= (2.54\,\text{kN/C})(\sin 30°\hat{\boldsymbol{i}} - \cos 30°\hat{\boldsymbol{j}}) \\ &= (1.27\,\text{kN/C})\hat{\boldsymbol{i}} + (-2.20\,\text{kN/C})\hat{\boldsymbol{j}}\end{aligned}$$

Substitute for $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_2$ in equation (1) to obtain:

$$\begin{aligned}\vec{\boldsymbol{E}}(6.0\,\text{m}, 2.0\,\text{m}) &= (3.67\,\text{kN/C})\hat{\boldsymbol{j}} + (1.27\,\text{kN/C})\hat{\boldsymbol{i}} + (-2.20\,\text{kN/C})\hat{\boldsymbol{j}} \\ &= \boxed{(1.3\,\text{kN/C})\hat{\boldsymbol{i}} + (1.5\,\text{kN/C})\hat{\boldsymbol{j}}}\end{aligned}$$

(*b*) Note that $\vec{\boldsymbol{E}}_1(6.0\,\text{m}, 5.0\,\text{m}) = \vec{\boldsymbol{E}}_1(6.0\,\text{m}, 2.0\,\text{m})$ so that:

$$\vec{\boldsymbol{E}}_1(6\,\text{m}, 5\,\text{m}) = \frac{\sigma}{2\in_0}\hat{\boldsymbol{j}} = \frac{65\,\text{nC/m}^2}{2(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{\boldsymbol{j}} = (3.67\,\text{kN/C})\hat{\boldsymbol{j}}$$

Note also that $\vec{\boldsymbol{E}}_2(6.0\,\text{m}, 5.0\,\text{m}) = -\vec{\boldsymbol{E}}_2(6.0\,\text{m}, 2.0\,\text{m})$ so that:

$$\vec{\boldsymbol{E}}_2(6.0\,\text{m}, 5.0\,\text{m}) = (-1.27\,\text{kN/C})\hat{\boldsymbol{i}} + (2.20\,\text{kN/C})\hat{\boldsymbol{j}}$$

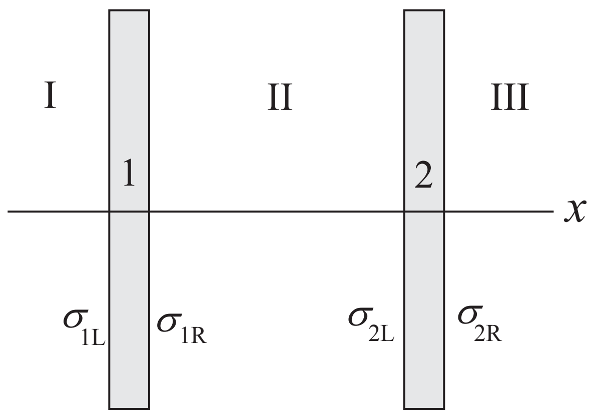
Substitute for $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_2$ in equation (1) to obtain:

$$\begin{aligned}\vec{\boldsymbol{E}}(6.0\,\text{m}, 5.0\,\text{m}) &= (3.67\,\text{kN/C})\hat{\boldsymbol{j}} + (-1.27\,\text{kN/C})\hat{\boldsymbol{i}} + (2.20\,\text{kN/C})\hat{\boldsymbol{j}} \\ &= \boxed{(-1.3\,\text{kN/C})\hat{\boldsymbol{i}} + (5.9\,\text{kN/C})\hat{\boldsymbol{j}}}\end{aligned}$$

71 •• Two identical square parallel metal plates each have an area of 500 cm^2. They are separated by 1.50 cm. They are both initially uncharged. Now a charge of +1.50 nC is transferred from the plate on the left to the plate on the right and the charges then establish electrostatic equilibrium. (Neglect edge effects.) (*a*) What is the electric field between the plates at a distance of 0.25 cm from the plate on the right? (*b*) What is the electric field between the plates a

distance of 1.00 cm from the plate on the left? (*c*) What is the electric field just to the left of the plate on the left? (*d*) What is the electric field just to the right of the plate to the right?

Picture the Problem The transfer of charge from the plate on the left to the plate on the right leaves the plates with equal but opposite charges. Because the metal plates are conductors, the charge on each plate is completely on the surface facing the other plate. The symbols for the four surface charge densities are shown in the figure. The *x* component of the electric field due to the charge on surface 1L is $-\sigma_{1L}/2\epsilon_0$ at points to the left of surface 1L and is $+\sigma_{1L}/2\epsilon_0$ at points to the right of surface 1L, where the +*x* direction is to the right. Similar expressions describe the electric fields due to the other three surface charges. We can use superposition of electric fields to find the electric field in each of the three regions.



Define σ_1 and σ_2 so that:

$$\sigma_1 = \sigma_{1L} + \sigma_{1R}$$
and
$$\sigma_2 = \sigma_{2L} + \sigma_{2R}$$

(*a*) and (*b*) In the region between the plates (region II):

$$E_{x,\text{II}} = \frac{\sigma_{1L}}{2\epsilon_0} + \frac{\sigma_{1R}}{2\epsilon_0} - \frac{\sigma_{2L}}{2\epsilon_0} - \frac{\sigma_{2R}}{2\epsilon_0}$$
$$= 0 + \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{2\epsilon_0} - 0 = \frac{\sigma_1 - \sigma_2}{\epsilon_0}$$

Let $\sigma_2 = -\sigma_1 = \sigma$. Then:

$$\sigma_1 - \sigma_2 = -\sigma - \sigma = -2\sigma$$

Substituting for $\sigma_1 - \sigma_2$ and using the definition of σ yields:

$$E_{x,\text{II}} = \frac{-2\sigma}{\epsilon_0} = -\frac{Q}{A\epsilon_0}$$

Substitute numerical values and evaluate $E_{x,\text{II}}$:

$$E_{x,\text{II}} = -\frac{1.50\,\text{nC}}{\left(8.854\times10^{-12}\,\dfrac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(500\times10^{-6}\,\text{m}^2\right)} = \boxed{339\text{ kN/C}} \text{ toward the left}$$

(*c*) The electric field strength just to the left of the plate on the left (region I) is given by:

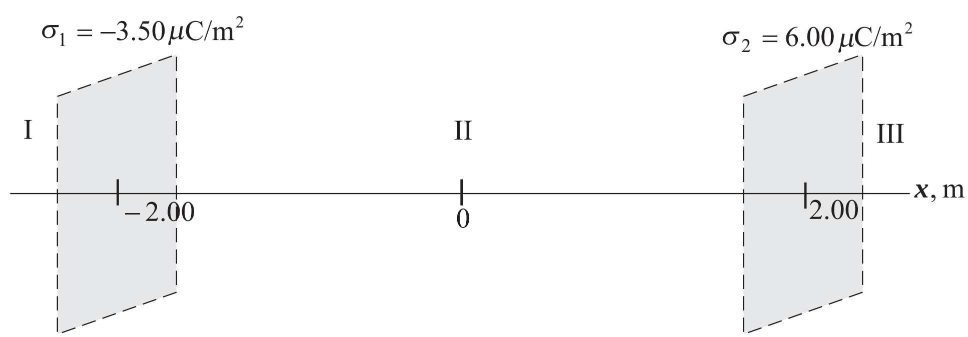
$$\begin{aligned} E_{x,\text{I}} &= -\frac{\sigma_{1\text{L}}}{2\in_0} - \frac{\sigma_{1\text{R}}}{2\in_0} - \frac{\sigma_{2\text{L}}}{2\in_0} - \frac{\sigma_{2\text{R}}}{2\in_0} \\ &= 0 - \frac{\sigma_1}{2\in_0} - \frac{\sigma_2}{2\in_0} - 0 \\ &= -\frac{-\sigma}{2\in_0} - \frac{\sigma}{2\in_0} = \boxed{0} \end{aligned}$$

(*d*) The electric field strength just to the right of the plate on the right (region III) is given by:

$$\begin{aligned} E_{x,\text{III}} &= \frac{\sigma_{1\text{L}}}{2\in_0} + \frac{\sigma_{1\text{R}}}{2\in_0} + \frac{\sigma_{2\text{L}}}{2\in_0} + \frac{\sigma_{2\text{R}}}{2\in_0} \\ &= 0 + \frac{\sigma_1}{2\in_0} + \frac{\sigma_2}{2\in_0} - 0 \\ &= \frac{-\sigma}{2\in_0} + \frac{\sigma}{2\in_0} = \boxed{0} \end{aligned}$$

72 •• Two infinite nonconducting uniformly charged planes lie parallel to each other and to the *yz* plane. One is at $x = -2.00$ m and has a surface charge density of $-3.50\ \mu\text{C/m}^2$. The other is at $x = 2.00$ m and has a surface charge density of $6.00\ \mu\text{C/m}^2$. Find the electric field in the region (*a*) $x < -2.00$ m, (*b*) $-2.00\text{ m} < x < 2.00$ m, and (*c*) $x > 2.00$ m.

Picture the Problem Let the numeral 1 refer to the infinite plane at $x = -2.00$ m and the numeral 2 to the plane at $x = 2.00$ m and let I, II, and III identify the regions to the left of plane 1, between the planes, and to the right of plane 2, respectively. We can use the expression for the electric field of in infinite plane of charge to express the electric field due to each plane of charge in each of the three regions. Their sum will be the resultant electric field in each region.



The resultant electric field is the sum of the fields due to planes 1 and 2:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \qquad (1)$$

(*a*) The field due to plane 1 in region I is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\epsilon_0}\left(-\hat{i}\right)$$

The field due to plane 2 in region I is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\epsilon_0}\left(-\hat{i}\right)$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\epsilon_0}\left(-\hat{i}\right) + \frac{\sigma_2}{2\epsilon_0}\left(-\hat{i}\right) = -\frac{\sigma_1 + \sigma_2}{2\epsilon_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\,\mu\text{C/m}^2 + 6.00\,\mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = \boxed{\left(-141\,\text{kN/C}\right)\hat{i}}$$

(*b*) The field due to plane 1 in region II is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\epsilon_0}\hat{i}$$

The field due to plane 2 in region II is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\epsilon_0}\left(-\hat{i}\right)$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\epsilon_0}\left(\hat{i}\right) + \frac{\sigma_2}{2\epsilon_0}\left(-\hat{i}\right) = \frac{\sigma_1 - \sigma_2}{2\epsilon_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\,\mu\text{C/m}^2 - 6.00\,\mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = \boxed{\left(-536\,\text{kN/C}\right)\hat{i}}$$

(*c*) The field due to plane 1 in region III is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\epsilon_0}\hat{i}$$

The field due to plane 2 in region III is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\epsilon_0}\hat{i}$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\epsilon_0}\left(\hat{i}\right) + \frac{\sigma_2}{2\epsilon_0}\left(\hat{i}\right) = \frac{\sigma_1 + \sigma_2}{2\epsilon_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\ \mu\text{C/m}^2 + 6.00\ \mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i}$$

$$= \boxed{(141\,\text{kN/C})\hat{i}}$$

73 ••• **[SSM]** A quantum-mechanical treatment of the hydrogen atom shows that the electron in the atom can be treated as a smeared-out distribution of negative charge of the form $\rho(r) = -\rho_0 e^{-2r/a}$. Here r represents the distance from the center of the nucleus and a represents the first *Bohr radius* which has a numerical value of 0.0529 nm. Recall that the nucleus of a hydrogen atom consists of just one proton and treat this proton as a positive point charge. (*a*) Calculate ρ_0, using the fact that the atom is neutral. (*b*) Calculate the electric field at any distance r from the nucleus.

Picture the Problem Because the atom is uncharged, we know that the integral of the electron's charge distribution over all of space must equal its charge q_e. Evaluation of this integral will lead to an expression for ρ_0. In (*b*) we can express the resultant electric field at any point as the sum of the electric fields due to the proton and the electron cloud.

(*a*) Because the atom is uncharged, the integral of the electron's charge distribution over all of space must equal its charge e:

$$e = \int_0^\infty \rho(r)dV = \int_0^\infty \rho(r)4\pi r^2 dr$$

Substitute for $\rho(r)$ and simplify to obtain:

$$e = -\int_0^\infty \rho_0 e^{-2r/a}4\pi r^2 dr$$

$$= -4\pi\rho_0\int_0^\infty r^2 e^{-2r/a}dr$$

Use integral tables or integration by parts to obtain:

$$\int_0^\infty r^2 e^{-2r/a}dr = \frac{a^3}{4}$$

Substitute for $\int_0^\infty r^2 e^{-2r/a}dr$ to obtain:

$$e = -4\pi\rho_0\left(\frac{a^3}{4}\right) = -\pi a^3\rho_0$$

Solving for ρ_0 yields:

$$\rho_0 = \boxed{-\frac{e}{\pi a^3}}$$

(*b*) The field will be the sum of the field due to the proton and that of the electron charge cloud:

$$E = E_{\text{p}} + E_{\text{cloud}}$$

Express the field due to the electron cloud:

$$E_{\text{cloud}}(r)=\frac{kQ(r)}{r^2}$$

where $Q(r)$ is the net negative charge enclosed a distance r from the proton.

Substitute for E_p and E_{cloud} to obtain:

$$E(r)=\frac{ke}{r^2}+\frac{kQ(r)}{r^2} \qquad (1)$$

$Q(r)$ is given by:

$$Q(r)=\int_0^r 4\pi r'^2\rho(r')dr'$$

$$=4\pi\int_0^r r'^2\rho_0 e^{-2r'/a}dr'$$

From Part (a), $\rho_0=\dfrac{-e}{\pi a^3}$:

$$Q(r)=4\pi\left(\frac{-e}{\pi a^3}\right)\int_0^r r'^2 e^{-2r'/a}dr'$$

$$=\frac{-4e}{a^3}\int_0^r r'^2 e^{-2r'/a}dr'$$

From a table of integrals:

$$\int_0^r x^2 e^{-2x/a}dx=\tfrac{1}{4}e^{-2r/a}a\left[\left(e^{-2r/a}-1\right)a^2-2ar-2r^2\right]$$

$$=\tfrac{1}{4}e^{-2r/a}a^3\left[\left(e^{-2r/a}-1\right)-2\frac{r}{a}-2\frac{r^2}{a^2}\right]$$

$$=\frac{a^3}{4}\left[\left(1-e^{-2r/a}\right)-2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$

Substituting for $\int_0^r r'^2 e^{-2r'/a}dr'$ in the expression for $Q(r)$ and simplifying yields:

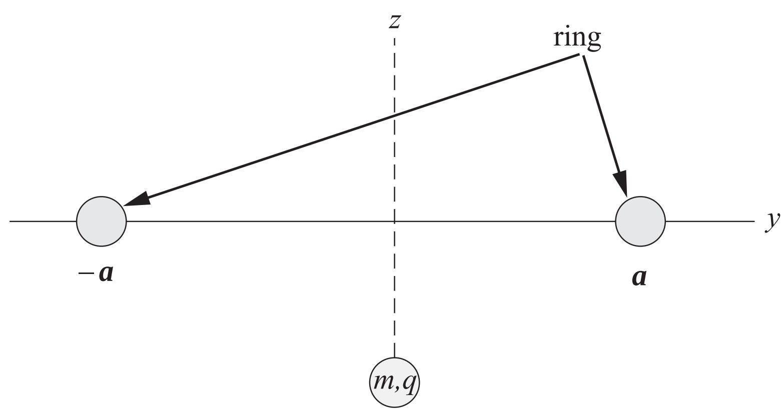
$$Q(r)=\frac{-e}{4}\left[\left(1-e^{-2r/a}\right)-2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$

Substitute for $Q(r)$ in equation (1) and simplify to obtain:

$$E(r) = \frac{ke}{r^2} - \frac{ke}{4r^2}\left[\left(1-e^{-2r/a}\right) - 2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$
$$= \boxed{\frac{ke}{r^2}\left(1-\frac{1}{4}\left[\left(1-e^{-2r/a}\right) - 2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]\right)}$$

74 •• A uniformly charged ring has a radius *a,* lies in a horizontal plane, and has a negative charge given by –*Q*. A small particle of mass *m* has a positive charge given by *q*. The small particle is located on the axis of the ring. (*a*) What is the minimum value of *q/m* such that the particle will be in equilibrium under the action of gravity and the electrostatic force? (*b*) If *q/m* is twice the value calculated in Part (*a*), where will the particle be when it is in equilibrium? Express your answer in terms of *a.*

Picture the Problem We can apply the condition for translational equilibrium to the particle and use the expression for the electric field on the axis of a ring charge to obtain an expression for *q/m*. Doing so will lead us to the conclusion that *q/m* will be a minimum when E_z is a maximum and that $z = -a/\sqrt{2}$ maximizes E_z.



(*a*) Apply $\sum F_z = 0$ to the particle:

$$qE_z - mg = 0 \Rightarrow \frac{q}{m} = \frac{g}{E_z} \qquad (1)$$

Note that this result tells us that the minimum value of *q/m* will be where the field due to the ring is greatest.

Express the electric field along the *z* axis due to the ring of charge:

$$E_z = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$$

Differentiate this expression with respect to z to obtain:

$$\frac{dE_x}{dz} = kQ\frac{d}{dz}\left[\frac{z}{\left(z^2+a^2\right)^{3/2}}\right] = kQ\frac{\left(z^2+a^2\right)^{3/2} - z\frac{d}{dx}\left(z^2+a^2\right)^{3/2}}{\left(z^2+a^2\right)^3}$$

$$= kQ\frac{\left(z^2+a^2\right)^{3/2} - z\left(\frac{3}{2}\right)\left(z^2+a^2\right)^{1/2}(2z)}{\left(z^2+a^2\right)^3} = kQ\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^3}$$

Set this expression equal to zero for extrema and simplify:

$$\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^3} = 0,$$

$$\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2} = 0,$$

and

$$z^2 + a^2 - 3z^2 = 0$$

Solve for z to obtain:

$$z = \pm\frac{a}{\sqrt{2}}$$

as candidates for maxima or minima.

You can either plot a graph of E_z or evaluate its second derivative at these points to show that it is a maximum at:

$$z = -\frac{a}{\sqrt{2}}$$

Substitute to obtain an expression $E_{z,\max}$:

$$E_{z,\max} = \left|\frac{kQ\left(-\frac{a}{\sqrt{2}}\right)}{\left(\left(-\frac{a}{\sqrt{2}}\right)^2 + a^2\right)^{3/2}}\right| = \frac{2kQ}{\sqrt{27}a^2}$$

Substitute in equation (1) to obtain:

$$\frac{q}{m} = \boxed{\frac{\sqrt{27}ga^2}{2kQ}}$$

(*b*) If q/m is twice as great as in (*a*), then the electric field should be half its value in (*a*), that is:

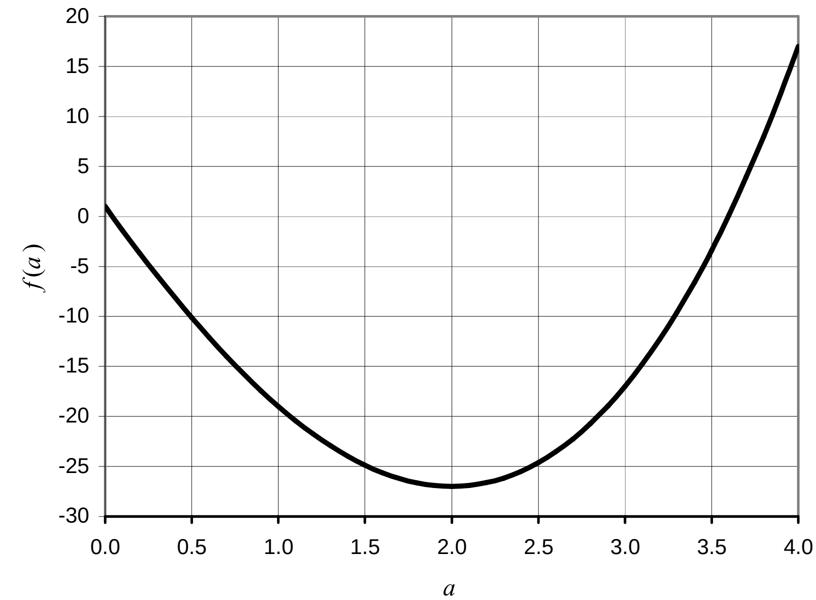
$$\frac{kQ}{\sqrt{27}a^2} = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$$

or

$$\frac{1}{27a^4} = \frac{z^2}{a^6\left(1+\frac{z^2}{a^2}\right)^3}$$

Let $u = z^2/a^2$ and simplify to obtain: $u^3 + 3u^2 - 24u + 1 = 0$

The following graph of $f(u) = u^3 + 3u^2 - 24u + 1$ was plotted using a spreadsheet program.

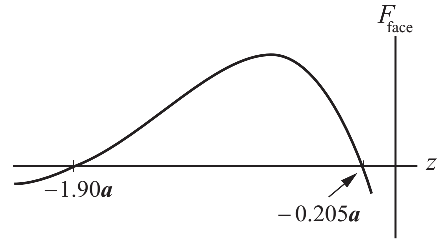


Use your graphing calculator or trial-and-error methods to obtain: $u = 0.04189$ and $u = 3.596$

The corresponding z values are: $z = -0.205a$ and $z = -1.90a$

The condition for a stable equilibrium position is that the particle, when displaced from its equilibrium position, experiences a restoring force; that is, a force that acts toward the equilibrium position. When the particle in this problem is just above its equilibrium position the net force on it must be downward and when it is just below the equilibrium position the net force on it must be upward. Note that the electric force is zero at the origin, so the net force there is downward and remains downward to the first equilibrium position as the weight force exceeds the electric force in this interval. The net force is upward between the first and second equilibrium positions as the electric force exceeds the weight force. The net force is downward below the second equilibrium position as the weight force exceeds the electric force. Thus, the first (higher) equilibrium position is stable and the second (lower) equilibrium position is unstable.

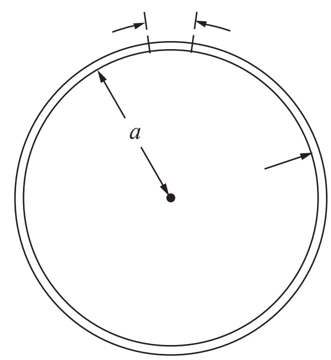
You might also find it instructive to use your graphing calculator to plot a graph of the electric force (the gravitational force is constant and only shifts the graph of the total force downward). Doing so will produce a graph similar to the one shown in the sketch to the right.



Note that the slope of the graph is negative on both sides of $-0.205a$ whereas it is positive on both sides of $-1.90a$; further evidence that $-0.205a$ is a position of stable equilibrium and $-1.90a$ a position of unstable equilibrium.

75 •• A long, thin, non-conducting plastic rod is bent into a circular loop that has a radius a. Between the ends of the rod a short gap of length ℓ, where $\ell << a$, remains. A positive charge of magnitude Q is evenly distributed on the loop. (a) What is the direction of the electric field at the center of the loop? Explain your answer. (b) What is the magnitude of the electric field at the center of the loop?

Picture the Problem The loop with the small gap is equivalent to a closed loop and a charge of $-Q\ell/(2\pi R)$ at the gap. The field at the center of a closed loop of uniform line charge is zero. Thus the field is entirely due to the charge $-Q\ell/(2\pi R)$.



(a) Express the field at the center of the loop:

$$\vec{E}_{\text{center}} = \vec{E}_{\text{loop}} + \vec{E}_{\text{gap}} \qquad (1)$$

Relate the field at the center of the loop to the charge in the gap:

$$\vec{E}_{\text{gap}} = -\frac{kq}{a^2}\hat{\boldsymbol{r}}$$

Use the definition of linear charge density to relate the charge in the gap to the length of the gap:

$$\lambda = \frac{q}{\ell} = \frac{Q}{2\pi a} \Rightarrow q = \frac{Q\ell}{2\pi a}$$

Substitute for q to obtain:

$$\vec{E}_{\text{gap}} = -\frac{kQ\ell}{2\pi a^3}\hat{\boldsymbol{r}}$$

Substituting in equation (1) yields:

$$\vec{E}_{\text{center}} = 0 - \frac{kQ\ell}{2\pi a^3}\hat{r} = -\frac{kQ\ell}{2\pi a^3}\hat{r}$$

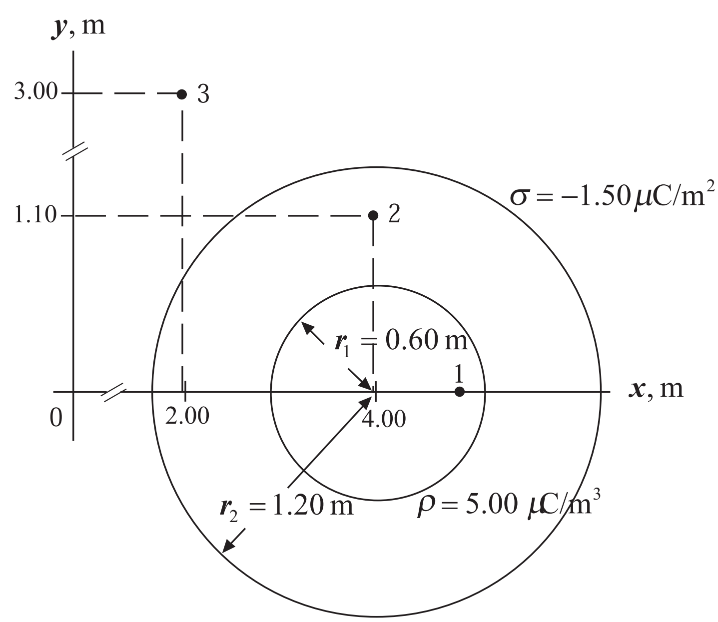
If Q is positive, the field at the origin points radially outward toward the gap.

(*b*) From our result in (*a*) we see that the magnitude of $\vec{E}_{\text{center}}$ is:

$$E_{\text{center}} = \boxed{\frac{kQ\ell}{2\pi a^3}}$$

76 •• A non-conducting solid sphere that is 1.20 m in diameter and has its center on the x axis at x = 4.00 m has a uniform volume charge of density of +5.00 μC/m^3. Concentric with the sphere is a thin non-conducting spherical shell that has a diameter of 2.40 m and a uniform surface charge density of –1.50 μC/m^2. Calculate the magnitude and direction of the electric field at (*a*) x = 4.50 m, y = 0, (*b*) x = 4.00 m, y = 1.10 m, and (*c*) x = 2.00 m, y = 3.00 m.

Picture the Problem We can find the electric fields at the three points of interest, labeled 1, 2, and 3 in the diagram, by adding the electric fields due to the charge distributions on the nonconducting sphere and the spherical shell.



Express the electric field due to the nonconducting sphere and the spherical shell at any point in space:

$$\vec{E} = \vec{E}_{\text{sphere}} + \vec{E}_{\text{shell}} \qquad (1)$$

(*a*) Because (4.50 m, 0) is inside the spherical shell:

$$\vec{E}_{\text{shell}}(4.50\,\text{m},0) = 0$$

Apply Gauss's law to a spherical surface inside the nonconducting sphere to obtain:

$$\vec{E}_{\text{sphere}}(r)=\frac{4\pi}{3}k\rho r\hat{\boldsymbol{i}}$$

Evaluate $\vec{E}_{\text{sphere}}(0.50\,\text{m})$:

$$\begin{aligned}\vec{E}_{\text{sphere}}(0.50\,\text{m})&=\frac{4\pi}{3}\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(5.00\,\mu\text{C/m}^2\right)(0.50\,\text{m})\hat{\boldsymbol{i}}\\&=(94.1\,\text{kN/C})\hat{\boldsymbol{i}}\end{aligned}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{E}(4.50\,\text{m},0)&=(94.1\,\text{kN/C})\hat{\boldsymbol{i}}+0\\&=(94.1\,\text{kN/C})\hat{\boldsymbol{i}}\end{aligned}$$

Find the magnitude and direction of $\vec{E}(4.50\,\text{m},0)$:

$$E(4.50\,\text{m},0)=\boxed{94\,\text{kN/C}}$$

and

$$\theta=\boxed{0°}$$

(*b*) Because (4.00 m, 1.10 m) is inside the spherical shell:

$$\vec{E}_{\text{shell}}(4.00\,\text{m},1.10\,\text{m})=0$$

Evaluate $\vec{E}_{\text{sphere}}(1.10\,\text{m})$:

$$\begin{aligned}\vec{E}_{\text{sphere}}(1.10\,\text{m})&=\frac{4\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(5.00\,\mu\text{C/m}^2\right)(0.600\,\text{m})^3}{3(1.10\,\text{m})^2}\hat{\boldsymbol{j}}\\&=(33.6\,\text{kN/C})\hat{\boldsymbol{j}}\end{aligned}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{E}(4.00\,\text{m},1.10\,\text{m})&=(33.6\,\text{kN/C})\hat{\boldsymbol{j}}+0\\&=(33.6\,\text{kN/C})\hat{\boldsymbol{j}}\end{aligned}$$

Find the magnitude and direction of $\vec{E}(4.00\,\text{m},1.10\,\text{m})$:

$$E(4.00\,\text{m},1.10\,\text{m})=\boxed{33.6\,\text{kN/C}}$$

and

$$\theta=\boxed{90°}$$

(*c*) Because (2.00 m, 3.00 m) outside the spherical shell:

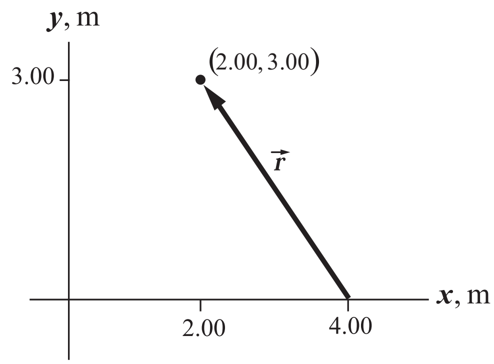
$$\vec{E}_{\text{shell}}(r)=\frac{kQ_{\text{shell}}}{r^2}\hat{\boldsymbol{r}}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing from (4.00 m, 0) to (2.00 m, 3.00 m).

Evaluate Q_{shell}:

$$Q_{\text{shell}} = \sigma A_{\text{shell}} = 4\pi\left(-1.50\,\mu\text{C/m}^2\right)\left(1.20\,\text{m}\right)^2 = -27.14\,\mu\text{C}$$

Refer to the following diagram to find $\hat{\boldsymbol{r}}$ and r:



$r = 3.606\,\text{m}$

and

$\hat{\boldsymbol{r}} = -0.5547\hat{\boldsymbol{i}} + 0.8321\hat{\boldsymbol{j}}$

Substitute numerical values and evaluate $\vec{E}_{\text{shell}}(2.00\,\text{m}, 3.00\,\text{m})$:

$$\vec{E}_{\text{shell}}(3.606\,\text{m}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(-27.14\,\mu\text{C}\right)}{\left(3.606\,\text{m}\right)^2}\hat{\boldsymbol{r}}$$
$$= \left(-18.77\,\text{kN/C}\right)\left(-0.5547\hat{\boldsymbol{i}} + 0.8321\hat{\boldsymbol{j}}\right)$$
$$= \left(10.41\,\text{kN/C}\right)\hat{\boldsymbol{i}} + \left(-15.61\,\text{kN/C}\right)\hat{\boldsymbol{j}}$$

Express the electric field due to the charged nonconducting sphere at a distance r from its center that is greater than its radius:

$$\vec{E}_{\text{sphere}}(r) = \frac{kQ_{\text{sphere}}}{r^2}\hat{\boldsymbol{r}}$$

Find the charge on the sphere:

$$Q_{\text{sphere}} = \rho V_{\text{sphere}} = \frac{4\pi}{3}\left(5.00\,\mu\text{C/m}^2\right)\left(0.600\,\text{m}\right)^3 = 4.524\,\mu\text{C}$$

Evaluate $\vec{E}_{\text{sphere}}(3.61\,\text{m})$:

$$\vec{E}_{\text{sphere}}(2.00\,\text{m}, 3.00\,\text{m}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(4.524\,\mu\text{C}\right)}{\left(3.606\,\text{m}\right)^2}\hat{\boldsymbol{r}} = \left(3.128\,\text{kN/C}\right)\hat{\boldsymbol{r}}$$
$$= \left(3.128\,\text{kN/C}\right)\left(-0.5547\hat{\boldsymbol{i}} + 0.8321\hat{\boldsymbol{j}}\right)$$
$$= \left(-1.735\,\text{kN/C}\right)\hat{\boldsymbol{i}} + \left(2.602\,\text{kN/C}\right)\hat{\boldsymbol{j}}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{E}(2.00\,\text{m},3.00\,\text{m}) &= (10.41\,\text{kN/C})\hat{i} + (-15.61\,\text{kN/C})\hat{j} + (-1.735\,\text{kN/C})\hat{i} \\ &\qquad + (2.602\,\text{kN/C})\hat{j} \\ &= (8.675\,\text{kN/C})\hat{i} + (-13.01\,\text{kN/C})\hat{j}\end{aligned}$$

Find the magnitude and direction of $\vec{E}(2.00\,\text{m},3.00\,\text{m})$:

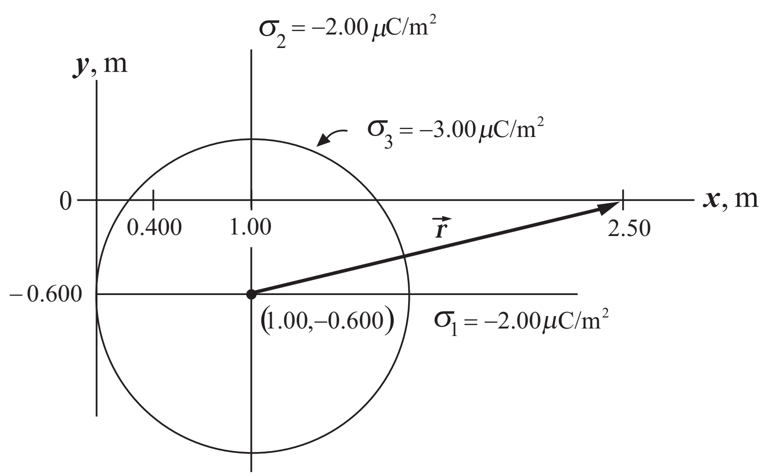
$$E(2.00\,\text{m},3.00\,\text{m}) = \sqrt{(8.675\,\text{kN/C})^2 + (-13.01\,\text{kN/C})^2} = \boxed{15.6\,\text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{-13.01\,\text{kN/C}}{8.675\,\text{kN/C}}\right) = \boxed{304°}$$

77 •• An infinite non-conducting plane sheet of charge that has a surface charge density + 3.00 μC/m^2 lies in the y = –0.600 m plane. A second infinite non-conducting plane sheet of charge that has a surface charge density of –2.00 μC/m^2 lies in the x = 1.00 m plane. Lastly, a non-conducting thin spherical shell of radius of 1.00 m and that has its center in the z = 0 plane at the intersection of the two charged planes has a surface charge density of –3.00 μC/m^2. Find the magnitude and direction of the electric field on the x axis at (a) x = 0.400 m and (b) x = 2.50 m.

Picture the Problem Let the numeral 1 refer to the infinite plane whose charge density is σ_1 and the numeral 2 to the infinite plane whose charge density is σ_2. We can find the electric fields at the two points of interest by adding the electric fields due to the charge distributions on the infinite planes and the sphere.



Express the electric field due to the infinite planes and the sphere at any point in space:

$$\vec{E} = \vec{E}_{\text{sphere}} + \vec{E}_1 + \vec{E}_2 \qquad (1)$$

(*a*) Because (0.400 m, 0) is inside the sphere:

$$\vec{E}_{\text{sphere}}(0.400\,\text{m},0) = 0$$

Find the field at (0.400 m, 0) due to plane 1:

$$\vec{E}_1(0.400\,\text{m},0) = \frac{\sigma_1}{2\,\epsilon_0}\hat{\boldsymbol{j}} = \frac{3.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{\boldsymbol{j}} = (169.4\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the field at (0.400 m, 0) due to plane 2:

$$\vec{E}_2(0.400\,\text{m},0) = \frac{\sigma_2}{2\,\epsilon_0}\left(-\hat{\boldsymbol{i}}\right) = \frac{-2.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\left(-\hat{\boldsymbol{i}}\right) = (112.9\,\text{kN/C})\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\vec{E}(0.400\,\text{m},0) = 0 + (169.4\,\text{kN/C})\hat{\boldsymbol{j}} + (112.9\,\text{kN/C})\hat{\boldsymbol{i}} = (112.9\,\text{kN/C})\hat{\boldsymbol{i}} + (169.4\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the magnitude and direction of $\vec{E}(0.400\,\text{m},0)$:

$$E(0.400\,\text{m},0) = \sqrt{(112.9\,\text{kN/C})^2 + (169.4\,\text{kN/C})^2} = 203.6\,\text{kN/C} = \boxed{204\,\text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{169.4\,\text{kN/C}}{112.9\,\text{kN/C}}\right) = 56.31° = \boxed{56.3°}$$

(*b*) Because (2.50 m, 0) is outside the sphere:

$$\vec{E}_{\text{sphere}}(0.400\,\text{m},0) = \frac{kQ_{\text{sphere}}}{r^2}\hat{\boldsymbol{r}}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing from (1.00 m, −0.600 m) to (2.50 m, 0).

Evaluate Q_{sphere}:

$$\begin{aligned} Q_{\text{sphere}} &= \sigma A_{\text{sphere}} = 4\pi\sigma R^2 \\ &= 4\pi(-3.00\,\mu\text{C/m}^2)(1.00\,\text{m})^2 \\ &= -37.70\,\mu\text{C} \end{aligned}$$

Referring to the diagram above, determine r and $\hat{\boldsymbol{r}}$:

$r = 1.616\,\text{m}$

and

$\hat{\boldsymbol{r}} = 0.9285\hat{\boldsymbol{i}} + 0.3714\hat{\boldsymbol{j}}$

Substitute and evaluate $\vec{\boldsymbol{E}}_{\text{sphere}}(2.50\,\text{m},0)$:

$$\vec{\boldsymbol{E}}_{\text{sphere}}(2.50\,\text{m},0) = \frac{(8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2)(-37.70\,\mu\text{C})}{(1.616\,\text{m})^2}\hat{\boldsymbol{r}}$$
$$= (-129.8\,\text{kN/C})(0.9285\hat{\boldsymbol{i}} + 0.3714\hat{\boldsymbol{j}})$$
$$= (-120.5\,\text{kN/C})\hat{\boldsymbol{i}} + (-48.22\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the field at (2.50 m, 0) due to plane 1:

$$\vec{\boldsymbol{E}}_1(2.50\,\text{m},0) = \frac{\sigma_1}{2\epsilon_0}\hat{\boldsymbol{j}} = \frac{3.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{\boldsymbol{j}} = (169.4\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the field at (2.50 m, 0) due to plane 2:

$$\vec{\boldsymbol{E}}_2(2.50\,\text{m},0) = \frac{\sigma_2}{2\epsilon_0}\hat{\boldsymbol{i}} = \frac{-2.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{\boldsymbol{i}} = (-112.9\,\text{kN/C})\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\vec{\boldsymbol{E}}(0.400\,\text{m},0) = (-120.5\,\text{kN/C})\hat{\boldsymbol{i}} + (-48.19\,\text{kN/C})\hat{\boldsymbol{j}} + (169.4\,\text{kN/C})\hat{\boldsymbol{j}}$$
$$+ (-112.9\,\text{kN/C})\hat{\boldsymbol{i}}$$
$$= (-233.5\,\text{kN/C})\hat{\boldsymbol{i}} + (121.2\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the magnitude and direction of $\vec{\boldsymbol{E}}(2.50\,\text{m},0)$:

$$E(2.50\,\text{m},0) = \sqrt{(-233.5\,\text{kN/C})^2 + (121.2\,\text{kN/C})^2} = \boxed{263\,\text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{121.2\,\text{kN/C}}{-233.5\,\text{kN/C}}\right) = \boxed{153^\circ}$$

78 •• An infinite non-conducting plane sheet lies in the $x = 2.00$ m plane and has a uniform surface charge density of $+2.00\ \mu\text{C/m}^2$. An infinite non-conducting line charge of uniform linear charge density 4.00 μC/m passes through the origin at an angle of 45.0º with the x axis in the xy plane. A solid non-conducting sphere

of volume charge density $-6.00\ \mu\text{C/m}^3$ and radius 0.800 m is centered on the x axis at $x = 1.00$ m. Calculate the magnitude and direction of the electric field in the $z = 0$ plane at $x = 1.50$ m, $y = 0.50$ m.

Picture the Problem Let P represent the point of interest at (1.50 m, 0.50 m). We can find the electric field at P by adding the electric fields due to the infinite plane, the infinite line, and the sphere. Once we've expressed the field at P in vector form, we can find its magnitude and direction.

Express the electric field at P:

$$\vec{E} = \vec{E}_{\text{plane}} + \vec{E}_{\text{line}} + \vec{E}_{\text{sphere}}$$

Find $\vec{E}_{\text{plane}}$ at P:

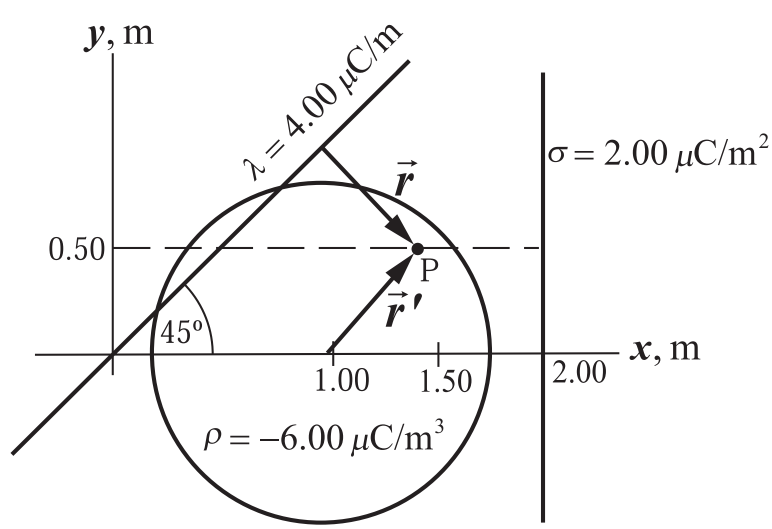
$$\vec{E}_{\text{plane}} = -\frac{\sigma}{2\in_0}\hat{i} = -\frac{2.00\ \mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = (-112.9\ \text{kN/C})\hat{i}$$

Express $\vec{E}_{\text{line}}$ at P:

$$\vec{E}_{\text{line}} = \frac{2k\lambda}{r}\hat{r}$$

Refer to the following figure to obtain:

$$\vec{r} = (0.50\,\text{m})\hat{i} - (0.50\,\text{m})\hat{j} \text{ and } \hat{r} = (0.707)\hat{i} - (0.707)\hat{j}$$



Substitute and simplify to obtain:

$$\vec{E}_{\text{line}} = \frac{2\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(4.00\ \mu\text{C/m}\right)}{0.707\ \text{m}}\left[(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}\right]$$
$$= (101.7\ \text{kN/C})\left[(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}\right] = (71.90\ \text{kN/C})\hat{\boldsymbol{i}} + (-71.90\ \text{kN/C})\hat{\boldsymbol{j}}$$

Letting r' represent the distance from the center of the sphere to P, apply Gauss's law to a spherical surface of radius r' centered at (1 m, 0) to obtain an expression for $\vec{E}_{\text{sphere}}$ at P:

$$\vec{E}_{\text{sphere}} = \frac{4\pi}{3}kr'\rho\hat{\boldsymbol{r}}'$$

where $\hat{\boldsymbol{r}}'$ is directed toward the center of the sphere.

Refer to the diagram used above to obtain:

$$\vec{\boldsymbol{r}}' = -(0.50\ \text{m})\hat{\boldsymbol{i}} - (0.50\ \text{m})\hat{\boldsymbol{j}}$$

and

$$\hat{\boldsymbol{r}}' = -(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}$$

Substitute numerical values and simplify to obtain:

$$\vec{E}_{\text{sphere}} = \frac{4\pi}{3}\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(0.707\ \text{m})\left(-6.00\ \mu\text{C/m}^3\right)\left[(0.707)\hat{\boldsymbol{i}} + (0.707)\hat{\boldsymbol{j}}\right]$$
$$= (-112.9\ \text{kN/C})\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) = (-112.9\ \text{kN/C})\hat{\boldsymbol{i}} + (-112.9\ \text{kN/C})\hat{\boldsymbol{j}}$$

Substitute for $\vec{E}_{\text{plane}}$, $\vec{E}_{\text{line}}$, and $\vec{E}_{\text{sphere}}$ and simplify to obtain:

$$\vec{E} = (-112.9\ \text{kN/C})\hat{\boldsymbol{i}} + (71.90\ \text{kN/C})\hat{\boldsymbol{i}} + (-71.90\ \text{kN/C})\hat{\boldsymbol{j}} + (-112.9\ \text{kN/C})\hat{\boldsymbol{i}}$$
$$+ (-112.9\ \text{kN/C})\hat{\boldsymbol{j}}$$
$$= (-154.0\ \text{kN/C})\hat{\boldsymbol{i}} + (-184.9\ \text{kN/C})\hat{\boldsymbol{j}}$$

Finally, find the magnitude and direction of $\vec{E}$:

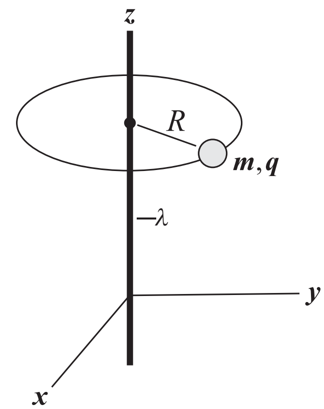
$$E = \sqrt{(-154.0\ \text{kN/C})^2 + (-184.9\ \text{kN/C})^2} = \boxed{241\ \text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{-154.0\ \text{kN/C}}{-184.9\ \text{kN/C}}\right) = \boxed{220°}$$

79 •• [SSM] A uniformly charged, infinitely long line of negative charge has a linear charge density of $-\lambda$ and is located on the z axis. A small positively charged particle that has a mass m and a charge q is in a circular orbit of radius R in the xy plane centered on the line of charge. (a) Derive an expression for the speed of the particle. (b) Obtain an expression for the period of the particle's orbit.

Picture the Problem (a) We can apply Newton's second law to the particle to express its speed as a function of its mass m, charge q, and the radius of its path R, and the strength of the electric field due to the infinite line charge E. (b) The period of the particle's motion is the ratio of the circumference of the circle in which it travels divided by its orbital speed.



(a) Apply Newton's second law to the particle to obtain:

$$\sum F_{\text{radial}} = qE = m\frac{v^2}{R}$$

where the inward direction is positive.

Solving for v yields:

$$v = \sqrt{\frac{qRE}{m}}$$

The strength of the electric field at a distance R from the infinite line charge is given by:

$$E = \frac{2k\lambda}{R}$$

Substitute for E and simplify to obtain:

$$v = \boxed{\sqrt{\frac{2kq\lambda}{m}}}$$

(b) The speed of the particle is equal to the circumference of its orbit divided by its period:

$$v = \frac{2\pi R}{T} \Rightarrow T = \frac{2\pi R}{v}$$

Substitute for v and simplify to obtain:

$$T = \boxed{\pi R\sqrt{\frac{2m}{kq\lambda}}}$$

80 •• A stationary ring of radius a that lies in the yz plane has a uniformly distributed positive charge Q. A small particle that has mass m and a negative charge q is located at the center of the ring. (a) Show that if $x << a$, the electric field along the axis of the ring is proportional to x. (b) Find the force on the particle of mass m as a function of x. (c) Show that if the particle is given a small displacement in the $+x$ direction, it will perform simple harmonic motion. (d) What is the frequency of that motion?

Picture the Problem Starting with the equation for the electric field on the axis of ring charge, we can factor the denominator of the expression to show that, for $x << a$, E_x is proportional to x. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's second law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the period of the motion from its angular frequency, which we can obtain from the differential equation of motion.

(a) Express the electric field on the axis of the ring of charge:

$$E_x = \frac{kQx}{\left(x^2 + a^2\right)^{3/2}}$$

Factor a^2 from the denominator of E_x to obtain:

$$E_x = \frac{kQx}{\left[a^2\left(1 + \frac{x^2}{a^2}\right)\right]^{3/2}} = \frac{kQx}{a^3\left(1 + \frac{x^2}{a^2}\right)^{3/2}} \approx \boxed{\frac{kQ}{a^3}x}$$

provided $x << a$.

(b) Express the force acting on the particle as a function of its charge and the electric field:

$$F_x = qE_x = \boxed{\frac{kqQ}{a^3}x}$$

(c) Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's second law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2} = -\frac{kqQ}{a^3}x$$

or

$$\boxed{\frac{d^2x}{dt^2} + \frac{kqQ}{ma^3}x = 0}$$

the differential equation of simple harmonic motion.

(*d*) Relate the frequency of the simple harmonic motion to its angular frequency:

$$f = \frac{\omega}{2\pi} \qquad (1)$$

From the differential equation we have:

$$\omega^2 = \frac{kqQ}{ma^3} \Rightarrow \omega = \sqrt{\frac{kqQ}{ma^3}}$$

Substitute for ω in equation (1) and simplify to obtain:

$$f = \boxed{\frac{1}{2\pi}\sqrt{\frac{kqQ}{ma^3}}}$$

81 •• [SSM] The charges Q and q of Problem 80 are +5.00 μC and –5.00 μC, respectively, and the radius of the ring is 8.00 cm. When the particle is given a small displacement in the x direction, it oscillates about its equilibrium position at a frequency of 3.34 Hz. (*a*) What is the particle's mass? (*b*) What is the frequency if the radius of the ring is doubled to 16.0 cm and all other parameters remain unchanged?

Picture the Problem Starting with the equation for the electric field on the axis of a ring charge, we can factor the denominator of the expression to show that, for $x << a$, E_x is proportional to x. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's second law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the angular frequency of the motion from the differential equation and use this expression to find the frequency of the motion when the radius of the ring is doubled and all other parameters remain unchanged.

(*a*) Express the electric field on the axis of the ring of charge:

$$E_x = \frac{kQx}{\left(x^2 + a^2\right)^{3/2}}$$

Factor a^2 from the denominator of E_x to obtain:

$$E_x = \frac{kQx}{\left[a^2\left(1+\frac{x^2}{a^2}\right)\right]^{3/2}} = \frac{kQx}{a^3\left(1+\frac{x^2}{a^2}\right)^{3/2}} \approx \frac{kQ}{a^3}x$$

provided $x << a$.

Express the force acting on the particle as a function of its charge and the electric field:

$$F_x = qE_x = \frac{kqQ}{a^3}x$$

Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's second law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2}=-\frac{kqQ}{a^3}x$$

or

$$\frac{d^2x}{dt^2}+\frac{kqQ}{ma^3}x=0$$

the differential equation of simple harmonic motion.

The angular frequency of the simple harmonic motion of the particle is given by:

$$\omega=\sqrt{\frac{kqQ}{ma^3}} \quad (1)$$

Solving for m yields:

$$m=\frac{kqQ}{\omega^2a^3}=\frac{kqQ}{4\pi^2f^2a^3}$$

Substitute numerical values and evaluate m:

$$m=\frac{\left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left|\left(-5.00\,\mu\text{C}\right)\right|\left(5.00\,\mu\text{C}\right)}{4\pi^2\left(3.34\,\text{s}^{-1}\right)^2\left(8.00\,\text{cm}\right)^3}=\boxed{0.997\,\text{kg}}$$

(*b*) Express the angular frequency of the motion if the radius of the ring is doubled:

$$\omega'=\sqrt{\frac{kqQ}{m(2a)^3}} \quad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{\omega'}{\omega}=\frac{2\pi f'}{2\pi f}=\frac{\sqrt{\frac{kqQ}{m(2a)^3}}}{\sqrt{\frac{kqQ}{ma^3}}}=\frac{1}{\sqrt{8}}$$

Solve for f' to obtain:

$$f'=\frac{f}{\sqrt{8}}=\frac{3.34\,\text{Hz}}{\sqrt{8}}=\boxed{1.18\,\text{Hz}}$$

82 •• If the radius of the ring in Problem 80 is doubled while keeping the linear charge density on the ring the same, does the frequency of oscillation of the particle change? If so, by what factor does it change?

Picture the Problem Starting with the equation for the electric field on the axis of a ring charge, we can factor the denominator of the expression to show that, for $x \ll a$, E_x is proportional to x. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's second law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the angular frequency of the motion from the differential equation and use this expression to find its value when the radius of the ring is doubled while keeping the linear charge density on the ring constant.

Express the electric field on the axis of the ring of charge:

$$E_x = \frac{kQx}{\left(x^2 + a^2\right)^{3/2}}$$

Factor a^2 from the denominator of E_x to obtain:

$$E_x = \frac{kQx}{\left[a^2\left(1 + \frac{x^2}{a^2}\right)\right]^{3/2}} = \frac{kQx}{a^3\left(1 + \frac{x^2}{a^2}\right)^{3/2}} \approx \frac{kQ}{a^3}x$$

provided $x \ll a$.

Express the force acting on the particle as a function of its charge and the electric field:

$$F_x = qE_x = \frac{kqQ}{a^3}x$$

Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's second law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2} = -\frac{kqQ}{a^3}x$$

or

$$\frac{d^2x}{dt^2} + \frac{kqQ}{ma^3}x = 0,$$

the differential equation of simple harmonic motion.

The angular frequency of the simple harmonic motion of the particle is given by:

$$\omega = \sqrt{\frac{kqQ}{ma^3}} \qquad (1)$$

Express the angular frequency of the motion if the radius of the ring is doubled while keeping the linear charge density constant (that is, doubling Q):

$$\omega' = \sqrt{\frac{kq(2Q)}{m(2a)^3}} \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{\omega'}{\omega} = \frac{2\pi f'}{2\pi f} = \frac{\sqrt{\dfrac{kq(2Q)}{m(2a)^3}}}{\sqrt{\dfrac{kqQ}{ma^3}}} = \frac{1}{2}$$

Yes. The frequency changes by a factor of 0.5.

83 ••• A uniformly charged non-conducting solid sphere of radius R has its center at the origin and has volume charge density of ρ. (a) Show that at a point within the sphere a distance r from the center $\vec{E} = \dfrac{\rho}{3\in_0} r\hat{\mathbf{r}}$. ($b$) Material is removed from the sphere leaving a spherical cavity of radius $b = R/2$ with its center at $x = b$ on the x axis (Figure 22-46). Calculate the electric field at points 1 and 2 shown in Figure 22-46. *Hint: Model the sphere-with-cavity as two uniform spheres of equal positive and negative charge densities.*

Picture the Problem In Part (a), you can apply Gauss's law to express $\vec{E}$ as a function of r for the uniformly charged nonconducting sphere with its center at the origin. In Part (b), you can use the hint to express the field at a generic point $P(x,y)$ in the cavity as the sum of the fields due to equal positive and negative charge densities and then evaluate this expression at points 1 and 2.

(a) The electric field at a distance r from the center of the uniformly charged nonconducting sphere is given by:

$$\vec{E}_\rho = E\hat{\mathbf{r}} \qquad (1)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward.

Apply Gauss's law to a spherical surface of radius r centered at the origin to obtain:

$$\oint_S E_n dA = E_\rho\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\in_0}$$

Relate Q_{inside} to the charge density ρ:

$$\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r^3} \Rightarrow Q_{\text{inside}} = \tfrac{4}{3}\rho\pi r^3$$

Substitute for Q_{inside}:

$$E_\rho\left(4\pi r^2\right) = \frac{4\rho\pi r^3}{3\in_0}$$

Solve for E_ρ to obtain:

$$E_\rho = \frac{\rho r}{3\in_0}$$

Substitute for E in equation (1) to obtain:

$$\vec{E}_\rho = \boxed{\frac{\rho}{3\in_0} r\hat{r}}$$

(*b*) The electric field at point $P(x,y)$ is the sum of the electric fields due to the two charge distributions:

$$\vec{E} = \vec{E}_\rho + \vec{E}_{-\rho} = E_\rho \hat{r} + E_{-\rho}\hat{r}' \quad (2)$$

where $\hat{r}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

Apply Gauss's law to a spherical surface of radius r' centered at $x = b = R/2$ to obtain:

$$\oint_S E_n dA = E_{-\rho}\left(4\pi r'^2\right) = \frac{Q_{\text{inside}}}{\in_0}$$

Relate Q_{inside} to the charge density $-\rho$:

$$-\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r'^3} \Rightarrow Q_{\text{inside}} = -\tfrac{4}{3}\rho\pi r'^3$$

Substitute for Q_{inside} to obtain:

$$E_{-\rho}\left(4\pi r'^2\right) = -\frac{4\pi r'^3\rho}{3\in_0}$$

Solving for $E_{-\rho}$ yields:

$$E_{-\rho} = -\frac{\rho}{3\in_0} r'$$

Substitute for E_ρ and $E_{-\rho}$ in equation (2) to obtain:

$$\vec{E} = \frac{\rho}{3\in_0} r\hat{r} - \frac{\rho}{3\in_0} r'\hat{r} \qquad (3)$$

The vectors $\vec{r} = r\hat{r}$ and $\vec{r}' = r'\hat{r}'$ are given by:

$$\vec{r} = x\hat{i} + y\hat{j} \text{ and } \vec{r}' = (x-b)\hat{i} + y\hat{j}$$

where x and y are the coordinates of any point in the cavity.

Substitute for $r\hat{r}$ and $r'\hat{r}'$ in equation (3) and simplify to obtain:

$$\vec{E} = \frac{\rho}{3\in_0}\left(x\hat{i} + y\hat{j}\right) - \frac{\rho}{3\in_0}\left[(x-b)\hat{i} + y\hat{j}\right] = \frac{\rho b}{3\in_0}\hat{i}$$

Because $\vec{E}$ is independent of x and y:

$$\vec{E}_1 = \vec{E}_2 = \boxed{\frac{\rho b}{3\in_0}\hat{i}}$$

84 ••• Show that the electric field throughout the cavity of Problem 83*b* is uniform and is given by $\vec{E} = \frac{\rho}{3\in_0} b\hat{i}$.

Picture the Problem The electric field in the cavity is the sum of the electric field due to the uniform and positive charge distribution of the sphere whose radius is a and the electric field due to any charge in the spherical cavity whose radius is b. You can use the hint given in Problem 83 to express the field at a generic point $P(x,y)$ in the cavity as the sum of the fields due to equal positive and negative charge densities to show that $\vec{E} = \frac{\rho}{3\in_0} b\hat{i}$.

The electric field at point $P(x,y)$ is the sum of the electric fields due to the two charge distributions:

$$\vec{E} = \vec{E}_\rho + \vec{E}_{-\rho} = E_\rho \hat{r} + E_{-\rho}\hat{r}' \qquad (1)$$

where $\hat{r}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

Apply Gauss's law to a spherical surface of radius r' centered at $x = b = R/2$ to obtain:

$$\oint_S E_n dA = E_{-\rho}\left(4\pi r'^2\right) = \frac{Q_{\text{inside}}}{\in_0}$$

Relate Q_{inside} to the charge density $-\rho$:

$$-\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r'^3} \Rightarrow Q_{\text{inside}} = -\tfrac{4}{3}\rho\pi r'^3$$

Substitute for Q_{inside} to obtain:

$$E_{-\rho}\left(4\pi r'^2\right) = -\frac{4\pi r'^3 \rho}{3\in_0}$$

Solving for $E_{-\rho}$ yields:

$$E_{-\rho} = -\frac{\rho}{3\in_0} r'$$

From Problem 83:

$$E_\rho = \frac{\rho}{3\in_0} r$$

Substitute for E_ρ and $E_{-\rho}$ in equation (1) to obtain:

$$\vec{E} = \frac{\rho}{3\in_0} r\hat{r} - \frac{\rho}{3\in_0} r'\hat{r} \qquad (2)$$

The vectors $\vec{r} = r\hat{r}$ and $\vec{r}' = r'\hat{r}'$ are given by:

$\vec{r} = x\hat{i} + y\hat{j}$ and $\vec{r}' = (x-b)\hat{i} + y\hat{j}$

where x and y are the coordinates of any point in the cavity.

Substitute for $r\hat{r}$ and $r'\hat{r}'$ in equation (2) and simplify to obtain:

$$\vec{E} = \frac{\rho}{3\in_0}\left(x\hat{i} + y\hat{j}\right) - \frac{\rho}{3\in_0}\left[(x-b)\hat{i} + y\hat{j}\right] = \boxed{\frac{\rho}{3\in_0} b\hat{i}}$$

85 ••• The cavity in Problem 83*b* is now filled with a uniformly charged non-conducting material with a total charge of Q. Calculate the new values of the electric field at points 1 and 2 shown in Figure 22-46.

Picture the Problem The electric field at a generic point $P(x,y)$ in the cavity is the sum of the fields due to the positive charge density and the total charge Q.

The electric field at point $P(x,y)$ is the sum of the electric fields due to the two charge distributions:

$$\vec{E} = \vec{E}_\rho + \vec{E}_{-\rho} + \vec{E}_Q \qquad (1)$$

where $\hat{r}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

From Problem 83:

$$\vec{E}_\rho + \vec{E}_{-\rho} = \frac{\rho b}{3\epsilon_0}\hat{i}$$

Substituting in equation (1) yields:

$$\vec{E} = \frac{\rho b}{3\epsilon_0}\hat{i} + \vec{E}_Q$$

Assuming that the cavity is filled with positive charge Q:

$$\vec{E} = \frac{\rho b}{3\epsilon_0}\hat{i} + \frac{Q}{4\pi\epsilon_0 b^3}r'\hat{r}$$

The vectors $\vec{r} = r\hat{r}$ and $\vec{r}' = r'\hat{r}'$ are given by:

$$\vec{r} = x\hat{i} + y\hat{j} \text{ and } \vec{r}' = (x-b)\hat{i} + y\hat{j}$$

where x and y are the coordinates of any point in the cavity.

Substitute for $r\hat{r}$ and $r'\hat{r}'$ and simplify to obtain:

$$\vec{E} = \frac{\rho b}{3\epsilon_0}\hat{i} + \frac{Q}{4\pi\epsilon_0 b^3}\left[(x-b)\hat{i} + y\hat{j}\right]$$

At point 1, $x = 2b$ and $y = 0$:

$$\vec{E}(2b,0) = \frac{\rho b}{3\epsilon_0}\hat{i} - \frac{Q}{4\pi\epsilon_0 b^3}\left[(2b-b)\hat{i}\right] = \boxed{\left(\frac{\rho b}{3\epsilon_0} + \frac{Q}{4\pi\epsilon_0 b^2}\right)\hat{i}}$$

At point 2, $x = 0$ and $y = 0$:

$$\vec{E}(0,0) = \frac{\rho b}{3\epsilon_0}\hat{i} + \frac{Q}{4\pi\epsilon_0 b^3}\left[(-b)\hat{i}\right] = \boxed{\left(\frac{\rho b}{3\epsilon_0} - \frac{Q}{4\pi\epsilon_0 b^2}\right)\hat{i}}$$

86 ••• A *small* Gaussian surface in the shape of a cube has faces parallel to the *xy*, *xz*, and *yz* planes (Figure 22-47) and is in a region in which the electric field is parallel to the *x* axis. (*a*) Using the differential approximation, show that the net electric flux of the electric field out of the Gaussian surface is given by $\phi_{\text{net}} \approx \frac{\partial E_x}{\partial x}\Delta V$, where ΔV is the volume enclosed by the Gaussian surface.

(*b*) Using Gauss's law and the results of Part (*a*) show that $\frac{\partial E_x}{\partial x} = \frac{\rho}{\epsilon_0}$, where ρ is the volume charge density inside the cube. (This equation is the one-dimensional version of the point form of Gauss's law.)

Picture the Problem Let the coordinates of one corner of the cube be (*x,y,z*), and assume that the sides of the cube are Δx, Δy, and Δz and compute the flux through the faces of the cube that are parallel to the *yz* plane. The net flux of the electric field out of the gaussian surface is the difference between the flux out of the surface and the flux into the surface.

(*a*) The net flux out of the cube is given by:

$$\phi_{\text{net}} = \phi(x+\Delta x) - \phi(x)$$

Use a Taylor series expansion to express the net flux through faces of the cube that are parallel to the *yz* plane:

$$\phi_{\text{net}} = \phi(x) + (\Delta x)\phi'(x) + \tfrac{1}{2}(\Delta x)^2\phi''(x) + \ldots - \phi(x) = (\Delta x)\phi'(x) + \tfrac{1}{2}(\Delta x)^2\phi''(x) + \ldots$$

Neglecting terms higher than first order we have:

$$\phi_{\text{net}} \approx \Delta x\phi'(x)$$

Because the electric field is in the *x* direction, $\phi(x)$ is:

$$\phi(x) = E_x\Delta y\Delta z$$

and

$$\phi'(x) = \frac{\partial E_x}{\partial x}\Delta y\Delta z$$

Substitute for $\phi'(x)$ to obtain:

$$\phi_{\text{net}} \approx \Delta x\frac{\partial E_x}{\partial x}(\Delta y\Delta z) = \frac{\partial E_x}{\partial x}(\Delta x\Delta y\Delta z) = \boxed{\frac{\partial E_x}{\partial x}\Delta V}$$

(*b*) From Gauss's law, the net flux through any surface is:

$$\phi_{\text{net}} = \frac{q_{\text{encl}}}{\epsilon_0} = \frac{\rho}{\epsilon_0}\Delta V$$

From Part (*a*):

$$\phi_{\text{net}} = \frac{\partial E_x}{\partial x}\Delta V$$

Equate these two expressions and simplify to obtain:

$$\frac{\partial E_x}{\partial x}\Delta V = \frac{\rho}{\epsilon_0}\Delta V \Rightarrow \frac{\partial E_x}{\partial x} = \boxed{\frac{\rho}{\epsilon_0}}$$

87 ••• [SSM] Consider a simple but surprisingly accurate model for the hydrogen molecule: two positive point charges, each having charge +*e*, are placed inside a uniformly charged sphere of radius *R*, which has a charge equal to –2*e*. The two point charges are placed symmetrically, equidistant from the center of the sphere (Figure 22-48). Find the distance from the center, *a*, where the net force on either point charge is zero.

Picture the Problem We can find the distance from the center where the net force on either charge is zero by setting the sum of the forces acting on either point charge equal to zero. Each point charge experiences two forces; one a Coulomb force of repulsion due to the other point charge, and the second due to that fraction of the sphere's charge that is between the point charge and the center of the sphere that creates an electric field at the location of the point charge.

Apply $\sum F = 0$ to either of the point charges:

$$F_{\text{Coulomb}} - F_{\text{field}} = 0 \qquad (1)$$

Express the Coulomb force on the proton:

$$F_{\text{Coulomb}} = \frac{ke^2}{(2a)^2} = \frac{ke^2}{4a^2}$$

The force exerted by the field *E* is:

$$F_{\text{field}} = eE$$

Apply Gauss's law to a spherical surface of radius *a* centered at the origin:

$$E\left(4\pi\, a^2\right) = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Relate the charge density of the electron sphere to Q_{enclosed}:

$$\frac{2e}{\frac{4}{3}\pi R^3} = \frac{Q_{\text{enclosed}}}{\frac{4}{3}\pi a^3} \Rightarrow Q_{\text{enclosed}} = \frac{2ea^3}{R^3}$$

Substitute for Q_{enclosed}:

$$E\left(4\pi a^2\right) = \frac{2ea^3}{\epsilon_0\, R^3}$$

Solve for *E* to obtain:

$$E = \frac{ea}{2\pi\,\epsilon_0\, R^3} \Rightarrow F_{\text{field}} = \frac{e^2 a}{2\pi\,\epsilon_0\, R^3}$$

Substitute for F_{Coulomb} and F_{field} in equation (1):

$$\frac{ke^2}{4a^2}-\frac{e^2a}{2\pi\in_0 R^3}=0$$

or

$$\frac{ke^2}{4a^2}-\frac{2ke^2a}{R^3}=0\Rightarrow a=\sqrt[3]{\frac{1}{8}}R=\boxed{\tfrac{1}{2}R}$$

88 ••• An electric dipole that has a dipole moment of $\vec{\boldsymbol{p}}$ is located at a perpendicular distance R from an infinitely long line charge that has a uniform linear charge density λ. Assume that the dipole moment is in the same direction as the field of the line of charge. Determine an expression for the electric force on the dipole.

Picture the Problem We can find the field due to the infinitely long line charge from $E=2k\lambda/r$ and the force that acts on the dipole using $F=p\,dE/dr$.

Express the force acting on the dipole:

$$F=p\frac{dE}{dr} \qquad (1)$$

The electric field at the location of the dipole is given by:

$$E=\frac{2k\lambda}{r}$$

Substitute for E in equation (1) to obtain:

$$F=p\frac{d}{dr}\left[\frac{2k\lambda}{r}\right]=\boxed{-\frac{2k\lambda p}{r^2}}$$

where the minus sign indicates that the dipole is attracted to the line charge.

Chapter 23
Electrical Potential

Conceptual Problems

1 • [SSM] A proton is moved to the left in a uniform electric field that points to the right. Is the proton moving in the direction of increasing or decreasing electric potential? Is the electrostatic potential energy of the proton increasing or decreasing?

Determine the Concept The proton is moving to a region of higher potential. The proton's electrostatic potential energy is increasing.

2 • An electron is moved to the left in a uniform electric field that points to the right. Is the electron moving in the direction of increasing or decreasing electric potential? Is the electrostatic potential energy of the electron increasing or decreasing?

Determine the Concept The electron is moving to a region of higher electric potential. The electron's electrostatic potential energy is decreasing.

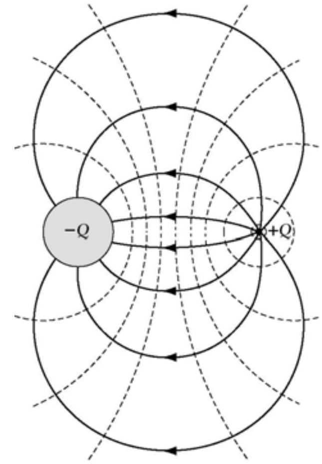
3 • If the electric potential is uniform throughout a region of space, what can be said about the electric field in that region?

Determine the Concept If V is constant, its gradient is zero; consequently the electric field is zero throughout the region.

4 • If V is known at only a single point in space, can $\vec{E}$ be found at that point? Explain your answer.

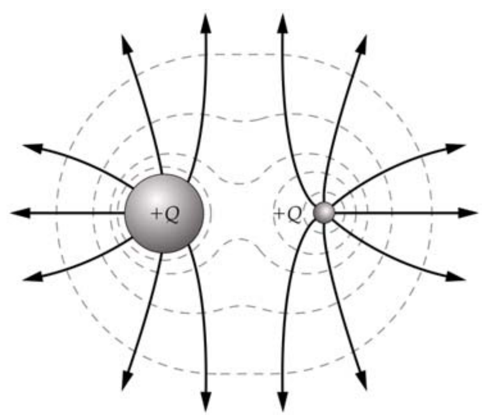
Determine the Concept No. $\vec{E}$ cannot be determined without knowing V at a continuum of points.

5 •• [SSM] Figure 23-29 shows a point particle that has a positive charge $+Q$ and a metal sphere that has a charge $-Q$. Sketch the electric field lines and equipotential surfaces for this system of charges.



Picture the Problem The electric field lines, shown as solid lines, and the equipotential surfaces (intersecting the plane of the paper), shown as dashed lines, are sketched in the adjacent figure. The point charge $+Q$ is the point at the right, and the metal sphere with charge $-Q$ is at the left. Near the two charges the equipotential surfaces are spheres, and the field lines are normal to the metal sphere at the sphere's surface.

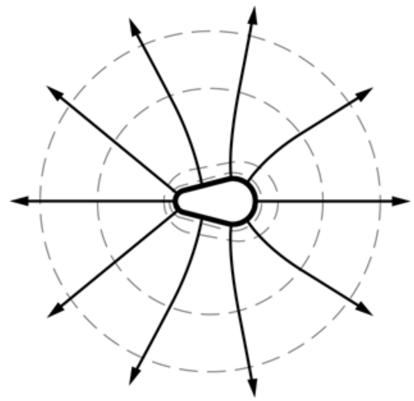
6 •• Figure 23-30 shows a point particle that has a negative charge $-Q$ and a metal sphere that has a charge $+Q$. Sketch the electric field lines and equipotential surfaces for this system of charges.



Picture the Problem The electric field lines, shown as solid lines, and the equipotential surfaces (intersecting the plane of the paper), shown as dashed lines, are sketched in the adjacent figure. The point charge $+Q$ is the point at the right, and the metal sphere with charge $+Q$ is at the left. Near the two charges the equipotential surfaces are spheres, and the field lines are normal to the metal sphere at the sphere's surface. Very far from both charges, the equipotential surfaces and field lines approach those of a point charge $2Q$ located at the midpoint.

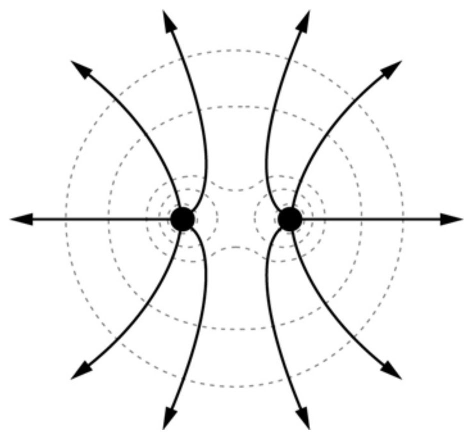
7 •• Sketch the electric field lines and equipotential surfaces for the region surrounding the charged conductor shown in Figure 23-31, assuming that the conductor has a net positive charge.

Picture the Problem The equipotential surfaces are shown with dashed lines, the field lines are shown in solid lines. It is assumed that the conductor carries a positive charge. Near the conductor the equipotential surfaces follow the conductor's contours; far from the conductor, the equipotential surfaces are spheres centered on the conductor. The electric field lines are perpendicular to the equipotential surfaces.



8 •• Two equal positive point charges are separated by a finite distance. Sketch the electric field lines and the equipotential surfaces for this system.

Picture the Problem The equipotential surfaces are shown with dashed lines, the electric field lines are shown with solid lines. Near each charge, the equipotential surfaces are spheres centered on each charge; far from the charges, the equipotential surface is a sphere centered at the midpoint between the charges. The electric field lines are perpendicular to the equipotential surfaces.



9 •• Two point charges are fixed on the x-axis. (a) Each has a positive charge q. One is at $x = -a$ and the other is at $x = +a$. At the origin, which of the following is true?

(1) $\vec{E} = 0$ and $V = 0$,
(2) $\vec{E} = 0$ and $V = 2kq/a$,
(3) $\vec{E} = \left(2kq/a^2\right)\hat{i}$ and $V = 0$,
(4) $\vec{E} = \left(2kq/a^2\right)\hat{i}$ and $V = 2kq/a$,
(5) None of the above.

(b) One has a positive charge $+q$ and the other has a negative charge $-q$. The positive point charge is at $x = -a$ and the negative point charge is at $x = +a$. At the origin, which of the following is true?

(1) $\vec{E} = 0$ and $V = 0$,

(2) $\vec{E} = 0$ and $V = 2kq/a$,
(3) $\vec{E} = \left(2kq/a^2\right)\hat{i}$ and $V = 0$,
(4) $\vec{E} = \left(2kq/a^2\right)\hat{i}$ and $V = 2kq/a$,
(5) None of the above.

Picture the Problem We can use Coulomb's law and the superposition of fields to find $\vec{E}$ at the origin and the definition of the electric potential due to a point charge to find V at the origin.

(*a*) Apply Coulomb's law and the superposition of fields to find the electric field $\vec{E}$ at the origin:

$$\vec{E} = \vec{E}_{+q\text{ at }-a} + \vec{E}_{+q\text{ at }+a} = \frac{kq}{a^2}\hat{i} - \frac{kq}{a^2}\hat{i} = 0$$

The potential V at the origin is given by:

$$V = V_{+q\text{ at }-a} + V_{+q\text{ at }+a} = \frac{kq}{a} + \frac{kq}{a} = \frac{2kq}{a}$$

and $\boxed{(2)}$ is correct.

(*b*) Apply Coulomb's law and the superposition of fields to find the electric field $\vec{E}$ at the origin:

$$\vec{E} = \vec{E}_{+q\text{ at }-a} + \vec{E}_{-q\text{ at }+a} = \left(\frac{kq}{a^2}\right)\hat{i} + \left(\frac{kq}{a^2}\right)\hat{i} = \left(\frac{2kq}{a^2}\right)\hat{i}$$

The potential V at the origin is given by:

$$V = V_{+q\text{ at }-a} + V_{-q\text{ at }+a} = \frac{kq}{a} + \frac{k(-q)}{a} = 0$$

and $\boxed{(3)}$ is correct.

10 •• The electrostatic potential (in volts) is given by $V(x, y, z) = 4.00|x| + V_0$, where V_0 is a constant, and x is in meters. (*a*) Sketch the electric field for this potential. (*b*) Which of the following charge distributions is most likely responsible for this potential: (1) A negatively charged flat sheet in the $z = 0$ plane, (2) a point charge at the origin, (3) a positively charged flat sheet in the $x = 0$ plane, or (4) a uniformly charged sphere centered at the origin. Explain your answer.

Picture the Problem We can use $\vec{E} = -\frac{\partial V}{\partial x}\hat{i}$ to find the electric field corresponding to the given potential and then compare its form to those produced by the four alternatives listed.

(*a*) Find the electric field corresponding to this potential function:

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} = -\frac{\partial}{\partial x}\left[4.00|x| + V_0\right]\hat{i}$$
$$= -4.00\frac{\partial}{\partial x}\left[|x|\right]\hat{i}$$

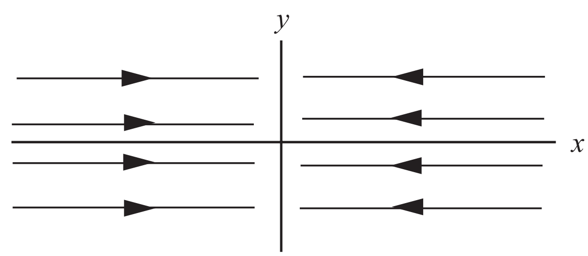
If $x > 0$, then $\frac{\partial}{\partial x}\left[|x|\right] = 1$ and:

$$\vec{E}_{x>0} = \left(-4.00\frac{\text{V}}{\text{m}}\right)\hat{i}$$

If $x < 0$, then $\frac{\partial}{\partial x}\left[|x|\right] = -1$ and:

$$\vec{E}_{x<0} = \left(4.00\frac{\text{V}}{\text{m}}\right)\hat{i}$$

A sketch of the electric field in this region follows:



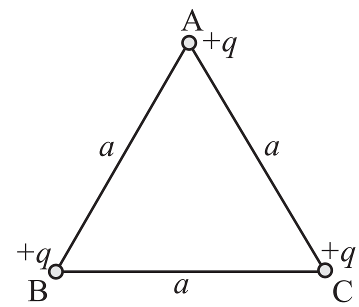
(*b*) $\boxed{(1)}$ is correct because field lines end on negative charges.

11 •• **[SSM]** The electric potential is the same everywhere on the surface of a conductor. Does this mean that the surface charge density is also the same everywhere on the surface? Explain your answer.

Determine the Concept No. The local surface charge density is proportional to the normal component of the electric field, not the potential on the surface.

12 •• Three identical positive point charges are located at the vertices of an equilateral triangle. If the length of each side of the triangle shrinks to one-fourth of its original length, by what factor does the electrostatic potential energy of this system change? (The electrostatic potential energy approaches zero if the length of each side of the triangle approaches infinity.)

Picture the Problem Points A, B, and C are at the vertices of an equilateral triangle of side a. The electrostatic potential energy of the system of the three equal positive point charges is the total work that must be done on the charges to bring them from infinity to this configuration.



The electrostatic potential energy is the work required to assemble the three charges at the vertices of the equilateral triangle:

$$U = W_{\infty \to \mathrm{A}} + W_{\infty \to \mathrm{B}} + W_{\infty \to \mathrm{C}} \qquad (1)$$

Place the first charge at point A. To accomplish this step, the work $W_{\infty \to \mathrm{A}}$ that is needed is zero:

$$W_{\infty \to \mathrm{A}} = 0$$

Bring the second charge to point B. The work required is $W_{\infty \to \mathrm{B}} = qV_{\mathrm{A}}$, where V_{A} is the potential at point B due to the first charge at point A a distance a away:

$$W_{\infty \to \mathrm{B}} = qV_{\mathrm{A}} = q\left(\frac{kq}{a}\right) = \frac{kq^2}{a}$$

Similarly, $W_{\infty \to \mathrm{C}}$ is given by:

$$W_{\infty \to \mathrm{C}} = qV_{\mathrm{C}} = q\left(\frac{kq}{a} + \frac{kq}{a}\right) = \frac{2kq^2}{a}$$

Substituting for $W_{\infty \to \mathrm{A}}$, $W_{\infty \to \mathrm{B}}$, and $W_{\infty \to \mathrm{C}}$ in equation (1) yields:

$$U = 0 + \frac{kq^2}{a} + \frac{2kq^2}{a} = \frac{3kq^2}{a}$$

If the length of each side of the triangle shrinks to one-fourth of its original length, its electrostatic potential energy U' becomes:

$$U' = \frac{3kq^2}{\frac{1}{4}a} = 4\left(\frac{3kq^2}{a}\right) = 4U$$

Hence the electrostatic potential energy of this system increases by a factor of $\boxed{4}$.

Estimation and Approximation Problems

13 • [SSM] Estimate maximum the potential difference between a thundercloud and Earth, given that the electrical breakdown of air occurs at fields of roughly 3.0×10^6 V/m.

Picture the Problem The potential difference between the thundercloud and Earth is the product of the electric field between them and their separation.

Express the potential difference between the cloud and Earth as a function of their separation d and electric field E between them:

$$V = Ed$$

Assuming that the thundercloud is at a distance of about 1 km above the surface of Earth, the potential difference is approximately:

$$V = \left(3.0\times10^{6}\ \text{V/m}\right)\left(10^{3}\ \text{m}\right) = \boxed{3.0\times10^{9}\ \text{V}}$$

Note that this is an upper bound, as there will be localized charge distributions on the thundercloud which raise the local electric field above the average value.

14 • The specifications for the gap width of typical automotive spark plug is approximately equal to the thickness of the cardboard used for matchbook covers. Because of the high compression of the air-gas mixture in the cylinder, the dielectric strength of the mixture is roughly 2.0×10^7 V/m. Estimate the maximum potential difference across the spark gap during operating conditions.

Picture the Problem The potential difference between the electrodes of the spark plug is the product of the electric field in the gap and the separation of the electrodes. We'll assume that the separation of the electrodes is 1.0 mm.

Express the potential difference between the electrodes of the spark plug as a function of their separation d and electric field E between them:

$$V = Ed$$

Substitute numerical values and evaluate V:

$$V = \left(2.0\times10^{7}\ \text{V/m}\right)\left(1.0\times10^{-3}\ \text{m}\right) = \boxed{20\,\text{kV}}$$

15 •• The radius of a proton is approximately 1.0×10^{-15} m. Suppose two protons having equal and opposite momenta undergo a head-on collision. Estimate the minimum kinetic energy (in MeV) required by each proton to allow the protons to overcome electrostatic repulsion and collide. *Hint: The rest energy of a proton is 938 MeV. If the kinetic energies of the protons are much less than this rest energy, then a non-relativistic calculation is justified.*

Picture the Problem We can use conservation of energy to relate the initial kinetic energy of the protons to their electrostatic potential energy when they have approached each other to the given "radius."

Apply conservation of energy to relate the initial kinetic energy of the protons to their electrostatic potential when they are separated by a distance r:

$$K_i + U_i = K_f + U_f$$

or, because $U_i = K_f = 0$,

$$K_i = U_f$$

Because each proton has kinetic energy K:

$$2K = \frac{ke^2}{r} \Rightarrow K = \frac{ke^2}{2r}$$

Substitute numerical values and evaluate K:

$$K = \frac{\left(8.988\times10^9\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left(1.602\times10^{-19}\ \mathrm{C}\right)^2}{2\left(1.0\times10^{-15}\ \mathrm{m}\right)} = 1.153\times10^{-13}\ \mathrm{J}\times\frac{1\,\mathrm{eV}}{1.602\times10^{-19}\ \mathrm{J}}$$

$$= \boxed{0.72\,\mathrm{MeV}}$$

Remarks: Because the kinetic energies of the protons is approximately 0.08% of their rest energy ($\mathbf{K/E_{rest} = 0.72\,MeV/938\,MeV \approx 0.08\%}$,) the non-relativistic calculation was justified.

16 • When you touch a friend after walking across a rug on a dry day, you typically draw a spark of about 2.0 mm. Estimate the potential difference between you and your friend just before the spark.

Picture the Problem The magnitude of the electric field for which dielectric breakdown occurs in air is about 3.0 MV/m. We can estimate the potential difference between you and your friend from the product of the length of the spark and the dielectric constant of air.

Express the product of the length of the spark and the dielectric constant of air:

$$V = (3.0\,\mathrm{MV/m})(2.0\,\mathrm{mm}) = \boxed{6.0\,\mathrm{kV}}$$

17 • Estimate the maximum surface charge density that can exist at the end of a sharp lightning rod so that no dielectric breakdown of air occurs.

Picture the Problem The maximum electric field E_{max} just outside the end of the lightning rod is related to the maximum surface charge density σ_{max}.

Express the maximum electric field just outside the end of the lightning rod as a function of the maximum surface charge density σ_{max}:

$$E_{max} = \frac{\sigma_{max}}{\epsilon_0} \Rightarrow \sigma_{max} = \epsilon_0 E_{max}$$

Substitute numerical values and evaluate σ_{max}:

$$\sigma_{max} = \left(8.854\times10^{-12}\,\frac{C^2}{N\cdot m^2}\right)\left(3.0\times10^6\,\frac{V}{m}\right) \approx \boxed{27\,\mu C/m^2}$$

18 •• The electric-field strength near the surface of Earth is about 300 V/m. (*a*) Estimate the magnitude of the charge density on the surface of Earth. (*b*) Estimate the total charge on Earth. (*c*) What is value of the electric potential at Earth′s surface? (Assume the potential is zero at infinity.) (*d*) If all Earth′s electrostatic potential energy could be harnessed and converted to electric energy at reasonable efficiency, how long could it be used to run the consumer households in the United States? Assume the average American household consumes about 500 kW·h of electric energy per month.

Picture the Problem (*a*) The charge density is the product of ϵ_0 and the magnitude of the electric field at the surface of Earth. (*b*) The total charge on Earth is the product of its surface charge density and its area. (*c*) The electric potential at Earth's surface is given by $V = kQ/R$ where Q is the charge on Earth and R is its radius. (*d*) We can estimate how long Earth's electrostatic energy could run households in the United States by dividing the energy available by the rate of consumption of electrical energy.

(*a*) The magnitude of the charge density σ on the surface of Earth is proportional to the electric field strength E at the surface of Earth:

$$\sigma = \epsilon_0 E$$

Substitute numerical values and evaluate σ:

$$\sigma = \left(8.854\times10^{-12}\,\frac{C^2}{N\cdot m^2}\right)\left(300\,\frac{V}{m}\right) = \boxed{2.66\,nC/m^2}$$

(*b*) The total charge on Earth is given by:

$$Q = \sigma A = 4\pi\sigma R^2$$

Substitute numerical values and evaluate Q:

$$Q = 4\pi\left(2.656\,\frac{\text{nC}}{\text{m}^2}\right)(6370\text{ km})^2 = \boxed{1.35\text{ MC}}$$

(c) The electric potential V at the surface of Earth is given by:

$$V = \frac{kQ}{R}$$

where R is the radius of Earth.

Multiplying and dividing by R and substituting for kQ/R^2 yields:

$$V = \frac{kQ}{R} = \left(\frac{kQ}{R^2}\right)R = ER$$

Substitute numerical values and evaluate V:

$$V = \left(300\,\frac{\text{V}}{\text{m}}\right)(6370\text{ km}) = \boxed{1.91\text{ GV}}$$

(d) Express the available energy:

$$E_{\text{avail}} = eU = e\left(\tfrac{1}{2}QV\right)$$

where e is the efficiency of energy conversion.

Assuming an efficiency of 1/3 yields:

$$E_{\text{avail}} = \tfrac{1}{3}\left(\tfrac{1}{2}QV\right) = \tfrac{1}{6}QV$$

Express the rate at which energy must be supplied to households in the United States:

$$P_{\text{req}} = P_{\substack{\text{req per}\\ \text{household}}} N_{\text{households}}$$

Assuming 80 million households in the United States, substitute numerical values and evaluate the lifetime of the electrical energy derived from Earth's electrostatic energy:

$$\frac{E_{\text{avail}}}{P_{\text{req}}} = \frac{\frac{1}{6}QV}{P_{\substack{\text{req per}\\ \text{household}}} N_{\text{households}}}$$

$$= \frac{(1.35\text{ MC})(1.91\text{ GV})}{6\left(500\,\dfrac{\text{kWh}\times\dfrac{3600\text{ s}}{\text{h}}}{\text{household}\cdot\text{month}}\times\dfrac{1\text{ month}}{30.4\text{ d}}\times\dfrac{1\text{ d}}{24\text{ h}}\right)\left(80\times10^{6}\text{ households}\right)}$$

$$\approx \boxed{2.2\text{ h}}$$

Electrostatic Potential Difference, Electrostatic Energy and Electric Field

19 • A point particle has a charge equal to +2.00 μC and is fixed at the origin. (*a*) What is the electric potential *V* at a point 4.00 m from the origin assuming that $V = 0$ at infinity? (*b*) How much work must be done to bring a second point particle that has a charge of +3.00 μC from infinity to a distance of 4.00 m from the +2.00-μC charge?

Picture the Problem The Coulomb potential at a distance *r* from the origin relative to $V = 0$ at infinity is given by $V = kq/r$ where *q* is the charge at the origin. The work that must be done by an outside agent to bring a charge from infinity to a position a distance *r* from the origin is the product of the magnitude of the charge and the potential difference due to the charge at the origin.

(*a*) The Coulomb potential of the charge is given by:

$$V = \frac{kq}{r}$$

Substitute numerical values and evaluate *V*:

$$V = \frac{\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(2.00\ \mu\text{C}\right)}{4.00\,\text{m}}$$

$$= 4.494\ \text{kV} = \boxed{4.49\,\text{kV}}$$

(*b*) The work that must be done is given by:

$$W = q\Delta V$$

Substitute numerical values and evaluate *W*:

$$W = \left(3.00\,\mu\text{C}\right)\left(4.494\,\text{kV}\right) = \boxed{13.5\,\text{mJ}}$$

20 •• The facing surfaces of two large parallel conducting plates separated by 10.0 cm have uniform surface charge densities that are equal in magnitude but opposite in sign. The difference in potential between the plates is 500 V. (*a*) Is the positive or the negative plate at the higher potential? (*b*) What is the magnitude of the electric field between the plates? (*c*) An electron is released from rest next to the negatively charged surface. Find the work done by the electric field on the electron as the electron moves from the release point to the positive plate. Express your answer in both electron volts and joules. (*d*) What is the change in potential energy of the electron when it moves from the release point plate to the positive plate? (*e*) What is its kinetic energy when it reaches the positive plate?

Picture the Problem Because the electric field is uniform, we can find its magnitude from $E = \Delta V/\Delta x$. We can find the work done by the electric field on the electron from the difference in potential between the plates and the charge of the

electron and find the change in potential energy of the electron from the work done on it by the electric field. We can use conservation of energy to find the kinetic energy of the electron when it reaches the positive plate.

(*a*) Because the electric force on a test charge is away from the positive plate and toward the negative plate, the positive plate is at the higher potential.

(*b*) Express the magnitude of the electric field between the plates in terms of their separation and the potential difference between them:

$$E = \frac{\Delta V}{\Delta x} = \frac{500\,\text{V}}{0.100\,\text{m}} = \boxed{5.00\,\text{kV/m}}$$

(*c*) Relate the work done by the electric field on the electron to the difference in potential between the plates and the charge of the electron:

$$W = q\Delta V = \left(1.602\times10^{-19}\,\text{C}\right)\left(500\text{V}\right) = \boxed{8.01\times10^{-17}\,\text{J}}$$

Converting 8.01×10^{-17} J to eV yields:

$$W = \left(8.01\times10^{-17}\,\text{J}\right)\left(\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right) = \boxed{500\,\text{eV}}$$

(*d*) Relate the change in potential energy of the electron to the work done on it as it moves from the negative plate to the positive plate:

$$\Delta U = -W = \boxed{-500\,\text{eV}}$$

(*e*) Apply conservation of energy to obtain:

$$\Delta K = -\Delta U = \boxed{500\,\text{eV}}$$

21 •• A uniform electric field that has a magnitude 2.00 kV/m points in the $+x$ direction. (*a*) What is the electric potential difference between the $x = 0.00$ m plane and the $x = 4.00$ m plane? A point particle that has a charge of $+3.00\ \mu$C is released from rest at the origin. (*b*) What is the change in the electric potential energy of the particle as it travels from the $x = 0.00$ m plane to the $x = 4.00$ m plane? (*c*) What is the kinetic energy of the particle when it arrives at the $x = 4.00$ m plane? (*d*) Find the expression for the electric potential $V(x)$ if its value is chosen to be zero at $x = 0$.

Picture the Problem (*a*) and (*b*) We can use the definition of potential difference to find the potential difference $V(4.00\text{ m}) - V(0)$ and (*c*) conservation of energy to find the kinetic energy of the charge when it is at $x = 4.00$ m. (*d*) We can find $V(x)$ if $V(x)$ is assigned various values at various positions from the definition of potential difference.

(*a*) Apply the definition of finite potential difference to obtain:

$$V(4.00\,\text{m}) - V(0) = -\int_a^b \vec{E}\cdot d\vec{\ell} = -\int_0^{4.00\,\text{m}} E d\ell = -(2.00\,\text{kN/C})(4.00\,\text{m}) = \boxed{-8.00\,\text{kV}}$$

(*b*) By definition, ΔU is given by:

$$\Delta U = q\Delta V = (3.00\,\mu\text{C})(-8.00\,\text{kV}) = \boxed{-24.0\,\text{mJ}}$$

(*c*) Use conservation of energy to relate ΔU and ΔK:

$$\Delta K + \Delta U = 0$$

or

$$K_{4\text{m}} - K_0 + \Delta U = 0$$

Because $K_0 = 0$:

$$K_{4\text{m}} = -\Delta U = \boxed{24.0\,\text{mJ}}$$

Use the definition of finite potential difference to obtain:

$$V(x) - V(x_0) = -E_x(x - x_0) = -(2.00\,\text{kV/m})(x - x_0)$$

(*d*) For $V(0) = 0$:

$$V(x) - 0 = -(2.00\,\text{kV/m})(x - 0)$$

or

$$V(x) = \boxed{-(2.00\,\text{kV/m})x}$$

22 •• In a potassium chloride molecule the distance between the potassium ion (K^+) and the chlorine ion (Cl^-) in a potassium chloride molecule is 2.80×10^{-10} m. (*a*) Calculate the energy (in eV) required to separate the two ions to an infinite distance apart. (Model the two ions as two point particles initially at rest.) (*b*) If twice the energy determined in Part (*a*) is actually supplied, what is the total amount of kinetic energy that the two ions have when they were an infinite distance apart?

Picture the Problem In general, the work done by an external agent in separating the two ions changes both their kinetic and potential energies. Here we're assuming that they are at rest initially and that they will be at rest when they are infinitely far apart. Because their potential energy is also zero when they are infinitely far apart, the energy W_{ext} required to separate the ions to an infinite

distance apart is the negative of their potential energy when they are a distance r apart.

(a) Express the energy required to separate the ions in terms of the work required by an external agent to bring about this separation:

$$W_{\text{ext}} = \Delta K + \Delta U = 0 - U_{\text{i}}$$
$$= -\frac{kq_{-}q_{+}}{r} = -\frac{k(-e)e}{r} = \frac{ke^2}{r}$$

Substitute numerical values and evaluate W_{ext}:

$$W_{\text{ext}} = \frac{(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2)(1.602\times10^{-19}\ \text{C})^2}{2.80\times10^{-10}\ \text{m}} = 8.238\times10^{-19}\ \text{J}$$

Convert this energy to eV:

$$W_{\text{ext}} = (8.238\times10^{-19}\ \text{J})\left(\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}}\right)$$
$$= \boxed{5.14\,\text{eV}}$$

(b) Apply the work-energy theorem to the system of ions to obtain:

$$2W_{\text{ext}} = \Delta K + \Delta U = K_{\text{f}} - U_{\text{i}}$$

where K_{f} is the kinetic energy of the ions when they are an infinite distance apart.

Solving for K_{f} yields:

$$K_{\text{f}} = 2W_{\text{ext}} + U_{\text{i}}$$

From Part (a), $U_{\text{i}} = -W_{\text{ext}}$:

$$K_{\text{f}} = 2W_{\text{ext}} - W_{\text{ext}} = W_{\text{ext}} = \boxed{5.14\,\text{eV}}$$

23 •• [SSM] Protons are released from rest in a Van de Graaff accelerator system. The protons initially are located where the electrical potential has a value of 5.00 MV and then they travel through a vacuum to a region where the potential is zero. (a) Find the final speed of these protons. (b) Find the accelerating electric-field strength if the potential changed *uniformly* over a distance of 2.00 m.

Picture the Problem We can find the final speeds of the protons from the potential difference through which they are accelerated and use $E = \Delta V/\Delta x$ to find the accelerating electric field.

(a) Apply the work-kinetic energy theorem to the accelerated protons:

$$W = \Delta K = K_{\text{f}} \Rightarrow e\Delta V = \tfrac{1}{2}mv^2$$

Solve for v to obtain:

$$v=\sqrt{\frac{2e\Delta V}{m}}$$

Substitute numerical values and evaluate v:

$$v=\sqrt{\frac{2\left(1.602\times10^{-19}\ \text{C}\right)\left(5.00\,\text{MV}\right)}{1.673\times10^{-27}\ \text{kg}}}$$

$$=\boxed{3.09\times10^{7}\ \text{m/s}}$$

(b) Assuming the same potential change occurred *uniformly* over the distance of 2.00 m, we can use the relationship between E, ΔV, and Δx express and evaluate E:

$$E=\frac{\Delta V}{\Delta x}=\frac{5.00\,\text{MV}}{2.00\,\text{m}}=\boxed{2.50\,\text{MV/m}}$$

24 •• The picture tube of a television set was, until recently, invariably a cathode-ray tube. In a typical cathode-ray tube, an electron "gun" arrangement is used to accelerate electrons from rest to the screen. The electrons are accelerated through a potential difference of 30.0 kV. (a) Which region is at a higher electric potential, the screen or the electron's starting location? Explain your answer. (b) What is the kinetic energy (in both eV and joules) of an electron as it reaches the screen?

Picture the Problem The work done on the electrons by the electric field changes their kinetic energy. Hence we can use the work-kinetic energy theorem to find the kinetic energy and the speed of impact of the electrons.

(a) Because positively charged objects are accelerated from higher potential to lower potential regions, the screen must be at the higher electric potential to accelerate electrons toward it.

(b) Use the work-kinetic energy theorem to relate the work done by the electric field to the change in the kinetic energy of the electrons:

$$W=\Delta K=K_\text{f}$$

or

$$K_\text{f}=e\Delta V \qquad (1)$$

Substitute numerical values and evaluate K_f:

$$K_\text{f}=(1e)(30.0\,\text{kV})=\boxed{3.00\times10^{4}\ \text{eV}}$$

Convert this energy to eV:

$$K_\text{f}=\left(3.00\times10^{4}\ \text{eV}\right)\left(\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)$$

$$=\boxed{4.81\times10^{-15}\ \text{J}}$$

25 ••• (*a*) A positively charged particle is on a trajectory to collide head-on with a massive positively charge nucleus that is initially at rest The particle initially has kinetic energy K_i. In addition, the particle is initially far from the nucleus. Derive an expression for the distance of closest approach. Your expression should be in terms of the initial kinetic energy K of the particle, the charge ze on the particle, and the charge Ze on the nucleus, where both z and Z are integers. (*b*) Find the numerical value for the distance of closest approach between a 5.00 MeV α-particle and between a 9.00 MeV α-particle and a stationary gold nucleus. (The values 5.00 MeV and 9.00 MeV are the initial kinetic energies of the alpha particles. Neglect the motion of the gold nucleus following the collisions.) (*c*) The radius of the gold nucleus is about 7×10^{-15} m. If α-particles approach the nucleus closer than 7×10^{-15} m, they experience the strong nuclear force in addition to the electric force of repulsion. In the early 20th century, before the strong nuclear force was known, Ernest Rutherford bombarded gold nuclei with α-particles that had kinetic energies of about 5 MeV. Would you expect this experiment to reveal the existence of this strong nuclear force? Explain your answer.

Picture the Problem We know that energy is conserved in the interaction between the α particle and the massive nucleus. Under the assumption that the recoil of the massive nucleus is negligible, we know that the initial kinetic energy of the α particle will be transformed into potential energy of the two-body system when the particles are at their distance of closest approach.

(*a*) Apply conservation of energy to the system consisting of the positively charged particle and the massive nucleus:

$$\Delta K + \Delta U = 0$$

or

$$K_f - K_i + U_f - U_i = 0$$

Because $K_f = U_i = 0$:

$$-K_i + U_f = 0$$

Letting r be the separation of the particles at closest approach, express U_f:

$$U_f = \frac{kq_{\text{nucleus}}q_{\text{particle}}}{r} = \frac{k(Ze)(ze)}{r} = \frac{kzZe^2}{r}$$

Substitute for U_f to obtain:

$$-K_i + \frac{kzZe^2}{r} = 0 \Rightarrow r = \boxed{\frac{kzZe^2}{K_i}}$$

(*b*) For a 5.00-MeV α particle and a stationary gold nucleus:

$$r_5 = \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(2)(79)(1.602\times10^{-19}\ \text{C})^2}{(5.00\,\text{MeV})(1.602\times10^{-19}\ \text{J/eV})} = \boxed{46\ \text{fm}}$$

For a 9.00-MeV α particle and a stationary gold nucleus:

$$r_9 = \frac{\left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)(2)(79)\left(1.602\times10^{-19}\ \mathrm{C}\right)^2}{(9.00\,\mathrm{MeV})\left(1.602\times10^{-19}\ \mathrm{J/eV}\right)} = \boxed{25\ \mathrm{fm}}$$

(*c*) No. The distance of closest approach for a 5-MeV alpha particle found above (46 fm) is much larger than the 7 fm radius of a gold nucleus. Hence the scattering was solely the result of the inverse-square Coulomb force.

Potential Due to a System of Point Charges

26 • Four point charges, each having a magnitude of 2.00 μC, are fixed at the corners of a square whose edges are 4.00-mlong. Find the electric potential at the center of the square if (*a*) all the charges are positive, (*b*) three of the charges are positive and one charge is negative, and (*c*) two charges are positive and two charges are negative. (Assume the potential is zero very far from all charges.)

Picture the Problem Let the numerals 1, 2, 3, and 4 denote the charges at the four corners of square and r the distance from each charge to the center of the square. The potential at the center of square is the algebraic sum of the potentials due to the four charges.

Express the potential at the center of the square:

$$V = \frac{kq_1}{r} + \frac{kq_2}{r} + \frac{kq_3}{r} + \frac{kq_4}{r} = \frac{k}{r}(q_1 + q_2 + q_3 + q_4) = \frac{k}{r}\sum_{i=1}^{4} q_i$$

(*a*) If the charges are positive:

$$V = \frac{8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}}{2.00\sqrt{2}\ \mathrm{m}}(4)(2.00\,\mu\mathrm{C}) = \boxed{25.4\,\mathrm{kV}}$$

(*b*) If three of the charges are positive and one is negative:

$$V = \frac{8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}}{2.00\sqrt{2}\ \mathrm{m}}(2)(2.00\,\mu\mathrm{C}) = \boxed{12.7\,\mathrm{kV}}$$

(*c*) If two are positive and two are negative:

$$V = \boxed{0}$$

27 • **[SSM]** Three point charges are fixed at locations on the x-axis: q_1 is at $x = 0.00$ m, q_2 is at $x = 3.00$ m, and q_3 is at $x = 6.00$ m. Find the electric potential at the point on the y axis at $y = 3.00$ m if (a) $q_1 = q_2 = q_3 = +2.00\ \mu$C, ($b$) $q_1 = q_2 = +2.00\ \mu$C and $q_3 = -2.00\ \mu$C, and (c) $q_1 = q_3 = +2.00\ \mu$C and $q_2 = -2.00\ \mu$C. (Assume the potential is zero very far from all charges.)

Picture the Problem The potential at the point whose coordinates are (0, 3.00 m) is the algebraic sum of the potentials due to the charges at the three locations given.

Express the potential at the point whose coordinates are (0, 3.00 m):

$$V = k\sum_{i=1}^{3}\frac{q_i}{r_i} = k\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}+\frac{q_3}{r_3}\right)$$

(a) For $q_1 = q_2 = q_3 = 2.00\ \mu$C:

$$V = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.00\ \mu\text{C})\left(\frac{1}{3.00\,\text{m}}+\frac{1}{3.00\sqrt{2}\,\text{m}}+\frac{1}{3.00\sqrt{5}\,\text{m}}\right)$$

$$= \boxed{12.9\,\text{kV}}$$

(b) For $q_1 = q_2 = 2.00\ \mu$C and $q_3 = -2.00\ \mu$C:

$$V = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.00\ \mu\text{C})\left(\frac{1}{3.00\,\text{m}}+\frac{1}{3.00\sqrt{2}\,\text{m}}-\frac{1}{3.00\sqrt{5}\,\text{m}}\right)$$

$$= \boxed{7.55\,\text{kV}}$$

(c) For $q_1 = q_3 = 2.00\ \mu$C and $q_2 = -2.00\ \mu$C:

$$V = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.00\ \mu\text{C})\left(\frac{1}{3.00\,\text{m}}-\frac{1}{3.00\sqrt{2}\,\text{m}}+\frac{1}{3.00\sqrt{5}\,\text{m}}\right)$$

$$= \boxed{4.43\,\text{kV}}$$

28 • Points A, B, and C are fixed at the vertices of an equilateral triangle whose edges are 3.00-m long. A point particle with a charge of +2.00 μC is fixed at each of vertices A and B. (a) What is the electric potential at point C? (Assume the potential is zero very far from all charges.) (b) How much work is required to move a point particle having a charge of +5.00 μC from a distance of infinity to point C? (c) How much additional work is required to move the +5.00-μC point particle from point C to the midpoint of side AB?

Picture the Problem (*a*) The potential at vertex *C* is the algebraic sum of the potentials due to the point charges at vertices *A* and *B*. (*b*) The work required to bring a charge from infinity to vertex *C* equals the change in potential energy of the system during this process. (*c*) The additional work required to move the +5.00-μC point particle from point *C* to the midpoint of side *AB* is the product of +5.00-μC and the difference in potential between point *C* and the midpoint of side *AB*.

(*a*) Express the potential at vertex *C* as the sum of the potentials due to the point charges at vertices *A* and *B*:

$$V_C = k\left(\frac{q_A}{r_A} + \frac{q_B}{r_B}\right)$$

Because $q_A = q_B = q_2$:

$$V_C = kq_2\left(\frac{1}{r_A} + \frac{1}{r_B}\right)$$

Substitute numerical values and evaluate V_C:

$$V_C = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(2.00\,\mu\text{C}\right)\left(\frac{1}{3.00\,\text{m}} + \frac{1}{3.00\,\text{m}}\right)$$
$$= 11.98\,\text{kV} = \boxed{12.0\,\text{kV}}$$

(*b*) Express the required work in terms of the change in the potential energy of the system:

$$W_{\infty\to C} = \Delta U = U_C - U_\infty = U_C$$

Substituting for U_C yields:

$$W_{\infty\to C} = q_5 V_C$$

Substitute numerical values and evaluate $W_{\infty\to C}$:

$$W_{\infty\to C} = \left(5.00\,\mu\text{C}\right)\left(11.98\,\text{kV}\right) = \boxed{59.9\,\text{mJ}}$$

(*c*) Express the required work in terms of the change in the potential energy of the system:

$$W_{C\to\text{midpoint}} = \Delta U = U_{\substack{\text{midpoint}\\ \text{of } AB}} - U_C$$

Substituting for $U_{\substack{\text{midpoint}\\ \text{of } AB}}$ and U_C yields:

$$W_{C\to\text{midpoint}} = q_5 V_{\substack{\text{midpoint}\\ \text{of } AB}} - q_5 V_C = q_5\left(V_{\substack{\text{midpoint}\\ \text{of } AB}} - V_C\right) \qquad (1)$$

The potential at the midpoint of *AB* is the sum of the potentials due to the point charges at vertices *A* and *B*:

$$V_{\substack{\text{midpoint}\\ \text{of } AB}} = k\left(\frac{q_A}{r_A} + \frac{q_B}{r_B}\right)$$

Because $q_A = q_B = q_2$ and $r_A = r_B = r$:

$$V_{\substack{\text{midpoint}\\ \text{of } AB}} = \frac{2kq_2}{r}$$

where r is the distance from vertex *A* (and vertex *B*) to the midpoint of side *AB* of the triangle.

Substituting in equation (1) and simplifying yields:

$$W_{C\to\text{midpoint}} = q_5\left(\frac{2kq_2}{r} - V_C\right)$$

Substitute numerical values and evaluate $W_{C\to\text{midpoint}}$:

$$W_{C\to\text{midpoint}} = (+5.00\ \mu\text{C})\left(\frac{2(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(+2.00\ \mu\text{C})}{1.50\ \text{m}} - 11.98\ \text{kV}\right)$$

$$= \boxed{59.9\ \text{mJ}}$$

29 •• Three identical point particles with charge q are at the vertices of an equilateral triangle that is circumscribed by a circle of radius a that lies in the $z = 0$ plane and is centered at the origin. The values of q and a are $+3.00\ \mu$C and 60.0 cm, respectively. (Assume the potential is zero very far from all charges.) (*a*) What is the electric potential at the origin? (*b*) What is the electric potential at the point on the z axis at $z = a$? (*c*) How would your answers to Parts (*a*) and (*b*) change if the charges were still on the circle but one is no longer at a vertex of the triangle? Explain your answer.

Picture the Problem The electric potential at the origin and at $z = a$ is the algebraic sum of the potentials at those points due to the individual charges distributed along the equator.

(*a*) Express the potential at the origin as the sum of the potentials due to the charges placed at 120° intervals along the equator of the sphere:

$$V_{\text{origin}} = k\sum_{i=1}^{3}\frac{q_i}{r_i} = 3\frac{kq}{a}$$

Substitute numerical values and evaluate V_{origin}:

$$V_{\text{origin}} = \frac{3\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.00\ \mu\text{C}\right)}{0.600\,\text{m}} = \boxed{135\,\text{kV}}$$

(*b*) Using geometry, find the distance from each charge to $z = a$:

$$a = 0.600\sqrt{2}\,\text{m}$$

Proceed as in (*a*) with $a = 0.600\sqrt{2}\,\text{m}$:

$$V = k\sum_{i=1}^{3}\frac{q_i}{r_i'} = 3\frac{kq}{a'}$$

$$= \frac{3\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.00\ \mu\text{C}\right)}{0.600\sqrt{2}\,\text{m}}$$

$$= \boxed{95.3\,\text{kV}}$$

(*c*) Because the two field points are equidistant from all points on the circle, the answers for Parts (*a*) and (*b*) would not change.

30 •• Two point charges q and q' are separated by a distance a. At a point $a/3$ from q and along the line joining the two charges the potential is zero. (Assume the potential is zero very far from all charges.) (*a*) Which of the following statements is true?

(1) The charges have the same sign.
(2) The charges have opposite signs.
(3) The relative signs of the charges can not be determined by using data given.

(*b*) Which of the following statements is true?

(1) $|q| > |q'|$.
(2) $|q| < |q'|$.
(3) $|q| = |q'|$.
(4) The relative magnitudes of the charges cannot be determined by using the data given.

(*c*) Find the ratio q/q'.

Picture the Problem We can use the fact that the electric potential at the point of interest is the algebraic sum of the potentials at that point due to the charges q and q' to find the ratio q/q'.

(a) The only way that, in the absence of other point charges, the potential can be zero at $a/3$ is if q and q' have opposite signs. $\boxed{2}$ is correct.

(b) Because the point of interest is closer to q, the magnitude of q must be less than the magnitude of q'. $\boxed{2}$ is correct.

(c) Express the potential at the point of interest as the sum of the potentials due to the two charges:

$$\frac{kq}{\frac{1}{3}a}+\frac{kq'}{\frac{2}{3}a}=0$$

Simplify to obtain:

$$q+\frac{q'}{2}=0 \Rightarrow \frac{q}{q'}=\boxed{-\frac{1}{2}}$$

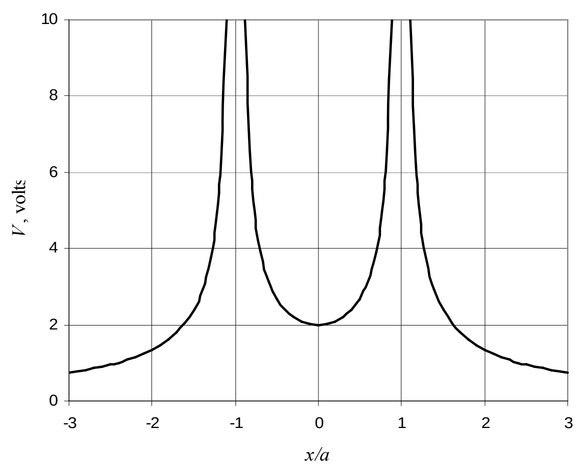
31 •• [SSM] Two identical positively charged point particles are fixed on the x-axis at $x = +a$ and $x = -a$. (a) Write an expression for the electric potential $V(x)$ as a function of x for all points on the x-axis. (b) Sketch $V(x)$ versus x for all points on the x axis.

Picture the Problem For the two charges, $r = |x-a|$ and $|x+a|$ respectively and the electric potential at x is the algebraic sum of the potentials at that point due to the charges at $x = +a$ and $x = -a$.

(a) Express $V(x)$ as the sum of the potentials due to the charges at $x = +a$ and $x = -a$:

$$V=\boxed{kq\left(\frac{1}{|x-a|}+\frac{1}{|x+a|}\right)}$$

(b) The following graph of V as a function of x/a was plotted using a spreadsheet program:



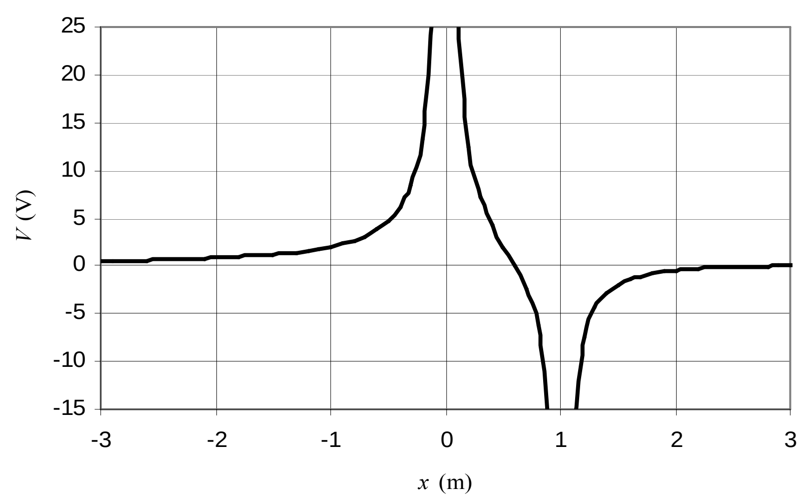
32 •• A point charge of $+3e$ is at the origin and a second point charge of $-2e$ is on the x-axis at $x = a$. (*a*) Sketch the potential function $V(x)$ versus x for all points on the x axis. (*b*) At what point or points, if any, is $V=$ zero on the x axis? (*c*) What point or points, if any, on the x-axis is the electric field zero? Are these locations the same locations found in Part (*b*)? Explain your answer. (*d*) How much work is needed to bring a third charge $+e$ to the point $x = \frac{1}{2}a$ on the x-axis?

Picture the Problem For the two charges, $r = |x-a|$ and $|x|$ respectively and the electric potential at x is the algebraic sum of the potentials at that point due to the charges at $x = a$ and $x = 0$. We can use the graph and the function found in Part (*a*) to identify the points at which $V(x) = 0$. We can find the work needed to bring a third charge $+e$ to the point $x = \frac{1}{2}a$ on the x axis from the change in the potential energy of this third charge.

(*a*) The potential at x is the sum of the potentials due to the point charges $+3e$ and $-2e$:

$$V(x) = \frac{k(3e)}{|x|} + \frac{k(-2e)}{|x-a|}$$

The following graph of $V(x)$ for $ke = 1$ and $a = 1$ was plotted using a spreadsheet program.



(*b*) From the graph we can see that $V(x) = 0$ when:

$$x = \boxed{\pm\infty}$$

Examining the function, we see that $V(x)$ is also zero provided:

$$\frac{3}{|x|} - \frac{2}{|x-a|} = 0$$

For $x > 0$, $V(x) = 0$ when:

$$x = \boxed{3a}$$

For $0 < x < a$, $V(x) = 0$ when:

$$x = \boxed{0.6a}$$

(*c*) The electric field at x is the sum of the electric fields due to the point charges $+3e$ and $-2e$:

$$E(x) = \frac{k(3e)}{x^2} + \frac{k(-2e)}{(x-r)^2}$$

Setting $E(x) = 0$ and simplifying yields:

$$x^2 - 6ax + 3a^2 = 0$$

Solve this equation to find the points on the x-axis where the electric field is zero:

$$x = \boxed{5.4a} \text{ and } x = \boxed{0.55a}$$

Note that the zeros of the electric field are different from the zeros of the electric potential. This is generally the case although, in special cases, they can be the same.

(*d*) Express the work that must be done in terms of the change in potential energy of the charge:

$$W = \Delta U = qV\left(\tfrac{1}{2}a\right)$$

Evaluate the potential at $x = \frac{1}{2}a$:

$$V\left(\tfrac{1}{2}a\right) = \frac{k(3e)}{\left|\frac{1}{2}a\right|} + \frac{k(-2e)}{\left|\frac{1}{2}a - a\right|} = \frac{6ke}{a} - \frac{4ke}{a} = \frac{2ke}{a}$$

Substitute for $V\left(\frac{1}{2}a\right)$ and q to obtain:

$$W = e\left(\frac{2ke}{a}\right) = \boxed{\frac{2ke^2}{a}}$$

33 ••• **[SSM]** A dipole consists of equal but opposite point charges $+q$ and $-q$. It is located so that its center is at the origin, and its axis is aligned with the y-axis (Figure 23-32) The distance between the charges is L. Let $\vec{r}$ be the vector from the origin to an arbitrary field point and θ be the angle that $\vec{r}$ makes with the $+y$ direction. (*a*) Show that at large distances from the dipole (that is for $r >> L$), the dipole's electric potential is given by $V(r,\theta) \approx k\vec{p}\cdot\hat{r}/r^2 = kp\cos\theta/r^2$, where $\vec{p}$ is the dipole moment of the dipole and θ is the angle between $\vec{r}$ and $\vec{p}$. (*b*) At what points in the region $r >> L$, other than at infinity, is the electric potential zero?

Picture the Problem The potential at the arbitrary field point is the sum of the potentials due to the equal but opposite point charges.

(*a*) Express the potential at the arbitrary field point at a large distance from the dipole:

$$V = V_+ + V_- = \frac{kq}{r_+} + \frac{k(-q)}{r_-} = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{r_- - r_+}{r_+ r_-}\right)$$

Referring to the figure, note that, for the far field ($r >> L$):

$$r_- - r_+ \approx L\cos\theta \text{ and } r_+ \approx r_- \approx r$$

Substituting and simplifying yields:

$$V(r,\theta) = kq\left(\frac{L\cos\theta}{r^2}\right) = \frac{kqL\cos\theta}{r^2}$$

or, because $p = qL$,

$$V(r,\theta) = \frac{kp\cos\theta}{r^2}$$

Finally, because $\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}} = p\cos\theta$:

$$V(r,\theta) = \boxed{\frac{k\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}}}{r^2}}$$

(*b*) $V(r,\theta) = 0$ where $\cos\theta = 0$:

$\theta = \cos^{-1} 0 = 90° \Rightarrow V = 0$ at points on the *z* axis. Note that these locations are equidistant from the two oppositely-charged ends of the dipole.

34 ••• A charge configuration consists of three point charges located on the *z* axis (Figure 23-33). One has a charge equal to $-2q$, and is located at the origin. The other two each have a charge equal to $+q$, one is located at $z = +L$ and the other is located at $z = -L$. This charge configuration can be modeled as two dipoles: one centered at $z = +L/2$ and with a dipole moment in the $+z$ direction, the other centered at $z = -L/2$ and with a dipole moment in the $-z$ direction. Each of these dipoles has a dipole moment that has a magnitude equal to qL. Two dipoles arranged in this fashion form a *linear electric quadrupole*. (There are other geometrical arrangements of dipoles that create quadrupoles but they are not linear.) (*a*) Using the result from Problem 33, show that at large distances from the quadrupole (that is for $r >> L$), the electric potential is given by $V_{\text{quad}}(r,\theta) = 2kB\cos^2\theta/r^3$, where $B = qL^2$. (*B* is the magnitude of the quadrupole moment of the charge configuration.) (*b*) Show that on the positive *z* axis, this potential gives an electric field (for $z >> L$) of $\vec{\boldsymbol{E}} = (6kB/z^4)\hat{\boldsymbol{k}}$. (*c*) Show that you get the result of Part (*b*) by adding the electric fields from the three point charges.

Picture the Problem (*a*) The electric potential due to the linear electric quadrupole is the sum of the potentials of the two dipoles. (*b*) The electric field on the *y* axis can be obtained from the electric potential on the *y* axis using $\vec{E}_{z\text{ axis}} = -\frac{\partial V_{z\text{ axis}}}{\partial z}\hat{k}$.

(*a*) The electric potential due to the linear electric quadrupole is given by:

$$V_{\text{quad}} = V_{\uparrow} + V_{\downarrow}$$

From Problem 23-33:

$$V_{\uparrow} = \frac{kqL\cos\theta}{r_{\uparrow}^2} \text{ and } V_{\downarrow} = -\frac{kqL\cos\theta}{r_{\downarrow}^2}$$

Substituting for $V_{\uparrow}$ and $V_{\downarrow}$ yields:

$$V_{\text{quad}} = \frac{kqL\cos\theta}{r_{\uparrow}^2} - \frac{kqL\cos\theta}{r_{\downarrow}^2}$$

Simplify to obtain:

$$V_{\text{quad}} = kqL\cos\theta\left(\frac{r_{\downarrow}^2 - r_{\uparrow}^2}{r_{\uparrow}^2 r_{\downarrow}^2}\right) = kqL\cos\theta\left(\frac{(r_{\downarrow} - r_{\uparrow})(r_{\downarrow} + r_{\uparrow})}{r_{\uparrow}^2 r_{\downarrow}^2}\right)$$

Referring to the figure, note that, for the far field ($r >> L$):

$r_{\downarrow} - r_{\uparrow} \approx L\cos\theta$, $r_{\uparrow} + r_{\downarrow} \approx 2r$, and $r_{\uparrow} \approx r_{\downarrow} \approx 2r$

Substitute and simplify to obtain:

$$V_{\text{quad}}(r,\theta) \approx kqL\cos\theta\left(\frac{2rL\cos\theta}{r^4}\right) = \frac{2kqL^2\cos^2\theta}{r^3}$$

Because $B = qL^2$:

$$V_{\text{quad}}(r,\theta) \approx \boxed{\frac{2kB\cos^2\theta}{r^3}}$$

(*b*) The electric field on the *z* axis is related to the electric potential on the *z* axis:

$$\vec{E}_{z\text{ axis}} = -\frac{\partial V_{z\text{ axis}}}{\partial z}\hat{k}$$

On the *y* axis, $\theta = 0$ and $\cos\boldsymbol{\theta} = 1$. Hence:

$$V_{y\text{ axis}} \approx \frac{2kB}{y^3}$$

Substituting for $V_{z\text{ axis}}$ yields:

$$\vec{E}_{z\text{ axis}} = -\frac{\partial}{\partial z}\left(\frac{2kB}{z^3}\right)\hat{k} = \boxed{\frac{6kB}{z^4}\hat{k}}$$

(*c*) The *E* field on the *z* axis is given by:

$$\vec{E}_{z\text{ axis}} = kq\left(\frac{1}{(z-L)^2} - \frac{2}{z^2} + \frac{1}{(z+L)^2}\right)\hat{k}$$

Letting $w = L^2/z^2$ yields:

$$\vec{E}_{z\text{ axis}} = \frac{kq}{z^2}\left(\frac{1}{(1-w)^2} - 2 + \frac{1}{(1+w)^2}\right)\hat{k}$$

Expand $(1-w)^{-2}$ and $(1+w)^{-2}$ binomially to obtain:

$$(1-w)^{-2} = 1 + 2w + 3w^2 + \text{higher-order terms}$$

and

$$(1+w)^{-2} = 1 - 2w + 3w^2 + \text{higher-order terms}$$

For $w << 1$:

$$(1-w)^{-2} \approx 1 + 2w + 3w^2$$

and

$$(1+w)^{-2} \approx 1 - 2w + 3w^2$$

Substituting in the expression for $\vec{E}_{z\text{ axis}}$ and simplifying yields:

$$\vec{E}_{z\text{ axis}} \approx \frac{kq}{z^2}\left(1 + 2w + 3w^2 - 2 + 1 - 2w + 3w^2\right)\hat{k} = \frac{kq}{z^2}\left(6w^2\right)\hat{k}$$

Finally, substitute for *w* to obtain:

$$\vec{E}_{z\text{ axis}} \approx \frac{kq}{z^2}\left(\frac{6L^2}{z^2}\right)\hat{k} = \frac{6kqL^2}{z^4}\hat{k}$$

$$= \boxed{\frac{6kB}{z^4}\hat{k}}$$

Computing the Electric Field from the Potential

35 • A uniform electric field is in the $-x$ direction. Points *a* and *b* are on the *x*-axis, with *a* at $x = 2.00$ m and *b* at $x = 6.00$ m. (*a*) Is the potential difference $V_b - V_a$ positive or negative? (*b*) If $|V_b - V_a|$ is 100 kV, what is the magnitude of the electric field?

Picture the Problem We can use the relationship $E_x = -(dV/dx)$ to decide the sign of $V_b - V_a$ and $E = \Delta V/\Delta x$ to find *E*.

(*a*) Because $E_x = -(dV/dx)$, V is greater for larger values of x. So:

$$\boxed{V_b - V_a \text{ is positive.}}$$

(*b*) Express E in terms of $V_b - V_a$ and the separation of points a and b:

$$E_x = \frac{\Delta V}{\Delta x} = \frac{V_b - V_a}{\Delta x}$$

Substitute numerical values and evaluate E_x:

$$E_x = \frac{100\,\text{kV}}{6.00\,\text{m} - 2.00\,\text{m}} = \boxed{25.0\ \text{kV/m}}$$

36 • An electric field is given by the expression $\vec{E} = bx^3\hat{i}$, where $b = 2.00\ \text{kV/m}^4$. Find the potential difference between the point at $x = 1.00$ m and the point at $x = 2.00$ m. Which of these points is at the higher potential?

Picture the Problem Because $V(x)$ and E_x are related through $E_x = -\,dV/dx$, we can find V from E by integration.

Separate variables in $E_x = -\,dV/dx$ and substitute for b to obtain:

$$dV = -E_x dx = -\left(2.00x^3\,\frac{\text{kV}}{\text{m}^4}\right)dx$$

Integrate V from V_1 to V_2 and x from $x = 1.00$ m to $x = 2.00$ m:

$$\int_{V_1}^{V_2} dV = V_2 - V_1$$
$$= -\left(2.00\,\frac{\text{kV}}{\text{m}^4}\right)\int_{1.00\,\text{m}}^{2.00\,\text{m}} x^3 dx$$
$$= -\left(2.00\,\frac{\text{kV}}{\text{m}^4}\right)\left[\tfrac{1}{4}x^4\right]_{1.00\,\text{m}}^{2.00\,\text{m}}$$

Simplify to obtain:

$$V_2 - V_1 = \boxed{-7.50\,\text{kV}}$$

Because $V_2 = V_1 + 7.50\,\text{kV}$, the point at $x = 2.00$ m is at the higher potential.

37 •• The electric field on the x axis due to a point charge fixed at the origin is given by $\vec{E} = \left(b/x^2\right)\hat{i}$, where $b = 6.00$ kV·m and $x \neq 0$. (*a*) Find the magnitude and sign of the point charge. (*b*) Find the potential difference between the points on the x-axis at $x = 1.00$ m and $x = 2.00$ m. Which of these points is at the higher potential.

Picture the Problem We can integrate $E_x = -dV_x/dx$ to obtain $V(x)$ and then use this function to find the electric potential difference between the given points.

(*a*) We know that this field is due to a point charge because it varies inversely with the square of the distance from the point charge. Because $\vec{E}_x$ is positive, the sign of the charge must be positive.

Because the given electric field is that due to a point charge, it follows that:

$$kq = 6.00\,\frac{\text{kV}}{\text{m}} \Rightarrow q = \frac{6.00\,\frac{\text{kV}}{\text{m}}}{k}$$

Substitute the numerical value of k and evaluate q:

$$q = \frac{6.00\,\frac{\text{kV}}{\text{m}}}{8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = \boxed{668\,\text{nC}}$$

(*b*) The potential difference between the points on the x-axis at $x = 1.00$ m and $x = 2.00$ m is given by:

$$\Delta V = V_1 - V_2 \qquad (1)$$

From $E_x = -\dfrac{dV_x}{dx}$ we have:

$$\int_\infty^x dV = -\int_\infty^x E_x dx = -kq\int_\infty^x x^{-2}dx$$

or

$$V(x) - V(\infty) = \frac{kq}{x}$$

Letting $V(\infty) = 0$ yields:

$$V(x) = \frac{kq}{x} = \frac{6.00\,\text{kV}\cdot\text{m}}{x}$$

Substituting in equation (1) yields:

$$V_1 - V_2 = \frac{6.00\,\text{kV}\cdot\text{m}}{1.00\,\text{m}} - \frac{6.00\,\text{kV}\cdot\text{m}}{2.00\,\text{m}} = \boxed{3.00\,\text{kV}}$$

Because $V_1 = V_2 + 3.00\,\text{kV}$, the point at $x = 2.00$ m is at the higher potential.

38 •• The electric potential due to a particular charge distribution is measured at many points along the x-axis. A plot of this data is shown in Figure 23-34. At what location (or locations) is the x component of the electric field equal to zero? At this location (or these locations) is the potential also equal to zero? Explain your answer.

Picture the Problem Because $E_x = -dV/dx$, we can find the point(s) at which $E_x = 0$ by identifying the values for x for which $dV/dx = 0$.

Examination of the graph indicates that $dV/dx = 0$ at approximately 4.5 m. Thus $E_x = 0$ at $x \approx \boxed{4.5\,\text{m}}$. At this location, the potential is not zero. The electric field is zero when the slope of the potential function is zero.

39 •• Three identical point charges, each with a charge equal to q, lie in the xy plane. Two of the charges are on the y-axis at $y = -a$ and $y = +a$, and the third charge is on the x-axis at $x = a$. (*a*) Find the potential as a function of position along the x axis. (*b*) Use the Part (*a*) result to obtain an expression for $E_x(x)$, the x component of the electric field as a function of x, on. Check your answers to Parts (*a*) and (*b*) at the origin and as x approaches ∞ to see if they yield the expected results.

Picture the Problem Let r_1 be the distance from $(0, a)$ to $(x, 0)$, r_2 the distance from $(0, -a)$, and r_3 the distance from $(a, 0)$ to $(x, 0)$. We can express $V(x)$ as the sum of the potentials due to the charges at $(0, a)$, $(0, -a)$, and $(a, 0)$ and then find E_x from $-dV/dx$.

(*a*) Express $V(x)$ as the sum of the potentials due to the charges at $(0, a)$, $(0, -a)$, and $(a, 0)$:

$$V(x) = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

where $q_1 = q_2 = q_3 = q$

At $x = 0$, the fields due to q_1 and q_2 cancel, so $E_x(0) = -kq/a^2$; this is also obtained from (*b*) if $x = 0$.

As $x\to\infty$, i.e., for $x >> a$, the three charges appear as a point charge $3q$, so $E_x = 3kq/x^2$; this is also the result one obtains from (*b*) for $x >> a$.

Substitute for the r_i to obtain:

$$V(x) = kq\left(\frac{1}{\sqrt{x^2+a^2}} + \frac{1}{\sqrt{x^2+a^2}} + \frac{1}{|x-a|}\right) = \boxed{kq\left(\frac{2}{\sqrt{x^2+a^2}} + \frac{1}{|x-a|}\right) \quad x \neq a}$$

(*b*) For $x > a$, $x - a > 0$ and:

$$|x-a| = x - a$$

Use $E_x = -dV/dx$ to find E_x:

$$E_x(x) = -\frac{d}{dx}\left[kq\left(\frac{2}{\sqrt{x^2+a^2}} + \frac{1}{x-a}\right)\right] = \boxed{\frac{2kqx}{\left(x^2+a^2\right)^{3/2}} + \frac{kq}{(x-a)^2} \quad x > a}$$

For $x < a$, $x - a < 0$ and:

$$|x-a| = -(x-a) = a - x$$

Use $E_x = -dV/dx$ to find E_x:

$$E_x(x) = -\frac{d}{dx}\left[kq\left(\frac{2}{\sqrt{x^2+a^2}} + \frac{1}{a-x}\right)\right] = \boxed{\frac{2kqx}{\left(x^2+a^2\right)^{3/2}} - \frac{kq}{(a-x)^2} \quad x < a}$$

Calculations of *V* for Continuous Charge Distributions

40 • A charge of +10.0 μC is uniformly distributed on a thin spherical shell of radius 12.0 cm. (Assume the potential is zero very far from all charges.) (*a*) What is the magnitude of the electric field just outside and just inside the shell? (*b*) What is the magnitude of the electric potential just outside and just inside the shell? (*c*) What is the electric potential at the center of the shell? (*d*) What is the magnitude of the electric field at the center of the shell?

Picture the Problem We can construct Gaussian surfaces just inside and just outside the spherical shell and apply Gauss's law to find the electric field at these locations. We can use the expressions for the electric potential inside and outside a spherical shell to find the potential at these locations.

(*a*) Apply Gauss's law to a spherical Gaussian surface of radius $r < 12.0$ cm:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0} = 0$$

because the charge resides on the outer surface of the spherical surface. Hence $\vec{E}_{r<12.0\,\text{cm}} = \boxed{0}$

Apply Gauss's law to a spherical Gaussian surface of radius $r > 12.0$ cm:

$$E\left(4\pi r^2\right) = \frac{q}{\epsilon_0}$$

and

$$E_{r>12.0\,\text{cm}} = \frac{q}{4\pi r^2 \epsilon_0} = \frac{kq}{r^2}$$

Substitute numerical values and evaluate $E_{r>12.0\,\text{cm}}$:

$$E_{r>12.0\,\text{cm}} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(10.0\,\mu\text{C})}{(0.120\,\text{m})^2} = \boxed{6.24\,\text{MV/m}}$$

(*b*) Express and evaluate the potential just inside the spherical shell:

$$V(r \le R) = \frac{kq}{R} = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(10.0\,\mu\text{C})}{0.120\,\text{m}} = \boxed{749\,\text{kV}}$$

Express and evaluate the potential just outside the spherical shell:

$$V(r \geq R) = \frac{kq}{r} = \frac{\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(10.0\ \mu\text{C})}{0.120\ \text{m}} = \boxed{749\ \text{kV}}$$

(*c*) The electric potential inside a uniformly charged spherical shell is constant and given by:

$$V(r \leq R) = \frac{kq}{R} = \frac{\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(10.0\ \mu\text{C})}{0.120\ \text{m}} = \boxed{749\ \text{kV}}$$

(*d*) In Part (*a*) we showed that: $\vec{E}_{r<12.0\,\text{cm}} = \boxed{0}$

41 • [SSM] An infinite line charge of linear charge density +1.50 μC/m lies on the *z*-axis. Find the electric potential at distances from the line charge of (*a*) 2.00 m, (*b*) 4.00 m, and (*c*) 12.0 m. Assume that we choose $V = 0$ at a distance of 2.50 m from the line of charge.

Picture the Problem We can use the expression for the potential due to a line charge $V(r) = -2k\lambda \ln\left(\frac{r}{a}\right)$, where $V = 0$ at some distance $r = a$, to find the potential at these distances from the line.

Express the potential due to a line charge as a function of the distance from the line:

$$V(r) = -2k\lambda \ln\left(\frac{r}{a}\right)$$

Because $V = 0$ at $r = 2.50$ m:

$$0 = -2k\lambda \ln\left(\frac{2.50\ \text{m}}{a}\right)$$

Equivalently:

$$0 = \ln\left(\frac{2.50\ \text{m}}{a}\right) \Rightarrow \frac{2.50\ \text{m}}{a} = \ln^{-1}(0) = 1$$

Solving for *a* gives:

$$a = 2.50\ \text{m}$$

Thus we have a = 2.50 m and:

$$V(r) = -2\left(8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.5\frac{\mu\text{C}}{\text{m}}\right)\ln\left(\frac{r}{2.50\,\text{m}}\right)$$
$$= -\left(2.696\times10^{4}\ \text{N}\cdot\text{m/C}\right)\ln\left(\frac{r}{2.50\,\text{m}}\right)$$

(a) Evaluate V(2.00 m):

$$V(2.00\,\text{m}) = -\left(2.696\times10^{4}\ \frac{\text{N}\cdot\text{m}}{\text{C}}\right)\ln\left(\frac{2.00\,\text{m}}{2.50\,\text{m}}\right) = \boxed{6.02\,\text{kV}}$$

(b) Evaluate V(4.00 m):

$$V(4.00\,\text{m}) = -\left(2.696\times10^{4}\ \frac{\text{N}\cdot\text{m}}{\text{C}}\right)\ln\left(\frac{4.00\,\text{m}}{2.50\,\text{m}}\right) = \boxed{-12.7\,\text{kV}}$$

(c) Evaluate V(12.0 m):

$$V(12.0\,\text{m}) = -\left(2.696\times10^{4}\ \frac{\text{N}\cdot\text{m}}{\text{C}}\right)\ln\left(\frac{12.0\,\text{m}}{2.50\,\text{m}}\right) = \boxed{-42.3\,\text{kV}}$$

42 • (a) Find the maximum net charge that can be placed on a spherical conductor of radius 16 cm before dielectric breakdown of the air occurs. (b) What is the electric potential of the sphere when it has this maximum charge? (Assume the potential is zero very far from all charges.)

Picture the Problem We can relate the dielectric strength of air (about 3.0 MV/m) to the maximum net charge that can be placed on a spherical conductor using the expression for the electric field at its surface. We can find the potential of the sphere when it carries its maximum charge using $V = kQ_{\text{max}}/R$.

(a) Express the dielectric strength of a spherical conductor in terms of the charge on the sphere:

$$E_{\text{breakdown}} = \frac{kQ_{\text{max}}}{R^2} \Rightarrow Q_{\text{max}} = \frac{E_{\text{breakdown}}R^2}{k}$$

Substitute numerical values and evaluate Q_{max}:

$$Q_{max} = \frac{(3.0\,\text{MV/m})(0.16\,\text{m})^2}{8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = 8.545\,\mu\text{C}$$

$$\approx \boxed{8.5\,\mu\text{C}}$$

(*b*) Because the charge carried by the sphere could be either positive or negative:

$$V_{max} = \pm\frac{kQ_{max}}{R}$$

$$= \pm\frac{\left(8.988\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(8.545\,\mu\text{C})}{0.16\,\text{m}}$$

$$\approx \boxed{\pm4.8\times10^5\text{ V}}$$

43 • Find the maximum surface charge density σ_{max} that can exist on the surface of any conductor before dielectric breakdown of the air occurs.

Picture the Problem We can solve the equation giving the electric field at the surface of a conductor for the greatest surface charge density that can exist before dielectric breakdown of the air occurs.

Relate the electric field at the surface of a conductor to the surface charge density:

$$E = \frac{\sigma}{\epsilon_0}$$

Solve for σ under dielectric breakdown of the air conditions:

$$\sigma_{max} = \epsilon_0\, E_{breaddown}$$

Substitute numerical values and evaluate σ_{max}:

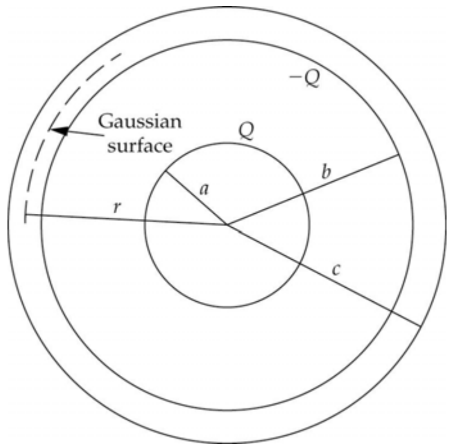
$$\sigma_{max} = \left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)(3\,\text{MV/m})$$

$$\approx \boxed{3\times10^{-5}\,\text{C/m}^2}$$

44 •• A conducting spherical shell of inner radius *b* and outer radius *c* is concentric with a small metal sphere of radius $a < b$. The metal sphere has a positive charge *Q*. The total charge on the conducting spherical shell is –*Q*. (Assume the potential is zero very far from all charges.) (*a*) What is the electric potential of the spherical shell? (*b*) What is the electric potential of the metal sphere?

Picture the Problem The diagram is a cross-sectional view showing the charges on the sphere and the spherical conducting shell. A portion of the Gaussian surface over which we'll integrate *E* in order to find *V* in the region $r > b$ is also shown. For $a < r < b$, the sphere acts like point charge *Q* and the potential of the metal sphere is the sum of the potential due to a point charge at its center and the

potential at its surface due to the charge on the inner surface of the spherical shell.



(*a*) Express $V_{r>b}$:

$$V_{r>b} = -\int E_{r>b}dr$$

Apply Gauss's law for $r > b$:

$$\oint_S \vec{\boldsymbol{E}}_r \cdot \hat{\boldsymbol{n}}dA = \frac{Q_{\text{enclosed}}}{\epsilon_0} = 0$$

and $E_{r>b} = 0$ because $Q_{\text{enclosed}} = 0$ for $r > b$.

Substitute for $E_{r>b}$ to obtain:

$$V_{r>b} = -\int (0)dr = \boxed{0}$$

(*b*) Express the potential of the metal sphere:

$$V_a = V_{Q\text{ at its center}} + V_{\text{surface}}$$

Express the potential at the surface of the metal sphere:

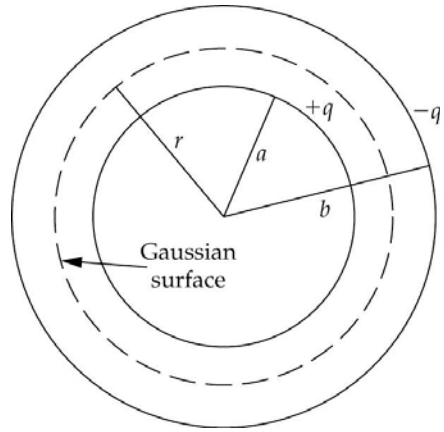
$$V_{\text{surface}} = \frac{k(-Q)}{b} = -\frac{kQ}{b}$$

Substitute and simplify to obtain:

$$V_a = \frac{kQ}{a} - \frac{kQ}{b} = \boxed{kQ\left(\frac{1}{a} - \frac{1}{b}\right)}$$

45 •• [SSM] Two coaxial conducting cylindrical shells have equal and opposite charges. The inner shell has charge $+q$ and an outer radius a, and the outer shell has charge $-q$ and an inner radius b. The length of each cylindrical shell is L, and L is very long compared with b. Find the potential difference, $V_a - V_b$ between the shells.

Picture the Problem The diagram is a cross-sectional view showing the charges on the inner and outer conducting shells. A portion of the Gaussian surface over which we'll integrate E in order to find V in the region $a < r < b$ is also shown. Once we've determined how E varies with r, we can find $V_a - V_b$ from $V_b - V_a = -\int E_r dr$.



Express the potential difference $V_b - V_a$:

$$V_b - V_a = -\int E_r dr \Rightarrow V_a - V_b = \int E_r dr$$

Apply Gauss's law to cylindrical Gaussian surface of radius r and length L:

$$\oint_S \vec{E} \cdot \hat{n} dA = E_r (2\pi r L) = \frac{q}{\epsilon_0}$$

Solving for E_r yields:

$$E_r = \frac{q}{2\pi \epsilon_0 rL}$$

Substitute for E_r and integrate from $r = a$ to b:

$$V_a - V_b = \frac{q}{2\pi \epsilon_0 L} \int_a^b \frac{dr}{r}$$

$$= \boxed{\frac{2kq}{L} \ln\left(\frac{b}{a}\right)}$$

46 •• Positive charge is placed on two conducting spheres that are very far apart and connected by a long very-thin conducting wire. The radius of the smaller sphere is 5.00 cm and the radius of the larger sphere is 12.0 cm. The electric field strength at the surface of the larger sphere is 200 kV/m. Estimate the surface charge density on each sphere.

Picture the Problem Let L and S refer to the larger and smaller spheres, respectively. We can use the fact that both spheres are at the same potential to estimate the electric fields near their surfaces. Knowing the electric fields, we can use $\sigma = \epsilon_0 E$ to estimate the surface charge density of each sphere.

Express the electric fields at the surfaces of the two spheres:

$$E_S = \frac{kQ_S}{R_S^2} \text{ and } E_L = \frac{kQ_L}{R_L^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{E_S}{E_L}=\frac{\frac{kQ_S}{R_S^2}}{\frac{kQ_L}{R_L^2}}=\frac{Q_S R_L^2}{Q_L R_S^2}$$

Because the potentials are equal at the surfaces of the spheres:

$$\frac{kQ_L}{R_L}=\frac{kQ_S}{R_S} \text{ and } \frac{Q_S}{Q_L}=\frac{R_S}{R_L}$$

Substitute for $\frac{Q_S}{Q_L}$ to obtain:

$$\frac{E_S}{E_L}=\frac{R_S R_L^2}{R_L R_S^2}=\frac{R_L}{R_S}\Rightarrow E_S=\frac{R_L}{R_S}E_L$$

Substitute numerical values and evaluate E_S:

$$E_S=\frac{12.0\,\text{cm}}{5.00\,\text{cm}}(200\,\text{kV/m})=480\,\text{kV/m}$$

Use $\sigma=\epsilon_0 E$ to estimate the surface charge density of each sphere:

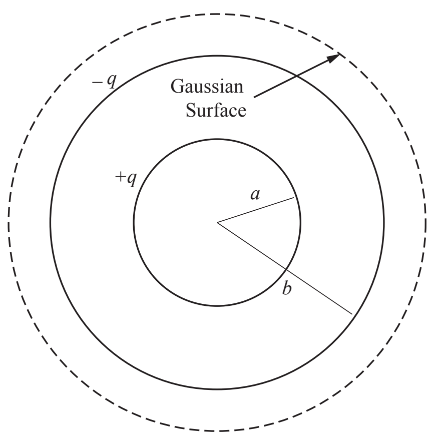
$$\sigma_{12\,\text{cm}}=\epsilon_0 E_{12\,\text{cm}}=\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(200\,\text{kV/m})=\boxed{1.77\,\mu\text{C/m}^2}$$

and

$$\sigma_{5\,\text{cm}}=\epsilon_0 E_{5\,\text{cm}}=\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(480\,\text{kV/m})=\boxed{4.25\,\mu\text{C/m}^2}$$

47 •• Two concentric conducting spherical shells have equal and opposite charges. The inner shell has outer radius a and charge $+q$; the outer shell has inner radius b and charge $-q$. Find the potential difference, $V_a - V_b$ between the shells.

Picture the Problem The diagram is a cross-sectional view showing the charges on the concentric spherical shells. The Gaussian surface over which we'll integrate E in order to find V in the region $r \geq b$ is also shown. We'll also find E in the region for which $a < r < b$. We can then use the relationship $V=-\int E dr$ to find V_a and V_b and their difference.



Express V_b:

$$V_b = -\int_{\infty}^{b} E_{r\geq b}\,dr$$

Apply Gauss's law for $r \geq b$:

$$\oint_S \vec{E}_r \cdot \hat{\mathbf{n}}\,dA = \frac{Q_{\text{enclosed}}}{\epsilon_0} = 0$$

and $E_{r\geq b} = 0$ because $Q_{\text{enclosed}} = 0$ for $r \geq b$.

Substitute for $E_{r\geq b}$ to obtain:

$$V_b = -\int_{\infty}^{b} (0)\,dr = 0$$

Express V_a:

$$V_a = -\int_{b}^{a} E_{r\geq a}\,dr$$

Apply Gauss's law for $r \geq a$:

$$E_{r\geq a}\left(4\pi r^2\right) = \frac{q}{\epsilon_0}$$

and

$$E_{r\geq a} = \frac{q}{4\pi \epsilon_0 r^2} = \frac{kq}{r^2}$$

Substitute for $E_{r\geq a}$ to obtain:

$$V_a = -kq\int_{b}^{a} \frac{dr}{r^2} = \frac{kq}{a} - \frac{kq}{b}$$

The potential difference between the shells is given by:

$$V_a - V_b = V_a = \boxed{kq\left(\frac{1}{a} - \frac{1}{b}\right)}$$

48 •• The electric potential at the surface of a uniformly charged sphere is 450 V. At a point outside the sphere at a (radial) distance of 20.0 cm from its surface, the electric potential is 150 V. (The potential is zero very far from the sphere.) What is the radius of the sphere, and what is the charge of the sphere?

Picture the Problem Let R be the radius of the sphere and Q its charge. We can express the potential at the two locations given and solve the resulting equations simultaneously for R and Q.

Relate the potential of the sphere at its surface to its radius:

$$\frac{kQ}{R} = 450\,\text{V} \qquad (1)$$

Express the potential at a distance of 20.0 cm from its surface:

$$\frac{kQ}{R+0.200\,\text{m}}=150\,\text{V} \qquad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{\dfrac{kQ}{R}}{\dfrac{kQ}{R+0.200\,\text{m}}}=\frac{450\,\text{V}}{150\,\text{V}}$$

or

$$\frac{R+0.200\,\text{m}}{R}=3\Rightarrow R=\boxed{10.0\,\text{cm}}$$

Solving equation (1) for Q yields:

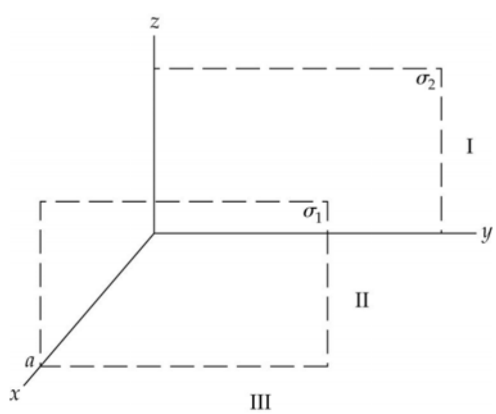
$$Q=(450\,\text{V})\frac{R}{k}$$

Substitute numerical values and evaluate Q:

$$Q=(450\,\text{V})\frac{(0.100\,\text{m})}{(8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2)}$$
$$=\boxed{5.01\,\text{nC}}$$

49 •• Consider two infinite parallel thin sheets of charge, one in the $x = 0$ plane and the other in the $x = a$ plane. The potential is zero at the origin. (*a*) Find the electric potential everywhere in space if the planes have equal positive charge densities $+\sigma$. (*b*) Find the electric potential everywhere in space if the sheet in the $x = 0$ plane has a charge density $+\sigma$ and the sheet in the $x = a$ plane has a charge density $-\sigma$.

Picture the Problem Let the charge density on the infinite plane at $x = a$ be σ_1 and that on the infinite plane at $x = 0$ be σ_2. Call that region in space for which $x < 0$, region I, the region for which $0 < x < a$, region II, and the region for which $a < x$, region III. We can integrate E due to the planes of charge to find the electric potential in each of these regions.



(*a*) Express the potential in region I in terms of the electric field in that region:

$$V_{\text{I}} = -\int_0^x \vec{E}_{\text{I}} \cdot d\vec{x}$$

Express the electric field in region I as the sum of the fields due to the charge densities σ_1 and σ_2:

$$\vec{E}_{\text{I}} = -\frac{\sigma_1}{2\in_0}\hat{i} - \frac{\sigma_2}{2\in_0}\hat{i} = -\frac{\sigma}{2\in_0}\hat{i} - \frac{\sigma}{2\in_0}\hat{i} = -\frac{\sigma}{\in_0}\hat{i}$$

Substitute for $\vec{E}_{\text{I}}$ and evaluate V_{I}:

$$V_{\text{I}} = -\int_0^x \left(-\frac{\sigma}{\in_0}\right)dx = \frac{\sigma}{\in_0}x + V(0) = \frac{\sigma}{\in_0}x + 0 = \boxed{\frac{\sigma}{\in_0}x}$$

Express the potential in region II in terms of the electric field in that region:

$$V_{\text{II}} = -\int \vec{E}_{\text{II}} \cdot d\vec{x} + V(0)$$

Express the electric field in region II as the sum of the fields due to the charge densities σ_1 and σ_2:

$$\vec{E}_{\text{II}} = -\frac{\sigma_1}{2\in_0}\hat{i} + \frac{\sigma_2}{2\in_0}\hat{i} = -\frac{\sigma}{2\in_0}\hat{i} + \frac{\sigma}{2\in_0}\hat{i} = 0$$

Substitute for $\vec{E}_{\text{II}}$ and evaluate V_{II}:

$$V_{\text{II}} = -\int_0^x (0)dx = 0 + V(0) = \boxed{0}$$

Express the potential in region III in terms of the electric field in that region:

$$V_{\text{III}} = -\int_a^x \vec{E}_{\text{III}} \cdot d\vec{x}$$

Express the electric field in region III as the sum of the fields due to the charge densities σ_1 and σ_2:

$$\vec{E}_{\text{III}} = \frac{\sigma_1}{2\in_0}\hat{i} + \frac{\sigma_2}{2\in_0}\hat{i} = \frac{\sigma}{2\in_0}\hat{i} + \frac{\sigma}{2\in_0}\hat{i} = \frac{\sigma}{\in_0}\hat{i}$$

Substitute for $\vec{E}_{\text{III}}$ and evaluate V_{III}:

$$V_{\text{III}} = -\int_a^x \left(\frac{\sigma}{\epsilon_0}\right)dx = -\frac{\sigma}{\epsilon_0}x + \frac{\sigma}{\epsilon_0}a = \boxed{-\frac{\sigma}{\epsilon_0}(x-a)}$$

(*b*) Proceed as in (*a*) with $\sigma_1 = -\sigma$ and $\sigma_2 = \sigma$ to obtain:

$$V_{\text{I}} = \boxed{0},\ V_{\text{II}} = \boxed{-\frac{\sigma}{\epsilon_0}x}, \text{ and}$$

$$V_{\text{III}} = \boxed{0}$$

These results are summarized in the following table:

Region	$x \le 0$	$0 \le x \le a$	$x \ge a$
Part (*a*)	$\frac{\sigma}{\epsilon_0}x$	0	$-\frac{\sigma}{\epsilon_0}(x-a)$
Part (*b*)	0	$-\frac{\sigma}{\epsilon_0}x$	0

50 ••• The expression for the potential along the axis of a thin uniformly charge disk is given by $V = 2\pi k\sigma|z|\left(\sqrt{1+\frac{R^2}{z^2}}-1\right)$ (Equation 23-20), where R and σ are the radius and the charge per unit area of the disk, respectively. Show that this expression reduces to $V = kQ/|z|$ for $|z| >> R$, where $Q = \sigma\pi R^2$ is the total charge on the disk. Explain why this result is expected. *Hint: Use the binomial theorem to expand the radical.*

Picture the Problem

Expand the radical expression binomially to obtain:

$$\sqrt{1+\frac{R^2}{z^2}} = 1 + \left(\frac{1}{2}\right)\left(\frac{R^2}{z^2}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{R^2}{z^2}\right)^2 + \text{higher order terms}$$

For $|z| >> R$:

$$\sqrt{1+\frac{R^2}{z^2}} \approx 1 + \left(\frac{1}{2}\right)\left(\frac{R^2}{z^2}\right)$$

and

$$\sqrt{1+\frac{R^2}{z^2}} - 1 \approx \frac{R^2}{2z^2}$$

Substituting in Equation 23-20 yields:

$$V = 2\pi k\sigma|z|\left(\frac{R^2}{2z^2}\right) = \pi k\sigma|z|\left(\frac{R^2}{z^2}\right)$$

The total charge on the disk is given by:

$$Q = \sigma\pi R^2 \Rightarrow \sigma = \frac{Q}{\pi R^2}$$

Substitute for σ and simplify to obtain:

$$V = \pi k\left(\frac{Q}{\pi R^2}\right)|z|\left(\frac{R^2}{z^2}\right) = \frac{k|z|Q}{z^2}$$

If $z \geq 0$, then $|z| = z$ and:

$$V = \frac{kzQ}{z^2} = \frac{kQ}{z} = \frac{kQ}{|z|}$$

If $z < 0$, then $|z| = -z$ and:

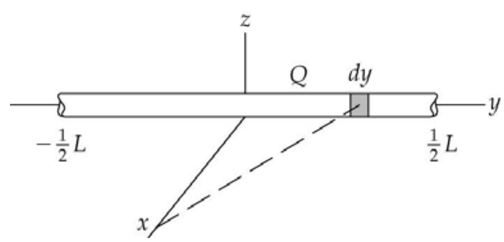
$$V = \frac{k(-z)Q}{z^2} = \frac{kQ}{-z} = \frac{kQ}{|z|}$$

Thus, for $|z| >> R$, Equation 23-20 reduces to:

$$V = \boxed{\frac{kQ}{|z|}}$$

51 •• [SSM] A rod of length L has a total charge Q uniformly distributed along its length. The rod lies along the y-axis with its center at the origin. (*a*) Find an expression for the electric potential as a function of position along the x-axis. (*b*) Show that the result obtained in Part (*a*) reduces to $V = kQ/|x|$ for $|x| >> L$. Explain why this result is expected.

Picture the Problem Let the charge per unit length be $\lambda = Q/L$ and dy be a line element with charge λdy. We can express the potential dV at any point on the x axis due to the charge element λdy and integrate to find $V(x, 0)$.



(*a*) Express the element of potential dV due to the line element dy:

$$dV = \frac{k\lambda}{r}dy$$

where $r = \sqrt{x^2 + y^2}$ and $\lambda = \frac{Q}{L}$.

Substituting for r and λ yields:

$$dV = \frac{kQ}{L}\frac{dy}{\sqrt{x^2 + y^2}}$$

Use a table of integrals to integrate dV from $y = -L/2$ to $y = L/2$:

$$V(x,0) = \frac{kQ}{L}\int_{-L/2}^{L/2}\frac{dy}{\sqrt{x^2+y^2}}$$

$$= \boxed{\frac{kQ}{L}\ln\left(\frac{\sqrt{x^2+\frac{1}{4}L^2}+\frac{1}{2}L}{\sqrt{x^2+\frac{1}{4}L^2}-\frac{1}{4}L^2}\right)}$$

(*b*) Factor x from the numerator and denominator within the parentheses to obtain:

$$V(x,0) = \frac{kQ}{L}\ln\left(\frac{\sqrt{1+\frac{L^2}{4x^2}}+\frac{L}{2x}}{\sqrt{1+\frac{L^2}{4x^2}}-\frac{L}{2x}}\right)$$

Use $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ to obtain:

$$V(x,0) = \frac{kQ}{L}\left\{\ln\left(\sqrt{1+\frac{L^2}{4x^2}}+\frac{L}{2x}\right) - \ln\left(\sqrt{1+\frac{L^2}{4x^2}}-\frac{L}{2x}\right)\right\}$$

Let $\varepsilon = \frac{L^2}{4x^2}$ and use $(1+\varepsilon)^{1/2} = 1+\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2+...$ to expand $\sqrt{1+\frac{L^2}{4x^2}}$:

$$\left(1+\frac{L^2}{4x^2}\right)^{1/2} = 1+\frac{1}{2}\frac{L^2}{4x^2}-\frac{1}{8}\left(\frac{L^2}{4x^2}\right)^2+... \approx 1 \text{ for } |x| >> L.$$

Substitute for $\left(1+\frac{L^2}{4x^2}\right)^{1/2}$ to obtain:

$$V(x,0) = \frac{kQ}{L}\left\{\ln\left(1+\frac{L}{2x}\right) - \ln\left(1-\frac{L}{2x}\right)\right\}$$

Let $\delta = \frac{L}{2x}$ and use $\ln(1+\delta) = \delta - \frac{1}{2}\delta^2+...$ to expand $\ln\left(1\pm\frac{L}{2x}\right)$:

$$\ln\left(1+\frac{L}{2x}\right) \approx \frac{L}{2x}-\frac{L^2}{4x^2} \text{ and } \ln\left(1-\frac{L}{2x}\right) \approx -\frac{L}{2x}-\frac{L^2}{4x^2} \text{ for } x >> L.$$

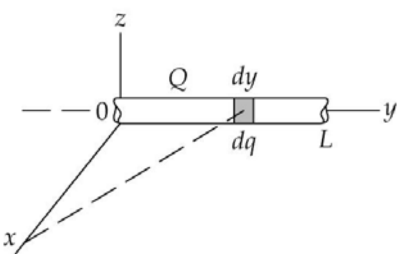
Substitute for $\ln\left(1+\frac{L}{2x}\right)$ and $\ln\left(1-\frac{L}{2x}\right)$ and simplify to obtain:

$$V(x,0)=\frac{kQ}{L}\left\{\frac{L}{2x}-\frac{L^2}{4x^2}-\left(-\frac{L}{2x}-\frac{L^2}{4x^2}\right)\right\}=\boxed{\frac{kQ}{x}}$$

Because, for $|x|>>L$, the charge carried by the rod is far enough away from the point of interest to look like a point charge, this result is what we would expect.

52 •• A rod of length L has a charge Q uniformly distributed along its length. The rod lies along the y-axis with one end at the origin. (*a*) Find an expression for the electric potential as a function of position along the x-axis. (*b*) Show that the result obtained in Part (*a*) reduces to $V=kQ/|x|$ for $|x|>>L$. Explain why this result is expected.

Picture the Problem Let the charge per unit length be $\lambda = Q/L$ and dy be a line element with charge λdy. We can express the potential dV at any point on the x axis due to λdy and integrate to find $V(x, 0)$.



(*a*) Express the element of potential dV due to the line element dy:

$$dV=\frac{k\lambda}{r}dy$$

where $r=\sqrt{x^2+y^2}$ and $\lambda=\frac{Q}{L}$.

Substituting for r and λ yields:

$$dV=\frac{kQ}{L}\frac{dy}{\sqrt{x^2+y^2}}$$

Use a table of integrals to integrate dV from $y = 0$ to $y = L$:

$$V(x,0)=\frac{kQ}{L}\int_0^L\frac{dy}{\sqrt{x^2+y^2}}$$
$$=\frac{kQ}{L}\ln\left(y+\sqrt{x^2+y^2}\right)_0^L$$
$$=\frac{kQ}{L}\left[\ln\left(L+\sqrt{x^2+L^2}\right)-\ln(x)\right]$$

Because $\ln a-\ln b=\ln\left(\frac{a}{b}\right)$:

$$V(x,0)=\boxed{\frac{kQ}{L}\left[\ln\left(\frac{L+\sqrt{x^2+L^2}}{x}\right)\right]} \quad (1)$$

(b) Factor x^2 under the radical to obtain:

$$\ln\left(\frac{L+\sqrt{x^2+L^2}}{x}\right)=\ln\left(\frac{L+\sqrt{x^2\left(1+\left(\frac{L}{x}\right)^2\right)}}{x}\right)=\ln\left(\frac{L+\sqrt{x^2}\sqrt{\left(1+\left(\frac{L}{x}\right)^2\right)}}{x}\right)$$

Because $|x|>>L$:

$$\ln\left(\frac{L+\sqrt{x^2+L^2}}{x}\right)\approx\ln\left(\frac{L+\sqrt{x^2}}{x}\right)$$
$$=\ln\left(1+\frac{L}{x}\right)$$

Expanding $\ln\left(1+\frac{L}{x}\right)$ binomially yields:

$$\ln\left(1+\frac{L}{x}\right)=\frac{L}{x}-\frac{1}{2}\left(\frac{L}{x}\right)^2$$
$$+\text{higher order terms}$$

Again, because $|x|>>L$:

$$\ln\left(1+\frac{L}{x}\right)\approx\frac{L}{x}$$

Substitute in equation (2) to obtain:

$$\ln\left(\frac{L+\sqrt{x^2+L^2}}{x}\right)\approx\frac{L}{x}$$

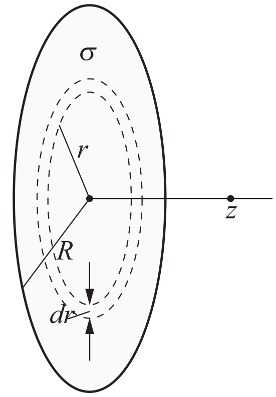
Finally, substituting in equation (1) yields:

$$V(x,0)=\frac{kQ}{L}\left[\frac{L}{x}\right]=\boxed{\frac{kQ}{x}}$$

Because, for $|x|>>L$, the charge carried by the rod is far enough away from the point of interest to look like a point charge, this result is what we would expect.

53 •• **[SSM]** A disk of radius R has a surface charge distribution given by $\sigma=\sigma_0 r^2/R^2$ where σ_0 is a constant and r is the distance from the center of the disk. (a) Find the total charge on the disk. (b) Find an expression for the electric potential at a distance z from the center of the disk on the axis that passes through the disk′s center and is perpendicular to its plane.

Picture the Problem We can find Q by integrating the charge on a ring of radius r and thickness dr from $r = 0$ to $r = R$ and the potential on the axis of the disk by integrating the expression for the potential on the axis of a ring of charge between the same limits.



(a) Express the charge dq on a ring of radius r and thickness dr:

$$dq = 2\pi r\sigma dr = 2\pi r\left(\sigma_0 \frac{r^2}{R^2}\right)dr$$

$$= \frac{2\pi\sigma_0}{R^2} r^3 dr$$

Integrate from $r = 0$ to $r = R$ to obtain:

$$Q = \frac{2\pi\sigma_0}{R^2}\int_0^R r^3 dr = \boxed{\tfrac{1}{2}\pi\sigma_0 R^2}$$

(b)Express the potential on the axis of the disk due to a circular element of charge $dq = \frac{2\pi\sigma_0}{R^2} r^3 dr$:

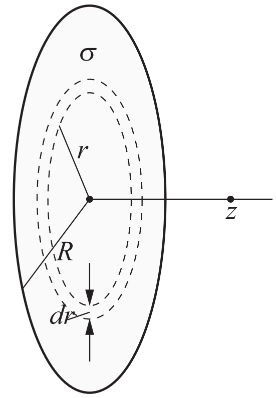
$$dV = \frac{kdq}{r'} = \frac{2\pi k\sigma_0}{R^2}\frac{r^3}{\sqrt{x^2 + r^2}}dr$$

Integrate from $r = 0$ to $r = R$ to obtain:

$$V = \frac{2\pi k\sigma_0}{R^2}\int_0^R \frac{r^3 dr}{\sqrt{x^2 + r^2}} = \boxed{\frac{2\pi k\sigma_0}{3R^2}\left[\left(R^2 - 2x^2\right)\sqrt{x^2 + R^2} + 2x^3\right]}$$

54 ••• A disk of radius R has a surface charge distribution given by $\sigma = \sigma_0 R/r$ where σ_0 is a constant and r is the distance from the center of the disk. (a) Find the total charge on the disk. (b) Find an expression for the electric potential at a distance x from the center of the disk on the axis that passes through the disk′s center and is perpendicular to its plane.

Picture the Problem We can find Q by integrating the charge on a ring of radius r and thickness dr from $r = 0$ to $r = R$ and the potential on the axis of the disk by integrating the expression for the potential on the axis of a ring of charge between the same limits.



(*a*) Express the charge dq on a ring of radius r and thickness dr:

$$dq = 2\pi r\sigma dr = 2\pi r\left(\sigma_0\frac{R}{r}\right)dr$$
$$= 2\pi\sigma_0 R dr$$

Integrate from $r = 0$ to $r = R$ to obtain:

$$Q = 2\pi\sigma_0 R\int_0^R dr = \boxed{2\pi\sigma_0 R^2}$$

(*b*) Express the potential on the axis of the disk due to a circular element of charge $dq = 2\pi r\sigma dr$:

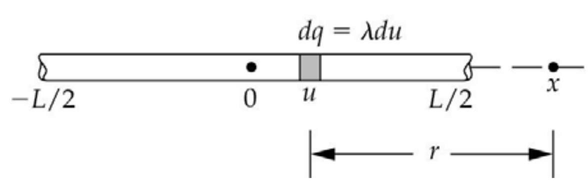
$$dV = \frac{kdq}{r'} = \frac{2\pi k\sigma_0 R dr}{\sqrt{x^2 + r^2}}$$

Integrate from $r = 0$ to $r = R$ to obtain:

$$V = 2\pi k\sigma_0 R\int_0^R \frac{dr}{\sqrt{x^2 + r^2}}$$
$$= \boxed{2\pi k\sigma_0 R\ln\left(\frac{R + \sqrt{x^2 + R^2}}{x}\right)}$$

55 •• A rod of length L has a total charge Q uniformly distributed along its length. The rod lies along the x-axis with its center at the origin. (*a*) What is the electric potential as a function of position along the x-axis for $x > L/2$? (*b*) Show that for $x >> L/2$, your result reduces to that due to a point charge Q.

Picture the Problem We can express the electric potential dV at x due to an elemental charge dq on the rod and then integrate over the length of the rod to find $V(x)$. In the second part of the problem we use a binomial expansion to show that, for $x >> L/2$, our result reduces to that due to a point charge Q.



(a) Express the potential at x due to the element of charge dq located at u:

$$dV = \frac{kdq}{r} = \frac{k\lambda du}{x-u}$$

or, because $\lambda = Q/L$,

$$dV = \frac{kQ}{L}\frac{du}{x-u}$$

Integrate V from $u = -L/2$ to $u = L/2$ and simplify to obtain:

$$\begin{aligned} V(x) &= \frac{kQ}{L}\int_{-L/2}^{L/2}\frac{du}{x-u} \\ &= -\frac{kQ}{L}\ln(x-u)\Big|_{-L/2}^{L/2} \\ &= \frac{kQ}{L}\left[-\ln\left(x-\tfrac{1}{2}L\right)+\ln\left(x+\tfrac{1}{2}L\right)\right] \\ &= \boxed{\frac{kQ}{L}\ln\left(\frac{x+\frac{1}{2}L}{x-\frac{1}{2}L}\right)} \end{aligned}$$

(b) From Part (a):

$$\begin{aligned} V(x) &= \frac{kQ}{L}\left[\ln\left(x+\tfrac{1}{2}L\right)-\ln\left(x-\tfrac{1}{2}L\right)\right] \\ &= \frac{kQ}{L}\left[\ln x\left(1+\frac{L}{2x}\right)-\ln x\left(1-\frac{L}{2x}\right)\right] \end{aligned}$$

Expanding the expression in the square brackets and simplifying gives:

$$V(x) = \frac{kQ}{L}\left[\ln x+\ln\left(1+\frac{L}{2x}\right)-\ln x+\ln\left(1-\frac{L}{2x}\right)\right] = \frac{kQ}{L}\left[\ln\left(1+\frac{L}{2x}\right)+\ln\left(1-\frac{L}{2x}\right)\right]$$

For $\dfrac{L}{2x} << 1$:

$$\ln\left(1\pm\frac{L}{2x}\right) \approx \pm\frac{L}{2x}$$

Substitute for $\ln\left(1\pm\dfrac{L}{2x}\right)$ to obtain:

$$V(x) \approx \frac{kQ}{L}\left[\frac{L}{2x}+\frac{L}{2x}\right] = \boxed{\frac{kQ}{x}},$$

the potential due to a point charge Q.

56 ••• A circle of radius a is removed from the center of a uniformly charged thin circular disk of radius b and charge per unit area σ. (a) Find an expression for the potential on the x axis a distance x from the center of the disk. (b) Show that for $x >> b$ the electric potential on the axis of the uniformly charged disk with cutout approaches kQ/x, where $Q = \sigma\pi(b^2 - a^2)$ is the total charge on the disk.

Picture the Problem The potential on the axis of the uniformly charged disk is the sum of the potential V_b due to the disk of radius *b* and the potential V_a due to the disk of radius *a* that has been removed. We can think of the charged disk that has been removed as having a negative charge density $-\sigma$. Note that if $x >> b$, then it is also true that $x >> a$.

(*a*)The potential on the axis of the circular disk is:

$$V(x) = V_b(x) + V_a(x)$$

where

$$V_b(x) = 2\pi k\sigma\left[\left(x^2 + b^2\right)^{1/2} - x\right]$$

and

$$V_a(x) = 2\pi k(-\sigma)\left[\left(x^2 + a^2\right)^{1/2} - x\right]$$

Substitute for $V_b(x)$ and $V_a(x)$ and simplify to obtain:

$$\begin{aligned} V(x) &= 2\pi k\sigma\left[\left(x^2 + b^2\right)^{1/2} - x\right] + 2\pi k(-\sigma)\left[\left(x^2 + a^2\right)^{1/2} - x\right] \\ &= 2\pi k\sigma\left[\left(x^2 + b^2\right)^{1/2} - \left(x^2 + a^2\right)^{1/2}\right] \\ &= \boxed{2\pi k\sigma\left(\sqrt{x^2 + b^2} - \sqrt{x^2 + a^2}\right)} \end{aligned}$$

(*b*) Expanding $\left(1 + \frac{b^2}{x^2}\right)^{1/2}$ binomially yields:

$$\left(1 + \frac{b^2}{x^2}\right)^{1/2} = 1 + \frac{b^2}{2x^2} + \text{higher order terms} \approx 1 + \frac{b^2}{2x^2}$$

Expanding $\left(1 + \frac{a^2}{x^2}\right)^{1/2}$ binomially yields:

$$\left(1 + \frac{a^2}{x^2}\right)^{1/2} \approx 1 + \frac{a^2}{2x^2}$$

Substituting in the expression for $V(x)$ and simplifying yields:

$$V(x) \approx 2\pi k\sigma x\left[1 + \frac{b^2}{2x^2} - \left(1 + \frac{a^2}{2x^2}\right)\right] = 2\pi k\sigma x\left[\frac{b^2}{2x^2} - \frac{a^2}{2x^2}\right] = \frac{\pi k\sigma\left(b^2 - a^2\right)}{x}$$

The total charge on the disk is:

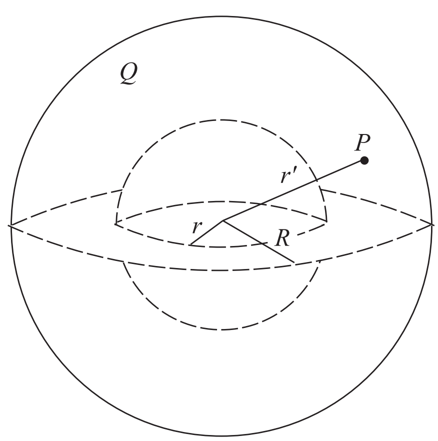
$$Q = \sigma\pi\left(b^2 - a^2\right) \Rightarrow \sigma = \frac{Q}{\pi\left(b^2 - a^2\right)}$$

Substituting for σ yields:

$$V(x) \approx \frac{\pi k\left(\dfrac{Q}{\pi\left(b^2 - a^2\right)}\right)\left(b^2 - a^2\right)}{x} = \boxed{\frac{kQ}{x}}$$

57 ••• The expression for the electric potential inside a uniformly charged solid sphere is given by $V(r) = \dfrac{kQ}{2R}\left(3 - \dfrac{r^2}{R^2}\right)$, where R is the radius of the sphere and r is the distance from the center. This expression was obtained in Example 23-12 by first finding the electric field. In this problem, you derive the same expression by modeling the sphere as a nested collection of thin spherical shells, and then adding the potentials of these shells at a field point inside the sphere. The potential dV that is a distance r' from the center of a uniformly charged thin spherical shell that has a radius r' and a charge dQ is given by $dV = kdQ/r'$ for $r' \geq r$ and $dV = kdQ/r$ for $r' \leq r$ (Equation 23-22). Consider a sphere of radius R containing a charge Q that is uniformly distributed and you want to find V at some point inside the sphere (that is for $r < R$). (*a*) Find an expression for the charge dQ on a spherical shell of radius r' and thickness dr'. (*b*) Find an expression for the potential dV at r due to the charge in a shell of radius r' and thickness dr', where $r \leq r' \leq R$. (*c*) Integrate your expression in Part (*b*) from $r' = r$ to $r' = R$ to find the potential at r due to all the charge in the region farther from r than the center of the sphere. (*d*) Find an expression for the potential dV at r due to the charge in a shell of radius r' and thickness dr', where $r' \leq r$. (*e*) Integrate your expression in Part (*d*) from $r' = 0$ to $r' = r$ to find the potential at r due to all the charge in the region closer than r to the center of the sphere. (*f*) Find the total potential V at r by adding your Part (*c*) and Part (*e*) results.

Picture the Problem The diagram shows a uniformly charged sphere of radius R and the field point P at which we wish to find the total potential. We can use the definition of charge density to find the charge inside a sphere of radius r and the potential at r due to this charge. We can express the potential at r due to the charge in a shell of radius r' and thickness dr' at $r' \geq r$ using $dV = kdq'/r$ and then integrate this expression from $r' = r$ to $r' = R$ to find V.



(*a*) The charge dQ in a shell of radius r' and thickness dr' at $r' > r$ is given by:

$$dQ = \rho dV' = \rho A' dr'$$

Because $A' = 4\pi r'^2$:

$$dQ = 4\pi r'^2 \rho dr'$$

Because the sphere is uniformly charged:

$$\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$$

Substituting for ρ and simplifying yields:

$$dQ = 4\pi r'^2 \left(\frac{3Q}{4\pi R^3}\right) dr' = \boxed{\frac{3Q}{R^3} r'^2 dr'}$$

(*b*) Express the potential dV in the interval $r \le r' \le R$ due to dQ:

$$dV = \frac{kdQ}{r'}$$

Substituting for dQ and simplifying yields:

$$dV = \frac{k}{r'}\left(\frac{3Q}{R^3} r'^2 dr'\right) = \boxed{\frac{3kQ}{R^3} r' dr'}$$

(*c*) Integrate dV from $r' = r$ to $r' = R$ to find V:

$$V = \frac{3kQ}{R^3}\int_r^R r' dr' = \boxed{\frac{3kQ}{2R^3}\left(R^2 - r^2\right)}$$

(*d*) The potential dV at r due to the charge in a shell of radius r' and thickness dr', where $r' \le r$, is given by:

$$dV = \frac{kdQ}{r} = \frac{k\rho dV'}{r} = \frac{k\rho A' dr'}{r}$$
$$= \frac{k\rho\left(4\pi r'^2\right)dr'}{r} = \frac{4\pi k\rho}{r} r'^2 dr'$$

Substituting for ρ and simplifying yields:

$$dV = \frac{4\pi k}{r}\left(\frac{Q}{\frac{4}{3}\pi R^3}\right) r'^2 dr'$$
$$= \boxed{\left(\frac{3kQ}{R^3 r}\right) r'^2 dr'}$$

(*e*) Integrate from $r' = 0$ to $r' = r$ to find the potential at r due to all the charge in the region closer than r to the center of the sphere:

$$V = \left(\frac{3kQ}{R^3 r}\right)\int_0^r r'^2 dr' = \boxed{\frac{kQ}{R^3} r^2}$$

(*f*) The sum of our results from Part (*c*) and Part (*e*) is:

$$V = \frac{kQ}{R^3}r^2 + \frac{3kQ}{2R^3}\left(R^2 - r^2\right) = \boxed{\frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)}$$

58 •• Calculate the electric potential at the point a distance $R/2$ from the center of a uniformly charged thin spherical shell of radius R and charge Q. (Assume the potential is zero far from the shell.)

Picture the Problem We can find the potential relative to infinity at a distance $R/2$ from the center of the spherical shell by integrating the electric field for 0 to ∞. We can apply Gauss's law to find the electric field both inside and outside the spherical shell.

The potential relative to infinity at a distance $R/2$ from the center of the spherical shell is given by:

$$V = -\int_{\infty}^{R/2} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{r}} = -\int_{\infty}^{R} E_{r>R}\,dr - \int_{R}^{R/2} E_{r<R}\,dr$$

Because there is no charge inside the spherical shell, $E_{r<R} = 0$ and:

$$\int_{R}^{R/2} E_{r<R}\,dr = 0 \text{ and } V = -\int_{\infty}^{R} E_{r>R}\,dr \quad (1)$$

Apply Gauss's law to a spherical surface of radius $r > R$ to obtain:

$$\oint_S E_n\,dA = E_{r>R}\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solving for $E_{r>R}$ yields:

$$E_{r>R} = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{kQ}{r^2}$$

Substitute for $E_{r>R}$ in equation (1) to obtain:

$$V = -kQ\int_{\infty}^{R} \frac{dr}{r^2}$$

Evaluating this integral yields:

$$V = -kQ\left[-\frac{1}{r}\right]_{\infty}^{R} = \boxed{\frac{kQ}{R}}$$

59 •• [SSM] A circle of radius a is removed from the center of a uniformly charged thin circular disk of radius b. Show that the potential at a point on the central axis of the disk a distance z from its geometrical center is given by $V(z) = 2\pi k\sigma\left(\sqrt{z^2 + b^2} - \sqrt{z^2 + a^2}\right)$, where σ is the charge density of the disk.

Picture the Problem We can find the electrostatic potential of the conducting washer by treating it as two disks with equal but opposite charge densities.

The electric potential due to a charged disk of radius R is given by:

$$V(x) = 2\pi k\sigma|x|\left(\sqrt{1+\frac{R^2}{x^2}}-1\right)$$

Superimpose the electrostatic potentials of the two disks with opposite charge densities and simplify to obtain:

$$\begin{aligned}
V(x) &= 2\pi k\sigma|x|\left(\sqrt{1+\frac{b^2}{x^2}}-1\right) - 2\pi k\sigma|x|\left(\sqrt{1+\frac{a^2}{x^2}}-1\right)\\
&= 2\pi k\sigma|x|\left(\left(\sqrt{1+\frac{b^2}{x^2}}-1\right)-\left(\sqrt{1+\frac{a^2}{x^2}}-1\right)\right)\\
&= 2\pi k\sigma|x|\left(\sqrt{1+\frac{b^2}{x^2}}-\sqrt{1+\frac{a^2}{x^2}}\right) = 2\pi k\sigma|x|\left(\sqrt{\frac{x^2+b^2}{x^2}}-\sqrt{\frac{x^2+a^2}{x^2}}\right)\\
&= 2\pi k\sigma|x|\left(\frac{\sqrt{x^2+b^2}}{\sqrt{x^2}}-\frac{\sqrt{x^2+a^2}}{\sqrt{x^2}}\right) = 2\pi k\sigma|x|\left(\frac{\sqrt{x^2+b^2}}{|x|}-\frac{\sqrt{x^2+a^2}}{|x|}\right)\\
&= 2\pi k\sigma\left(\sqrt{x^2+b^2}-\sqrt{x^2+a^2}\right)
\end{aligned}$$

The charge density σ is given by:

$$\sigma = \frac{Q}{\pi\left(b^2-a^2\right)}$$

Substituting for σ yields:

$$V(x) = \boxed{\frac{2kQ}{\left(b^2-a^2\right)}\left(\sqrt{x^2+b^2}-\sqrt{x^2+a^2}\right)}$$

Equipotential Surfaces

60 • An infinite flat sheet of charge has a uniform surface charge density equal to 3.50 μC/m^2. How far apart are the equipotential surfaces whose potentials differ by 100 V?

Picture the Problem We can equate the expression for the electric field due to an infinite plane of charge and $-\Delta V/\Delta x$ and solve the resulting equation for the separation of the equipotential surfaces.

Express the electric field due to the infinite plane of charge:

$$E = \frac{\sigma}{2\in_0}$$

Relate the electric field to the potential:

$$E = -\frac{\Delta V}{\Delta x}$$

Equate these expressions and solve for Δx to obtain:

$$\Delta x = \frac{2\in_0 \Delta V}{\sigma}$$

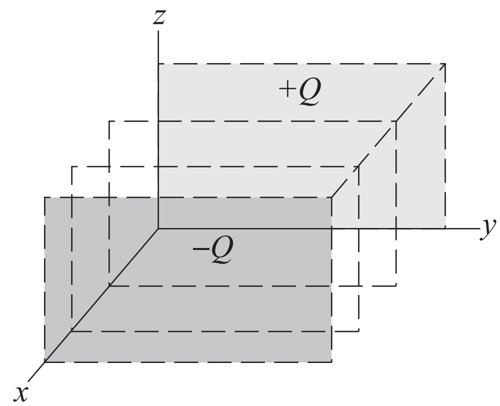
Substitute numerical values and evaluate $|\Delta x|$:

$$|\Delta \boldsymbol{x}| = \frac{2\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}\right)(100\,\mathrm{V})}{3.50\,\mu\mathrm{C/m^2}}$$

$$= \boxed{0.506\,\mathrm{mm}}$$

61 •• [SSM] Consider two parallel uniformly charged infinite planes that are equal but oppositely charged. (*a*) What is (are) the shape(s) of the equipotentials in the region between them? Explain your answer. (*b*) What is (are) the shape(s) of the equipotentials in the regions not between them? Explain your answer.

Picture the Problem The two parallel planes, with their opposite charges, are shown in the pictorial representation.



(*a*) Because the electric field between the charged plates is uniform and perpendicular to the plates, the equipotential surfaces are planes parallel to the charged planes.

(*b*) The regions to either side of the two charged planes are equipotential regions, so any surface in either of these regions is an equipotential surface.

62 •• A Geiger tube consists of two elements, a long metal cylindrical shell and a long straight metal wire running down its central axis. Model the tube as if both the wire and cylinder are infinitely long. The central wire is positively charged and the outer cylinder is negatively charge. The potential difference between the wire and the cylinder is 1.00 kV. (*a*) What is the direction of the electric field inside the tube? (*b*) Which element is at a higher electric potential?

(*c*) What is (are) the shape(s) of the equipotentials inside the tube? (*d*) Consider two equipotentials described in Part (*c*). Suppose they differ in electric potential by 10 V. Do two such equipotentials near the central wire have the same spacing as they would near the outer cylinder? If not, where in the tube are the equipotentials that are more widely spaced? Explain your answer.

Determine the Concept
(*a*) The direction of the electric field inside the tube is the direction of the force the electric field exerts on a positively charged object. Because the wire is positively charged and the tube is negatively charged, and because of the cylindrical geometry of the Geiger tube, the electric field is directed radially away from the central wire.

(*b*) Because it would require more work to bring a positively charged object from infinity to the surface of the wire than it would to bring this test object to the surface of the cylindrical tube, the central wire is at the higher electric potential.

(*c*) Because of the cylindrical geometry of the Geiger tube, the equipotential surfaces are cylinders concentric with the central wire.

(*d*) No. Because the magnitude of the electric field, which is the rate of change with distance (also known as the gradient) of the potential decreases with distance from the wire, the spacing between adjacent equipotential surfaces having the same potential difference between them decreases as you get farther from the central wire.

63 •• [SSM] Suppose the cylinder in the Geiger tube in Problem 62 has an inside diameter of 4.00 cm and the wire has a diameter of 0.500 mm. The cylinder is grounded so its potential is equal to zero. (*a*) What is the radius of the equipotential surface that has a potential equal to 500 V? Is this surface closer to the wire or to the cylinder? (*b*) How far apart are the equipotential surfaces that have potentials of 200 and 225 V? (*c*) Compare your result in Part (*b*) to the distance between the two surfaces that have potentials of 700 and 725 V respectively. What does this comparison tell you about the electric field strength as a function of the distance from the central wire?

Picture the Problem If we let the electric potential of the cylinder be zero, then the surface of the central wire is at +1000 V and we can use Equation 23-23 to find the electric potential at any point between the outer cylinder and the central wire.

(*a*) From Equation 23-23 we have:

$$V(r) = 2k\lambda \ln\left(\frac{R_{\text{ref}}}{r}\right) \qquad (1)$$

where R_{ref} is the radius of the outer cylinder and r is the distance from the center of the central wire and $r < R_{\text{ref}}$.

Solving for $2k\lambda$ yields:

$$2k\lambda = \frac{V(r)}{\ln\left(\frac{R_{\text{ref}}}{r}\right)}$$

At the surface of the wire, $V = 1000$ V and $r = 0.250$ mm. Hence:

$$2k\lambda = \frac{1000\text{ V}}{\ln\left(\frac{2.00\text{ cm}}{0.250\text{ mm}}\right)} = 228.2\text{ V}$$

and

$$V(r) = (228.2\text{ V})\ln\left(\frac{2.00\text{ cm}}{r}\right)$$

Setting $V = 500$ V yields:

$$500\text{ V} = (228.2\text{ V})\ln\left(\frac{2.00\text{ cm}}{r}\right)$$

or

$$\ln\left(\frac{2.00\text{ cm}}{r}\right) = 2.191 \Rightarrow \frac{2.00\text{ cm}}{r} = e^{2.191}$$

Solve for r to obtain:

$r = \dfrac{2.00\text{ cm}}{e^{2.191}} = \boxed{0.224\text{ cm}}$, closer to the wire.

(*b*) The separation of the equipotential surfaces that have potential values of 200 and 225 V is:

$$\Delta r = \left|r_{225\text{ V}} - r_{200\text{ V}}\right| \qquad (2)$$

Solving equation (1) for r yields:

$$r = R_{\text{ref}}e^{-\frac{V}{2k\lambda}} = (2.00\text{ cm})e^{-\frac{V}{228.2\text{ V}}}$$

Substitute for the radii in equation (2), simplify, and evaluate Δr to obtain:

$$\Delta r = \left|(2.00\text{ cm})e^{-\frac{225\text{ V}}{228.2\text{ V}}} - (2.00\text{ cm})e^{-\frac{200\text{ V}}{228.2\text{ V}}}\right| = (2.00\text{ cm})\left|e^{-\frac{225\text{ V}}{228.2\text{ V}}} - e^{-\frac{200\text{ V}}{228.2\text{ V}}}\right|$$

$$= \boxed{0.864\text{ mm}}$$

(*c*) The distance between the 700 V and the 725 V equipotentials is:

$$\Delta r = (2.00\text{ cm})\left| e^{-\frac{725\text{ V}}{228.2\text{ V}}} - e^{-\frac{700\text{ V}}{228.2\text{ V}}} \right|$$

$$= \boxed{0.0966\text{ mm}}$$

This closer spacing of these two equipotential surfaces was to be expected. Close to the central wire, two equipotential surfaces with the same difference in potential should be closer together to reflect the fact that the higher electric field strength is greater closer to the wire.

64 •• A point particle that has a charge of $+11.1$ nC is at the origin. (*a*) What is (are) the shapes of the equipotential surfaces in the region around this charge? (*b*) Assuming the potential to be zero at $r = \infty$, calculate the radii of the five surfaces that have potentials equal to 20.0 V, 40.0 V, 60.0 V, 80.0 V and 100.0 V, and sketch them to scale centered on the charge. (*c*) Are these surfaces equally spaced? Explain your answer. (*d*) Estimate the electric field strength between the 40.0-V and 60.0-V equipotential surfaces by dividing the difference between the two potentials by the difference between the two radii. Compare this estimate to the exact value at the location midway between these two surfaces.

Picture the Problem We can integrate the expression for the electric field due to a point charge to find an expression for the electric potential of the point particle.

(*a*) The equipotential surfaces are spheres centered on the charge.

(*b*) From the relationship between the electric potential due to the point charge and the electric field of the point charge we have:

$$\int_a^b dV = -\int_{r_a}^{r_b} \vec{\boldsymbol{E}} \cdot d\vec{\mathbf{r}} = -kQ\int_{r_a}^{r_b} r^{-2}\,dr$$

or

$$V_b - V_a = kQ\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Taking the potential to be zero at $r_a = \infty$ yields:

$$V_b - 0 = kQ\left(\frac{1}{r_b}\right) \Rightarrow V = \frac{kQ}{r} \Rightarrow r = \frac{kQ}{V}$$

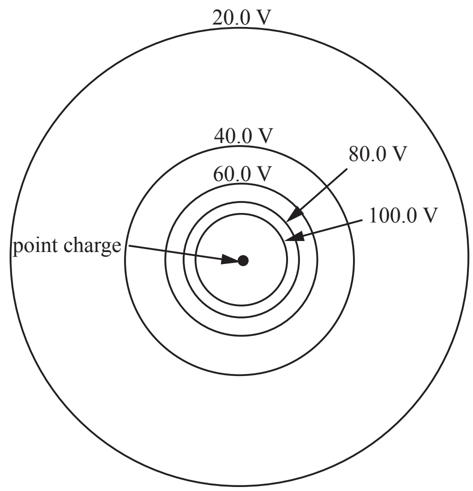
Because $Q = +1.11 \times 10^{-8}$ C:

$$r = \frac{\left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.11\times10^{-8}\text{ C}\right)}{V} = \frac{99.77\,\frac{\text{N}\cdot\text{m}^2}{\text{C}}}{V} \quad (1)$$

Use equation (1) to complete the following table:

V (V)	20.0	40.0	60.0	80.0	100.0
r (m)	4.99	2.49	1.66	1.25	1.00

The equipotential surfaces are shown in cross-section to the right:



(c) No. The equipotential surfaces are closest together where the electric field strength is greatest.

(d) The average value of the magnitude of the electric field between the 40.0-V and 60.0-V equipotential surfaces is given by:

$$E = -\frac{\Delta V}{\Delta r} = -\frac{40\text{ V} - 60\text{ V}}{\Delta r}$$

Drop perpendiculars to the r axis from 40.0 V and 60.0 V to approximate the radii corresponding to each of these potential surfaces:

$$E_{\text{est}} \approx -\frac{40\text{ V} - 60\text{ V}}{2.4\text{ m} - 1.7\text{ m}} = 29\frac{\text{V}}{\text{m}}$$

The exact value of the electric field at the location midway between these two surfaces is given by $E = kQ/r^2$, where r is the average of the radii of the 40.0-V and 60.0-V equipotential surfaces. Substitute numerical values and evaluate E_{exact}.

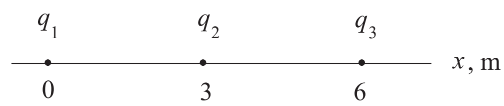
$$E_{\text{exact}} = \frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.11\times10^{-8}\text{ C}\right)}{\left(\frac{1.66\text{ m} + 2.49\text{ m}}{2}\right)^2} = 23\frac{\text{V}}{\text{m}}$$

The estimated value for E differs by about 21% from the exact value.

Electrostatic Potential Energy

65 • Three point charges are on the x-axis: q_1 is at the origin, q_2 is at $x = +3.00$ m, and q_3 is at $x = +6.00$ m. Find the electrostatic potential energy of this system of charges for the following charge values: (a) $q_1 = q_2 = q_3 = +2.00\ \mu$C; ($b$) $q_1 = q_2 = +2.00\ \mu$C and $q_3 = -2.00\ \mu$C; and (c) $q_1 = q_3 = +2.00\ \mu$C and $q_2 = -2.00\ \mu$C. (Assume the potential energy is zero when the charges are very far from each other.)

The electrostatic potential energy of this system of three point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram.



Express the work required to assemble this system of charges:

$$U = \frac{kq_1q_2}{r_{1,2}} + \frac{kq_1q_3}{r_{1,3}} + \frac{kq_2q_3}{r_{2,3}} = k\left(\frac{q_1q_2}{r_{1,2}} + \frac{q_1q_3}{r_{1,3}} + \frac{q_2q_3}{r_{2,3}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, and $r_{2,3}$:

$$r_{1,2} = 3\,\text{m}, r_{2,3} = 3\,\text{m}, \text{and}\, r_{1,3} = 6\,\text{m}$$

(a) Evaluate U for $q_1 = q_2 = q_3 = 2.00\ \mu$C:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(\frac{(2.00\,\mu\text{C})(2.00\,\mu\text{C})}{3.00\,\text{m}} + \frac{(2.00\,\mu\text{C})(2.00\,\mu\text{C})}{6.00\,\text{m}} + \frac{(2.00\,\mu\text{C})(2.00\,\mu\text{C})}{3.00\,\text{m}}\right)$$

$$= \boxed{30.0\,\text{mJ}}$$

(b) Evaluate U for $q_1 = q_2 = 2.00\ \mu$C and $q_3 = -2.00\ \mu$C:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(\frac{(2.00\,\mu\text{C})(2.00\,\mu\text{C})}{3.00\,\text{m}} + \frac{(2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{6.00\,\text{m}} + \frac{(2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{3.00\,\text{m}}\right)$$

$$= \boxed{-5.99\,\text{mJ}}$$

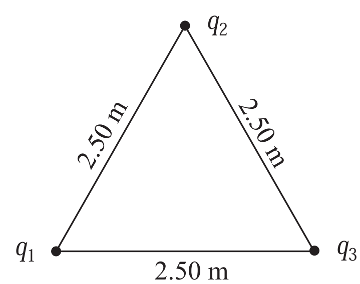
(c) Evaluate U for $q_1 = q_3 = 2.00\ \mu$C and $q_2 = -2.00\ \mu$C:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(\frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{3.00\,\text{m}} + \frac{(2.00\ \mu\text{C})(2.00\ \mu\text{C})}{6.00\,\text{m}} + \frac{(-2.00\ \mu\text{C})(2.00\ \mu\text{C})}{3.00\,\text{m}}\right)$$

$$= \boxed{-18.0\,\text{mJ}}$$

66 • Point charges q_1, q_2, and q_3 are fixed at the vertices of an equilateral triangle whose sides are 2.50 m-long. Find the electrostatic potential energy of this system of charges for the following charge values:
(a) $q_1 = q_2 = q_3 = +4.20\ \mu$C, ($b$) $q_1 = q_2 = +4.20\ \mu$C and $q_3 = -4.20\ \mu$C; and (c) $q_1 = q_2 = -4.20\ \mu$C and $q_3 = +4.20\ \mu$C. (Assume the potential energy is zero when the charges are very far from each other.)

Picture the Problem The electrostatic potential energy of this system of three point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram.



Express the work required to assemble this system of charges:

$$U = \frac{kq_1q_2}{r_{1,2}} + \frac{kq_1q_3}{r_{1,3}} + \frac{kq_2q_3}{r_{2,3}} = k\left(\frac{q_1q_2}{r_{1,2}} + \frac{q_1q_3}{r_{1,3}} + \frac{q_2q_3}{r_{2,3}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, and $r_{2,3}$:

$$r_{1,2} = r_{2,3} = r_{1,3} = 2.50\,\text{m}$$

(a) Evaluate U for $q_1 = q_2 = q_3 = 4.20\ \mu$C:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(4.20\ \mu\text{C})(4.20\ \mu\text{C})}{2.50\,\text{m}} + \frac{(4.20\ \mu\text{C})(4.20\ \mu\text{C})}{2.50\,\text{m}} + \frac{(4.20\ \mu\text{C})(4.20\ \mu\text{C})}{2.50\,\text{m}}\right]$$

$$= \boxed{190\,\text{mJ}}$$

(*b*) Evaluate U for $q_1 = q_2 = 4.20\ \mu\text{C}$ and $q_3 = -4.20\ \mu\text{C}$:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(4.20\,\mu\text{C})(4.20\,\mu\text{C})}{2.50\,\text{m}} + \frac{(4.20\,\mu\text{C})(-4.20\,\mu\text{C})}{2.50\,\text{m}} + \frac{(4.20\,\mu\text{C})(-4.20\,\mu\text{C})}{2.50\,\text{m}}\right]$$

$$= \boxed{-63.4\,\text{mJ}}$$

(*c*) Evaluate U for $q_1 = q_2 = -4.20\ \mu\text{C}$ and $q_3 = +4.20\ \mu\text{C}$:

$$U = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(-4.20\,\mu\text{C})(-4.20\,\mu\text{C})}{2.50\,\text{m}} + \frac{(-4.20\,\mu\text{C})(4.20\,\mu\text{C})}{2.50\,\text{m}} + \frac{(-4.20\,\mu\text{C})(4.20\,\mu\text{C})}{2.50\,\text{m}}\right]$$

$$= \boxed{-63.4\,\text{mJ}}$$

67 •• [SSM] (*a*) How much charge is on the surface of an isolated spherical conductor that has a 10.0-cm radius and is charged to 2.00 kV? (*b*)What is the electrostatic potential energy of this conductor? (Assume the potential is zero far from the sphere.)

Picture the Problem The potential of an isolated spherical conductor is given by $V = kQ/r$, where Q is its charge and r its radius, and its electrostatic potential energy by $U = \frac{1}{2}QV$. We can combine these relationships to find the sphere's electrostatic potential energy.

(*a*) The potential of the isolated spherical conductor at its surface is related to its radius:

$$V = \frac{kQ}{R} \Rightarrow Q = \frac{RV}{k}$$

where R is the radius of the spherical conductor.

Substitute numerical values and evaluate Q:

$$Q = \frac{(10.0\,\text{cm})(2.00\,\text{kV})}{8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}}$$

$$= 22.25\ \text{nC} = \boxed{22.3\,\text{nC}}$$

(*b*) Express the electrostatic potential energy of the isolated spherical conductor as a function of its charge Q and potential V:

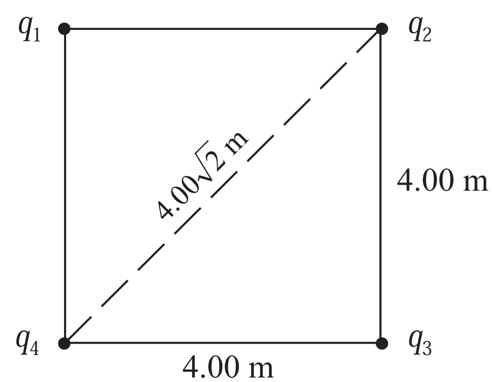
$$U = \tfrac{1}{2}QV$$

Substitute numerical values and evaluate U:

$$U = \tfrac{1}{2}(22.25\ \text{nC})(2.00\ \text{kV}) = \boxed{22.3\ \mu\text{J}}$$

68 ••• Four point charges, each having a charge with a magnitude of 2.00 μC, are at the corners of a square whose sides are 4.00 m-long. Find the electrostatic potential energy of this system under the following conditions: (*a*) all of the charges are negative, (*b*) three of the charges are positive and one of the charges is negative, and (*c*) the charges at two adjacent corners are positive and the other two charges are negative. (*d*) the charges at two opposite corners are positive and the other two charges are negative. (Assume the potential energy is zero when the point charges are very far from each other.)

Picture the Problem The electrostatic potential energy of this system of four point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram.



The work required to assemble this system of charges equals the potential energy of the assembled system:

$$U = \frac{kq_1q_2}{r_{1,2}} + \frac{kq_1q_3}{r_{1,3}} + \frac{kq_1q_4}{r_{1,4}} + \frac{kq_2q_3}{r_{2,3}} + \frac{kq_2q_4}{r_{2,4}} + \frac{kq_3q_4}{r_{3,4}}$$
$$= k\left(\frac{q_1q_2}{r_{1,2}} + \frac{q_1q_3}{r_{1,3}} + \frac{q_1q_4}{r_{1,4}} + \frac{q_2q_3}{r_{2,3}} + \frac{q_2q_4}{r_{2,4}} + \frac{q_3q_4}{r_{3,4}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, $r_{1,4}$, $r_{2,3}$, $r_{2,4}$, and $r_{3,4}$,:

$$r_{1,2} = r_{2,3} = r_{3,4} = r_{1,4} = 4.00\ \text{m}$$

and

$$r_{1,3} = r_{2,4} = 4.00\sqrt{2}\ \text{m}$$

(*a*) Evaluate U for $q_1 = q_2 = q_3 = q_4 = -2.00\ \mu\text{C}$:

$$U = \left(8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}}\right]$$

$$= \boxed{48.7\ \text{mJ}}$$

(*b*) Evaluate U for $q_1 = q_2 = q_3 = 2\ \mu\text{C}$ and $q_4 = -2.00\ \mu\text{C}$:

$$U = \left(8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(2.00\ \mu\text{C})(2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}}\right]$$

$$= \boxed{0}$$

(*c*) Let $q_1 = q_2 = 2.00\ \mu\text{C}$ and $q_3 = q_4 = -2.00\ \mu\text{C}$:

$$U = \left(8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(2.00\ \mu\text{C})(2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}} + \frac{(2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\sqrt{2}\ \text{m}} + \frac{(-2.00\ \mu\text{C})(-2.00\ \mu\text{C})}{4.00\ \text{m}}\right]$$

$$= \boxed{-12.7\ \text{mJ}}$$

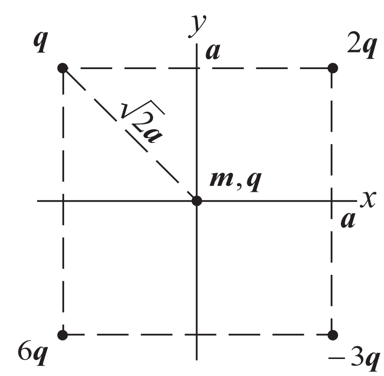
(d) Let $q_1 = q_3 = 2.00\ \mu\text{C}$ and $q_2 = q_4 = -2.00\ \mu\text{C}$:

$$U = \left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{4.00\,\text{m}} + \frac{(2.00\,\mu\text{C})(2.00\,\mu\text{C})}{4.00\sqrt{2}\,\text{m}} + \frac{(2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{4.00\,\text{m}} + \frac{(-2.00\,\mu\text{C})(2.00\,\mu\text{C})}{4.00\,\text{m}} + \frac{(-2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{4.00\sqrt{2}\,\text{m}} + \frac{(2.00\,\mu\text{C})(-2.00\,\mu\text{C})}{4.00\,\text{m}}\right]$$

$$= \boxed{-23.2\,\text{mJ}}$$

69 •• [SSM] Four point charges are fixed at the corners of a square centered at the origin. The length of each side of the square is $2a$. The charges are located as follows: $+q$ is at $(-a, +a)$, $+2q$ is at $(+a, +a)$, $-3q$ is at $(+a, -a)$, and $+6q$ is at $(-a, -a)$. A fifth particle that has a mass m and a charge $+q$ is placed at the origin and released from rest. Find its speed when it is a very far from the origin.

Picture the Problem The diagram shows the four point charges fixed at the corners of the square and the fifth charged particle that is released from rest at the origin. We can use conservation of energy to relate the initial potential energy of the particle to its kinetic energy when it is at a great distance from the origin and the electrostatic potential at the origin to express U_i.



Use conservation of energy to relate the initial potential energy of the particle to its kinetic energy when it is at a great distance from the origin:

$$\Delta K + \Delta U = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0$$

Express the initial potential energy of the particle to its charge and the electrostatic potential at the origin:

$$U_i = qV(0)$$

Substitute for K_f and U_i to obtain:

$$\tfrac{1}{2}mv^2 - qV(0) = 0 \Rightarrow v = \sqrt{\frac{2qV(0)}{m}}$$

Express the electrostatic potential at the origin:

$$V(0)=\frac{kq}{\sqrt{2}a}+\frac{2kq}{\sqrt{2}a}+\frac{-3kq}{\sqrt{2}a}+\frac{6kq}{\sqrt{2}a}=\frac{6kq}{\sqrt{2}a}$$

Substitute for $V(0)$ and simplify to obtain:

$$v=\sqrt{\frac{2q}{m}\left(\frac{6kq}{\sqrt{2}a}\right)}=\boxed{q\sqrt{\frac{6\sqrt{2}k}{ma}}}$$

70 •• Consider two point particles that each have charge $+e$, are at rest, and are separated by 1.50×10^{-15} m. (*a*) How much work was required to bring them together from a very large separation distance? (*b*) If they are released, how much kinetic energy will each have when they are separated by twice their separation at release? (*c*) The mass of each particle is 1.00 u (1.00 AMU). What speed will each have when they are very far from each other?

Picture the Problem (*a*) In the absence of other charged bodies, no work is required to bring the first proton from infinity to its initial position. We can use the work- energy theorem to find the work required to bring the second proton to a position 1.50×10^{-15} m away from the first proton. (*b*) and (*c*) We can apply conservation of mechanical energy to the two-proton system to find the kinetic energy of each proton when they are separated by twice their separation at release and when they are separated by a large distance.

(*a*) Apply the work-energy theorem to the second proton to obtain:

$$W_{\text{ext}}=\Delta K+\Delta U=0+\frac{ke^2}{r}=\frac{ke^2}{r}$$

Substitute numerical values and evaluate W_{ext}:

$$W_{\text{ext}}=\frac{\left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\text{ C}\right)^2}{1.50\times10^{-15}\text{ m}}=1.538\times10^{-13}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J}}=\boxed{960\,\text{keV}}$$

(*b*) Apply conservation of mechanical energy to the separating protons to obtain:

$$\Delta K+\Delta U=0\Rightarrow K_{\text{f}}-K_{\text{i}}+U_{\text{f}}-U_{\text{i}}=0$$

or, because $K_{\text{i}}=0$,

$$K_{\text{f}}+U_{\text{f}}-U_{\text{i}}=0$$

Substituting for U_{i} and U_{f} and simplifying yields:

$$K_{\text{f}}=U_{\text{i}}-U_{\text{f}}=\frac{ke^2}{r_{\text{i}}}-\frac{ke^2}{r_{\text{f}}}=\frac{ke^2}{r}-\frac{ke^2}{2r}=\frac{ke^2}{2r}$$

Remembering that K_f is the kinetic energy of both protons, substitute numerical values and evaluate $K_{\text{f, each proton}}$:

$$K_{\text{f, each proton}} = \frac{1}{2}\frac{\left(8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{2\left(1.50\times10^{-15}\ \text{m}\right)}$$
$$= 3.844\times10^{-14}\ \text{J}\times\frac{1\ \text{eV}}{1.602\times10^{-19}\ \text{J}}$$
$$= \boxed{240\ \text{keV}}$$

(*c*) Apply conservation of mechanical energy to the separating protons to obtain:

$$\Delta K + \Delta U = 0 \Rightarrow K_f - K_i + U_f - U_i = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0 \Rightarrow \tfrac{1}{2}m_p v_\infty^2 - U_i = 0$$

Solving for v_∞ yields:

$$v_\infty = \sqrt{\frac{2U_i}{m_p}}$$ where U_i is half the initial potential energy of the two-proton system.

Substitute numerical values and evaluate v_∞:

$$v_\infty = \sqrt{\frac{2\left(480\ \text{keV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)}{1.673\times10^{-27}\ \text{kg}}}$$
$$= \boxed{9.59\times10^{6}\ \text{m/s}}$$

71 ••• Consider an electron and a proton that are initially at rest and are separated by 2.00 nm. Neglecting any motion of the much more massive proton, what is the minimum (*a*) kinetic energy and (*b*) speed that the electron must be projected at so it reaches a point a distance of 12.0 nm from the proton? Assume the electron's velocity is directed radially away from the proton. (*c*) How far will the electron travel away from the proton if it has twice that initial kinetic energy?

Picture the Problem We can apply the conservation of mechanical energy to the electron-proton system to find the minimum initial kinetic energy (that is, the initial kinetic energy that corresponds to a final kinetic energy of zero) required in Part (*a*) and, in Part (*c*), to find how far the electron will travel away from the proton if it has twice the initial kinetic energy found in Part (*a*).

(*a*) Apply conservation of mechanical energy to the electron-stationary proton system with $K_f = 0$ to obtain:

$$\Delta K + \Delta U = 0 \Rightarrow -K_{i,\,min} + U_f - U_i = 0$$

Solving for $K_{i,\,min}$ and substituting for U_f and U_i yields:

$$K_{i,\,min} = U_f - U_i = -\frac{ke^2}{r_f} - \left(-\frac{ke^2}{r_i}\right)$$

or, simplifying further,

$$K_{i,\,min} = ke^2\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

Substitute numerical values and evaluate $K_{i,\,min}$:

$$K_{i,\,min} = \left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\ \text{C}\right)^2\left(\frac{1}{2.00\ \text{nm}} - \frac{1}{12.0\ \text{nm}}\right)$$
$$= 9.611\times10^{-20}\ \text{J} = \boxed{9.61\times10^{-20}\ \text{J}}$$

(*b*) Relate the minimum initial kinetic energy of the electron to its initial speed:

$$K_{i,\,min} = \tfrac{1}{2}m_e v_i^2 \Rightarrow v_i = \sqrt{\frac{2K_{i,\,min}}{m_e}}$$

Substitute numerical values and evaluate v_i:

$$v_i = \sqrt{\frac{2\left(9.611\times10^{-20}\ \text{J}\right)}{9.109\times10^{-31}\ \text{kg}}} = \boxed{4.59\times10^5\ \text{m/s}}$$

(*c*) Apply conservation of mechanical energy to the electron- proton system to obtain:

$$\Delta K + \Delta U = 0 \Rightarrow K_f - K_i + U_f - U_i = 0$$

Assuming, as we did in Part (*a*), that $K_f = 0$ yields:

$$-2K_{i,\,min} + U_f - U_i = 0$$

Substituting for U_f and U_i yields:

$$-2K_{i,\,min} - \frac{ke^2}{r_f} + \frac{ke^2}{r_i} = 0$$

Solve for r_f to obtain:

$$r_f = \frac{r_i}{1 - \dfrac{2r_iK_{i,\,min}}{ke^2}}$$

Substituting numerical values and evaluating r_f yields a negative value; a result that is not physical and suggests that, contrary to our assumption, $K_f \neq 0$. To confirm that this is the case, assume that the electron escapes from the proton (it's final electrostatic potential will then be equal to zero) and find the initial kinetic energy required for this to occur.

If the electron is to escape the influence of the proton, its final electrostatic potential energy will be zero and:

$$K_{\text{i, escape}} = -U_i = -\left(-\frac{ke^2}{r_i}\right) = \frac{ke^2}{r_i}$$

Substitute numerical values and evaluate $K_{\text{i, escape}}$:

$$K_{\text{i, escape}} = \frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{2.00\ \text{nm}} = 1.15\times10^{-19}\ \text{J}$$

Because $2K_{\text{i, min}} > K_{\text{i, escape}}$, the electron escapes from the proton with residual kinetic energy.

General Problems

72 • A positive point charge equal to 4.80×10^{-19} C is separated from a negative point charge of the same magnitude by 6.40×10^{-10} m. What is the electric potential at a point 9.20×10^{-10} m from each of the two charges?

Picture the Problem Because the charges are point charges, we can use the expression for the Coulomb potential to find the field at any distance from them.

Using the expression for the potential due to a system of point charges, express the potential at the point 9.20×10^{-10} m from each of the two charges:

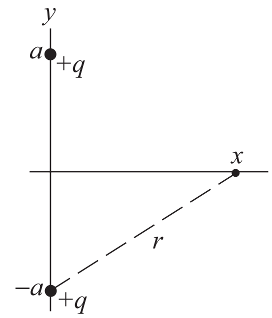
$$V = \frac{kq_+}{r} + \frac{kq_-}{r} = \frac{k}{r}\left(q_+ + q_-\right)$$

Because $q_+ = -q_-$:

$$V = \boxed{0}$$

73 • **[SSM]** Two positive point charges, each have a charge of $+q$, and are fixed on the y-axis at $y = +a$ and $y = -a$. (*a*) Find the electric potential at any point on the x-axis. (*b*) Use your result in Part (*a*) to find the electric field at any point on the x-axis.

Picture the Problem The potential V at any point on the x axis is the sum of the Coulomb potentials due to the two point charges. Once we have found V, we can use $\vec{E} = -(\partial V_x/\partial x)\hat{i}$ to find the electric field at any point on the x axis.



(*a*) Express the potential due to a system of point charges:

$$V = \sum_i \frac{kq_i}{r_i}$$

Substitute to obtain:

$$V(x) = V_{\text{charge at }+a} + V_{\text{charge at }-a} = \frac{kq}{\sqrt{x^2+a^2}} + \frac{kq}{\sqrt{x^2+a^2}} = \boxed{\frac{2kq}{\sqrt{x^2+a^2}}}$$

(*b*) The electric field at any point on the x axis is given by:

$$\vec{E}(x) = -\frac{\partial V_x}{\partial x}\hat{i} = -\frac{d}{dx}\left[\frac{2kq}{\sqrt{x^2+a^2}}\right]\hat{i} = \boxed{\frac{2kqx}{(x^2+a^2)^{3/2}}\hat{i}}$$

74 • If a conducting sphere is to be charged to a potential of 10.0 kV, what is the smallest possible radius of the sphere so that the electric field near the surface of the sphere will not exceed the dielectric strength of air?

Picture the Problem The radius of the sphere is related to the electric field and the potential at its surface. The dielectric strength of air is about 3 MV/m.

Relate the electric field at the surface of a conducting sphere to the potential at the surface of the sphere:

$$E_r = \frac{V(r)}{r} \Rightarrow r = \frac{V(r)}{E_r}$$

When E is a maximum, r is a minimum:

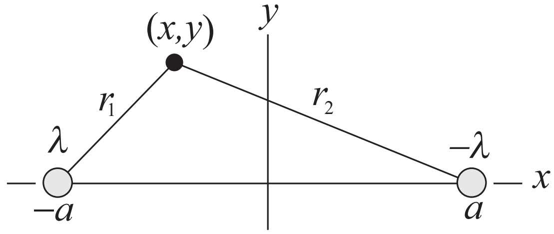
$$r_{\text{min}} = \frac{V(r)}{E_{\text{max}}}$$

Substitute numerical values and evaluate r_{min}:

$$r_{\text{min}} = \frac{10.0\,\text{kV}}{3\,\text{MV/m}} \approx \boxed{3\,\text{mm}}$$

75 •• **[SSM]** Two infinitely long parallel wires have a uniform charge per unit length λ and $-\lambda$ respectively. The wires are parallel with the z-axis. The positively charged wire intersects the x-axis at $x = -a$, and the negatively charged wire intersects the x-axis at $x = +a$. (a) Choose the origin as the reference point where the potential is zero, and express the potential at an arbitrary point (x, y) in the xy plane in terms of x, y, λ, and a. Use this expression to solve for the potential everywhere on the y axis. (b) Using a = 5.00 cm and λ = 5.00 nC/m, obtain the equation for the equipotential in the xy plane that passes through the point $x = \frac{1}{4}a$, $y = 0$. (c) Use a **spreadsheet** program to plot the equipotential found in part (b).

Picture the Problem The geometry of the wires is shown below. The potential at the point whose coordinates are (x, y) is the sum of the potentials due to the charge distributions on the wires.



(a) Express the potential at the point whose coordinates are (x, y):

$$V(x,y) = V_{\text{wire at }-a} + V_{\text{wire at }a}$$
$$= 2k\lambda \ln\left(\frac{r_{\text{ref}}}{r_1}\right) + 2k(-\lambda)\ln\left(\frac{r_{\text{ref}}}{r_2}\right)$$
$$= 2k\lambda\left[\ln\left(\frac{r_{\text{ref}}}{r_1}\right) - \ln\left(\frac{r_{\text{ref}}}{r_2}\right)\right]$$
$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right)$$

where $V(0) = 0$.

Because $r_1 = \sqrt{(x+a)^2 + y^2}$ and $r_2 = \sqrt{(x-a)^2 + y^2}$:

$$V(x,y) = \boxed{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{(x-a)^2+y^2}}{\sqrt{(x+a)^2+y^2}}\right)}$$

On the y-axis, $x = 0$ and:

$$V(0,y) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{a^2+y^2}}{\sqrt{a^2+y^2}}\right)$$
$$= \frac{\lambda}{2\pi\epsilon_0}\ln(1) = \boxed{0}$$

(b) Evaluate the potential at $\left(\frac{1}{4}a,0\right)=\left(1.25\,\text{cm},0\right)$:

$$V\left(\tfrac{1}{4}a,0\right)=\frac{\lambda}{2\pi\in_0}\ln\left(\frac{\sqrt{\left(\frac{1}{4}a-a\right)^2}}{\sqrt{\left(\frac{1}{4}a+a\right)^2}}\right)$$

$$=\frac{\lambda}{2\pi\in_0}\ln\left(\frac{3}{5}\right)$$

Equate $V(x,y)$ and $V\left(\frac{1}{4}a,0\right)$:

$$\frac{3}{5}=\frac{\sqrt{(x-5)^2+y^2}}{\sqrt{(x+5)^2+y^2}}$$

Solve for y to obtain:

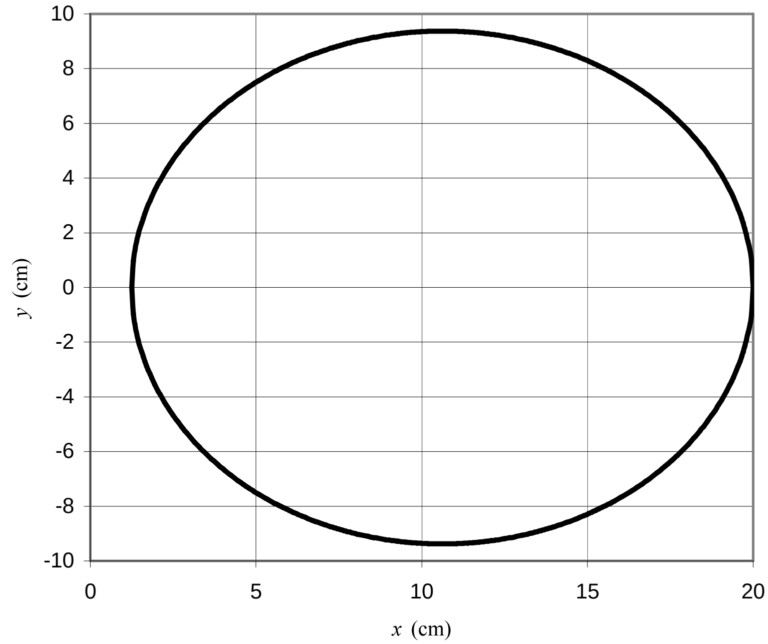
$$y=\boxed{\pm\sqrt{21.25x-x^2-25}}$$

(c) A spreadsheet program to plot $y=\pm\sqrt{21.25x-x^2-25}$ is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	1.25	$\frac{1}{4}a$
A3	A2 + 0.05	$x+\Delta x$
B2	SQRT(21.25*A2 – A2^2 – 25)	$y=\sqrt{21.25x-x^2-25}$
B4	–B2	$y=-\sqrt{21.25x-x^2-25}$

	A	B	C
1	x	y_{pos}	y_{neg}
2	1.25	0.00	0.00
3	1.30	0.97	–0.97
4	1.35	1.37	–1.37
5	1.40	1.67	–1.67
373	19.80	1.93	–1.93
374	19.85	1.67	–1.67
375	19.90	1.37	–1.37
376	19.95	0.97	–0.97

The following graph shows the equipotential curve in the xy plane for $V\left(\frac{1}{4}a,0\right)=\dfrac{\lambda}{2\pi\in_0}\ln\left(\dfrac{3}{5}\right)$.



76 •• The equipotential curve graphed in Problem 75 should be a circle. (*a*) Show mathematically that it is a circle. (*b*) The equipotential circle in the xy plane is the intersection of a three-dimensional equipotential surface and the xy plane. Describe the three-dimensional surface using one or two sentences.

Picture the Problem We can use the expression for the potential at any point in the xy plane to show that the equipotential curve is a circle.

(*a*) Equipotential surfaces must satisfy the condition:

$$V=\frac{\lambda}{2\pi\in_0}\ln\left(\frac{r_2}{r_1}\right)$$

Solving for r_2/r_1 yields:

$$\frac{r_2}{r_1}=e^{\frac{2\pi\in_0 V}{\lambda}}=C\ \text{ or }\ r_2=Cr_1$$

where C is a constant.

Substitute for r_1 and r_2 to obtain:

$$(x-a)^2+y^2=C^2\left[(x+a)^2+y^2\right]$$

Expand this expression, combine like terms, and simplify to obtain:

$$x^2+2a\frac{C^2+1}{C^2-1}x+y^2=-a^2$$

Complete the square by adding $\left[a^2\left(\frac{C^2+1}{C^2-1}\right)^2\right]$ to both sides of the equation:

$$x^2+2a\frac{C^2+1}{C^2-1}x+\left[a^2\left(\frac{C^2+1}{C^2-1}\right)^2\right]+y^2=\left[a^2\left(\frac{C^2+1}{C^2-1}\right)^2\right]-a^2=\frac{4a^2C^2}{\left(C^2-1\right)^2}$$

Letting $\alpha=2a\frac{C^2+1}{C^2-1}$ and $\beta=2a\frac{C}{C^2-1}$ yields:

$\boxed{(x+\alpha)^2+y^2=\beta^2,}$ the equation of circle in the xy plane with its center at $(-\alpha,0)$.

(b) The three-dimensional surfaces are cylinders whose axes are parallel to the wires and are in the $y = 0$ plane.

77 ••• The hydrogen atom in its ground state can be modeled as a positive point charge of magnitude $+e$ (the proton) surrounded by a negative charge distribution that has a charge density (the electron) that varies with the distance from the center of the proton r as: $\rho(r)=-\rho_0e^{-2r/a}$ (a result obtained from quantum mechanics), where a = 0.523 nm is the most probable distance of the electron from the proton. (a) Calculate the value of ρ_0 needed for the hydrogen atom to be neutral. (b) Calculate the electrostatic potential (relative to infinity) of this system as a function of the distance r from the proton.

Picture the Problem (a) Expressing the charge dq in a spherical shell of volume $4\pi r^2dr$ within a distance r of the proton and setting the integral of this expression equal to the charge of the electron will allow us to solve for the value of ρ_0 needed for charge neutrality. (b) The electrostatic potential of this system is the sum of the electrostatic potentials due to the proton and electron's charge density. The potential due to the proton is ke/r. We can use the given charge density to express the potential function due to the electron's charge distribution and then integrate this function to find the potential due to the electron.

(a) Express the charge dq in a spherical shell of volume $dV = 4\pi r^2dr$ at a distance r from the proton:

$$dq=\rho dV=\left(-\rho_0e^{-2r/a}\right)\left(4\pi r^2dr\right)=-4\pi\rho_0r^2e^{-2r/a}dr$$

Express the condition for charge neutrality:

$$e=-4\pi\rho_0\int_0^\infty r^2e^{-2r/a}dr$$

From a table of integrals we have:

$$\int x^2e^{bx}dx=\frac{e^{bx}}{b^3}\left(b^2x^2-2bx+2\right)$$

Using this result yields:

$$\int_0^\infty r^2 e^{-2r/a}\,dr = \frac{a^3}{4}$$

Substitute in the expression for e to obtain:

$$e = -4\pi\rho_0 \frac{a^3}{4} = -\pi\rho_0 a^3 \Rightarrow \rho_0 = -\frac{e}{\pi a^3}$$

Substitute numerical values and evaluate ρ_0:

$$\rho_0 = -\frac{1.602\times10^{-19}\ \text{C}}{\pi(0.523\ \text{nm})^3} = \boxed{-3.56\times10^{8}\ \text{C/m}^3}$$

(*b*) The electrostatic potential of this proton-electron system is the sum of the electrostatic potentials due to the proton and the electron's charge density:

$$V = V_1 + V_2 \qquad (1)$$

where

$$V_1 = \frac{ke}{r} + \frac{kQ_1}{r},$$

$$V_2 = \int_r^\infty \frac{k\rho(r')4\pi r'^2\,dr'}{r'} \qquad (2)$$

and

$$Q_1 = \int_0^r \rho(r')4\pi r'^2\,dr'$$

Substituting for $\rho(r')$ in the expression for Q_1 yields:

$$Q_1 = 4\pi\rho_0 \int_0^r r'^2 e^{-2r'/a}\,dr'$$

From a table of integrals we have:

$$\int x^2 e^{bx}\,dx = \frac{e^{bx}}{b^3}\left(b^2x^2 - 2bx + 2\right)$$

Using this result to evaluate $\int_0^r r'^2 e^{-2r'/a}\,dr'$ yields:

$$\begin{aligned}\int_0^r x^2 e^{-2x/a}\,dr' &= -\frac{a^3 e^{-2x/a}}{8}\left(\frac{4}{a^2}x^2 + 2\left(\frac{2}{a}\right)x + 2\right)\Bigg|_0^r \\ &= -\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) - \left(-\frac{a^3}{8}(2)\right) \\ &= -\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\end{aligned}$$

and

$$Q_1 = 4\pi\rho_0\left[-\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]$$

Substituting for Q_1 in the expression for V_1 yields:

$$V_1 = \frac{ke}{r} + \frac{4\pi k\rho_0}{r}\left[-\frac{a^3e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]$$

Substitute for ρ_0 from Part (a) and simplify to obtain:

$$V_1 = \frac{ke}{r} + \frac{4\pi k\left(-\frac{e}{\pi a^3}\right)}{r}\left[-\frac{a^3e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]$$
$$= \frac{ke}{r}e^{-2r/a}\left(\frac{2}{a^2}r^2 + \frac{2}{a}r + 1\right)$$

Substituting for $\rho(r')$ in equation (2) and simplifying yields:

$$V_2 = \int_r^\infty \frac{k\rho_0 e^{-2r/a}4\pi r'^2\,dr'}{r'}$$
$$= 4\pi k\rho_0\int_r^\infty e^{-2r/a}r'\,dr'$$

From a table of integrals we have:

$$\int xe^{bx}dx = \frac{e^{bx}}{b^2}(bx-1)$$

Using this result to evaluate $\int_r^\infty e^{-2r/a}r'\,dr'$ yields:

$$\int_r^\infty e^{-2x/a}x\,dx = \frac{a^2}{4}e^{-2x/a}\left(\frac{2}{a}x+1\right)\Bigg|_r^\infty$$
$$= -\frac{a^2}{4}e^{-2x/a}\left(\frac{2}{a}r+1\right)$$

Substitute for $\int_r^\infty e^{-2r/a}r'\,dr'$ and ρ_0 in the expression for V_2 to obtain:

$$V_2 = 4\pi k\left(-\frac{e}{\pi a^3}\right)e^{-2x/a}\left[-\frac{a^2}{4}\left(\frac{2}{a}r+1\right)\right]$$
$$= ke\left(\frac{1}{a}\right)e^{-2r/a}\left(\frac{2}{a}r+1\right)$$

Substituting for V_1 and V_2 in equation (1) and simplifying yields:

$$V = \frac{ke}{r}e^{-2r/a}\left(\frac{2}{a^2}r^2 + \frac{2}{a}r + 1\right) + \frac{ke}{a}e^{-2r/a}\left(\frac{2}{a}r+1\right) = \boxed{ke\left(\frac{1}{a}+\frac{1}{r}\right)e^{-2r/a}}$$

78 •• Charge is supplied to the metal dome of a Van de Graaff generator by the belt at the rate of 200 μC/s when the potential difference between the belt and the dome is 1.25 MV. The dome transfers charge to the atmosphere at the same rate, so the 1.25 MV potential difference is maintained. What minimum power is needed to drive the moving belt and maintain the 1.25 MV potential difference?

Picture the Problem We can use the definition of power and the expression for the work done in moving a charge through a potential difference to find the minimum power needed to drive the moving belt.

Relate the power needed to drive the moving belt to the rate at which the generator is doing work:

$$P = \frac{dW}{dt}$$

Express the work done in moving a charge q through a potential difference ΔV:

$$W = q\Delta V$$

Substitute for W to obtain:

$$P = \frac{d}{dt}[q\Delta V] = \Delta V\frac{dq}{dt}$$

Substitute numerical values and evaluate P:

$$P = (1.25\,\text{MV})(200\,\mu\text{C/s}) = \boxed{250\text{ W}}$$

79 •• A positive point charge $+Q$ is located on the x axis at $x = -a$. (a) How much work is required to bring an identical point charge from infinity to the point on the x axis at $x = +a$? (b) With the two identical point charges in place at $x = -a$ and $x = +a$, how much work is required to bring a third point charge $-Q$ from infinity to the origin? (c) How much work is required to move the charge $-Q$ from the origin to the point on the x axis at $x = 2a$ along the semicircular path shown (Figure 23-35)?

Picture the Problem We can use $W_{q\to\text{final position}} = Q\Delta V_{i\to f}$ to find the work required to move these charges between the given points.

(a) Express the required work in terms of the charge being moved and the potential due to the charge at $x = +a$ and simplify to obtain:

$$W_{+Q\to+a} = Q\Delta V_{\infty\to+a} = Q[V(a) - V(\infty)]$$
$$= QV(a) = Q\left(\frac{kQ}{2a}\right) = \boxed{\frac{kQ^2}{2a}}$$

(*b*) Express the required work in terms of the charge being moved and the potentials due to the charges at $x = +a$ and $x = -a$ and simplify to obtain:

$$W_{-Q\to 0} = -Q\Delta V_{\infty\to 0} = -Q[V(0) - V(\infty)]$$
$$= -QV(0) = -Q\left[V_{\substack{\text{charge}\\ \text{at } -a}} + V_{\substack{\text{charge}\\ \text{at } +a}}\right]$$
$$= -Q\left(\frac{kQ}{a} + \frac{kQ}{a}\right) = \boxed{\frac{-2kQ^2}{a}}$$

(*c*) Express the required work in terms of the charge being moved and the potentials due to the charges at $x = +a$ and $x = -a$ and simplify to obtain:

$$W_{-Q\to 2a} = -Q\Delta V_{0\to 2a}$$
$$= -Q[V(2a) - V(0)]$$
$$= -Q\left[V_{\substack{\text{charge}\\ \text{at } -a}} + V_{\substack{\text{charge}\\ \text{at } +a}} - V(0)\right]$$
$$= -Q\left(\frac{kQ}{3a} + \frac{kQ}{a} - \frac{2kQ}{a}\right)$$
$$= \boxed{\frac{2kQ^2}{3a}}$$

80 •• A charge of +2.00 nC is uniformly distributed on a ring of radius 10.0 cm that lies in the $x = 0$ plane and is centered at the origin. A point charge of +1.00 nC is initially located on the x axis at $x = 50.0$ cm. Find the work required to move the point charge to the origin.

Picture the Problem Let q represent the charge being moved from $x = 50.0$ cm to the origin, Q the ring charge, and a the radius of the ring. We can use $W_{q\to\text{final position}} = q\Delta V_{i\to f}$, where V is the expression for the axial field due to a ring charge, to find the work required to move q from $x = 50.0$ cm to the origin.

Express the required work in terms of the charge being moved and the potential due to the ring charge at $x = 50.0$ cm and $x = 0$:

$$W = q\Delta V = q[V(0) - V(0.500\,\text{m})]$$

The potential on the axis of a uniformly charged ring is given by:

$$V(x) = \frac{kQ}{\sqrt{x^2 + a^2}}$$

At $x = 50.0$ cm:

$$V(0.500\,\text{m}) = \frac{kQ}{\sqrt{(0.500\,\text{m})^2 + a^2}}$$

At $x = 0$:

$$V(0) = \frac{kQ}{\sqrt{a^2}} = \frac{kQ}{a}$$

Substituting for $V(0)$ and $V(0.500\text{ m})$ yields:

$$W = q\left[\frac{kQ}{a} - \frac{kQ}{\sqrt{(0.500\text{ m})^2 + a^2}}\right] = kQq\left[\frac{1}{a} - \frac{1}{\sqrt{(0.500\text{ m})^2 + a^2}}\right]$$

Substitute numerical values and evaluate W:

$$W = \left(8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(2.00\,\text{nC})(1.00\,\text{nC}) \times\left[\frac{1}{0.100\text{ m}} - \frac{1}{\sqrt{(0.500\text{ m})^2 + (0.100\text{ m})^2}}\right]$$

$$= 1.445\times10^{-7}\text{ J} = \boxed{1.4\times10^{-7}\text{ J}} = 1.445\times10^{-7}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J}}$$

$$= \boxed{9.0\times10^{11}\text{ eV}}$$

81 •• Two metal spheres each have a radius of 10.0 cm. The centers of the two spheres are 50.0 cm apart. The spheres are initially neutral, but a charge Q is transferred from one sphere to the other, creating a potential difference between the spheres of 100 V. A proton is released from rest at the surface of the positively charged sphere and travels to the negatively charged sphere. (*a*) What is the proton′s kinetic energy just as it strikes the negatively charged sphere? (*b*) At what speed does it strike the sphere?

Picture the Problem The proton′s kinetic energy just as it strikes the negatively charged sphere is the product of its charge and the potential difference through which it has been accelerated. We can find the speed of the proton as it strikes the negatively charged sphere from its kinetic energy and, in turn, its kinetic energy from the potential difference through which it is accelerated.

(*a*) Apply the work-kinetic energy theorem to the proton to obtain:

$$W_{\text{net}} = \Delta K = K_{\text{f}} - K_{\text{i}}$$

or, because $K_{\text{i}} = 0$,

$$W_{\text{net}} = K_{\text{f}} = K_{\text{p}}$$

The net work done on the proton is given by:

$$W_{\text{net}} = q_{\text{p}}V$$

Equating W_{net} and K_{p} yields:

$$K_{\text{p}} = q_{\text{p}}V = e(100\text{ V}) = \boxed{100\text{ eV}}$$

(*b*) Use the definition of kinetic energy to express the speed of the proton when it strikes the negatively charged sphere:

$$K_p = \tfrac{1}{2}m_p v^2 \Rightarrow v = \sqrt{\frac{2K_p}{m_p}} = \sqrt{\frac{2\Delta V}{m_p}}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{\frac{2\left(1.602\times10^{-19}\ \text{C}\right)\left(100\ \text{V}\right)}{1.673\times10^{-27}\ \text{kg}}} = \boxed{1.38\times10^{5}\ \text{m/s}}$$

82 •• (*a*) Using a **spreadsheet** program, graph $V(z)$ versus z for a uniformly charged ring in the $z = 0$ plane and centered at the origin. The potential on the z axis is given by $V(z) = kQ\big/\sqrt{a^2+z^2}$ (Equation 23-19). (*b*) Use your graph to estimate the points on the z axis where the electric field strength is greatest.

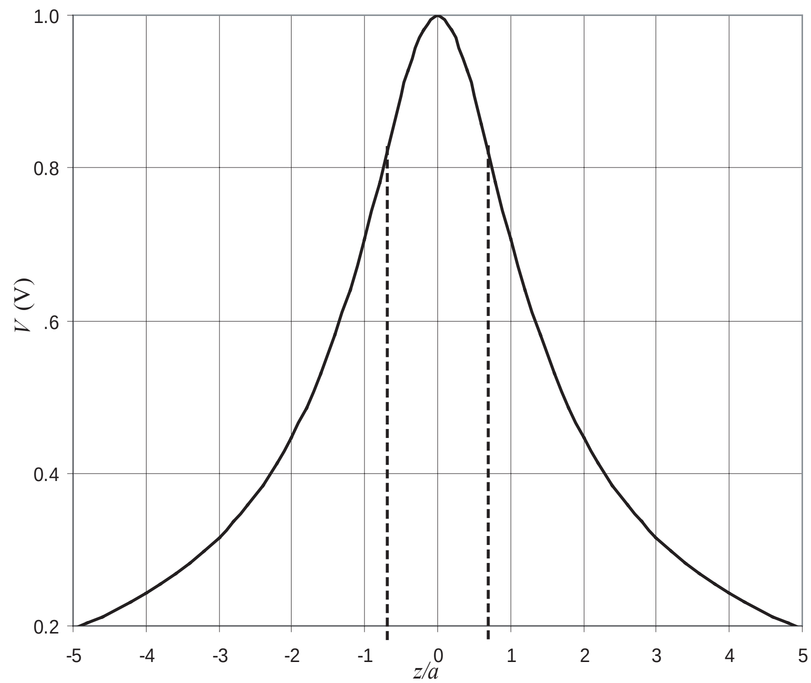
Picture the Problem (*b*) The electric field strength is greatest where the magnitude of the slope of the graph of electric potential is greatest.

(*a*) A spreadsheet solution is shown below for $kQ = a = 1$. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A4	A3 + 0.1	$z + \Delta z$
B3	1/(1+A3^2)^(1/2)	$\dfrac{kQ}{\sqrt{a^2+z^2}}$

	A	B
1		
2	z/a	$V(z/a)$
3	−5.0	0.196
4	−4.8	0.204
5	−4.6	0.212
6	−4.4	0.222
7	−4.2	0.232
8	−4.0	0.243
9	−3.8	0.254
49	4.2	0.232
50	4.4	0.222
51	4.6	0.212
52	4.8	0.204
53	5.0	0.196

The following graph, plotted using a spreadsheet program, shows V as a function of z/a:



Examining the graph we see that the magnitude of the slope is maximum at $z/a \approx \boxed{0.7}$ and at $z/a \approx \boxed{-0.7}$.

83 •• A spherical conductor of radius R_1 is charged to 20 kV. When it is connected by a long very-thin conducting wire to a second conducting sphere far away, its potential drops to 12 kV. What is the radius of the second sphere?

Picture the Problem Let R_2 be the radius of the second sphere and Q_1 and Q_2 the charges on the spheres when they have been connected by the wire. When the spheres are connected, the charge initially on the sphere of radius R_1 will redistribute until the spheres are at the same potential.

Express the common potential of the spheres when they are connected:

$$12\,\text{kV} = \frac{kQ_1}{R_1} \qquad (1)$$

and

$$12\,\text{kV} = \frac{kQ_2}{R_2} \qquad (2)$$

Express the potential of the first sphere before it is connected to the second sphere:

$$20\,\text{kV} = \frac{k(Q_1 + Q_2)}{R_1} \quad (3)$$

Solve equation (1) for Q_1:

$$Q_1 = \frac{(12\,\text{kV})R_1}{k}$$

Solve equation (2) for Q_2:

$$Q_2 = \frac{(12\,\text{kV})R_2}{k}$$

Substitute for Q_1 and Q_2 in equation (3) and simplify to obtain:

$$20\,\text{kV} = \frac{k\left(\frac{(12\,\text{kV})R_1}{k} + \frac{(12\,\text{kV})R_2}{k}\right)}{R_1} = 12\,\text{kV} + 12\,\text{kV}\left(\frac{R_2}{R_1}\right)$$

or

$$8 = 12\left(\frac{R_2}{R_1}\right) \Rightarrow R_2 = \boxed{\frac{2}{3}R_1}$$

84 •• A metal sphere centered at the origin has a surface charge density that has a magnitude of 24.6 nC/m^2 and a radius less than 2.00 m. A distance of 2.00 m from the origin, the electric potential is 500 V and the electric field strength is 250 V/m. (Assume the potential is zero very far from the sphere.) (*a*) What is the radius of the metal sphere? (*b*) What is the sign of the charge on the sphere? Explain your answer.

Picture the Problem We can use the definition of surface charge density to relate the radius R of the sphere to its charge Q and the potential function $V(r) = kQ/r$ to relate Q to the potential at r = 2.00 m.

(*a*) Use its definition, relate the surface charge density σ to the charge Q on the sphere and the radius R of the sphere:

$$\sigma = \frac{Q}{4\pi R^2} \Rightarrow R = \sqrt{\frac{Q}{4\pi\sigma}}$$

Relate the potential to the charge on the sphere:

$$V(r) = \frac{kQ}{r} \Rightarrow Q = \frac{rV(r)}{k}$$

Substitute for Q in the expression for R and simplify to obtain:

$$R=\sqrt{\frac{rV(r)}{4\pi k\sigma}}=\sqrt{\frac{4\pi\in_0 rV(r)}{4\pi\sigma}}=\sqrt{\frac{\in_0 rV(r)}{\sigma}}$$

Substitute numerical values and evaluate R:

$$R=\sqrt{\frac{(8.854\times10^{-12}\ \mathrm{C^2/N\cdot m^2})(2.00\ \mathrm{m})(500\ \mathrm{V})}{24.6\ \mathrm{nC/m^2}}}=\boxed{60.0\ \mathrm{cm}}$$

(b) The charge on the sphere is positive. The formula for the electric potential outside a uniformly charged spherical shell is $V=kQ/r$. If V is positive, then so is Q.

85 •• Along the central axis of a uniformly charged disk, at a point 0.60 m from the center of the disk, the potential is 80 V and the magnitude of the electric field is 80 V/m. At a distance of 1.5 m, the potential is 40 V and the magnitude of the electric field is 23.5 V/m. (Assume the potential is zero very far from the sphere.) Find the total charge on the disk.

Picture the Problem We can use the definition of surface charge density to relate the radius R of the disk to its charge Q and the potential function $V(r)=kQ/r$ to relate Q to the potential at r = 1.5 m.

Use its definition, relate the surface charge density σ to the charge Q on the disk and the radius R of the disk:

$$\sigma=\frac{Q}{\pi R^2}\Rightarrow Q=\pi\sigma R^2 \qquad (1)$$

Relate the potential at r to the charge on the disk:

$$V(r)=2\pi k\sigma\left(\sqrt{x^2+R^2}-x\right)$$

Substitute V(0.60 m) = 80 V:

$$80\ \mathrm{V}=2\pi k\sigma\left(\sqrt{(0.60\ \mathrm{m})^2+R^2}-0.60\ \mathrm{m}\right)$$

Substitute V(1.5 m) = 40 V:

$$40\ \mathrm{V}=2\pi k\sigma\left(\sqrt{(1.5\ \mathrm{m})^2+R^2}-1.5\ \mathrm{m}\right)$$

Divide the first of these equations by the second to obtain:

$$2=\frac{\sqrt{(0.60\ \mathrm{m})^2+R^2}-0.60\ \mathrm{m}}{\sqrt{(1.5\ \mathrm{m})^2+R^2}-1.5\ \mathrm{m}}$$

Solving for R yields:

$$R=0.80\ \mathrm{m}$$

Express the electric field on the axis of a disk charge:

$$E_x = 2\pi k\sigma\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

Solving for σ yields:

$$\sigma = \frac{E_x}{2\pi k\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)} = \frac{2\in_0 E_x}{1-\frac{x}{\sqrt{x^2+R^2}}}$$

Substitute for σ in equation (1) to obtain:

$$Q = \frac{2\pi\in_0 R^2 E_x}{1-\frac{x}{\sqrt{x^2+R^2}}}$$

Substitute numerical values and evaluate Q:

$$Q = \frac{2\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(0.80\,\text{m})^2(23.5\,\text{V/m})}{1-\frac{1.5\,\text{m}}{\sqrt{(1.5\,\text{m})^2+(0.80\,\text{m})^2}}} = \boxed{7.1\,\text{nC}}$$

86 •• A radioactive ^{210}Po nucleus emits an α-particle that has a charge $+2e$. When the α-particle is a large distance from the nucleus it has a kinetic energy of 5.30 MeV. Assume that the α-particle had negligible kinetic energy as it left at the surface of the nucleus. The "daughter" (or residual) nucleus ^{206}Pb has a charge $+82e$. Determine the radius of the ^{206}Pb nucleus. (Neglect the radius of the α particle and assume the ^{206}Pb nucleus remains at rest.)

Picture the Problem We can use $U = kq_1q_2/r$ to relate the electrostatic potential energy of the particles to their separation.

Express the electrostatic potential energy of the two particles in terms of their charge and separation:

$$U = \frac{kq_1q_2}{r} \Rightarrow r = \frac{kq_1q_2}{U}$$

Substitute numerical values and evaluate r:

$$r = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)(2)(82)\left(1.602\times10^{-19}\,\text{C}\right)^2}{5.30\,\text{MeV}\times\frac{1.602\times10^{-19}\,\text{C}}{\text{eV}}} = \boxed{44.6\,\text{fm}}$$

87 ••• [SSM] Configuration *A* consists of two point particles; one particle has a charge of $+q$ and is on the x axis at $x = +d$ and the other particle has a charge of $-q$ and is at $x = -d$ (Figure 23-36*a*). (*a*) Assuming the potential is zero at large distances from these charged particles, show that the potential is also zero everywhere on the $x = 0$ plane. (*b*) Configuration *B* consists of a flat metal plate of infinite extent and a point particle located a distance *d* from the plate (Figure 23-36*b*) The point particle has a charge equal to $+q$ and the plate is grounded. (Grounding the plate forces its potential to equal zero.) Choose the line perpendicular to the plate and through the point charge as the x axis, and choose the origin at the surface of the plate nearest the particle. (These choices put the particle on the x axis at $x = +d$.) For configuration *B*, the electric potential is zero both at all points in the half-space $x \geq 0$ that are very far from the particle and at all points on the $x = 0$ plane—just as was the case for configuration *A*. (*c*) A theorem, called the *uniqueness theorem*, implies that throughout the half-space $x \geq 0$ the potential function *V*—and thus the electric field $\vec{E}$—for the two configurations are identical. Using this result, obtain the electric field $\vec{E}$ at every point in the $x = 0$ plane in the configuration *B*. (The uniqueness theorem tells us that in configuration *B* the electric field at each point in the $x = 0$ plane is the same as it is in configuration *A*.) Use this result to find the surface charge density σ at each point in the conducting plane (in configuration *B*).

Picture the Problem We can use the relationship between the potential and the electric field to show that this arrangement is equivalent to replacing the plane by a point charge of magnitude $-q$ located a distance *d* beneath the plane. In (*b*) we can first find the field at the plane surface and then use $\sigma = \epsilon_0 E$ to find the surface charge density. In (*c*) the work needed to move the charge to a point 2*d* away from the plane is the product of the potential difference between the points at distances 2*d* and 3*d* from $-q$ multiplied by the separation Δx of these points.

(*a*) The potential anywhere on the plane is 0 and the electric field is perpendicular to the plane in both configurations, so they must give the same potential everywhere in the *xy* plane. Also, because the net charge is zero, the potential at infinity is zero.

(*b*) The surface charge density is given by:

$$\sigma = \epsilon_0 E \qquad (1)$$

At any point on the plane, the electric field points in the negative x direction and has magnitude:

$$E=\frac{kq}{d^2+r^2}\cos\theta$$

where θ is the angle between the horizontal and a vector pointing from the positive charge to the point of interest on the xz plane and r is the distance along the plane from the origin (that is, directly to the left of the charge).

Because $\cos\theta=\dfrac{d}{\sqrt{d^2+r^2}}$:

$$E=\frac{kq}{d^2+r^2}\frac{d}{\sqrt{d^2+r^2}}=\frac{kqd}{\left(d^2+r^2\right)^{3/2}}$$
$$=\frac{qd}{4\pi\in_0\left(d^2+r^2\right)^{3/2}}$$

Substitute for E in equation (1) and simplify to obtain:

$$\sigma=\boxed{\frac{qd}{4\pi\left(d^2+r^2\right)^{3/2}}}$$

88 ••• A particle that has a mass m and a positive charge q is constrained to move along the x-axis. At $x = –L$ and $x = L$ are two ring charges of radius L (Figure 23-38). Each ring is centered on the x-axis and lies in a plane perpendicular to it. Each ring has a total positive charge Q uniformly distributed on it. (*a*) Obtain an expression for the potential $V(x)$ on the x axis due to the charge on the rings. (*b*) Show that $V(x)$ has a minimum at $x = 0$. (*c*) Show that for $|x|<<L$, the potential approaches the form $V(x) = V(0) + \alpha x^2$. (*d*) Use the result of Part (*c*) to derive an expression for the angular frequency of oscillation of the mass m if it is displaced slightly from the origin and released. (Assume the potential equals zero at points far from the rings.)

Picture the Problem We can express the potential due to the ring charges as the sum of the potentials due to each of the ring charges. To show that $V(x)$ is a minimum at $x = 0$, we must show that the first derivative of $V(x) = 0$ at $x = 0$ and that the second derivative is positive. In Part (*c*) we can use a Taylor expansion to show that, for $|x|<<L$, the potential approaches the form $V(x) = V(0) + \alpha x^2$. In Part (*d*) we can obtain the potential energy function from the potential function and, noting that it is quadratic in x, find the "spring" constant and the angular frequency of oscillation of the particle provided its displacement from its equilibrium position is small.

(a) Express the potential due to the ring charges as the sum of the potentials due to each of their charges:

$$V(x) = V_{\substack{\text{ring to}\\ \text{the left}}} + V_{\substack{\text{ring to}\\ \text{the right}}}$$

The potential for a ring of charge is:

$$V(x) = \frac{kQ}{\sqrt{x^2 + a^2}}$$

where a is the radius of the ring and Q is its charge.

For the ring to the left we have:

$$V_{\substack{\text{ring to}\\ \text{the left}}} = \frac{kQ}{\sqrt{(x+L)^2 + L^2}}$$

For the ring to the right we have:

$$V_{\substack{\text{ring to}\\ \text{the right}}} = \frac{kQ}{\sqrt{(x-L)^2 + L^2}}$$

Substitute for $V_{\substack{\text{ring to}\\ \text{the left}}}$ and $V_{\substack{\text{ring to}\\ \text{the right}}}$ to obtain:

$$V(x) = \boxed{\frac{kQ}{\sqrt{(x+L)^2 + L^2}} + \frac{kQ}{\sqrt{(x-L)^2 + L^2}}}$$

(b) Evaluate dV/dx to obtain:

$$\frac{dV}{dx} = kQ\left\{\frac{L-x}{\left[(L-x)^2 + L^2\right]^{3/2}} - \frac{L+x}{\left[(L+x)^2 + L^2\right]^{3/2}}\right\} = 0 \text{ for extrema}$$

Solving for x yields:

$$x = 0$$

Evaluate d^2V/dx^2 to obtain:

$$\frac{d^2V}{dx^2} = kQ\left\{\frac{3(L-x)^2}{\left[(L-x)^2 + L^2\right]^{5/2}} - \frac{1}{\left[(L-x)^2 + L^2\right]^{3/2}} + \frac{3(L+x)^2}{\left[(L+x)^2 + L^2\right]^{5/2}} - \frac{1}{\left[(L-x)^2 + L^2\right]^{3/2}}\right\}$$

Evaluating this expression for $x = 0$ yields:

$$\frac{d^2V(0)}{dx^2} = \frac{kQ}{2\sqrt{2}L^3} > 0$$

$$\Rightarrow \boxed{V(x) \text{ is a maximum at } x = 0.}$$

(*c*) The Taylor expansion of $V(x)$ is:

$$V(x) = V(0) + V'(0)x + \tfrac{1}{2}V''(0)x^2 + \text{higher order terms}$$

For $x << L$:

$$V(x) \approx V(0) + V'(0)x + \tfrac{1}{2}V''(0)x^2$$

Substitute our results from Parts (*a*) and (*b*) to obtain:

$$V(x) = \frac{\sqrt{2}kQ}{L} + (0)x + \tfrac{1}{2}\left(\frac{kQ}{2\sqrt{2}L^3}\right)x^2 = \frac{\sqrt{2}kQ}{L} + \frac{kQ}{4\sqrt{2}L^3}x^2$$

or

$$V(x) = \boxed{V(0) + \alpha x^2}$$

where

$$V(0) = \boxed{\frac{\sqrt{2}kQ}{L}} \text{ and } \alpha = \boxed{\frac{kQ}{4\sqrt{2}L^3}}$$

(*d*) Express the angular frequency of oscillation of a simple harmonic oscillator:

$$\omega = \sqrt{\frac{k'}{m}}$$

where k' is the restoring constant.

From our result for Part (*c*) and the definition of electric potential:

$$U(x) = qV(0) + \frac{1}{2}\left(\frac{kqQ}{2\sqrt{2}L^3}\right)x^2 = qV(0) + \tfrac{1}{2}k'x^2$$

where $k' = \dfrac{kqQ}{2\sqrt{2}L^3}$

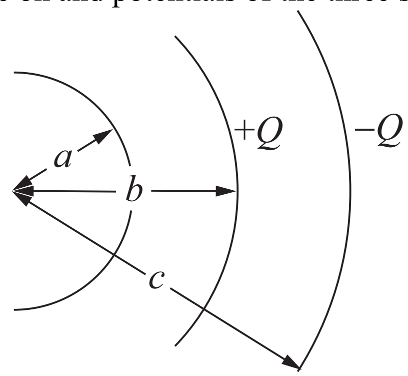
Substituting for k' in the expression for ω yields:

$$\omega = \boxed{\sqrt{\frac{kqQ}{2m\sqrt{2}L^3}}}$$

89 ••• Three concentric conducting thin spherical shells have radii a, b, and c so that $a < b < c$. Initially, the inner shell is uncharged, the middle shell has a positive charge $+Q$, and the outer shell has a charge $-Q$. (Assume the potential equals zero at points far from the shells.) (*a*) Find the electric potential of each of the three shells. (*b*) If the inner and outer shells are now connected by a conducting wire that is insulated as it passes through a small hole in the middle

shell, what is the electric potential of each of the three shells, and what is the final charge on each shell?

Picture the Problem The diagram shows part of the shells in a cross-sectional view under the conditions of Part (*a*) of the problem. We can use Gauss's law to find the electric field in the regions defined by the three surfaces and then find the electric potentials from the electric fields. In Part (*b*) we can use the redistributed charges to find the charge on and potentials of the three surfaces.



(*a*) Apply Gauss's law to a spherical Gaussian surface of radius $r \geq c$ to obtain:

$$E_r\left(4\pi r^2\right)=\frac{Q_{\text{enclosed}}}{\epsilon_0}=0$$

and $E_r = 0$ because the net charge enclosed by the Gaussian surface is zero.

Because $E_r(c) = 0$:

$$V(c)=\boxed{0}$$

Apply Gauss's law to a spherical Gaussian surface of radius $b < r < c$ to obtain:

$$E_r\left(4\pi r^2\right)=\frac{Q}{\epsilon_0}$$

and

$$E_r(b<r<c)=\frac{kQ}{r^2}$$

Use $E_r(b<r<c)$ to find the potential difference between c and b:

$$V(b)-V(c)=-kQ\int_c^b\frac{dr}{r^2}$$

$$=kQ\left(\frac{1}{b}-\frac{1}{c}\right)$$

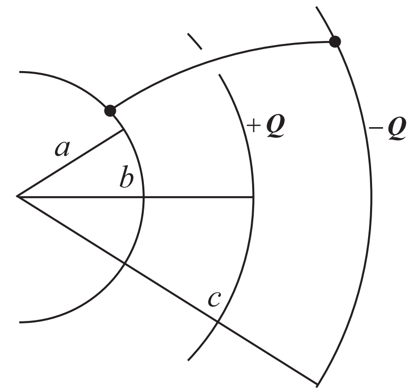
Because $V(c) = 0$:

$$V(b)=\boxed{kQ\left(\frac{1}{b}-\frac{1}{c}\right)}$$

The inner shell carries no charge, so the field between $r = a$ and $r = b$ is zero and:

$$V(a) = V(b) = \boxed{kQ\left(\frac{1}{b} - \frac{1}{c}\right)}$$

(b) When the inner and outer shells are connected their potentials become equal as a consequence of the redistribution of charge.



The charges on surfaces a and c are related according to:

$$Q_a + Q_c = -Q \tag{1}$$

Q_b does not change with the connection of the inner and outer shells:

$$Q_b = \boxed{Q}$$

Express the potentials of shells a and c:

$$V(a) = V(c) = \boxed{0}$$

In the region between the $r = a$ and $r = b$, the field is kQ_a/r^2 and the potential at $r = b$ is then:

$$V(b) = kQ_a\left(\frac{1}{b} - \frac{1}{a}\right) \tag{2}$$

The enclosed charge for $b < r < c$ is $Q_a + Q$, and by Gauss's law the field in this region is:

$$E_{b<r<c} = \frac{k(Q_a + Q)}{r^2}$$

Express the potential difference between b and c:

$$V(c) - V(b) = k(Q_a + Q)\left(\frac{1}{c} - \frac{1}{b}\right)$$
$$= -V(b)$$

because $V(c) = 0$.

Solve for $V(b)$ to obtain:

$$V(b) = k(Q_a + Q)\left(\frac{1}{b} - \frac{1}{c}\right) \tag{3}$$

Equate equations (2) and (3) and solve for Q_a to obtain:

$$Q_a = \boxed{-Q\frac{a(c-b)}{b(c-a)}} \qquad (4)$$

Substitute equation (4) in equation (1) and solve for Q_c to obtain:

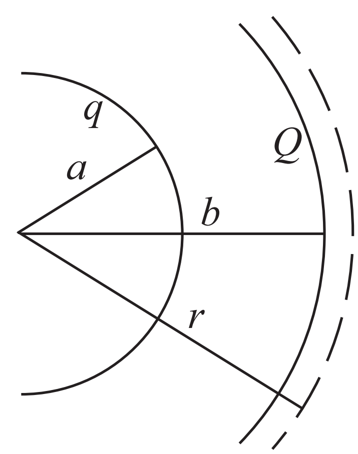
$$Q_c = \boxed{-Q\frac{c(b-a)}{b(c-a)}} \qquad (5)$$

Substitute equations (4) and (5) in equation (3) to obtain:

$$V(b) = \boxed{kQ\frac{(c-b)(b-a)}{b^2(c-a)}}$$

90 ••• Consider two concentric spherical thin metal shells of radii a and b, where $b > a$. The outer shell has a charge Q, but the inner shell is grounded. This means that the potential on the inner shell is the same as the potential at points far from the shells. Find the charge on the inner shell.

Picture the Problem The diagram shows a cross-sectional view of a portion of the concentric spherical shells. Let the charge on the inner shell be q. The dashed line represents a spherical Gaussian surface over which we can integrate $\vec{E}\cdot\hat{n}dA$ in order to find E_r for $r \geq b$. We can find $V(b)$ from the integral of E_r between $r = \infty$ and $r = b$. We can obtain a second expression for $V(b)$ by considering the potential difference between a and b and solving the two equations simultaneously for the charge q on the inner shell.



Apply Gauss's law to a spherical surface of radius $r \geq b$:

$$E_r\left(4\pi r^2\right) = \frac{Q+q}{\epsilon_0} \Rightarrow E_r = \frac{k(Q+q)}{r^2}$$

Use E_r to find $V(b)$:

$$V(b) = -k(Q+q)\int_{\infty}^{b}\frac{dr}{r^2} = \frac{k(Q+q)}{b}$$

We can also determine $V(b)$ by considering the potential difference between a, i.e., 0 and b:

$$V(b) = kq\left(\frac{1}{b} - \frac{1}{a}\right)$$

Equate these expressions for $V(b)$ to obtain:

$$\frac{k(Q+q)}{b} = ka\left(\frac{1}{b}-\frac{1}{a}\right) \Rightarrow q = \boxed{-\frac{a}{b}Q}$$

91 ••• [SSM] Show that the total work needed to assemble a uniformly charged sphere that has a total charge of Q and radius R is given by $3Q^2/(20\pi \in_0 R)$. Energy conservation tells us that this result is the same as the resulting electrostatic potential energy of the sphere. *Hint: Let ρ be the charge density of the sphere that has charge Q and radius R. Calculate the work dW to bring in charge dq from infinity to the surface of a uniformly charged sphere that has radius r (r < R) and charge density ρ. (No additional work is required to smear dq throughout a spherical shell of radius r, thickness dr, and charge density ρ. Why?)*

Picture the Problem We can use the hint to derive an expression for the electrostatic potential energy dU required to bring in a layer of charge of thickness dr and then integrate this expression from $r = 0$ to R to obtain an expression for the required work.

If we build up the sphere in layers, then at a given radius r the net charge on the sphere will be given by:

$$Q(r) = Q\left(\frac{r}{R}\right)^3$$

When the radius of the sphere is r, the potential relative to infinity is:

$$V(r) = \frac{Q(r)}{4\pi \in_0 r} = \frac{Q}{4\pi \in_0} \frac{r^2}{R^3}$$

Express the work dW required to bring in charge dQ from infinity to the surface of a uniformly charged sphere of radius r:

$$\begin{aligned} dW = dU &= V(r)dQ \\ &= \frac{Q}{4\pi \in_0}\frac{r^2}{R^3}\left(4\pi r^2 \frac{3Q}{4\pi R^3}dr\right) \\ &= \frac{3Q^2}{4\pi \in_0 R^6} r^4 dr \end{aligned}$$

Integrate dW from 0 to R to obtain:

$$\begin{aligned} W = U &= \frac{3Q^2}{4\pi \in_0 R^6}\int_0^R r^4 dr \\ &= \frac{3Q^2}{4\pi \in_0 R^6}\left[\frac{r^5}{5}\right]_0^R = \frac{3Q^2}{20\pi \in_0 R} \\ &= \boxed{\frac{3Q^2}{20\pi \in_0 R}} \end{aligned}$$

92 ••• (*a*) Use the result of Problem 91 to calculate the *classical electron radius*, the radius of a uniform sphere that has a charge $-e$ has and an electrostatic potential energy equal to the rest energy of the electron (5.11×10^5 eV). Comment

on the shortcomings of this model for the electron. (*b*) Repeat the calculation in Part (*a*) for a proton using its rest energy of 938 MeV. Experiments indicate the proton has an approximate radius of about 1.2×10^{-15} m. Is your result close to this value?

Picture the Problem We can equate the rest energy of an electron and the result of Problem 91 in order to obtain an expression that we can solve for the classical electron radius.

(*a*) From Problem 91 we have:

$$U = \frac{3e^2}{20\pi \in_0 R}$$

The rest energy of the electron is given by:

$$E_0 = m_0 c^2$$

Equate these energies to obtain:

$$\frac{3e^2}{20\pi \in_0 R} = m_0 c^2$$

Solving for R yields:

$$R = \frac{3e^2}{20\pi \in_0 m_0 c^2} \qquad (1)$$

Substitute numerical values and evaluate R:

$$R = \frac{3\left(1.602\times10^{-19}\ \text{C}\right)^2}{20\pi\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(5.11\times10^5\ \text{eV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)} = \boxed{1.69\times10^{-15}\ \text{m}}$$

This model does not explain how the electron holds together against its own mutual repulsion.

(*b*) For a proton, equation (1) yields:

$$R = \frac{3\left(1.602\times10^{-19}\ \text{C}\right)^2}{20\pi\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(938\ \text{MeV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)} = \boxed{9.21\times10^{-19}\ \text{m}}$$

This result is way too small to agree with the experimental value of 1.2×10^{-15} m.

93 ••• **[SSM]** (*a*) Consider a uniformly charged sphere that has radius R and charge Q and is composed of an incompressible fluid, such as water. If the sphere fissions (splits) into two halves of equal volume and equal charge, and if these halves stabilize into uniformly charged spheres, what is the radius R' of each? (*b*) Using the expression for potential energy shown in Problem 90,

calculate the change in the total electrostatic potential energy of the charged fluid. Assume that the spheres are separated by a large distance.

Picture the Problem Because the post-fission volumes of the fission products are equal, we can express the post-fission radii in terms of the radius of the pre-fission sphere.

(*a*) Relate the initial volume V of the uniformly charged sphere to the volumes V' of the fission products:

$$V = 2V'$$

Substitute for V and V':

$$\tfrac{4}{3}\pi R^3 = 2\left(\tfrac{4}{3}\pi R'^3\right)$$

Solving for R' yields:

$$R' = \frac{1}{\sqrt[3]{2}}R = \boxed{0.794R}$$

(*b*) Express the difference ΔE in the total electrostatic energy as a result of fissioning:

$$\Delta E = E' - E$$

From Problem 91 we have:

$$E = \frac{3Q^2}{20\pi \in_0 R}$$

After fissioning:

$$E' = 2\left(\frac{3Q'^2}{20\pi \in_0 R'}\right) = 2\left[\frac{3\left(\frac{1}{2}Q\right)^2}{20\pi \in_0 \frac{1}{\sqrt[3]{2}}R}\right]$$

$$= \frac{\sqrt[3]{2}}{2}\left(\frac{3Q^2}{20\pi \in_0 R}\right) = 0.630E$$

Substitute for E and E' to obtain:

$$\Delta E = 0.630E - E = \boxed{-0.370E}$$

94 ••• Problem 93 can be modified to be used as a very simple model for nuclear fission. When a ^{235}U nucleus absorbs a neutron, it can fission into the fragments ^{140}Xe, ^{94}Sr, and 2 neutrons. The ^{235}U has 92 protons, while ^{140}Xe has 54 protons and ^{94}Sr has 38 protons. Estimate the energy released during this fission process (in MeV), assuming that the mass density of the nucleus is constant and has a value of 4×10^{17} kg/m^3.

Picture the Problem We can use the definition of density to express the radius R of a nucleus as a function of its atomic mass N. We can then use the result derived in Problem 93 to express the electrostatic energies of the ^{235}U nucleus and the nuclei of the fission fragments ^{140}Xe and ^{94}Sr.

The energy released by this fission process is:

$$\Delta E = U_{^{235}\text{U}} - \left(U_{^{140}\text{Xe}} + U_{^{94}\text{Sr}}\right) \quad (1)$$

Express the mass of a nucleus in terms of its density and volume:

$$Nm = \tfrac{4}{3}\rho\pi R^3 \Rightarrow R = \sqrt[3]{\frac{3Nm}{4\pi\rho}}$$

where N is the nuclear number.

Substitute numerical values and evaluate R as a function of N:

$$R = \sqrt[3]{\frac{3\left(1.660\times10^{-27}\ \text{kg}\right)}{4\pi\left(4\times10^{17}\ \text{kg/m}^3\right)}}N^{1/3} = \left(9.97\times10^{-16}\ \text{m}\right)N^{1/3}$$

The 'radius' of the ^{235}U nucleus is therefore:

$$R_U = \left(9.97\times10^{-16}\ \text{m}\right)(235)^{1/3} = 6.15\times10^{-15}\ \text{m}$$

From Problem 91 we have:

$$U = \frac{3Q^2}{20\pi \in_0 R}$$

Substitute numerical values and evaluate the electrostatic energy of the ^{235}U nucleus:

$$U_{^{235}\text{U}} = \frac{3\left(92\times1.602\times10^{-19}\ \text{C}\right)^2}{20\pi\left(8.854\times10^{-12}\ \dfrac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(6.15\times10^{-15}\ \text{m}\right)}$$
$$= 1.90\times10^{-10}\ \text{J}\times\frac{1\text{eV}}{1.602\times10^{-19}\ \text{J/eV}} = 1189\,\text{MeV}$$

The radii of ^{140}Xe and ^{94}Sr are:

$$R_{^{149}\text{Xe}} = \left(9.97\times10^{-16}\ \text{m}\right)(140)^{1/3} = 5.177\times10^{-15}\ \text{m}$$

and

$$R_{^{94}\text{Sr}} = \left(9.97\times10^{-16}\ \text{m}\right)(94)^{1/3} = 4.533\times10^{-15}\ \text{m}$$

Proceed as above to find the electrostatic energy of the fission fragments ${}^{140}\text{Xe}$ and ${}^{94}\text{Sr}$:

$$U_{{}^{140}\text{Xe}} = \frac{3\left(54\times1.602\times10^{-19}\text{ C}\right)^2}{20\pi\left(8.854\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2\right)\left(5.18\times10^{-15}\text{ m}\right)}$$

$$= 7.791\times10^{-11}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J/eV}} = 486\,\text{MeV}$$

and

$$U_{{}^{94}\text{Sr}} = \frac{3\left(38\times1.602\times10^{-19}\text{ C}\right)^2}{20\pi\left(8.854\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2\right)\left(4.53\times10^{-15}\text{ m}\right)}$$

$$= 4.412\times10^{-11}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J/eV}} = 275\,\text{MeV}$$

Substitute for $U_{{}^{235}\text{U}}$, $U_{{}^{140}\text{Xe}}$, and $U_{{}^{94}\text{Sr}}$ in equation (1) and evaluate ΔE:

$$\Delta E = 1189\,\text{MeV} - \left(486\,\text{MeV} + 275\,\text{MeV}\right) \approx \boxed{428\,\text{MeV}}$$

Chapter 24
Capacitance

Conceptual Problems

1 • If the voltage across a parallel-plate capacitor is doubled, its capacitance (*a*) doubles (*b*) drops by half (*c*) remains the same.

Determine the Concept The capacitance of a parallel-plate capacitor is a function of the surface area of its plates, the separation of these plates, and the electrical properties of the matter between them. The capacitance is, therefore, independent of the voltage across the capacitor. $\boxed{(c)}$ is correct.

2 • If the charge on an isolated spherical conductor is doubled, its self-capacitance (*a*) doubles (*b*) drops by half (*c*) remains the same.

Determine the Concept The capacitance of an isolated spherical capacitor is given by $C = 4\pi \in_0 R$, where R is its radius. The capacitance is, therefore, independent of the charge of the capacitor. $\boxed{(c)}$ is correct.

3 • True or false: The electrostatic energy density is uniformly distributed in the region between the conductors of a cylindrical capacitor.

Determine the Concept False. The electrostatic energy density is not uniformly distributed the electric field strength is not uniformly distributed,

4 • If the distance between the plates of a charged and isolated parallel-plate capacitor is doubled, what is the ratio of the final stored energy to the initial stored energy?

Determine the Concept The energy stored in the electric field of a parallel-plate capacitor is related to the capacitance of the capacitor and the potential difference across the capacitor by $U = \frac{1}{2}QV$. Because the capacitor is isolated, Q is constant and U is directly proportional to V. Because, for a parallel-plate capacitor, $V = Ed$ and E is constant, U is directly proportional to d. Hence doubling the distance between the plates doubles the stored energy and the ratio of the initial stored energy to the final stored energy is $\boxed{2}$.

5 • **[SSM]** A parallel-plate capacitor is connected to a battery. The space between the two plates is empty. If the separation between the capacitor plates is tripled while the capacitor remains connected to the battery, what is the ratio of the final stored energy to the initial stored energy?

Determine the Concept The energy stored in a capacitor is given by $U = \frac{1}{2}QV$ and the capacitance of a parallel-plate capacitor by $C = \epsilon_0 A/d$. We can combine these relationships, using the definition of capacitance and the condition that the potential difference across the capacitor is constant, to express *U* as a function of *d*.

Express the energy stored in the capacitor:

$$U = \tfrac{1}{2}QV \qquad (1)$$

Use the definition of capacitance to express the charge of the capacitor:

$$Q = CV$$

Express the capacitance of a parallel-plate capacitor in terms of the separation *d* of its plates:

$$C = \frac{\epsilon_0 A}{d}$$

where *A* is the area of one plate.

Substituting for *Q* and *C* in equation (1) yields:

$$U = \frac{\epsilon_0 AV^2}{2d}$$

Because $U \propto \dfrac{1}{d}$, tripling the separation of the plates will reduce the energy stored in the capacitor to one-third its previous value. Hence the ratio of the final stored energy to the initial stored energy is $\boxed{1/3}$.

6 • If the capacitor of Problem 5 is disconnected from the battery before the separation between the plates is tripled, what is the ratio of the final stored energy to the initial stored energy?

Picture the Problem Let *V* represent the initial potential difference between the plates, *U* the energy stored in the capacitor initially, *d* the initial separation of the plates, and *V* ′, *U* ′, and *d* ′ these physical quantities when the plate separation has been tripled. We can use $U = \frac{1}{2}QV$ to relate the energy stored in the capacitor to the potential difference across it and $V = Ed$ to relate the potential difference to the separation of the plates.

Express the energy stored in the capacitor before the tripling of the separation of the plates:

$$U = \tfrac{1}{2}QV$$

Express the energy stored in the capacitor after the tripling of the separation of the plates:

$U' = \frac{1}{2}QV'$
because the charge on the plates does not change.

Express the ratio of U' to U and simplify to obtain:

$$\frac{U'}{U} = \frac{\frac{1}{2}QV'}{\frac{1}{2}QV} = \frac{V'}{V}$$

The potential differences across the capacitor plates before and after the plate separation, in terms of the electric field E between the plates, are given by:

$V = Ed$
and
$V' = Ed'$
because E depends solely on the charge on the plates and, as observed above, the charge does not change during the separation process.

Substituting for V and V' to obtain:

$$\frac{U'}{U} = \frac{Ed'}{Ed} = \frac{d'}{d}$$

For $d' = 3d$:

$\frac{U'}{U} = \frac{3d}{d} = 3 \Rightarrow$ The ratio of the final stored energy to the initial stored energy is $\boxed{3}$.

7 • True or false:

(*a*) The equivalent capacitance of two capacitors in parallel is always greater than the larger of the two capacitance values.
(*b*) The equivalent capacitance of two capacitors in series is always less than the least of the two capacitance values if the charges on the two plates that are connected by an otherwise isolated conductor sum to zero.

(*a*) True. The equivalent capacitance of two capacitors in parallel is the sum of the individual capacitances.

(*b*) True. The equivalent capacitance of two capacitors in series is the reciprocal of the sum of the reciprocals of the individual capacitances.

8 • Two uncharged capacitors have capacitances C_0 and $2C_0$, respectively, and are connected in series. This series combination is then connected across the terminals a battery. Which of the following is true?

(a) The capacitor $2C_0$ has twice the charge of the other capacitor.
(b) The voltage across each capacitor is the same.
(c) The energy stored by each capacitor is the same.
(d) The equivalent capacitance is $3C_0$.
(e) The equivalent capacitance is $2C_0/3$.

(a) False. Capacitors connected in series carry the same charge Q.

(b) False. The voltage V across a capacitor whose capacitance is C_0 is Q/C_0 and the voltage across the second capacitor is $Q/(2C_0)$.

(c) False. The energy stored in a capacitor is given by $\frac{1}{2}QV$.

(d) False. This would be the equivalent capacitance if they were connected in parallel.

(e) True. Taking the reciprocal of the sum of the reciprocals of C_0 and $2C_0$ yields $C_{eq} = 2C_0/3$.

9 • [SSM] A dielectric is inserted between the plates of a parallel-plate capacitor, completely filling the region between the plates. Air initially filled the region between the two plates. The capacitor was connected to a battery during the entire process. True or false:

(a) The capacitance value of the capacitor increases as the dielectric is inserted between the plates.
(b) The charge on the capacitor plates decreases as the dielectric is inserted between the plates.
(c) The electric field between the plates does not change as the dielectric is inserted between the plates.
(d) The energy storage of the capacitor decreases as the dielectric is inserted between the plates.

Determine the Concept The capacitance of the capacitor is given by $C = \dfrac{\kappa \in_0 A}{d}$, the charge on the capacitor is given by $Q = CV$, and the energy stored in the capacitor is given by $U = \frac{1}{2}CV^2$.

(a) True. As the dielectric material is inserted, κ increases from 1 (air) to its value for the given dielectric material.

(b) False. Q does not change. C increases and V decreases, but their product ($Q = CV$) is constant.

(c) False. Because $E = V/d$ and V decreases, E must decrease.

(d) False. The energy stored in the capacitor is given by $U = \frac{1}{2}CV^2$. Because V is constant and C increases, U increases.

10 •• Capacitors A and B (Figure 24-33) have identical plate areas and gap separations. The space between the plates of each capacitor is half-filled with a dielectric as shown. Which has the larger capacitance, capacitor A or capacitor B? Explain your answer.

Picture the Problem We can treat configuration A as two capacitors in parallel and configuration B as two capacitors in series. Finding the equivalent capacitance of each configuration and examining their ratio will allow us to decide whether A or B has the greater capacitance. In both cases, we'll let C_1 be the capacitance of the dielectric-filled capacitor and C_2 be the capacitance of the air capacitor.

In configuration A we have:

$$C_\mathrm{A} = C_1 + C_2$$

Express C_1 and C_2:

$$C_1 = \frac{\kappa \in_0 A_1}{d_1} = \frac{\kappa \in_0 \frac{1}{2} A}{d} = \frac{\kappa \in_0 A}{2d}$$

and

$$C_2 = \frac{\in_0 A_2}{d_2} = \frac{\in_0 \frac{1}{2} A}{d} = \frac{\in_0 A}{2d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_\mathrm{A} = \frac{\kappa \in_0 A}{2d} + \frac{\in_0 A}{2d} = \frac{\in_0 A}{2d}(\kappa + 1)$$

In configuration B we have:

$$\frac{1}{C_\mathrm{B}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_b = \frac{C_1 C_2}{C_1 + C_2}$$

Express C_1 and C_2:

$$C_1 = \frac{\in_0 A_1}{d_1} = \frac{\in_0 A}{\frac{1}{2} d} = \frac{2 \in_0 A}{d}$$

and

$$C_2 = \frac{\kappa \in_0 A_2}{d_2} = \frac{\kappa \in_0 A}{\frac{1}{2} d} = \frac{2\kappa \in_0 A}{d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_B = \frac{\left(\frac{2\in_0 A}{d}\right)\left(\frac{2\kappa\in_0 A}{d}\right)}{\frac{2\in_0 A}{d}+\frac{2\kappa\in_0 A}{d}} = \frac{\left(\frac{2\in_0 A}{d}\right)\left(\frac{2\kappa\in_0 A}{d}\right)}{\frac{2\in_0 A}{d}(\kappa+1)} = \frac{2\in_0 A}{d}\left(\frac{\kappa}{\kappa+1}\right)$$

Divide $\boldsymbol{C}_B$ by $\boldsymbol{C}_A$

$$\frac{C_B}{C_A} = \frac{\frac{2\in_0 A}{d}\left(\frac{\kappa}{\kappa+1}\right)}{\frac{\in_0 A}{2d}(\kappa+1)} = \frac{4\kappa}{(\kappa+1)^2}$$

Because $\frac{4\kappa}{(\kappa+1)^2} < 1$ for $\kappa > 1$:

$$\boxed{C_A > C_B}$$

11 •• [SSM] (*a*) Two identical capacitors are connected in parallel. This combination is then connected across the terminals of a battery. How does the total energy stored in the parallel combination of the two capacitors compare to the total energy stored if just one of the capacitors were connected across the terminals of the same battery? (*b*) Two identical capacitors that have been discharged are connected in series. This combination is then connected across the terminals of a battery. How does the total energy stored in the series combination of the two capacitors compare to the total energy stored if just one of the capacitors were connected across the terminals of the same battery?

Picture the Problem The energy stored in a capacitor whose capacitance is C and across which there is a potential difference V is given by $U = \frac{1}{2}CV^2$. Let C_0 represent the capacitance of the each of the two identical capacitors.

(*a*) The energy stored in the parallel system is given by:

$$U_{\text{parallel}} = \tfrac{1}{2}C_{\text{eq}}V^2$$

When the capacitors are connected in parallel, their equivalent capacitance is:

$$C_{\text{parallel}} = C_0 + C_0 = 2C_0$$

Substituting for C_{eq} and simplifying yields:

$$U_{\text{parallel}} = \tfrac{1}{2}(2C_0)V^2 = C_0V^2 \qquad (1)$$

If just one capacitor is connected to the same battery the stored energy is:

$$U_{1\,\text{capacitor}} = \tfrac{1}{2}C_0V^2 \qquad (2)$$

Dividing equation (1) by equation (2) and simplifying yields:

$$\frac{U_{\text{parallel}}}{U_{1\,\text{capacitor}}} = \frac{C_0V^2}{\tfrac{1}{2}C_0V^2} = 2$$

or

$$U_{\text{parallel}} = \boxed{2U_{1\,\text{capacitor}}}$$

(*b*) The energy stored in the series system is given by:

$$U_{\text{series}} = \tfrac{1}{2}C_{eq}V^2$$

When the capacitors are connected in series, their equivalent capacitance is:

$$C_{\text{series}} = \tfrac{1}{2}C_0$$

Substituting for C_{eq} and simplifying yields:

$$U_{\text{series}} = \tfrac{1}{2}\left(\tfrac{1}{2}C_0\right)V^2 = \tfrac{1}{4}C_0V^2 \quad (3)$$

Dividing equation (3) by equation (2) and simplifying yields:

$$\frac{U_{\text{series}}}{U_{1\,\text{capacitor}}} = \frac{\tfrac{1}{4}C_0V^2}{\tfrac{1}{2}C_0V^2} = \tfrac{1}{2}$$

or

$$U_{\text{series}} = \boxed{\tfrac{1}{2}U_{1\,\text{capacitor}}}$$

12 •• Two identical capacitors that have been discharged are connected in series across the terminals of a 100-V battery. When only one of the capacitors is connected across the terminals of this battery, the energy stored is U_0. What is the total energy stored in the two capacitors when the series combination is connected to the battery? (*a*) $4U_0$, (*b*) $2U_0$, (*c*) U_0, (*d*) $U_0/2$, (*e*) $U_0/4$

Picture the Problem We can use the expression $U = \tfrac{1}{2}CV^2$ to express the ratio of the energy stored in the single capacitor and in the identical-capacitors-in-series combination.

Express the energy stored in the capacitors when they are connected to the 100-V battery:

$$U = \tfrac{1}{2}C_{eq}V^2$$

Express the equivalent capacitance of the two identical capacitors connected in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C} + \frac{1}{C} \Rightarrow C_{\text{eq}} = \tfrac{1}{2}C$$

Substitute for C_{eq} to obtain:

$$U = \tfrac{1}{2}\left(\tfrac{1}{2}C\right)V^2 = \tfrac{1}{4}CV^2$$

Express the energy stored in one capacitor when it is connected to the 100-V battery:

$$U_0 = \tfrac{1}{2}CV^2$$

Express the ratio of U to U_0:

$$\frac{U}{U_0} = \frac{\tfrac{1}{4}CV^2}{\tfrac{1}{2}CV^2} = \frac{1}{2} \Rightarrow U = \tfrac{1}{2}U_0$$

$\boxed{(d)}$ is correct.

Estimation and Approximation

13 •• [SSM] Disconnect the coaxial cable from a television or other device and estimate the diameter of the inner conductor and the diameter of the shield. Assume a plausible value (see Table 24–1) for the dielectric constant of the dielectric separating the two conductors and estimate the capacitance per unit length of the cable.

Picture the Problem The outer diameter of a "typical" coaxial cable is about 5 mm, while the inner diameter is about 1 mm. From Table 24-1 we see that a reasonable range of values for κ is 3-5. We can use the expression for the capacitance of a cylindrical capacitor to estimate the capacitance per unit length of a coaxial cable.

The capacitance of a cylindrical dielectric-filled capacitor is given by:

$$C = \frac{2\pi\kappa \in_0 L}{\ln\left(\frac{R_2}{R_1}\right)}$$

where L is the length of the capacitor, R_1 is the radius of the inner conductor, and R_2 is the radius of the second (outer) conductor.

Divide both sides by L to obtain an expression for the capacitance per unit length of the cable:

$$\frac{C}{L} = \frac{2\pi\kappa \in_0}{\ln\left(\frac{R_2}{R_1}\right)} = \frac{\kappa}{2k\ln\left(\frac{R_2}{R_1}\right)}$$

If $\kappa = 3$:

$$\frac{C}{L} = \frac{3}{2\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\ln\left(\dfrac{2.5\,\text{mm}}{0.5\,\text{mm}}\right)} \approx 0.1\,\text{nF/m}$$

If $\kappa = 5$:

$$\frac{C}{L} = \frac{5}{2\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\ln\left(\dfrac{2.5\,\text{mm}}{0.5\,\text{mm}}\right)} \approx 0.2\,\text{nF/m}$$

A reasonable range of values for C/L, corresponding to $3 \le \kappa \le 5$, is:

$$\boxed{0.1\,\text{nF/m} \le \frac{C}{L} \le 0.2\,\text{nF/m}}$$

14 •• You are part of an engineering research team that is designing a pulsed nitrogen laser. To create the high-energy densities needed to operate such a laser, the electrical discharge from a high-voltage capacitor is used. Typically, the energy requirement per pulse (i.e., per discharge) is 100 J. Estimate the capacitance required if the discharge is to creates a spark across a gap of about 1.0 cm. Assume that the dielectric breakdown of nitrogen is the same as the value for normal air.

Picture the Problem The energy stored in a capacitor is given by $U = \frac{1}{2}CV^2$.

Relate the energy stored in a capacitor to its capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2 \Rightarrow C = \frac{2U}{V^2}$$

The potential difference across the spark gap is related to the width of the gap d and the electric field E in the gap:

$$V = Ed$$

Substitute for V in the expression for C to obtain:

$$C = \frac{2U}{E^2d^2}$$

Substitute numerical values and evaluate C:

$$C = \frac{2(100\,\text{J})}{\left(3\times10^{6}\ \text{V/m}\right)^2(1.0\,\text{cm})^2} = \boxed{22\,\mu\text{F}}$$

15 •• [SSM] Estimate the capacitance of the Leyden jar shown in the Figure 24-34. The figure of a man is one-tenth the height of an average man.

Picture the Problem Modeling the Leyden jar as a parallel-plate capacitor, we can use the equation for the capacitance of a dielectric-filled parallel-plate capacitor that relates its capacitance to the area A of its plates and their separation (the thickness of the glass) d to estimate the capacitance of the jar. See Table 24-1 for the dielectric constants of various materials.

The capacitance of a dielectric-filled parallel-plate capacitor is given by:

$$C = \frac{\kappa \in_0 A}{d}$$

where κ is the dielectric constant.

Let the plate area be the sum of the area of the lateral surface of the jar and its base:

$$A = A_{\substack{\text{lateral}\\\text{area}}} + A_{\text{base}} = 2\pi Rh + \pi R^2$$

where h is the height of the jar and R is its inside radius.

Substitute for A and simplify to obtain:

$$C = \frac{\kappa \in_0 \left(2\pi Rh + \pi R^2\right)}{d} = \frac{\pi\kappa \in_0 R(2h + R)}{d}$$

If the glass of the Leyden jar is Bakelite of thickness 2.0 mm and the radius and height of the jar are 4.0 cm and 40 cm, respectively, then:

$$C = \frac{\pi(4.9)(4.0\text{ cm})\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left[2(40\text{ cm}) + 4.0\text{ cm}\right]}{2.0\text{ mm}} = \boxed{2.3\text{ nF}}$$

Capacitance

16 • An isolated conducting sphere that has a 10.0 cm radius has an electric potential of 2.00 kV (the potential far from the sphere is zero). (*a*) How much charge is on the sphere? (*b*) What is the self-capacitance of the sphere? (*c*) By how much does the self-capacitance change if the sphere's electric potential is increased to 6.00 kV?

Picture the Problem The charge on the spherical conductor is related to its radius and potential according to $V = kQ/r$ and we can use the definition of capacitance to find the self-capacitance of the sphere.

(*a*) Relate the potential *V* of the spherical conductor to the charge on it and to its radius:

$$V = \frac{kQ}{r} \Rightarrow Q = \frac{rV}{k}$$

Substitute numerical values and evaluate *Q*:

$$Q = \frac{(10.0\,\text{cm})(2.00\,\text{kV})}{8.988\times 10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = 22.252\text{ nC}$$

$$= \boxed{22.3\,\text{nC}}$$

(*b*) Use the definition of capacitance to relate the self-capacitance of the sphere to its charge and potential:

$$C = \frac{Q}{V} = \frac{22.252\,\text{nC}}{2.00\,\text{kV}} = \boxed{11.1\,\text{pF}}$$

(*c*) It doesn't. The self-capacitance of a sphere is a function of its radius.

17 • The charge on one plate of a capacitor is +30.0 μC and the charge on the other plate is –30.0 μC. The potential difference between the plates is 400 V. What is the capacitance of the capacitor?

Picture the Problem We can use its definition to find the capacitance of this capacitor.

Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{30.0\,\mu\text{C}}{400\,\text{V}} = \boxed{75.0\,\text{nF}}$$

18 •• Two isolated conducting spheres of equal radius *R* have charges +*Q* and –*Q*, respectively. Their centers are separated by a distance *d* that is large compared to their radius. Estimate the capacitance of this unusual capacitor.

Picture the Problem Let the separation of the spheres be *d* and their radii be *R*. Outside the two spheres the electric field is approximately the field due to point charges of +*Q* and −*Q*, each located at the centers of spheres, separated by distance *d*. We can derive an expression for the potential at the surface of each sphere and then use the potential difference between the spheres and the definition of capacitance and to estimate the capacitance of the two-sphere system.

The capacitance of the two-sphere system is given by:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the potential difference between the spheres.

The potential at any point outside the two spheres is:

$$V=\frac{k(+Q)}{r_1}+\frac{k(-Q)}{r_2}$$

where r_1 and r_2 are the distances from the given point to the centers of the spheres.

For a point on the surface of the sphere with charge $+Q$:

$r_1=R$ and $r_2=d+\delta$
where $|\delta|<R$

Substitute to obtain:

$$V_{+Q}=\frac{k(+Q)}{R}+\frac{k(-Q)}{d+\delta}$$

For $\delta \ll d$:

$$V_{+Q}=\frac{kQ}{R}-\frac{kQ}{d}$$

and

$$V_{-Q}=-\frac{kQ}{R}+\frac{kQ}{d}$$

The potential difference between the spheres is:

$$\begin{aligned}\Delta V&=V_Q-V_{-Q}\\&=\frac{kQ}{R}-\frac{kQ}{d}-\left(\frac{-kQ}{R}+\frac{kQ}{d}\right)\\&=2kQ\left(\frac{1}{R}-\frac{1}{d}\right)\end{aligned}$$

Substitute for ΔV in the expression for C to obtain:

$$\begin{aligned}C&=\frac{Q}{2kQ\left(\frac{1}{R}-\frac{1}{d}\right)}=\frac{2\pi\in_0}{\left(\frac{1}{R}-\frac{1}{d}\right)}\\&=\frac{2\pi\in_0 R}{1-\frac{R}{d}}\end{aligned}$$

For $d \gg R$:

$$C=\boxed{2\pi\in_0 R}$$

The Storage of Electrical Energy

19 • **[SSM]** (*a*) The potential difference between the plates of a 3.00-μF capacitor is 100 V. How much energy is stored in the capacitor? (*b*) How much additional energy is required to increase the potential difference between the plates from 100 V to 200 V?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to C and V is $U = \frac{1}{2}CV^2$.

(*a*) Express the energy stored in the capacitor as a function of C and V:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate U:

$$U = \tfrac{1}{2}(3.00\,\mu\text{F})(100\,\text{V})^2 = \boxed{15.0\,\text{mJ}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor at 200 V and the energy stored at 100 V:

$$\begin{aligned}\Delta U &= U(200\,\text{V}) - U(100\,\text{V})\\ &= \tfrac{1}{2}(3.00\,\mu\text{F})(200\,\text{V})^2 - 15.0\,\text{mJ}\\ &= \boxed{45.0\,\text{mJ}}\end{aligned}$$

20 • The charges on the plates of a 10-μF capacitor are ±4.0 μC. (*a*) How much energy is stored in the capacitor? (*b*) If charge is transferred until the charges on the plates are equal to ±2.0 μC, how much stored energy remains?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to Q and C is $U = \frac{1}{2}\frac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of C and Q:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate U:

$$U = \frac{1}{2}\frac{(4.0\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.80\,\mu\text{J}}$$

(*b*) Express the energy remaining when half the charge is removed:

$$U\left(\tfrac{1}{2}Q\right) = \frac{1}{2}\frac{(2.0\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.20\,\mu\text{J}}$$

21 • (*a*) Find the energy stored in a 20.0-nF capacitor when to the charges on the plates are ±5.00 μC. (*b*) How much additional energy is stored if charges are increased from ±5.00 μC to ±10.0 μC?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to Q and C is $U = \frac{1}{2}\frac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of *C* and *Q*:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate *U*:

$$U(5.00\,\mu\text{C}) = \frac{1}{2}\frac{(5.00\,\mu\text{C})^2}{20.0\,\text{nF}} = \boxed{0.625\,\text{mJ}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor when its charge is 5 μC and when its charge is 10 μC:

$$\begin{aligned}\Delta U &= U(10.0\,\mu\text{C}) - U(5.00\,\mu\text{C})\\ &= \frac{1}{2}\frac{(10.0\,\mu\text{C})^2}{20.0\,\text{nF}} - 0.625\,\text{mJ}\\ &= 2.50\,\text{mJ} - 0.625\,\text{mJ} = \boxed{1.88\,\text{mJ}}\end{aligned}$$

22 • What is the maximum electric energy density in a region containing dry air at standard conditions?

Picture the Problem The energy per unit volume in an electric field varies with the square of the electric field according to $u = \frac{1}{2}\epsilon_0 E^2$. Under standard conditions, dielectric breakdown occurs at approximately E = 3.0 MV/m.

Express the energy per unit volume in an electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute numerical values and evaluate *u*:

$$\begin{aligned}u &= \tfrac{1}{2}\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(3.0\,\frac{\text{MV}}{\text{m}}\right)^2\\ &\approx \boxed{40\,\text{J/m}^3}\end{aligned}$$

23 •• **[SSM]** An air-gap parallel-plate capacitor that has a plate area of 2.00 m^2 and a separation of 1.00 mm is charged to 100 V. (*a*) What is the electric field between the plates? (*b*) What is the electric energy density between the plates? (*c*) Find the total energy by multiplying your answer from Part (*b*) by the volume between the plates. (*d*) Determine the capacitance of this arrangement. (*e*) Calculate the total energy from $U = \frac{1}{2}CV^2$, and compare your answer with your result from Part (*c*).

Picture the Problem Knowing the potential difference between the plates, we can use $E = V/d$ to find the electric field between them. The energy per unit volume is given by $u = \frac{1}{2}\epsilon_0 E^2$ and we can find the capacitance of the parallel-plate capacitor using $C = \epsilon_0 A/d$.

(*a*) Express the electric field between the plates in terms of their separation and the potential difference between them:

$$E=\frac{V}{d}=\frac{100\,\text{V}}{1.00\,\text{mm}}=\boxed{100\,\text{kV/m}}$$

(*b*) Express the energy per unit volume in an electric field:

$$u=\tfrac{1}{2}\in_0 E^2$$

Substitute numerical values and evaluate *u*:

$$u=\tfrac{1}{2}\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(100\,\text{kV/m})^2$$
$$=44.27\ \text{mJ/m}^3=\boxed{44.3\,\text{mJ/m}^3}$$

(*c*) The total energy is given by:

$$U=uV=uAd$$
$$=\left(44.27\,\text{mJ/m}^3\right)\left(2.00\,\text{m}^2\right)(1.00\,\text{mm})$$
$$=\boxed{88.5\,\mu J}$$

(*d*) The capacitance of a parallel-plate capacitor is given by:

$$C=\frac{\in_0 A}{d}$$

Substitute numerical values and evaluate *C*:

$$C=\frac{\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(2.00\,\text{m}^2\right)}{1.00\,\text{mm}}$$
$$=17.71\,\text{nF}=\boxed{17.7\,\text{nF}}$$

(*e*) The total energy is given by:

$$U=\tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U=\tfrac{1}{2}(17.71\text{nF})(100\,\text{V})^2$$
$$=\boxed{88.5\,\mu\text{J, in agreement with }(c).}$$

24 •• A solid metal sphere has radius of 10.0 cm and a concentric metal spherical shell has an inside radius of 10.5 cm. The solid sphere has a charge 5.00 nC. (*a*) Estimate the energy stored in the electric field in the region between the spheres. *Hint: You can treat the spheres essentially as parallel flat slabs separated by 0.5 cm*. (*b*) Estimate the capacitance of this two-sphere system. (*c*) Estimate the total energy stored in the electric field from $\tfrac{1}{2}Q^2/C$ and compare it to your answer in Part (*a*).

Picture the Problem The total energy stored in the electric field is the product of the energy density in the space between the spheres and the volume of this space.

(*a*) The total energy U stored in the electric field is given by:

$U = uV$
where u is the energy density and V is the volume between the spheres.

The energy density of the field is:

$u = \frac{1}{2} \in_0 E^2$
where E is the field between the spheres.

The volume between the spheres is approximately:

$V \approx 4\pi r_1^2 (r_2 - r_1)$

Substitute for u and V to obtain:

$$U = 2\pi \in_0 E^2 r_1^2 (r_2 - r_1) \qquad (1)$$

The magnitude of the electric field between the concentric spheres is the sum of the electric fields due to each charge distribution:

$E = E_Q + E_{-Q}$

Because the two surfaces are so close together, the electric field between them is approximately the sum of the fields due to two plane charge distributions:

$$E = \frac{\sigma_Q}{2 \in_0} + \frac{\sigma_{-Q}}{2 \in_0} = \frac{\sigma_Q}{\in_0}$$

Substitute for σ_Q to obtain:

$$E \approx \frac{Q}{4\pi r_1^2 \in_0}$$

Substitute for E in equation (1) and simplify:

$$U = 2\pi \in_0 \left(\frac{Q}{4\pi r_1^2 \in_0}\right)^2 r_1^2 (r_2 - r_1) = \frac{Q^2}{8\pi \in_0} \frac{r_2 - r_1}{r_1^2}$$

Substitute numerical values and evaluate U:

$$U = \frac{(5.00\,\text{nC})^2 (10.5\,\text{cm} - 10.0\,\text{cm})}{8\pi \left(8.854 \times 10^{-12} \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(10.0\,\text{cm})^2} = 56.2 \times 10^{-9}\ \text{J} = \boxed{0.06\,\mu\text{J}}$$

(*b*) The capacitance of the two-sphere system is given by:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the potential difference between the two spheres.

The electric potentials at the surfaces of the spheres are:

$$V_1 = \frac{Q}{4\pi \in_0 r_1} \text{ and } V_2 = \frac{Q}{4\pi \in_0 r_2}$$

Substitute for ΔV and simplify to obtain:

$$C = \frac{Q}{\frac{Q}{4\pi \in_0 r_1} - \frac{Q}{4\pi \in_0 r_2}} = 4\pi \in_0 \frac{r_1 r_2}{r_2 - r_1}$$

Substitute numerical values and evaluate C:

$$C = 4\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\frac{(10.0\,\text{cm})(10.5\,\text{cm})}{10.5\,\text{cm}-10.0\,\text{cm}} = 0.2337\text{ nF} = \boxed{0.2\,\text{nF}}$$

(*c*) Use $\frac{1}{2}Q^2/C$ to find the total energy stored in the electric field between the spheres:

$$U = \frac{1}{2}\left[\frac{(5.00\,\text{nC})^2}{0.2337\,\text{nF}}\right] = \boxed{0.05\,\mu\text{J}}$$

a result that agrees to within 5% with the exact result obtained in (*a*).

25 •• A parallel-plate capacitor has plates of area 500 cm^2 and is connected across the terminals of a battery. After some time has passed, the capacitor is disconnected from the battery. When the plates are then moved 0.40 cm farther apart, the charge on each plate remains constant but the potential difference between the plates increases by 100 V. (*a*) What is the magnitude of the charge on each plate? (*b*) Do you expect the energy stored in the capacitor to increase, decrease, or remain constant as the plates are moved this way? Explain your answer. (*c*) Support your answer to Part (*b*), by determining the change in stored energy in the capacitor due to the movement of the plates.

Picture the Problem (*a*) We can relate the charge Q on the positive plate of the capacitor to the charge density of the plate σ using its definition. The charge density, in turn, is related to the electric field between the plates according to $\sigma = \in_0 E$ and the electric field can be found from $E = \Delta V/\Delta d$. We can use $\Delta U = \frac{1}{2}Q\Delta V$ in Part (*c*) to find the increase in the energy stored due to the movement of the plates.

(*a*) Express the charge Q on the positive plate of the capacitor in terms of the plate's charge density σ and surface area A:

$$Q = \sigma A$$

Relate σ to the electric field E between the plates of the capacitor:

$$\sigma = \epsilon_0 E$$

Express E in terms of the change in V as the plates are separated a distance Δd:

$$E = \frac{\Delta V}{\Delta d}$$

Substitute for σ and E to obtain:

$$Q = \epsilon_0 EA = \epsilon_0 A\frac{\Delta V}{\Delta d}$$

Substitute numerical values and evaluate Q:

$$Q = \left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)\left(500\,\text{cm}^2\right)\frac{100\,\text{V}}{0.40\,\text{cm}} = 11.1\,\text{nC} = \boxed{11\,\text{nC}}$$

(*b*) Because work has to be done to pull the plates farther apart, you would expect the energy stored in the capacitor to increase.

(*c*) Express the change in the electrostatic energy in terms of the change in the potential difference:

$$\Delta U = \tfrac{1}{2}Q\Delta V$$

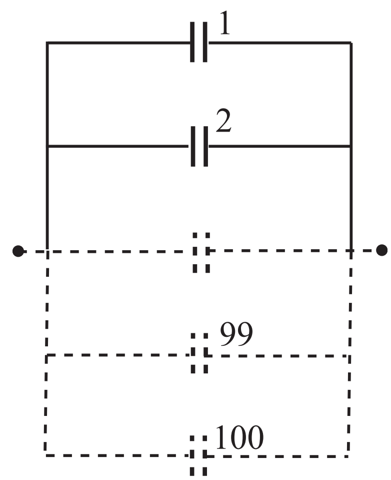
Substitute numerical values and evaluate ΔU:

$$\Delta U = \tfrac{1}{2}(11.1\,\text{nC})(100\,\text{V}) = \boxed{0.55\,\mu\text{J}}$$

Combinations of Capacitors

26 • (*a*) How many 1.00-μF capacitors connected in parallel would it take to store a total charge of 1.00 mC if the potential difference of across each capacitor is 10.0 V? Diagram the parallel combination. (*b*) What would be the potential difference across this parallel combination? (*c*) If the capacitors in Part (*a*) are discharged, connected in series, and then energized until the potential difference across each is equal to 10.0 V, find the charge on each capacitor and the potential difference across the connection.

Picture the Problem We can apply the properties of capacitors connected in parallel to determine the number of 1.00-μF capacitors connected in parallel it would take to store a total charge of 1.00 mC with a potential difference of 10.0 V across each capacitor. Knowing that the capacitors are connected in parallel (Parts (*a*) and (*b*)) we determine the potential difference across the combination. In Part (*c*) we can use our knowledge of how potential differences add in a series circuit to find the potential difference across the combination and the definition of capacitance to find the charge on each capacitor.



(*a*) Express the number of capacitors n in terms of the charge q on each and the total charge Q:

$$n = \frac{Q}{q}$$

Relate the charge q on one capacitor to its capacitance C and the potential difference across it:

$$q = CV$$

Substitute for q to obtain:

$$n = \frac{Q}{CV}$$

Substitute numerical values and evaluate n:

$$n = \frac{1.00\,\text{mC}}{(1.00\,\mu\text{F})(10.0\,\text{V})} = \boxed{100}$$

(*b*) Because the capacitors are connected in parallel the potential difference across the combination is the same as the potential difference across each of them:

$$V_{\substack{\text{parallel}\\ \text{combination}}} = V = \boxed{10.0\,\text{V}}$$

(*c*) With the capacitors connected in series, the potential difference across the combination will be the sum of the potential differences across the 100 capacitors:

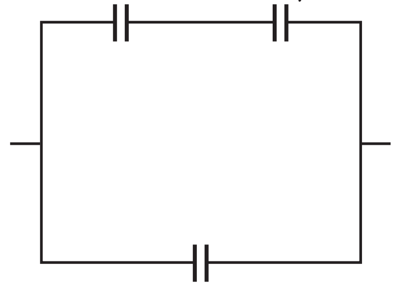
$$V_{\substack{\text{series}\\\text{combination}}} = 100(10.0\,\text{V})$$
$$= \boxed{1.00\,\text{kV}}$$

Use the definition of capacitance to find the charge on each capacitor:

$$q = CV = (1\,\mu\text{F})(10\,\text{V}) = \boxed{10.0\,\mu\text{C}}$$

27 • A 3.00-μF capacitor and a 6.00-μF capacitor are discharged and then connected in series, and the series combination is then connected in parallel with an 8.00-μF capacitor. Diagram this combination. What is the equivalent capacitance of this combination?

Picture the Problem The capacitor array is shown in the diagram. We can find the equivalent capacitance of this combination by first finding the equivalent capacitance of the 3.00-μF and 6.00-μF capacitors in series and then the equivalent capacitance of this capacitor with the 8.00-μF capacitor in parallel.



Express the equivalent capacitance for the 3.00-μF and 6.00-μF capacitors in series:

$$\frac{1}{C_{3+6}} = \frac{1}{3.00\,\mu\text{F}} + \frac{1}{6.00\,\mu\text{F}}$$

Solve for C_{3+6}:

$$C_{3+6} = 2.00\,\mu\text{F}$$

Find the equivalent capacitance of a 2.00-μF capacitor in parallel with an 8.00-μF capacitor:

$$C_{2+8} = 2.00\,\mu\text{F} + 8.00\,\mu\text{F} = \boxed{10.00\,\mu\text{F}}$$

28 • Three capacitors are connected in a triangle as shown in Figure 24-35. Find an expression for the equivalent capacitance between points *a* and *c* in terms of the three capacitance values.

Picture the Problem Because we're interested in the equivalent capacitance across terminals *a* and *c*, we need to recognize that capacitors C_1 and C_3 are in series with each other and in parallel with capacitor C_2.

Find the equivalent capacitance of C_1 and C_3 in series:

$$\frac{1}{C_{1+3}} = \frac{1}{C_1} + \frac{1}{C_3} \Rightarrow C_{1+3} = \frac{C_1C_3}{C_1 + C_3}$$

Find the equivalent capacitance of C_{1+3} and C_2 in parallel:

$$C_{\text{eq}} = C_2 + C_{1+3} = \boxed{C_2 + \frac{C_1C_3}{C_1 + C_3}}$$

29 •• A 10.0-μF capacitor and a 20.0-μF capacitor are connected in parallel across the terminals of a 6.00-V battery. (*a*) What is the equivalent capacitance of this combination? (*b*) What is the potential difference across each capacitor? (*c*) Find the charge on each capacitor. (*d*) Find the energy stored in each capacitor.

Picture the Problem Because the capacitors are connected in parallel we can add their capacitances to find the equivalent capacitance of the combination. Also, because they are in parallel, they have a common potential difference across them. We can use the definition of capacitance to find the charge on each capacitor.

(*a*) Find the equivalent capacitance of the two capacitors in parallel:

$$C_{\text{eq}} = 10.0\,\mu\text{F} + 20.0\,\mu\text{F} = \boxed{30.0\,\mu\text{F}}$$

(*b*) Because capacitors in parallel have a common potential difference across them:

$$V = V_{10} = V_{20} = \boxed{6.00\,\text{V}}$$

(*c*) Use the definition of capacitance to find the charge on each capacitor:

$$Q_{10} = C_{10}V = (10.0\,\mu\text{F})(6.00\,\text{V}) = \boxed{60.0\,\mu\text{C}}$$

and

$$Q_{20} = C_{20}V = (20.0\,\mu\text{F})(6.00\,\text{V}) = \boxed{120\,\mu\text{C}}$$

(*d*) Use $\boldsymbol{U} = \frac{1}{2}\boldsymbol{QV}$ to find the energy stored in each capacitor:

$$U_{10} = \tfrac{1}{2}Q_{10}V_{10} = \tfrac{1}{2}(60.0\,\mu\text{C})(6.00\text{ V}) = \boxed{180\,\mu\text{J}}$$

and

$$U_{20} = \tfrac{1}{2}Q_{20}V_{20} = \tfrac{1}{2}(120.0\,\mu\text{C})(6.00\text{ V}) = \boxed{360\,\mu\text{J}}$$

30 •• A 10.0-μF capacitor and a 20.0-μF capacitor are connected in parallel across the terminals of a 6.00-V battery. (*a*) What is the equivalent capacitance of this combination? (*b*) What is the potential difference across each capacitor?

(*c*) Find the charge on each capacitor. (*d*) Find the energy stored in each capacitor.

Picture the Problem We can use the properties of capacitors in series to find the equivalent capacitance and the charge on each capacitor. We can then apply the definition of capacitance to find the potential difference across each capacitor.

(*a*) Because the capacitors are connected in series they have equal charges:

$$Q_{10} = Q_{20} = C_{eq}V$$

Express the equivalent capacitance of the two capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{10.0\,\mu\text{F}} + \frac{1}{20.0\,\mu\text{F}}$$

Solve for C_{eq} to obtain:

$$C_{eq} = \frac{(10.0\,\mu\text{F})(20.0\,\mu\text{F})}{10.0\,\mu\text{F} + 20.0\,\mu\text{F}} = \boxed{6.67\,\mu\text{F}}$$

(*b*) Because the capacitors are in series, they have the same charge. Substitute numerical values to obtain:

$$Q_{10} = Q_{20} = (6.67\,\mu\text{F})(6.00\,\text{V}) = \boxed{40.0\,\mu\text{C}}$$

(*c*) Apply the definition of capacitance to find the potential difference across each capacitor:

$$V_{10} = \frac{Q_{10}}{C_{10}} = \frac{40.0\,\mu\text{C}}{10.0\,\mu\text{F}} = \boxed{4.00\,\text{V}}$$

and

$$V_{20} = \frac{Q_{20}}{C_{20}} = \frac{40.0\,\mu\text{C}}{20.0\,\mu\text{F}} = \boxed{2.00\,\text{V}}$$

(*d*) Use $U = \frac{1}{2}QV$ to find the energy stored in each capacitor:

$$U_{10} = \tfrac{1}{2}Q_{10}V_{10} = \tfrac{1}{2}(40.0\,\mu\text{C})(4.00\,\text{V}) = \boxed{80.0\,\mu\text{J}}$$

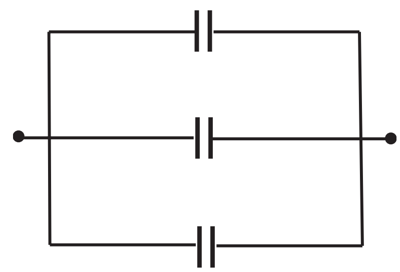
and

$$U_{20} = \tfrac{1}{2}Q_{20}V_{20} = \tfrac{1}{2}(40.0\,\mu\text{C})(2.00\,\text{V}) = \boxed{40.0\,\mu\text{J}}$$

31 •• Three identical capacitors are connected so that their maximum equivalent capacitance, which is 15.0 μF, is obtained. (*a*) Determine how the capacitors are connected and diagram the combination. (*b*) There are three additional ways to connect all three capacitors. Diagram these three ways and determine the equivalent capacitances for each arrangement.

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitances for various connection combinations.

(*a*) If their capacitance is to be a maximum, the capacitors must be connected in parallel:



Find the capacitance of each capacitor:

$$C_{eq} = 3C = 15.0\,\mu\text{F} \Rightarrow C = 5.00\,\mu\text{F}$$

(*b*) (1) Connect the three capacitors in series:

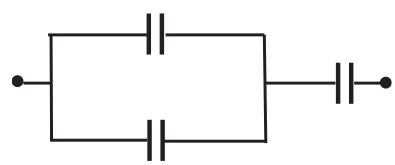
Because the capacitors are in series, their equivalent capacitance is the reciprocal of the sum of their reciprocals:

$$\frac{1}{C_{eq}} = \frac{1}{5.00\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}}$$

and

$$C_{eq} = \boxed{1.67\,\mu\text{F}}$$

(2) Connect two in parallel, with the third in series with that combination:



Find the equivalent capacitance of the two capacitors that are in parallel and then the equivalent capacitance of the network of three capacitors:

$$C_{\substack{\text{eq, two}\\ \text{in parallel}}} = 2(5.00\,\mu\text{F}) = 10.0\,\mu\text{F}$$

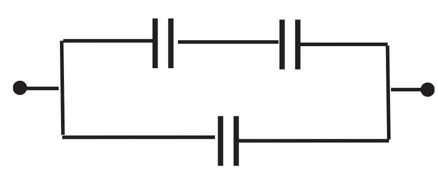
and

$$\frac{1}{C_{eq}} = \frac{1}{10.0\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}}$$

Solving for C_{eq} yields:

$$C_{eq} = \boxed{3.33\,\mu\text{F}}$$

(3) Connect two in series, with the third in parallel with that combination:



Find the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq, two in series}}}=\frac{1}{5.00\,\mu\text{F}}+\frac{1}{5.00\,\mu\text{F}}$$

or

$$C_{\text{eq, two in series}}=2.50\,\mu\text{F}$$

Find the capacitance equivalent to 2.50 μF and 5.00 μF in parallel:

$$C_{\text{eq}}=2.50\,\mu\text{F}+5.00\,\mu\text{F}=\boxed{7.50\,\mu\text{F}}$$

32 •• For the circuit shown in Figure 24-36, the capacitors were each discharged before being connected to the voltage source. Find (*a*) the equivalent capacitance of the combination, (*b*) the charge stored on the positively charge plate of each capacitor, (*c*) the voltage across each capacitor, and (*d*) the energy stored in each capacitor.

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitance between the terminals and these properties and the definition of capacitance to find the charge on each capacitor.

(*a*) Relate the equivalent capacitance of the two capacitors in series to their individual capacitances:

$$\frac{1}{C_{4+15}}=\frac{1}{4.00\,\mu\text{F}}+\frac{1}{15.0\,\mu\text{F}}$$

Solving for C_{4+15} yields:

$$C_{4+15}=3.158\,\mu\text{F}$$

Find the equivalent capacitance of C_{4+15} in parallel with the 12.0-μF capacitor:

$$C_{\text{eq}}=3.158\,\mu\text{F}+12.0\,\mu\text{F}$$
$$=15.16\,\mu\text{F}=\boxed{15.2\,\mu\text{F}}$$

(*b*) Use the definition of capacitance to find the charge stored on the 12-μF capacitor:

$$Q_{12}=C_{12}V_{12}=C_{12}V=(12.0\,\mu\text{F})(200\,\text{V})$$
$$=\boxed{2.40\,\text{mC}}$$

Because the capacitors in series have the same charge:

$$Q_4=Q_{15}=C_{4+15}V=(3.158\,\mu\text{F})(200\,\text{V})$$
$$=0.6316\,\text{mC}=\boxed{0.632\,\text{mC}}$$

(*c*) Because the 12.0-μF capacitor is connected directly across the source, the voltage across it is:

$$V_{12}=\boxed{200\,\text{V}}$$

Use the definition of capacitance to find V_4 and V_{15}:

$$V_4 = \frac{Q_4}{C_4} = \frac{0.6316\text{ mC}}{4.00\ \mu\text{F}} = \boxed{158\text{ V}}$$

and

$$V_{15} = \frac{Q_{15}}{C_{15}} = \frac{0.6316\text{ mC}}{15.0\ \mu\text{F}} = \boxed{42\text{ V}}$$

(*d*) Use $U = \frac{1}{2}QV$ to find the energy stored in each capacitor:

$$U_4 = \tfrac{1}{2}Q_4V_4 = \tfrac{1}{2}(0.6316\text{ mC})(158\text{ V}) = \boxed{49.9\text{ mJ}}$$

$$U_{15} = \tfrac{1}{2}Q_{15}V_{15} = \tfrac{1}{2}(0.6316\text{ mC})(42\text{ V}) = \boxed{13.3\text{ mJ}}$$

and

$$U_{12} = \tfrac{1}{2}Q_{12}V_{12} = \tfrac{1}{2}(2.40\text{ mC})(200\text{ V}) = \boxed{240\text{ mJ}}$$

33 •• (*a*) Show that the equivalent capacitance of two capacitors in series can be written

$$C_{\text{eq}} = \frac{C_1C_2}{C_1 + C_2}$$

(*b*) Using only this formula and some algebra, show that C_{eq} must always be less than C_1 and C_2, and hence must be less than the smaller of the two values.
(*c*) Show that the equivalent capacitance of three capacitors in series is can be written

$$C_{\text{eq}} = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}$$

(*d*) Using only this formula and some algebra, show that C_{eq} must always be less than each of C_1, C_2 and C_3, and hence must be less than the least of the three values.

Picture the Problem We can use the properties of capacitors in series to establish the results called for in this problem.

(*a*) Express the equivalent capacitance of two capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_2 + C_1}{C_1C_2}$$

Solve for C_{eq} by taking the reciprocal of both sides of the equation to obtain:

$$C_{\text{eq}} = \boxed{\frac{C_1C_2}{C_1 + C_2}}$$

(*b*) Divide numerator and denominator of this expression by C_1 to obtain:

$$C_{eq} = \frac{C_2}{1+\frac{C_2}{C_1}}$$

Because $1+\frac{C_2}{C_1} > 1$:

$$C_{eq} < C_2$$

Divide numerator and denominator of this expression by C_2 to obtain:

$$C_{eq} = \frac{C_1}{1+\frac{C_1}{C_2}}$$

Because $1+\frac{C_1}{C_2} > 1$:

$C_{eq} < C_1$, showing that C_{eq} must always be less than both C_1 and C_2, and hence must be less than the smaller of the two values.

(*c*) Using our result from part (*a*) for two of the capacitors, add a third capacitor C_3 in series to obtain:

$$\frac{1}{C_{eq}} = \frac{C_1+C_2}{C_1C_2} + \frac{1}{C_3} = \frac{C_1C_3 + C_2C_3 + C_1C_2}{C_1C_2C_3}$$

Take the reciprocal of both sides of the equation to obtain:

$$C_{eq} = \boxed{\frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}}$$

(*d*) Rewrite the result of Part (*c*) as follows:

$$C_{eq} = \left(\frac{C_1C_2}{C_1C_2 + C_2C_3 + C_1C_3}\right)C_3$$

Divide numerator and denominator of this expression by C_1C_2 to obtain:

$$C_{eq} = \left(\frac{1}{\frac{C_1C_2}{C_1C_2} + \frac{C_2C_3}{C_1C_2} + \frac{C_1C_3}{C_1C_2}}\right)C_3 = \left(\frac{1}{1+\frac{C_3}{C_1}+\frac{C_3}{C_2}}\right)C_3$$

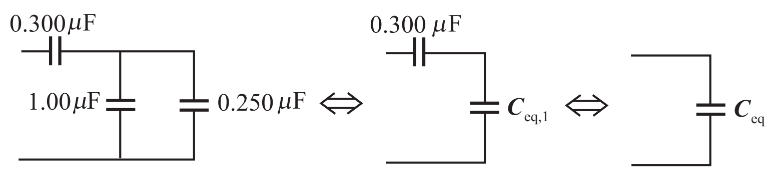
Because $1+\frac{C_3}{C_1}+\frac{C_3}{C_2} > 1$:

$$C_{eq} < C_3$$

Proceed similarly to show that $C_{eq} < C_1$ and $C_{eq} < C_2$. This shows that C_{eq} must always be less than C_1, C_2 and C_3, and hence must be less than the smaller of the three values.

34 •• For the circuit shown in Figure 24-37 find (*a*) the equivalent capacitance between the terminals, (*b*) the charge stored on the positively charge plate of each capacitor, (c) the voltage across each capacitor, and (*d*) the total stored energy.

Picture the Problem Let C_{eq1} represent the equivalent capacitance of the parallel combination and C_{eq} the total equivalent capacitance between the terminals. We can use the equations for capacitors in parallel and then in series to find C_{eq}. Because the charge on C_{eq} is the same as on the 0.300-μF capacitor and C_{eq1}, we'll know the charge on the 0.300-μF capacitor when we have found the total charge Q_{eq} stored by the circuit. We can find the charges on the 1.00-μF and 0.250-μF capacitors by first finding the potential difference across them and then using the definition of capacitance.



(*a*) Find the equivalent capacitance for the parallel combination:

$$C_{eq1} = 1.00\ \mu\text{F} + 0.250\ \mu\text{F} = 1.25\ \mu\text{F}$$

The 0.300-μF capacitor is in series with C_{eq1}. Their equivalent capacitance C_{eq} is given by:

$$\frac{1}{C_{eq}} = \frac{1}{0.300\ \mu\text{F}} + \frac{1}{1.25\ \mu\text{F}}$$

and

$$C_{eq} = 0.24194\ \mu\text{F} = \boxed{0.242\ \mu\text{F}}$$

(*b*) Express the total charge stored by the circuit Q_{eq}:

$$\begin{aligned} Q_{eq} &= Q_{0.300} = Q_{1.25} = C_{eq}V \\ &= (0.24194\ \mu\text{F})(10.0\ \text{V}) \\ &= 2.4194\ \mu\text{C} \end{aligned}$$

The 1.00-μF and 0.250-μF capacitors, being in parallel, have a common potential difference. Express this potential difference in terms of the 10.0 V across the system and the potential difference across the 0.300-μF capacitor:

$$V_{1.25} = 10.0\,\text{V} - V_{0.3} = 10.0\,\text{V} - \frac{Q_{0.3}}{C_{0.3}} = 10.0\,\text{V} - \frac{2.4194\,\mu\text{C}}{0.300\,\mu\text{F}} = 1.935\,\text{V}$$

Using the definition of capacitance, find the charge on the 1.00-μF and 0.250-μF capacitors:

$$Q_{1.00} = C_{1.00}V_{1.00} = (1.00\,\mu\text{F})(1.935\,\text{V}) = \boxed{1.9\,\mu\text{C}}$$

and

$$Q_{0.250} = C_{0.250}V_{0.250} = (0.250\,\mu\text{F})(1.935\,\text{V}) = \boxed{0.48\,\mu\text{C}}$$

Because the voltage across the parallel combination of the 1.00-μF and 0.250-μF capacitors is 1.935 V, the voltage across the 0.300-μF capacitor is 8.065 V and:

$$Q_{0.300} = C_{0.300}V_{0.300} = (0.300\,\mu\text{F})(8.065\,\text{V}) = \boxed{2.42\,\mu\text{C}}$$

(*c*) From (*b*), the voltages across the 0.300-μF capacitor and the parallel combination of the 1.00-μF and 0.250-μF capacitors is:

$$V_{0.300} = \boxed{8.06\,\text{V}}$$

and

$$V_{1.00} = V_{0.250} = 10.0\,\text{V} - 8.06\,\text{V} = \boxed{1.9\,\text{V}}$$

(*d*) The total stored energy is given by:

$$U = \tfrac{1}{2}C_{\text{eq}}V^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(0.2419\,\mu\text{F})(10.0\,\text{V})^2 = \boxed{12.1\,\mu\text{J}}$$

35 •• Five identical capacitors of capacitance C_0 are connected in a so-called "bridge" network, as shown in Figure 24-38. (*a*) What is the equivalent capacitance between points *a* and *b*? (*b*) Find the equivalent capacitance between points *a* and *b* if the capacitor at the center is replaced by a capacitor that has a capacitance of 10 C_0.

Picture the Problem Note that there are three parallel paths between *a* and *b*. We can find the equivalent capacitance of the capacitors connected in series in the upper and lower branches and then find the equivalent capacitance of three capacitors in parallel.

(*a*) Find the equivalent capacitance of the series combination of capacitors in the upper and lower branch:

$$\frac{1}{C_{eq}}=\frac{1}{C_0}+\frac{1}{C_0}\Rightarrow \mathrm{C}_{eq}=\tfrac{1}{2}C_0$$

Now we have two capacitors with capacitance $\frac{1}{2}\boldsymbol{C}_0$ in parallel with a capacitor whose capacitance is C_0. Find their equivalent capacitance:

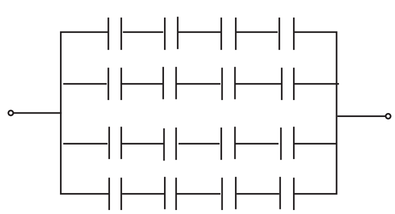
$$C'_{eq}=\tfrac{1}{2}C_0+C_0+\tfrac{1}{2}C_0=\boxed{2C_0}$$

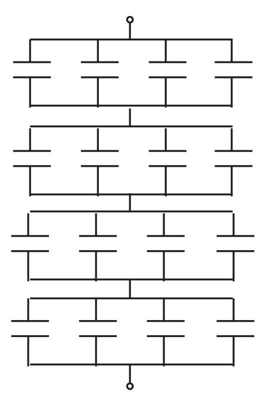
(*b*) If the central capacitance is $10C_0$, then:

$$C'_{eq}=\tfrac{1}{2}C_0+10C_0+\tfrac{1}{2}C_0=\boxed{11C_0}$$

36 •• You and your laboratory team have been given a project by your electrical engineering professor. Your team must design a network of capacitors that has a equivalent capacitance of 2.00 μF and breakdown voltage of 400 V. The restriction is that your team must use only 2.00-μF capacitors that have individual breakdown voltages of 100 V. Diagram the combination.

Picture the Problem Place four of the capacitors in series. Then the potential across each is 100 V when the potential across the combination is 400 V. The equivalent capacitance of the series combination is 0.500 μF. If we place four such series combinations in parallel, as shown in the diagram to the left, the total capacitance between the terminals is 2.00 μF. An alternative arrangement of the capacitors is shown in the diagram to the right. Placing four of the 2.00-μF capacitors in parallel results in an equivalent capacitance of 8.00 μF. Four such combinations connected in series yields a total capacitance of 2.00 μF.





37 •• Find the different equivalent capacitances that can be obtained by using two or three of the following capacitors: a 1.00-μF capacitor, a 2.00-μF capacitor, and a 4.00-μF capacitor.

Picture the Problem We can connect two capacitors in parallel, all three in parallel, two in series, three in series, two in parallel in series with the third, and two in series in parallel with the third.

Connect 2 in parallel to obtain:

$$C_{eq} = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} = \boxed{3.00\,\mu\text{F}}$$

or

$$C_{eq} = 1.00\,\mu\text{F} + 4.00\,\mu\text{F} = \boxed{5.00\,\mu\text{F}}$$

or

$$C_{eq} = 2.00\,\mu\text{F} + 4.00\,\mu\text{F} = \boxed{6.00\,\mu\text{F}}$$

Connect all three in parallel to obtain:

$$C_{eq} = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}$$
$$= \boxed{7.00\,\mu\text{F}}$$

Connect two in series:

$$C_{eq} = \frac{(1.00\,\mu\text{F})(2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F}} = \boxed{0.667\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1.00\,\mu\text{F})(4.00\,\mu\text{F})}{1.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{0.800\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2.00\,\mu\text{F})(4.00\,\mu\text{F})}{2.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{1.33\,\mu\text{F}}$$

Connecting all three in series yields:

$$C_{eq} = \frac{(1.00\,\mu\text{F})(2.00\,\mu\text{F})(4.00\,\mu\text{F})}{(1.00\,\mu\text{F})(2.00\,\mu\text{F}) + (2.00\,\mu\text{F})(4.00\,\mu\text{F}) + (1.00\,\mu\text{F})(4.00\,\mu\text{F})} = \boxed{0.571\,\mu\text{F}}$$

Connect two in parallel, and the parallel combination in series with the third:

$$C_{eq} = \frac{(4.00\,\mu\text{F})(1.00\,\mu\text{F} + 2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}}$$
$$= \boxed{1.71\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1.00\,\mu\text{F})(4.00\,\mu\text{F} + 2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}}$$
$$= \boxed{0.857\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2.00\,\mu F)(4.00\,\mu F + 1.00\,\mu F)}{1.00\,\mu F + 2.00\,\mu F + 4.00\,\mu F} = \boxed{1.43\,\mu F}$$

Connect two in series, and the series combination in parallel with the third:

$$C_{eq} = \frac{(1.00\,\mu F)(2.00\,\mu F)}{1.00\,\mu F + 2.00\,\mu F} + 4.00\,\mu F = \boxed{4.67\,\mu F}$$

or

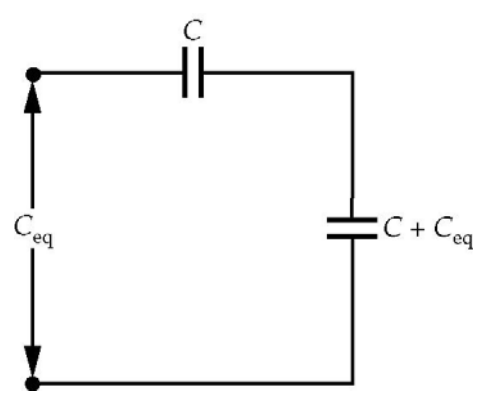
$$C_{eq} = \frac{(4.00\,\mu F)(2.00\,\mu F)}{4.00\,\mu F + 2.00\,\mu F} + 1.00\,\mu F = \boxed{2.33\,\mu F}$$

or

$$C_{eq} = \frac{(1.00\,\mu F)(4.00\,\mu F)}{1.00\,\mu F + 4.00\,\mu F} + 2.00\,\mu F = \boxed{2.80\,\mu F}$$

38 ••• What is the equivalent capacitance (in terms of C which is the capacitance of one of the capacitors) of the infinite ladder of capacitors shown in Figure 24-39?

Picture the Problem Let C be the capacitance of each capacitor in the ladder and let C_{eq} be the equivalent capacitance of the infinite ladder less the series capacitor in the first rung. Because the capacitance is finite and non-zero, adding one more stage to the ladder will not change the capacitance of the network. The capacitance of the two capacitor combination shown to the right is the equivalent of the infinite ladder, so it has capacitance C_{eq} also.



The equivalent capacitance of the parallel combination of C and C_{eq} is: $C + C_{eq}$

The equivalent capacitance of the series combination of C and $(C + C_{eq})$ is C_{eq}, so:

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{C + C_{eq}}$$

Simply this expression to obtain a quadratic equation in C_{eq}:

$$C_{eq}^2 + CC_{eq} - C^2 = 0$$

Solving for the positive value of C_{eq} yields:

$$C_{eq} = \left(\frac{\sqrt{5} - 1}{2}\right)C = \boxed{0.618C}$$

Parallel-Plate Capacitors

39 • A parallel-plate capacitor has a capacitance of 2.00 μF and a plate separation of 1.60 mm. (*a*) What is the maximum potential difference between the plates, so that dielectric breakdown of the air between the plates does not occur? (*b*) How much charge is stored at this potential difference?

Picture the Problem The potential difference *V* across a parallel-plate capacitor, the electric field *E* between its plates, and the separation *d* of the plates are related according to $V = Ed$. We can use this relationship to find V_{max} corresponding to dielectric breakdown and the definition of capacitance to find the maximum charge on the capacitor.

(*a*) Express the potential difference *V* across the plates of the capacitor in terms of the electric field between the plates *E* and their separation *d*:

$$V = Ed$$

V_{max} corresponds to E_{max}:

$$V_{max} = (3.00\,\text{MV/m})(1.60\,\text{mm}) = \boxed{4.80\,\text{kV}}$$

(*b*) Using the definition of capacitance, find the charge *Q* stored at this maximum potential difference:

$$Q = CV_{max} = (2.00\,\mu\text{F})(4.80\,\text{kV}) = \boxed{9.60\,\text{mC}}$$

40 • An electric field of 2.00×10^4 V/m exists between the circular plates of a parallel-plate capacitor that has a plate separation of 2.00 mm. (*a*) What is the potential difference across the capacitor plates? (*b*) What plate radius is required if the positively charged plate is to have a charge of 10.0 μC?

Picture the Problem The potential difference V across a parallel-plate capacitor, the electric field E between its plates, and the separation d of the plates are related according to $V = Ed$. In Part (*b*) we can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to find the required plate radius.

(*a*) Express the potential difference V across the plates of the capacitor in terms of the electric field between the plates E and their separation d:

$$V = Ed$$

Substitute numerical values and evaluate V:

$$V = \left(2.00\times10^{4}\ \text{V/m}\right)\left(2.00\,\text{mm}\right) = \boxed{40.0\,\text{V}}$$

(*b*) Use the definition of capacitance to relate the capacitance of the capacitor to its charge and the potential difference across it:

$$C = \frac{Q}{V}$$

The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \pi R^2}{d}$$

where R is the radius of the circular plates.

Equate these two expressions for C:

$$\frac{\epsilon_0 \pi R^2}{d} = \frac{Q}{V} \Rightarrow R = \sqrt{\frac{Qd}{\epsilon_0 \pi V}}$$

Substitute numerical values and evaluate R:

$$R = \sqrt{\frac{\left(10.0\,\mu\text{C}\right)\left(2.00\,\text{mm}\right)}{\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(40.0\,\text{V}\right)}} = \boxed{4.24\,\text{m}}$$

41 •• A parallel-plate, air-gap capacitor has a capacitance of 0.14 μF. The plates are 0.50 mm apart. (*a*) What is the area of each plate? (*b*) What is the potential difference between the plates if the positively charged plate has a charge of 3.2 μC? (*c*) What is the stored energy? (*d*) What is the maximum energy this capacitor can store before dielectric breakdown of the air between the plates occurs?

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor to find the area of each plate and the definition of capacitance to find the potential difference when the capacitor is charged to 3.2 μC. We can find the stored energy using $U = \frac{1}{2}CV^2$ and the definition of capacitance and the relationship between the potential difference across a parallel-plate capacitor and the electric field between its plates to find the charge at which dielectric breakdown occurs. Recall that $E_{\text{max, air}} = 3.00$ MV/m.

(*a*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} \Rightarrow A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate *A*:

$$A = \frac{(0.14\,\mu\text{F})(0.50\,\text{mm})}{8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}} = 7.906\,\text{m}^2$$

$$= \boxed{7.9\,\text{m}^2}$$

(*b*) Using the definition of capacitance, find the potential difference across the capacitor when it is charged to 3.2 μC:

$$V = \frac{Q}{C} = \frac{3.2\,\mu\text{C}}{0.14\,\mu\text{F}} = 22.9\,\text{V} = \boxed{23\,\text{V}}$$

(*c*) Express the stored energy as a function of the capacitor's capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(0.14\,\mu\text{F})(22.9\,\text{V})^2 = 36.7\,\mu\text{J}$$

$$= \boxed{37\,\mu\text{J}}$$

(*d*) The maximum energy this capacitor can store before dielectric breakdown of the air between the plates occurs is given by:

$$U_{\text{max}} = \tfrac{1}{2}CV_{\text{max}}^2$$

Relate the maximum potential difference to the maximum electric field between the plates:

$$V_{\text{max}} = E_{\text{max}}d$$

Substituting for V_{max} yields:

$$U_{\text{max}} = \tfrac{1}{2}Cd^2E_{\text{max}}^2$$

Substitute numerical values and evaluate $U_{\max}$:

$$U_{\max} = \tfrac{1}{2}(0.14\,\mu\text{F})(0.50\,\text{mm})^2(3.00\,\text{MV/m})^2 = \boxed{0.16\,\text{J}}$$

42 •• Design a 0.100-μF parallel-plate capacitor that has air between its plates and that can be charged to a maximum potential difference of 1000 V before dielectric breakdown occurs. (*a*) What is the minimum possible separation between the plates? (*b*) What minimum area must each plate of the capacitor have?

Picture the Problem The potential difference across the capacitor plates V is related to their separation d and the electric field between them according to $V = Ed$. We can use this equation with $E_{\max}$ = 3.00 MV/m to find $d_{\min}$. In Part (*b*) we can use the expression for the capacitance of a parallel-plate capacitor to find the required area of the plates.

(*a*) Use the relationship between the potential difference across the plates and the electric field between them to find the minimum separation of the plates:

$$d_{\min} = \frac{V}{E_{\max}} = \frac{1000\,\text{V}}{3.00\,\text{MV/m}} = \boxed{0.333\,\text{mm}}$$

(*b*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} \Rightarrow A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.100\,\mu\text{F})(0.333\,\text{mm})}{8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{3.76\,\text{m}^2}$$

Cylindrical Capacitors

43 • In preparation for an experiment that you will do in your introductory nuclear physics lab, you are shown the inside of a Geiger tube. You measure the radius and the length of the central wire of the Geiger tube to be 0.200 mm and 12.0 cm, respectively. The outer surface of the tube is a conducing cylindrical shell that has an inner radius of 1.50 cm. The shell is coaxial with the wire and has the same length (12.0 cm). Calculate (*a*) the capacitance of your tube, assuming that the gas in the tube has a dielectric constant of 1.00, and (*b*) the value of the linear charge density on the wire when the potential difference between the wire and shell is 1.20 kV?

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa \in_0 L/\ln(R_2/R_1)$ where L is its length and R_1 and R_2 the radii of the inner and outer conductors.

(*a*) The capacitance of the coaxial cylindrical shell is given by:

$$C = \frac{2\pi\kappa \in_0 L}{\ln\left(\frac{R_2}{R_1}\right)}$$

Substitute numerical values and evaluate C:

$$C = \frac{2\pi(1.00)(8.854\times10^{-12}\ \mathrm{C^2/N\cdot m^2})(0.120\ \mathrm{m})}{\ln\left(\frac{1.50\ \mathrm{cm}}{0.200\ \mathrm{mm}}\right)} = 1.546\ \mathrm{pF} = \boxed{1.55\ \mathrm{pF}}$$

(*b*) Use the definition of capacitance to express the charge per unit length:

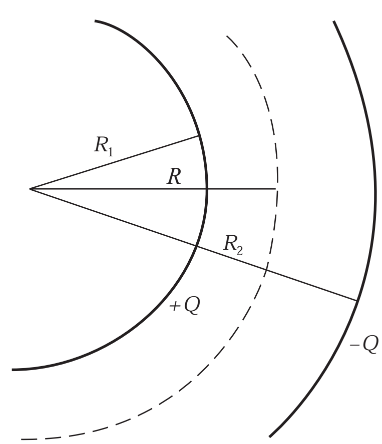
$$\lambda = \frac{Q}{L} = \frac{CV}{L}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{(1.546\ \mathrm{pF})(1.20\ \mathrm{kV})}{0.120\ \mathrm{m}} = \boxed{15.5\ \mathrm{nC/m}}$$

44 •• A cylindrical capacitor consists of a long wire that has a radius R_1, a length L and a charge $+Q$. The wire is enclosed by a coaxial outer cylindrical shell that has a inner radius R_2, length L, and charge $-Q$. (*a*) Find expressions for the electric field and energy density as a function of the distance R from the axis. (*b*) How much energy resides in a region between the conductors that has a radius R, a thickness dR, and a volume $2\pi rL\ dR$? (*c*) Integrate your expression from Part (*b*) to find the total energy stored in the capacitor. Compare your result with that obtained by using the formula $U = Q^2/(2C)$ in conjunction with the known expression for the capacitance of a cylindrical capacitor.

Picture the Problem The diagram shows a partial cross-sectional view of the inner wire and the outer cylindrical shell. By symmetry, the electric field is radial in the space between the wire and the concentric cylindrical shell. We can apply Gauss's law to cylindrical surfaces of radii $R < R_1$, $R_1 < R < R_2$, and $R > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a cylindrical surface of radius $R < R_1$ and length L to obtain:

$$E_{R<R_1}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{R<R_1} = \boxed{0}$$

Because $E = 0$ for $R < R_1$:

$$u_{R<R_1} = \boxed{0}$$

Apply Gauss's law to a cylindrical surface of radius $R_1 < R < R_2$ and length L to obtain:

$$E_{R_1<R<R_2}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

where λ is the linear charge density.

Solve for $E_{R_1<R<R_2}$ to obtain:

$$E_{R_1<R<R_2} = \frac{\lambda}{2\pi \epsilon_0 R} = \boxed{\frac{2kQ}{RL}}$$

The energy density in the region $R_1 < R < R_2$ is given by:

$$u_{R_1<R<R_2} = \tfrac{1}{2}\epsilon_0 E^2_{R_1<R<R_2}$$

Substituting for $E_{R_1<R<R_2}$ and simplifying yields:

$$u_{R_1<R<R_2} = \tfrac{1}{2}\epsilon_0\left(\frac{2k\lambda}{R}\right)^2 = \tfrac{1}{2}\epsilon_0\left(\frac{2kQ}{RL}\right)^2$$
$$= \boxed{\frac{2k^2\epsilon_0 Q^2}{R^2L^2}}$$

Apply Gauss's law to a cylindrical surface of radius $R > R_2$ and length L to obtain:

$$E_{R>R_2}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{R>R_2} = \boxed{0}$$

Because $E = 0$ for $R > R_2$:

$$u_{R>R_2} = \boxed{0}$$

(*b*) Express the energy residing in a cylindrical shell between the conductors of radius R, thickness dR, and volume $2\pi RL\,dR$:

$$dU = 2\pi RLu(R)dR$$
$$= 2\pi RL\left(\frac{2k^2\epsilon_0 Q^2}{R^2L^2}\right)dr$$
$$= \boxed{\frac{kQ^2}{RL}dR}$$

(c) Integrate dU from $R = R_1$ to $R = R_2$ to obtain:

$$U = \frac{kQ^2}{L}\int_{R_1}^{R_2}\frac{dR}{R} = \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)}$$

Use $U = \frac{1}{2}CV^2$ and the expression for the capacitance of a cylindrical capacitor to obtain:

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2\left(\dfrac{2\pi \in_0 L}{\ln\left(\dfrac{R_2}{R_1}\right)}\right)} = \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)}$$

in agreement with the result from Part (b).

45 ••• Three concentric, thin long conducting cylindrical shells have radii of 2.00 mm, 5.00 mm, and 8.00 mm. The space between the shells is filled with air. The innermost and outermost shells are connected at one end by a conducting wire. Find the capacitance per unit length of this configuration.

Picture the Problem Note that with the innermost and outermost cylinders connected together the system corresponds to two cylindrical capacitors connected in parallel. We can use $C = \dfrac{2\pi \in_0 \kappa L}{\ln(R_2/R_1)}$ to express the capacitance per unit length and then calculate and add the capacitances per unit length of each of the cylindrical shell capacitors.

Relate the capacitance of a cylindrical capacitor to its length L and inner and outer radii R_1 and R_2:

$$C = \frac{2\pi \in_0 \kappa L}{\ln(R_2/R_1)}$$

Divide both sides of the equation by L to express the capacitance per unit length:

$$\frac{C}{L} = \frac{2\pi \in_0 \kappa}{\ln(R_2/R_1)}$$

Express the capacitance per unit length of the cylindrical system:

$$\frac{C}{L} = \left(\frac{C}{L}\right)_{\text{outer}} + \left(\frac{C}{L}\right)_{\text{inner}} \qquad (1)$$

Substitute numerical values and evaluate the capacitance per unit length of the outer cylindrical shell combination:

$$\left(\frac{C}{L}\right)_{\text{outer}} = \frac{2\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(1.00)}{\ln(0.800\,\text{cm}/0.500\,\text{cm})} = 118.4\,\text{pF/m}$$

Substitute numerical values and evaluate capacitance per unit length of the inner cylindrical shell combination:

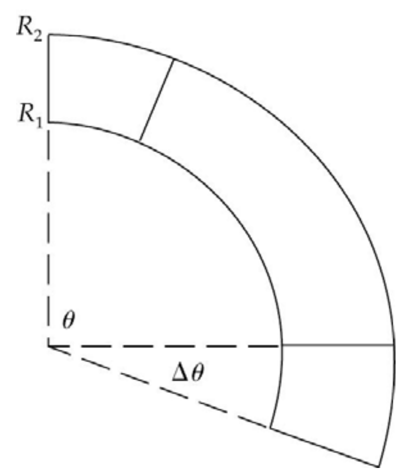
$$\left(\frac{C}{L}\right)_{\text{inner}} = \frac{2\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(1.00)}{\ln(0.500\,\text{cm}/0.200\,\text{cm})} = 60.7\,\text{pF/m}$$

Substituting numerical results in equation (1) yields:

$$\frac{C}{L} = 118.4\,\text{pF/m} + 60.7\,\text{pF/m} = \boxed{179\,\text{pF/m}}$$

46 ••• A *goniometer* is a precise instrument for measuring angles. A *capacitive goniometer* is shown in Figure 24-40*a*. Each plate of the variable capacitor (Figure 24-40*b*) consists of a flat metal semicircle that has an inner radius R_1 and an outer radius R_2. The plates share a common rotation axis, and the width of the air gap separating the plates is d. Calculate the capacitance as a function of the angle θ and the parameters given.

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor of variable area and the geometry of the figure to express the capacitance of the goniometer.



The capacitance of the parallel-plate capacitor is given by:

$$C = \frac{\in_0 (A - \Delta A)}{d}$$

The area of the plates is:

$$A = \pi\left(R_2^2 - R_1^2\right)\frac{\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\theta}{2}$$

If the top plate rotates through an angle $\Delta\theta$, then the area is reduced by:

$$\Delta A = \pi\left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}$$

Substitute for A and ΔA in the expression for C and simplify to obtain:

$$C = \frac{\epsilon_0}{d}\left[\left(R_2^2 - R_1^2\right)\frac{\theta}{2} - \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}\right]$$

$$= \boxed{\frac{\epsilon_0\left(R_2^2 - R_1^2\right)}{2d}(\theta - \Delta\theta)}$$

47 ••• A *capacitive pressure gauge* is shown in Figure 24-41. Each plate has an area A. The plates are separated by a material that has a dielectric constant κ, a thickness d, and a Young's modulus Y. If a pressure increase of ΔP is applied to the plates, derive an expression for the change in capacitance.

Picture the Problem Because A is constant, we can differentiate the expression for the capacitance of a dielectric-filled parallel-plate capacitor with respect to the plate separation d and use a differential approximation to find the change in its capacitance ΔC resulting from a pressure increase ΔP. We can then use the definition of Young's modulus to eliminate the plate separation (thickness of the dielectric material) from this expression.

The capacitance of a dielectric-filled parallel-plate capacitor is given by:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

Differentiating this expression with respect to d gives:

$$\frac{dC}{dd} = -\frac{\kappa \epsilon_0 A}{d^2}$$

or, using a differential approximation,

$$\frac{\Delta C}{\Delta d} \approx -\frac{\kappa \epsilon_0 A}{d^2}$$

Solving for ΔC and simplifying yields:

$$\Delta C \approx -\frac{\kappa \epsilon_0 A}{d^2}\Delta d = -\frac{\kappa \epsilon_0 A}{d}\left(\frac{\Delta d}{d}\right)$$

$$= -C\left(\frac{\Delta d}{d}\right)$$

From the definition of Young's modulus we have:

$$Y = \frac{P}{\Delta d/d} \Rightarrow \frac{\Delta d}{d} = -\frac{P}{Y}$$

Substituting for $\Delta d/d$ yields:

$$\Delta C = \boxed{-C\left(\frac{P}{Y}\right)}$$

Spherical Capacitors

48 •• Model Earth as a conducting sphere. (*a*) What is its self-capacitance? (*b*) Assume the magnitude of the electric field at Earth's surface is 150 V/m. What charge density does this correspond to? Express this value in fundamental charge units *e* per square centimeter

Picture the Problem (*a*) We can use the definition of capacitance and the expression for the electric potential at the surface of Earth to find Earth's self-capacitance. In Part (*b*) we can use $E = \sigma/\epsilon_0$ to find Earth's surface charge density.

(*a*) The self-capacitance of Earth is given by:

$C = \dfrac{Q}{V}$ where Q is the charge on Earth and V is the potential at its surface.

Because $V = \dfrac{kQ}{R}$ where R is the radius of Earth:

$$C = \frac{Q}{\frac{kQ}{R}} = \frac{R}{k}$$

Substitute numerical values and evaluate C:

$$C = \frac{6370\text{ km}}{8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = 0.7087\text{ mF}$$

$$= \boxed{0.709\text{ mF}}$$

(*b*) The electric field at the surface of Earth is related to Earth's charge density:

$$E = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = \epsilon_0 E$$

Substitute numerical values and evaluate σ:

$$\sigma = \left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(150\ \frac{\text{V}}{\text{m}}\right) = 1.328\frac{\text{nC}}{\text{m}^2}\times\left(\frac{1\text{ m}}{10^2\text{ cm}}\right)^2\times\frac{1e}{1.602\times10^{-19}\text{ C}}$$

$$= \boxed{829\times10^{3}\ \frac{e}{\text{cm}^2}}$$

49 •• **[SSM]** A spherical capacitor consists of a thin spherical shell that has a radius R_1 and a thin, concentric spherical shell that has a radius R_2, where $R_2 > R_1$. (*a*) Show that the capacitance is given by $C = 4\pi\epsilon_0 R_1R_2/(R_2 - R_1)$.

(*b*) Show that when the radii of the shells are nearly equal, the capacitance approximately is given by the expression for the capacitance of a parallel-plate capacitor, $C = \epsilon_0 A/d$, where A is the area of the sphere and $d = R_2 - R_1$.

Picture the Problem We can use the definition of capacitance and the expression for the potential difference between charged concentric spherical shells to show that $C = 4\pi\epsilon_0 R_1R_2/(R_2 - R_1)$.

(*a*) Using its definition, relate the capacitance of the concentric spherical shells to their charge Q and the potential difference V between their surfaces:

$$C = \frac{Q}{V}$$

Express the potential difference between the conductors:

$$V = kQ\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = kQ\frac{R_2 - R_1}{R_1R_2}$$

Substitute for V and simplify to obtain:

$$C = \frac{Q}{kQ\dfrac{R_2 - R_1}{R_1R_2}} = \frac{R_1R_2}{k(R_2 - R_1)} = \boxed{\frac{4\pi\epsilon_0 R_1R_2}{R_2 - R_1}}$$

(*b*) Because $R_2 = R_1 + d$:

$$\begin{aligned} R_1R_2 &= R_1(R_1 + d) \\ &= R_1^2 + R_1d \\ &\approx R_1^2 = R^2 \end{aligned}$$

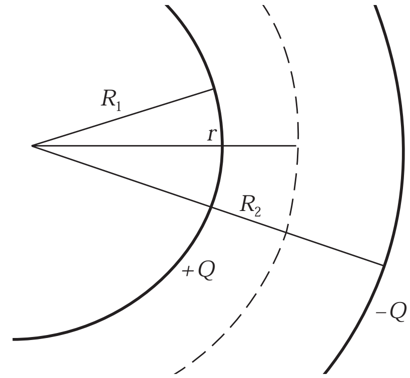
because d is small.

Substitute to obtain:

$$C \approx \frac{4\pi\epsilon_0 R^2}{d} = \boxed{\frac{\epsilon_0 A}{d}}$$

50 •• A spherical capacitor is composed of an inner sphere which has a radius R_1 and a charge $+Q$ and an outer concentric spherical thin shell which has a radius R_2 and a charge $-Q$. (*a*) Find the electric field and the energy density as a function of r, where r is the distance from the center of the sphere, for $0 \le r < \infty$. (*b*) Calculate the energy associated with the electrostatic field in a spherical shell between the conductors that has a radius r, a thickness dr, and a volume $4\pi r^2\,dr$? (*c*) Integrate your expression from Part (*b*) to find the total energy and compare your result with the result obtained using $U = \frac{1}{2}QV$.

Picture the Problem The diagram shows a partial cross-sectional view of the inner and outer spherical shells. By symmetry, the electric field is radial. We can apply Gauss's law to spherical surfaces of radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a spherical surface of radius $r < R_1$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and, because $Q_{\text{inside}} = 0$,

$$E_r = \boxed{0}$$

Because $E = 0$ for $r < R_1$:

$$u = \boxed{0}$$

Apply Gauss's law to a spherical surface of radius $R_1 < r < R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for E_r to obtain:

$$E_{r_2} = \frac{Q}{4\pi \epsilon_0 r^2} = \boxed{\frac{kQ}{r^2}} \quad R_1 \le r \le R_2$$

Express the energy density in the region $R_1 < r < R_2$:

$$u = \tfrac{1}{2}\epsilon_0 E^2_{R_1<r<R_2} = \tfrac{1}{2}\epsilon_0\left(\frac{kQ}{r^2}\right)^2$$

$$= \boxed{\frac{k^2 \epsilon_0 Q^2}{2r^4}} \quad R_1 \le r \le R_2$$

Apply Gauss's law to a cylindrical surface of radius $r > R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and, because $Q_{\text{inside}} = 0$,

$$E_r = \boxed{0}$$

Because $E = 0$ for $r > R_2$:

$$u_r = \boxed{0}$$

(*b*) Express the energy in the electrostatic field in a spherical shell of radius *r*, thickness *dr*, and volume $4\pi r^2 dr$ between the conductors:

$$dU = 4\pi r^2 u(r)dr = 4\pi r^2\left(\frac{k^2 \in_0 Q^2}{2r^4}\right)dr$$

$$= \boxed{\frac{kQ^2}{2r^2}dr}$$

(*c*) Integrate *dU* from $r = R_1$ to R_2 to obtain:

$$U = \frac{kQ^2}{2}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{kQ^2(R_2 - R_1)}{2R_1R_2}$$

$$= \boxed{\frac{1}{2}Q^2\left(\frac{R_2 - R_1}{4\pi \in_0 R_1R_2}\right)}$$

Note that the quantity in parentheses is $1/C$, so we have $U = \frac{1}{2}Q^2/C$.

51 ••• An isolated conducting sphere of radius *R* has a charge *Q* distributed uniformly over its surface. Find the distance *R*′ from the center of the sphere such that half the total electrostatic energy of the system is associated with the electric field beyond that distance.

Picture the Problem We know, from Gauss's law, that the field inside the shell is zero. We can then express the energy in the electrostatic field in a spherical shell of radius *r*, thickness *dr*, and volume $4\pi r^2 dr$ outside the given spherical shell and find the total energy in the electric field by integrating from *R* to ∞. If we then integrate the same expression from *R* to *R*′ we can find the distance *R*′ from the center of the sphere such that half the total electrostatic field energy of the system is within that distance.

Apply Gauss's law to a spherical shell of radius $r > R$ to obtain:

$$E_{r>R}\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\in_0} = \frac{Q}{\in_0}$$

Solve for $E_{r>R}$ outside the spherical shell:

$$E_{r>R} = \frac{kQ}{r^2}$$

Express the energy density in the region $r > R$:

$$u = \tfrac{1}{2}\in_0 E_{r>R}^2 = \tfrac{1}{2}\in_0\left(\frac{kQ}{r^2}\right)^2 = \frac{k^2 \in_0 Q^2}{2r^4}$$

Express the energy in the electrostatic field in a spherical shell of radius *r*, thickness *dr*, and volume $4\pi r^2 dr$ outside the spherical shell:

$$dU = 4\pi r^2 u(r)dr$$

$$= 4\pi r^2\left(\frac{k^2 \in_0 Q^2}{2r^4}\right)dr$$

$$= \frac{kQ^2}{2r^2}dr$$

Integrate dU from R to ∞ to obtain:

$$U_{\text{tot}} = \frac{kQ^2}{2}\int_R^\infty \frac{dr}{r^2} = \frac{kQ^2}{2R}$$

Integrate dU from R to R' to obtain:

$$U = \frac{kQ^2}{2}\int_R^{R'} \frac{dr}{r^2} = \frac{kQ^2}{2}\left(\frac{1}{R} - \frac{1}{R'}\right)$$

Set $U = \frac{1}{2}U_{\text{tot}}$ to obtain:

$$\frac{kQ^2}{2}\left(\frac{1}{R} - \frac{1}{R'}\right) = \frac{kQ^2}{4R} \Rightarrow R' = \boxed{2R}$$

Disconnected and Reconnected Capacitors

52 •• A 2.00-μF capacitor is energized to a potential difference of 12.0 V. The wires connecting the capacitor to the battery are then disconnected from the battery and connected across a second, initially uncharged, capacitor. The potential difference across the 2.00-μF capacitor then drops to 4.00 V. What is the capacitance of the second capacitor?

Picture the Problem Let C_1 represent the capacitance of the 2.00-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as described in the problem statement they are in parallel and, hence, share a common potential difference V'. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor.

Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (2.00\,\mu\text{F})(12.0\,\text{V}) = 24.0\,\mu\text{C}$$

Express the equivalent capacitance of the two-capacitor system and solve for C_2:

$$C_{\text{eq}} = C_1 + C_2 \Rightarrow C_2 = C_{\text{eq}} - C_1$$

Using the definition of capacitance, express C_{eq} in terms of the total charge Q' of the two-capacitor system and the common potential difference V' across the capacitors:

$$C_{\text{eq}} = \frac{Q'}{V'}$$

Substitute for C_{eq} to obtain:

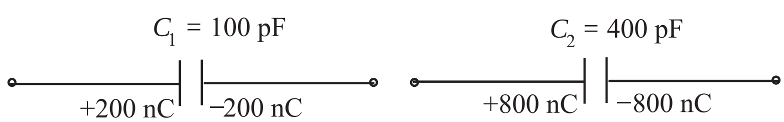
$$C_2 = \frac{Q'}{V'} - C_1$$

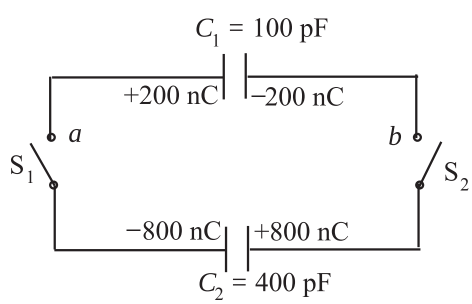
Substitute numerical values and evaluate C_2:

$$C_2 = \frac{24.0\,\mu\text{C}}{4.00\,\text{V}} - 2.00\,\mu\text{F} = \boxed{4.00\,\mu\text{F}}$$

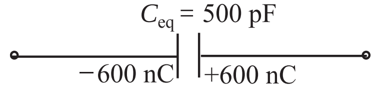
53 •• [SSM] A 100-pF capacitor and a 400-pF capacitor are both charged to 2.00 kV. They are then disconnected from the voltage source and are connected together, positive plate to negative plate and negative plate to positive plate. (*a*) Find the resulting potential difference across each capacitor. (*b*) Find the energy dissipated when the connections are made.

Picture the Problem (*a*) Just after the two capacitors are disconnected from the voltage source, the 100-pF capacitor carries a charge of 200 nC and the 400-pF capacitor carries a charge of 800 nC. After switches S_1 and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b*, and the equivalent capacitance of the system is $C_{eq} = C_{100} + C_{400}$. The plates to the right in the diagram below form a single conductor with a charge of 600 nC, and the plates to the left form a conductor with charge $-Q = -600$ nC. The potential difference across each capacitor is $V = Q/C_{eq}$. In Part (*b*) we can find the energy dissipated when the connections are made by subtracting the energy stored in the system after they are connected from the energy stored in the system before they are connected.





(*a*) When the switches are closed and the capacitors are connected together, their initial charges redistribute and the final charge on the two-capacitor system is 600 nC and the equivalent capacitance is 500 pF:



The potential difference across each capacitor is the potential difference across the equivalent capacitor:

$$V = \frac{Q}{C_{eq}} = \frac{600\text{ nC}}{500\text{ pF}} = \boxed{1.20\text{ kV}}$$

(*b*) The energy dissipated when the capacitors are connected is the difference between the energy stored after they are connected and the energy stored before they were connected:

$$U_{dissipated} = U_{before} - U_{after} \quad (1)$$

U_{before} is given by:

$$U_{before} = U_1 + U_2 = \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2 = \tfrac{1}{2}(Q_1 + Q_2)V_i$$

where V_i is the charging voltage.

U_{after} is given by:

$$U_{after} = \tfrac{1}{2}QV$$

where Q is the total charge stored after the capacitors have been connected and V is the voltage found in Part (*a*).

Substitute for U_{before} and U_{after} in equation (1) and simplify to obtain:

$$U_{dissipated} = \tfrac{1}{2}(Q_1 + Q_2)V_i - \tfrac{1}{2}QV$$

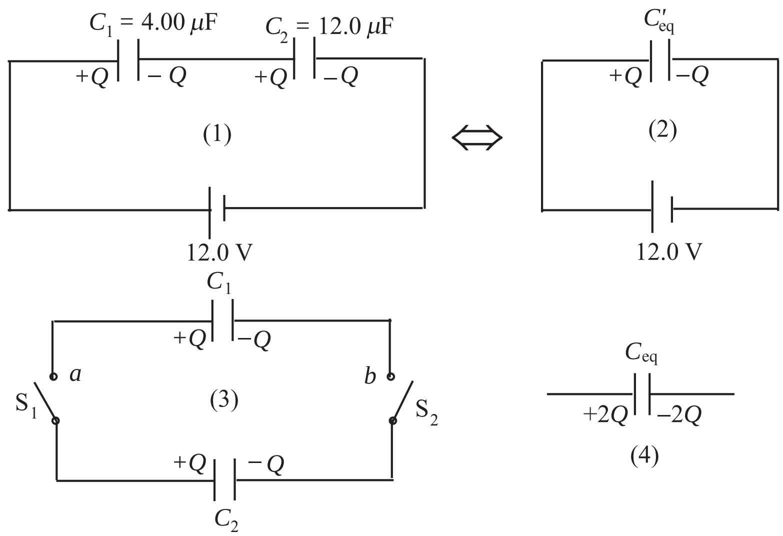
Substitute numerical values and evaluate $U_{dissipated}$:

$$U_{dissipated} = \tfrac{1}{2}(200\text{ nC} + 800\text{ nC})(2.00\text{ kV}) - \tfrac{1}{2}(600\text{ nC})(1.20\text{ kV}) = \boxed{640\ \mu\text{J}}$$

54 •• Two capacitors, one that has a capacitance of 4.00 μF and one that has a capacitance of 12.0 μF, are first discharged and then are connected in series. The series combination is then connected across the terminals of a 12.0-V battery. Next they are carefully disconnected so that they are not discharged and they are then reconnected to each other—positive plate to positive plate and negative plate to negative plate. (*a*) Find the potential difference across each capacitor after they are reconnected. (*b*) Find the energy stored in the capacitors before they are disconnected from the battery, and find the energy stored after they are reconnected.

Picture the Problem Let C_1 represent the capacitance of the 4.00-μF capacitor and C_2 the capacitance of the 12.0-μF capacitor. (*a*) Just after the two capacitors are disconnected from the battery, they both carry charge Q. After switches S_1

and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b,* and each will have the charge it acquired while they were connected in series across the battery. We can use the definition of capacitance and the equivalent capacitance of the two capacitors to find the common potential difference across them. In Part (*b*) we can use $U = \frac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.



(*a*) From diagram (4):

$$V_{4.00} = V_{12.0} = \frac{2Q}{C_{eq}} \quad (1)$$

where Q is the charge on each capacitor *before* they are disconnected.

Find the equivalent capacitance of the two capacitors when they are connected in parallel:

$$C_{eq} = C_1 + C_2 = 4.00\,\mu\text{F} + 12.0\,\mu\text{F} = 16.0\,\mu\text{F}$$

Express the charge Q on each capacitor before they are disconnected:

$$Q = C'_{eq}V$$

Express the equivalent capacitance of the two capacitors connected in series:

$$C'_{eq} = \frac{C_1C_2}{C_1 + C_2} = \frac{(4.00\,\mu\text{F})(12.0\,\mu\text{F})}{4.00\,\mu\text{F} + 12.0\,\mu\text{F}} = 3.00\,\mu\text{F}$$

Substitute to find Q:

$$Q = (3.00\,\mu\text{F})(12.0\,\text{V}) = 36.0\,\mu\text{C}$$

Substitute numerical values for Q and C_{eq} in equation (1) and evaluate $V_{4.00} = V_{12.0}$:

$$V_{4.00} = V_{12.0} = \frac{2(36.0\,\mu\text{C})}{16.0\,\mu\text{F}} = \boxed{4.50\,\text{V}}$$

(*b*) The energy stored in the capacitors initially is:

$$U_i = \tfrac{1}{2}C'_{eq}V_i^2 = \tfrac{1}{2}(3.00\,\mu\text{F})(12.0\,\text{V})^2 = \boxed{216\,\mu\text{J}}$$

The energy stored in the capacitors when they have been reconnected is:

$$U_f = \tfrac{1}{2}C_{eq}V_f^2 = \tfrac{1}{2}(16.0\,\mu\text{F})(4.50\,\text{V})^2 = \boxed{162\,\mu\text{J}}$$

55 •• A 1.2-μF capacitor is charged to 30 V. After charging, the capacitor is disconnected from the voltage source and is connected across the terminals of a second capacitor that had previously been discharged. The final voltage across the 1.2-μF capacitor is 10 V. (*a*) What is the capacitance of the second capacitor? (*b*) How much energy was dissipated when the connection was made?

Picture the Problem Let C_1 represent the capacitance of the 1.2-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as described in the problem statement they are in parallel and, hence, share a common potential difference. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor. In Part (*b*) we can use $U = \tfrac{1}{2}CV^2$ to find the energy before and after the connection was made and, hence, the energy dissipated when the connection was made.

(*a*) Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (1.2\,\mu\text{F})(30\,\text{V}) = 36\,\mu\text{C}$$

Because the capacitors are in parallel:

$$C_{eq} = C_1 + C_2 \Rightarrow C_2 = C_{eq} - C_1 \quad (1)$$

Using the definition of capacitance, express C_{eq} in terms of the charge Q_2 on the second capacitor and the common potential difference V_2 across the two capacitors:

$$C_{eq} = \frac{Q_2}{V_2} = \frac{Q_1}{V_2}$$

Substituting for C_{eq} in equation (1) yields:

$$C_2 = \frac{Q_1}{V_2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{36\,\mu\text{C}}{10\,\text{V}} - 1.2\,\mu\text{F} = \boxed{2.4\,\mu\text{F}}$$

(*b*) The energy dissipated when the connections are made in terms of the energy stored in the capacitors before and after their connection is given by:

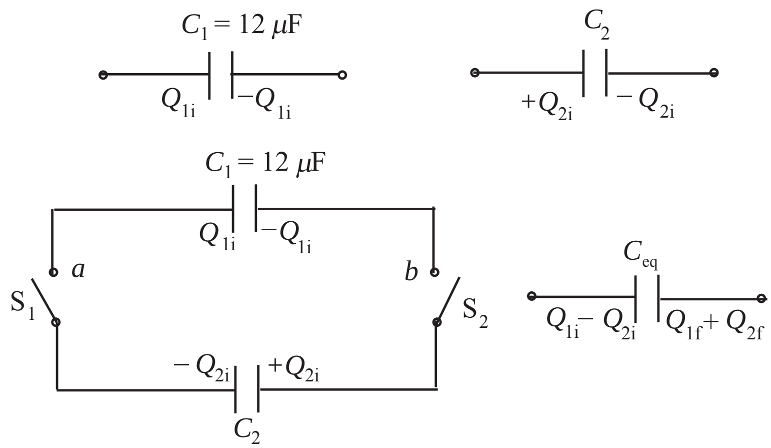
$$\begin{aligned} U_{\text{dissipated}} &= U_{\text{before}} - U_{\text{after}} \\ &= \tfrac{1}{2}C_1V_1^2 - \tfrac{1}{2}C_{\text{eq}}V_{\text{f}}^2 \\ &= \tfrac{1}{2}\left(C_1V_1^2 - C_{\text{eq}}V_{\text{f}}^2\right) \end{aligned}$$

Substitute numerical values and evaluate $U_{\text{dissipated}}$:

$$U_{\text{dissipated}} = \tfrac{1}{2}\left[(1.2\,\mu\text{F})(30\,\text{V})^2 - (3.6\,\mu\text{F})(10\,\text{V})^2\right] = \boxed{0.4\,\text{mJ}}$$

56 •• A 12-μF capacitor and a capacitor of unknown capacitance are both charged to 2.00 kV. After charging, the capacitors are disconnected from the voltage source. The capacitors are then connected to each other—positive plate to negative plate and negative plate to positive plate. The final voltage across the 12-μF capacitor is 1.00 kV. (*a*) What is the capacitance of the second capacitor? (*b*) How much energy was dissipated when the connection was made?

Picture the Problem Let C_1 represent the capacitance of the 12-μF capacitor and C_2 the capacitance of the second capacitor. (*a*) Just after the two capacitors are disconnected from the voltage source, the 12-μF capacitor carries a charge of Q_{1i} = 24 mC and the unknown capacitor carries a charge Q_{2i}. After switches S_1 and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b,* and the equivalent capacitance of the system is $C_{\text{eq}} = C_1 + C_2$. The plates to the right in the diagram below form a single conductor with a charge of $Q_{1i} - Q_{2i}$, and the plates to the left form a conductor with charge $Q_{1f} + Q_{2f}$, where Q_{1i}, Q_{2i}, Q_{1f} and Q_{2f} are all positive . Because they are in parallel, the potential difference across both C_1 and C_2 when they are connected is 1.00 kV. In Part (*b*) we can find the energy dissipated when the connections are made by subtracting the energy stored in the system after they are connected from the energy stored in the system before they are connected.



(*a*) When the switches are closed and the capacitors are connected together, their initial charges redistribute. Apply conservation of charge to obtain:

$$Q_{1i} - Q_{2i} = Q_{1f} + Q_{2f}$$

or

$$C_1V_i - C_2V_i = C_1V_f + C_2V_f$$

Solving for C_2 yields:

$$C_2 = \frac{V_i - V_f}{V_i + V_f}C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \left(\frac{2.00\ \text{kV} - 1.00\ \text{kV}}{2.00\ \text{kV} + 1.00\ \text{kV}}\right)(12\ \mu\text{F}) = \boxed{4.0\ \mu\text{F}}$$

(*b*) The energy dissipated when the connections are made is the difference between the initial and final energies stored by the system:

$$U_{\text{dissipated}} = U_i - U_f \qquad (1)$$

U_i and U_f are given by:

$$U_i = \tfrac{1}{2}C_1V_i^2 + \tfrac{1}{2}C_2V_i^2 = \tfrac{1}{2}(C_1 + C_2)V_i^2$$

and

$$U_f = \tfrac{1}{2}C_{eq}V_f^2$$

Because C_1 and C_2 are in parallel:

$$C_{eq} = C_1 + C_2$$

Substituting for C_{eq} yields:

$$U_f = \tfrac{1}{2}(C_1 + C_2)V_f^2$$

Substitute for U_i and U_f in equation (1) to obtain:

$$U_{\text{dissipated}} = \tfrac{1}{2}(C_1 + C_2)V_i^2 - \tfrac{1}{2}(C_1 + C_2)V_f^2 = \tfrac{1}{2}(C_1 + C_2)(V_i^2 - V_f^2)$$

Substitute numerical values and evaluate $U_{\text{dissipated}}$:

$$U_{\text{dissipated}} = \tfrac{1}{2}(12\,\mu\text{F} + 4.0\,\mu\text{F})\left[(2.00\,\text{kV})^2 - (1.00\,\text{kV})^2\right] = \boxed{24\,\text{J}}$$

57 •• Two capacitors, one that has a capacitance of 4.00 μF and one that has a capacitance of 12.0 μF, are connected in parallel. The parallel combination is then connected across the terminals of a 12.0-V battery. Next they are carefully disconnected so that they are not discharged. They are then reconnected to each other—the positive plate of each capacitor connected to the negative plate of the other. (*a*) Find the potential difference across each capacitor after they are reconnected. (*b*) Find the energy stored in the capacitors before they are disconnected from the battery, and find the energy stored after they are reconnected.

Picture the Problem When the capacitors are reconnected, each will have a charge equal to the difference between the charges they acquired while they were connected in parallel across the 12.0-V battery. We can use the definition of capacitance and their equivalent capacitance to find the common potential difference across them. In Part (*b*) we can use $U = \frac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.

(*a*) Using the definition of capacitance, express the potential difference across the capacitors when they are reconnected:

$$V_{\text{f}} = V_{4.00} = V_{12.0} = \frac{Q_{\text{f}}}{C_{\text{eq}}} = \frac{Q_{\text{f}}}{C_1 + C_2} \quad (1)$$

where Q_{f} is the common charge on the capacitors *after* they are reconnected.

Express the final charge Q_{f} on each capacitor:

$$Q_{\text{f}} = Q_2 - Q_1$$

Use the definition of capacitance to substitute for Q_2 and Q_1:

$$Q_{\text{f}} = C_2V - C_1V = (C_2 - C_1)V$$

Substitute in equation (1) to obtain:

$$V_{\text{f}} = \frac{C_2 - C_1}{C_1 + C_2}V$$

Substitute numerical values and evaluate V_{f}:

$$V_{\text{f}} = V_{4.00} = V_{12.0} = \frac{12.0\,\mu\text{F} - 4.00\,\mu\text{F}}{12.0\,\mu\text{F} + 4.00\,\mu\text{F}}(12.0\,\text{V}) = \boxed{6.0\,\text{V}}$$

(*b*) The energy stored in the capacitors initially is given by:

$$U_i = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 = \tfrac{1}{2}V^2(C_1 + C_2)$$

Substitute numerical values and evaluate U_i:

$$U_i = \tfrac{1}{2}(12.0\,\text{V})^2(12.0\,\mu\text{F} + 4.00\,\mu\text{F}) = \boxed{1.15\,\text{mJ}}$$

The energy stored in the capacitors when they have been reconnected is given by:

$$U_f = \tfrac{1}{2}C_1V_f^2 + \tfrac{1}{2}C_2V_f^2 = \tfrac{1}{2}V_f^2(C_1 + C_2)$$

Substitute numerical values and evaluate U_f:

$$U_f = \tfrac{1}{2}(6.0\,\text{V})^2(12.0\,\mu\text{F} + 4.00\,\mu\text{F}) = \boxed{0.29\,\text{mJ}}$$

58 •• A 20-pF capacitor is charged to 3.0 kV and then removed from the battery and connected to an uncharged 50-pF capacitor. (*a*) What is the new charge on each capacitor? (*b*) Find the energy stored in the 20-pF capacitor before it is disconnected from the battery, and the energy stored in the two capacitors after they are connected to each other. Does the stored energy increase or decrease when the two capacitors are connected to each other?

Picture the Problem Let the numeral 1 refer to the 20-pF capacitor and the numeral 2 to the 50-pF capacitor. We can use conservation of charge and the fact that the connected capacitors will have the same potential difference across them to find the charge on each capacitor. We can decide whether stored energy increases or decreases when the two capacitors are connected by calculating the change in the electrostatic energy during this process.

(*a*) The final charges on the capacitors are given by:

$$Q_{1f} = C_1V_f \qquad (1)$$

and

$$Q_{2f} = C_2V_f \qquad (2)$$

Using the fact that charge is conserved when the capacitors are connected, relate the charge Q_{1i} initially on the 20-pF capacitor to the charges on the two capacitors when they have been connected:

$$Q_{1i} = Q_{1f} + Q_{2f}$$

or

$$C_1V_{1i} = C_1V_f + C_2V_f$$

Solving for V_f yields:

$$V_f = \frac{C_1}{C_1 + C_2}V_{1i}$$

Substitute for V_f in equations (1) and (2) to obtain:

$$Q_{1f} = \frac{C_1^2}{C_1 + C_2} V_{1i}$$

and

$$Q_{2f} = \frac{C_1 C_2}{C_1 + C_2} V_{1i}$$

Substitute numerical values and evaluate Q_{1f}:

$$Q_{1f} = \left(\frac{(20\text{ pF})^2}{20\text{ pF} + 50\text{ pF}}\right)(3.0\text{ kV}) = 17.1\text{ nC} = \boxed{17\text{ nC}}$$

Substitute numerical values and evaluate Q_{2f}:

$$Q_{2f} = \left(\frac{(20\text{ pF})(50\text{ pF})}{20\text{ pF} + 50\text{ pF}}\right)(3.0\text{ kV}) = 42.9\text{ nC} = \boxed{43\text{ nC}}$$

(*b*) The energy stored in the 20-pF capacitor before it is disconnected from the battery is given by:

$$U_i = U_{1i} = \tfrac{1}{2} C_1 V_{1i}^2 = \tfrac{1}{2}(20\text{ pF})(3.0\text{ kV})^2 = \boxed{90\ \mu\text{J}}$$

The energy stored in the two capacitors after they are connected to each other is given by:

$$U_f = U_{1f} + U_{2f} = \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2}$$

Substitute numerical values and evaluate U_f:

$$U_f = \frac{(17.1\text{ nC})^2}{2(20\text{ pF})} + \frac{(42.9\text{ nC})^2}{2(50\text{ pF})} = \boxed{26\ \mu\text{J}}$$

Because $U_f < U_i$, the stored energy decreases when the two capacitors are connected.

59 •• [SSM] Capacitors 1, 2 and 3, have capacitances equal to 2.00 μF, 4.00 μF, and 6.00 μF, respectively. The capacitors are connected in parallel, and the parallel combination is connected across the terminals of a 200-V source. The capacitors are then disconnected from both the voltage source and each other, and are connected to three switches as shown in Figure 24-42. (*a*) What is the potential difference across each capacitor when switches S_1 and S_2 are closed but switch S_3 remains open? (*b*) After switch S_3 is closed, what is the final charge on the leftmost plate of each capacitor? (*c*) Give the final potential difference across each capacitor after switch S_3 is closed.

Picture the Problem Let lower case *q*s refer to the charges before S_3 is closed and upper case *Q*s refer to the charges after this switch is closed. We can use conservation of charge to relate the charges on the capacitors before S_3 is closed

to their charges when this switch is closed. We also know that the sum of the potential differences around the circuit when S_3 is closed must be zero and can use this to obtain a fourth equation relating the charges on the capacitors after the switch is closed to their capacitances. Solving these equations simultaneously will yield the charges Q_1, Q_2, and Q_3. Knowing these charges, we can use the definition of capacitance to find the potential difference across each of the capacitors.

(*a*) With S_1 and S_2 closed, but S_3 open, the charges on and the potential differences across the capacitors do not change. Hence:

$$V_1 = V_2 = V_3 = \boxed{200\,\text{V}}$$

(*b*) When S_3 is closed, the charges can redistribute; express the conditions on the charges that must be satisfied as a result of this redistribution:

$q_2 - q_1 = Q_2 - Q_1$,
$q_3 - q_2 = Q_3 - Q_2$,
and
$q_1 - q_3 = Q_1 - Q_3$.

Express the condition on the potential differences that must be satisfied when S_3 is closed:

$V_1 + V_2 + V_3 = 0$
where the subscripts refer to the three capacitors.

Use the definition of capacitance to eliminate the potential differences:

$$\frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} = 0 \qquad (1)$$

Use the definition of capacitance to find the initial charge on each capacitor:

$q_1 = C_1V = (2.00\,\mu\text{F})(200\,\text{V}) = 400\,\mu\text{C}$,
$q_2 = C_2V = (4.00\,\mu\text{F})(200\,\text{V}) = 800\,\mu\text{C}$,
and
$q_3 = C_3V = (6.00\,\mu\text{F})(200\,\text{V}) = 1200\,\mu\text{C}$

Let $q = q_1$. Then:

$q_2 = 2Q$ and $q_3 = 3Q$

Express Q_2 and Q_3 in terms of Q_1 and Q:

$$Q_2 = Q + Q_1 \qquad (2)$$
and
$$Q_3 = Q_1 + 2Q \qquad (3)$$

Substitute in equation (1) to obtain:

$$\frac{Q_1}{C_1}+\frac{Q+Q_1}{C_2}+\frac{Q_1+2Q}{C_3}=0$$

or

$$\frac{Q_1}{2.00\,\mu\text{F}}+\frac{Q+Q_1}{4.00\,\mu\text{F}}+\frac{Q_1+2Q}{6.00\,\mu\text{F}}=0$$

Solve for and evaluate Q_1 to obtain:

$$Q_1=-\tfrac{7}{11}Q=-\tfrac{7}{11}(400\,\mu\text{C})=\boxed{-255\,\mu\text{C}}$$

Substitute in equation (2) to obtain:

$$Q_2=400\,\mu\text{C}-255\,\mu\text{C}=\boxed{145\,\mu\text{C}}$$

Substitute in equation (3) to obtain:

$$Q_3=-255\,\mu\text{C}+2(400\,\mu\text{C})=\boxed{545\,\mu\text{C}}$$

(*c*) Use the definition of capacitance to find the potential difference across each capacitor with S_3 closed:

$$V_1=\frac{Q_1}{C_1}=\frac{-255\,\mu\text{C}}{2.00\,\mu\text{F}}=\boxed{-127\,\text{V}},$$

$$V_2=\frac{Q_2}{C_2}=\frac{145\,\mu\text{C}}{4.00\,\mu\text{F}}=\boxed{36.3\,\text{V}},$$

and

$$V_3=\frac{Q_3}{C_3}=\frac{545\,\mu\text{C}}{6.00\,\mu\text{F}}=\boxed{90.8\,\text{V}}$$

60 •• A capacitor has a capacitance *C* and a charge *Q* on its positively charged plate. A student connects one terminal of the capacitor to a terminal of an identical capacitor whose plates are electrically neutral. When the remaining two terminals are connected, charge flows until electrostatic equilibrium is reestablished and both capacitors have charge *Q*/2 on them. Compare the total energy initially stored in the one capacitor to the total energy stored in the two when electrostatic equilibrium is reestablished. If there is less energy afterward, where do you think the missing energy went? *Hint: Wires that transport charge can heat up, which is called Joule heating and is discussed in detail in Chapter 25.*

Picture the Problem We can use the expression for the energy stored in a capacitor to express the ratio of the energy stored in the system after the discharge of the first capacitor to the energy stored in the system prior to the discharge.

Express the energy *U* initially stored in the capacitor whose capacitance is *C*:

$$U=\frac{Q^2}{2C}$$

The energy U' stored in the two capacitors after the first capacitor has discharged is:

$$U' = \frac{\left(\frac{Q}{2}\right)^2}{2C} + \frac{\left(\frac{Q}{2}\right)^2}{2C} = \frac{Q^2}{4C}$$

Express the ratio of U' to U:

$$\frac{U'}{U} = \frac{\frac{Q^2}{4C}}{\frac{Q^2}{2C}} = \frac{1}{2} \Rightarrow U' = \boxed{\tfrac{1}{2}U}$$

The missing energy was converted into thermal energy by the resistance of the connecting wires.

Dielectrics

61 • You are a laboratory assistant in a physics department that has budget problems. Your supervisor wants to construct cheap parallel-plate capacitors for use in introductory laboratory experiments. The design uses polyethylene, which has a dielectric constant of 2.30, between two sheets of aluminum foil. The area of each sheet of foil is 400 cm^2 and the thickness of the polyethylene is 0.300 mm. Find the capacitance of this arrangement.

Picture the Problem The capacitance of a parallel-plate capacitor filled with a dielectric of constant κ is given by $C = \frac{\kappa \in_0 A}{d}$.

Relate the capacitance of the parallel-plate capacitor to the area of its plates, their separation, and the dielectric constant of the material between the plates:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute numerical values and evaluate C:

$$C = \frac{(2.30)\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(400\,\text{cm}^2\right)}{0.300\,\text{mm}} = \boxed{2.72\,\text{nF}}$$

62 •• The radius and the length of the central wire in a Geiger tube are 0.200 mm and 12.0 cm, respectively. The outer surface of the tube is a conducing cylindrical shell that has an inner radius of 1.50 cm. The shell is coaxial with the wire and has the same length (12.0 cm).The tube is filled with a gas that has a dielectric constant of 1.08 and a dielectric strength of 2.00×10^6 V/m. (*a*) What is

the maximum potential difference that can be maintained between the wire and shell? (*b*) What is the maximum charge per unit length on the wire?

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa \in_0 L/\ln(r_2/r_1)$, where L is its length and r_1 and r_2 the radii of the inner and outer conductors. We can use this expression, in conjunction with the definition of capacitance, to express the potential difference between the wire and the cylindrical shell in the Geiger tube. Because the electric field E in the tube is related to the linear charge density λ on the wire according to $E = 2k\lambda/\kappa r$, we can use this expression to find $2k\lambda/\kappa$ for $E = E_{max}$. In Part (*b*) we'll use this relationship to find the charge per unit length λ on the wire.

(*a*) Use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to express the potential difference between the wire and the cylindrical shell in the tube:

$$\Delta V = \frac{Q}{C} = \frac{Q}{\dfrac{2\pi\kappa \in_0 L}{\ln(R/r)}} = \frac{2\lambda}{4\pi \in_0 \kappa}\ln\left(\frac{R}{r}\right) = \frac{2k\lambda}{\kappa}\ln\left(\frac{R}{r}\right)$$

where λ is the linear charge density, κ is the dielectric constant of the gas in the Geiger tube, r is the radius of the wire, and R the radius of the coaxial cylindrical shell of length L.

Express the electric field at a distance r greater than its radius from the center of the wire:

$$E = \frac{2k\lambda}{\kappa r} \Rightarrow \frac{2k\lambda}{\kappa} = Er \qquad (1)$$

Substituting for $\dfrac{2k\lambda}{\kappa}$ yields:

$$\Delta V = Er\ln\left(\frac{R}{r}\right)$$

Noting that E is a maximum at $r = 0.200$ mm, substitute numerical values and evaluate ΔV_{max}:

$$\Delta V_{max} = (2.00\times10^6\ \text{V/m})(0.200\,\text{mm})\ln\left(\frac{1.50\,\text{cm}}{0.200\,\text{mm}}\right) = \boxed{1.73\,\text{kV}}$$

(*b*) Solve equation (1) for λ:

$$\lambda = \frac{E_{max}\kappa r}{2k}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{(1.08)(2.00\times10^6\ \text{V/m})(0.200\,\text{mm})}{2(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)} = \boxed{24.0\,\text{nC/m}}$$

63 •• You are a materials science engineer and your group has fabricated a new dielectric, that has an exceptionally large dielectric constant of 24 and a dielectric strength of 4.0×10^7 V/m. Suppose you want to use this material to construct a 0.10-μF parallel plate capacitor that can withstand a potential difference of 2.0 kV. (*a*) What is the minimum plate separation required to do this? (*b*) What is the area of each plate at this separation?

Picture the Problem We can use the relationship between the electric field between the plates of a capacitor, their separation, and the potential difference between them to find the minimum plate separation. We can use the expression for the capacitance of a dielectric-filled parallel-plate capacitor to determine the necessary area of the plates.

(*a*) Relate the electric field of the capacitor to the potential difference across its plates:

$$E = \frac{V}{d} \Rightarrow d = \frac{V}{E}$$

where d is the plate separation.

Noting that d_{min} corresponds to E_{max}, evaluate d_{min}:

$$d_{\text{min}} = \frac{V}{E_{\text{max}}} = \frac{2000\,\text{V}}{4.0\times10^7\ \text{V/m}} = \boxed{50\,\mu\text{m}}$$

(*b*) Relate the capacitance of a parallel-plate capacitor to the area of its plates:

$$C = \frac{\kappa \in_0 A}{d} \Rightarrow A = \frac{Cd}{\kappa \in_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.10\,\mu\text{F})(50\,\mu\text{m})}{24(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)} = 2.35\times10^{-2}\ \text{m}^2 = \boxed{240\,\text{cm}^2}$$

64 •• A parallel-plate capacitor has plates separated by a distance d. The capacitance of this capacitor is C_0 when no dielectric is in the space between the plates. However, the space between the plates is completely filled by two different dielectrics. One dielectric has a thickness $\frac{1}{4}d$ and a dielectric constant κ_1, and the other dielectric has a thickness $\frac{3}{4}d$ and a dielectric constant κ_2. Find the capacitance of this capacitor.

Picture the Problem We can model this system as two capacitors in series, one of thickness $\frac{1}{4}d$ and the other of thickness $\frac{3}{4}d$ and use the equation for the equivalent capacitance of two capacitors connected in series. Let the capacitance of the capacitor whose dielectric constant is κ_1 be C_1 and the capacitance of the capacitor whose dielectric constant is κ_2 be C_2.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq}}}=\frac{1}{C_1}+\frac{1}{C_2}\Rightarrow C_{\text{eq}}=\frac{C_1C_2}{C_1+C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1=\frac{\kappa_1 \epsilon_0 A}{\frac{1}{4}d}=\frac{4\kappa_1 \epsilon_0 A}{d}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2=\frac{\kappa_2 \epsilon_0 A}{\frac{3}{4}d}=\frac{4\kappa_2 \epsilon_0 A}{3d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{\text{eq}}=\frac{\left(\frac{4\kappa_1 \epsilon_0 A}{d}\right)\left(\frac{4\kappa_2 \epsilon_0 A}{3d}\right)}{\frac{4\kappa_1 \epsilon_0 A}{d}+\frac{4\kappa_2 \epsilon_0 A}{3d}}=\frac{\left(\frac{\kappa_1}{d}\right)\left(\frac{4\kappa_2}{3d}\right)}{\frac{3\kappa_1}{3d}+\frac{\kappa_2}{3d}}\epsilon_0 A=\frac{\frac{4\kappa_1\kappa_2}{d}}{3\kappa_1+\kappa_2}\epsilon_0 A$$

$$=\frac{4\kappa_1\kappa_2}{3\kappa_1+\kappa_2}\left(\frac{\epsilon_0 A}{d}\right)=\boxed{\left(\frac{4\kappa_1\kappa_2}{3\kappa_1+\kappa_2}\right)C_0}$$

65 •• Two capacitors each have two conducting plates of surface area A and an air gap of width d. They are connected in parallel, as shown in Figure 24-43 and each has a charge Q on the positively charged plate. A slab that has a width d, an area A, and a dielectric constant κ is inserted between the plates of *one* of the capacitors. Calculate the new charge Q' on the positively charged plate of that capacitor after electrostatic equilibrium is re-established.

Picture the Problem Let the charge on the capacitor with the air gap be Q_1 and the charge on the capacitor with the dielectric gap be Q_2. If the capacitances of the capacitors were initially C, then the capacitance of the capacitor with the dielectric inserted is $C' = \kappa C$. We can use the conservation of charge and the equivalence of the potential difference across the capacitors to obtain two equations that we can solve simultaneously for Q_1 and Q_2.

Apply conservation of charge during the insertion of the dielectric to obtain:

$$Q_1 + Q_2 = 2Q \qquad (1)$$

Because the capacitors have the same potential difference across them:

$$\frac{Q_1}{C} = \frac{Q_2}{\kappa C} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$Q_1 = \boxed{\frac{2Q}{1+\kappa}} \text{ and } Q_2 = \boxed{\frac{2Q\kappa}{1+\kappa}}$$

66 •• A parallel-plate capacitor has a plate separation d has a capacitance equal to C_0 where there is only empty space in the space between the plates. A slab of thickness t, where $t < d$, that has a dielectric constant κ is placed in the space between the plates completely covering one of the plates. What is the capacitance with the slab inserted?

Picture the Problem We can model this system as two capacitors in series, C_1 of thickness t and C_2 of thickness $d - t$ and use the equation for the equivalent capacitance of two capacitors connected in series.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \text{ or } C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1 = \frac{\kappa \in_0 A}{t}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2 = \frac{\in_0 A}{d-t}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{\text{eq}} = \frac{\left(\frac{\kappa \in_0 A}{t}\right)\left(\frac{\in_0 A}{d-t}\right)}{\frac{\kappa \in_0 A}{t} + \frac{\in_0 A}{d-t}} = \frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t} + \frac{1}{d-t}} \in_0 A = \frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t} + \frac{1}{d-t}} \in_0 A$$

$$= \frac{\kappa}{\kappa(d-t)+t} \in_0 A = \boxed{\left(\frac{\kappa d}{\kappa(d-t)+t}\right) C_0}$$

67 •• The membrane of the axon of a nerve cell can be modeled as a thin cylindrical shell of radius 1.00×10^{-5} m, having a length of 10.0 cm and a thickness of 10.0 nm. The membrane has a positive charge on one side and a negative charge on the other, and the membrane acts as a parallel-plate capacitor of area $2\pi rL$ and separation d. Assume the membrane is filled with a material whose dielectric constant is 3.00. (*a*) Find the capacitance of the membrane. If the potential difference across the membrane is 70.0 mV, find (*b*) the charge on the positively charged side of the membrane.

Picture the Problem Because $d << r$, we can model the membrane as a parallel-plate capacitor. We can use the definition of capacitance to find the charge on each side of the membrane in Part (*b*).

(*a*) Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute for the area of the plates:

$$C = \frac{2\pi\kappa \in_0 rL}{d} = \frac{\kappa rL}{2kd}$$

Substitute numerical values and evaluate C:

$$C = \frac{(3.00)(1.00\times10^{-5}\text{ m})(0.100\text{ m})}{2(8.988\times10^{9}\text{ N}\cdot\text{m}^2/\text{C}^2)(10.0\text{ nm})}$$
$$= 16.69\text{ nF} = \boxed{16.7\text{ nF}}$$

(*b*) Use the definition of capacitance to find the charge on each side of the membrane:

$$Q = CV = (16.69\text{nF})(70.0\text{mV})$$
$$= \boxed{1.17\text{nC}}$$

68 •• The space between the plates of the capacitor that is connected across the terminals of a battery is filled with a dielectric material. Determine the dielectric constant of the material if the induced bound–charge-per-unit-area on it is (*a*) 80 percent of the free–charge-per-unit-area on the plates, (*b*) 20 percent of the free–charge-per-unit-area on the plates, and (*c*) 98 percent of the free–charge-per-unit-area on the plates.

Picture the Problem The bound charge density is related to the dielectric constant and the free charge density by the equation $\sigma_b = \left(1 - \frac{1}{\kappa}\right)\sigma_f$.

Solve the equation relating σ_b, σ_f, and κ for κ to obtain:

$$\kappa = \frac{1}{1 - \sigma_b/\sigma_f}$$

(a) Evaluate this expression for $\sigma_b/\sigma_f = 0.80$:

$$\kappa = \frac{1}{1-0.80} = \boxed{5.0}$$

(b) Evaluate this expression for $\sigma_b/\sigma_f = 0.20$:

$$\kappa = \frac{1}{1-0.20} = \boxed{1.3}$$

(c) Evaluate this expression for $\sigma_b/\sigma_f = 0.98$:

$$\kappa = \frac{1}{1-0.98} = \boxed{50}$$

69 •• The positively charge plate of a parallel-plate capacitor has a charge equal to Q. When the space between the plates is evacuated of air, the electric field strength between the plates is 2.5×10^5 V/m. When the space is filled with a certain dielectric material, the field strength between the plates is reduced to 1.2×10^5 V/m. (a) What is the dielectric constant of the material? (b) If $Q = 10$ nC, what is the area of the plates? (c) What is the total induced bound charge on either face of the dielectric material?

Picture the Problem We can use the definition of the dielectric constant to find its value. In Part (b) we can use the expression for the electric field in the space between the charged capacitor plates to find the area of the plates and in Part (c) we can relate the surface charge densities to the induced charges on the plates.

(a) Using the definition of the dielectric constant, relate the electric field without a dielectric E_0 to the field with a dielectric E:

$$E = \frac{E_0}{\kappa} \Rightarrow \kappa = \frac{E_0}{E}$$

Solve for and evaluate κ:

$$\kappa = \frac{2.5\times10^5\ \text{V/m}}{1.2\times10^5\ \text{V/m}} = 2.08 = \boxed{2.1}$$

(b) Relate the electric field in the region between the plates to the surface charge density of the plates:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} \Rightarrow A = \frac{Q}{E_0\,\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{10\,\text{nC}}{\left(2.5\times10^5\ \text{V/m}\right)\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}$$

$$= 4.52\times10^{-3}\ \text{m}^2 = \boxed{45\,\text{cm}^2}$$

(*c*) Relate the surface charge densities to the induced charges on the plates:

$$\sigma_b = \left(1 - \frac{1}{\kappa}\right)\sigma_f$$

or

$$\frac{\sigma_b}{\sigma_f} = \frac{Q_b}{Q_f} = 1 - \frac{1}{\kappa} \Rightarrow Q_b = \left(1 - \frac{1}{\kappa}\right)Q_f$$

Substitute numerical values and evaluate Q_b:

$$Q_b = \left(1 - \frac{1}{2.08}\right)(10\,\text{nC}) = \boxed{5.2\,\text{nC}}$$

70 •• Find the capacitance of the parallel-plate capacitor shown in Figure 24-44.

Picture the Problem We can model this parallel-plate capacitor as a combination of two capacitors C_1 and C_2 in series with capacitor C_3 in parallel.

Express the capacitance of two series-connected capacitors in parallel with a third:

$$C = C_3 + C_s \qquad (1)$$

where

$$C_s = \frac{C_1 C_2}{C_1 + C_2} \qquad (2)$$

Express each of the capacitances C_1, C_2, and C_3 in terms of the dielectric constants, plate areas, and plate separations:

$$C_1 = \frac{\kappa_1 \in_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_1 \in_0 A}{d},$$

$$C_2 = \frac{\kappa_2 \in_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_2 \in_0 A}{d},$$

and

$$C_3 = \frac{\kappa_3 \in_0 \left(\frac{1}{2}A\right)}{d} = \frac{\kappa_3 \in_0 A}{2d}$$

Substitute in equation (2) to obtain:

$$C_s = \frac{\left(\frac{\kappa_1 \in_0 A}{d}\right)\left(\frac{\kappa_2 \in_0 A}{d}\right)}{\frac{\kappa_1 \in_0 A}{d} + \frac{\kappa_2 \in_0 A}{d}} = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\in_0 A}{d}\right)$$

Substitute in equation (1) to obtain:

$$C = \frac{\kappa_3 \in_0 A}{2d} + \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\in_0 A}{d}\right) = \boxed{\left(\kappa_3 + \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right)\left(\frac{\in_0 A}{2d}\right)}$$

General Problems

71 • You are given four identical capacitors and a 100-V battery. When only one of the capacitors is connected to this battery the energy stored is U_0. Combine the four capacitors in such a way that the total energy stored in all four capacitors is U_0? Describe the combination and explain your answer.

Picture the Problem We can use the expression $U_0 = \frac{1}{2}C_{eq}V^2$ to express the total energy stored in the combination of four capacitors in terms of their equivalent capacitance C_{eq}.

The energy stored in one capacitor when it is connected to the 100-V battery is:

$$U_0 = \tfrac{1}{2}CV^2$$

When the four capacitors are connected to the battery in some combination, the total energy stored in them is:

$$U = \tfrac{1}{2}C_{eq}V^2$$

Equate U and U_0 and solve for C_{eq}:

$$\tfrac{1}{2}C_{eq}V^2 = \tfrac{1}{2}CV^2 \Rightarrow C_{eq} = C$$

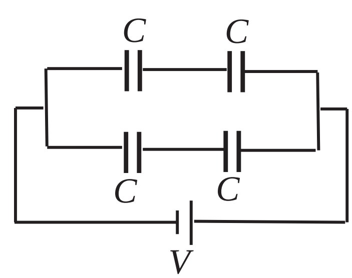
The equivalent capacitance C' of two capacitors of capacitance C connected in series is their product divided by their sum:

$$C' = \frac{C^2}{C+C} = \tfrac{1}{2}C$$

If we connect two of the capacitors in series in parallel with the other two capacitors connected in series, their equivalent capacitance will be:

$$C_{eq} = C' + C' = \tfrac{1}{2}C + \tfrac{1}{2}C = C$$

A series combination of two of the capacitors connected in parallel with a series combination of the other two capacitors will result in total energy U_0 stored in all four capacitors. The circuit diagram is shown to the right.



72 • Three capacitors have capacitances of 2.00 μF, 4.00 μF, and 8.00 μF. Find the equivalent capacitance if (*a*) the capacitors are connected in parallel and (*b*) if the capacitors are connected in series.

Picture the Problem We can use the equations for the equivalent capacitance of three capacitors connected in parallel and in series to find these equivalent capacitances.

(*a*) Express the equivalent capacitance of three capacitors connected in parallel:

$$C_{\text{eq}} = C_1 + C_2 + C_3$$

Substitute numerical values and evaluate C_{eq}:

$$C_{\text{eq}} = 2.00\,\mu\text{F} + 4.00\,\mu\text{F} + 8.00\,\mu\text{F} = \boxed{14.00\,\mu\text{F}}$$

(*b*) The equivalent capacitance of the three capacitors connected in series is given by:

$$C_{\text{eq}} = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}$$

Substitute numerical values and evaluate C_{eq}:

$$C_{\text{eq}} = \frac{(2.00\,\mu\text{F})(4.00\,\mu\text{F})(8.00\,\mu\text{F})}{(2.00\,\mu\text{F})(4.00\,\mu\text{F}) + (4.00\,\mu\text{F})(8.00\,\mu\text{F}) + (2.00\,\mu\text{F})(8.00\,\mu\text{F})} = \boxed{1.14\,\mu\text{F}}$$

73 • A 1.00-μF capacitor is connected in parallel with a 2.00-μF capacitor, and this combination is connected in series with a 6.00-μF capacitor. What is the equivalent capacitance of this combination?

Picture the Problem We can first use the equation for the equivalent capacitance of two capacitors connected in parallel and then the equation for two capacitors connected in series to find the equivalent capacitance.

Find the equivalent capacitance of a 1.00-μF capacitor connected in parallel with a 2.00-μF capacitor:

$$C_{\text{eq},1} = C_1 + C_2 = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} = 3.00\,\mu\text{F}$$

Find the equivalent capacitance of a 3.00-μF capacitor connected in series with a 6.00-μF capacitor:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_6}{C_{\text{eq},1} + C_6} = \frac{(3.00\,\mu\text{F})(6.00\,\mu\text{F})}{3.00\,\mu\text{F} + 6.00\,\mu\text{F}} = \boxed{2.00\,\mu\text{F}}$$

74 • The voltage across a parallel-plate capacitor that has a plate separation equal to 0.500 mm is 1.20 kV. The capacitor is disconnected from the voltage source and the separation between the plates is increased until the energy stored in the capacitor has been doubled. Determine the final separation between the plates.

Picture the Problem The charge Q and the charge density σ are independent of the separation of the plates and do not change during the process described in the problem statement. Because the electric field E depends on σ, it too is constant. We can use $U = \frac{1}{2}CV^2$ and the relationship between V and E, together with the expression for the capacitance of a parallel-plate capacitor, to show that $U \propto d$.

Express the energy stored in the capacitor in terms of its capacitance C and the potential difference across its plates:

$$U = \tfrac{1}{2}CV^2 \qquad (1)$$

Express V in terms of E:

$$V = Ed$$

where d is the separation of the plates.

Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute for C and V in equation (1) to obtain:

$$U = \tfrac{1}{2}\frac{\kappa \in_0 A}{d}(Ed)^2 = \left(\tfrac{1}{2}\kappa \in_0 AE^2\right)d$$

Because $U \propto d$, to double U one must double d. Hence:

$$d_{\text{f}} = 2d = 2(0.500\,\text{mm}) = \boxed{1.00\,\text{mm}}$$

75 •• Determine the equivalent capacitance, in terms of C_0, of each of the combinations of capacitors shown in Figure 24-45.

Picture the Problem We can use the equations for the equivalent capacitance of capacitors connected in parallel and in series to find the single capacitor that will store the same amount of charge as each of the networks shown above.

(*a*) Find the capacitance of the two capacitors in parallel:

$$C_{\text{eq,1}} = C_0 + C_0 = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with C_0:

$$C_{\text{eq,2}} = \frac{C_{\text{eq,1}}C_0}{C_{\text{eq,1}} + C_0} = \frac{(2C_0)C_0}{2C_0 + C_0} = \boxed{\tfrac{2}{3}C_0}$$

(*b*) Find the capacitance of two capacitors of capacitance C_0 in parallel:

$$C_{\text{eq},1} = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with $2C_0$:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_0}{C_{\text{eq},1} + C_0} = \frac{(2C_0)(2C_0)}{2C_0 + 2C_0} = \boxed{C_0}$$

(*c*) Find the equivalent capacitance of three equal capacitors connected in parallel:

$$C_{\text{eq}} = C_0 + C_0 + C_0 = \boxed{3C_0}$$

76 •• Figure 24-46 shows four capacitors connected in the arrangement known as a capacitance bridge. The capacitors are initially uncharged. What must the relation between the four capacitances be so that the potential difference between points *c* and *d* remains zero when a voltage *V* is applied between points *a* and *b*?

Picture the Problem Note that with *V* applied between *a* and *b*, C_1 and C_3 are in series, and so are C_2 and C_4. Because in a series combination the potential differences across the two capacitors are inversely proportional to the capacitances, we can establish proportions involving the capacitances and potential differences for the left- and right-hand side of the network and then use the condition that $V_c = V_d$ to eliminate the potential differences and establish the relationship between the capacitances.

Letting *Q* represent the charge on capacitors 1 and 2, relate the potential differences across the capacitors to their common charge and capacitances:

$$V_1 = \frac{Q}{C_1} \text{ and } V_3 = \frac{Q}{C_3}$$

Divide the first of these equations by the second to obtain:

$$\frac{V_1}{V_3} = \frac{C_3}{C_1} \quad (1)$$

Proceed similarly to obtain:

$$\frac{V_2}{V_4} = \frac{C_4}{C_2} \quad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{V_1V_4}{V_3V_2} = \frac{C_3C_2}{C_1C_4} \quad (3)$$

If $V_c = V_d$ then we must have: $V_1 = V_2$ and $V_3 = V_4$

Substitute in equation (3) and rearrange to obtain: $\boxed{C_2C_3 = C_1C_4}$

77 •• The plates of a parallel-plate capacitor are separated by distance *d*, and each plate has area *A*. The capacitor is charged to a potential difference *V* and then disconnected from the voltage source. The plates are then pulled apart until the separation is 3*d*. Find (*a*) the new capacitance, (*b*) the new potential difference, and (*c*) the new stored energy. (*d*) How much work was required to change the plate separation from *d* to 3*d*?

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor as a function of *A* and *d* to determine the effect on the capacitance of doubling the plate separation. We can use $V = Ed$ to determine the effect on the potential difference across the capacitor of doubling the plate separation. Finally, we can use $U = \frac{1}{2}CV^2$ to determine the effect of doubling the plate separation on the energy stored in the capacitor.

(*a*) The capacitance of a capacitor whose plates are separated by a distance 3*d* is given by:

$$C_{\text{new}} = \boxed{\frac{\epsilon_0 A}{3d}}$$

(*b*) Express the potential difference across a parallel-plate capacitor whose plates are separated by a distance *d*:

$$V = Ed$$

where the electric field *E* depends solely on the charge on the capacitor plates.

Express the new potential difference across the plates resulting from tripling their separation:

$$V_{\text{new}} = E(3d) = 3(Ed) = \boxed{3V}$$

(*c*) Relate the energy stored in a parallel-plate capacitor to the separation of the plates:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2$$

When the plate separation is tripled we have:

$$U_{\text{new}} = \frac{1}{2}\frac{\epsilon_0 A}{3d}(3V)^2 = \boxed{\frac{3\epsilon_0 AV^2}{2d}}$$

(*d*) Relate the work required to change the plate separation from *d* to 3*d* to the change in the electrostatic potential energy of the system:

$$W = U_{\text{new}} - U_{\text{i}} = \frac{3 \in_0 AV^2}{2d} - \frac{1}{2}\frac{\in_0 AV^2}{d} = \boxed{\frac{\in_0 AV^2}{d}}$$

78 •• A parallel-plate capacitor has capacitance C_0 when there is no dielectric in the space between the plates. The space between the plates is then filled with a material that has a dielectric constant of κ. When a second capacitor of capacitance C' is connected in series with the first, the capacitance of the series combination is C_0. Find C' in terms of C_0.

Picture the Problem We can use the equation for the equivalent capacitance of two capacitors in series to relate C_0 to C' and the capacitance of the dielectric-filled parallel-plate capacitor and then solve the resulting equation for C'.

Express the equivalent capacitance of the system in terms of C' and C, where C is the dielectric-filled capacitor:

$$C_0 = \frac{C'C}{C'+C} \Rightarrow C' = \frac{C_0 C}{C - C_0}$$

Express the capacitance of the dielectric-filled capacitor:

$$C = \frac{\kappa \in_0 A}{d} = \kappa C_0$$

Substitute for C in the equation for C' and simplify to obtain:

$$C' = \frac{C_0(\kappa C_0)}{\kappa C_0 - C_0} = \boxed{\frac{\kappa}{\kappa - 1}C_0}$$

79 •• **[SSM]** A parallel combination of two identical 2.00-μF parallel-plate capacitors (no dielectric is in the space between the plates) is connected to a 100-V battery. The battery is then removed and the separation between the plates of one of the capacitors is doubled. Find the charge on the positively charged plate of each of the capacitors.

Picture the Problem When the battery is removed, after having initially charged both capacitors, and the separation of one of the capacitors is doubled, the charge is redistributed subject to the condition that the total charge remains constant; that is, $Q = Q_1 + Q_2$ where Q is the initial charge on both capacitors and Q_2 is the charge on the capacitor whose plate separation has been doubled. We can use the conservation of charge during the plate separation process and the fact that, because the capacitors are in parallel, they share a common potential difference.

Find the equivalent capacitance of the two 2.00-μF parallel-plate capacitors connected in parallel:

$$C_{eq} = 2.00\,\mu\text{F} + 2.00\,\mu\text{F} = 4.00\,\mu\text{F}$$

Use the definition of capacitance to find the charge on the equivalent capacitor:

$$Q = C_{eq}V = (4.00\,\mu\text{F})(100\,\text{V}) = 400\,\mu\text{C}$$

Relate this total charge to charges distributed on capacitors 1 and 2 when the battery is removed and the separation of the plates of capacitor 2 is doubled:

$$Q = Q_1 + Q_2 \qquad (1)$$

Because the capacitors are in parallel:

$$V_1 = V_2 \text{ and } \frac{Q_1}{C_1} = \frac{Q_2}{C_2'} = \frac{Q_2}{\frac{1}{2}C_2} = \frac{2Q_2}{C_2}$$

Solve for Q_1 to obtain:

$$Q_1 = 2\left(\frac{C_1}{C_2}\right)Q_2 \qquad (2)$$

Substitute equation (2) in equation (1) and solve for Q_2 to obtain:

$$Q_2 = \frac{Q}{2(C_1/C_2)+1}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \frac{400\,\mu\text{C}}{2(2.00\,\mu\text{F}/2.00\,\mu\text{F})+1} = \boxed{133\,\mu\text{C}}$$

Substitute numerical values in equation (1) or equation (2) and evaluate Q_1:

$$Q_1 = \boxed{267\,\mu\text{C}}$$

80 •• An parallel-plate capacitor with no dielectric in the space between the plates has a capacitance C_0 and a plate separation d. Two dielectric slabs that have dielectric constants of κ_1 and κ_2 respectively are then inserted between the plates as shown in Figure 24-47. Each slab is has a thickness $\frac{1}{2}d$ and has area A, the same area as each capacitor plate. When the charge on the positively charged capacitor plate is Q, find (*a*) the electric field in each dielectric, and (*b*) the potential difference between the plates. (*c*) Show that the capacitance of the system after the slabs are inserted is given by $\left[2\kappa_1\kappa_2/(\kappa_1+\kappa_2)\right]C_o$. (*d*) Show that $\left[2\kappa_1\kappa_2/(\kappa_1+\kappa_2)\right]C_o$ is the equivalent capacitance of a series combination of two capacitors, each having plates of area A and a gap width equal to d/2. The space between the plates of one is filled with a material that has a dielectric constant

equal to κ_1 and the space between the plates of the other is filled with a material that has a dielectric constant equal to κ_2.

Picture the Problem We can relate the electric field in the dielectric to the electric field between the capacitor's plates in the absence of a dielectric using $E = E_0/\kappa$. In Part (*b*) we can express the potential difference between the plates as the sum of the potential differences across the dielectrics and then express the potential differences in terms of the electric fields in the dielectrics. In Part (*c*) we can use our result from (*b*) and the definition of capacitance to express the capacitance of the dielectric-filled capacitor. In Part (*d*) we can confirm the result of Part (*c*) by using the addition formula for capacitors in series.

(*a*) Express the electric field E in a dielectric of constant κ in terms of the electric field E_0 in the absence of the dielectric:

$$E = \frac{E_0}{\kappa}$$

Express the electric field E_0 in the absence of the dielectrics:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Substitute for E_0 to obtain:

$$E = \frac{Q}{\kappa \epsilon_0 A}$$

Use this relationship to express the electric fields in dielectrics whose constants are κ_1 and κ_2:

$$E_1 = \boxed{\frac{Q}{\kappa_1 \epsilon_0 A}} \text{ and } E_2 = \boxed{\frac{Q}{\kappa_2 \epsilon_0 A}}$$

(*b*) Express the potential difference between the plates as the sum of the potential differences across the dielectrics:

$$V = V_1 + V_2$$

Relate the potential differences to the electric fields and the thicknesses of the dielectrics:

$$V_1 = E_1 \frac{d}{2} = \frac{Qd}{2\kappa_1 \epsilon_0 A}$$

and

$$V_2 = E_2 \frac{d}{2} = \frac{Qd}{2\kappa_2 \epsilon_0 A}$$

Substitute for V_1 and V_2 and simplify to obtain:

$$V = \frac{Qd}{2\kappa_1 \in_0 A} + \frac{Qd}{2\kappa_2 \in_0 A}$$
$$= \boxed{\frac{Qd}{2 \in_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)}$$

(*c*) Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{Q}{\dfrac{Qd}{2 \in_0 A}\left(\dfrac{1}{\kappa_1} + \dfrac{1}{\kappa_2}\right)}$$
$$= \frac{2 \in_0 A}{d\left(\dfrac{\kappa_1 + \kappa_2}{\kappa_1\kappa_2}\right)} = \frac{2 \in_0 A}{d}\left(\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)$$
$$= \boxed{2C_0\left(\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)}$$

where $C_0 = \in_0 A/d$.

(*d*) Express the equivalent capacitance C of capacitors C_1 and C_2 in series:

$$C = \frac{C_1C_2}{C_1 + C_2}$$

Express C_1:

$$C_1 = \frac{\kappa_1 \in_0 A}{d/2} = \frac{2\kappa_1 \in_0 A}{d} = 2\kappa_1C_0$$

Express C_2:

$$C_2 = \frac{\kappa_2 \in_0 A}{d/2} = \frac{2\kappa_2 \in_0 A}{d} = 2\kappa_2C_0$$

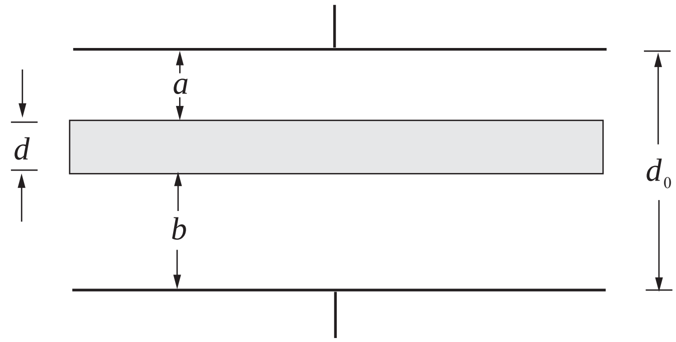
Substitute for C_1 and C_2 and simplify to obtain:

$$C = \frac{(2\kappa_1C_0)(2\kappa_2C_0)}{2\kappa_1C_0 + 2\kappa_2C_0} = \boxed{2C_0\left(\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)},$$

a result in agreement with Part (*c*).

81 •• The plates of a parallel-plate capacitor are separated by distance d_0, and each plate has area *A*. A metal slab of thickness *d* and area *A* is inserted between the plates in such a way that the slab is parallel with the capacitor plates. (*a*) Show that the new capacitance is given by $\in_0 A/(d_0 - d)$, regardless of the distance between the metal slab and the positively charged plate. (*b*) Show that this arrangement can be modeled as a capacitor that has plate separation *a* in series with a capacitor of separation *b*, where $a + b + d = d_0$.

Picture the Problem The following diagram shows the arrangement described in the problem statement.



(*a*) and (*b*) Modeling the arrangement as a capacitor that has plate separation *a* in series with a capacitor of separation *b* gives:

$$\frac{1}{C}=\frac{a}{\epsilon_0 A}+\frac{b}{\epsilon_0 A}=\frac{a+b}{\epsilon_0 A}$$

From the diagram:

$$a+b+d=d_0 \Rightarrow a+b=d_0-d$$

Substituting for $a+b$ yields:

$$\frac{1}{C}=\frac{d_0-d}{\epsilon_0 A}\Rightarrow C=\boxed{\frac{\epsilon_0 A}{d_0-d}}$$

82 •• A parallel-plate capacitor that has plate area *A* is filled with two dielectrics of equal size, as shown in Figure 24-48. (*a*) Show that this system can be modeled as two capacitors that are connected in parallel and each have an area $\frac{1}{2}A$. (*b*) Show that the capacitance is given by $\frac{1}{2}(\kappa_1+\kappa_2)C_0$, where C_0 is the capacitance if there were no dielectric materials in the space between the plates.

Picture the Problem We can express the ratio of C_{eq} to C_0 to show that the capacitance with the dielectrics in place is $(\kappa_1+\kappa_2)/2$ times greater than that of the capacitor in the absence of the dielectrics.

(*a*) Because the capacitor plates are conductors, the potentials are the same across the entire upper and lower plates. Hence, the system is equivalent to two capacitors, each of area $\frac{1}{2}A$, in parallel.

(*b*) Relate the capacitance C_0, in the absence of the dielectrics, to the plate area and separation:

$$C_0=\frac{\epsilon_0 A}{d}$$

Express the equivalent capacitance of capacitors C_1 and C_2, each with plate area $\frac{1}{2}A$, connected in parallel, and simplify to obtain:

$$C_{\text{eq}} = C_1 + C_2$$
$$= \frac{\kappa_1 \epsilon_0 \left(\frac{1}{2}A\right)}{d} + \frac{\kappa_2 \epsilon_0 \left(\frac{1}{2}A\right)}{d}$$
$$= \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(\kappa_1 + \kappa_2)$$
$$= \boxed{\frac{1}{2}(\kappa_1 + \kappa_2)C_0}$$

83 •• A parallel-plate capacitor with no dielectric in the space between the plates has a plate area A and a gap width x. A charge Q is on the positively charged plate. (*a*) Find the stored electrostatic energy as a function of x. (*b*) Find the increase in energy dU due to an increase in plate separation dx from $dU = (dU/dx)\,dx$. (*c*) If F is the force exerted by one plate on the other, the work needed to move one plate a distance dx is $F\,dx = dU$. Show that $F = Q_2/(2\epsilon_0 A)$. (*d*) Show that the force in Part (*c*) equals $\frac{1}{2}EQ$, where Q is the charge on one plate and E is the electric field between the plates. Give a conceptual explanation for the factor $\frac{1}{2}$ in this result.

Picture the Problem We can use $U = Q^2/2C$ and the expression for the capacitance as a function of plate separation to express U as a function of x. Differentiation of this result with respect to x will yield dU. Because the work done in increasing the plate separation a distance dx equals the change in the electrostatic potential energy of the capacitor, we can evaluate F from dU/dx. Finally, we can express F in terms of Q and E by relating E to x using $E = V/x$ and using the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor.

(*a*) Relate the electrostatic energy U stored in the capacitor to its capacitance C:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Express the capacitance as a function of the plate separation:

$$C = \frac{\epsilon_0 A}{x}$$

Substitute for C to obtain:

$$U = \boxed{\frac{Q^2}{2\epsilon_0 A}x}$$

(*b*) Use the result obtained in (*a*) to evaluate dU:

$$dU = \frac{dU}{dx}dx = \frac{d}{dx}\left[\frac{Q^2}{2\epsilon_0 A}x\right]dx$$
$$= \boxed{\frac{Q^2}{2\epsilon_0 A}dx}$$

(*c*) Relate the work needed to move one plate a distance *dx* to the change in the electrostatic potential energy of the system:

$$W = dU = Fdx$$

Solve for and evaluate *F*:

$$F = \frac{dU}{dx} = \frac{d}{dx}\left[\frac{Q^2}{2\in_0 A}x\right] = \boxed{\frac{Q^2}{2\in_0 A}}$$

(*d*) Express the electric field between the plates in terms of their separation and their potential difference:

$$E = \frac{V}{x}$$

Use the definition of capacitance to eliminate *V*:

$$E = \frac{Q}{Cx}$$

Use the expression for the capacitance of a parallel-plate capacitor to eliminate *C*:

$$E = \frac{Q}{\frac{\in_0 A}{x}x} = \frac{Q}{\in_0 A}$$

Substitute in our result from Part (*c*) to obtain:

$$F = \frac{Q(\in_0 AE)}{2\in_0 A} = \boxed{\tfrac{1}{2}QE}$$

The field *E* is due to the sum of the fields from charges $+Q$ and $-Q$ on the opposite plates of the capacitor. Each plate produces a field $\frac{1}{2}E$. and the force is the product of charge *Q* and the field $\frac{1}{2}E$.

84 •• A rectangular parallel-plate capacitor that has a length *a* and a width *b* has a dielectric that has a width *b* partially inserted a distance *x* between the plates, as shown in Figure 24-49. (*a*) Find the capacitance as a function of *x*. Neglect edge effects. (*b*) Show that your answer gives the expected results for $x = 0$ and $x = a$.

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Express the equivalent capacitance of the two capacitors in parallel:

$$C(x) = C_1 + C_2 \qquad (1)$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_1:

$$C_1 = \frac{\kappa \in_0 A_1}{d} = \frac{\kappa \in_0 bx}{d}$$

Express the capacitance C_0 of the capacitor with the dielectric removed; that is, $x = 0$:

$$C_0 = \frac{\in_0 ab}{d}$$

Divide C_1 by C_0 and simplify to obtain:

$$\frac{C_1}{C_0} = \frac{\frac{\kappa \in_0 bx}{d}}{\frac{\in_0 ab}{d}} = \frac{\kappa x}{a} \Rightarrow C_1 = \frac{\kappa x}{a} C_0$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_2:

$$C_2 = \frac{\in_0 A_2}{d} = \frac{\in_0 b(a-x)}{d}$$

Divide C_2 by C_0 to obtain:

$$\frac{C_2}{C_0} = \frac{\frac{\in_0 b(a-x)}{d}}{\frac{\in_0 ab}{d}} = \frac{a-x}{a}$$

or, solving for C_2,

$$C_2 = \frac{a-x}{a} C_0$$

Substitute for C_1 and C_2 in equation (1) and simplify to obtain:

$$\begin{aligned} C(x) &= \frac{\kappa x}{a} C_0 + \frac{a-x}{a} C_0 \\ &= \frac{C_0}{a}\left[a + (\kappa - 1)x\right] \\ &= \boxed{\frac{\in_0 b}{d}\left[a + (\kappa - 1)x\right]} \end{aligned}$$

(*b*) Evaluate C for $x = 0$:

$$C(0) = \frac{\in_0 b}{d}[a] = \frac{\in_0 ab}{d} = \boxed{C_0}$$

as expected.

Evaluate C for $x = a$:

$$\begin{aligned} C(a) &= \frac{\in_0 b}{d}\left[a + (\kappa - 1)a\right] \\ &= \boxed{\frac{\kappa \in_0 ab}{d}} \end{aligned}$$ *as expected.*

85 ••• **[SSM]** An electrically isolated capacitor that has charge Q on its positively charged plate is partly filled with a dielectric substance as shown in Figure 24-51. The capacitor consists of two rectangular plates that have edge lengths a and b and are separated by distance d. The dielectric is inserted into the gap a distance x. (*a*) What is the energy stored in the capacitor? *Hint: the capacitor can be modeled as two capacitors connected in parallel*. (*b*) Because the energy of the capacitor decreases as x increases, the electric field must be doing work on the dielectric, meaning that there must be an electric force pulling it in. Calculate this force by examining how the stored energy varies with x. (*c*) Express the force in terms of the capacitance and potential difference V between the plates. (*d*) From where does this force originate?

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel, one with an air gap and other filled with a dielectric of constant κ. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Using the hint, express the energy stored in the capacitor as a function of the equivalent capacitance C_{eq}:

$$U = \frac{1}{2}\frac{Q^2}{C_{\text{eq}}}$$

The capacitances of the two capacitors are:

$$C_1 = \frac{\kappa \in_0 ax}{d} \text{ and } C_2 = \frac{\in_0 a(a-x)}{d}$$

Because the capacitors are in parallel, C_{eq} is the sum of C_1 and C_2:

$$\begin{aligned} C_{eq} &= C_1 + C_2 = \frac{\kappa \in_0 ax}{d} + \frac{\in_0 a(a-x)}{d} \\ &= \frac{\in_0 a}{d}(\kappa x + a - x) \\ &= \frac{\in_0 a}{d}[(\kappa - 1)x + a] \end{aligned}$$

Substitute for C_{eq} in the expression for U and simplify to obtain:

$$U = \boxed{\frac{Q^2 d}{2 \in_0 a[(\kappa-1)x + a]}}$$

(*b*) The force exerted by the electric field is given by:

$$F = -\frac{dU}{dx}$$
$$= -\frac{d}{dx}\left[\frac{1}{2}\frac{Q^2 d}{\in_0 a[(\kappa-1)x+a]}\right]$$
$$= -\frac{Q^2 d}{2\in_0 a}\frac{d}{dx}\left\{[(\kappa-1)x+a]^{-1}\right\}$$
$$= \boxed{\frac{(\kappa-1)Q^2 d}{2a\in_0 [(\kappa-1)x+a]^2}}$$

(*c*) Rewrite the result in (*b*) to obtain:

$$F = \frac{(\kappa-1)Q^2\left(\frac{a\in_0}{d}\right)}{2\left(\frac{a\in_0}{d}\right)^2[(\kappa-1)x+a]^2}$$
$$= \frac{(\kappa-1)Q^2\left(\frac{a\in_0}{d}\right)}{2C_{eq}^2}$$
$$= \boxed{\frac{(\kappa-1)a\in_0 V^2}{2d}}$$

Note that this expression is independent of *x*.

(*d*) The force originates from the fringing fields around the edges of the capacitor. The effect of the force is to pull the polarized dielectric into the space between the capacitor plates.

86 ••• A spherical capacitor consists of an solid conducting sphere that has a radius *a* and a charge +*Q* and an concentric conducting spherical shell that has an inner radius *b* and a charge of *Q*. The space between the two is filled with two different dielectric materials of dielectric constants κ_1 and κ_2. The boundary between the two dielectrics occurs a distance $\frac{1}{2}(a+b)$ from the center.
(*a*) Calculate the electric field in the regions $a < r < \frac{1}{2}(a+b)$ and $\frac{1}{2}(a+b) < r < b$. (*b*) Integrate the expression $dV = -\vec{E}\cdot d\vec{\ell}$ to obtain the potential difference, *V*, between the two conductors. (*c*) Use $C = Q/V$ to obtain an expression for the capacitance of this system. (*d*) Show that your answer from Part (*c*) simplifies to the expected one if the κ_1 equals κ_2.

Picture the Problem (*a*) Gauss's law tells us that, in the absence of dielectric materials, the electric field between the conductors is $E_0 = kQ/r^2$. In the dielectric materials, the field is reduced by the appropriate dielectric constant.

(*a*) The electric fields in the regions
$a < r < (a + b)/2$
and
$(a + b)/2 < r < b$
are given by:

$$E_1 = \boxed{\frac{kQ}{\kappa_1 r^2}} \text{ and } E_2 = \boxed{\frac{kQ}{\kappa_2 r^2}}$$

(*b*) Because the electric field is directed radially away from the center, the potential of the outer spherical shell is less than the potential of the inner sphere. Hence the difference in potential between the spheres is given by:

$$\begin{aligned}|V| &= |V_1 - V_2| \\ &= \left| -\int_a^{\frac{1}{2}(a+b)} E_1 dr - \left(-\int_{\frac{1}{2}(a+b)}^{b} E_2 dr \right) \right| \\ &= \left| -\frac{kQ}{\kappa_1} \int_a^{\frac{1}{2}(a+b)} r^{-2} dr + \frac{kQ}{\kappa_2} \int_{\frac{1}{2}(a+b)}^{b} r^{-2} dr \right|\end{aligned}$$

Evaluating the integrals and simplifying yields:

$$\begin{aligned}|V| &= \left| \frac{kQ}{\kappa_1}\frac{1}{r}\bigg]_a^{\frac{1}{2}(a+b)} + \frac{kQ}{\kappa_2}\frac{1}{r}\bigg]_{\frac{1}{2}(a+b)}^{b} \right| = \left| \frac{kQ}{\kappa_1}\left(\frac{2}{a+b} - \frac{1}{a}\right) + \frac{kQ}{\kappa_2}\left(\frac{1}{b} - \frac{2}{a+b}\right) \right| \\ &= \left| \frac{kQ(a-b)}{a+b}\frac{\kappa_1 a + \kappa_2 b}{\kappa_1\kappa_2 ab} \right| = \boxed{\frac{kQ(b-a)}{a+b}\frac{\kappa_1 a + \kappa_2 b}{\kappa_1\kappa_2 ab}}\end{aligned}$$

(*c*) Use the definition of capacitance and simplify to obtain:

$$\begin{aligned}C &= \frac{Q}{|V|} = \frac{Q}{\dfrac{kQ(a-b)}{a+b}\dfrac{\kappa_1 a + \kappa_2 b}{\kappa_1\kappa_2 ab}} \\ &= \boxed{\frac{\kappa_1\kappa_2 ab(a+b)}{k(b-a)(\kappa_1 a + \kappa_2 b)}}\end{aligned}$$

(*d*) Letting $\kappa_1 = \kappa_2 = \kappa$, the result in Part (*c*) reduces to the expression for a spherical capacitor filled with just one dielectric material. In particular, if the capacitor is air filled ($\kappa = 1.00$), the expression for *C* is that for an air-filled spherical capacitor.

$$\begin{aligned}C &= \frac{\kappa\kappa ab(a+b)}{k(b-a)(\kappa a + \kappa b)} \\ &= \frac{\kappa ab}{k(b-a)} \\ &= \boxed{\frac{4\pi \epsilon_0 \kappa ab}{b-a}}\end{aligned}$$

87 ••• A capacitance balance is shown in Figure 24-50. The balance has a weight attached to one side and a capacitor that has a variable gap width on the other side. Assume the upper plate of the capacitor has negligible mass. When the capacitor potential difference between the plates is V_0, the attractive force between

the plates balances the weight of the hanging mass. (*a*) Is the balance stable? That is, if we balance it out, and then move the plates a little closer together, will they snap shut or move back to the equilibrium point? (*b*) Calculate the value of V_0 required to balance an object of mass M, assuming the plates are separated by distance d_0 and have area A. *HINT: A useful relation is that the force between the plates is equal to the derivative of the stored electrostatic energy with respect to the plate separation.*

Picture the Problem To avoid have to write dd (as in $F = -dE/dd$) in relating the force on the electrostatic balance plates to the electric field in the region between them, let ℓ be the variable separation of the plates. We can use the definition of the work done in charging the capacitor to relate the force on the upper plate to the energy stored in the capacitor. Solving this expression for the force and substituting for the energy stored in a parallel-plate capacitor will yield an expression that we can use to decide whether the balance is stable. We can use this same expression and a condition for equilibrium to find the voltage required to balance the object whose mass is M.

(*a*) Express the work done in charging the capacitor (the energy stored in it) in terms of the force between the plates:

$$dW = dE = -Fd\ell \Rightarrow F = -\frac{dE}{d\ell}$$

The energy stored in the capacitor is given by:

$$E = \frac{1}{2}CV_0^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{\ell}\right)V_0^2$$

Differentiate E with respect to ℓ to obtain:

$$F = -\frac{d}{d\ell}\left[\frac{1}{2}\left(\frac{\epsilon_0 A}{\ell}\right)V_0^2\right] = \left(\frac{\epsilon_0 A}{2\ell^2}\right)V_0^2$$

or, changing back to the variable d,

$$F = \left(\frac{\epsilon_0 A}{2d^2}\right)V_0^2$$

Because F increases as ℓ decreases, a decrease in plate separation will unbalance the system. Hence, the balance is unstable.

(*b*) Apply $\sum F = 0$ to the object whose mass is M when the plate separation is d_0 to obtain:

$$Mg - \left(\frac{\epsilon_0 A}{2d_0^2}\right)V^2 = 0 \Rightarrow V = \boxed{d_0\sqrt{\frac{2Mg}{\epsilon_0 A}}}$$

88 ••• You are an intern at an engineering company that makes capacitors used for energy storage in pulsed lasers. Your manger asks your team to construct a parallel-plate, air-gap capacitor that will store 100 kJ of energy. (*a*) What

minimum volume is required between the plates of the capacitor? (*b*) Suppose you have developed a dielectric that has a dielectric strength of 3.00×10^8 V/m and a dielectric constant of 5.00. What volume of this dielectric, between the plates of the capacitor, is required for it to be able to store 100 kJ of energy?

Picture the Problem Recall that the dielectric strength of air is 3.00 MV/m. We can express the maximum energy to be stored in terms of the capacitance of the air-gap capacitor and the maximum potential difference between its plates. This maximum potential can, in turn, be expressed in terms of the maximum electric field (dielectric strength) possible in the air gap. We can solve the resulting equation for the volume of the space between the plates. In Part (*b*) we can modify the equation we derive in Part (*a*) to accommodate a dielectric with a constant other than 1.

(*a*) Express the energy stored in the capacitor in terms of its capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2$$

Express the capacitance of the air-gap parallel-plate capacitor:

$$C = \frac{\epsilon_0 A}{d}$$

Relate the potential difference across the plates to the electric field between them:

$$V = Ed$$

Substitute for C and V in the expression for U to obtain:

$$U = \tfrac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2$$

$$= \tfrac{1}{2}\epsilon_0 (Ad)E^2 = \tfrac{1}{2}\epsilon_0 \upsilon E^2$$

where $\upsilon = Ad$ is the volume between the plates.

Solving for $\upsilon = \upsilon_{max}$ yields:

$$\upsilon_{max} = \frac{2U_{max}}{\epsilon_0 E_{max}^2} \qquad (1)$$

Substitute numerical values and evaluate υ_{max}:

$$\upsilon_{max} = \frac{2(100\,\mathrm{kJ})}{(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2})(3\,\mathrm{MV/m})^2}$$

$$= \boxed{2.51\times10^3\ \mathrm{m^3}}$$

(*b*) With the dielectric in place equation (1) becomes:

$$\upsilon_{max} = \frac{2U_{max}}{\kappa \in_0 E_{max}^2} \quad (2)$$

Evaluate equation (2) with $\kappa = 5.00$ and $E_{max} = 3.00 \times 10^8$ V/m:

$$\upsilon_{max} = \frac{2(100\,\text{kJ})}{(5.00)\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(3.00\times10^8\,\frac{\text{V}}{\text{m}}\right)^2} = \boxed{5.02\times10^{-2}\ \text{m}^3}$$

89 ••• Consider two parallel-plate capacitors, C_1 and C_2, that are connected in parallel. The capacitors are identical except that C_2 has a dielectric inserted between its plates. A 200 V battery is connected across the combination until electrostatic equilibrium is established, and then the battery is disconnected. (*a*) What is the charge on each capacitor? (*b*) What is the total stored energy of the capacitors? (*c*) The dielectric is removed from C_2. What is the final stored energy of the capacitors? (*d*) What is the final voltage across the two capacitors?

Picture the Problem We can use the definition of capacitance to find the charge on each capacitor in Part (*a*). In Part (*b*) we can express the total energy stored as the sum of the energy stored on the two capacitors by using our result from (*a*) for the charge on each capacitor. When the dielectric is removed in Part (*c*) each capacitor will carry half the charge carried by the capacitor system previously and we can proceed as in (*b*). Knowing the total charge stored by the capacitors, we can use the definition of capacitance to find the final voltage across the two capacitors in Part (*d*).

(*a*) Use the definition of capacitance to express the charge on each capacitor as a function of its capacitance:

$$Q_1 = C_! V = \boxed{(200\,\text{V})C_1}$$

and

$$Q_2 = C_2 V = \kappa C_1 V = \boxed{(200\,\text{V})\kappa C_1}$$

(*b*) Express the total stored energy of the capacitors as the sum of stored energy in each capacitor:

$$\begin{aligned} U &= U_1 + U_2 = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 \\ &= \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}\kappa C_1V^2 \\ &= \tfrac{1}{2}C_1V^2(1+\kappa) \\ &= \tfrac{1}{2}(200\,\text{V})^2 C_1(1+\kappa) \\ &= \boxed{(2.00\times10^4\,\text{V}^2)(1+\kappa)C_1} \end{aligned}$$

(*c*) With the dielectric removed, each capacitor carries charge $Q/2$. Express the final energy stored by the capacitors under this condition:

$$U_f = \frac{1}{2}\frac{Q_1^2}{C_1} + \frac{1}{2}\frac{Q_2^2}{C_2} = \frac{1}{2}\frac{Q^2}{4C_1} + \frac{1}{2}\frac{Q^2}{4C_1}$$
$$= \frac{Q^2}{4C_1}$$

Using the definition of capacitance, express the total charge carried by the capacitors with the dielectric in place in C_2:

$$Q = Q_1 + Q_2 = C_1V + C_2V$$
$$= C_1V + \kappa C_1V = C_1V(1+\kappa)$$
$$= (200\text{ V})C_1(1+\kappa)$$

Substitute for Q in the expression for U_f to obtain:

$$U_f = \frac{[(200\text{ V})C_1(1+\kappa)]^2}{4C_1}$$
$$= \boxed{(1.00\times10^4\text{ V}^2)C_1(1+\kappa)^2}$$

(*d*) Use the definition of capacitance to express the final voltage across the capacitors:

$$V_f = \frac{Q}{C_{eq}} = \frac{(200\text{ V})C_1(1+\kappa)}{2C_1}$$
$$= \boxed{100(1+\kappa)\text{V}}$$

90 ••• A capacitor is constructed of two coaxial conducting thin cylindrical shells of radii a and b ($b > a$), which have a length $L >> b$. A charge of $+Q$ is on the inner cylinder, and a charge of $-Q$ is on the outer cylinder. The region between the two cylinders is filled with a material that has a dielectric constant κ. (*a*) Find the potential difference between the cylinders. (*b*) Find the density of the free charge σ_f on the inner cylinder and the outer cylinder. (*c*) Find the bound charge density σ_b on the inner cylindrical surface of the dielectric and on the outer cylindrical surface of the dielectric. (*d*) Find the total stored energy. (*e*) If the dielectric will move without friction, how much mechanical work is required to remove the dielectric cylindrical shell?

Picture the Problem We can use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to find the potential difference between the cylinders. In Part (*b*) we can apply the definition of surface charge density to find the density of the free charge σ_f on the inner and outer cylindrical surfaces. We can use the fact that that Q and Q_b are proportional to E and E_b to express Q_b at a and b and then apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$. In Part (*d*) we can use $U = \frac{1}{2}QV$ to find the total stored electrostatic energy and in (*e*) find the mechanical work required from the change in energy of the system resulting from the removal of the dielectric cylindrical shell.

(*a*) Using the definition of capacitance, relate the potential difference between the cylinders to their charge and capacitance:

$$V = \frac{Q}{C}$$

Express the capacitance of a cylindrical capacitor as a function of its radii *a* and *b* and length *L*:

$$C = \frac{2\pi \in_0 \kappa L}{\ln(b/a)}$$

Substituting for *C* and simplifying yields:

$$V = \frac{Q\ln(b/a)}{2\pi \in_0 \kappa L} = \boxed{\frac{2kQ\ln(b/a)}{\kappa L}}$$

(*b*) Apply the definition of surface charge density to obtain:

$$\sigma_f(a) = \boxed{\frac{Q}{2\pi aL}}$$

and

$$\sigma_f(b) = \boxed{\frac{-Q}{2\pi bL}}$$

(*c*) Noting that Q and Q_b are proportional to E and E_b, express Q_b at *a* and *b*:

$$Q_b(a) = \frac{-Q(\kappa-1)}{\kappa}$$

and

$$Q_b(b) = \frac{Q(\kappa-1)}{\kappa}$$

Apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$:

$$\sigma_b(a) = \frac{Q_b(a)}{A} = \frac{\frac{-Q(\kappa-1)}{\kappa}}{2\pi aL} = \boxed{\frac{-Q(\kappa-1)}{2\pi aL\kappa}}$$

and

$$\sigma_b(b) = \frac{Q_b(b)}{A} = \frac{\frac{Q(\kappa-1)}{\kappa}}{2\pi bL} = \boxed{\frac{Q(\kappa-1)}{2\pi bL\kappa}}$$

(*d*) Express the total stored energy in terms of the charge stored and the potential difference between the cylinders:

$$U = \tfrac{1}{2}QV = \tfrac{1}{2}Q\left[\frac{2kQ\ln(b/a)}{\kappa L}\right] = \boxed{\frac{kQ^2\ln(b/a)}{\kappa L}}$$

(*e*) Express the work required to remove the dielectric cylindrical shell in terms of the change in the potential energy of the system:

$$W = \Delta U = U' - U$$

where $U' = \kappa U$ is the potential energy of the system with the dielectric shell in place.

Substitute for U and U' and simplify to obtain:

$$W = \kappa U - U = U(\kappa - 1)$$
$$= \boxed{\frac{kQ^2(\kappa - 1)\ln(b/a)}{\kappa L}}$$

91 ••• Before Switch S is closed, as shown in Figure 24-51 the voltage across the terminals of the switch is 120 V and the voltage across the capacitor labeled C_1 is 40.0 V. The capacitance of C_1 is 0.200 μF. The total energy stored in the two capacitors is 1.44 mJ. After closing the switch, the voltage across each capacitor is 80.0 V, and the energy stored by the two capacitors has dropped to 960 μJ. Determine the capacitance of C_2 and the charge on that capacitor before the switch was closed.

Picture the Problem Note that, with switch S closed, C_1 and C_2 are in parallel and we can use $U_{\text{closed}} = \frac{1}{2}C_{\text{eq}}V^2$ and $C_{\text{eq}} = C_1 + C_2$ to obtain an equation we can solve for C_2. We can use the definition of capacitance to express Q_2 in terms of V_2 and C_2 and $U_{\text{open}} = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$ to obtain an equation from which we can determine V_2.

Express the energy stored in the capacitors after the switch is closed:

$$U_{\text{closed}} = \tfrac{1}{2}C_{\text{eq}}V^2$$

Express the equivalent capacitance of C_1 and C_2 in parallel:

$$C_{\text{eq}} = C_1 + C_2$$

Substitute for C_{eq} to obtain:

$$U_{\text{closed}} = \tfrac{1}{2}(C_1 + C_2)V^2$$

Solving for C_2 yields:

$$C_2 = \frac{2U_{\text{closed}}}{V^2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{2(960\ \mu\text{J})}{(80\ \text{V})^2} - 0.200\ \mu\text{F} = \boxed{0.100\ \mu\text{F}}$$

Express the charge on C_2 when the switch is open:

$$Q_2 = C_2V_2 \qquad (1)$$

Express the energy stored in the capacitors with the switch open:

$$U_{\text{open}} = \tfrac{1}{2}C_1V_1^2 + \tfrac{1}{2}C_2V_2^2$$

Solving for V_2 yields:

$$V_2 = \sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}}$$

Substitute for V_2 in equation (1) to obtain:

$$Q_2 = C_2\sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}} = \sqrt{C_2\left(2U_{\text{open}} - C_1V_1^2\right)}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \sqrt{(0.100\,\mu\text{F})\left[2(1.44\,\text{mJ}) - (0.200\,\mu\text{F})(40.0\,\text{V})^2\right]} = \boxed{16.0\,\mu\text{C}}$$

92 ••• An air-filled parallel-plate capacitor that has gap-width d has plates which each have an area A. The capacitor is charged to a potential difference V and is then removed from the voltage source. A dielectric slab that has a dielectric constant of 2.00, a thickness d, and an area $\frac{1}{2}A$ is then inserted, as shown in Figure24-52. Let σ_1 be the free charge density at the conductor–dielectric surface, and let σ_2 be the free charge density at the conductor–air surface. (*a*) Explain why the electric field must have the same value inside the dielectric as in the free space between the plates. (*b*) Show that $\sigma_1 = 2\sigma_2$. (*c*) Show that the final capacitance (after the slab is inserted) is 1.50 times the capacitance when the capacitor is filled with air. (*d*) Show that the final potential difference is $\frac{2}{3}V$. (*e*) Show that energy stored after the slab is inserted is only two-thirds of the energy stored before insertion.

Picture the Problem (*b*) We can express the electric fields in the dielectric and in the free space in terms of the charge densities and then use the fact that the electric field has the same value inside the dielectric as in the free space between the plates to establish that $\sigma_1 = 2\sigma_2$. In Parts (*c*) and (*d*) we can model the system as two capacitors in parallel to show that the equivalent capacitance is $3\epsilon_0A/(2d)$ and then use the definition of capacitance to show that the new potential difference is $\frac{2}{3}V$.

(*a*) The potential difference between the plates is the same for both halves (the plates are equipotential surfaces). Therefore, $E = V/d$ must be the same everywhere between the plates.

(*b*) Relate the electric field in each region to σ and κ:

$$E = \frac{\sigma}{\kappa \in_0} \Rightarrow \sigma = \kappa \in_0 E$$

Express σ_1 and σ_2:

$$\sigma_1 = \kappa_1 \in_0 E_1 = 2 \in_0 E_1$$

and

$$\sigma_2 = \kappa_2 \in_0 E_2 = \in_0 E_1$$

Divide the 1st of these equations by the 2nd and simplify to obtain:

$$\boxed{\sigma_1 = 2\sigma_2}$$

(*c*) Model the partially dielectric-filled capacitor as two capacitors in parallel to obtain:

$$C_{eq} = C_1 + C_2$$

where

$$C_1 = \frac{\kappa \in_0 \left(\frac{1}{2}A\right)}{d} = \frac{\kappa \in_0 A}{2d}$$

and

$$C_2 = \frac{\in_0 \left(\frac{1}{2}A\right)}{d} = \frac{\in_0 A}{2d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{eq} = \frac{\kappa \in_0 A}{2d} + \frac{\in_0 A}{2d} = \frac{2 \in_0 A}{2d} + \frac{\in_0 A}{2d}$$

$$= \frac{3 \in_0 A}{2d} = \boxed{1.50 C_{\text{air-filled}}}$$

(*d*) Use the definition of capacitance to relate V_f, Q_f, and C_f:

$$V_f = \frac{Q_f}{C_f}$$

Because the capacitors are in parallel:

$$Q_f = Q_i = VC_i = \frac{V \in_0 A}{d}$$

Substitute for Q_f and C_f and simplify to obtain:

$$V_f = \frac{V \in_0 A}{C_f d} = \frac{V \in_0 A}{\left(\frac{3 \in_0 A}{2d}\right)d} = \boxed{\tfrac{2}{3}V}$$

(*e*) The energy stored after the slab is inserted is given by:

$$U_f = \tfrac{1}{2} C_f V_f^2$$

Substituting for C_f and V_f and simplifying yields:

$$U_f = \tfrac{1}{2}\left(\tfrac{3}{2}C_i\right)\left(\tfrac{2}{3}V\right)^2 = \tfrac{1}{3}C_i V^2 = \boxed{\tfrac{2}{3}U_i}$$

The presence of the dielectric slab reduces the potential difference between the capacitor plates and, hence, the energy stored in the capacitor.

93 ••• A capacitor has rectangular plates of length a and width b. The top plate is inclined at a small angle, as shown in Figure 24-53. The plate separation varies from y_0 at the left to $2y_0$ at the right, where y_0 is much less than a or b. Calculate the capacitance of this arrangement. *HINT*: *Break the problem up into a parallel combination. Choose strips of width dx and length b to approximate small (differential) capacitors (each having a value of dC). Each will have a plate area of bdx and separation distance $y_0 + (y_0/a)x$. Then argue that these differential capacitors are connected in parallel.*

Picture the Problem Choose a coordinate system in which the +x direction is the right and the origin is at the left edge of the capacitor. We can express an element of capacitance dC and then integrate this expression to find C for this capacitor.

Express an element of capacitance dC of length b, width dx and separation $d = y_0 + (y_0/a)x$:

$$dC = \frac{\epsilon_0\, b}{d}dx = \frac{\epsilon_0\, b}{y_0(1+x/a)}dx$$

These elements are all in parallel, so the total capacitance is obtained by integration:

$$C = \frac{\epsilon_0\, b}{y_0}\int_0^{y_0}\frac{1}{1+x/a}dx = \boxed{\frac{\epsilon_0\, ab}{y_0}\ln(2)}$$

94 ••• Not all dielectrics that separate the plates of a capacitor are rigid. For example, the membrane of a nerve axon is a bi-lipid layer that has a finite compressibility. Consider a parallel-plate capacitor whose plate separation is maintained by a material that has a dielectric constant of 3.00, a dielectric strength of 40.0 kV/mm and a Young's modulus[1] for compressive stress of 5.00×10^6 N/m^2. When the potential difference between the capacitor plates is zero, the thickness of the dielectric is equal to 0.200 mm and the capacitance of the capacitor is given by C_0. (*a*) Derive an expression for the capacitance, as a function of the potential difference between the capacitor plates. (*b*) What is the maximum value of this potential difference? (Assume that the dielectric constant and the dielectric strength do *not* change under compression.)

Picture the Problem The diagram below and to the left shows the dielectric-filled parallel-plate capacitor before compression and the diagram to the right shows the capacitor when the plate separation has been reduced to x. We can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to derive an expression for the capacitance as a function of voltage across the capacitor. We can find the maximum voltage that can be applied from the dielectric strength of the dielectric and the separation of the plates. In Part (*c*) we can find the fraction of the total energy that is electrostatic field energy and

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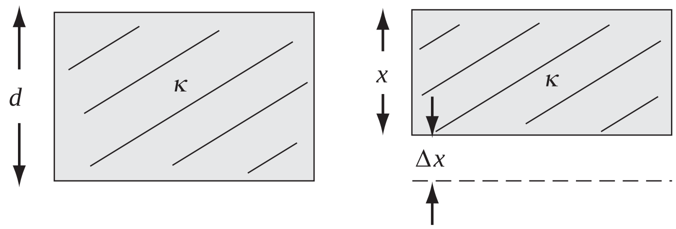
[1] Young's modulus is discussed in Section 12-8.

the fraction that is mechanical stress energy by expressing either of these as a fraction of their sum.



(*a*) Use its definition to express the capacitance as a function of the voltage across the capacitor:

$$C(V)=\frac{Q}{V} \qquad (1)$$

The limiting value of the capacitance is:

$$C_0=\frac{\kappa \in_0 A}{d}$$

Substitute numerical values and evaluate C_0:

$$C_0=\frac{3\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}\right)A}{0.200\,\mathrm{mm}}$$

$$=0.133\times10^{-6}\,A\frac{\mathrm{C}^2}{\mathrm{N\cdot m^3}}$$

Let x be the variable separation. Because κ is independent of x:

$$C(x)=\frac{\kappa \in_0 A}{x}$$

and

$$Q(x)=C(x)V=\frac{\kappa \in_0 A}{x}V$$

Substitute in equation (1) to obtain:

$$C(V)=\frac{\frac{\kappa \in_0 A}{x}V}{V}=\frac{\kappa \in_0 A}{x} \qquad (2)$$

$$=\frac{\kappa \in_0 A}{d-\Delta x}$$

The force of attraction between the plates is given in Problem 95 (*c*):

$$F=\frac{Q^2(x)}{2\kappa \in_0 A}$$

Substitute to obtain:

$$F=\frac{\left(\frac{\kappa\in_0 A}{x}V\right)^2}{2\kappa\in_0 A}=-\frac{\kappa\in_0 AV^2}{2x^2}$$

where the minus sign is used to indicate that the force acts to decrease the plate separation *x*.

Apply Hooke's law to relate the stress to the strain:

$$Y=\frac{F/A}{\Delta x/x}\text{ or }\frac{\Delta x}{x}=\frac{F}{YA}$$

Substitute for *F* to obtain:

$$\frac{\Delta x}{x}=-\frac{\kappa\in_0 V^2}{2Yx^2}$$

and

$$\Delta x=-\frac{\kappa\in_0 V^2}{2Yx}=-\frac{\kappa\in_0 V^2}{2Yd}\qquad(3)$$

provided $\Delta x \ll d$

The voltage across the capacitor is:

$$V=E_{max}d=(40.0\,\text{kV/mm})(0.200\,\text{mm})=8.00\,\text{kV}$$

Substitute numerical values in equation (3) and evaluate Δx:

$$\Delta\boldsymbol{x}=-\frac{3\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(8.00\,\text{kV})^2}{2\left(5.00\times10^6\,\text{N/m}^2\right)(0.200\,\text{mm})}=8.50\times10^{-7}\,\text{m}=8.50\times10^{-4}\,\text{mm}$$

Substitute in equation (2) to obtain:

$$C(V)=\frac{\kappa\in_0 A}{d-\frac{\kappa\in_0 V^2}{2Yd}}=\frac{\kappa\in_0 A}{d\left(1-\frac{\kappa\in_0 V^2}{2Yd^2}\right)}=C_0\left(1-\frac{\kappa\in_0 V^2}{2Yd^2}\right)^{-1}\approx\boxed{C_0\left(1+\frac{\kappa\in_0 V^2}{2Yd^2}\right)}$$

provided $\Delta x \ll d$.

(*b*) Express the maximum voltage that can be applied in terms of the maximum electric field:

$$V_{max}=E_{max}(d-\Delta x)=(40.0\,\text{kV/mm})\left(0.200\,\text{mm}-8.50\times10^{-4}\,\text{mm}\right)=\boxed{7.97\,\text{kV}}$$

Chapter 25
Electric Current and Direct-Current Circuits

Conceptual Problems

1 • In our study of electrostatics, we concluded that no electric field exists within the material of a conductor in electrostatic equilibrium. Why can we discuss electric fields within the material of conductors in this chapter?

Determine the Concept In earlier chapters the conductors are constrained to be in electrostatic equilibrium. In this chapter this constraint is no longer in place.

2 • Figure 25-12 shows a mechanical analog of a simple electric circuit. Devise another mechanical analog in which the current is represented by a flow of water instead of marbles. In the water circuit, what would be analogous to the battery? What would be analogous to the wire? What would be analogous to the resistor?

Determine the Concept The analog is a wind-up water pump that pumps water through a tube with a necked down section. One end of the tube is connected to the output port of the pump, and the other end of the tube is connected to the input port of the pump. The pump, including the spring, is analogous to the battery. The tube is analogous to the wires. The necked down section is analogous to the resistor.

3 • Wires A and B are both made of copper. The wires are connected in series, so we know they carry the same current. However, the diameter of wire A is twice the diameter of wire B. Which wire has the higher number density (number per unit volume) of charge carriers? (*a*) A, (*b*) B, (*c*) They have the same number density of charge carriers.

Determine the Concept Equation 25-3 ($I = qnAv_d$) relates the current I in a wire to the charge q of the charge carriers, the number density n of charge carriers, the cross-sectional area A of the wire, and the drift speed v_d of the charge carriers. Both the number density of the charge carriers and the resistivity are properties of the substance, and do not depend on the current. Because the number density of carriers may, like resistivity, vary slowly with temperature and the drift speeds are not equal in the two wires, the correct answer is $\boxed{(c)}$.

4 • The diameters of copper wires A and B are equal. The current carried by wire A is twice the current carried by wire B. In which wire do the charge carriers have the higher drift speed? (*a*) A (*b*) B (*c*) They have the same drift speed.

Determine the Concept Equation 25-3 ($I = qnAv_d$) relates the current I in a wire to the charge q of the charge carriers, the number density n of charge carriers, the cross-sectional area A of the wire, and the drift velocity v_d of the charge carriers. Both the number density of the charge carriers and the resistivity are properties of the substance, and do not depend on the current. Because the cross-sectional areas, the charge of the charge carriers, and the number densities of the charge carriers are the same for the two wires, the drift speed of the charge carriers will be higher in the wire carrying the larger current. Because wire A carries the larger current, $\boxed{(a)}$ is correct.

5 • Wire A and wire B are identical copper wires. The current carried by wire A is twice the current carried by wire B. Which wire has the higher current density? (*a*) A, (*b*) B, (*c*) They have the same current density. (*d*) None of the above

Determine the Concept The current density is the ratio of the current to the cross-sectional area of the conductor. Because the wires are identical, their cross-sectional areas are the same and their current densities are directly proportional to their currents. Because wire A carries the larger current, $\boxed{(a)}$ is correct.

6 • Consider a metal wire that has each end connected to a different terminal of the same battery. Your friend argues that no matter how long the wire is, the drift speed of the charge carriers in the wire is the same. Evaluate your friend's claim.

Determine the Concept The longer the wire is, the higher its resistance is. Thus, the longer the wire is the smaller the current in the wire is. The smaller the current is, the smaller the drift speed of the charge carriers is. The claim that the drift speed is independent of length is not true.

7 • In a resistor, the direction of the current must always be in the ″downhill″ direction, that is, in the direction of decreasing electric potential. Is it also the case that in a battery, the direction of the current must always be ″downhill″? Explain your answer.

Determine the Concept No, it is not necessarily true for a battery. Under normal operating conditions, the current in the battery is in the direction away from the negative battery terminal and toward the positive battery terminal. That is, it is opposite to the direction of the electric field.

8 • Discuss the distinction between an emf and a potential difference.

Determine the Concept An emf is a source of energy that gives rise to a potential difference between two points and may result in current flow if there is a conducting path whereas a potential difference is the consequence of two points in space being at different potentials.

9 • Wire A and wire B are made of the same material and have the same length. The diameter of wire A is twice the diameter of wire B. If the resistance of wire B is R, then what is the resistance of wire A? (Neglect any effects that temperature may have on resistance.) (*a*) R, (*b*) $2R$, (*c*) $R/2$, (*d*) $4R$, (*e*) $R/4$,

Picture the Problem The resistances of the wires are given by $R = \rho L/A$, where L is the length of the wire and A is its cross-sectional area. Because the resistance of a wire varies inversely with its cross-sectional area and its cross-sectional area varies with the square of its diameter, the wire that has the larger diameter will have a resistance that is one fourth the resistance of the wire that has the smaller diameter. Because wire A has a diameter that is twice that of wire B, its resistance is one-fourth the resistance of wire B. $\boxed{(e)}$ is correct.

10 • Two cylindrical copper wires have the same mass. Wire A is twice as long as wire B. (Neglect any effects that temperature may have on resistance.) Their resistances are related by (*a*) $R_A = 8R_B$, (*b*) $R_A = 4R_B$, (*c*) $R_A = 2R_B$, (*d*) $R_A = R_B$.

Determine the Concept The resistance of a wire is given by $R = \rho L/A$, where L is the length of the wire and A is its cross-sectional area. Because the wires have the same masses, their volumes must be the same. Because their volumes are the same, the cross-sectional area of wire A must be half the cross-sectional area of wire B. So wire A must have a resistance, based solely on its cross-sectional area, that is twice the resistance of wire B. Because it is also twice as long as wire B, its resistance will be four times the resistance of wire B. $\boxed{(b)}$ is correct.

11 • If the current in a resistor is I, the power delivered to the resistor is P. If the current in the resistor is increased to $3I$, what is the power then delivered to the resistor? (Assume the resistance of the resistor does not change.) (*a*) P, (*b*) $3P$, (*c*) $P/3$, (*d*) $9P$, (*e*) $P/9$

Picture the Problem The power dissipated in the resistor is given by $P = I^2R$. Because of this quadratic dependence of the power dissipated on the current, tripling the current in the resistor increases its resistance by a factor of 9. $\boxed{(d)}$ is correct.

12 • If the potential drop across the resistor is V, the power delivered to the resistor is P. If the potential drop is increased to $2V$, what is the power delivered to the resistor then equal to? (*a*) P, (*b*) $2P$, (*c*) $4P$, (*d*) $P/2$, (*e*) $P/4$

Picture the Problem Assuming the current (which depends on the resistance) to be constant, the power dissipated in a resistor is directly proportional to the square of the potential drop across it. Hence, doubling the potential drop across a resistor increases the power delivered to the resistor by a factor of 4. $\boxed{(c)}$ is correct.

13 • **[SSM]** A heater consists of a variable resistor (a resistor whose resistance can be varied) connected across an ideal voltage supply. (An ideal voltage supply is one that has a constant emf and a negligible internal resistance.) To increase the heat output, should you decrease the resistance or increase the resistance? Explain your answer.

Determine the Concept You should decrease the resistance. The heat output is given by $P = V^2/R$. Because the voltage across the resistor is constant, decreasing the resistance will increase P.

14 • One resistor has a resistance R_1 and another resistor has a resistance R_2. The resistors are connected in parallel. If $R_1 >> R_2$, the equivalent resistance of the combination is approximately (*a*) R_1, (*b*) R_2, (*c*) 0, (*d*) infinity.

Determine the Concept The equivalent resistance of the two resistors connected in parallel is given by $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 >> R_2$, then the reciprocal of R_1 is very small and $\frac{1}{R_{eq}} \approx \frac{1}{R_2}$. Hence $R_{eq} \approx R_2$ and $\boxed{(b)}$ is correct.

15 • One resistor has a resistance R_1 and another resistor has a resistance R_2. The resistors are connected in series. If $R_1 >> R_2$, the equivalent resistance of the combination is approximately (*a*) R_1, (*b*) R_2, (*c*) 0, (*d*) infinity.

Determine the Concept The equivalent resistance of the two resistors connected in series is given by $R_{eq} = R_1 + R_2 = R_1\left(1 + \frac{R_2}{R_1}\right)$. If $R_1 >> R_2$, then R_2/R_1 is very small and $R_{eq} \approx R_1$. Hence $\boxed{(a)}$ is correct.

16 • A parallel combination consisting of resistors A and B is connected across the terminals of a battery. The resistor A has twice the resistance of resistor B. If the current carried by resistor A is I, then what is the current carried by resistor B? (*a*) I, (*b*) $2I$, (*c*) $I/2$, (*d*) $4I$, (*e*) $I/4$

Picture the Problem Ohm's law states that the current in a resistor is proportional to the potential drop across the resistor. Because the potential difference across resistors A and connected in parallel is the same for each resistor, the resistor with half the resistance of the other resistor will carry twice the current carried by the resistor with the larger resistance. Because resistor A has twice the resistance of resistor B, resistor B will carry twice the current of resistor A. $\boxed{(b)}$ is correct.

17 • A series combination consisting of resistors *A* and *B* is connected across the terminals of a battery. The resistor A has twice the resistance of resistor B. If the current carried by resistor A is *I*, then what is the current carried by resistor B? (*a*) *I,* (*b*) 2*I*, (*c*) *I*/2, (*d*) 4*I*, (*e*) *I*/4

Determine the Concept In a series circuit, because there are no alternative pathways, all resistors carry the same current. $\boxed{(a)}$ is correct.

18 • Kirchhoff's junction rule is considered to be a consequence of (*a*) conservation of charge, (*b*) conservation of energy, (*c*) Newton's laws, (*d*) Coulomb's law, (*e*) quantization of charge.

Determine the Concept While Kirchhoff's junction rule is a statement about current, recall that current is the rate at which charge passes some point in space. Hence, the junction rule is actually a statement that charge is conserved. $\boxed{(a)}$ is correct.

19 • True or False:

(*a*) An ideal voltmeter has a zero internal resistance.
(*b*) An ideal ammeter has a zero internal resistance.
(*c*) An ideal voltage source has a zero internal resistance

(*a*) False. An ideal voltmeter would have infinite resistance. A voltmeter consists of a galvanometer movement connected in series with a large resistance. The large resistor accomplishes two purposes; 1) it protects the galvanometer movement by limiting the current drawn by it, and 2) minimizes the loading of the circuit by the voltmeter by placing a large resistance in parallel with the circuit element across which the potential difference is being measured.

(*b*) True. An ideal ammeter would have zero resistance. An ammeter consists of a very small resistance in parallel with a galvanometer movement. The small resistance accomplishes two purposes: 1) It protects the galvanometer movement by shunting most of the current in the circuit around the galvanometer movement, and 2) It minimizes the loading of the circuit by the ammeter by minimizing the resistance of the ammeter.

(*c*) True. An ideal voltage source would have zero internal resistance. The terminal potential difference of a voltage source is given by $V = \mathcal{E} - Ir,$ where $\mathcal{E}$ is the emf of the source, I is the current drawn from the source, and r is the internal resistance of the source.

20 • Before you and your classmates run an experiment, your professor lectures about safety. She reminds you that to measure the voltage across a resistor you connect a voltmeter in parallel with the resistor, and to measure the current in a resistor you connect an ammeter in series with the resistor. She also states that connecting a voltmeter in series with a resistor will not measure the voltage across the resistor, but also cannot do any damage to the circuit or the instrument. In addition, connecting an ammeter in parallel with a resistor will not measure the current in the resistor, but could cause significant damage to the circuit and the instrument. Explain why connecting a voltmeter in series with a resistor causes no damage while connecting an ammeter in parallel with a resistor can cause significant damage.

Determine the Concept Because of the voltmeter's high resistance, if you connect a voltmeter in series with a circuit element, the current, both in the voltmeter and in the rest of the circuit, will be very small. This means that there is little chance of heating the voltmeter and causing damage. However, because of the ammeter's low resistance, if you connect an ammeter in parallel with a circuit element, the current, both in the ammeter and in the entire circuit, excluding any elements in parallel with the ammeter, will be very large. This means that there is a good chance of overheating and causing damage, maybe even a fire. For this reason, ammeters are often equipped with fuses or circuit breakers.

21 • The capacitor in Figure 25-49 is initially uncharged. Just after the switch S is closed, (*a*) the voltage across C equals $\mathcal{E}$, (*b*) the voltage across R equals $\mathcal{E}$, (*c*) the current in the circuit is zero, (*d*) both (*a*) and (*c*) are correct.

Determine the Concept If we apply Kirchhoff's loop rule with the switch closed, we obtain $\mathcal{E} - IR - V_C = 0$. Immediately after the switch is closed, $I = I_{\max}$ and $V_C = 0$. Hence $\mathcal{E} = I_{\max} R$. $\boxed{(b)}$ is correct.

22 •• A capacitor is discharging through a resistor. If it takes a time T for the charge on a capacitor to drop to half its initial value, how long (in terms of T) does it take for the stored energy to drop to half its initial value?

Picture the Problem We can express the variation of charge on the discharging capacitor as a function of time to find the time T it takes for the charge on the capacitor to drop to half its initial value. We can also express the energy remaining in the electric field of the discharging capacitor as a function of time and find the time t' for the energy to drop to half its initial value in terms of T.

Express the time-dependence of the charge stored on a capacitor:

$$Q(t) = Q_0 e^{-t/\tau}$$

where $\tau = RC$.

For $Q(t) = \frac{1}{2}Q_0$:

$$\tfrac{1}{2}Q_0 = Q_0 e^{-T/\tau} \text{ or } \tfrac{1}{2} = e^{-T/\tau}$$

Take the natural logarithm of both sides of the equation and solve for T to obtain:

$$T = \tau \ln(2)$$

Express the dependence of the energy stored in a capacitor on the potential difference V_C across its terminals:

$$U(t) = \tfrac{1}{2}CV_C^2 \qquad (1)$$

Express the potential difference across a discharging capacitor as a function of time:

$$V_C = V_0 e^{-t/RC}$$

Substitute for V_C in equation (1) and simplify to obtain:

$$U(t) = \tfrac{1}{2}C\left(V_0 e^{-t/RC}\right)^2 = \tfrac{1}{2}CV_0^2 e^{-2t/RC} = U_0 e^{-2t/RC}$$

For $U(t) = \frac{1}{2}U_0$:

$$\tfrac{1}{2}U_0 = U_0 e^{-2t'/RC} \Rightarrow \tfrac{1}{2} = e^{-2t'/RC}$$

Take the natural logarithm of both sides of the equation and solve for t' to obtain:

$$t' = \tfrac{1}{2}\tau \ln(2) = \boxed{\tfrac{1}{2}T}$$

23 •• [SSM] In Figure 25-50, the values of the resistances are related as follows: $R_2 = R_3 = 2R_1$. If power P is delivered to R_1, what is the power delivered to R_2 and R_3?

Determine the Concept The power delivered to a resistor varies with the square of the current in the resistor and is directly proportional to the resistance of the resistor ($P = I^2R$).

The power delivered to R_1 is: $P = I_1^2 R_1$

The power delivered to R_2 is: $P_2 = I_2^2 R_2 = I_2^2(2R_1)$

Because $R_2 = R_3$: $I_2 = I_3 = \frac{1}{2}I_1$

Substituting for I_2 and simplifying gives: $P_2 = \left(\frac{1}{2}I_1\right)^2(2R_1) = \frac{1}{2}I_1^2R_1 = \boxed{\frac{1}{2}P}$

Similarly: $P_3 = I_3^2R_3 = \left(\frac{1}{2}I_1\right)^2(2R_1) = \frac{1}{2}I_1^2R_1 = \boxed{\frac{1}{2}P}$

24 •• The capacitor in Figure 25-49 is initially uncharged. The switch *S* is closed and remains closed for a very long time. During this time, (*a*) the energy supplied by the battery is $\frac{1}{2}C\mathcal{E}^2$, (*b*) the energy dissipated in the resistor is $\frac{1}{2}C\mathcal{E}^2$, (*c*) energy in the resistor is dissipated at a constant rate, (*d*) the total charge passing through the resistor is $\frac{1}{2}C\mathcal{E}$.

Determine the Concept The energy stored in the fully charged capacitor is $U = \frac{1}{2}C\mathcal{E}^2$. During the charging process, a total charge $Q_f = \mathcal{E}C$ flows through the battery. The battery therefore does work $W = Q_f\mathcal{E} = C\mathcal{E}^2$. The energy dissipated in the resistor is the difference between W and U. $\boxed{(b)}$ is correct.

Estimation and Approximation

25 •• It is not a good idea to stick the ends of a paper clip into the two rectangular slots of a household electrical wall outlet in the United States. Explain why by estimating the current that a paper clip would carry until either the fuse blows or the breaker trips.

Picture the Problem The current drawn by the paper clip is the ratio of the potential difference (120 V) between the two holes and the resistance of the paper clip. We can use $R = \rho L/A$ to find the resistance of the paper clip. Paper clips are made of steel, which has a resistivity in the range 10 to 100 × 10^{-8} Ω· m,

From the definition of resistance: $I = \frac{\mathcal{E}}{R}$

The resistance of the paper clip is given by:

$$R = \rho\frac{L}{A}$$

Substituting for R in the expression for I and simplifying yields

$$I = \frac{\mathcal{E}}{\rho\frac{L}{A}} = \frac{\mathcal{E}A}{\rho L} = \frac{\mathcal{E}\pi r^2}{\rho L} = \frac{\mathcal{E}\pi d^2}{4\rho L}$$

Assuming that the length of a paper clip is 10 cm and that its diameter is 1.0 mm, substitute numerical values and evaluate I:

$$I = \frac{(120\text{ V})\pi(1.0\text{ mm})^2}{4(50\times10^{-8}\ \Omega\cdot\text{m})(10\text{ cm})} = \boxed{1.9\text{ kA}}$$

26 •• (*a*) Estimate the resistance of an automobile jumper cable. (*b*) Look up the current required to start a typical car. At that current, what is the potential drop that occurs across the jumper cable? (*c*) How much power is dissipated in the jumper cable when it carries this current?

Picture the Problem (*a*) We can use the definition of resistivity to find the resistance of the jumper cable. In Part (*b*), the application of Ohm's law will yield the potential difference across the jumper cable when it is starting a car, and, in Part (*c*), we can use the expression for the power dissipated in a conductor to find the power dissipation in the jumper cable.

(*a*) Noting that a jumper cable has two leads, express the resistance of the cable in terms of the wire's resistivity and the cable's length, and cross-sectional area:

$$R = \rho\frac{L}{A}$$

Assuming the length of the jumper cables to be 3.0 m and its cross-sectional area to be 10 mm^2, substitute numerical values (see Table 25-1 for the resistivity of copper) and evaluate R:

$$R = (1.7\times10^{-8}\ \Omega\cdot\text{m})\frac{3.0\text{ m}}{10\text{ mm}^2}$$
$$= \boxed{5.1\text{ m}\Omega}$$

(*b*) Apply Ohm's law to the cable with a starting current of 90 A to obtain:

$$V = IR = (90\text{ A})(5.1\text{ m}\Omega) = 459\text{ mV}$$
$$= \boxed{0.46\text{ V}}$$

(*c*) Use the expression for the power dissipated in a conductor to obtain:

$$P = IV = (90\,\mathrm{A})(459\,\mathrm{mV}) = \boxed{41\,\mathrm{W}}$$

27 •• Your manager wants you to design a new super-insulated water heater for the residential market. A coil of Nichrome wire is to be used as the heating element. Estimate the length of wire required. *HINT: You will need to determine the size of a typical hot water heater and a reasonable time period for creating hot water.*

Picture the Problem We can combine the expression for the rate at which energy is delivered to the water to warm it ($P = \mathcal{E}^2/R$) and the expression for the resistance of a conductor ($R = \rho L/A$) to obtain an expression for the required length L of Nichrome wire.

From Equation 25-10, the length of Nichrome wire required is given by:

$L = \dfrac{RA}{\rho_{\mathrm{Ni}}}$ where A is the cross-sectional area of the wire.

Use an expression for the power dissipated in a resistor to relate the required resistance to rate at which energy is delivered to generate the warm water:

$$P = \frac{\mathcal{E}^2}{R} \Rightarrow R = \frac{\mathcal{E}^2}{P}$$

Substituting for R and simplifying yields:

$$L = \frac{\mathcal{E}^2 A}{\rho_{\mathrm{Ni}} P}$$

The power required to heat the water is the rate at which the wire will deliver energy to the water:

$$P = \frac{\Delta E}{\Delta t} = \frac{\Delta(mc\Delta T)}{\Delta t} = mc\frac{\Delta T}{\Delta t}$$

Substitute for P to obtain:

$$L = \frac{\mathcal{E}^2 A}{\rho_{\mathrm{Ni}} mc\dfrac{\Delta T}{\Delta t}}$$

The mass of water to be heated is the product of its density and the volume of the water tank:

$m = \rho_{\mathrm{H_2O}} V = \rho_{\mathrm{H_2O}} A'd$

where A' is the cross-sectional area of the water tank.

Finally, substituting for m and simplifying yields:

$$L=\frac{\mathcal{E}^2 A}{\rho_{\text{Ni}}\rho_{\text{H}_2\text{O}}A'dc\dfrac{\Delta T}{\Delta t}}=\frac{\mathcal{E}^2\pi r_{\text{wire}}^2\Delta t}{\rho_{\text{Ni}}\rho_{\text{H}_2\text{O}}\pi r_{\text{tank}}^2 dc\Delta T}=\frac{\mathcal{E}^2 r_{\text{wire}}^2\Delta t}{\rho_{\text{Ni}}\rho_{\text{H}_2\text{O}} r_{\text{tank}}^2 dc\Delta T}$$

The diameter of a typical 40-gal water heater is about 50 cm and its height is approximately 1.3 m. Assume that the diameter of the Nichrome wire is 2.0 mm and that the water, initially at 20°C, is to be heated to 80°C in 1.0 h. Substitute numerical values (see Table 25-1 for the resistivity of Nichrome and Table 18-1 for the heat capacity of water) and evaluate L:

$$L=\frac{(120\,\text{V})^2\left(2.0\times10^{-3}\,\text{m}\right)^2\left(1.0\,\text{h}\times\dfrac{3600\,\text{s}}{\text{h}}\right)}{\left(100\times10^{-8}\,\Omega\cdot\text{m}\right)\left(1000\dfrac{\text{kg}}{\text{m}^3}\right)(0.25\ \text{m})^2(0.50\,\text{m})\left(4.18\dfrac{\text{kJ}}{\text{kg}\cdot\text{C}^\circ}\right)(60\text{C}^\circ)}=\boxed{26\,\text{m}}$$

28 •• A compact fluorescent light bulb costs about \$6.00 each and has a typical lifetime of 10 000 h. These bulbs use 20 W of power, but produce illumination equivalent to that of 75-W incandescent bulbs. An incandescent bulb costs about \$1.50 and has a typical lifetime of 1000 h. Your family wonders whether it should buy fluorescent light bulbs. Estimate the amount of money your household would save each year by using compact fluorescent light bulbs instead of the incandescent bulbs.

Picture the Problem We can find the annual savings by taking into account the costs of the two types of bulbs, the rate at which they consume energy and the cost of that energy, and their expected lifetimes. We'll assume that the household lights are on for an average of 8 hours per day.

Express the yearly savings:

$$\Delta\$=\text{Cost}_{\text{incandescent}}-\text{Cost}_{\text{fluorescent}}\qquad(1)$$

Express the annual costs with the incandescent and fluorescent bulbs:

$$\text{Cost}_{\text{incandescent}}=\text{Cost}_{\text{bulbs}}+\text{Cost}_{\text{energy}}$$

and

$$\text{Cost}_{\text{fluorescent}}=\text{Cost}_{\text{bulbs}}+\text{Cost}_{\text{energy}}$$

The annual cost of the incandescent bulbs is the product of the number of bulbs in use, the annual consumption of bulbs, and the cost per bulb:

$$\text{Cost}_{\text{bulbs}} = (6)\left(\frac{365.24\,\text{d} \times \dfrac{8\,\text{h}}{\text{d}}}{1000\,\text{h}}\right)(\$1.50) = \$26.30$$

The cost of operating the incandescent bulbs for one year is the product of the energy consumed and the cost per unit of energy:

$$\text{Cost}_{\text{energy}} = 6(75\,\text{W})\left(365.24\frac{\text{d}}{\text{y}}\right)\left(\frac{8\,\text{h}}{\text{d}}\right)\left(\frac{\$0.115}{\text{kW}\cdot\text{h}}\right) = \$151.21$$

The annual cost of the fluorescent bulbs is the product of the number of bulbs in use, the annual consumption of bulbs, and the cost per bulb:

$$\text{Cost}_{\text{bulbs}} = (6)\left(\frac{365.24\,\text{d} \times \dfrac{8\,\text{h}}{\text{d}}}{10000\,\text{h}}\right)(\$6) = \$10.52$$

The cost of operating the fluorescent bulbs for one year is the product of the energy consumed and the cost per unit of energy:

$$\text{Cost}_{\text{energy}} = 6(20\,\text{W})\left(365.24\,\text{d} \times \frac{8\,\text{h}}{\text{d}}\right)\left(\frac{\$0.115}{\text{kW}\cdot\text{h}}\right) = \$40.32$$

Substitute in equation (1) and evaluate the cost savings Δ$:

$$\begin{aligned}\Delta\$ &= \text{Cost}_{\text{incandescent}} - \text{Cost}_{\text{fluorescent}} = (\$26.30 + \$151.20) - (\$10.52 + \$40.32) \\ &= \boxed{\$127}\end{aligned}$$

29 •• The wires in a house must be large enough in diameter so that they do not get hot enough to start a fire. While working for a building contractor during the summer, you are involved in remodeling a house. The local building code states that the joule heating of the wire used in houses should not exceed 2.0 W/m. Estimate the maximum gauge of the copper wire that you can use during the rewiring of the house with 20-A circuits.

Picture the Problem We can use an expression for the power dissipated in a resistor to relate the Joule heating in the wire to its resistance and the definition of resistivity to relate the resistance to the length and cross-sectional area of the wire. We can find the diameter of the wire from its cross-sectional area and then use Table 25-2 to find the maximum gauge you can use during the rewiring of the house.

Express the power the wires must dissipate in terms of the current they carry and their resistance:

$$P = I^2 R$$

Divide both sides of the equation by L to express the power dissipation per unit length:

$$\frac{P}{L} = \frac{I^2 R}{L}$$

The resistance of the wire is given by:

$$R = \rho\frac{L}{A} = \rho\frac{L}{\frac{\pi}{4}d^2} = \frac{4\rho L}{\pi d^2}$$

Substitute for R to obtain:

$$\frac{P}{L} = \frac{4\rho I^2}{\pi d^2} \Rightarrow d = 2I\sqrt{\frac{\rho}{\pi(P/L)}}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper wire) and evaluate d:

$$d = 2(20\,\text{A})\sqrt{\frac{1.7\times10^{-8}\,\Omega\cdot\text{m}}{\pi(2.0\,\text{W/m})}} = 2.1\,\text{mm}$$

Consulting Table 25-2, we note that the maximum gauge you can use is $\boxed{12}$.

30 •• A laser diode used in making a laser pointer is a highly nonlinear circuit element. Its behavior is as follows: for any voltage drop across it that is less than about 2.30 V, it behaves as if it has an infinite internal resistance, but for voltages higher than 2.30 V it has a very low internal resistance—effectively zero. (*a*) A laser pointer is made by putting two 1.55 V watch batteries in series across the laser diode. If the batteries each have an internal resistance between 1.00 Ω and 1.50 Ω, estimate the current in the laser beam. (*b*) About half of the power delivered to the laser diode goes into radiant energy. Using this fact, estimate the power of the laser beam, and compare this value to typical quoted values of about 3.00 mW. (*c*) If the batteries each have a capacity of 20.0-mA·h (i.e., they can deliver a constant current of 20.0 mA for approximately one hour before discharging), estimate how long one can continuously operate the laser pointer before replacing the batteries.

Picture the Problem Let r be the internal resistance of each battery and apply Kirchhoff's loop rule to the circuit consisting of the two batteries and the laser diode to relate the current in laser diode to r and the potential differences across the batteries and the diode. We can find the power of the laser diode from the product of the potential difference across the internal resistance of the batteries and the current I delivered by them and the time-to-discharge from the combined capacities of the two batteries and I.

(*a*) Apply Kirchhoff's loop rule to the circuit consisting of the two batteries and the laser diode to obtain:

$$2\mathcal{E} - 2Ir - V_{\substack{\text{laser}\\ \text{diode}}} = 0$$

Solving for I yields:

$$I = \frac{\mathcal{E} - V_{\substack{\text{laser}\\ \text{diode}}}}{2r}$$

Assuming that $r = 125\ \Omega$, substitute numerical values and evaluate I:

$$I = \frac{2(1.55) - 2.30\,\text{V}}{2(125\,\Omega)} = 3.20\,\text{mA}$$
$$= \boxed{3.2\,\text{mA}}$$

(*b*) The power delivered by the batteries is given by:

$$P = IV = (3.20\,\text{mA})(2.30\,\text{V}) = 7.36\,\text{mW}$$

The power of the laser is half this value:

$$P_{\text{laser}} = \tfrac{1}{2}P = \tfrac{1}{2}(7.36\,\text{mW}) = 3.68\,\text{mW}$$
$$= \boxed{3.7\,\text{mW}}$$

Express the ratio of P_{laser} to P_{quoted}:

$$\frac{P_{\text{laser}}}{P_{\text{quoted}}} = \frac{3.68\,\text{mW}}{3.00\,\text{mW}} = 1.23$$

or

$$P_{\text{laser}} = \boxed{1.23P_{\text{quoted}}}$$

(*c*) Express the time-to-discharge:

$$\Delta t_{\text{discharge}} = \frac{\text{Capacity}}{I}$$

Because each battery has a capacity of 20.0 mA·h, the series combination has a capacity of 40.0 mA·h and:

$$\Delta t_{\text{discharge}} = \frac{40.0\,\text{mA}\cdot\text{h}}{3.20\,\text{mA}} \approx \boxed{12.5\,\text{h}}$$

Current, Current Density, Drift Speed and the Motion of Charges

31 • [SSM] A 10-gauge copper wire carries a current equal to 20 A. Assuming copper has one free electron per atom, calculate the drift speed of the free electrons in the wire.

Picture the Problem We can relate the drift velocity of the electrons to the current density using $I = nev_\text{d}A$. We can find the number density of charge carriers n using $n = \rho N_\text{A}/M$, where ρ is the mass density, N_A Avogadro's number, and M the molar mass. We can find the cross-sectional area of 10-gauge wire in Table 25-2.

Use the relation between current and drift velocity to relate I and n:

$$I = nev_\text{d}A \Rightarrow v_\text{d} = \frac{I}{neA}$$

The number density of charge carriers n is related to the mass density ρ, Avogadro's number N_A, and the molar mass M:

$$n = \frac{\rho N_\text{A}}{M}$$

For copper, $\rho = 8.93\ \text{g/cm}^3$ and $M = 63.55$ g/mol. Substitute and evaluate n:

$$n = \frac{(8.93\,\text{g/cm}^3)(6.022\times10^{23}\ \text{atoms/mol})}{63.55\,\text{g/mol}}$$

$$= 8.459\times10^{28}\ \text{atoms/m}^3$$

Using Table 25-2, find the cross-sectional area of 10-gauge wire:

$$A = 5.261\,\text{mm}^2$$

Substitute numerical values and evaluate v_d:

$$v_\text{d} = \frac{20\,\text{A}}{(8.459\times10^{28}\ \text{m}^{-3})(1.602\times10^{-19}\ \text{C})(5.261\text{mm}^2)} = \boxed{0.28\,\text{mm/s}}$$

32 •• A thin nonconducting ring that has a radius a and a linear charge density λ rotates with angular speed ω about an axis through its center and perpendicular to the plane of the ring. Find the current of the ring.

Picture the Problem We can use the definition of current, the definition of charge density, and the relationship between period and frequency to derive an expression for the current as a function of a, λ, and ω.

Use the definition of current to relate the charge ΔQ associated with a segment of the ring to the time Δt it takes the segment to pass a given point:

$$I = \frac{\Delta Q}{\Delta t}$$

Because each segment carries a charge ΔQ and the time for one revolution is T:

$$I = \frac{\Delta Q}{T} = \Delta Q f \qquad (1)$$

Use the definition of the charge density λ to relate the charge ΔQ to the radius a of the ring:

$$\lambda = \frac{\Delta Q}{2\pi a} \Rightarrow \Delta Q = 2\pi a\lambda$$

Substitute for ΔQ in equation (1) to obtain:

$$I = 2\pi a\lambda f$$

Because $\omega = 2\pi f$:

$$I = \boxed{a\lambda\omega}$$

33 •• **[SSM]** A length of 10-gauge copper wire and a length of 14-gauge copper wire are welded together end to end. The wires carry a current of 15 A. (*a*) If there is one free electron for each copper atom in each wire, find the drift speed of the electrons in each wire. (*b*) What is the ratio of the magnitude of the current density in the length of10-gauge wire to the magnitude of the current density in the length of 14-gauge wire?

Picture the Problem (*a*) The current will be the same in the two wires and we can relate the drift velocity of the electrons in each wire to their current densities and the cross-sectional areas of the wires. We can find the number density of charge carriers n using $n = \rho N_\text{A}/M$, where ρ is the mass density, N_A Avogadro's number, and M the molar mass. We can find the cross-sectional area of 10- and 14-gauge wires in Table 25-2. In Part (*b*) we can use the definition of current density to find the ratio of the magnitudes of the current densities in the 10-gauge and 14-gauge wires.

(*a*) Relate the current density to the drift velocity of the electrons in the 10-gauge wire:

$$\frac{I_{10\text{ gauge}}}{A_{10\text{ gauge}}} = nev_{\text{d},10} \Rightarrow v_{\text{d},10} = \frac{I_{10\text{ gauge}}}{neA_{10\text{ gauge}}}$$

The number density of charge carriers n is related to the mass density ρ, Avogadro's number N_A, and the molar mass M:

$$n = \frac{\rho N_A}{M}$$

For copper, $\rho = 8.93\ \text{g/cm}^3$ and $M = 63.55$ g/mol. Substitute numerical values and evaluate n:

$$n = \frac{\left(8.93\,\frac{\text{g}}{\text{cm}^3}\right)\left(6.022\times10^{23}\,\frac{\text{atoms}}{\text{mol}}\right)}{63.55\,\frac{\text{g}}{\text{mol}}} = 8.462\times10^{28}\,\frac{\text{atoms}}{\text{m}^3}$$

Use Table 25-2 to find the cross-sectional area of 10-gauge wire:

$$A_{10} = 5.261\,\text{mm}^2$$

Substitute numerical values and evaluate $v_{d.10}$:

$$v_{d,10} = \frac{15\,\text{A}}{\left(8.462\times10^{28}\,\text{m}^{-3}\right)\left(1.602\times10^{-19}\,\text{C}\right)\left(5.261\,\text{mm}^2\right)} = 0.210\,\text{mm/s} = \boxed{0.21\,\text{mm/s}}$$

Express the continuity of the current in the two wires:

$$I_{10\,\text{gauge}} = I_{14\,\text{gauge}}$$

or

$$nev_{d,10}A_{10\,\text{gauge}} = nev_{d,14}A_{14\,\text{gauge}}$$

Solve for $v_{d,14}$ to obtain:

$$v_{d,14} = v_{d,10}\frac{A_{10\,\text{gauge}}}{A_{14\,\text{gauge}}}$$

Use Table 25-2 to find the cross-sectional area of 14-gauge wire:

$$A_{14} = 2.081\,\text{mm}^2$$

Substitute numerical values and evaluate $v_{d,14}$:

$$v_{d,14} = (0.210\,\text{mm/s})\frac{5.261\,\text{mm}^2}{2.081\,\text{mm}^2} = \boxed{0.53\,\text{mm/s}}$$

(*b*) The ratio of current density, 10-gauge wire to 14-gauge wire, is given by:

$$\frac{J_{10}}{J_{14}} = \frac{\frac{I_{10}}{A_{10}}}{\frac{I_{14}}{A_{14}}} = \frac{I_{10}A_{14}}{I_{14}A_{10}}$$

Because $I_{10} = I_{14}$:

$$\frac{J_{10}}{J_{14}} = \frac{A_{14}}{A_{10}}$$

Substitute numerical values and evaluate $\frac{J_{10}}{J_{14}}$:

$$\frac{J_{10}}{J_{14}} = \frac{2.081\,\text{mm}^2}{5.261\,\text{mm}^2} = \boxed{0.396}$$

34 •• An accelerator produces a beam of protons with a circular cross section that is 2.0 mm in diameter and has a current of 1.0 mA. The current density is uniformly distributed through the beam. The kinetic energy of each proton is 20 MeV. The beam strikes a metal target and is absorbed by the target. (*a*) What is the number density of the protons in the beam? (*b*) How many protons strike the target each minute? (*c*) What is the magnitude of the current density in this beam?

Picture the Problem We can use $I = neAv$ to relate the number n of protons per unit volume in the beam to current I. We can find the speed of the particles in the beam from their kinetic energy. In Part (*b*) we can express the number of protons N striking the target per unit time as the product of the number of protons per unit volume n in the beam and the volume of the cylinder containing those protons that will strike the target in an elapsed time Δt and solve for N. Finally, we can use the definition of current to express the charge arriving at the target as a function of time.

(*a*) Use the relation between current and drift velocity (Equation 25-3) to relate I and n:

$$I = neAv \Rightarrow n = \frac{I}{eAv}$$

The kinetic energy of the protons is given by:

$$K = \tfrac{1}{2}m_{\text{p}}v^2 \Rightarrow v = \sqrt{\frac{2K}{m_{\text{p}}}}$$

Relate the cross-sectional area A of the beam to its diameter D:

$$A = \tfrac{1}{4}\pi D^2$$

Substitute for v and A and simplify to obtain:

$$n = \frac{I}{\tfrac{1}{4}\pi e D^2\sqrt{\frac{2K}{m_{\text{p}}}}} = \frac{4I}{\pi e D^2}\sqrt{\frac{m_{\text{p}}}{2K}}$$

Substitute numerical values and evaluate n:

$$n = \frac{4(1.0\,\text{mA})}{\pi(1.602\times10^{-19}\,\text{C})(2\,\text{mm})^2}\sqrt{\frac{1.673\times10^{-27}\,\text{kg}}{2(20\,\text{MeV})(1.602\times10^{-19}\,\text{J/eV})}}$$
$$= 3.21\times10^{13}\,\text{m}^{-3} = \boxed{3.2\times10^{13}\,\text{m}^{-3}}$$

(b) Express the number of protons N striking the target per unit time as the product of the number n of protons per unit volume in the beam and the volume of the cylinder containing those protons that will strike the target in an elapsed time Δt and solve for N:

$$\frac{N}{\Delta t} = n(vA) \Rightarrow N = nvA\Delta t$$

Substitute for v and A to obtain:

$$N = \tfrac{1}{4}\pi D^2 n\Delta t\sqrt{\frac{2K}{m_\text{p}}}$$

Substitute numerical values and evaluate N:

$$N = \tfrac{1}{4}\pi(2\,\text{mm})^2(3.21\times10^{13}\,\text{m}^{-3})(1\,\text{min})\sqrt{\frac{2(20\,\text{MeV})(1.602\times10^{-19}\,\text{J/eV})}{1.673\times10^{-27}\,\text{kg}}}$$
$$= \boxed{3.7\times10^{17}}$$

(c) The magnitude of the current density in this beam is given by:

$$J = \frac{I}{A} = \frac{1.0\,\text{mA}}{\pi(1.0\times10^{-3}\,\text{m})^2}$$
$$= \boxed{0.32\,\text{kA/m}^2}$$

35 •• [SSM] In one of the colliding beams of a planned proton *supercollider*, the protons are moving at nearly the speed of light and the beam current is 5.00-mA. The current density is uniformly distributed throughout the beam. (a) How many protons are there per meter of length of the beam? (b) If the cross-sectional area of the beam is 1.00 x 10^{-6} m^2, what is the number density of protons? (c) What is the magnitude of the current density in this beam?

Picture the Problem We can relate the number of protons per meter N to the number n of free charge-carrying particles per unit volume in a beam of cross-sectional area A and then use the relation between current and drift velocity to relate n to I.

(*a*) Express the number of protons per meter N in terms of the number n of free charge-carrying particles per unit volume in a beam of cross-sectional area A:

$$N = nA \qquad (1)$$

Use the relation between current and drift velocity to relate I and n:

$$I = enAv \Rightarrow n = \frac{I}{eAv}$$

Substitute for n and simplify to obtain:

$$N = \frac{IA}{eAv} = \frac{I}{ev}$$

Substitute numerical values and evaluate N:

$$N = \frac{5.00\,\text{mA}}{\left(1.602\times10^{-19}\ \text{C}\right)\left(2.998\times10^{8}\ \text{m/s}\right)} = 1.041\times10^{8}\ \text{m}^{-1} = \boxed{1.04\times10^{8}\ \text{m}^{-1}}$$

(*b*) From equation (1) we have:

$$n = \frac{N}{A} = \frac{1.041\times10^{8}\ \text{m}^{-1}}{1.00\times10^{-6}\ \text{m}^{2}} = \boxed{1.04\times10^{14}\ \text{m}^{-3}}$$

(*c*) The magnitude of the current density in this beam is given by:

$$J = \frac{I}{A} = \frac{5.00\ \text{mA}}{1.00\times10^{-6}\,\text{m}^{2}} = \boxed{5.00\,\text{kA/m}^{2}}$$

36 •• The *solar wind* consists of protons from the Sun moving toward Earth (the wind actually consists of about 95% protons). The number density of protons at a distance from the Sun equal to the orbital radius of Earth is about 7.0 protons per cubic centimeter. Your research team monitors a satellite that is in orbit around the Sun at a distance from the Sun equal to equal to Earth's orbital radius. You are in charge of the satellite's *mass spectrometer*, an instrument used to measure the composition and intensity of the solar wind. The aperture of your spectrometer is a circle of radius 25 cm. The rate of collection of protons by the spectrometer is such that they constitute a measured current of 85 nA. What is the speed of the protons in the solar wind? (Assume the protons enter the aperture at normal incidence.)

Picture the Problem We can use Equation 25-3 to express the drift speed of the protons in the solar wind in terms of the proton current, the number density of the protons, and the cross-sectional area of the aperture of the spectrometer.

From Equation 25-3, the drift speed of the protons in the solar winds is given by:

$$v_d = \frac{I}{qnA}$$

Substitute numerical values and evaluate v_d:

$$v_d = \frac{85\ \text{nA}}{\left(1.602\times10^{-19}\ \text{C}\right)\left(\dfrac{7.0}{\text{cm}^3}\right)\pi\left(25\ \text{cm}\right)^2} = \boxed{3.9\times10^5\ \text{m/s}}$$

37 •• A gold wire has a 0.10-mm-diameter cross section. Opposite ends of this wire are connected to the terminals of a 1.5-V battery. If the length of the wire is 7.5 cm, how much time, on average, is required for electrons leaving the negative terminal of the battery to reach the positive terminal? Assume the resistivity of gold is $2.44\times10^{-8}\ \Omega\cdot\text{m}$.

Picture the Problem We can use the definition of average speed, Equation 25-3 ($I = qnAv_d$), Equation 25-10 ($R = \rho L/A$) and the relationship between the number density of electrons, Avogadro's number, and the molar mass of gold to derive an expression for the average travel time of the electrons.

The average transit time for the electrons is the ratio of the distance they travel to their drift speed:

$\Delta t = \dfrac{L}{v_d}$ where L is the length of the gold wire.

From Equation 25-3:

$$v_d = \frac{I}{qnA}$$

Substituting for v_d and I and simplifying yields:

$$\Delta t = \frac{L}{\dfrac{I}{qnA}} = \frac{qnAL}{\dfrac{V}{R}} = \frac{qnALR}{V}$$

Use Equation 25-10 to express the resistance of the gold wire:

$$R = \rho_{Au}\frac{L}{A}$$

Substitute for R and simplify to obtain:

$$\Delta t = \frac{qnAL\rho_{Au}\dfrac{L}{A}}{V} = \frac{qnL^2\rho_{Au}}{V}$$

The number density of free electrons is given by:

$$n = \left(\frac{\text{mass}}{\text{molar mass}}\right)\left(\frac{1}{\text{volume}}\right)N_A$$

$$= \left(\frac{\rho_{\text{density, Au}}}{M_{\text{mol, Au}}}\right)N_A = \left(\frac{N_A}{M_{\text{mol, Au}}}\right)\rho_{\text{density, Au}}$$

Finally, substituting for n yields:

$$\Delta t = \frac{q\left(\dfrac{N_A}{M_{\text{mol, Au}}}\right)\rho_{\text{density, Au}} L^2 \rho_{\text{Au}}}{V}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \left(1.602\times10^{-19}\ \text{C}\right)\left(\frac{6.022\times10^{23}\ \text{mol}^{-1}}{0.19697\,\dfrac{\text{kg}}{\text{mol}}}\right)\left(19.3\times10^{3}\,\frac{\text{kg}}{\text{m}^3}\right)(7.5\ \text{cm})^2$$
$$\times\frac{\left(2.44\times10^{-8}\ \Omega\cdot\text{m}\right)}{1.5\ \text{V}} = \boxed{0.86\ \text{s}}$$

Resistance, Resistivity and Ohm’s Law

38 • A 10-m-long wire has a resistance equal to 0.20 Ω and carries a current equal to 5.0 A. (*a*) What is the potential difference across the entire length of the wire? (*b*) What is the electric-field strength in the wire?

Picture the Problem We can use Ohm’s law to find the potential difference between the ends of the wire and $V = EL$ to find the magnitude of the electric field in the wire.

(*a*) Apply Ohm’s law to obtain:

$$V = RI = (0.20\,\Omega)(5.0\ \text{A}) = \boxed{1.0\ \text{V}}$$

(*b*) Relate the electric field to the potential difference across the wire and the length of the wire:

$$E = \frac{V}{L} = \frac{1.0\ \text{V}}{10\ \text{m}} = \boxed{0.10\ \text{V/m}}$$

39 • **[SSM]** A potential difference of 100 V across the terminals of a resistor produces a current of 3.00 A in the resistor. (*a*) What is the resistance of the resistor? (*b*) What is the current in the resistor when the potential difference is only 25.0 V? (Assume the resistance of the resistor remains constant.)

Picture the Problem We can apply Ohm’s law to both parts of this problem, solving first for R and then for I.

(*a*) Apply Ohm's law to obtain:

$$R = \frac{V}{I} = \frac{100\,\text{V}}{3.00\,\text{A}} = \boxed{33.3\,\Omega}$$

(*b*) Apply Ohm's law a second time to obtain:

$$I = \frac{V}{R} = \frac{25.0\,\text{V}}{33.3\,\Omega} = \boxed{0.750\,\text{A}}$$

40 • A block of carbon is 3.0 cm long and has a square cross-section whose sides are 0.50-cm long. A potential difference of 8.4 V is maintained across its length. (*a*) What is the resistance of the block? (*b*) What is the current in this resistor?

Picture the Problem We can use $R = \rho L/A$ to find the resistance of the block and Ohm's law to find the current in it for the given potential difference across its length.

(*a*) Relate the resistance of the block to its resistivity ρ, cross-sectional area A, and length L:

$$R = \rho\frac{L}{A}$$

Substitute numerical values (see Table 25-1 for the resistivity of carbon) and evaluate R:

$$R = \left(3500\times10^{-8}\,\Omega\cdot\text{m}\right)\frac{3.0\,\text{cm}}{(0.50\,\text{cm})^2}$$

$$= 42.0\,\text{m}\Omega = \boxed{42\,\text{m}\Omega}$$

(*b*) Apply Ohm's law to obtain:

$$I = \frac{V}{R} = \frac{8.4\,\text{V}}{42.0\,\text{m}\Omega} = \boxed{0.20\,\text{kA}}$$

41 • **[SSM]** An extension cord consists of a pair of 30-m-long 16-gauge copper wires. What is the potential difference that must be applied across one of the wires if it is to carry a current of 5.0 A?

Picture the Problem We can use Ohm's law in conjunction with $R = \rho L/A$ to find the potential difference across one wire of the extension cord.

Using Ohm's law, express the potential difference across one wire of the extension cord:

$$V = IR$$

Relate the resistance of the wire to its resistivity ρ, cross-sectional area A, and length L:

$$R = \rho\frac{L}{A}$$

Substitute for R to obtain:

$$V = \rho \frac{LI}{A}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper and Table 25-2 for the cross-sectional area of 16-gauge wire) and evaluate V:

$$V = \left(1.7\times10^{-8}\ \Omega\cdot\text{m}\right)\frac{(30\,\text{m})(5.0\,\text{A})}{1.309\,\text{mm}^2}$$
$$= \boxed{1.9\,\text{V}}$$

42 • (*a*) How long is a 14-gauge copper wire that has a resistance of 12.0 Ω? (*b*) How much current will it carry if a 120-V potential difference is applied across its length?

Picture the Problem We can use $R = \rho L/A$ to find the length of a 14-gauge copper wire that has a resistance of 12.0 Ω and Ohm's law to find the current in the wire.

(*a*) Relate the resistance of the wire to its resistivity ρ, cross-sectional area A, and length L:

$$R = \rho\frac{L}{A} \Rightarrow L = \frac{RA}{\rho}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper and Table 25-2 for the cross-sectional area of 14-gauge wire) and evaluate L:

$$L = \frac{(12.0\,\Omega)\left(2.081\,\text{mm}^2\right)}{1.7\times10^{-8}\ \Omega\cdot\text{m}} = \boxed{1.5\,\text{km}}$$

(*b*) Apply Ohm's law to find the current in the wire:

$$I = \frac{V}{R} = \frac{120\ \text{V}}{12.0\,\Omega} = \boxed{10.0\ \text{A}}$$

43 • A cylinder of glass is 1.00 cm long and has a resistivity of 1.01×10^{12} Ω·m. What length of copper wire that has the same cross-sectional area will have the same resistance as the glass cylinder?

Picture the Problem We can use $R = \rho L/A$ to express the resistances of the glass cylinder and the copper wire. Expressing their ratio will eliminate the common cross-sectional areas and leave us with an expression we can solve for the length of the copper wire.

Relate the resistance of the glass cylinder to its resistivity, cross-sectional area, and length:

$$R_{\text{glass}} = \rho_{\text{glass}}\frac{L_{\text{glass}}}{A_{\text{glass}}}$$

Relate the resistance of the copper wire to its resistivity, cross-sectional area, and length:

$$R_{\text{Cu}} = \rho_{\text{Cu}} \frac{L_{\text{Cu}}}{A_{\text{Cu}}}$$

Divide the second of these equations by the first to obtain:

$$\frac{R_{\text{Cu}}}{R_{\text{glass}}} = \frac{\rho_{\text{Cu}} \dfrac{L_{\text{Cu}}}{A_{\text{Cu}}}}{\rho_{\text{glass}} \dfrac{L_{\text{glass}}}{A_{\text{glass}}}} = \frac{\rho_{\text{Cu}}}{\rho_{\text{glass}}} \cdot \frac{L_{\text{Cu}}}{L_{\text{glass}}}$$

Because $A_{\text{glass}} = A_{\text{Cu}}$ and $R_{\text{Cu}} = R_{\text{glass}}$:

$$1 = \frac{\rho_{\text{Cu}}}{\rho_{\text{glass}}} \cdot \frac{L_{\text{Cu}}}{L_{\text{glass}}} \Rightarrow L_{\text{Cu}} = \frac{\rho_{\text{glass}}}{\rho_{\text{Cu}}} L_{\text{glass}}$$

Substitute numerical values (see Table 25-1 for the resistivities of glass and copper) and evaluate L_{Cu}:

$$L_{\text{Cu}} = \frac{1.01\times10^{12}\ \Omega\cdot\text{m}}{1.7\times10^{-8}\ \Omega\cdot\text{m}}(1.00\,\text{cm})$$

$$= \boxed{5.94\times10^{17}\ \text{m}} \times \frac{1\,c\cdot\text{y}}{9.461\times10^{15}\ \text{m}}$$

$$= \boxed{63\,c\cdot\text{y}}$$

44 •• While remodeling your garage, you need to temporarily splice, end to end, an 80-m-long copper wire that is 1.00 mm in diameter with a 49-m-long aluminum wire that has the same diameter. The maximum current in the wires is 2.00 A. (*a*) Find the potential drop across each wire of this system when the current is 2.00 A. (*b*) Find the electric field in each wire when the current is 2.00 A.

Picture the Problem We can use Ohm's law to relate the potential differences across the two wires to their resistances and $R = \rho L/A$ to relate their resistances to their lengths, resistivities, and cross-sectional areas. Once we've found the potential differences across each wire, we can use $E = V/L$ to find the electric field in each wire.

(*a*) Apply Ohm's law to express the potential drop across each wire:

$$V_{\text{Cu}} = IR_{\text{Cu}} \text{ and } V_{\text{Fe}} = IR_{\text{Fe}}$$

Relate the resistances of the wires to their resistivity, cross-sectional area, and length:

$$R_{\text{Cu}} = \rho_{\text{Cu}} \frac{L_{\text{Cu}}}{A_{\text{Cu}}} \text{ and } R_{\text{Fe}} = \rho_{\text{Cu}} \frac{L_{\text{Fe}}}{A_{\text{Fe}}}$$

Substitute for R_{Cu} and R_{Fe} to obtain:

$$V_{\text{Cu}} = \frac{\rho_{\text{Cu}} l_{\text{Cu}}}{A_{\text{Cu}}} I \text{ and } V_{\text{Fe}} = \frac{\rho_{\text{Fe}} l_{\text{Fe}}}{A_{\text{Fe}}} I$$

Substitute numerical values (see Table 25-1 for the resistivities of copper and iron) and evaluate the potential differences:

$$V_{\text{Cu}} = \frac{(1.7\times10^{-8}\ \Omega\cdot\text{m})(80\ \text{m})}{\frac{1}{4}\pi(1.00\ \text{mm})^2}(2.00\ \text{A}) = 3.463\ \text{V} = \boxed{3.5\ \text{V}}$$

and

$$V_{\text{Fe}} = \frac{(10\times10^{-8}\ \Omega\cdot\text{m})(49\ \text{m})}{\frac{1}{4}\pi(1.00\ \text{mm})^2}(2.00\ \text{A}) = 12.48\ \text{V} = \boxed{12\ \text{V}}$$

(*b*) Express the electric field in each conductor in terms of its length and the potential difference across it:

$$E_{\text{Cu}} = \frac{V_{\text{Cu}}}{l_{\text{Cu}}} \text{ and } E_{\text{Fe}} = \frac{V_{\text{Fe}}}{l_{\text{Fe}}}$$

Substitute numerical values and evaluate the electric fields:

$$E_{\text{Cu}} = \frac{3.463\ \text{V}}{80\ \text{m}} = \boxed{43\ \text{mV/m}}$$

and

$$E_{\text{Fe}} = \frac{12.48\ \text{V}}{49\ \text{m}} = \boxed{0.25\ \text{V/m}}$$

45 •• [SSM] A 1.00-m-long wire has a resistance equal to 0.300 Ω. A second wire made of identical material has a length of 2.00 m and a mass equal to the mass of the first wire. What is the resistance of the second wire?

Picture the Problem We can use $R = \rho L/A$ to relate the resistance of the wires to their lengths, resistivities, and cross-sectional areas. To find the resistance of the second wire, we can use the fact that the volumes of the two wires are the same to relate the cross-sectional area of the first wire to the cross-sectional area of the second wire.

Relate the resistance of the first wire to its resistivity, cross-sectional area, and length:

$$R = \rho\frac{L}{A}$$

Relate the resistance of the second wire to its resistivity, cross-sectional area, and length:

$$R' = \rho\frac{L'}{A'}$$

Divide the second of these equations by the first to obtain:

$$\frac{R'}{R}=\frac{\rho\frac{L'}{A'}}{\rho\frac{L}{A}}=\frac{L'}{L}\frac{A}{A'}\Rightarrow R'=2\frac{A}{A'}R \quad (1)$$

Express the relationship between the volume V of the first wire and the volume V' of the second wire:

$$V=V' \text{ or } LA=L'A'\Rightarrow\frac{A}{A'}=\frac{L'}{L}=2$$

Substituting for $\frac{A}{A'}$ in equation (1) yields:

$$R'=2(2)R=4R$$

Because R = 3.00 Ω:

$$R'=4(0.300\,\Omega)=\boxed{1.20\,\Omega}$$

46 •• A 10-gauge copper wire can safely carry currents up to 30.0 A. (*a*) What is the resistance of a 100-m length of the wire? (*b*) What is the electric field in the wire when the current is 30.0 A? (*c*) How long does it take for an electron to travel 100 m in the wire when the current is 30.0 A?

Picture the Problem We can use $R=\rho L/A$ to find the resistance of the wire from its length, resistivity, and cross-sectional area. The electric field can be found using $E = V/L$ and Ohm's law to eliminate V. The time for an electron to travel the length of the wire can be found from $L = v_d\Delta t$, with v_d expressed in term of I using $I=neAv_d$.

(*a*) Relate the resistance of the unstretched wire to its resistivity, cross-sectional area, and length:

$$R=\rho\frac{L}{A}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper and Table 25-2 for the cross-sectional area of 10-gauge wire) and evaluate R:

$$R=\left(1.7\times10^{-8}\,\Omega\cdot\text{m}\right)\frac{100\,\text{m}}{5.261\,\text{mm}^2}$$
$$=0.323\,\Omega=\boxed{0.32\,\Omega}$$

(*b*) Relate the electric field in the wire to the potential difference between its ends:

$$E=\frac{V}{L}$$

Use Ohm's law to substitute for V:

$$E=\frac{IR}{L}$$

Substitute numerical values and evaluate E:

$$E = \frac{(30.0\,\text{A})(0.323\,\Omega)}{100\,\text{m}} = \boxed{97\,\text{mV/m}}$$

(c) Express the time Δt for an electron to travel a distance L in the wire in terms of its drift speed v_d:

$$\Delta t = \frac{L}{v_\text{d}}$$

Relate the current in the wire to the drift speed of the charge carriers:

$$I = neAv_\text{d} \Rightarrow v_\text{d} = \frac{I}{neA}$$

Substitute for v_d to obtain:

$$\Delta t = \frac{neAL}{I}$$

Substitute numerical values (in Example 25-1 it is shown that $n = 8.47\times10^{28}\ \text{m}^{-3}$) and evaluate Δt:

$$\Delta t = \frac{(8.47\times10^{28}\ \text{m}^{-3})(1.602\times10^{-19}\ \text{C})(5.261\,\text{mm}^2)(100\,\text{m})}{30.0\,\text{A}}$$

$$= \boxed{2.38\times10^5\ \text{s}} = \boxed{2.75\ \text{d}}$$

47 •• A cube of copper has edges that are 2.00-cm long. If copper in the cube is drawn to form a length of 14-gauge wire, what will the resistance of the length of wire be? Assume the density of the copper does not change.

Picture the Problem We can use $R = \rho L/A$ to find express the resistance of the wire in terms of its length, resistivity, and cross-sectional area. The fact that the volume of the copper does not change as the cube is drawn out to form the wire will allow us to eliminate either the length or the cross-sectional area of the wire and solve for its resistance.

Express the resistance of the wire in terms of its resistivity, cross-sectional area, and length:

$$R = \rho\frac{L}{A}$$

Relate the volume V of the wire to its length and cross-sectional area:

$$V = AL \Rightarrow L = \frac{V}{A}$$

Substitute for L to obtain:

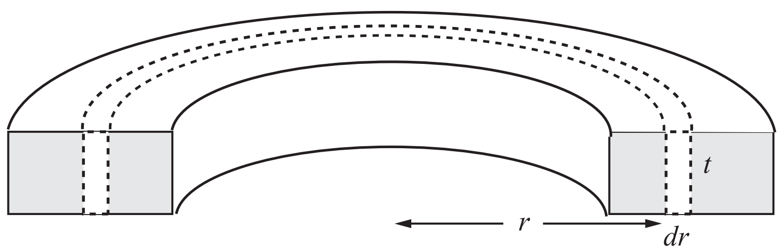
$$R = \rho\frac{V}{A^2}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper and Table 25-2 for the cross-sectional area of 14-gauge wire) and evaluate R:

$$R = \left(1.7\times10^{-8}\ \Omega\cdot\text{m}\right)\frac{\left(2.00\,\text{cm}\right)^3}{\left(2.081\,\text{mm}^2\right)^2} = \boxed{31\,\text{m}\Omega}$$

48 ••• Find an expression for the resistance between the ends of the half ring shown in Figure 25-51. The resistivity of the material constituting the half-ring is ρ. *Hint: Model the half ring as a parallel combination of a large number of thin half-rings. Assume the current is uniformly distributed on a cross section of the half ring.*

Picture the Problem We can use, as our element of resistance, the semicircular strip of height t, radius r, and thickness dr shown below. Then $dR = \left(\pi r\rho/t\right)dr$. Because the strips are in parallel, integrating over them will give us the reciprocal of the resistance of half ring.



The resistance dR between the ends of the infinitesimal strip is given by:

$$dR = \rho\frac{\pi r}{t\,dr}$$

Because the infinitesimal strips are in parallel, their equivalent resistance is:

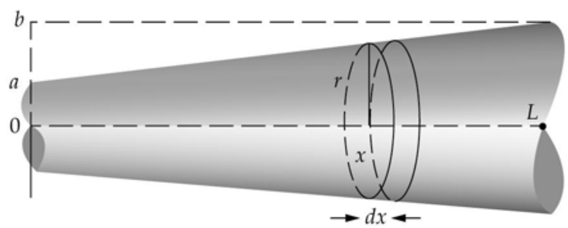
$$\frac{1}{R} = \frac{t}{\pi\rho}\int_a^b \frac{dr}{r}$$

Integrating dR_{eq} from $r = a$ to $r = b$ yields:

$$\frac{1}{R} = \frac{t}{\pi\rho}\ln\left(\frac{b}{a}\right) \Rightarrow R = \boxed{\frac{\rho\pi}{t\ln\left(\frac{b}{a}\right)}}$$

49 ••• **[SSM]** Consider a wire of length L in the shape of a truncated cone. The radius of the wire varies with distance x from the narrow end according to $r = a + [(b - a)/L]x$, where $0 < x < L$. Derive an expression for the resistance of this wire in terms of its length L, radius a, radius b and resistivity ρ. *Hint: Model the wire as a series combination of a large number of thin disks. Assume the current is uniformly distributed on a cross section of the cone.*

Picture the Problem The element of resistance we use is a segment of length dx and cross-sectional area $\pi[a + (b - a)x/L]^2$. Because these resistance elements are in series, integrating over them will yield the resistance of the wire.



Express the resistance of the chosen element of resistance:

$$dR = \rho\frac{dx}{A} = \frac{\rho}{\pi[a+(b-a)(x/L)]^2}dx$$

Integrate dR from $x = 0$ to $x = L$ and simplify to obtain:

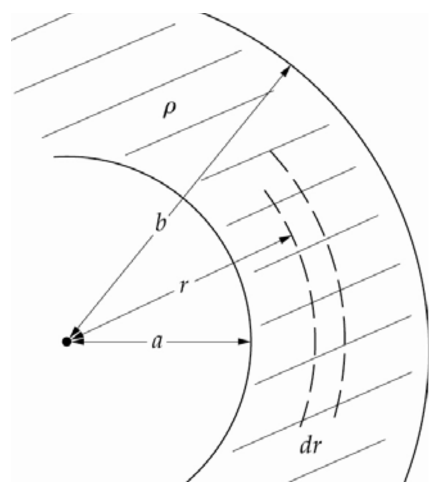
$$R = \frac{\rho}{\pi}\int_0^L \frac{dx}{[a+(b-a)(x/L)]^2}$$

$$= \frac{\rho L}{\pi(b-a)}\left(\frac{1}{a} - \frac{1}{a+(b-a)}\right)$$

$$= \boxed{\frac{\rho L}{\pi ab}}$$

50 ••• The space between two metallic concentric spherical shells is filled with a material that has a resistivity of $3.50 \times 10^{-5}\ \Omega\cdot\text{m}$. If the inner metal shell has an outer radius of 1.50 cm and the outer metal shell has an inner radius of 5.00 cm, what is the resistance between the conductors? *Hint: Model the material as a series combination of a large number of thin spherical shells.*

Picture the Problem The diagram shows a cross-sectional view of the concentric spheres of radii a and b as well as a spherical-shell element of radius r. We can express the resistance dR of the spherical-shell element and then integrate over the volume filled with the material whose resistivity ρ is given to find the resistance between the conductors. Note that the elements of resistance are in series.



Express the element of resistance dR:

$$dR = \rho\frac{dr}{A} = \rho\frac{dr}{4\pi r^2}$$

Integrate dR from $r = a$ to $r = b$ to obtain:

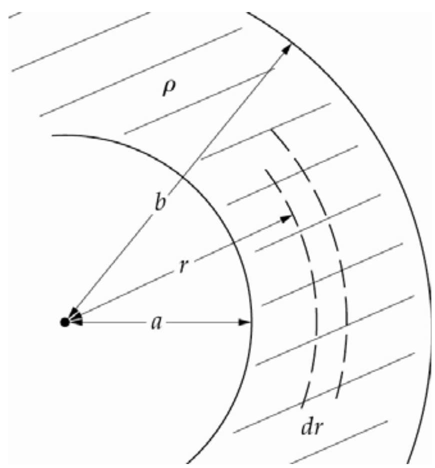
$$R = \frac{\rho}{4\pi}\int_a^b \frac{dr}{r^2} = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Substitute numerical values and evaluate R:

$$R = \frac{3.50\times10^{-5}\ \Omega\cdot\text{m}}{4\pi}\left(\frac{1}{1.50\,\text{cm}} - \frac{1}{5.00\,\text{cm}}\right) = \boxed{130\ \mu\Omega}$$

51 ••• The space between two metallic coaxial cylinders that have the same length L is completely filled with a nonmetalic material having a resistivity ρ. The inner metal shell has an outer radius a and the outer metal shell has an inner radius b. (a) What is the resistance between the two cylinders? *Hint: Model the material as a series combination of a large number of thin cylindrical shells.* (b) Find the current between the two metallic cylinders if $\rho = 30.0\ \Omega\text{·m}$, $a = 1.50$ cm, $b = 2.50$ cm, $L = 50.0$ cm, and a potential difference of 10.0 V is maintained between the two cylinders.

Picture the Problem The diagram shows a cross-sectional view of the coaxial cylinders of radii a and b as well as a cylindrical-shell element of radius r. We can express the resistance dR of the cylindrical-shell element and then integrate over the volume filled with the material whose resistivity ρ is given to find the resistance between the two cylinders. Note that the elements of resistance are in series.



(a) Express the element of resistance dR:

$$dR = \rho\frac{dr}{A} = \rho\frac{dr}{2\pi rL}$$

Integrate dR from $r = a$ to $r = b$ to obtain:

$$R = \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} = \boxed{\frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)}$$

(*b*) Apply Ohm's law to obtain:

$$I = \frac{V}{R} = \frac{V}{\frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)} = \frac{2\pi LV}{\rho\ln\left(\frac{b}{a}\right)}$$

Substitute numerical values and evaluate I:

$$I = \frac{2\pi(50.0\,\text{cm})(10.0\,\text{V})}{(30.0\,\Omega\cdot\text{m})\ln\left(\frac{2.50\,\text{cm}}{1.50\,\text{cm}}\right)} = \boxed{2.05\,\text{A}}$$

Temperature Dependence of Resistance

52 • A tungsten rod is 50 cm long and has a square cross-section that has 1.0-mm-long edges. (*a*) What is its resistance at 20°C? (*b*) What is its resistance at 40°C?

Picture the Problem We can use $R = \rho L/A$ to find the resistance of the rod at 20°C. Ignoring the effects of thermal expansion, we can we apply the equation defining the temperature coefficient of resistivity, α, to relate the resistance at 40°C to the resistance at 20°C.

(*a*) Express the resistance of the rod at 20°C as a function of its resistivity, length, and cross-sectional area:

$$R_{20} = \rho_{20}\frac{L}{A}$$

Substitute numerical values and evaluate R_{20}:

$$R_{20} = \left(5.5\times10^{-8}\,\Omega\cdot\text{m}\right)\frac{0.50\,\text{m}}{(1.0\,\text{mm})^2}$$
$$= 27.5\,\text{m}\Omega = \boxed{28\,\text{m}\Omega}$$

(*b*) Express the resistance of the rod at 40°C as a function of its resistance at 20°C and the temperature coefficient of resistivity α:

$$R_{40} = \rho_{40}\frac{L}{A}$$
$$= \rho_{20}\left[1+\alpha\left(t_C - 20\text{C}^\circ\right)\right]\frac{L}{A}$$
$$= \rho_{20}\frac{L}{A} + \rho_{20}\frac{L}{A}\alpha\left(t_C - 20\text{C}^\circ\right)$$
$$= R_{20}\left[1+\alpha\left(t_C - 20\text{C}^\circ\right)\right]$$

Substitute numerical values (see Table 25-1 for the temperature coefficient of resistivity of tungsten) and evaluate R_{40}:

$$R_{40} = (27.5\,\text{m}\Omega)\left[1+\left(4.5\times10^{-3}\,\text{K}^{-1}\right)\left(20\,\text{C}^\circ\right)\right] = \boxed{30\,\text{m}\Omega}$$

53 • **[SSM]** At what temperature will the resistance of a copper wire be 10 percent greater than its resistance at 20°C?

Picture the Problem The resistance of the copper wire increases with temperature according to $R_{t_C} = R_{20}\left[1+\alpha\left(t_C - 20\text{C}°\right)\right]$. We can replace R_{t_C} by $1.1R_{20}$ and solve for t_C to find the temperature at which the resistance of the wire will be 110% of its value at 20°C.

Express the resistance of the wire at $1.10R_{20}$:

$$1.10R_{20} = R_{20}\left[1+\alpha\left(t_C - 20\text{C}°\right)\right]$$

Simplifying this expression yields:

$$0.10 = \alpha\left(t_C - 20\text{C}°\right)$$

Solve to t_C to obtain:

$$t_C = \frac{0.10}{\alpha} + 20\text{C}°$$

Substitute numerical values (see Table 25-1 for the temperature coefficient of resistivity of copper) and evaluate t_C:

$$t_C = \frac{0.10}{3.9\times10^{-3}\ \text{K}^{-1}} + 20\text{C}° = \boxed{46°\text{C}}$$

54 •• You have a toaster that uses a Nichrome wire as a heating element. You need to determine the temperature of the Nichrome wire under operating conditions. First, you measure the resistance of the heating element at 20°C and find it to be 80.0 Ω. Then you measure the current immediately after you plug the toaster into a wall outlet—before the temperature of the Nichrome wire increases significantly. You find this startup current to be 8.70 A. When the heating element reaches its operating temperature, you measure the current to be 7.00 A. Use your data to determine the maximum operating temperature of the heating element.

Picture the Problem Let the primed quantities denote the current and resistance at the final temperature of the heating element. We can express R' in terms of R_{20} and the final temperature of the wire t_C using $R' = R_{20}\left[1+\alpha\left(t_C - 20°\text{C}\right)\right]$ and relate I', R', I_{20}, and R_{20} using Ohm's law.

Express the resistance of the heating element at its final temperature as a function of its resistance at 20°C and the temperature coefficient of resistivity for Nichrome:

$$R' = R_{20}\left[1+\alpha\left(t_C - 20°\text{C}\right)\right] \qquad (1)$$

Apply Ohm's law to the heating element when it is first turned on:

$$V = I_{20}R_{20}$$

Apply Ohm's law to the heating element when it has reached its final temperature:

$$V = I'R'$$

Because the voltage is constant, we have:

$$I'R' = I_{20}R_{20} \text{ or } R' = \frac{I_{20}}{I'}R_{20}$$

Substitute in equation (1) to obtain:

$$\frac{I_{20}}{I'}R_{20} = R_{20}\left[1+\alpha\left(t_C - 20°C\right)\right]$$

or

$$\frac{I_{20}}{I'} = 1+\alpha\left(t_C - 20°C\right)$$

Solve for t_C to obtain:

$$t_C = \frac{\frac{I_{20}}{I'} - 1}{\alpha} + 20°C$$

Substitute numerical values (see Table 25-1 for the temperature coefficient of resistivity of Nichrome) and evaluate t_C:

$$t_C = \frac{\frac{8.70\,A}{7.00\,A} - 1}{0.4\times10^{-3}\,K^{-1}} + 20°C \approx \boxed{6\times10^2\,°C}$$

where the answer has only one significant figure because the temperature coefficient is only given to one significant figure.

Remarks: At 600°C the wire glows red.

55 •• Your electric space heater has a Nichrome heating element that has a resistance of 8.00 Ω at 20.0°C. When 120 V are applied, the electric current heats the Nichrome wire to 1000°C. (*a*) What is the initial current in the heating element at 20.0°C? (*b*) What is the resistance of the heating element at 1000°C? (*c*) What is the operating power of this heater?

Picture the Problem We can apply Ohm's law to find the initial current drawn by the cold heating element. The resistance of the wire at 1000°C can be found using $R_{1000} = R_{20}\left[1+\alpha\left(t_C - 20.0°C\right)\right]$ and the power consumption of the heater at this temperature from $P = V^2/R_{1000}$.

(*a*) Apply Ohm's law to find the initial current I_{20} drawn by the heating element:

$$I = \frac{V}{R_{20}} = \frac{120\,\text{V}}{8.00\,\Omega} = \boxed{15.0\,\text{A}}$$

(*b*) Express the resistance of the heating element at 1000°C as a function of its resistance at 20.0°C and the temperature coefficient of resistivity for Nichrome:

$$R_{1000} = R_{20}\left[1 + \alpha\left(t_\text{C} - 20.0°\text{C}\right)\right]$$

Substitute numerical values (see Table 25-1 for the temperature coefficient of resistivity of Nichrome) and evaluate R_{1000}:

$$R_{1000} = (8.00\,\Omega)\left[1 + \left(0.4\times10^{-3}\,\text{K}^{-1}\right)\times\left(1000°\text{C} - 20.0°\text{C}\right)\right] = \boxed{11.1\,\Omega}$$

(*c*) The operating power of this heater at 1000°C is:

$$P = \frac{V^2}{R_{1000}} = \frac{(120\,\text{V})^2}{11.1\,\Omega} = \boxed{1.30\,\text{kW}}$$

56 •• A 10.0-Ω Nichrome resistor is wired into an electronic circuit using copper leads (wires) that have diameters equal to 0.600 mm. The copper leads have a total length of 50.0 cm. (*a*) What additional resistance is due to the copper leads? (*b*) What percentage error in the total resistance is produced by neglecting the resistance of the copper leads? (*c*) What change in temperature would produce a change in resistance of the Nichrome wire equal to the resistance of the copper leads? Assume that the Nichrome section is the only one whose temperature is changed.

Picture the Problem We can find the resistance of the copper leads using $R_\text{Cu} = \rho_\text{Cu} L/A$ and express the percentage error in neglecting the resistance of the leads as the ratio of R_Cu to R_Nichrome. In Part (*c*) we can express the change in resistance in the Nichrome wire corresponding to a change Δt_C in its temperature and then find Δt_C by substitution of the resistance of the copper wires in this equation.

(*a*) Relate the resistance of the copper leads to their resistivity, length, and cross-sectional area:

$$R_\text{Cu} = \rho_\text{Cu}\frac{L}{A}$$

Substitute numerical values (see Table 25-1 for the resistivity of copper) and evaluate R_Cu:

$$R_\text{Cu} = \left(1.7\times10^{-8}\,\Omega\cdot\text{m}\right)\frac{50.0\,\text{cm}}{\frac{1}{4}\pi(0.600\,\text{mm})^2} = 30.1\,\text{m}\Omega = \boxed{30\,\text{m}\Omega}$$

(b) Express the percentage error as the ratio of R_{Cu} to $R_{Nichrome}$:

$$\%\,\text{error} = \frac{R_{Cu}}{R_{Nichrome}} = \frac{30.1\,\text{m}\Omega}{10.0\,\Omega} = \boxed{0.30\%}$$

(c) Express the change in the resistance of the Nichrome wire as its temperature changes from t_C to t_C':

$$\begin{aligned}\Delta R &= R' - R \\ &= R_{20}\left[1+\alpha\left(t_C' - 20°\text{C}\right)\right] \\ &\quad - R_{20}\left[1+\alpha\left(t_C - 20°\text{C}\right)\right] \\ &= R_{20}\alpha\Delta t_C\end{aligned}$$

Solve for Δt_C to obtain:

$$\Delta t_C = \frac{\Delta R}{R_{20}\alpha}$$

Set ΔR equal to the resistance of the copper wires (see Table 25-1 for the temperature coefficient of resistivity of Nichrome wire) and evaluate Δt_C:

$$\Delta t_C = \frac{30.1\,\text{m}\Omega}{(10.0\,\Omega)\left(0.4\times10^{-3}\,\text{K}^{-1}\right)} = \boxed{8°\text{C}}$$

57 ••• [SSM] A wire that has a cross-sectional area A, a length L_1, a resistivity ρ_1, and a temperature coefficient α_1 is connected end to end to a second wire that has the same cross-sectional area, a length L_2, a resistivity ρ_2, and a temperature coefficient α_2, so that the wires carry the same current. (a) Show that if $\rho_1L_1\alpha_1 + \rho_2L_2\alpha_2 = 0$, then the total resistance is independent of temperature for small temperature changes. (b) If one wire is made of carbon and the other wire is made of copper, find the ratio of their lengths for which the total resistance is approximately independent of temperature.

Picture the Problem Expressing the total resistance of the two current-carrying (and hence warming) wires connected in series in terms of their resistivities, temperature coefficients of resistivity, lengths and temperature change will lead us to an expression in which, if $\rho_1L_1\alpha_1 + \rho_2L_2\alpha_2 = 0$, the total resistance is temperature independent. In Part (b) we can apply the condition that

$$\rho_1L_1\alpha_1 + \rho_2L_2\alpha_2 = 0$$

to find the ratio of the lengths of the carbon and copper wires.

(a) Express the total resistance of these two wires connected in series:

$$\begin{aligned}R = R_1 + R_2 &= \rho_1\frac{L_1}{A}\left(1+\alpha_1\Delta T\right) + \rho_2\frac{L_2}{A}\left(1+\alpha_2\Delta T\right) \\ &= \frac{1}{A}\left[\rho_1L_1\left(1+\alpha_1\Delta T\right) + \rho_2L_2\left(1+\alpha_2\Delta T\right)\right]\end{aligned}$$

Expand and simplify this expression to obtain:

$$R = \frac{1}{A}\left[\rho_1 L_1 + \rho_2 L_2 + \left(\rho_1 L_1 \alpha_1 + \rho_1 L_1 \alpha_2\right)\Delta T\right]$$

If $\rho_1 L_1 \alpha_1 + \rho_2 L_2 \alpha_2 = 0$, then:

$R = \boxed{\frac{1}{A}\left[\rho_1 L_1 + \rho_2 L_2\right]}$ independently of the temperature.

(*b*) Apply the condition for temperature independence obtained in (*a*) to the carbon and copper wires:

$$\rho_C L_C \alpha_C + \rho_{Cu} L_{Cu} \alpha_{Cu} = 0$$

Solve for the ratio of L_{Cu} to L_C:

$$\frac{L_{Cu}}{L_C} = -\frac{\rho_C \alpha_C}{\rho_{Cu} \alpha_{Cu}}$$

Substitute numerical values (see Table 25-1 for the temperature coefficient of resistivity of carbon and copper) and evaluate the ratio of L_{Cu} to L_C:

$$\frac{L_{Cu}}{L_C} = -\frac{\left(3500\times10^{-8}\ \Omega\cdot\text{m}\right)\left(-0.5\times10^{-3}\ \text{K}^{-1}\right)}{\left(1.7\times10^{-8}\ \Omega\cdot\text{m}\right)\left(3.9\times10^{-3}\ \text{K}^{-1}\right)} \approx \boxed{3\times10^{2}}$$

58 ••• The resistivity of tungsten increases approximately linearly with temperature from 56.0 nΩ·m at 293 K to 1.10 $\mu\Omega$·m at 3500 K. A light bulb is powered by a a 100-V dc power supply. Under these operating conditions the temperature of the tungsten filament is 2500 K, the length of the filament is equal to 5.00 cm and the power delivered to the filament is 40-W. Estimate (*a*) the resistance of the filament and (*b*) the diameter of the filament.

Picture the Problem We can use the relationship between the rate at which energy is transformed into heat and light in the filament and the resistance of and potential difference across the filament to estimate the resistance of the filament. The linear dependence of the resistivity on temperature will allow us to find the resistivity of the filament at 2500 K. We can then use the relationship between the resistance of the filament, its resistivity, and cross-sectional area to find its diameter.

(*a*) Express the wattage of the lightbulb as a function of its resistance R and the voltage V supplied by the source:

$$P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P}$$

Substitute numerical values and evaluate R:

$$R = \frac{(100\,\text{V})^2}{40\,\text{W}} = \boxed{0.25\,\text{k}\Omega}$$

(*b*) Relate the resistance R of the filament to its resistivity ρ, radius r, and length ℓ:

$$R = \frac{\rho\ell}{\pi r^2} \Rightarrow r = \sqrt{\frac{\rho\ell}{\pi R}}$$

The diameter d of the filament is:

$$d = 2\sqrt{\frac{\rho\ell}{\pi R}} \qquad (1)$$

Because the resistivity varies linearly with temperature, we can use a proportion to find its value at 2500 K:

$$\frac{\rho_{2500\,\text{K}} - \rho_{293\,\text{K}}}{\rho_{3500\,\text{K}} - \rho_{293\,\text{K}}} = \frac{2500\,\text{K} - 293\,\text{K}}{3500\,\text{K} - 293\,\text{K}} = \frac{2207}{3207}$$

Solve for $\rho_{2500\,\text{K}}$ to obtain:

$$\rho_{2500\,\text{K}} = \frac{2207}{3207}\left(\rho_{3500\,\text{K}} - \rho_{293\,\text{K}}\right) + \rho_{293\,\text{K}}$$

Substitute numerical values and evaluate $\rho_{2500\,\text{K}}$:

$$\rho_{2500\,\text{K}} = \frac{2207}{3207}\left(1.10\,\mu\Omega\cdot\text{m} - 56.0\,\text{n}\Omega\cdot\text{m}\right) + 56.0\,\text{n}\Omega\cdot\text{m} = 774.5\,\mu\Omega\cdot\text{m}$$

Substitute numerical values in equation (1) and evaluate d:

$$d = 2\sqrt{\frac{(774.5\,\text{n}\Omega\cdot\text{m})(5.00\,\text{cm})}{\pi(250\,\Omega)}} = \boxed{14.0\,\mu\text{m}}$$

59 ••• A 5.00-V light bulb used in an electronics class has a carbon filament that has a length of 3.00 cm and a diameter of 40.0 μm. At temperatures between 500 K and 700 K, the resistivity of the carbon used in making small light bulb filaments is about 3.00×10^{-5} Ω·m. (*a*) Assuming that the bulb is a perfect blackbody radiator, calculate the temperature of the filament under operating conditions. (*b*) One concern about carbon filament bulbs, unlike tungsten filament bulbs, is that the resistivity of carbon decreases with increasing temperature. Explain why this decrease in resistivity is a concern.

Picture the Problem We can use the relationship between the rate at which an object radiates and its temperature to find the temperature of the bulb.

(*a*) At a temperature T, the power emitted by a perfect blackbody is:

$$P = \sigma A T^4$$

where $\sigma = 5.67\times10^{-8}$ W/m^2·K^4 is the Stefan-Boltzmann constant.

Solve for T and substitute for P to obtain:

$$T = \sqrt[4]{\frac{P}{\sigma A}} = \sqrt[4]{\frac{P}{\sigma \pi dL}} = \sqrt[4]{\frac{V^2}{\sigma \pi dLR}}$$

Relate the resistance R of the filament to its resistivity ρ:

$$R = \frac{\rho L}{A} = \frac{4\rho L}{\pi d^2}$$

Substitute for R in the expression for T to obtain:

$$T = \sqrt[4]{\frac{V^2}{\sigma \pi dL \dfrac{4\rho L}{\pi d^2}}} = \sqrt[4]{\frac{V^2 d}{4\sigma L^2 \rho}}$$

Substitute numerical values and evaluate T:

$$T = \sqrt[4]{\frac{(5.00\,\text{V})^2(40.0\times10^{-6}\,\text{m})}{4(5.67\times10^{-8}\,\text{W/m}^2\cdot\text{K}^4)(0.0300\,\text{m})^2(3.00\times10^{-5}\,\Omega\cdot\text{m})}} = \boxed{636\,\text{K}}$$

(b) As the filament heats up, its resistance decreases. This results in more power being dissipated, further heat, higher temperature, etc. If not controlled, this thermal runaway can burn out the filament.

Energy in Electric Circuits

60 • A 1.00-kW heater is designed to operate at 240 V. (a) What is the heater's resistance and what is the current in the wires that supply power to the heater? (b) What is the power delivered to the heater if it operates at 120 V? Assume that its resistance remains the same.

Picture the Problem We can use $P = V^2/R$ to find the resistance of the heater and Ohm's law to find the current it draws.

(a) Express the power output of the heater in terms of its resistance and its operating voltage:

$$P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P}$$

Substitute numerical values and evaluate R:

$$R = \frac{(240\,\text{V})^2}{1.00\,\text{kW}} = 57.60\,\Omega = \boxed{57.6\,\Omega}$$

Apply Ohm's law to find the current drawn by the heater:

$$I = \frac{V}{R} = \frac{240\,\text{V}}{57.60\,\Omega} = \boxed{4.17\,\text{A}}$$

(b) The power delivered to the heater operating at 120 V is:

$$P = \frac{(120\,\text{V})^2}{57.60\,\Omega} = \boxed{250\,\text{W}}$$

61 • A battery has an emf of 12 V. How much work does it do in 5.0 s if it delivers a current of 3.0 A?

Picture the Problem We can use the definition of power and the relationship between the battery's power output and its emf to find the work done by it under the given conditions.

Use the definition of power to relate the work done by the battery to the time current is drawn from it:

$$P = \frac{\Delta W}{\Delta t} \Rightarrow \Delta W = P\Delta t$$

Express the power output of the battery as a function of the battery's emf:

$$P = \mathcal{E}I$$

Substituting for P yields:

$$\Delta W = \mathcal{E}I\Delta t$$

Substitute numerical values and evaluate ΔW:

$$\Delta W = (12\,\text{V})(3.0\,\text{A})(5.0\,\text{s}) = \boxed{0.18\,\text{kJ}}$$

62 • An automotive battery has an emf of 12.0 V. When supplying power to the starter motor, the current in the battery is 20.0 A and the terminal voltage of the battery is 11.4 V. What is the internal resistance of the battery?

Picture the Problem We can relate the terminal voltage of the battery to its emf, internal resistance, and the current delivered by it and then solve this relationship for the internal resistance.

Express the terminal potential difference of the battery in terms of its emf and internal resistance:

$$V_a - V_b = \mathcal{E} - Ir \Rightarrow r = \frac{\mathcal{E} - (V_a - V)_b}{I}$$

Substitute numerical values and evaluate r:

$$r = \frac{12.0\,\text{V} - 11.4\,\text{V}}{20.0\,\text{A}} = \boxed{0.03\,\Omega}$$

63 • **[SSM]** (*a*) How much power is delivered by the battery in Problem 62 due to the chemical reactions within the battery when the current in the battery is 20 A? (*b*) How much of this power is delivered to the starter when the current in the battery is 20 A? (*c*) By how much does the chemical energy of the battery decrease if the current in the starter is 20 A for 7.0 s? (*d*) How much energy is dissipated in the battery during these 7.0 seconds?

Picture the Problem We can find the power delivered by the battery from the product of its emf and the current it delivers. The power delivered to the battery can be found from the product of the potential difference across the terminals of the starter (or across the battery when current is being drawn from it) and the current being delivered to it. In Part (*c*) we can use the definition of power to relate the decrease in the chemical energy of the battery to the power it is delivering and the time during which current is drawn from it. In Part (*d*) we can use conservation of energy to relate the energy delivered by the battery to the heat developed in the battery and the energy delivered to the starter

(*a*) Express the power delivered by the battery as a function of its emf and the current it delivers:

$$P = \mathcal{E}I = (12.0\,\text{V})(20\,\text{A}) = 240\,\text{W} = \boxed{0.24\,\text{kW}}$$

(*b*) Relate the power delivered to the starter to the potential difference across its terminals:

$$P_{\text{starter}} = V_{\text{starter}}I = (11.4\,\text{V})(20\,\text{A}) = 228\,\text{W} = \boxed{0.23\,\text{kW}}$$

(*c*) Use the definition of power to express the decrease in the chemical energy of the battery as it delivers current to the starter:

$$\Delta E = P\Delta t = (240\,\text{W})(7.0\,\text{s}) = 1680\,\text{J} = \boxed{1.7\,\text{kJ}}$$

(*d*) Use conservation of energy to relate the energy delivered by the battery to the heat developed in the battery and the energy delivered to the starter:

$$E_{\substack{\text{delivered}\\ \text{by battery}}} = E_{\substack{\text{transformed}\\ \text{into heat}}} + E_{\substack{\text{delivered}\\ \text{to starter}}} = Q + E_{\substack{\text{delivered}\\ \text{to starter}}}$$

Express the energy delivered by the battery and the energy delivered to the starter in terms of the rate at which this energy is delivered:

$$P\Delta t = Q + P_s\Delta t \Rightarrow Q = (P - P_s)\Delta t$$

Substitute numerical values and evaluate Q:

$$Q = (240\,\text{W} - 228\,\text{W})(7.0\,\text{s}) = \boxed{84\,\text{J}}$$

64 • A battery that has an emf of 6.0 V and an internal resistance of 0.30 Ω is connected to a variable resistor with resistance R. Find the current and power delivered by the battery when R is (*a*) 0, (*b*) 5.0 Ω, (*c*) 10 Ω, and (*d*) infinite.

Picture the Problem We can use conservation of energy to relate the emf of the battery to the potential differences across the variable resistor and the energy converted to heat within the battery. Solving this equation for I will allow us to find the current for the four values of R and we can use $P = I^2R$ to find the power delivered by the battery for the four values of R.

Apply conservation of energy (Kirchhoff's loop rule) to the circuit:

$$\mathcal{E} = IR + Ir \Rightarrow I = \frac{\mathcal{E}}{R+r}$$

Express the power delivered by the battery as a function of the current drawn from it:

$$P = I^2R$$

(a) For $R = 0$:

$$I = \frac{\mathcal{E}}{R+r} = \frac{6.0\,\text{V}}{0.30\,\Omega} = \boxed{20\,\text{A}}$$

and

$$P = (20\,\text{A})^2(0) = \boxed{0\,\text{W}}$$

(b) For $R = 5.0\ \Omega$:

$$I = \frac{\mathcal{E}}{R+r} = \frac{6.0\,\text{V}}{5.0\,\Omega + 0.30\,\Omega} = 1.13\,\text{A}$$

$$= \boxed{1.1\,\text{A}}$$

and

$$P = (1.13\,\text{A})^2(5.0\,\Omega) = \boxed{6.4\,\text{W}}$$

(c) For $R = 10\ \Omega$:

$$I = \frac{\mathcal{E}}{R+r} = \frac{6.0\,\text{V}}{10\,\Omega + 0.30\,\Omega} = 0.583\,\text{A}$$

$$= \boxed{0.58\,\text{A}}$$

and

$$P = (0.583\text{A})^2(10\,\Omega) = \boxed{3.4\,\text{W}}$$

(d) For $R = \infty$:

$$I = \frac{\mathcal{E}}{R+r} = \lim_{R\to\infty} \frac{6.0\,\text{V}}{R + 0.30\,\Omega} = \boxed{0\,\text{A}}$$

and

$$P = \boxed{0\,\text{W}}$$

65 •• A 12.0-V automobile battery that has a negligible internal resistance can deliver a total charge of 160 A·h. (*a*) What is the amount of energy stored in the battery? (*b*) After studying all night for a calculus test, you try to drive to class to take the test. However, you find that the car's battery is "dead" because you had left the headlights on! Assuming the battery was able to produce current at a constant rate until it died, how long were your lights on? Assume the pair of headlights together operates at a power of 150 W.

Picture the Problem We can express the total stored energy ΔU in the battery in terms of its emf and the product $I\Delta t$ of the current it can deliver for a period of time Δt. We can apply the definition of power to relate the lifetime of the battery to the rate at which it is providing energy to the pair of headlights.

(*a*) Express ΔU in terms of $\mathcal{E}$ and the product $I\Delta t$:

$$\Delta U = \mathcal{E} I \Delta t$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = (12.0\,\text{V})(160\,\text{A}\cdot\text{h}) = 1.92\,\text{kW}\cdot\text{h}$$
$$= 1.92\,\text{kW}\cdot\text{h} \times \frac{3.6\,\text{MJ}}{\text{kW}\cdot\text{h}}$$
$$= \boxed{6.9\,\text{MJ}}$$

(*b*) Use the definition of power to relate the lifetime of the battery to the rate at which it is providing energy to the pair of headlights:

$$\Delta t = \frac{\Delta U}{P}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1.92\,\text{kW}\cdot\text{h}}{150\,\text{W}} = \boxed{12.8\,\text{h}}$$

66 •• The measured current in a circuit in your uncle's house is 12.5 A. In this circuit, the only appliance that is on is a space heater that is being used to heat the bathroom. A pair of 12-gauge copper wires carries the current from the supply panel in your basement to the wall outlet in the bathroom, a distance of 30.0 m. You measure the voltage at the supply panel to be exactly 120 V. What is the voltage at the wall outlet in the bathroom that the space heater is connected to?

Picture the Problem We can use conservation of energy (Kirchhoff's loop rule) to relate the emf at the fuse box and the voltage drop in the wires to the voltage at the wall outlet in the bathroom.

Apply Kirchhoff's loop rule to the circuit to obtain:

$$\mathcal{E} - V_{\text{wires}} - V_{\text{outlet}} = 0$$

or

$$\mathcal{E} - IR_{\text{wires}} - V_{\text{outlet}} = 0$$

Solve for V_{outlet} to obtain:

$$V_{\text{outlet}} = \mathcal{E} - IR_{\text{wires}}$$

Relate the resistance of the copper wires to the resistivity of copper, the length of the wires, and the cross-sectional area of 12-gauge wire:

$$R_{\text{wires}} = \rho_{\text{Cu}} \frac{L}{A}$$

Substituting for R_{wires} yields:

$$V_{\text{outlet}} = \mathcal{E} - \frac{I\rho_{\text{Cu}}L}{A}$$

Substitute numerical values (see Table 25 -1 for the resistivity of copper and Table 25-2 for the cross-sectional area of 12-gauge wire) and evaluate V_{outlet}:

$$V_{\text{outlet}} = 120\,\text{V} - \frac{(12.5\,\text{A})(1.7\times10^{-8}\,\Omega\cdot\text{m})(60.0\,\text{m})}{3.309\,\text{mm}^2} = \boxed{116\,\text{V}}$$

67 •• [SSM] A lightweight electric car is powered by a series combination of ten 12.0-V batteries, each having negligible internal resistance. Each battery can deliver a charge of 160 A·h before needing to be recharged. At a speed of 80.0 km/h, the average force due to air drag and rolling friction is 1.20 kN. (*a*) What must be the minimum power delivered by the electric motor if the car is to travel at a speed of 80.0 km/h? (*b*) What is the total charge, in coulombs, that can be delivered by the series combination of ten batteries before recharging is required? (*c*) What is the total electrical energy delivered by the ten batteries before recharging? (*d*) How far can the car travel (at 80.0 km/h) before the batteries must be recharged? (*e*) What is the cost per kilometer if the cost of recharging the batteries is 9.00 cents per kilowatt-hour?

Picture the Problem We can use $P = fv$ to find the power the electric motor must develop to move the car at 80 km/h against a frictional force of 1200 N. We can find the total charge that can be delivered by the 10 batteries using $\Delta Q = NI\Delta t$. The total electrical energy delivered by the 10 batteries before recharging can be found using the definition of emf. We can find the distance the car can travel from the definition of work and the cost per kilometer of driving the car this distance by dividing the cost of the required energy by the distance the car has traveled.

(*a*) Express the power the electric motor must develop in terms of the speed of the car and the friction force:

$$P = fv = (1.20\,\text{kN})(80.0\,\text{km/h}) = \boxed{26.7\,\text{kW}}$$

(*b*) Because the batteries are in series, the total charge that can be delivered before charging is the same as the charge from a single battery:

$$\Delta Q = I\Delta t = (160\,\text{A}\cdot\text{h})\left(\frac{3600\,\text{s}}{\text{h}}\right) = \boxed{576\,\text{kC}}$$

(*c*) Use the definition of emf to express the total electrical energy available in the batteries:

$$W = Q\mathcal{E} = 10(576\,\text{kC})(12.0\,\text{V}) = 69.12\,\text{MJ} = \boxed{69.1\,\text{MJ}}$$

(*d*) Relate the amount of work the batteries can do to the work required to overcome friction:

$$W = fd \Rightarrow d = \frac{W}{f}$$

Substitute numerical values and evaluate *d*:

$$d = \frac{69.12\,\text{MJ}}{1.20\,\text{kN}} = \boxed{57.6\,\text{km}}$$

(*e*) The cost per kilometer is the ratio of the cost of the energy to the distance traveled before recharging:

$$\text{Cost/km} = \frac{\left(\dfrac{\$0.0900}{\text{kW}\cdot\text{h}}\right)\mathcal{E}\,I\,t}{d}$$

Substitute numerical values and calculate the cost per kilometer:

$$\text{Cost/km} = \frac{\left(\dfrac{\$0.0900}{\text{kW}\cdot\text{h}}\right)(120\,\text{V})(160\,\text{A}\cdot\text{h})}{5.76\,\text{km}} = \boxed{\$0.300/\text{km}}$$

68 ••• A 100-W heater is designed to operate with an applied voltage of 120 V. (*a*) What is the heater's resistance, and what current does the heater carry? (*b*) Show that if the potential difference V across the heater changes by a small amount ΔV, the power P changes by a small amount ΔP, where $\Delta P/P \approx 2\Delta V/V$. *Hint*: *Approximate the changes by modeling them as differentials, and assume the resistance is constant.* (*c*) Using the Part-(*b*) result, find the approximate power dissipated in the heater, if the potential difference is decreased to 115 V. Compare your result to the exact answer.

Picture the Problem We can use the definition of power to find the current drawn by the heater and Ohm's law to find its resistance. In Part (*b*) we'll use the hint to show that $\Delta P/P \approx 2\Delta V/V$ and in Part (*c*) use this result to find the approximate power dissipated in the heater if the potential difference is decreased to 115 V.

(*a*) Use the definition of power to relate the current I drawn by the heater to its power rating P and the potential difference across it V:

$$P = IV \Rightarrow I = \frac{P}{V}$$

Substitute numerical values and evaluate I:

$$I = \frac{100\text{ W}}{120\text{ V}} = \boxed{0.833\text{ A}}$$

Apply Ohm's law to relate the resistance of the heater to the voltage across it and the current it draws:

$$R = \frac{V}{I} = \frac{120\text{ V}}{0.833\text{ A}} = \boxed{144\,\Omega}$$

(*b*) Approximating dP/dV as a differential yields:

$$\frac{dP}{dV} \approx \frac{\Delta P}{\Delta V} \Rightarrow \Delta P \approx \frac{dP}{dV}\Delta V$$

Express the dependence of P on V:

$$P = \frac{V^2}{R}$$

Assuming R to be constant, evaluate dP/dV:

$$\frac{dP}{dV} = \frac{d}{dV}\left[\frac{V^2}{R}\right] = \frac{2V}{R}$$

Substitute to obtain:

$$\Delta P \approx \frac{2V}{R}\Delta V = 2\frac{V^2}{R}\frac{\Delta V}{V} = 2P\frac{\Delta V}{V}$$

Divide both sides of the equation by P to obtain:

$$\frac{\Delta P}{P} = \boxed{2\frac{\Delta V}{V}}$$

(*c*) Express the approximate power dissipated in the heater as the sum of its power consumption and the change in its power dissipation when the voltage is decreased by ΔV:

$$\begin{aligned} P &\approx P_0 + \Delta P \\ &= P_0 + 2P_0\frac{\Delta V}{V} \qquad (1) \\ &= P_0\left(1 + 2\frac{\Delta V}{V}\right) \end{aligned}$$

Assuming that the difference between 120 V and 115 V is good to three significant figures, substitute numerical values and evaluate P:

$$P \approx (100\,\text{W})\left(1 + 2\left(\frac{-5.00\,\text{V}}{120\,\text{V}}\right)\right) = \boxed{91.7\text{ W}}$$

The exact power dissipated in the heater is:

$$P_{\text{exact}} = \frac{V'^2}{R} = \frac{(115\,\text{V})^2}{144\,\Omega} = 91.8\text{ W}$$

The power calculated using the approximation is 0.1 percent less than the power calculated exactly.

Combinations of Resistors

69 • **[SSM]** If the potential drop from point *a* to point *b* (Figure 25-52) is 12.0 V, find the current in each resistor.

Picture the Problem We can apply Ohm's law to find the current through each resistor.

Apply Ohm's law to each of the resistors to find the current flowing through each:

$$I_4 = \frac{V}{4.00\,\Omega} = \frac{12.0\,\text{V}}{4.00\,\Omega} = \boxed{3.00\,\text{A}}$$

$$I_3 = \frac{V}{3.00\,\Omega} = \frac{12.0\,\text{V}}{3.00\,\Omega} = \boxed{4.00\,\text{A}}$$

and

$$I_6 = \frac{V}{6.00\,\Omega} = \frac{12.0\,\text{V}}{6.00\,\Omega} = \boxed{2.00\,\text{A}}$$

Remarks: You would find it instructive to use Kirchhoff's junction rule (conservation of charge) to confirm our values for the currents through the three resistors.

70 • If the potential drop between point *a* and point *b* (Figure 25-53) is 12.0 V, find the current in each resistor.

Picture the Problem We can simplify the network by first replacing the resistors that are in parallel by their equivalent resistance. We can then add that resistance and the 3.00-Ω resistance to find the equivalent resistance between points *a* and *b*. Denoting the currents through each resistor with subscripts corresponding to the resistance through which the current flows, we can apply Ohm's law to find those currents.

Express the equivalent resistance of the two resistors in parallel and solve for $R_{\text{eq},1}$:

$$\frac{1}{R_{\text{eq},1}}=\frac{1}{R_6}+\frac{1}{R_2}=\frac{1}{6.00\,\Omega}+\frac{1}{2.00\,\Omega}$$

and

$$R_{\text{eq},1}=1.50\,\Omega$$

Because the 3.00-Ω resistor is in series with $R_{\text{eq},1}$:

$$R_{\text{eq}}=R_3+R_{\text{eq},1}=3.00\,\Omega+1.50\,\Omega=4.50\,\Omega$$

Apply Ohm's law to the network to find I_3:

$$I_3=\frac{V_{ab}}{R_{\text{eq}}}=\frac{12.0\,\text{V}}{4.50\,\Omega}=2.667\text{ A}=\boxed{2.67\,\text{A}}$$

Find the potential difference across the parallel resistors:

$$V_{6\,\&\,2}=V_{ab}-V_3=12.0\,\text{V}-(2.667\,\text{A})(3.00\,\Omega)=3.999\,\text{V}$$

Use the common potential difference across the resistors in parallel to find the current through each of them:

$$I_6=\frac{V_6}{R_6}=\frac{3.999\,\text{V}}{6.00\Omega}=\boxed{0.667\,\text{A}}$$

and

$$I_2=\frac{V_{6\&2}}{R_2}=\frac{3.999\,\text{V}}{2.00\Omega}=\boxed{2.00\,\text{A}}$$

Remarks: We could have found the currents through the 6.00-Ω and 2.00-Ω resistors by using the fact that the current at the junction between the 3.00-Ω resistor and the parallel branch divides inversely with the resistance in each branch of the parallel network.

71 • (*a*) Show that the equivalent resistance between point *a* and point *b* in Figure 25-54 is *R*. (*b*) How would adding a fifth resistor that has resistance *R* between point *c* and point *d* effect the equivalent resistance between point *a* and point *b*?

Picture the Problem Note that the resistors between *a* and *c* and between *c* and *b* are in series as are the resistors between *a* and *d* and between *d* and *b*. Hence, we have two branches in parallel, each branch consisting of two resistors *R* in series. In Part (*b*) it will be important to note that the potential difference between point *c* and point *d* is zero.

(*a*) Express the equivalent resistance between points *a* and *b* in terms of the equivalent resistances between *acb* and *adb*:

$$\frac{1}{R_{\text{eq}}}=\frac{1}{R_{acb}}+\frac{1}{R_{adb}}=\frac{1}{2R}+\frac{1}{2R}$$

Solve for R_{eq} to obtain:

$$R_{\text{eq}}=\boxed{R}$$

(*b*) It would not affect it. Because the potential difference between points c and d is zero, no current would flow through the resistor connected between these two points, and the addition of that resistor would not change the network.

72 •• The battery in Figure 25-55 has negligible internal resistance. Find (*a*) the current in each resistor and (*b*) the power delivered by the battery.

Picture the Problem Note that the 2.00-Ω resistors are in parallel with each other and with the 4.00-Ω resistor. We can Apply Kirchhoff's loop rule to relate the current I_3 drawn from the battery to the emf of the battery and equivalent resistance R_{eq} of the resistor network. We can find the current through the resistors connected in parallel by applying Kirchhoff's loop rule a second time. In Part (*b*) we can find the power delivered by the battery from the product of its emf and the current it delivers to the circuit.

(*a*) Apply Kirchhoff's loop rule to obtain:

$$\mathcal{E}-I_3R_{\text{eq}}=0\Rightarrow I_3=\frac{\mathcal{E}}{R_{\text{eq}}}\qquad(1)$$

Find the equivalent resistance of the three resistors in parallel:

$$\frac{1}{R_{\text{eq},1}}=\frac{1}{R_2}+\frac{1}{R_2}+\frac{1}{R_4}=\frac{1}{2.00\,\Omega}+\frac{1}{2.00\,\Omega}+\frac{1}{4.00\,\Omega}$$

and

$$R_{\text{eq},1}=0.800\,\Omega$$

Find the equivalent resistance of $R_{\text{eq},1}$ and R_3 in series:

$$R_{\text{eq}}=R_3+R_{\text{eq},1}=3.00\,\Omega+0.800\,\Omega=3.80\,\Omega$$

Substitute numerical values in equation (1) and evaluate I_3:

$$I_3=\frac{6.00\,\text{V}}{3.80\,\Omega}=1.579\text{ V}=\boxed{1.58\,\text{A}}$$

Express the current through each of the parallel resistors in terms of their common potential difference V:

$$I_2=\frac{V}{R_2}\text{ and }I_4=\frac{V}{R_4}$$

Apply Kirchhoff's loop rule to obtain:

$$\mathcal{E} - I_3R_3 - V = 0 \Rightarrow V = \mathcal{E} - I_3R_3$$

Substituting for V in the equations for I_2 and I_4 yields:

$$I_2 = \frac{\mathcal{E} - I_3R_3}{R_2} \text{ and } I_4 = \frac{\mathcal{E} - I_3R_3}{R_4}$$

Substitute numerical values and evaluate I_2 and I_4:

$$I_2 = \frac{6.00\,\text{V} - (1.579\,\text{A})(3.00\,\Omega)}{2.00\,\Omega}$$

$$= \boxed{0.632\,\text{A}}$$

and

$$I_4 = \frac{6.00\,\text{V} - (1.579\,\text{A})(3.00\,\Omega)}{4.00\,\Omega}$$

$$= \boxed{0.316\,\text{A}}$$

(b) P is the product of $\mathcal{E}$ and I_3:

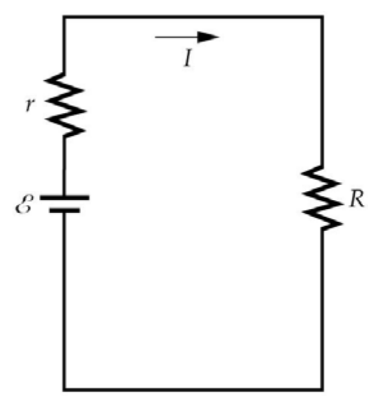
$$P = \mathcal{E}I_3 = (6.00\,\text{V})(1.579\,\text{A})$$

$$= \boxed{9.47\,\text{W}}$$

Remarks: Note that the currents I_3, I_2, and I_4 satisfy Kirchhoff's junction rule.

73 •• [SSM] A 5.00-V power supply has an internal resistance of 50.0 Ω. What is the smallest resistor that can be put in series with the power supply so that the voltage drop across the resistor is larger than 4.50 V?

Picture the Problem Let r represent the resistance of the internal resistance of the power supply, $\mathcal{E}$ the emf of the power supply, R the resistance of the external resistor to be placed in series with the power supply, and I the current drawn from the power supply. We can use Ohm's law to express the potential difference across R and apply Kirchhoff's loop rule to express the current through R in terms of $\mathcal{E}$, r, and R.



Express the potential difference across the resistor whose resistance is R:

$$V_R = IR \qquad (1)$$

Apply Kirchhoff's loop rule to the circuit to obtain:

$$\mathcal{E} - Ir - IR = 0 \Rightarrow I = \frac{\mathcal{E}}{r+R}$$

Substitute in equation (1) to obtain:

$$V_R = \left(\frac{\mathcal{E}}{r+R}\right)R \Rightarrow R = \frac{V_R r}{\mathcal{E} - V_R}$$

Substitute numerical values and evaluate R:

$$R = \frac{(4.50\,\text{V})(50.0\,\Omega)}{5.00\,\text{V} - 4.50\,\text{V}} = \boxed{0.45\,\text{k}\Omega}$$

74 •• You have been handed an unknown battery. Using your multimeter, you determine that when a 5.00-Ω resistor is connected across the battery's terminals, the current in the battery is 0.500 A. When this resistor is replaced by an 11.0-Ω resistor, the current drops to 0.250 A. From this data, find (*a*) the emf and (*b*) internal resistance of your battery.

Picture the Problem We can apply Kirchhoff's loop rule to the two circuits described in the problem statement and solve the resulting equations simultaneously for r and $\mathcal{E}$.

(*a*) and (*b*) Apply Kirchhoff's loop rule to the two circuits to obtain:

$$\mathcal{E} - I_1 r - I_1 R_5 = 0$$

and

$$\mathcal{E} - I_2 r - I_2 R_{11} = 0$$

Substitute numerical values and simplify to obtain:

$$\mathcal{E} - (0.500\,\text{A})r = 2.50\,\text{V} \qquad (1)$$

and

$$\mathcal{E} - (0.250\,\text{A})r = 2.75\,\text{V} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$\mathcal{E} = \boxed{3.00\,\text{V}} \text{ and } r = \boxed{1.00\,\Omega}$$

75 •• (*a*) Find the equivalent resistance between point *a* and point *b* in Figure 25-56. (*b*) If the potential drop between point *a* and point *b* is 12.0 V, find the current in each resistor.

Picture the Problem We can simplify the network by first replacing the resistors that are in parallel by their equivalent resistance. We'll then have a parallel network with two resistors in series in each branch and can use the expressions for resistors in series to simplify the network to two resistors in parallel. The equivalent resistance between points *a* and *b* will be the single resistor equivalent to these two resistors. In Part (*b*) we'll use the fact that the potential difference across the upper branch is the same as the potential difference across the lower branch, in conjunction with Ohm's law, to find the currents through each resistor.

(*a*) The equivalent resistance of the two 6.00-Ω resistors in parallel is given by:

$$R_{\text{eq},1} = \frac{R_6 R_6}{R_6 + R_6} = \frac{(6.00\,\Omega)^2}{6.00\,\Omega + 6.00\,\Omega} = 3.00\,\Omega$$

Find the equivalent resistance of the 6.00-Ω resistor is in series with $R_{\text{eq},1}$:

$$R_{\text{eq},2} = R_6 + R_{\text{eq},1} = 6.00\,\Omega + 3.00\,\Omega = 9.00\,\Omega$$

Find the equivalent resistance of the 12.0-Ω resistor in series with the 6.00-Ω resistor:

$$R_{\text{eq},3} = R_6 + R_{12} = 6.00\,\Omega + 12.0\,\Omega = 18.0\,\Omega$$

Finally, find the equivalent resistance of $R_{\text{eq},3}$ in parallel with $R_{\text{eq},2}$:

$$R_{\text{eq}} = \frac{R_{\text{eq},2} R_{\text{eq},3}}{R_{\text{eq},2} + R_{\text{eq},3}} = \frac{(9.00\,\Omega)(18.0\,\Omega)}{9.00\,\Omega + 18.0\,\Omega} = \boxed{6.00\,\Omega}$$

(*b*) Apply Ohm's law to the upper branch to find the current $I_{\substack{\text{upper}\\\text{branch}}} = I_{12} = I_6$:

$$I_{\substack{\text{upper}\\\text{branch}}} = I_{12} = I_6 = \frac{V_{ab}}{R_{\text{eq},3}} = \frac{12.0\text{ V}}{18.0\,\Omega} = \boxed{667\text{ mA}}$$

Apply Ohm's law to the lower branch to find the current $I_{\substack{\text{lower}\\\text{branch}}} = I_{\substack{6.00\text{-}\Omega\text{ resistor}\\\text{in series}}}$:

$$I_{\substack{\text{lower}\\\text{branch}}} = I_{\substack{6.00\text{-}\Omega\text{ resistor}\\\text{in series}}} = \frac{V_{ab}}{R_{\text{eq},2}} = \frac{12.0\text{ V}}{9.00\,\Omega} = \boxed{1.33\text{ A}}$$

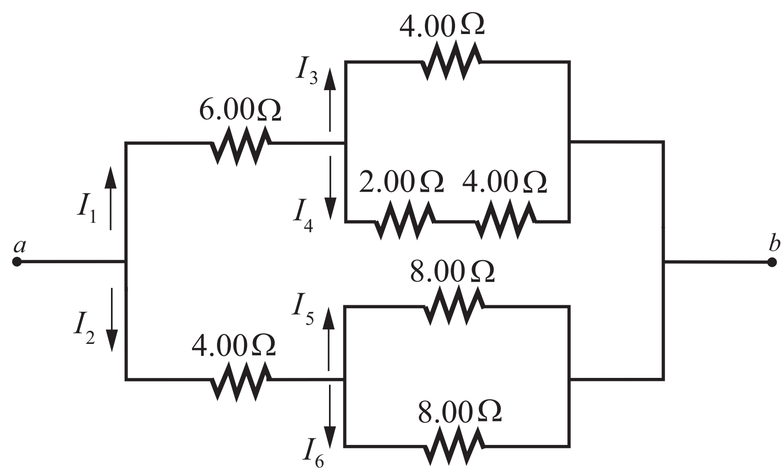
Express the current through the 6.00-Ω resistors in parallel:

$$I_{\substack{6.00\text{-}\Omega\text{ resistors}\\\text{in parallel}}} = \tfrac{1}{2} I_l = \tfrac{1}{2}(1.33\text{ A}) = \boxed{667\text{ mA}}$$

In summary, the current in both the 6.00-Ω and the 12.0-Ω resistor in the upper branch is 667 mA. The current in each 6.00-Ω resistor in the parallel combination in the lower branch is 667 mA. The current in the 6.00-Ω resistor on the right in the lower branch is 1.33 A.

76 •• (*a*) Find the equivalent resistance between point *a* and point *b* in Figure 25-57. (*b*) If the potential drop between point *a* and point *b* is 12.0 V, find the current in each resistor.

Picture the Problem Assign currents in each of the resistors as shown in the diagram. We can simplify the network by first replacing the resistors that are in parallel by their equivalent resistance. We'll then have a parallel network with two resistors in series in each branch and can use the expressions for resistors in series to simplify the network to two resistors in parallel. The equivalent resistance between points *a* and *b* will be the single resistor equivalent to these two resistors. In Part (*b*) we'll use the fact that the potential difference across the upper branch is the same as the potential difference across the lower branch, in conjunction with Ohm's law, to find the currents through each resistor.



(*a*) Find the equivalent resistance of the resistors in parallel in the upper branch:

$$R_{\text{eq},1} = \frac{(R_2 + R_4)R_4}{(R_2 + R_4) + R_4}$$
$$= \frac{(6.00\,\Omega)(4.00\,\Omega)}{2.00\,\Omega + 4.00\,\Omega + 4.00\,\Omega}$$
$$= 2.40\,\Omega$$

Find the equivalent resistance of the 6.00-Ω resistor is in series with $R_{\text{eq},1}$:

$$R_{\text{eq},2} = R_6 + R_{\text{eq},1}$$
$$= 6.00\,\Omega + 2.40\,\Omega = 8.40\,\Omega$$

Express and evaluate the equivalent resistance of the resistors in parallel in the lower branch and solve for $R_{\text{eq},2}$:

$$R_{\text{eq},2} = \frac{R_8 R_8}{R_8 + R_8} = \tfrac{1}{2}R_8 = 4.00\Omega$$

Find the equivalent resistance of the 4.00-Ω resistor is in series with $R_{\text{eq},2}$:

$$R_{\text{eq},3} = R_4 + R_{\text{eq},2}$$
$$= 4.00\,\Omega + 4.00\,\Omega = 8.00\,\Omega$$

Finally, find the equivalent resistance of $R_{\text{eq},2}$ in parallel with $R_{\text{eq},3}$:

$$R_{\text{eq}}=\frac{R_{\text{eq},2}R_{\text{eq},3}}{R_{\text{eq},2}+R_{\text{eq},3}}=\frac{(8.40\,\Omega)(8.00\,\Omega)}{8.40\,\Omega+8.00\,\Omega}=\boxed{4.10\,\Omega}$$

(*b*) Apply Ohm's law to the upper branch to find the current I_1 through the 6.00-Ω resistor:

$$I_1=\frac{V_{ab}}{R_{\text{eq},2}}=\frac{12.0\,\text{V}}{8.40\,\Omega}=\boxed{1.43\,\text{A}}$$

Find the potential difference across the 4.00-Ω and 6.00-Ω parallel combination in the upper branch:

$$\begin{aligned}V_{4\text{ and }6}&=12.0\,\text{V}-V_6=12.0\,\text{V}-I_1R_6\\&=12.0\,\text{V}-(1.43\,\text{A})(6.00\,\Omega)\\&=3.43\,\text{V}\end{aligned}$$

Apply Ohm's law to find I_4:

$$I_4=\frac{V_6}{R_6}=\frac{3.43\,\text{V}}{6.00\,\Omega}=\boxed{0.57\,\text{A}}$$

Apply Ohm's law to find I_3:

$$I_3=\frac{V_4}{R_4}=\frac{3.43\,\text{V}}{4.00\,\Omega}=\boxed{0.86\,\text{A}}$$

Apply Ohm's law to the lower branch to find I_2:

$$I_2=\frac{V_{ab}}{R_{\text{eq},2}}=\frac{12.0\,\text{V}}{8.00\,\Omega}=\boxed{1.50\,\text{A}}$$

Find the potential difference across the 8.00-Ω and 8.00-Ω parallel combination in the lower branch:

$$\begin{aligned}V_{8\text{ and }8}&=12.0\,\text{V}-I_2R_4\\&=12.0\,\text{V}-(1.50\,\text{A})(4.00\,\Omega)\\&=6.0\,\text{V}\end{aligned}$$

Apply Ohm's law to find $I_5=I_6$:

$$I_5=I_6=\frac{V_{8\text{ and }8}}{8.00\,\Omega}=\frac{6.0\,\text{V}}{8.00\,\Omega}=\boxed{0.75\,\text{A}}$$

In summary, the current through the 6.00-Ω resistor is 1.43 A, the current through the lower 4.00-Ω resistor is 1.50 A, the current through the 4.00-Ω resistor that is in parallel with the 2.00-Ω and 4.00-Ω resistors is 0.86 A, the current through the 2.00-Ω and 4.00-Ω resistors that are in series is 0.57 A, and the currents through the two 8.00-Ω resistors are 0.75 A.

77 •• [SSM] A length of wire has a resistance of 120 Ω. The wire is cut into pieces that have the same length, and then the wires are connected in parallel. The resistance of the parallel arrangement is 1.88 Ω. Find the number of pieces into which the wire was cut.

Picture the Problem We can use the equation for N identical resistors connected in parallel to relate N to the resistance R of each piece of wire and the equivalent resistance R_{eq}.

Express the resistance of the N pieces connected in parallel:

$$\frac{1}{R_{eq}} = \frac{N}{R}$$

where R is the resistance of one of the N pieces.

Relate the resistance of one of the N pieces to the resistance of the wire:

$$R = \frac{R_{wire}}{N}$$

Substitute for R to obtain:

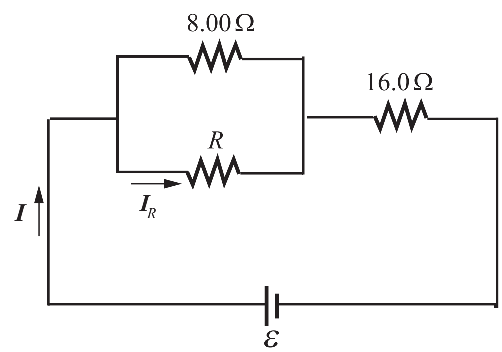
$$\frac{1}{R_{eq}} = \frac{N^2}{R_{wire}} \Rightarrow N = \sqrt{\frac{R_{wire}}{R_{eq}}}$$

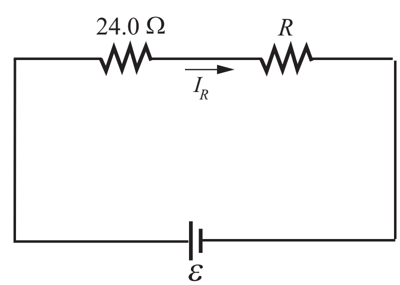
Substitute numerical values and evaluate N:

$$N = \sqrt{\frac{120\,\Omega}{1.88\,\Omega}} = \boxed{8 \text{ pieces}}$$

78 •• A parallel combination of an 8.00-Ω resistor and a resistor of unknown resistance is connected in series with a 16.0-Ω resistor and an ideal battery. This circuit is then disassembled and the three resistors are then connected in series with each other and the same battery. In both arrangements, the current through the 8.00-Ω resistor is the same. What is the resistance of the unknown resistor?

Picture the Problem The same current I_R flows in R when the resistors are connected as shown in the following diagrams. Applying Kirchhoff's loop rule to these circuits will yield expressions for I_R that we can solve simultaneously for R.





Applying Kirchhoff's loop rule to the circuit to the left gives:

$$\varepsilon - I(16.0\,\Omega) - \frac{(8.00\,\Omega)R}{8.00\,\Omega + R} = 0$$

Solve for I to obtain:

$$I=\frac{\varepsilon}{16.0\,\Omega+\dfrac{(8.00\,\Omega)R}{8.00\,\Omega+R}}$$

Applying Kirchhoff's loop rule to the loop that includes the source, the 16.0-Ω resistor and R yields:

$$\varepsilon-I(16.0\,\Omega)-I_R R=0$$

Solve for I_R to obtain:

$$I_R=\frac{\varepsilon-I(16.0\,\Omega)}{R}$$

Substituting for I and simplifying gives:

$$I_R=\frac{\varepsilon-\dfrac{\varepsilon}{16.0\,\Omega+\dfrac{(8.00\,\Omega)R}{8.00\,\Omega+R}}(16.0\,\Omega)}{R}$$

$$=\frac{\varepsilon}{R}-\frac{\varepsilon}{R}\left(\frac{1}{1+\dfrac{R}{16.0\,\Omega+2R}}\right)$$

Apply Kirchhoff's loop rule to the circuit to the right to obtain:

$$\varepsilon-I_R(24.0\,\Omega)-I_R R=0$$

Solving for I_R gives:

$$I_R=\frac{\varepsilon}{24.0\,\Omega+R}$$

Equating the two expressions for I_R yields:

$$\frac{\varepsilon}{R}-\frac{\varepsilon}{R}\left(\frac{1}{1+\dfrac{R}{16.0\,\Omega+2R}}\right)=\frac{\varepsilon}{24.0\,\Omega+R}$$

Simplify to obtain:

$$\frac{\varepsilon}{R}-\frac{\varepsilon}{R}\left(\frac{1}{1+\dfrac{R}{16.0\,\Omega+2R}}\right)=\frac{\dfrac{\varepsilon}{R}}{\dfrac{24.0\,\Omega}{R}+1}$$

or

$$1-\frac{1}{1+\dfrac{R}{16.0\,\Omega+2R}}=\frac{1}{\dfrac{24.0\,\Omega}{R}+1}$$

Solving for R yields:

$$R=\boxed{4.00\,\Omega}$$

79 •• For the network shown in Figure 25-58, let R_{ab} denote the equivalent resistance between terminals *a* and *b*. Find (*a*) R_3, so that $R_{ab} = R_1$, (*b*) R_2, so that $R_{ab} = R_3$; and (*c*) R_1, so that $R_{ab} = R_1$.

Picture the Problem We can find the equivalent resistance R_{ab} between points *a* and *b* and then set this resistance equal, in turn, to R_1, R_3, and R_1 and solve for R_3, R_2, and R_1, respectively.

(*a*) Express the equivalent resistance between points *a* and *b*:

$$R_{ab} = \frac{R_1 R_2}{R_1 + R_2} + R_3$$

Equate this expression to R_1 and solve for R_3 to obtain:

$$R_1 = \frac{R_1 R_2}{R_1 + R_2} + R_3 \Rightarrow R_3 = \boxed{\frac{R_1^2}{R_1 + R_2}}$$

(*b*) Set R_3 equal to R_{ab}:

$$R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3$$

Solve for R_2 to obtain:

$$R_2 = \boxed{0}$$

(*c*) Set R_1 equal to R_{ab} and rewrite the result as a quadratic equation in R_1:

$$R_1 = \frac{R_1 R_2}{R_1 + R_2} + R_3$$

or

$$R_1^2 - R_3 R_1 - R_2 R_3 = 0$$

Solving the quadratic equation for R_1 yields:

$$R_1 = \boxed{\frac{R_3 + \sqrt{R_3^2 + 4R_2 R_3}}{2}}$$

where we've used the positive sign because resistance is a non-negative quantity.

80 •• Check your results for Problem 79 using the following specific values: (*a*) $R_1 = 4.00\ \Omega$, $R_2 = 6.00\ \Omega$; (*b*) $R_1 = 4.00\ \Omega$, $R_3 = 3.00\ \Omega$; and (*c*) $R_2 = 6.00\ \Omega$, $R_3 = 3.00\ \Omega$.

Picture the Problem We can substitute the given resistances in the equations derived in Problem 79.

(a) For $R_1 = 4.00\ \Omega$ and $R_2 = 6.00\ \Omega$:

$$R_3 = \frac{R_1^2}{R_1 + R_2} = \frac{(4.00\,\Omega)^2}{4.00\,\Omega + 6.00\,\Omega} = \boxed{1.60\,\Omega}$$

and

$$R_{ab} = \frac{R_1 R_2}{R_1 + R_2} + R_3 = \frac{(4.00\,\Omega)(6.00\,\Omega)}{4.00\,\Omega + 6.00\,\Omega} + 1.60\,\Omega = \boxed{4.00\,\Omega}$$

(b) For $R_1 = 4.00\ \Omega$ and $R_3 = 3.00\ \Omega$:

$$R_2 = \boxed{0}$$

and

$$R_{ab} = \frac{R_1(0)}{R_1 + 0} + R_3 = 0 + 3.00\,\Omega = \boxed{3.00\,\Omega}$$

(c) For $R_2 = 6.00\ \Omega$ and $R_3 = 3.00\ \Omega$:

$$R_1 = \frac{3.00\,\Omega + \sqrt{(3.00\,\Omega)^2 + 4(6.00\,\Omega)(3.00\,\Omega)}}{2} = \frac{3.00\,\Omega + 9.00\,\Omega}{2} = \boxed{6.00\,\Omega}$$

and

$$R_{ab} = \frac{R_1 R_2}{R_1 + R_2} + R_3 = \frac{(6.00\,\Omega)(6.00\,\Omega)}{6.00\,\Omega + 6.00\,\Omega} + 3.00\,\Omega = \boxed{6.00\,\Omega}$$

Kirchhoff's Rules

81 • [SSM] In Figure 25-59, the battery's emf is 6.00 V and R is 0.500 Ω. The rate of Joule heating in R is 8.00 W. (a) What is the current in the circuit? (b) What is the potential difference across R? (c) What is the resistance r?

Picture the Problem We can relate the current provided by the source to the rate of Joule heating using $P = I^2R$ and use Ohm's law and Kirchhoff's rules to find the potential difference across R and the value of r.

(a) Relate the current I in the circuit to rate at which energy is being dissipated in the form of Joule heat:

$$P = I^2R \Rightarrow I = \sqrt{\frac{P}{R}}$$

Substitute numerical values and evaluate I:

$$I = \sqrt{\frac{8.00\text{ W}}{0.500\,\Omega}} = \boxed{4.00\text{ A}}$$

(*b*) Apply Ohm's law to find V_R:

$$V_R = IR = (4.00\text{ A})(0.500\,\Omega) = \boxed{2.00\text{ V}}$$

(*c*) Apply Kirchhoff's loop rule to obtain:

$$\mathcal{E} - Ir - IR = 0 \Rightarrow r = \frac{\mathcal{E} - IR}{I} = \frac{\mathcal{E}}{I} - R$$

Substitute numerical values and evaluate r:

$$r = \frac{6.00\text{ V}}{4.00\text{ A}} - 0.500\,\Omega = \boxed{1.00\,\Omega}$$

82 • The batteries in the circuit in Figure 25-60 have negligible internal resistance. (*a*) Find the current using Kirchhoff's loop rule. (*b*) Find the power delivered to or supplied by each battery. (*c*) Find the rate of Joule heating in each resistor.

Picture the Problem Assume that the current flows clockwise in the circuit and let $\mathcal{E}_1$ represent the 12.0-V source and $\mathcal{E}_2$ the 6.00-V source. We can apply Kirchhoff's loop rule (conservation of energy) to this series circuit to relate the current to the emfs of the sources and the resistance of the circuit. In Part (*b*) we can find the power delivered or absorbed by each source using $P = \mathcal{E}I$ and in Part (*c*) the rate of Joule heating using $P = I^2R$.

(*a*) Apply Kirchhoff's loop rule to the circuit to obtain:

$$\mathcal{E}_1 - IR_2 - \mathcal{E}_2 - IR_4 = 0$$

Solving for I yields:

$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_2 + R_4}$$

Substitute numerical values and evaluate I:

$$I = \frac{12.0\text{ V} - 6.00\text{ V}}{2.00\,\Omega + 4.00\,\Omega} = \boxed{1.0\text{ A}}$$

(*b*) Express the power delivered /absorbed by each source in terms of its emf and the current drawn from or forced through it:

$$P_{12} = \mathcal{E}_{12}I = (12.0\text{ V})(1.0\text{ A}) = \boxed{12\text{ W}}$$

and

$$P_6 = \mathcal{E}_6 I = (-6.00\text{ V})(1.0\text{ A}) = \boxed{-6.0\text{ W}}$$

where the minus sign means that this source is absorbing power.

(*c*) Express the rate of Joule heating

$$P_2 = I^2R_2 = (1.0\text{ A})^2(2.00\,\Omega) = \boxed{2.0\text{ W}}$$

in terms of the current through and the resistance of each resistor:

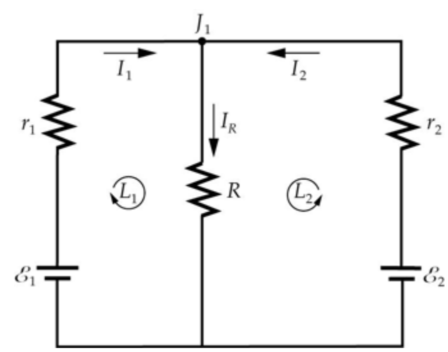
and

$$P_4 = I^2 R_4 = (1.0\,\text{A})^2(4.00\,\Omega) = \boxed{4.0\,\text{W}}$$

83 •• An old car battery that has an emf of $\mathcal{E}$ = 11.4 V and an internal resistance of 50.0 mΩ is connected to a 2.00-Ω resistor. In an attempt to recharge the battery, you connect a second battery that has an emf of $\mathcal{E}$ = 12.6 V and an internal resistance of 10.0 mΩ in parallel with the first battery and the resistor with a pair of jumper cables. (*a*) Draw a diagram of your circuit. (*b*) Find the current in each branch of this circuit. (*c*) Find the power supplied by the second battery and discuss where this power is delivered. Assume that the emfs and internal resistances of both batteries remain constant.

Picture the Problem The circuit is shown in the diagram for Part (*a*). (*b*) Let $\mathcal{E}_1$ and r_1 denote the emf of the old battery and its internal resistance, $\mathcal{E}_2$ and r_2 the emf of the second battery and its internal resistance, and R the load resistance. Let I_1, I_2, and I_R be the currents. We can apply Kirchhoff's rules to determine the unknown currents. (*c*) The power supplied by the second battery is given by $P_2 = (\mathcal{E}_2 - I_2 r_2)I_2$.

(*a*) The circuit diagram is shown to the right:



(*b*) Apply Kirchhoff's junction rule to junction 1 to obtain:

$$I_1 + I_2 = I_R$$

Apply Kirchhoff's loop rule to loop 1 to obtain:

$$\mathcal{E}_1 - r_1 I_1 - RI_R = 0$$

or

$$11.4\,\text{V} - (0.0500\,\Omega)I_1 - (2.00\,\Omega)I_R = 0$$

Apply Kirchhoff's loop rule to loop 2 to obtain:

$$\mathcal{E}_2 - r_2 I_2 - RI_R = 0$$

or

$$12.6\,\text{V} - (0.0100\,\Omega)I_2 - (2.00\,\Omega)I_R = 0$$

Solve the equations in I_1, I_2, and I_R simultaneously to obtain:

$$I_1 = -18.971\ \text{A} = \boxed{-19.0\ \text{A}},$$

$$I_2 = 25.145\ \text{A} = \boxed{25.1\ \text{A}}, \text{ and}$$

$$I_R = 6.174\ \text{A} = \boxed{6.17\ \text{A}}$$

where the minus sign for I_1 means that the current flows in the direction opposite to the direction we arbitrarily chose, i.e., the battery is being charged.

(c) The power supplied by the second battery is given by:

$$P_2 = (\mathcal{E}_2 - I_2 r_2) I_2$$

Substitute numerical values and evaluate P_2:

$$P_2 = [12.6\ \text{V} - (25.145\ \text{A})(10.0\ \text{m}\Omega)] \times (25.145\ \text{A}) = \boxed{311\ \text{W}}$$

The power delivered to the first battery is given by:

$$P_1 = \mathcal{E}_1 I_1 + I_1^2 r_1$$

Substitute numerical values and evaluate P_1:

$$P_1 = (11.4\ \text{V})(18.971\ \text{A}) + (18.971\ \text{A})^2 (0.0500\ \Omega) = 216\ \text{W} + 18.0\ \text{W} = 234\ \text{W}$$

Find the power dissipated in the load resistance R:

$$P_R = I_R^2 R = (6.174\ \text{A})^2 (2.00\ \Omega) = 76.2\ \text{W}$$

In summary, battery 2 supplies 311 W. 234 W is delivered to Battery 1, and of that 234 W, 216 W goes into recharging battery 1 and 18.0 W is dissipated by the internal resistance. In addition, 76.2 W is delivered to the 2.00-Ω resistor.

Remarks: Note that the sum of the power dissipated in the internal and load resistances and that absorbed by the second battery is the same (within round off errors) as that delivered by the first battery … just as we would expect from conservation of energy.

84 •• In the circuit in Figure 25-61, the reading of the ammeter is the same when both switches are open and when both switches are closed. What is the unknown resistance R?

Picture the Problem Note that when both switches are closed the 50.0-Ω resistor is shorted. With both switches open, we can apply Kirchhoff's loop rule to find the current I in the 100-Ω resistor. With the switches closed, the 100-Ω resistor and R are in parallel. Hence, the potential difference across them is the same and we can express the current I_{100} in terms of the current I_{tot} flowing into the parallel branch whose resistance is R, and the resistance of the 100-Ω resistor. I_{tot}, in turn, depends on the equivalent resistance of the closed-switch circuit, so we can express $I_{100} = I$ in terms of R and solve for R.

Apply Kirchhoff's loop rule to a loop around the outside of the circuit with both switches open:

$$\mathcal{E} - (300\,\Omega)I - (100\,\Omega)I - (50.0\,\Omega)I = 0$$

Solving for I yields:

$$I = \frac{\mathcal{E}}{450\,\Omega} = \frac{1.50\,\text{V}}{450\,\Omega} = 3.33\,\text{mA}$$

Relate the potential difference across the 100-Ω resistor to the potential difference across R when both switches are closed:

$$(100\,\Omega)I_{100} = RI_R \qquad (1)$$

Apply Kirchhoff's junction rule at the junction to the left of the 100-Ω resistor and R:

$$I_{tot} = I_{100} + I_R$$
or
$$I_R = I_{tot} - I_{100}$$
where I_{tot} is the current drawn from the source when both switches are closed.

Substituting for I_R in equation (1) yields:

$$(100\,\Omega)I_{100} = R(I_{tot} - I_{100})$$
or
$$I_{100} = \frac{RI_{tot}}{R + 100\,\Omega} \qquad (2)$$

Express the current I_{tot} drawn from the source with both switches closed:

$$I_{tot} = \frac{\mathcal{E}}{R_{eq}}$$

Express the equivalent resistance when both switches are closed:

$$R_{eq} = \frac{(100\,\Omega)R}{R + 100\,\Omega} + 300\,\Omega$$

Substitute for R_{eq} to obtain:

$$I_{tot} = \frac{1.5\,\text{V}}{\dfrac{(100\,\Omega)R}{R + 100\,\Omega} + 300\,\Omega}$$

Substituting for I_{tot} in equation (2) yields:

$$I_{100}=\frac{R}{R+100\,\Omega}\left(\frac{1.50\,\text{V}}{\dfrac{(100\,\Omega)R}{R+100\,\Omega}+300\,\Omega}\right)$$

$$=\frac{(1.50\,\text{V})R}{(400\,\Omega)R+30{,}000\,\Omega^2}=3.33\,\text{mA}$$

Solving for R yields:

$$R=\boxed{600\,\Omega}$$

Remarks: Note that we can also obtain the result in the third step by applying Kirchhoff's loop rule to the parallel branch of the circuit.

85 •• [SSM] In the circuit shown in Figure 25-62, the batteries have negligible internal resistance. Find (*a*) the current in each branch of the circuit, (*b*) the potential difference between point *a* and point *b*, and (*c*) the power supplied by each battery.

Picture the Problem Let I_1 be the current delivered by the left battery, I_2 the current delivered by the right battery, and I_3 the current through the 6.00-Ω resistor, directed down. We can apply Kirchhoff's rules to obtain three equations that we can solve simultaneously for I_1, I_2, and I_3. Knowing the currents in each branch, we can use Ohm's law to find the potential difference between points *a* and *b* and the power delivered by both the sources.

(*a*) Apply Kirchhoff's junction rule at junction *a*:

$$I_{4\Omega}+I_{3\Omega}=I_{6\Omega}\qquad(1)$$

Apply Kirchhoff's loop rule to a loop around the outside of the circuit to obtain:

$$12.0\,\text{V}-(4.00\,\Omega)I_{4\Omega}+(3.00\,\Omega)I_{3\Omega}-12.0\,\text{V}=0$$

or

$$-(4.00\,\Omega)I_{4\Omega}+(3.00\,\Omega)I_{3\Omega}=0\qquad(2)$$

Apply Kirchhoff's loop rule to a loop around the left-hand branch of the circuit to obtain:

$$12.0\,\text{V}-(4.00\,\Omega)I_{4\Omega}-(6.00\,\Omega)I_{6\Omega}=0\qquad(3)$$

Solving equations (1), (2), and (3) simultaneously yields:

$I_{4\Omega}=\boxed{0.667\,\text{A}}$, $I_{3\Omega}=\boxed{0.889\,\text{A}}$,

and

$I_{6\Omega}=\boxed{1.56\,\text{A}}$

(*b*) Apply Ohm's law to find the potential difference between points *a* and *b*:

$$V_{ab} = (6.00\,\Omega)I_{6\Omega} = (6.00\,\Omega)(1.56\,\text{A}) = \boxed{9.36\,\text{V}}$$

(*c*) Express the power delivered by the 12.0-V battery in the left-hand branch of the circuit:

$$P_{\text{left}} = \mathcal{E}\,I_{4\Omega} = (12.0\,\text{V})(0.667\,\text{A}) = \boxed{8.00\,\text{W}}$$

Express the power delivered by the 12.0-V battery in the right-hand branch of the circuit:

$$P_{\text{right}} = \mathcal{E}\,I_{3\Omega} = (12.0\,\text{V})(0.889\,\text{A}) = \boxed{10.7\,\text{W}}$$

86 •• In the circuit shown in Figure 25-63, the batteries have negligible internal resistance. Find (*a*) the current in each branch of the circuit, (*b*) the potential difference between point *a* and point *b*, and (*c*) the power supplied by each battery.

Picture the Problem Let $I_{2\Omega}$ be the current delivered by the 7.00-V battery, $I_{3\Omega}$ the current delivered by the 5-V battery, and $I_{1\Omega}$, directed up, the current through the 1.00-Ω resistor. We can apply Kirchhoff's rules to obtain three equations that we can solve simultaneously for I_1, I_2, and I_3. Knowing the currents in each branch, we can use Ohm's law to find the potential difference between points *a* and *b* and the power delivered by both the sources.

(*a*) Apply Kirchhoff's junction rule at junction *a*:

$$I_{2\Omega} = I_{3\Omega} + I_{1\Omega} \qquad (1)$$

Apply Kirchhoff's loop rule to a loop around the outside of the circuit to obtain:

$$7.00\,\text{V} - (2.00\,\Omega)I_{2\Omega} - (1.00\,\Omega)I_{1\Omega} = 0 \qquad (2)$$

Apply Kirchhoff's loop rule to a loop around the left-hand branch of the circuit to obtain:

$$7.00\,\text{V} - (2.00\,\Omega)I_{2\Omega} - (3.00\,\Omega)I_{3\Omega} + 5.00\,\text{V} = 0$$

or

$$(2.00\,\Omega)I_{2\Omega} + (3.00\,\Omega)I_{3\Omega} = 12.0\,\text{V} \qquad (3)$$

Solve equations (1), (2), and (3) simultaneously to obtain:

$$I_{2\Omega} = \boxed{3.00\,\text{A}},\ I_{3\Omega} = \boxed{2.00\,\text{A}},$$

and

$$I_{1\Omega} = \boxed{1.00\,\text{A}}$$

(*b*) Apply Ohm's law to find the potential difference between points *a* and *b*:

$$V_{ab} = -5.00\,\text{V} + (3.00\,\Omega)I_{3\Omega}$$
$$= -5.00\,\text{V} + (3.00\,\Omega)(2.00\,\text{A})$$
$$= \boxed{1.00\,\text{V}}$$

(*c*) Express the power delivered by the 7.00-V battery:

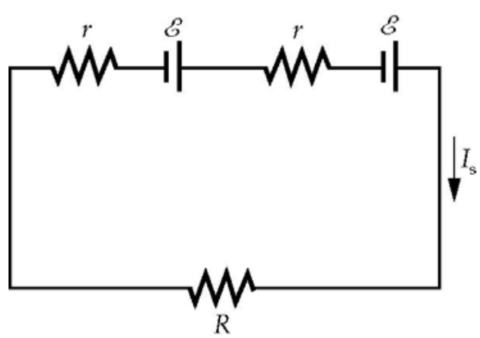
$$P_{7\,\text{V}} = \mathcal{E}\,I_{2\Omega} = (7.00\,\text{V})(3.00\,\text{A})$$
$$= \boxed{21.0\,\text{W}}$$

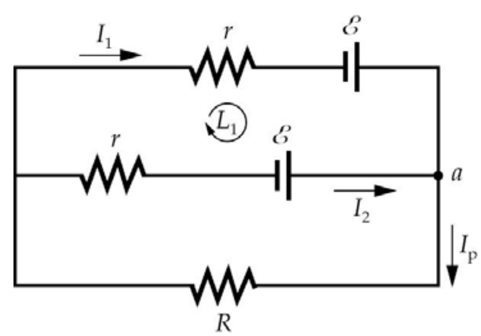
(*c*) Express the power delivered by the 5.00-V battery:

$$P_{5\,\text{V}} = \mathcal{E}\,I_{3\Omega} = (5.00\,\text{V})(2.00\,\text{A})$$
$$= \boxed{10.0\,\text{W}}$$

87 ••• Two identical batteries, each having an emf $\mathcal{E}$ and an internal resistance *r*, can be connected across a resistance *R* with the batteries connected either in series or in parallel. In each situation, determine explicitly whether the power supplied to *R* is greater when *R* is less than *r* or when *R* is greater than *r*.

Picture the Problem The series and parallel connections are shown below. The power supplied to a resistor whose resistance is *R* when the current through it is *I* is given by $P = I^2R$. We can use Kirchhoff's rules to find the currents I_p and I_s and, hence, P_s and P_p, and then use the inequalities $R > r$ and $R < r$ to show that if $R > r$, then $P_p < P_s$ and if $R < r$, then $P_p > P_s$. The steps in the proofs that follow were motivated by having started with the desired outcome and then working backward; for example, assuming that $P_p < P_s$ and showing that $R > r$.





The power P_s supplied to *R* in the series circuit is given by:

$$P_s = I_s^2 R$$

Apply Kirchhoff's loop rule to obtain:

$$-rI_s + \mathcal{E} - rI_s + \mathcal{E} - RI_s = 0$$

Solving for I_s yields:

$$I_s = \frac{2\mathcal{E}}{2r+R}$$

Substitute for I_s and simplify to obtain:

$$P_s = \left(\frac{2\mathcal{E}}{2r+R}\right)^2 R = \frac{4\mathcal{E}^2 R}{(2r+R)^2} \quad (1)$$

The power P_p supplied to R in the series circuit is given by:

$$P_p = I_p^2 R$$

Apply Kirchhoff's junction rule to point a to obtain:

$$I_p = I_1 + I_2$$

Apply Kirchhoff's loop rule to loop 1 to obtain:

$$-rI_1 + \mathcal{E} - \mathcal{E} + rI_2 = 0$$

or

$$I_1 = I_2 = \tfrac{1}{2} I_p$$

Apply Kirchhoff's loop rule to the outer loop to obtain:

$$\mathcal{E} - RI_p - rI_1 = 0$$

or

$$\mathcal{E} - RI_p - \tfrac{1}{2} rI_p = 0 \Rightarrow I_p = \frac{\mathcal{E}}{\tfrac{1}{2}r + R}$$

Substituting for I_p in the expression for P_p yields:

$$P_p = \left(\frac{\mathcal{E}}{\tfrac{1}{2}r+R}\right)^2 R = \frac{\mathcal{E}^2 R}{\left(\tfrac{1}{2}r+R\right)^2} \quad (2)$$

To prove that if $R > r$, then $P_p < P_s$, suppose that $R > r$. Then:

$$3R^2 > 3r^2$$

Adding $r^2 + 4rR + R^2$ to both sides of the inequality yields:

$$r^2 + 4rR + 4R^2 > 4r^2 + 4rR + R^2$$

Factor 4 from the left-hand side and recognize that the right-hand side is the square of a binomial to obtain:

$$4\left(\tfrac{1}{2}r+R\right)^2 > (2r+R)^2$$

or

$$\left(\tfrac{1}{2}r+R\right)^2 > \frac{(2r+R)^2}{4}$$

Taking the reciprocal of both sides of the inequality reverses the sense of the inequality:

$$\frac{1}{\left(\tfrac{1}{2}r+R\right)^2} < \frac{4}{(2r+R)^2}$$

Multiplying both sides of the inequality by $\mathcal{E}^2 R$ yields:

$$\frac{\mathcal{E}^2 R}{\left(\frac{1}{2}r+R\right)^2}<\frac{4\mathcal{E}^2 R}{\left(2r+R\right)^2}$$

For the series combination, the power delivered to the load is greater if $R > r$ and is greatest when $R = 2r$. If $R = r$, both arrangements provide the same power to the load.

To prove that if $R < r$, then $P_{\text{p}} > P_{\text{s}}$, suppose that $R < r$. Then:

$$3R^2 < 3r^2$$

Adding $r^2 + 4rR + R^2$ to both sides of the inequality yields:

$$r^2 + 4rR + 4R^2 < 4r^2 + 4rR + R^2$$

Factor 4 from the left-hand side and recognize that the right-hand side is the square of a binomial to obtain:

$$4\left(\tfrac{1}{2}r+R\right)^2<\left(2r+R\right)^2$$

or

$$\left(\tfrac{1}{2}r+R\right)^2<\frac{\left(2r+R\right)^2}{4}$$

Taking the reciprocal of both sides of the inequality reverses the sense of the inequality:

$$\frac{1}{\left(\frac{1}{2}r+R\right)^2}>\frac{4}{\left(2r+R\right)^2}$$

Multiplying both sides of the inequality by $\mathcal{E}^2 R$ yields:

$$\frac{\mathcal{E}^2 R}{\left(\frac{1}{2}r+R\right)^2}>\frac{4\mathcal{E}^2 R}{\left(2r+R\right)^2}$$

For the parallel combination, the power delivered to the load is greater if $R < r$ and is a maximum when $R = \frac{1}{2}r$.

88 •• The circuit fragment shown in Figure 25-64 is called a *voltage divider*. (*a*) If R_{load} is not attached, show that $V_{\text{out}} = VR_2/(R_1 + R_2)$. (*b*) If $R_1 = R_2 = 10\ \text{k}\Omega$, what is the smallest value of R_{load} that can be used so that V_{out} drops by less than 10 percent from its unloaded value? (V_{out} is measured with respect to ground.)

Picture the Problem Let the current drawn from the source be *I.* We can use Ohm's law in conjunction with Kirchhoff's loop rule to express the output voltage as a function of *V,* R_1, and R_2. In (*b*) we can use the result of (*a*) to express the condition on the output voltages in terms of the effective resistance of the loaded output and the resistances R_1 and R_2.

(*a*) Use Ohm's law to express V_{out} in terms of R_2 and *I*:

$$V_{\text{out}} = IR_2$$

Apply Kirchhoff's loop rule to the circuit to obtain:

$$V - IR_1 - IR_2 = 0 \Rightarrow I = \frac{V}{R_1 + R_2}$$

Substitute for I in the expression for V_{out} to obtain:

$$V_{\text{out}} = \left(\frac{V}{R_1 + R_2}\right) R_2 = \boxed{V\left(\frac{R_2}{R_1 + R_2}\right)}$$

(*b*) Relate the effective resistance of the loaded circuit R_{eff} to R_2 and R_{load}:

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_2} + \frac{1}{R_{\text{load}}}$$

Solving for R_{load} yields:

$$R_{\text{load}} = \frac{R_2 R_{\text{eff}}}{R_2 - R_{\text{eff}}} \qquad (1)$$

Letting V'_{out} represent the output voltage under load, express the condition that V_{out} drops by less than 10 percent of its unloaded value:

$$\frac{V_{\text{out}} - V'_{\text{out}}}{V_{\text{out}}} = 1 - \frac{V'_{\text{out}}}{V_{\text{out}}} < 0.1 \qquad (2)$$

Using the result from (*a*), express V'_{out} in terms of the effective output load R_{eff}:

$$V'_{\text{out}} = V\left(\frac{R_{\text{eff}}}{R_1 + R_{\text{eff}}}\right)$$

Substitute for V_{out} and V'_{out} in equation (2) and simplify to obtain:

$$1 - \frac{\dfrac{R_{\text{eff}}}{R_1 + R_{\text{eff}}}}{\dfrac{R_2}{R_1 + R_2}} < 0.1$$

or

$$1 - \frac{R_{\text{eff}}(R_1 + R_2)}{R_2(R_1 + R_{\text{eff}})} < 0.1$$

Solving for R_{eff} yields:

$$R_{\text{eff}} > \frac{0.9 R_1 R_2}{R_1 + 0.1 R_2}$$

Substitute numerical values and evaluate R_{eff}:

$$R_{\text{eff}} > \frac{(0.90)(10\,\text{k}\Omega)(10\,\text{k}\Omega)}{10\,\text{k}\Omega + (0.10)(10\,\text{k}\Omega)} = 8.18\,\text{k}\Omega$$

Finally, substitute numerical values in equation (1) and evaluate R_{load}:

$$R_{\text{load}} < \frac{(10\,\text{k}\Omega)(8.18\,\text{k}\Omega)}{10\,\text{k}\Omega - 8.18\,\text{k}\Omega} = \boxed{45\,\text{k}\Omega}$$

89 ••• [SSM] For the circuit shown in Figure 25-65, find the potential difference between point *a* and point *b*.

Picture the Problem Let I_1 be the current in the left branch resistor, directed up; let I_3 be the current, directed down, in the middle branch; and let I_2 be the current in the right branch, directed up. We can apply Kirchhoff's rules to find I_3 and then the potential difference between points *a* and *b*.

Relate the potential at *a* to the potential at *b*:

$$V_a - R_4 I_3 - 4.00\,\text{V} = V_b$$

or

$$V_a - V_b = R_4 I_3 + 4.00\,\text{V} \qquad (1)$$

Apply Kirchhoff's junction rule at *a* to obtain:

$$I_1 + I_2 = I_3 \qquad (2)$$

Apply the loop rule to a loop around the outside of the circuit to obtain:

$$2.00\,\text{V} - (1.00\,\Omega)I_1 + (1.00\,\Omega)I_2 - 2.00\,\text{V} + (1.00\,\Omega)I_2 - (1.00\,\Omega)I_1 = 0$$

or

$$I_1 - I_2 = 0 \qquad (3)$$

Apply the loop rule to the left side of the circuit to obtain:

$$2.00\,\text{V} - (1.00\,\Omega)I_1 - (4.00\,\Omega)I_3 - 4.00\,\text{V} - (1.00\,\Omega)I_1 = 0$$

or

$$-(1.00\,\Omega)I_1 - (2.00\,\Omega)I_3 = 1.00\,\text{V} \quad (4)$$

Solve equations (2), (3), and (4) simultaneously to obtain:

$I_1 = -0.200\,\text{A}$, $I_2 = -0.200\,\text{A}$, and $I_3 = -0.400\,\text{A}$

where the minus signs indicate that the currents flow in opposite directions to the directions chosen.

Substitute numerical values in equation (1) to obtain:

$$V_a - V_b = (4.00\,\Omega)(-0.400\,\text{A}) + 4.00\,\text{V} = \boxed{2.40\,\text{V}}$$

Remarks: Note that point *a* is at the higher potential.

90 ••• For the circuit shown in Figure 25-66, find (*a*) the current in each resistor, (*b*) the power supplied by each source of emf, and (*c*) the power delivered to each resistor.

Picture the Problem Let $I_{1,2\Omega}$ be the current in the 1.00-Ω and 2.00-Ω resistors either side of the 4.00-V source, directed to the right; let $I_{2\Omega}$ be the current, directed up, in the middle branch; and let $I_{6\Omega}$ be the current in the 6.00-Ω resistor, directed down. We can apply Kirchhoff's rules to find these currents, the power supplied by each source, and the power dissipated in each resistor.

(*a*) Apply Kirchhoff's junction rule at the top junction to obtain:

$$I_{1,2\Omega} + I_{2\Omega} = I_{6\Omega} \quad (1)$$

Apply Kirchhoff's loop rule to the outside loop of the circuit to obtain:

$$8.00\,\text{V} - (1.00\,\Omega)I_{1,2\Omega} + 4.00\,\text{V} - (2.00\,\Omega)I_{1,2\Omega} - (6.00\,\Omega)I_{6\Omega} = 0$$

or

$$(3.00\,\Omega)I_{1,2\Omega} + (6.00\,\Omega)I_{6\Omega} = 12.0\,\text{V} \quad (2)$$

Apply the loop rule to the inside loop at the left-hand side of the circuit to obtain:

$$8.00\,\text{V} - (1.00\,\Omega)I_{1,2\Omega} + 4.00\,\text{V} - (2.00\,\Omega)I_{1,2\Omega} + (2.00\,\Omega)I_{2\Omega} - 4.00\,\text{V} = 0$$

or

$$8.00\,\text{V} - (3.00\,\Omega)I_{1,2\Omega} + (2.00\,\Omega)I_{2\Omega} = 0 \quad (3)$$

Solve equations (1), (2), and (3) simultaneously to obtain:

$I_{1,2\Omega} = \boxed{2.00\,\text{A}}$, $I_{2\Omega} = \boxed{-1.00\,\text{A}}$, and $I_{6\Omega} = \boxed{1.00\,\text{A}}$

where the minus sign indicates that the current flows downward rather than upward as we had assumed.

(*b*) The power delivered by the 8.00-V source is:

$$P_{8\,\text{V}} = \mathcal{E}_{8\,\text{V}} I_{1,2\Omega} = (8.00\,\text{V})(2.00\,\text{A}) = \boxed{16.0\,\text{W}}$$

The power delivered by the 4.00-V source is:

$$P_{4\,\text{V}} = \mathcal{E}_{4\,\text{V}} I_{2\Omega} = (4.00\,\text{V})(-1.00\,\text{A}) = \boxed{-4.00\,\text{W}}$$

where the minus sign indicates that this source is having current forced through it and is absorbing power.

(*c*) Express the power dissipated in the 1.00-Ω resistor:

$$P_{1\Omega} = I_{1,2\Omega}^2 R_{1\Omega} = (2.00\,\text{A})^2(1.00\,\Omega) = \boxed{4.00\,\text{W}}$$

Express the power dissipated in the 2.00-Ω resistor in the left branch:

$$P_{2\Omega,\text{left}} = I_{1,2\Omega}^2 R_{2\Omega} = (2.00\,\text{A})^2(2.00\,\Omega) = \boxed{8.00\,\text{W}}$$

Express the power dissipated in the 2.00-Ω resistor in the middle branch:

$$P_{2\Omega,\text{middle}} = I_{2\Omega}^2 R_{2\Omega} = (1.00\,\text{A})^2(2.00\,\Omega) = \boxed{2.00\,\text{W}}$$

Express the power dissipated in the 6.00-Ω resistor:

$$P_{6\Omega} = I_{6\Omega}^2 R_{6\Omega} = (1.00\,\text{A})^2(6.00\,\Omega) = \boxed{6.00\,\text{W}}$$

Ammeters and Voltmeters

91 •• [SSM] The voltmeter shown in Figure 25-67 can be modeled as an ideal voltmeter (a voltmeter that has an infinite internal resistance) in parallel with a 10.0 MΩ resistor. Calculate the reading on the voltmeter when (*a*) R = 1.00 kΩ, (*b*) R = 10.0 kΩ, (*c*) R = 1.00 MΩ, (*d*) R = 10.0 MΩ, and (*e*) R = 100 MΩ. (*f*) What is the largest value of R possible if the measured voltage is to be within 10 percent of the *true* voltage (that is, the voltage drop across R without the voltmeter in place)?

Picture the Problem Let I be the current drawn from source and R_{eq} the resistance equivalent to R and 10 MΩ connected in parallel and apply Kirchhoff's loop rule to express the measured voltage V across R as a function of R.

The voltage measured by the voltmeter is given by:

$$V = IR_{eq} \qquad (1)$$

Apply Kirchhoff's loop rule to the circuit to obtain:

$$10.0\,\text{V} - IR_{eq} - I(2R) = 0$$

Solving for I yields:

$$I = \frac{10.0\,\text{V}}{R_{eq} + 2R}$$

Express R_{eq} in terms of R and 10.0-MΩ resistance in parallel with it:

$$\frac{1}{R_{eq}} = \frac{1}{10.0\,\text{M}\Omega} + \frac{1}{R}$$

Solving for R_{eq} yields:

$$R_{eq} = \frac{(10.0\,\text{M}\Omega)R}{R + 10.0\,\text{M}\Omega}$$

Substitute for I in equation (1) and simplify to obtain:

$$V = \left(\frac{10.0\,\text{V}}{R_{eq} + 2R}\right)R_{eq} = \frac{10.0\,\text{V}}{1 + \dfrac{2R}{R_{eq}}}$$

Substitute for R_{eq} and simplify to obtain:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{R + 15.0\,\text{M}\Omega} \qquad (2)$$

(a) Evaluate equation (2) for $R = 1.00\ \text{k}\Omega$:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{1.00\,\text{k}\Omega + 15.0\,\text{M}\Omega} = \boxed{3.3\,\text{V}}$$

(b) Evaluate equation (2) for $R = 10.0\ \text{k}\Omega$:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{10.0\,\text{k}\Omega + 15.0\,\text{M}\Omega} = \boxed{3.3\,\text{V}}$$

(c) Evaluate equation (2) for $R = 1.00\ \text{M}\Omega$:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{1.00\,\text{M}\Omega + 15.0\,\text{M}\Omega} = \boxed{3.1\,\text{V}}$$

(d) Evaluate equation (2) for $R = 10.0\ \text{M}\Omega$:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{10.0\,\text{M}\Omega + 15.0\,\text{M}\Omega} = \boxed{2.0\,\text{V}}$$

(e) Evaluate equation (2) for $R = 100\ \text{M}\Omega$:

$$V = \frac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{100\,\text{M}\Omega + 15.0\,\text{M}\Omega} = \boxed{0.43\,\text{V}}$$

(f) Express the condition that the measured voltage to be within 10 percent of the *true* voltage V_{true}:

$$\frac{V_{true} - V}{V_{true}} = 1 - \frac{V}{V_{true}} < 0.1$$

Substitute for V and V_{true} to obtain:

$$1 - \frac{\dfrac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{R + 15.0\,\text{M}\Omega}}{IR} < 0.1$$

Because $I = 10.0\ \text{V}/3R$:

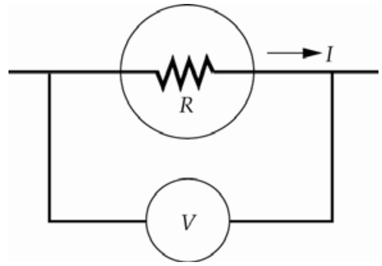
$$1 - \frac{\dfrac{(10.0\,\text{V})(5.0\,\text{M}\Omega)}{R + 15.0\,\text{M}\Omega}}{\dfrac{10.0}{3}\text{V}} < 0.1$$

Solving for R yields:

$$R < \frac{1.5\,\text{M}\Omega}{0.90} = \boxed{1.67\,\text{M}\Omega}$$

92 •• You are given a D'Arsonval galvanometer that will deflect full scale if a current of 50.0 μA runs through the galvanometer. At this current, there is a voltage drop of 0.250 V across the meter. What is the internal resistance of the galvanometer?

Picture the Problem The diagram shows a voltmeter connected in parallel with a galvanometer movement whose internal resistance is *R*. We can apply Kirchhoff's loop rule to express *R* in terms of *I* and *V*.



Apply Kirchhoff's loop rule to the loop that includes the galvanometer movement and the voltmeter:

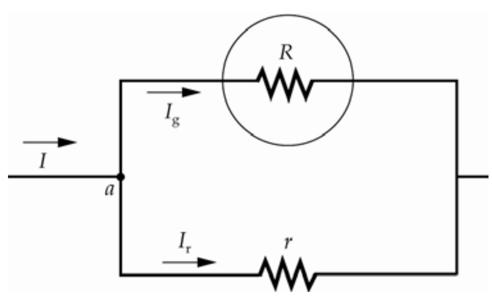
$$V - IR = 0 \Rightarrow R = \frac{V}{I}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{0.250\,\text{V}}{50.0\,\mu\text{A}} = \boxed{5.00\,\text{k}\Omega}$$

93 •• You are given a D'Arsonval galvanometer that will deflect full scale if a current of 50.0 μA runs through the galvanometer. At this current, there is a voltage drop of 0.250 V across the meter. You wish to use this galvanometer to construct an ammeter that can measure currents up to 100 mA. Show that this can be done by placing a resistor in parallel with the meter, and find the value of its resistance.

Picture the Problem When there is a voltage drop of 0.250 V across this galvanometer, the meter reads full scale. The diagram shows the galvanometer movement with a resistor of resistance *r* in parallel. The purpose of this resistor is to limit the current through the movement to I_g = 50.0 μA. We can apply Kirchhoff's loop rule to the circuit fragment containing the galvanometer movement and the shunt resistor to derive an expression for *r*.



Apply Kirchhoff's loop rule to the circuit fragment to obtain:

$$-RI_g + rI_r = 0$$

Apply Kirchhoff's junction rule at point *a* to obtain:

$$I_r = I - I_g$$

Substitute for I_r in the loop equation:

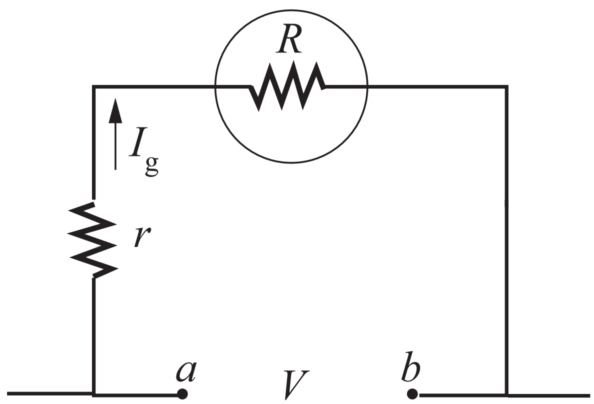
$$-RI_g + r(I - I_g) = 0 \Rightarrow r = \frac{RI_g}{I - I_g}$$

Noting that $RI_g = 0.250$ V, substitute numerical values and evaluate r:

$$r = \frac{0.250\,\text{V}}{100\,\text{mA} - 50.0\,\mu\text{A}} = \boxed{2.5\,\Omega}$$

94 •• You are given a D'Arsonval galvanometer that will deflect full scale if a current of 50.0 μA runs through the galvanometer. At this current, there is a voltage drop of 0.250 V across the meter. You wish to use this galvanometer to construct a voltmeter that can measure potential differences up to 10.0 V. Show that this can be done by placing a large resistance in series with the meter movement, and find the resistance needed.

Picture the Problem The circuit diagram shows a fragment of a circuit in which a resistor of resistance r is connected in series with the meter movement of Problem 92. The purpose of this resistor is to limit the current through the galvanometer movement to 50 μA and to produce a deflection of the galvanometer movement that is a measure of the potential difference V. We can apply Kirchhoff's loop rule to express r in terms of V_g, I_g, and R.



Apply Kirchhoff's loop rule to the circuit fragment to obtain:

$$V - rI_g - RI_g = 0$$

Solving for r yields:

$$r = \frac{V - RI_g}{I_g} = \frac{V}{I_g} - R \qquad (1)$$

Use Ohm's law to relate the current I_g through the galvanometer movement to the potential difference V_g across it:

$$I_g = \frac{V_g}{R} \Rightarrow R = \frac{V_g}{I_g}$$

Use the values for V_g and I_g given in Problem 114 to evaluate R:

$$R = \frac{0.250\,\text{V}}{50.0\,\mu\text{A}} = 5000\,\Omega$$

Substitute numerical values in equation (1) and evaluate r:

$$r = \frac{10.0\,\text{V}}{50.0\,\mu\text{A}} - 5000\,\Omega = \boxed{195\,\text{k}\Omega}$$

Remarks: The total series resistance is the sum of *r* and *R* or 200 kΩ.

RC Circuits

95 • For the circuit shown in Figure 25-68, $C = 6.00\text{-}\mu\text{F}$, $\mathcal{E} = 100$ V and $R = 500\ \Omega$. After having been at contact a for a long time, the switch throw is rotated to contact b. (a) What is the charge on the upper plate capacitor just as the switch throw is moved to contact b? (b) What is the current just after the switch throw is rotated to contact b? (c) What is the time constant of this circuit? (d) How much charge is on the upper plate of the capacitor 6.00 ms after the switch throw is rotated to contact b?

Picture the Problem We can use the definition of capacitance to find the initial charge on the capacitor and Ohm's law to find the initial current in the circuit. We can find the time constant of the circuit using its definition and the charge on the capacitor after 6.00 ms using $Q(t) = Q_0 e^{-t/\tau}$.

(a) Use the definition of capacitance to find the initial charge on the capacitor:

$$Q_0 = CV_0 = (6.00\,\mu\text{F})(100\,\text{V}) = \boxed{600\,\mu\text{C}}$$

(b) Apply Ohm's law to the resistor to obtain:

$$I_0 = \frac{V_0}{R} = \frac{100\,\text{V}}{500\,\Omega} = \boxed{0.200\,\text{A}}$$

(c) Use its definition to find the time constant of the circuit:

$$\tau = RC = (500\,\Omega)(6.00\,\mu\text{F}) = \boxed{3.00\,\text{ms}}$$

(d) Express the charge on the capacitor as a function of time:

$$Q(t) = Q_0 e^{-t/\tau}$$

Substitute numerical values and evaluate Q(6 ms):

$$Q(6.00\,\text{ms}) = (600\,\mu\text{C})e^{-6.00\,\text{ms}/3.00\,ms} = \boxed{81.2\,\mu\text{C}}$$

96 • At $t = 0$ the switch throw in Figure 25-68 is rotated to contact b after having been at contact a for a long time. (a) How much energy is stored in the capacitor at $t = 0$? (b) For $t > 0$, find the energy stored in the capacitor as a function of time, and (c) sketch a plot of the energy stored in the capacitor versus time t.

Picture the Problem (*a*) The initial energy stored in the capacitor is given by $U_0 = \frac{1}{2}C\mathcal{E}^2$. (*b*) The energy stored in the capacitor as a function of time is given by $U(t) = \frac{1}{2}C(V_C(t))^2$ where $V_C(t) = V_0 e^{-t/\tau}$.

(*a*) The initial energy stored in the capacitor is given by:

$$U_0 = \tfrac{1}{2}(6.00\ \mu\text{F})(100\ \text{V})^2 = \boxed{30.0\ \text{mJ}}$$

(*b*) Express the energy stored in the discharging capacitor as a function of time:

$$U(t) = \tfrac{1}{2}C(V_C(t))^2$$

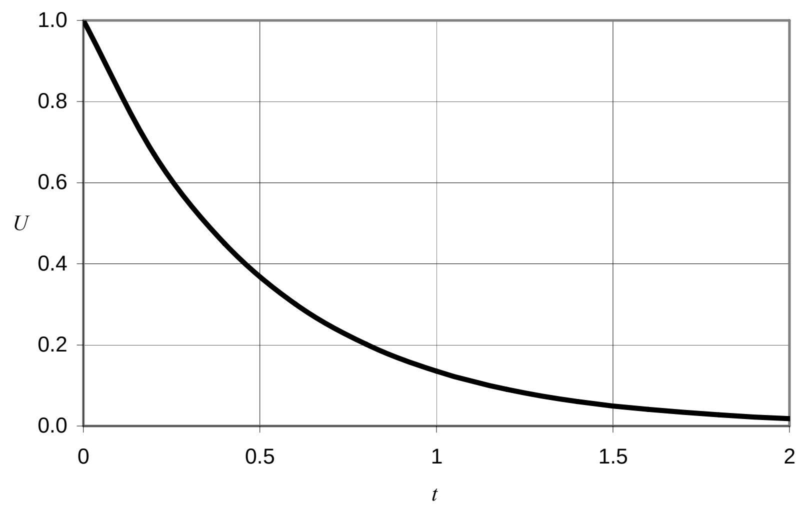
where

$$V_C(t) = \mathcal{E}\, e^{-t/\tau}$$

Substitute for $V_C(t)$ and simplify to obtain:

$$U(t) = \tfrac{1}{2}C\left(\mathcal{E}\, e^{-t/\tau}\right)^2 = \tfrac{1}{2}C\mathcal{E}^2 e^{-2t/\tau} = \boxed{U_0 e^{-2t/\tau}}$$

(*c*) A graph of *U* versus *t* is shown below. *U* is in units of U_0 and *t* is in units of τ.



97 •• **[SSM]** In the circuit in Figure 25-69, the emf equals 50.0 V and the capacitance equals 2.00 μF. Switch *S* is opened after having been closed for a long time, and 4.00 s later the voltage drop across the resistor is 20.0 V. Find the resistance of the resistor.

Picture the Problem We can find the resistance of the circuit from its time constant and use Ohm's law and the expression for the current in a charging *RC* circuit to express τ as a function of time, V_0 and $V(t)$.

Express the resistance of the resistor in terms of the time constant of the circuit:

$$R = \frac{\tau}{C} \qquad (1)$$

Using Ohm's law, express the voltage drop across the resistor as a function of time:

$$V(t) = I(t)R$$

Express the current in the circuit as a function of the elapsed time after the switch is closed:

$$I(t) = I_0 e^{-t/\tau}$$

Substitute for $I(t)$ to obtain:

$$V(t) = I_0 e^{-t/\tau} R = (I_0 R)e^{-t/\tau} = V_0 e^{-t/\tau}$$

Take the natural logarithm of both sides of the equation and solve for τ to obtain:

$$\tau = -\frac{t}{\ln\left[\frac{V(t)}{V_0}\right]}$$

Substitute for τ in equation (1) to obtain:

$$R = -\frac{t}{C\ln\left[\frac{V(t)}{V_0}\right]}$$

Substitute numerical values and evaluate R using the data given for $t = 4.00$ s:

$$R = -\frac{4.00\,\text{s}}{(2.00\,\mu\text{F})\ln\left(\frac{20.0\,\text{V}}{50.0\,\text{V}}\right)} = \boxed{2.18\,\text{M}\Omega}$$

98 •• For the circuit shown in Figure 25-68, $C = 0.120\ \mu$F and $\mathcal{E} = 100$ V. The switch throw is rotated to contact b after having been at contact a for a long time, and 4.00 s later the potential difference across the capacitor is equal to $\frac{1}{2}\mathcal{E}$. What is the value of R?

Picture the Problem We can find the resistance of the circuit from its time constant and use the expression for the charge on a discharging capacitor as a function of time to express τ as a function of time, V_0, and $V(t)$.

Express the effective resistance across the capacitor in terms of the time constant of the circuit:

$$R = \frac{\tau}{C} \qquad (1)$$

Express the voltage across the capacitor as a function of the elapsed time after the switch is closed:

$$V(t) = V_0 e^{-t/\tau}$$

Take the natural logarithm of both sides of the equation and solve for τ to obtain:

$$\tau = -\frac{t}{\ln\left(\frac{V(t)}{V_0}\right)}$$

Substitute for τ in equation (1) to obtain:

$$R = -\frac{t}{C\ln\left(\frac{V(t)}{V_0}\right)}$$

Substitute numerical values and evaluate R:

$$R = -\frac{4.00\,\text{s}}{(0.120\,\mu\text{F})\ln\left(\frac{\frac{1}{2}V_0}{V_0}\right)} = \boxed{48.1\,\text{M}\Omega}$$

99 •• In the circuit in Figure 25-70, the emf equals 6.00 V and has negligible internal resistance. The capacitance equals 1.50 μF and the resistance equals 2.00-MΩ. Switch S has been closed for a long time. Switch S is opened. After a time interval equal to one time constant of the circuit has elapsed, find: (*a*) the charge on the capacitor plate on the right, (*b*) the rate at which the charge is increasing, (*c*) the current, (*d*) the power supplied by the battery, (*e*) the power delivered to the resistor, and (*f*) the rate at which the energy stored in the capacitor is increasing.

Picture the Problem We can use $Q(t) = Q_\text{f}\left(1 - e^{-t/\tau}\right) = C\mathcal{E}\left(1 - e^{-t/\tau}\right)$ to find the charge on the capacitor at $t = \tau$ and differentiate this expression with respect to time to find the rate at which the charge is increasing (the current). The power supplied by the battery is given by $P_\tau = I_\tau \mathcal{E}$ and the power dissipated in the resistor by $P_{R,\tau} = I_\tau^2 R$. In Part (*f*) we can differentiate $U(t) = Q^2(t)/2C$ with respect to time and evaluate the derivative at $t = \tau$ to find the rate at which the energy stored in the capacitor is increasing.

(*a*) Express the charge Q on the capacitor as a function of time:

$$Q(t) = Q_\text{f}\left(1 - e^{-t/\tau}\right) = C\mathcal{E}\left(1 - e^{-t/\tau}\right) \quad (1)$$

where $\tau = RC$.

Evaluate $Q(\tau)$ to obtain:

$$Q(\tau) = (1.50\,\mu\text{F})(6.00\,\text{V})\left(1 - e^{-1}\right) = 5.689\,\mu\text{C} = \boxed{5.69\,\mu\text{C}}$$

(*b*) and (*c*) Differentiate equation (1) with respect to *t* to obtain:

$$\frac{dQ(t)}{dt} = I(t) = I_0 e^{-t/\tau}$$

Apply Kirchhoff's loop rule to the circuit just after the circuit is completed to obtain:

$$\mathcal{E} - RI_0 - V_{C0} = 0$$

Because $V_{C0} = 0$ we have:

$$I_0 = \frac{\mathcal{E}}{R}$$

Substituting for I_0 yields:

$$\frac{dQ(t)}{dt} = I(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}$$

Substitute numerical values and evaluate $\frac{dQ(t)}{dt} = I(t)$:

$$\frac{dQ(t)}{dt} = I(t) = \frac{6.00\,\text{V}}{2.00\,\text{M}\Omega} e^{-1}$$

$$= 1.104\,\mu\text{C/s} = \boxed{1.10\,\mu\text{C/s}}$$

(*d*) Express the power supplied by the battery as the product of its emf and the current drawn from it at $t = \tau$:

$$P(\tau) = I(\tau)\mathcal{E} = (1.104\,\mu\text{A})(6.00\,\text{V})$$

$$= \boxed{6.62\,\mu\text{W}}$$

(*e*) The power dissipated in the resistor is given by:

$$P_R(\tau) = I^2(\tau)R = (1.104\,\mu\text{A})^2(2.00\,\text{M}\Omega)$$

$$= \boxed{2.44\,\mu\text{W}}$$

(*f*) Express the energy stored in the capacitor as a function of time:

$$U(t) = \frac{Q^2(t)}{2C}$$

Differentiate this expression with respect to time to obtain:

$$\frac{dU(t)}{dt} = \frac{1}{2C}\frac{d}{dt}\left[Q^2(t)\right]$$

$$= \frac{1}{2C}(2Q(t))\frac{dQ(t)}{dt}$$

$$= \frac{Q(t)}{C} I(t)$$

Evaluate $\frac{dU(t)}{dt}$ when $t = \tau$ to obtain:

$$\frac{dU(\tau)}{dt} = \frac{5.689\,\mu\text{C}}{1.50\,\mu\text{F}}(1.104\,\mu\text{A})$$

$$= \boxed{4.19\,\mu\text{W}}$$

Remarks: Note that our answer for Part (*f*) is the difference between the power delivered by the battery at $t = \tau$ and the rate at which energy is dissipated in the resistor at the same time.

100 •• A constant charge of 1.00 mC is on the positively charged plate of the 5.00-μF capacitor in the circuit shown in Figure 25-70. Find (*a*) the battery current, and (*b*) the resistances R_1, R_2, and R_3.

Picture the Problem We can apply Kirchhoff's junction rule to find the current in each branch of this circuit and then use the loop rule to obtain equations solvable for R_1, R_2, and R_3.

(*a*) Apply Kirchhoff's junction rule at the junction of the 5.00-μF capacitor and the 10.0-Ω and 50.0-Ω resistors under steady-state conditions:

$$I_{\text{bat}} = I_{10\,\Omega} + 5.00\,\text{A} \qquad (1)$$

Because the potential differences across the 5.00-μF capacitor and the 10.0-Ω resistor are the same:

$$I_{10\Omega} = \frac{V_{10\,\Omega}}{10.0\,\Omega} = \frac{V_{\text{C}}}{10.0\,\Omega}$$

Express the potential difference across the capacitor to its steady-state charge:

$$V_{\text{C}} = \frac{Q_{\text{f}}}{C}$$

Substitute for V_{C} to obtain:

$$I_{10\Omega} = \frac{Q_{\text{f}}}{(10.0\,\Omega)C}$$

Substitute in equation (1) to obtain:

$$I_{\text{bat}} = \frac{Q_{\text{f}}}{(10.0\,\Omega)C} + 5.00\,\text{A}$$

Substitute numerical values and evaluate I_{bat}:

$$I_{\text{bat}} = \frac{1000\,\mu\text{C}}{(10.0\,\Omega)(5.00\,\mu\text{F})} + 5.00\,\text{A} = \boxed{25.0\,\text{A}}$$

(*b*) Use Kirchhoff's junction rule to find the currents $I_{5\,\Omega}$, I_{R3}, and I_{R1}:

$I_{5\Omega} = 10.0\,\text{A}$, $I_{R3} = 15.0\,\text{A}$, and
$I_{R1} = I_{\text{bat}} = 25.0\,\text{A}$

Apply the loop rule to the loop that includes the battery, R_1, and the 50.0-Ω and 5.00-Ω resistors:

$$310\,\text{V} - (25.0\,\text{A})R_1 - (5.00\,\text{A})(50.0\,\Omega) - (10.0\,\text{A})(5.00\,\Omega) = 0$$

Solve for R_1 to obtain:

$$R_1 = \boxed{0.400\,\Omega}$$

Apply the loop rule to the loop that includes the battery, R_1, the 10.0-Ω resistor and R_3:

$$310\,\text{V} - (25.0\,\text{A})(0.400\,\Omega) - (20.0\,\text{A})(10.0\,\Omega) - (15.0\,\text{A})R_3 = 0$$

Solving for R_3 yields:

$$R_3 = \boxed{6.67\,\Omega}$$

Apply the loop rule to the loop that includes the 10.0-Ω and 50.0-Ω resistors and R_2:

$$-(20.0\,\text{A})(10.0\,\Omega) - (5.00\,\text{A})R_2 + (5.00\,\text{A})(50.0\,\Omega) = 0$$

Solving for R_2 yields:

$$R_2 = \boxed{10.0\,\Omega}$$

101 •• Show that Equation 25-39 can be rearranged and written as $\dfrac{dQ}{\mathcal{E}C - Q} = \dfrac{dt}{RC}$. Integrate this equation to derive the solution given by Equation 25-40.

Picture the Problem We can separate the variables in Equation 25-39 to obtain the equation given in the problem statement. Integrating this differential equation will yield Equation 25-40.

Solve Equation 25-39 for dQ/dt to obtain:

$$\frac{dQ}{dt} = \frac{\mathcal{E}C - Q}{RC}$$

Separate the variables to obtain:

$$\boxed{\frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}}$$

Integrate dQ' from 0 to Q and dt' from 0 to t:

$$\int_0^Q \frac{dQ'}{\mathcal{E}C - Q'} = \frac{1}{RC}\int_0^t dt'$$

and

$$\ln\left(\frac{\mathcal{E}C}{\mathcal{E}C - Q}\right) = \frac{t}{RC}$$

Transform from logarithmic to exponential form to obtain:

$$\frac{\mathcal{E}C}{\mathcal{E}C-Q}=e^{\frac{t}{RC}}$$

Solve for Q to obtain Equation 25-40:

$$Q=\mathcal{E}C\left(1-e^{-t/RC}\right)=\boxed{Q_{\mathrm{f}}\left(1-e^{-t/RC}\right)}$$

102 •• Switch S, shown in Figure 25-71, is closed after having been open for a long time. (*a*) What is the initial value of the battery current just after switch S is closed? (*b*) What is the battery current a long time after switch S is closed? (*c*) What are the charges on the plates of the capacitors a long time after switch S is closed? (*d*) Switch S is reopened. What are the charges on the plates of the capacitors a long time after switch *S* is reopened?

Picture the Problem When the switch is closed, the initial potential differences across the capacitors are zero (they have no charge) and the resistors in the bridge portion of the circuit are in parallel. When a long time has passed, the current through the capacitors will be zero and the resistors will be in series. In both cases, the application of Kirchhoff's loop rule to the entire circuit will yield the current in the circuit. To find the final charges on the capacitors we can use the definition of capacitance and apply Kirchhoff's loop rule to the loops containing two resistors and a capacitor to find the potential differences across the capacitors.

(*a*) Apply Kirchhoff's loop rule to the circuit immediately after the switch is closed:

$$50.0\,\mathrm{V}-I_0(10.0\,\Omega)-I_0R_{\mathrm{eq}}=0$$

Solving for I_0 yields:

$$I_0=\frac{50.0\,\mathrm{V}}{10.0\,\Omega+R_{\mathrm{eq}}}$$

Find the equivalent resistance of 15.0 Ω, 12.0 Ω, and 15.0 Ω in parallel:

$$\frac{1}{R_{\mathrm{eq}}}=\frac{1}{15.0\,\Omega}+\frac{1}{12.0\,\Omega}+\frac{1}{15.0\,\Omega}$$

and

$$R_{\mathrm{eq}}=4.615\,\Omega$$

Substitute for R_{eq} and evaluate I_0:

$$I_0=\frac{50.0\,\mathrm{V}}{10.0\,\Omega+4.615\,\Omega}=\boxed{3.42\,\mathrm{A}}$$

(*b*) Apply Kirchhoff's loop rule to the circuit a long time after the switch is closed:

$$50.0\,\mathrm{V}-I_\infty(10.0\,\Omega)-I_\infty R_{\mathrm{eq}}=0$$

Solving for I_∞ yields:

$$I_\infty = \frac{50.0\,\text{V}}{10.0\,\Omega + R_{\text{eq}}}$$

Find the equivalent resistance of 15.0 Ω, 12.0 Ω, and 15.0 Ω in series:

$$R_{\text{eq}} = 15.0\,\Omega + 12.0\,\Omega + 15.0\,\Omega = 42.0\,\Omega$$

Substitute for R_{eq} and evaluate I_∞:

$$I_\infty = \frac{50.0\,\text{V}}{10.0\,\Omega + 42.0\,\Omega} = \boxed{0.962\,\text{A}}$$

(*c*) Using the definition of capacitance, express the charge on the capacitors in terms of their final potential differences:

$$Q_{10\,\mu\text{F}} = C_{10\,\mu\text{F}} V_{10\,\mu\text{F}} \qquad (1)$$

and

$$Q_{5\,\mu\text{F}} = C_{5\,\mu\text{F}} V_{5\,\mu\text{F}} \qquad (2)$$

Apply Kirchhoff's loop rule to the loop containing the 15.0-Ω and 12.0-Ω resistors and the 10.0 μF capacitor to obtain:

$$V_{10\,\mu\text{F}} - (15.0\,\Omega)I_\infty - (12.0\,\Omega)I_\infty = 0$$

Solving for $V_{10\,\mu\text{F}}$ yields:

$$V_{10\,\mu\text{F}} = (27.0\,\Omega)I_\infty$$

Substitute numerical values in equation (1) and evaluate $Q_{10\,\mu\text{F}}$:

$$Q_{10\,\mu\text{F}} = (10.0\,\mu\text{F})(27.0\,\Omega)(0.962\,\text{A}) = \boxed{260\,\mu\text{C}}$$

Apply Kirchhoff's loop rule to the loop containing the 15.0-Ω and 12.0-Ω resistors and the 5.00 μF capacitor to obtain:

$$V_{5\,\mu\text{F}} - (15.0\,\Omega)I_\infty - (12.0\,\Omega)I_\infty = 0$$

Solve for $V_{5\,\mu\text{F}}$:

$$V_{5\,\mu\text{F}} = (27.0\,\Omega)I_\infty$$

Substitute numerical values in equation (2) and evaluate $Q_{5\,\mu\text{F}}$:

$$Q_{5\,\mu\text{F}} = (5.00\,\mu\text{F})(27.0\,\Omega)(0.962\,\text{A}) = \boxed{130\,\mu\text{C}}$$

(*d*) The charges on the plates of the capacitors a long time after switch S is reopened will be 0.

103 ••• In the circuit shown in Figure 25-72, switch S has been open for a long time. At time $t = 0$ the switch is then closed. (*a*) What is the battery current just after switch S is closed? (*b*) What is the battery current a long time after switch S is closed? (*c*) What is the current in the 600-Ω resistor as a function of time?

Picture the Problem Let $R_1 = 200\ \Omega$, $R_2 = 600\ \Omega$, I_1 and I_2 their currents, and I_3 the current into the capacitor. We can apply Kirchhoff's loop rule to find the initial battery current I_0 and the battery current I_∞ a long time after the switch is closed. In Part (*c*) we can apply both the loop and junction rules to obtain equations that we can use to obtain a linear differential equation with constant coefficients describing the current in the 600-Ω resistor as a function of time. We can solve this differential equation by assuming a solution of a given form, differentiating this assumed solution and substituting it and its derivative in the differential equation. Equating coefficients, requiring the solution to hold for all values of the assumed constants, and invoking an initial condition will allow us to find the constants in the assumed solution.

(*a*) Apply Kirchhoff's loop rule to the circuit at the instant the switch is closed:

$$\mathcal{E} - (200\,\Omega)I_0 - V_{C0} = 0$$

Because the capacitor is initially uncharged:

$$V_{C0} = 0$$

Solve for and evaluate I_0:

$$I_0 = \frac{\mathcal{E}}{200\,\Omega} = \frac{50.0\,\text{V}}{200\,\Omega} = \boxed{0.250\,\text{A}}$$

(*b*) Apply Kirchhoff's loop rule to the circuit after a long time has passed:

$$50.0\,\text{V} - (200\,\Omega)I_\infty - (600\,\Omega)I_\infty = 0$$

Solve for I_∞ to obtain:

$$I_\infty = \frac{50.0\,\text{V}}{800\,\Omega} = \boxed{62.5\,\text{mA}}$$

(*c*) Apply the junction rule at the junction between the 200-Ω resistor and the capacitor to obtain:

$$I_1 = I_2 + I_3 \qquad (1)$$

Apply the loop rule to the loop containing the source, the 200-Ω resistor and the capacitor to obtain:

$$\mathcal{E} - R_1 I_1 - \frac{Q}{C} = 0 \qquad (2)$$

Apply the loop rule to the loop containing the 600-Ω resistor and the capacitor to obtain:

$$\frac{Q}{C} - R_2 I_2 = 0 \qquad (3)$$

Differentiate equation (2) with respect to time to obtain:

$$\frac{d}{dt}\left[\mathcal{E} - R_1 I_1 - \frac{Q}{C}\right] = 0 - R_1\frac{dI_1}{dt} - \frac{1}{C}\frac{dQ}{dt}$$

$$= -R_1\frac{dI_1}{dt} - \frac{1}{C}I_3 = 0$$

or

$$R_1\frac{dI_1}{dt} = -\frac{1}{C}I_3 \qquad (4)$$

Differentiate equation (3) with respect to time to obtain:

$$\frac{d}{dt}\left[\frac{Q}{C} - R_2 I_2\right] = \frac{1}{C}\frac{dQ}{dt} - R_2\frac{dI_2}{dt} = 0$$

or

$$R_2\frac{dI_2}{dt} = \frac{1}{C}I_3 \qquad (5)$$

Using equation (1), substitute for I_3 in equation (5) to obtain:

$$\frac{dI_2}{dt} = \frac{1}{R_2 C}\left(I_1 - I_2\right) \qquad (6)$$

Solve equation (2) for I_1:

$$I_1 = \frac{\mathcal{E} - Q/C}{R_1} = \frac{\mathcal{E} - R_2 I_2}{R_1}$$

Substitute for I_1 in equation (6) and simplify to obtain the differential equation for I_2:

$$\frac{dI_2}{dt} = \frac{1}{R_2 C}\left(\frac{\mathcal{E} - R_2 I_2}{R_1} - I_2\right)$$

$$= \frac{\mathcal{E}}{R_1 R_2 C} - \left(\frac{R_1 + R_2}{R_1 R_2 C}\right) I_2$$

To solve this linear differential equation with constant coefficients we can assume a solution of the form:

$$I_2(t) = a + be^{-t/\tau} \qquad (7)$$

Differentiate $I_2(t)$ with respect to time to obtain:

$$\frac{dI_2}{dt} = \frac{d}{dt}\left[a + be^{-t/\tau}\right] = -\frac{b}{\tau}e^{-t/\tau}$$

Substitute for I_2 and dI_2/dt to obtain:

$$-\frac{b}{\tau}e^{-t/\tau} = \frac{\varepsilon}{R_1R_2C} - \left(\frac{R_1+R_2}{R_1R_2C}\right)\left(a + be^{-t/\tau}\right)$$

Equate coefficients of $e^{-t/\tau}$ to obtain:

$$\tau = \frac{R_1R_2C}{R_1+R_2}$$

Requiring the equation to hold for all values of a yields:

$$a = \frac{\varepsilon}{R_1+R_2}$$

If I_2 is to be zero when $t = 0$:

$$0 = a + b \Rightarrow b = -a = -\frac{\varepsilon}{R_1+R_2}$$

Substitute in equation (7) to obtain:

$$I_2(t) = \frac{\varepsilon}{R_1+R_2} - \frac{\varepsilon}{R_1+R_2}e^{-t/\tau}$$
$$= \frac{\varepsilon}{R_1+R_2}\left(1 - e^{-t/\tau}\right)$$

where

$$\tau = \frac{R_1R_2C}{R_1+R_2} = \frac{(200\,\Omega)(600\,\Omega)(5.00\,\mu\text{F})}{200\,\Omega + 600\,\Omega}$$
$$= 0.750\,\text{ms}$$

Substitute numerical values and evaluate $I_2(t)$:

$$I_2(t) = \frac{50.0\,\text{V}}{200\,\Omega + 600\,\Omega}\left(1 - e^{-t/0.750\,\text{ms}}\right)$$
$$= \boxed{(62.5\,\text{mA})\left(1 - e^{-t/0.750\,\text{ms}}\right)}$$

104 ••• In the circuit shown in Figure 25-72, switch S has been open for a long time. At time t =0 the switch is then closed. (*a*) What is the battery current just after switch S is closed? (*b*) What is the battery current a long time after switch S is closed? (*c*) The switch has been closed for a long time. At t =0 the switch is then opened. Find the current through the 600-kΩ resistor as a function of time.

Picture the Problem Let R_1 represent the 1.20-MΩ resistor and R_2 the 600-kΩ resistor. Immediately after switch S is closed, the capacitor has zero charge and so the potential difference across it (and the 600 kΩ-resistor) is zero. A long time after the switch is closed, the capacitor will be fully charged and the potential difference across it will be given by both Q/C and $I_\infty R_2$. When the switch is opened after having been closed for a long time, both the source and the

1.20-MΩ resistor will be out of the circuit and the fully charged capacitor will discharge through R_2. We can use Kirchhoff's loop to find the currents drawn from the source immediately after the switch is closed and a long time after the switch is closed, as well as the current in the RC circuit when the switch is again opened and the capacitor discharges through R_2.

(*a*) Apply Kirchhoff's loop rule to the circuit immediately after the switch is closed to obtain:

$$\mathcal{E} - I_0R_1 - V_{C0} = 0$$

or, because $V_{C0} = 0$,

$$\mathcal{E} - I_0R_1 = 0 \Rightarrow I_0 = \frac{\mathcal{E}}{R_1}$$

Substitute numerical values and evaluate I_0:

$$I_0 = \frac{50.0\,\text{V}}{1.20\,\text{M}\Omega} = \boxed{41.7\,\mu\text{A}}$$

(*b*) Apply Kirchhoff's loop rule to the circuit a long time after the switch is closed to obtain:

$$\mathcal{E} - I_\infty R_1 - I_\infty R_2 = 0 \Rightarrow I_\infty = \frac{\mathcal{E}}{R_1 + R_2}$$

Substitute numerical values and evaluate I_∞:

$$I_\infty = \frac{50.0\,\text{V}}{1.20\,\text{M}\Omega + 600\,\text{k}\Omega} = \boxed{27.8\,\mu\text{A}}$$

(*c*) Apply Kirchhoff's loop rule to the RC circuit sometime after the switch is opened and solve for $I(t)$ to obtain:

$$V_C(t) - R_2I(t) = 0 \Rightarrow I(t) = \frac{V_C(t)}{R_2}$$

Substituting for $V_C(t)$ yields:

$$I(t) = \frac{V_{C\infty}}{R_2}e^{-t/\tau} = I_\infty e^{-t/\tau}$$

where $\tau = R_2C$.

Substitute numerical values to obtain:

$$I(t) = (27.8\,\mu\text{A})e^{-t/(600\,\text{k}\Omega)(2.50\,\mu\text{F})}$$
$$= \boxed{(27.8\,\mu\text{A})e^{-t/1.50\,\text{s}}}$$

105 ••• [SSM] In the circuit shown in Figure 25-74, the capacitor has a capacitance of 2.50 μF and the resistor has a resistance of 0.500 MΩ. Before the switch is closed, the potential drop across the capacitor is 12.0 V, as shown. Switch S is closed at $t = 0$. (*a*) What is the current immediately after switch S is closed? (*b*) At what time t is the voltage across the capacitor 24.0 V?

Picture the Problem We can apply Kirchhoff's loop rule to the circuit immediately after the switch is closed in order to find the initial current I_0. We

can find the time at which the voltage across the capacitor is 24.0 V by again applying Kirchhoff's loop rule to find the voltage across the resistor when this condition is satisfied and then using the expression $I(t)=I_0e^{-t/\tau}$ for the current through the resistor as a function of time and solving for t.

(*a*) Apply Kirchhoff's loop rule to the circuit immediately after the switch is closed:

$$\mathcal{E}-12.0\,\text{V}-I_0R=0$$

Solving for I_0 yields:

$$I_0=\frac{\mathcal{E}-12.0\,\text{V}}{R}$$

Substitute numerical values and evaluate I_0:

$$I_0=\frac{36.0\,\text{V}-12.0\,\text{V}}{0.500\,\text{M}\Omega}=\boxed{48.0\,\mu\text{A}}$$

(*b*) Apply Kirchhoff's loop rule to the circuit when V_C = 24.0 V and solve for V_R:

$$36.0\,\text{V}-24.0\,\text{V}-I(t)R=0$$

and

$$I(t)R=12.0\,\text{V}$$

Express the current through the resistor as a function of I_0 and τ:

$I(t)=I_0e^{-t/\tau}$ where $\tau=RC$.

Substitute to obtain:

$$RI_0e^{-t/\tau}=12.0\,\text{V}\Rightarrow e^{-t/\tau}=\frac{12.0\,\text{V}}{RI_0}$$

Take the natural logarithm of both sides of the equation to obtain:

$$-\frac{t}{\tau}=\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)$$

Solving for t yields:

$$t=-\tau\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)=-RC\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)$$

Substitute numerical values and evaluate t:

$$t=-(0.500\,\text{M}\Omega)(2.50\,\mu\text{F})\ln\left[\frac{12.0\,\text{V}}{(0.500\,\text{M}\Omega)(48.0\,\mu\text{A})}\right]=\boxed{0.866\,\text{s}}$$

106 ••• Repeat Problem 105 if the initial polarity of the capacitor opposite to that shown in Figure 25-74.

Picture the Problem We can apply Kirchhoff's loop rule to the circuit immediately after the switch is closed in order to find the initial current I_0. We can find the time at which the voltage across the capacitor is 24.0 V by again applying Kirchhoff's loop rule to find the voltage across the resistor when this condition is satisfied and then using the expression $I(t)=I_0e^{-t/\tau}$ for the current through the resistor as a function of time and solving for t.

(*a*) Apply Kirchhoff's loop rule to the circuit immediately after the switch is closed:

$$\mathcal{E}+12.0\,\text{V}-I_0R=0 \Rightarrow I_0=\frac{\mathcal{E}+12.0\,\text{V}}{R}$$

Substitute numerical values and evaluate I_0:

$$I_0=\frac{36.0\,\text{V}+12.0\,\text{V}}{0.500\,\text{M}\Omega}=\boxed{96.0\,\mu\text{A}}$$

(*b*) Apply Kirchhoff's loop rule to the circuit when V_C = 24.0 V and solve for V_R:

$$36.0\,\text{V}-24.0\,\text{V}-I(t)R=0$$

and

$$I(t)R=12.0\,\text{V}$$

Express the current through the resistor as a function of I_0 and τ:

$$I(t)=I_0e^{-t/\tau} \text{ where } \tau=RC.$$

Substitute for $I(t)$ to obtain:

$$RI_0e^{-t/\tau}=12.0\,\text{V} \Rightarrow e^{-t/\tau}=\frac{12.0\,\text{V}}{RI_0}$$

Taking the natural logarithm of both sides of the equation yields:

$$-\frac{t}{\tau}=\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)$$

Solving for t yields:

$$t=-\tau\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)=-RC\ln\left(\frac{12.0\,\text{V}}{RI_0}\right)$$

Substitute numerical values and evaluate t:

$$t=-(0.500\,\text{M}\Omega)(2.50\,\mu\text{F})\ln\left[\frac{12.0\,\text{V}}{(0.500\,\text{M}\Omega)(96.0\,\mu\text{A})}\right]=\boxed{1.73\,\text{s}}$$

General Problems

107 •• **[SSM]** In Figure 25-75, R_1 = 4.00 Ω, R_2 = 6.00 Ω, R_3 = 12.0 Ω, and the battery emf is 12.0 V. Denote the currents through these resistors as I_1, I_2 and

I_3, respectively, (*a*) Decide which of the following inequalities holds for this circuit. Explain your answer conceptually. (1) $I_1 > I_2 > I_3$, (2) $I_2 = I_3$, (3) $I_3 > I_2$, (4) None of the above (*b*) To verify that your answer to Part (*a*) is correct, calculate all three currents.

Determine the Concept We can use Kirchhoff's rules in Part (*b*) to confirm our choices in Part (*a*).

(*a*) 1. The potential drops across R_2 and R_3 are equal, so $I_2 > I_3$. The current in R_1 equals the sum of the currents I_2 and I_3, so I_1 is greater than either I_2 or I_3.

(*b*) Apply Kirchhoff's junction rule to obtain:

$$I_1 - I_2 - I_3 = 0 \quad (1)$$

Applying Kirchhoff's loop rule in the clockwise direction to the loop defined by the two resistors in parallel yields:

$$R_3 I_3 - R_2 I_2 = 0$$

or

$$0I_1 - R_2 I_2 + R_3 I_3 = 0 \quad (2)$$

Apply Kirchhoff's loop rule in the clockwise direction to the loop around the perimeter of the circuit to obtain:

$$\mathcal{E} - R_1 I_1 - R_2 I_2 = 0$$

or

$$R_1 I_1 + R_2 I_2 - 0I_3 = \mathcal{E} \quad (3)$$

Substituting numerical values in equations (1), (2), and (3) yields:

$$I_1 - I_2 - I_3 = 0$$
$$0I_1 - (6.00\,\Omega)I_2 + (12.0\,\Omega)I_3 = 0$$
$$(4.00\,\Omega)I_1 + (6.00\,\Omega)I_2 - 0I_3 = 12.0\text{ V}$$

Solve this system of three equations in three unknowns for the currents in the branches of the circuit to obtain:

$I_1 = \boxed{1.50\text{ A}}$, $I_2 = \boxed{1.00\text{ A}}$, and $I_3 = \boxed{0.50\text{ A}}$, confirming our choice in Part (*a*).

108 •• A 120-V, 25.0-W light bulb is connected in series with a 120-V, 100-W light bulb and a potential difference of 120 V is placed across the combination. Assume the bulbs have constant resistance. (*a*) Which bulb should be brighter under these conditions? Explain your answer conceptually. *Hint: What does the phrase "25.0-W light bulb" mean? That is, under what conditions is 25-W of power delivered to the bulb?* (*b*) Determine the power delivered to each bulb under these conditions. Do your results support your answer to Part (*a*)?

Picture the Problem In Part (*b*) we need to find the current drawn by the series combination of the two light bulbs in order to calculate the actual power output of each bulb. To find this current, we first need to determine the resistances of the bulbs. These resistances are given by $R = \mathcal{E}^2/P$, one of the three forms of Equation 25-14.

(*a*) The 25-W bulb will be brighter. The more power delivered to a bulb the brighter the bulb. The resistance of the 25-W bulb is 4 times greater than that of the 100-W bulb, and in the series combination, the same current *I* flows through the bulbs. Hence, $I^2R_{25} > I^2R_{100}$.

(*b*) The actual power output of each bulb is given by:

$$P_{25} = I^2R_{25} \quad (1)$$

and

$$P_{100} = I^2R_{100} \quad (2)$$

We need to find the current drawn by the series combination of the two light bulbs. To find this current, we first need to determine the resistances of the bulbs. There resistances are given by:

$$R_{25} = \frac{\mathcal{E}^2}{P_{25}} = \frac{(120\ \text{V})^2}{25\ \text{W}} = 576\Omega$$

and

$$R_{100} = \frac{\mathcal{E}^2}{P_{100}} = \frac{(120\ \text{V})^2}{100\ \text{W}} = 144\Omega$$

Use Ohm's law to express the current drawn by the series combination of the light bulbs:

$$I = \frac{\mathcal{E}}{R_{\text{eq}}} = \frac{\mathcal{E}}{R_{25} + R_{100}}$$

Substitute numerical values and evaluate *I*:

$$I = \frac{120\ \text{V}}{576\ \Omega + 144\ \Omega} = 0.1667\ \text{A}$$

Substitute numerical values in equations (1) and (2) and evaluate P_{25} and P_{100}:

$$P_{25} = (0.1667\ \text{A})^2(576\ \Omega) = \boxed{16\ \text{W}}$$

and

$$P_{100} = (0.1667\ \text{A})^2(144\ \Omega) = \boxed{4\ \text{W}}$$

confirming our choice in Part (*a*).

109 •• The circuit shown in Figure 25-76 is a *Wheatstone bridge*, and the variable resistor is being used as a slide-wire potentiometer. The resistance R_0 is known. This "bridge" is used to determine an unknown resistance R_x. The resistances R_1 and R_2 comprise a wire 1.00 m long. Point *a* is a sliding contact that is moved along the wire to vary these resistances. Resistance R_1 is proportional to the distance from the left end of the wire (labeled 0.00 cm) to point *a*, and R_2 is proportional to the distance from point *a* to the right end of the wire (labeled

100 cm). The sum of R_1 and R_2 remains constant. When points *a* and *b* are at the same potential, there is no current in the galvanometer and the bridge is said to be balanced. (Because the galvanometer is used to detect the absence of a current, it is called a *null detector*.) If the fixed resistance $R_0 = 200\ \Omega$, find the unknown resistance R_x if (*a*) the bridge balances at the 18.0-cm mark, (*b*) the bridge balances at the 60.0-cm mark, and (*c*) the bridge balances at the 95.0-cm mark.

Picture the Problem Let the current flowing through the galvanometer be I_G. By applying Kirchhoff's rules to the loops including 1) R_1, the galvanometer, and R_x, and 2) R_2, the galvanometer, and R_0, we can obtain two equations relating the unknown resistance to R_1, R_2 and R_0. Using $R = \rho L/A$ will allow us to express R_x in terms of the length of wire L_1 that corresponds to R_1 and the length of wire L_2 that corresponds to R_2.

Apply Kirchhoff's loop rule to the loop that includes R_1, the galvanometer, and R_x to obtain:

$$-R_1I_1 + R_xI_2 = 0 \quad (1)$$

Apply Kirchhoff's loop rule to the loop that includes R_2, the galvanometer, and R_0 to obtain:

$$-R_2(I_1 - I_G) + R_0(I_2 + I_G) = 0 \quad (2)$$

When the bridge is balanced, $I_G = 0$ and equations (1) and (2) become:

$$R_1I_1 = R_xI_2 \quad (3)$$

and

$$R_2I_1 = R_0I_2 \quad (4)$$

Divide equation (3) by equation (4) and solve for R_x to obtain:

$$R_x = R_0\frac{R_1}{R_2} \quad (5)$$

Express R_1 and R_2 in terms of their lengths, cross-sectional areas, and the resistivity of their wire:

$$R_1 = \rho\frac{L_1}{A} \text{ and } R_2 = \rho\frac{L_2}{A}$$

Substitute in equation (5) to obtain:

$$R_x = R_0\frac{L_1}{L_2}$$

(*a*) When the bridge balances at the 18.0-cm mark, $L_1 = 18.0$ cm, $L_2 = 82.0$ cm and:

$$R_x = (200\,\Omega)\frac{18.0\,\text{cm}}{82.0\,\text{cm}} = \boxed{43.9\,\Omega}$$

(*b*) When the bridge balances at the 60.0-cm mark, L_1 = 60.0 cm, L_2 = 40.0 cm and:

$$R_x = (200\,\Omega)\frac{60.0\,\text{cm}}{40.0\,\text{cm}} = \boxed{300\,\Omega}$$

(*c*) When the bridge balances at the 95.0-cm mark, L_1 = 95.0 cm, L_2 = 5.0 cm and:

$$R_x = (200\,\Omega)\frac{95.0\,\text{cm}}{5.0\,\text{cm}} = \boxed{3.8\,\text{k}\Omega}$$

110 •• For the Wheatstone bridge in Problem 109, suppose the bridge balances at the 98.0-cm mark. (*a*) What is the unknown resistance? (*b*) What is the percentage error in the measured value of R_x if there is an error of 2.00 mm in the location of the balance point? (*c*) To what value should R_0 be changed to so that the balance point for this unknown resistor will be nearer the 50.0-cm mark? (*d*) If the balance point is at the 50.0-cm mark, what is the percentage error in the measured value of R_x if there is an error of 2.00 mm in the location of the balance point?

Picture the Problem Let the current flowing through the galvanometer by I_G. By applying Kirchhoff's rules to the loops including 1) R_1, the galvanometer, and R_x, and 2) R_2, the galvanometer, and R_0, we can obtain two equations relating the unknown resistance to R_1, R_2 and R_0. Using $R = \rho L/A$ will allow us to express R_x in terms of the length of wire L_1 that corresponds to R_1 and the length of wire L_2 that corresponds to R_2. To find the effect of an error of 2.00 mm in the location of the balance point we can use the relationship $\Delta R_x = (dR_x/dL)\Delta L$ to determine ΔR_x and then divide by $R_x = R_0\,L/(1-L)$ to find the fractional change (error) in R_x resulting from a given error in the determination of the balance point.

Apply Kirchhoff's loop rule to the loop that includes R_1, the galvanometer, and R_x to obtain:

$$-R_1I_1 + R_xI_2 = 0 \quad (1)$$

Apply Kirchhoff's loop rule to the loop that includes R_2, the galvanometer, and R_0 to obtain:

$$-R_2(I_1 - I_G) + R_0(I_2 + I_G) = 0 \quad (2)$$

When the bridge is balanced, $I_G = 0$ and equations (1) and (2) become:

$$R_1I_1 = R_xI_2 \quad (3)$$

and

$$R_2I_1 = R_0I_2 \quad (4)$$

Divide equation (3) by equation (4) and solve for R_x to obtain:

$$R_x = R_0\frac{R_1}{R_2} \quad (5)$$

Express R_1 and R_2 in terms of their lengths, cross-sectional areas, and the resistivity of their wire:

$$R_1 = \rho\frac{L_1}{A} \text{ and } R_2 = \rho\frac{L_2}{A}$$

Substitute in equation (5) to obtain:

$$R_x = R_0\frac{L_1}{L_2} \qquad (6)$$

(*a*) When the bridge balances at the 98.0-cm mark, $L_1 = 98.0$ cm, $L_2 = 2.0$ cm and:

$$R_x = (200\,\Omega)\frac{98.0\,\text{cm}}{2.0\,\text{cm}} = \boxed{9.8\,\text{k}\Omega}$$

(*b*) Express R_x in terms of the distance to the balance point:

$$R_x = R_0\frac{L}{1-L}$$

Express the error ΔR_x in R_x resulting from an error ΔL in L:

$$\Delta R_x = \frac{dR_x}{dL}\Delta L = R_0\frac{d}{dL}\left[\frac{L}{1-L}\right]\Delta L$$

$$= R_0\frac{1}{(1-L)^2}\Delta L$$

Divide ΔR_x by R_x to obtain:

$$\frac{\Delta R_x}{R_x} = \frac{R_0\dfrac{1}{(1-L)^2}\Delta L}{R_0\dfrac{L}{1-L}} = \frac{1}{1-L}\frac{\Delta L}{L}$$

Evaluate $\Delta R_x/R_x$ for $L = 98.0$ cm and $\Delta L = 2.00$ mm:

$$\frac{\Delta R_x}{R_x} = \left(\frac{1.00\,\text{m}}{1.00\,\text{m} - 0.98\,\text{m}}\right)\left(\frac{2.00\,\text{mm}}{1.00\,\text{m}}\right)$$

$$= \boxed{10\%}$$

(*c*) Solve equation (6) for the ratio of L_1 to L_2:

$$\frac{L_1}{L_2} = \frac{R_x}{R_0}$$

For $L_1 = 50.0$ cm, $L_2 = 50.0$ cm, $R_0 = R_x = 9.8$ kΩ. Hence, a resistor of approximately 10 kΩ will cause the bridge to balance near the 50.0-cm mark.

(*d*) From Part (*b*) we have:

$$\frac{\Delta R_x}{R_x} = \frac{1}{1-L}\frac{\Delta L}{L}$$

Evaluate $\Delta R_x/R_x$ for L = 50.0 cm and ΔL = 2.00 mm:

$$\frac{\Delta R_x}{R_x}=\left(\frac{1.00\,\text{m}}{1.00\,\text{m}-0.50\,\text{m}}\right)\left(\frac{2.00\,\text{mm}}{1.00\,\text{m}}\right)=\boxed{0.40\%}$$

111 •• **[SSM]** You are running an experiment that uses an accelerator that produces a 3.50-μA proton beam. Each proton in the beam has 60.0-MeV of kinetic energy. The protons impinge upon, and come to rest inside, a 50.0-g copper target within a vacuum chamber. You are concerned that the target will get too hot and melt the solder on some connecting wires that are crucial to the experiment. (*a*) Determine the number of protons that strike the target per second. (*b*) Find the amount of energy delivered to the target each second. (*c*) Determine how much time elapses before the target temperature increases to 300°C? (Neglect any heat released by the target.)

Picture the Problem Knowing the beam current and charge per proton, we can use $I = ne$ to determine the number of protons striking the target per second. The energy deposited per second is the power delivered to the target and is given by $P = IV$. We can find the elapsed time before the target temperature rises 300°C using $\Delta Q = P\Delta t = mc_{\text{Cu}}\Delta T$.

(*a*) Relate the current to the number of protons per second n arriving at the target:

$$I = ne \Rightarrow n=\frac{I}{e}$$

Substitute numerical values and evaluate n:

$$n=\frac{3.50\,\mu\text{A}}{1.602\times10^{-19}\,\text{C}}=\boxed{2.18\times10^{13}\,\text{s}^{-1}}$$

(*b*) Express the power of the beam in terms of the beam current and energy:

$$P = IV = (3.50\,\mu\text{A})\left(60.0\frac{\text{MeV}}{\text{proton}}\right)=\boxed{210\text{ W}}$$

(*c*) Relate the energy delivered to the target to its heat capacity and temperature change:

$$\Delta Q = P\Delta t = C_{\text{Cu}}\Delta T = mc_{\text{Cu}}\Delta T$$

Solving for Δt yields:

$$\Delta t=\frac{mc_{\text{Cu}}\Delta T}{P}$$

Substitute numerical values (see Table 19-1 for the specific heat of copper)and evaluate Δt:

$$\Delta t = \frac{(50.0\,\text{g})(0.386\,\text{kJ/kg}\cdot\text{K})(300°\text{C})}{210\,\text{J/s}} = \boxed{27.6\,\text{s}}$$

112 •• The belt of a Van de Graaff generator carries a surface charge density of 5.00 mC/m^2. The belt is 0.500 m wide and moves at 20.0 m/s. (*a*) What current does the belt carry? (*b*) If the potential of the dome of the generator is 100 kV above ground, what is the minimum power of the motor needed to drive the belt?

Picture the Problem We can use the definition of current to express the current delivered by the belt in terms of the surface charge density, width, and speed of the belt. The minimum power needed to drive the belt can be found from $P = IV$.

(*a*) Use its definition to express the current carried by the belt:

$$I = \frac{dQ}{dt} = \sigma w \frac{dx}{dt} = \sigma w v$$

Substitute numerical values and evaluate I:

$$I = (5.00\,\text{mC/m}^2)(0.500\,\text{m})(20.0\,\text{m/s}) = \boxed{50.0\,\text{mA}}$$

(*b*) Express the minimum power of the motor in terms of the current delivered and the potential of the charge:

$$P = IV$$

Substitute numerical values and evaluate P:

$$P = (50.0\,\text{mA})(100\,\text{kV}) = \boxed{5.00\,\text{kW}}$$

113 •• Large conventional electromagnets use water cooling to prevent excessive heating of the magnet coils. A large laboratory electromagnet carries a current equal to 100 A when a voltage of 240 V is applied to the terminals of the energizing coils. To cool the coils, water at an initial temperature of 15°C is circulated around the coils. How many liters of water must circulate by the coils each second if the temperature of the coils is not to exceed 50°C?

Picture the Problem We can differentiate the expression relating the amount of heat required to produce a given temperature change with respect to time to express the mass flow-rate required to maintain the temperature of the coils at 50°C. We can then use the definition of density to find the necessary volume flow rate.

Express the heat that must be dissipated in terms of the specific heat and mass of the water and the desired temperature change of the water:

$$Q = mc_{\text{water}}\Delta T$$

Differentiate this expression with respect to time to obtain an expression for the power dissipation:

$$P = \frac{dQ}{dt} = \frac{dm}{dt}c_{\text{water}}\Delta T$$

Solving for $\frac{dm}{dt}$ yields:

$$\frac{dm}{dt} = \frac{P}{c_{\text{water}}\Delta T}$$

Substitute for the power dissipated to obtain:

$$\frac{dm}{dt} = \frac{IV}{c_{\text{water}}\Delta T}$$

Substitute numerical values and evaluate $\frac{dm}{dt}$:

$$\frac{dm}{dt} = \frac{(100\,\text{A})(240\,\text{V})}{\left(4.18\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(50°\text{C} - 15°\text{C})} = 0.164\,\text{kg/s}$$

Using the definition of density, express the volume flow rate in terms of the mass flow rate to obtain:

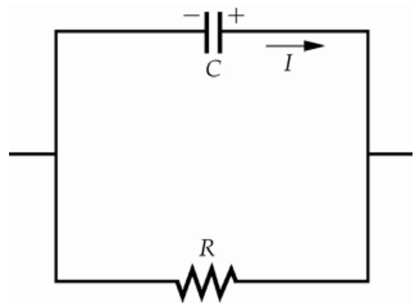
$$\frac{dV}{dt} = \frac{1}{\rho}\frac{dm}{dt}$$

Substitute numerical values and evaluate $\frac{dV}{dt}$:

$$\frac{dV}{dt} = \left(\frac{0.164\frac{\text{kg}}{\text{s}}}{1.00\times10^{3}\,\frac{\text{kg}}{\text{m}^3}}\right)\left(\frac{1\,\text{L}}{10^{-3}\,\text{m}^3}\right) = \boxed{0.16\,\text{L/s}}$$

114 ••• (*a*) Give support to the assertion that a leaky capacitor (one for which the resistance of the dielectric is finite) can be modeled as a capacitor that has an infinite resistance in parallel with a resistor. (*b*) Show that the time constant for discharging this capacitor is given by $\tau = \kappa\epsilon_0\rho$. (For simplicity, assume the capacitor is a parallel plate variety filled completely with a leaky dielectric.) (*c*) Mica has a dielectric constant equal to about 5.0 and a resistivity equal to about 9.0×10^{13} Ω·m. Calculate the time it takes for the charge of a mica-filled capacitor to decrease to 10 percent of its initial value.

Picture the Problem We'll assume that the capacitor is fully charged initially and apply Kirchhoff's loop rule to the circuit fragment to obtain the differential equation describing the discharge of the leaky capacitor. We'll show that the solution to this equation is the familiar expression for an exponential decay with time constant $\tau = \epsilon_0 \rho\kappa$.



(*a*) If we think of the leaky capacitor as a resistor/capacitor combination, the voltage drop across the resistor must be the same as the voltage drop across the capacitor. Hence they must be in parallel.

(*b*) Assuming that the capacitor is initially fully charged, apply Kirchhoff's loop rule to the circuit fragment to obtain:

$$\frac{Q}{C} - RI = 0$$

or, because $I = -\frac{dQ}{dt}$,

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0$$

Separate variables in this differential equation to obtain:

$$\frac{dQ}{Q} = -\frac{1}{RC}dt \qquad (1)$$

The capacitance of a dielectric-filled parallel-plate capacitor with plate area A and plate separation d is given by:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

The resistance of a conductor with the same dimensions is given by:

$$R = \frac{\rho d}{A}$$

The product of R and C is:

$$RC = \epsilon_0 \rho\kappa$$

Substituting for RC in the differential equation yields:

$$\frac{dQ}{Q} = -\frac{1}{\epsilon_0 \rho\kappa}dt$$

Integrate this equation from $Q' = Q_0$ to Q to obtain:

$$Q = Q_0 e^{-t/\tau} \text{ where } \tau = \boxed{\epsilon_0 \rho\kappa}$$

(*c*) Because $Q/Q_0 = 0.10$:

$$e^{-t/\tau} = 0.10$$

Solve for t by taking the natural logarithm of both sides of the equation:

$$-\frac{t}{\tau}=\ln(0.10)\Rightarrow t=-\epsilon_0\ \rho\kappa\ln(0.10)$$

Substitute numerical values and evaluate t:

$$t=-\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(9.0\times10^{13}\ \Omega\cdot\text{m}\right)(5)\ln(0.10)=9.17\times10^3\ \text{s}\approx\boxed{2.5\,\text{h}}$$

115 ••• Figure 25-77 shows the basis of the sweep circuit used in an oscilloscope. Switch S is an electronic switch that closes whenever the potential across the switch *S* increases to a value V_c and opens when the potential across switch S decreases to 0.200 V. The emf $\mathcal{E}$], which is much greater than V_c, charges the capacitor C through a resistor R_1. The resistor R_2 represents the small but finite resistance of the electronic switch. In a typical circuit, $\mathcal{E}$ = 800 V, V_c = 4.20 V, R_2 = 1.00 mΩ, R_1 = 0.500 MΩ, and C = 20.0 nF. (*a*) What is the time constant for charging of the capacitor C? (*b*) Show that as the potential across switch S increases from 0.200 V to 4.20 V, the potential across the capacitor increases almost linearly with time. *Hint*: *Use the approximation* $e^x \approx 1 + x$, for $|x| << 1$. *(This approximation of* e^x *can be derived using the differential approximation.)* (*c*) What should the value of R_1 be changed to so that the capacitor charges from 0.200 V to 4.20 V in 0.100 s? (*d*) How much time elapses during the discharge of the capacitor when switch S closes? (*e*) At what average rate is energy dissipated in the resistor R_1 during charging and in the switch resistance R_2 during discharge?

Picture the Problem We can use its definition to find the time constant of the charging circuit in Part (*a*). In Part (*b*) we can use the expression for the potential difference as a function of time across a charging capacitor and utilize the hint given in the problem statement to show that the voltage across the capacitor increases almost linearly over the time required to bring the potential across the switch to its critical value. In Part (*c*) we can use the result derived in Part (*b*) to find the value of R_1 such that C charges from 0.200 V to 4.20 V in 0.100 s. In Part (*d*) we can use the expression for the potential difference as a function of time across a discharging capacitor to find the discharge time. Finally, in Part (*e*) we can integrate I^2R_1 over the discharge time to find the rate at which energy is dissipated in R_1 during the discharge of the capacitor and use the difference in the energy stored in the capacitor initially and when the switch opens to find the rate of energy dissipation in resistance of the capacitor.

(a) When the capacitor is charging, the switch is open and the resistance in the charging circuit is R_1. Hence:

$$\tau = R_1 C = (0.500\,\text{M}\Omega)(20.0\,\text{nF}) = \boxed{10.0\,\text{ms}}$$

(b) Express the voltage across the charging capacitor as a function of time:

$$V(t) = \mathcal{E}\left(1 - e^{-t/\tau}\right)$$

Use the power series for e^x to expand $e^{t/\tau}$:

$$e^{t/\tau} = 1 + \frac{t}{\tau} + \frac{1}{2!}\left(\frac{t}{\tau}\right)^2 + \ldots \approx 1 + \frac{1}{\tau}t$$

provided $t/\tau << 1$.

Substitute for $e^{t/\tau}$ to obtain:

$$V(t) = \mathcal{E}\left\{1 - \left(1 + \frac{1}{\tau}t\right)\right\} = \boxed{\frac{\mathcal{E}}{\tau}t}$$

(c) Using the result derived in (b), relate the time Δt required to change the voltage across the capacitor by an amount ΔV to $\Delta V(t)$:

$$\Delta V(t) = \frac{\mathcal{E}}{\tau}\Delta t$$

or

$$\tau = R_1 C = \frac{\mathcal{E}}{\Delta V(t)}\Delta t \Rightarrow R_1 = \frac{\mathcal{E}}{C\Delta V(t)}\Delta t$$

Substitute numerical values and evaluate R_1:

$$R_1 = \frac{(800\,\text{V})(0.100\,\text{s})}{(20.0\,\text{nF})(4.20\,\text{V} - 0.200\,\text{V})} = \boxed{1.00\,\text{G}\Omega}$$

(d) Express the potential difference across the capacitor as a function of time:

$$V_\text{C}(t) = V_\text{C0}e^{-t/\tau'}$$

where

$\tau' = R_2C$.

Solve for t to obtain:

$$t = -\tau' \ln\left(\frac{V_\text{C}(t)}{V_\text{C0}}\right) = -R_2 C \ln\left(\frac{V_\text{C}(t)}{V_\text{C0}}\right)$$

Substitute numerical values and evaluate t:

$$t = -(1.00\,\text{m}\Omega)(20.0\,\text{nF})\ln\left(\frac{0.200\,\text{V}}{4.20\,\text{V}}\right) = 60.89\,\text{ps} = \boxed{60.9\,\text{ps}}$$

(*e*) Express the rate at which energy is dissipated in R_1 as a function of its resistance and the current through it:

$$P_1 = \frac{\Delta E_1}{\Delta t} = I^2 R_1$$

Because the current varies with time, we need to integrate over time to find ΔE_1:

$$\Delta E_1 = \int_{t_1}^{t_2} I^2 R_1 dt = \int_{t_1}^{t_2} \left(\frac{V(t)}{R_1}\right)^2 R_1 dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\mathcal{E}t}{\tau R_1}\right)^2 R_1 dt$$

$$= \left(\frac{\mathcal{E}}{\tau}\right)^2 \frac{1}{R_1} \int_{0.005\text{ s}}^{0.105\text{ s}} t^2 dt$$

$$= \left(\frac{800\,\text{V}}{20.0\,\text{s}}\right)^2 \frac{1}{1.00\,\text{G}\Omega} \left[\frac{t^3}{3}\right]_{0.00500\text{ s}}^{0.105\text{ s}}$$

$$= 6.17\times10^{-10}\text{ J}$$

Substitute numerical values and evaluate P_1:

$$P_1 = \frac{6.17\times10^{-10}\text{ J}}{0.100\,\text{s}} = \boxed{6.17\,\text{nW}}$$

Express the rate at which energy is dissipated in the switch resistance:

$$P_2 = \frac{\Delta U_\text{C}}{\Delta t} = \frac{U_\text{Ci} - U_\text{Cf}}{\Delta t}$$

$$= \frac{\frac{1}{2}CV_\text{i}^2 - \frac{1}{2}CV_\text{f}^2}{\Delta t} = \frac{\frac{1}{2}C\left(V_\text{i}^2 - V_\text{f}^2\right)}{\Delta t}$$

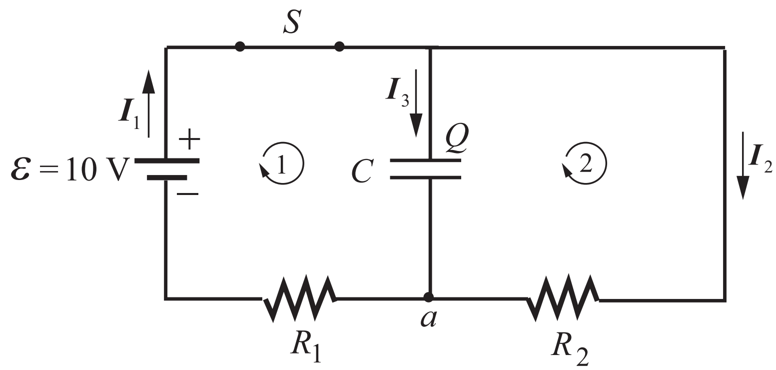
Substitute numerical values and evaluate P_2:

$$P_2 = \frac{\frac{1}{2}(20.0\,\text{nF})\left[(4.20\,\text{V})^2 - (0.200\,\text{V})^2\right]}{60.89\,\text{ps}} = \boxed{2.89\,\text{kW}}$$

116 ••• In the circuit shown in Figure 25-78, $R_1 = 2.00$ MΩ, $R_2 = 5.00$ MΩ, and $C = 1.00$ μF. The capacitor is initially without charge on either plate. At $t = 0$, switch S is closed, and at $t = 2.00$ s switch S is opened. (*a*) Sketch a graph of the voltage across C and the current in R_2 between $t = 0$ and $t = 10.0$ s. (*b*) Find the voltage across the capacitor at $t = 2.00$ s and at $t = 8.00$ s.

Picture the Problem We can apply both the loop and junction rules to obtain equations that we can use to obtain a linear differential equation with constant coefficients describing the current in R_2 as a function of time. We can solve this differential equation by assuming a solution of an appropriate form, differentiating this assumed solution and substituting it and its derivative in the differential equation. Equating coefficients, requiring the solution to hold for all

values of the assumed constants, and invoking an initial condition will allow us to find the constants in the assumed solution. Once we know how the current varies with time in R_2, we can express the potential difference across it (as well as across C because they are in parallel). To find the voltage across the capacitor at $t = 8.00$ s, we can express the dependence of the voltage on time for a discharging capacitor (C is discharging after $t = 2.00$ s) and evaluate this function, with a time constant differing from that found in (a), at $t = 6.00$ s. The diagram shows the circuit shortly after the switch is closed. The directions of the currents in the resistors and the capacitor have been chosen as shown.



(a) Apply the junction rule at junction a to obtain:

$$I_1 = I_2 + I_3 \quad (1)$$

Apply the loop rule to loop 1 to obtain:

$$\mathcal{E} - \frac{Q}{C} - R_1 I_1 = 0 \quad (2)$$

Apply the loop rule to loop 2 to obtain:

$$\frac{Q}{C} - R_2 I_2 = 0 \quad (3)$$

Differentiate equation (2) with respect to time to obtain:

$$\frac{d}{dt}\left[\mathcal{E} - R_1 I_1 - \frac{Q}{C}\right] = 0 - R_1\frac{dI_1}{dt} - \frac{1}{C}\frac{dQ}{dt}$$

$$= -R_1\frac{dI_1}{dt} - \frac{1}{C}I_3 = 0$$

or

$$R_1\frac{dI_1}{dt} = -\frac{1}{C}I_3 \quad (4)$$

Differentiate equation (3) with respect to time to obtain:

$$\frac{d}{dt}\left[\frac{Q}{C} - R_2 I_2\right] = \frac{1}{C}\frac{dQ}{dt} - R_2\frac{dI_2}{dt} = 0$$

or

$$R_2\frac{dI_2}{dt} = \frac{1}{C}I_3 \quad (5)$$

Using equation (1), substitute for I_3 in equation (5) to obtain:

$$\frac{dI_2}{dt}=\frac{1}{R_2C}\left(I_1-I_2\right) \qquad (6)$$

Solve equation (2) for I_1:

$$I_1=\frac{\mathcal{E}-Q/C}{R_1}=\frac{\mathcal{E}-R_2I_2}{R_1}$$

Substitute for I_1 in equation (6) and simplify to obtain the differential equation for I_2:

$$\frac{dI_2}{dt}=\frac{1}{R_2C}\left(\frac{\mathcal{E}-R_2I_2}{R_1}-I_2\right)=\frac{\mathcal{E}}{R_1R_2C}-\left(\frac{R_1+R_2}{R_1R_2C}\right)I_2$$

To solve this linear differential equation with constant coefficients we can assume a solution of the form:

$$I_2(t)=a+be^{-t/\tau} \qquad (7)$$

Differentiate $I_2(t)$ with respect to time to obtain:

$$\frac{dI_2}{dt}=\frac{d}{dt}\left[a+be^{-t/\tau}\right]=-\frac{b}{\tau}e^{-t/\tau}$$

Substitute for I_2 and dI_2/dt to obtain:

$$-\frac{b}{\tau}e^{-t/\tau}=\frac{\mathcal{E}}{R_1R_2C}-\left(\frac{R_1+R_2}{R_1R_2C}\right)\left(a+be^{-t/\tau}\right)$$

Equate coefficients of $e^{-t/\tau}$ to obtain:

$$\tau=\frac{R_1R_2C}{R_1+R_2}$$

Requiring the equation to hold for all values of a yields:

$$a=\frac{\mathcal{E}}{R_1+R_2}$$

If I_2 is to be zero when $t = 0$:

$$0=a+b\Rightarrow b=-a=-\frac{\mathcal{E}}{R_1+R_2}$$

Substitute in equation (7) to obtain:

$$I_2(t)=\frac{\mathcal{E}}{R_1+R_2}-\frac{\mathcal{E}}{R_1+R_2}e^{-t/\tau}=\frac{\mathcal{E}}{R_1+R_2}\left(1-e^{-t/\tau}\right)$$

where $\tau=\dfrac{R_1R_2C}{R_1+R_2}$

Substitute numerical values and evaluate τ:

$$\tau = \frac{(2.00\,\mathrm{M\Omega})(5.00\,\mathrm{M\Omega})(1.00\,\mu\mathrm{F})}{2.00\,\mathrm{M\Omega}+5.00\,\mathrm{M\Omega}} = 1.43\,\mathrm{s}$$

Substitute numerical values and evaluate $I_2(t)$:-

$$I_2(t) = \frac{10.0\,\mathrm{V}}{2.00\,\mathrm{M\Omega}+5.00\,\mathrm{M\Omega}}\left(1-e^{-t/1.429\,\mathrm{s}}\right) = (1.429\,\mu\mathrm{A})\left(1-e^{-t/1.43\,\mathrm{s}}\right)$$

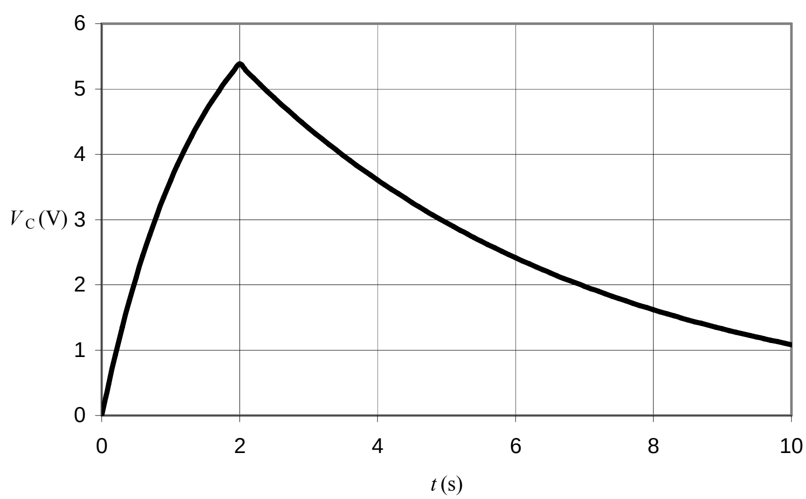
Because C and R_2 are in parallel, they have a common potential difference given by:

$$V_C(t) = V_2(t) = I_2(t)R_2 = (1.429\,\mu\mathrm{A})(5.00\,\mathrm{M\Omega})\left(1-e^{-t/1.429\mathrm{s}}\right) = (7.143\,\mathrm{V})\left(1-e^{-t/1.429\mathrm{s}}\right)$$

Evaluate V_C at $t = 2.00$ s:

$$V_C(2\,\mathrm{s}) = (7.143\,\mathrm{V})\left(1-e^{-2.00\,\mathrm{s}/1.429\,\mathrm{s}}\right) = 5.381\,\mathrm{V}$$

The voltage across the capacitor as a function of time is shown in the following graph. The current through the 5.00-MΩ resistor R_2 follows the same time course, its value being $V_C/(5.00\times10^6)$ A.



(b) The value of V_C at $t = 2.00$ s has already been determined to be:

$$V_C(2.00\,\mathrm{s}) = 5.381\,\mathrm{V} = \boxed{5.38\,\mathrm{V}}$$

When S is opened at $t = 2.00$ s, C discharges through R_2 with a time constant given by:

$$\tau' = R_2C = (5.00\,\mathrm{M\Omega})(1.00\,\mu\mathrm{F}) = 5.00\,\mathrm{s}$$

Express the potential difference across C as a function of time:

$$V_C(t) = V_{C0}e^{-(t-2.00\,\text{s})/\tau'} = (5.381\,\text{V})e^{-(t-2.00\,\text{s})/5.00\,\text{s}}$$

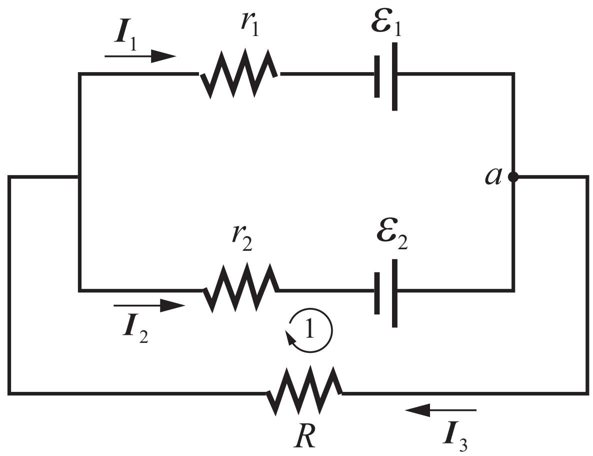
Evaluate V_C at t = 8.00 s to obtain:

$$V_C(8.00\,\text{s}) = (5.381\,\text{V})e^{-6.00\,\text{s}/5.00\,\text{s}} = \boxed{1.62\,\text{V}}$$

in good agreement with the graph.

117 ••• Two batteries that have emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ and internal resistances r_1 and r_2 are connected in parallel. Prove that if a resistor of resistance R is connected in parallel with combination, the optimal load resistance (the value of R at which maximum power is delivered) is given by $R = r_1r_2/(r_1 + r_2)$.

Picture the Problem Let I_1 be the current supplied by the battery whose emf is $\mathcal{E}_1$, I_2 the current supplied by the battery whose emf is $\mathcal{E}_2$, and I_3 the current through the resistor R. We can apply Kirchhoff's rules to obtain three equations in the unknowns I_1, I_2, and I_3 that we can solve simultaneously to find I_3. We can then express the power delivered by the sources to R. Setting the derivative of this expression equal to zero will allow us to solve for the value of R that maximizes the power delivered by the sources.



Apply Kirchhoff's junction rule at a to obtain:

$$I_1 + I_2 = I_3 \tag{1}$$

Apply the loop rule around the outside of the circuit to obtain:

$$\mathcal{E}_1 - I_3R - r_1I_1 = 0 \tag{2}$$

Apply the loop rule to loop 1 to obtain:

$$\mathcal{E}_2 - I_3R - r_2I_2 = 0 \tag{3}$$

Eliminate I_1 from equations (1) and (2) to obtain:

$$\mathcal{E}_1 - I_3 R - r_1 (I_3 - I_2) = 0 \qquad (4)$$

Solving equation (3) for I_2 yields:

$$I_2 = \frac{\mathcal{E}_2 - I_3 R}{r_2}$$

Substitute for I_2 in equation (4) to obtain:

$$\mathcal{E}_1 - I_3 R - r_1\left(I_3 - \frac{\mathcal{E}_2 - I_3 R}{r_2}\right) = 0$$

Solving for I_3 yields:

$$I_3 = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 r_2 + R(r_1 + r_2)}$$

Express the power delivered to R:

$$P = I_3^2 R = \left(\frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 r_2 + R(r_1 + r_2)}\right)^2 R$$

$$= \left(\frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 + r_2}\right)^2 \left[\frac{R}{(R + A)^2}\right]$$

where $A = \dfrac{r_1 r_2}{r_1 + r_2}$

Noting that the quantity in parentheses is independent of R and that therefore we can ignore it, differentiate P with respect to R and set the derivative equal to zero:

$$\frac{dP}{dR} = \frac{d}{dR}\left[\frac{R}{(R + A)^2}\right]$$

$$= \frac{(R + A)^2 - R\dfrac{d}{dR}(R + A)^2}{(R + A)^4}$$

$$= \frac{(R + A)^2 - 2R(R + A)}{(R + A)^4}$$

$$= 0 \text{ for extrema}$$

Solving for R yields:

$$R = A = \frac{r_1 r_2}{r_1 + r_2}$$

To establish that this value for R corresponds to a maximum, we need to evaluate the second derivative of P with respect to R at $R = A$ and show that this quantity is negative; that is, concave downward:

$$\frac{d^2P}{dR^2} = \frac{d}{dR}\left[\frac{(R+A)^2 - 2R(R+A)}{(R+A)^4}\right] = \frac{2R-4A}{(R+A)^4}$$

and

$$\left.\frac{d^2P}{dR^2}\right|_{R=A} = \frac{-2A}{(R+A)^4} < 0$$

Because $\left.\frac{d^2P}{dR^2}\right|_{R=A} < 0$ we can conclude that:

$R = \boxed{\frac{r_1r_2}{r_1+r_2}}$ maximizes the power delivered by the sources.

118 ••• Capacitors C_1 and C_2 are connected to a resistor of resistance R and an ideal battery that has terminal voltage V_0 as shown in Figure 25-79. Initially the throw of switch S is at contact a and both capacitors are without charge. The throw is then rotated to contact b and left there for a long time. Finally, at time $t = 0$, the throw is returned to contact a. (*a*) Quantitatively compare the total energy stored in the two capacitors at $t = 0$ and a long time later. (*b*) Find the current through R as a function of time. (*c*) Find the energy dissipated in the resistor as a function of time. (*d*) Find the total energy dissipated in the resistor and compare it with the loss of stored energy found in Part (*a*).

Picture the Problem (*a*) Let Q_1 and Q_2 represent the final charges on the capacitors C_1 and C_2. Knowing that charge is conserved as it is redistributed to the two capacitors and that the final-state potential differences across the two capacitors will be the same, we can obtain two equations in the unknowns Q_1 and Q_2 that we can solve simultaneously. We can compare the initial and final energies stored in this system by examining their ratio. (*b*) Let q_1 and q_2 be the time-dependent charges on the two capacitors after the switches are closed. We can use Kirchhoff's loop rule and the conservation of charge to obtain a first-order linear differential equation describing the current I_2 through R after the switches are closed. We can solve this differential equation by assuming a solution of the form $q_2(t) = a + be^{-t/\tau}$ and requiring that the solution satisfy the initial condition $q_2(0) = 0$ and the differential equation be satisfied for all values of t. (*c*) Once we know I_2, we can find the energy dissipated in the resistor as a function of time and (*d*) the total energy dissipated in the resistor.

(*a*) The initial and final energies stored in the capacitors are:

$$U_i = \frac{1}{2}C_1V_0^2 \qquad (1)$$

and

$$U_f = \frac{1}{2}\frac{Q_1^2}{C_1} + \frac{1}{2}\frac{Q_2^2}{C_2} \qquad (2)$$

Relate the total charge stored initially to the final charges Q_1 and Q_2 on C_1 and C_2:

$$Q = C_1V_0 = Q_1 + Q_2 \qquad (3)$$

Because, in their final state, the potential differences across the two capacitors will be the same:

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \qquad (4)$$

Solve equation (4) for Q_1 and substitute in equation (3) to obtain:

$$\frac{C_1}{C_2}Q_2 + Q_2 = C_1V_0 \Rightarrow Q_2 = \frac{C_1C_2}{C_1 + C_2}V_0$$

Substitute for Q_2 in either equation (3) or equation (4) and solve for Q_1 to obtain:

$$Q_1 = \frac{C_1^2}{C_1 + C_2}V_0$$

Substitute for Q_1 and Q_2 in equation (2) and simplify to obtain:

$$U_f = \frac{1}{2}\frac{\left(\frac{C_1^2}{C_1 + C_2}V_0\right)^2}{C_1} + \frac{1}{2}\frac{\left(\frac{C_1C_2}{C_1 + C_2}V_0\right)^2}{C_2} = \frac{1}{2}\frac{C_1^2}{C_1 + C_2}V_0^2$$

Expressing the difference between U_i and U_f yields:

$$U_i - U_f = \frac{1}{2}C_1V_0^2 - \frac{1}{2}\frac{C_1^2}{C_1 + C_2}V_0^2 = \boxed{\frac{1}{2}\frac{C_1C_2}{C_1 + C_2}V_0^2}$$

Note that, if we express the ratio of U_i and U_f we obtain:

$$\frac{U_i}{U_f}=\frac{C_1V_0^2}{\dfrac{\left(\dfrac{C_1^2}{C_1+C_2}V_0\right)^2}{C_1}+\dfrac{\left(\dfrac{C_1C_2}{C_1+C_2}V_0\right)^2}{C_2}}$$

$$=1+\frac{C_2}{C_1}$$

That is, U_i is greater than U_f by a factor of $1 + C_2/C_1$.

(b) Apply Kirchhoff's loop rule to the circuit when the switch is in position a to obtain:

$$\frac{q_1}{C_1}-IR-\frac{q_2}{C_2}=0$$

or, because $I=\dfrac{dq_2}{dt}$,

$$\frac{q_1}{C_1}-R\frac{dq_2}{dt}-\frac{q_2}{C_2}=0$$

Apply conservation of charge during the redistribution of charge to obtain:

$$q_1=Q-q_2=C_1V_0-q_2$$

Substituting for q_1 yields:

$$V_0-\frac{q_2}{C_1}-R\frac{dq_2}{dt}-\frac{q_2}{C_2}=0$$

Rearrange to obtain a first- order differential equation:

$$R\frac{dq_2}{dt}+\left(\frac{C_1+C_2}{C_1C_2}\right)q_2=V_0$$

Assume a solution of the form:

$$q_2(t)=a+be^{-t/\tau} \qquad (4)$$

Differentiate the assumed solution with respect to time to obtain:

$$\frac{dq_2(t)}{dt}=\frac{d}{dt}\left[a+be^{-t/\tau}\right]=-\frac{b}{\tau}e^{-t/\tau}$$

Substitute for dq_2/dt and q_2 in the differential equation to obtain:

$$R\left(-\frac{b}{\tau}e^{-t/\tau}\right)+\left(\frac{C_1+C_2}{C_1C_2}\right)\left(a+be^{-t/\tau}\right)=V_0$$

Rearranging yields:

$$\left[-\frac{R}{\tau}e^{-t/\tau}\right]b+\left(\frac{C_1+C_2}{C_1C_2}\right)a+\left[\left(\frac{C_1+C_2}{C_1C_2}\right)e^{-t/\tau}\right]b=V_0$$

If this equation is to be satisfied for all values of t:

$$a=\frac{C_1C_2}{C_1+C_2}V_0=C_{\text{eq}}V_0$$

and

$$\left[-\frac{R}{\tau}e^{-t/\tau}\right]b+\left[\left(\frac{C_1+C_2}{C_1C_2}\right)e^{-t/\tau}\right]b=0$$

Simplifying further yields:

$$-\frac{R}{\tau}+\frac{C_1+C_2}{C_1C_2}=0$$

Solve for τ to obtain:

$$\tau=R\frac{C_1C_2}{C_1+C_2}=RC_{\text{eq}}$$

Applying the initial condition $q_2(0)=0$ to equation (4) yields:

$$0=a+b$$

or

$$b=-a=-C_{\text{eq}}V_0$$

Substitute for a and b in equation (4) to obtain:

$$q_2(t)=C_{\text{eq}}V_0-C_{\text{eq}}V_0\,e^{-t/\tau}=C_{\text{eq}}V_0\left(1-e^{-t/\tau}\right)$$

Differentiate $q_2(t)$ with respect to time to find the current through R as a function of time:

$$I(t)=\frac{dq_2(t)}{dt}=C_{\text{eq}}V_0\frac{d}{dt}\left(1-e^{-t/\tau}\right)=C_{\text{eq}}V_0\left(-e^{-t/\tau}\right)\left(-\frac{1}{\tau}\right)=\frac{C_{\text{eq}}V_0}{\tau}e^{-t/\tau}=\boxed{\frac{V_0}{R}e^{-t/\tau}}$$

where $\tau=\boxed{R\frac{C_1C_2}{C_1+C_2}}$.

(*c*) Express the energy dissipated in the resistor as a function of time:

$$P(t)=I^2R=\left(\frac{V_0}{R}e^{-t/\tau}\right)^2R=\boxed{\frac{V_0^2}{R}e^{-2t/\tau}}$$

where $\tau=\boxed{R\frac{C_1C_2}{C_1+C_2}}$

(*d*) The total energy dissipated in the resistor is the integral of $P(t)$ between $t = 0$ and $t = \infty$:

$$E = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC_{\text{eq}}}\,dt = \tfrac{1}{2}V_0^2 C_{\text{eq}}$$

Because the capacitors are in series:

$$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$$

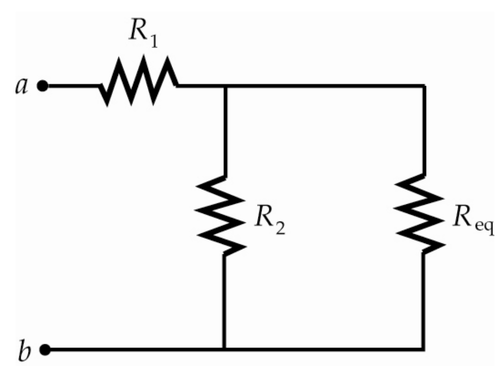
Substituting for C_{eq} yields:

$$E = \boxed{\frac{1}{2}\frac{C_1 C_2}{C_1 + C_2}V_0^2}$$

This energy, dissipated as Joule heating in the resistor, is exactly the difference between the initial and final energies found in Part (*a*).

119 ••• (*a*) Calculate the equivalent resistance (in terms of R, the resistance of each individual resistor) between points *a* and *b* for the infinite ladder of resistors shown in Figure 25-80 assuming the resistors are identical. That is assuming $R = R_1 + R_2$. (*b*) Repeat Part (*a*) but do not assume that $R_1 = R_2$ and express your answer in terms of R_1 and R_2. (*c*), Check your results by showing that your result from Part (*b*) agrees with your result from Part (*a*) if you substitute R for both R_1 and R_2.

Picture the Problem Let R be the resistance of each resistor in the ladder and let R_{eq} be the equivalent resistance of the infinite ladder. If the resistance is finite and non-zero, then adding one or more stages to the ladder will not change the resistance of the network. We can apply the rules for resistance combination to the diagram shown to the right to obtain a quadratic equation in R_{eq} that we can solve for the equivalent resistance between points *a* and *b*.



(*a*) The equivalent resistance of the series combination of R and ($R \parallel R_{\text{eq}}$) is R_{eq}, so:

$$R_{\text{eq}} = R + R\|R_{\text{eq}} = R + \frac{RR_{\text{eq}}}{R + R_{\text{eq}}}$$

Simplify to obtain:

$$R_{\text{eq}}^2 - RR_{\text{eq}} - R^2 = 0$$

Solve for R_{eq} to obtain:

$$R_{\text{eq}} = \left(\frac{1+\sqrt{5}}{2}\right)R \approx \boxed{1.618R}$$

(*b*) The equivalent resistance of the series combination of R_1 and ($R_2 \parallel R_{eq}$) is R_{eq}, so:

$$R_{eq} = R_1 + R_2 \| R_{eq} = R_1 + \frac{R_2 R_{eq}}{R_2 + R_{eq}}$$

Simplify to obtain:

$$R_{eq}^2 - R_1 R_{eq} - R_1 R_2 = 0$$

Solve for the positive value of R_{eq} to obtain:

$$R_{eq} = \boxed{\frac{R_1 + \sqrt{R_1^2 + 4R_1 R_2}}{2}}$$

(*c*) If $R_1 = R_2 = R$, then:

$$R_{eq} = \frac{R + \sqrt{R^2 + 4RR}}{2} = \boxed{\left(\frac{1+\sqrt{5}}{2}\right)R}$$

in agreement with Part (*a*).

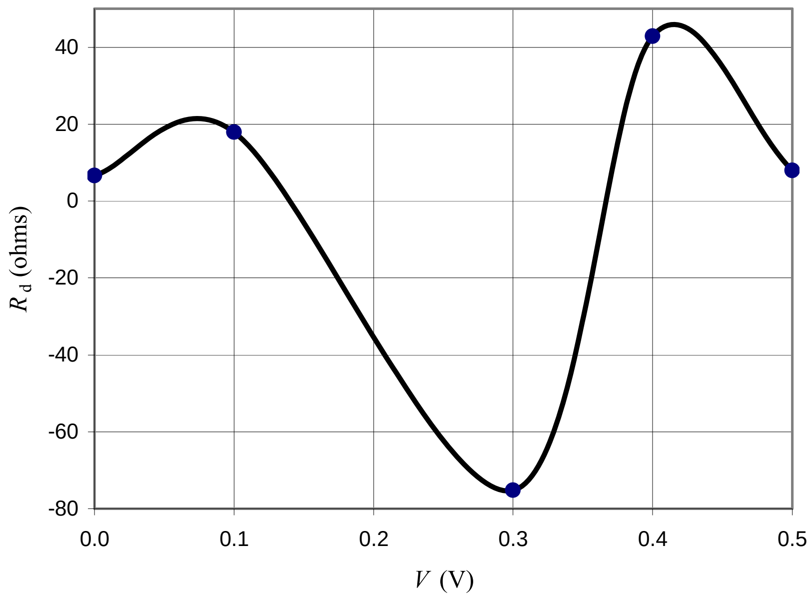
120 ••• A graph of current as a function of voltage for an Esaki diode is shown in Figure 25-81. (*a*) Make a graph of the differential resistance of the diode as a function of voltage. The differential resistance R_d of a circuit element is defined as $R_d = dV/dI$, where V is the voltage drop across the element and I is the current in the element. (*b*) At what value of the voltage drop does the differential resistance become negative? (*c*) What is the maximum differential resistance for this diode in the range shown and at what voltage does it occur? (*d*) Are there any places in the voltage range shown where the diode exhibits a differential resistance equal to zero? If so, under value(s) of the voltage does this (do these) occur?

Picture the Problem We can approximate the slope of the graph in Figure 25-81 and take its reciprocal to obtain values for R_d that we can plot as a function of V.

(*a*) Use the graph in Figure 25-81 to complete the table to the right.

V (V)	R_d (Ω)
0	6.67
0.1	17.9
0.3	−75.2
0.4	42.9
0.5	8

The following graph was plotted using a spreadsheet program.



(*b*) The differential resistance becomes negative at approximately 0.14 V.

(*c*) The maximum differential resistance for this diode is approximately 45 Ω and occurs at about 0.42 V.

(*d*) The diode exhibits no resistance where the curve crosses the V axis; that is at $V = 0.14$ V and 0.36 V.

Chapter 26
The Magnetic Field

Conceptual Problems

1 • [SSM] When the axis of a cathode-ray tube is horizontal in a region in which there is a magnetic field that is directed vertically upward, the electrons emitted from the cathode follow one of the dashed paths to the face of the tube in Figure 26-30. The correct path is (*a*) 1, (*b*) 2, (*c*) 3, (*d*) 4, (*e*) 5.

Determine the Concept Because the electrons are initially moving at 90° to the magnetic field, they will be deflected in the direction of the magnetic force acting on them. Use the right-hand rule based on the expression for the magnetic force acting on a moving charge $\vec{F} = q\vec{v} \times \vec{B}$, remembering that, for a negative charge, the force is in the direction opposite that indicated by the right-hand rule, to convince yourself that the particle will follow the path whose terminal point on the screen is 2. $\boxed{(b)}$ is correct.

2 •• We define the direction of the electric field to be the same as the direction of the force on a positive test charge. Why then do we *not* define the direction of the magnetic field to be the same as the direction of the magnetic force on a moving positive test charge?

Determine the Concept The direction of the force depends on the direction of the velocity. We do not define the direction of the magnetic field to be in the direction of the force because if we did, the magnetic field would be in a different direction each time the velocity was in a different direction. If this were the case, the magnetic field would not be a useful construct.

3 • [SSM] A *flicker bulb* is a light bulb that has a long, thin flexible filament. It is meant to be plugged into an ac outlet that delivers current at a frequency of 60 Hz. There is a small permanent magnet inside the bulb. When the bulb is plugged in the filament oscillates back and forth. At what frequency does it oscillate? Explain.

Determine the Concept Because the alternating current running through the filament is changing direction every 1/60 s, the filament experiences a force that changes direction at the frequency of the current.

4 • In a cyclotron, the potential difference between the dees oscillates with a period given by $T = 2\pi m/(qB)$. Show that the expression to the right of the equal sign has units of seconds if q, B and m have units of coulombs, teslas and kilograms, respectively.

Determine the Concept Substituting the SI units for q, B, and m yields:

$$\frac{\mathrm{C\cdot T}}{\mathrm{kg}}=\frac{\mathrm{C}\left(\frac{\mathrm{N}}{\mathrm{A\cdot m}}\right)}{\mathrm{kg}}=\frac{\mathrm{C}\left(\frac{\mathrm{N\cdot s}}{\mathrm{C\cdot m}}\right)}{\mathrm{kg}}=\frac{\frac{\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}}\cdot\mathrm{s}}{\mathrm{m}}}{\mathrm{kg}}=\frac{\frac{\mathrm{kg\cdot m}}{\mathrm{s}}}{\mathrm{kg\cdot m}}=\boxed{\frac{1}{\mathrm{s}}}$$

5 • A ^{7}Li nucleus has a charge equal to $+3e$ and a mass that is equal to the mass of seven protons. A ^{7}Li nucleus and a proton are both moving perpendicular to a uniform magnetic field $\vec{B}$. The magnitude of the momentum of the proton is equal to the magnitude of the momentum of the nucleus. The path of the proton has a radius of curvature equal to R_p and the path of the ^{7}Li nucleus has a radius of curvature equal to R_Li. The ratio $R_\mathrm{p}/R_\mathrm{Li}$ is closest to (*a*) 3/1, (*b*) 1/3, (*c*) 1/7, (*d*) 7/1, (*e*) 3/7, (*f*) 7/3.

Determine the Concept We can use Newton's second law for circular motion to express the radius of curvature R of each particle in terms of its charge, momentum, and the magnetic field. We can then divide the proton's radius of curvature by that of the ^{7}Li nucleus to decide which of these alternatives is correct.

Apply Newton's second law to the lithium nucleus to obtain:

$$qvB=m\frac{v^2}{R}\Rightarrow R=\frac{mv}{qB}$$

For the ^{7}Li nucleus this becomes:

$$R_\mathrm{Li}=\frac{p_\mathrm{Li}}{3eB} \qquad (1)$$

For the proton we have:

$$R_\mathrm{p}=\frac{p_\mathrm{p}}{eB} \qquad (2)$$

Divide equation (2) by equation (1) and simplify to obtain:

$$\frac{R_\mathrm{p}}{R_\mathrm{Li}}=\frac{\frac{p_\mathrm{p}}{eB}}{\frac{p_\mathrm{Li}}{3eB}}=3\frac{p_\mathrm{p}}{p_\mathrm{Li}}$$

Because the momenta are equal:

$$\frac{R_\mathrm{p}}{R_\mathrm{Li}}=3\Rightarrow\boxed{(a)}\text{ is correct.}$$

6 • An electron moving in the $+x$ direction enters a region that has a uniform magnetic field in the $+y$ direction. When the electron enters this region, it will (*a*) be deflected toward the $+y$ direction, (*b*) be deflected toward the $-y$ direction, (*c*) be deflected toward the $+z$ direction, (*d*) be deflected toward the $-z$ direction, (*e*) continue undeflected in the $+x$ direction.

Determine the Concept Application of the right-hand rule indicates that a positively charged body would experience a downward force and, in the absence of other forces, be deflected downward. Because the direction of the magnetic force on an electron is opposite that of the force on a positively charged object, an electron will be deflected upward. $\boxed{(c)}$ is correct.

7 • **[SSM]** In a velocity selector, the speed of the undeflected charged particle is given by the ratio of the magnitude of the electric field to the magnitude of the magnetic field. Show that E/B in fact does have the units of m/s if E and B are in units of volts per meter and teslas, respectively.

Determine the Concept Substituting the SI units for E and B yields:

$$\frac{\dfrac{\text{N}}{\text{C}}}{\dfrac{\text{N}}{\text{A}\cdot\text{m}}}=\frac{\text{A}\cdot\text{m}}{\text{C}}==\frac{\dfrac{\text{C}}{\text{s}}\cdot\text{m}}{\text{C}}=\boxed{\frac{\text{m}}{\text{s}}}$$

8 • How are the properties of magnetic field lines similar to the properties of electric field lines? How are they different?

Similarities	Differences
1. Their density on a surface perpendicular to the lines is a measure of the strength of the field 2. The lines point in the direction of the field 3. The lines do not cross.	1. Magnetic field lines neither begin nor end. Electric field lines begin on positive charges and end on negative charges. 2. Electric forces are parallel or anti-parallel to the field lines. Magnetic forces are perpendicular to the field lines.

9 • True or false:

(*a*) The magnetic moment of a bar magnet points from its north pole to its south pole.
(*b*) Inside the material of a bar magnet, the magnetic field due to the bar magnet points from the magnet's south pole toward its north pole.
(*c*) If a current loop simultaneously has its current doubled and its area cut in half, then the magnitude of its magnetic moment remains the same.
(*d*) The maximum torque on a current loop placed in a magnetic field occurs when the plane of the loop is perpendicular to the direction of the magnetic field.

(*a*) False. By definition, the magnetic moment of a small bar magnet points from its south pole to its north pole.

(*b*) True. The external magnetic field of a bar magnet points from the north pole of the magnet to south pole. Because magnetic field lines are continuous, the magnet's internal field lines point from the south pole to the north pole.

(*c*) True. Because the magnetic dipole moment of a current loop is given by $\vec{\boldsymbol{\mu}} = NIA\vec{\boldsymbol{n}}$, simultaneously doubling the current and halving its area leaves the magnetic dipole moment unchanged.

(*d*) False. The magnitude of the torque acting on a magnetic dipole moment is given $\tau = \mu B\sin\theta$ where θ is the angle between the axis of the current loop and the direction of the magnetic field. When the plane of the loop is perpendicular to the field direction $\theta = 0$ and the torque is zero.

10 •• Show that the von Klitzing constant, h/e^2, gives the SI unit for resistance (the ohm) h and e are in units of joule seconds and coulombs, respectively.

Determine the Concept The von Klitzing resistance is related to the Hall resistance according to $R_K = nR_H$ where $R_K = h/e^2$.

Substituting the SI units of h and e yields: $\dfrac{\mathrm{J}\cdot\mathrm{s}}{\mathrm{C}^2} = \dfrac{\mathrm{J/C}}{\mathrm{C/s}} = \mathrm{V/A} = \boxed{\Omega}$

11 ••• The theory of relativity states that no law of physics can be described using the absolute velocity of an object, which is in fact impossible to define due to a lack of an absolute reference frame. Instead, the behavior of interacting objects can only be described by the relative velocities between the objects. New physical insights result from this idea. For example, in Figure 26-31 a magnet moving at high speed relative to some observer passes by an electron that is at rest relative to the same observer. Explain why you are sure that a force must be acting on the electron. In what direction will the force point at the instant the north pole of the magnet passes directly underneath the electron? Explain your answer.

Determine the Concept From relativity; this is equivalent to the electron moving from right to left at speed v with the magnet stationary. When the electron is directly over the magnet, the field points directly up, so there is a force directed out of the page on the electron.

Estimation and Approximation

12 • Estimate the maximum magnetic force per meter that Earth's magnetic field could exert on a current-carrying wire in a 20-A circuit in your house.

Picture the Problem Because the magnetic force on a current-carrying wire is given by $\vec{F} = I\vec{L} \times \vec{B}$, the maximum force occurs when θ = 90°. Under this condition, $F_{\text{max}} = ILB$.

The maximum force per unit length that Earth's magnetic field could exert on a current-carrying wire in your home is given by:

$$\left.\frac{F}{L}\right]_{\text{max}} = IB$$

For a 20-A circuit and $B = 0.5 \times 10^{-4}$ T:

$$\left.\frac{F}{L}\right]_{\text{max}} = (20\ \text{A})(0.5 \times 10^{-4}\ \text{T}) = \boxed{1\ \text{mN/m}}$$

13 •• Your friend wants to be magician and intends to use Earth's magnetic field to suspend a current-carrying wire above the stage. He asks you to estimate the minimum current needed to suspend the wire just above Earth's surface at the equator (where Earth's magnetic field is horizontal). Assume the wire has a linear mass density of 10 g/m. Would you advise him to proceed with his plans for this act?

Picture the Problem Because the magnetic force on a current-carrying wire is given by $\vec{F} = I\vec{L} \times \vec{B}$, the maximum force occurs when θ = 90°. Under this condition, $F_{\text{max}} = ILB$. In order to suspend the wire, this magnetic force would have to be equal in magnitude to the gravitational force exerted by Earth on the wire:

Letting the upward direction be the $+y$ direction, apply $\sum F_y = 0$ to the wire to obtain:

$$F_{\text{m}} - F_{\text{g}} = 0$$

or,

$$ILB - mg = 0$$

Solving for I yields:

$$I = \frac{mg}{LB} = \left(\frac{m}{L}\right)\frac{g}{B}$$

where m/L is the linear density of the wire.

Substitute numerical values and evaluate I:

$$I = (10\text{ g/m})\frac{9.81\text{ m/s}^2}{0.5\times10^{-4}\text{ T}} \approx \boxed{2\text{ kA}}$$

You should advise him to develop some other act. A current of 2000 A would overheat the wire (which is a gross understatement).

The Force Exerted by a Magnetic Field

14 • Find the magnetic force on a proton moving in the $+x$ direction at a speed of 0.446 Mm/s in a uniform magnetic field of 1.75 T in the $+z$ direction.

Picture the Problem The magnetic force acting on a charge is given by $\vec{F} = q\vec{v}\times\vec{B}$. We can express $\vec{v}$ and $\vec{B}$, form their vector ("cross") product, and multiply by the scalar q to find $\vec{F}$.

The magnetic force acting on the proton is given by:

$$\vec{F} = q\vec{v}\times\vec{B}$$

Express $\vec{v}$:

$$\vec{v} = (0.446\text{ Mm/s})\hat{i}$$

Express $\vec{B}$:

$$\vec{B} = (1.75\text{ T})\hat{k}$$

Substitute numerical values and evaluate $\vec{F}$:

$$\vec{F} = (1.602\times10^{-19}\text{ C})\left[(0.446\text{ Mm/s})\hat{i}\times(1.75\text{ T})\hat{k}\right] = \boxed{-(0.125\text{ pN})\hat{j}}$$

15 • A point particle has a charge equal to –3.64 nC and a velocity equal to 2.75×10^3 m/s $\hat{i}$. Find the force on the charge if the magnetic field is (*a*) $0.38\text{ T}\,\hat{j}$, (*b*) $0.75\text{ T}\,\hat{i} + 0.75\text{ T}\,\hat{j}$, (*c*) $0.65\text{ T}\,\hat{i}$, and (*d*) $0.75\text{ T}\,\hat{i} + 0.75\text{ T}\,\hat{k}$.

Picture the Problem The magnetic force acting on the charge is given by $\vec{F} = q\vec{v}\times\vec{B}$. We can express $\vec{v}$ and $\vec{B}$, form their vector (also known as the "cross") product, and multiply by the scalar q to find $\vec{F}$.

Express the force acting on the charge:

$$\vec{F} = q\vec{v}\times\vec{B}$$

Substitute numerical values to obtain:

$$\vec{F} = (-3.64\text{ nC})\left[(2.75\times10^3\text{ m/s})\hat{i}\times\vec{B}\right]$$

(a) Evaluate $\vec{F}$ for $\vec{B} = 0.38\ \text{T}\ \hat{j}$:

$$\vec{F} = (-3.64\,\text{nC})\left[(2.75\times10^3\ \text{m/s})\hat{i}\times(0.38\,\text{T})\hat{j}\right] = \boxed{-(3.8\,\mu\text{N})\hat{k}}$$

(b) Evaluate $\vec{F}$ for $\vec{B} = 0.75\ \text{T}\ \hat{i} + 0.75\ \text{T}\ \hat{j}$:

$$\vec{F} = (-3.64\,\text{nC})\left[(2.75\times10^3\ \text{m/s})\hat{i}\times\left\{(0.75\,\text{T})\hat{i}+(0.75\,\text{T})\hat{j}\right\}\right] = \boxed{-(7.5\,\mu\text{N})\hat{k}}$$

(c) Evaluate $\vec{F}$ for $\vec{B} = 0.65\ \text{T}\ \hat{i}$:

$$\vec{F} = (-3.64\,\text{nC})\left[(2.75\times10^3\ \text{m/s})\hat{i}\times(0.65\,\text{T})\hat{i}\right] = \boxed{0}$$

(d) Evaluate $\vec{F}$ for $\vec{B} = 0.75\ \text{T}\ \hat{i} + 0.75\ \text{T}\hat{k}$:

$$\vec{F} = (-3.64\,\text{nC})\left[(2.75\times10^3\ \text{m/s})\hat{i}\times(0.75\,\text{T})\hat{i}+(0.75\,\text{T})\hat{k}\right] = \boxed{(7.5\,\mu\text{N})\hat{j}}$$

16 • A uniform magnetic field equal to 1.48 T is in the +z direction. Find the force exerted by the field on a proton if the velocity of the proton is (a) $2.7\ \text{km/s}\ \hat{i}$, ($b$) $3.7\ \text{km/s}\ \hat{j}$, ($c$) $6.8\ \text{km/s}\ \hat{k}$, and ($d$) $4.0\ \text{km/s}\ \hat{i} + 3.0\ \text{km/s}\ \hat{j}$.

Picture the Problem The magnetic force acting on the proton is given by $\vec{F} = q\vec{v}\times\vec{B}$. We can express $\vec{v}$ and $\vec{B}$, form their vector product, and multiply by the scalar q to find $\vec{F}$.

The magnetic force acting on the proton is given by:

$$\vec{F} = q\vec{v}\times\vec{B}$$

(a) Evaluate $\vec{F}$ for $\vec{v} = 2.7\ \text{km/s}\ \hat{i}$:

$$\vec{F} = (1.602\times10^{-19}\ \text{C})\left[(2.7\,\text{km/s})\hat{i}\times(1.48\,\text{T})\hat{k}\right] = \boxed{-(6.4\times10^{-16}\ \text{N})\hat{j}}$$

(b) Evaluate $\vec{F}$ for $\vec{v} = 3.7\ \text{km/s}\ \hat{j}$:

$$\vec{F} = (1.602\times10^{-19}\ \text{C})\left[(3.7\,\text{km/s})\hat{j}\times(1.48\,\text{T})\hat{k}\right] = \boxed{(8.8\times10^{-16}\ \text{N})\hat{i}}$$

(c) Evaluate $\vec{F}$ for $\vec{v}$ = 6.8 km/s $\hat{k}$:

$$\vec{F} = \left(1.602\times10^{-19}\text{ C}\right)\left[\left(6.8\text{ km/s}\right)\hat{k}\times\left(1.48\text{ T}\right)\hat{k}\right] = \boxed{0}$$

(d) Evaluate $\vec{F}$ for $\vec{v} = 4.0\text{ km/s}\,\hat{i} + 3.0\text{ km/s}\,\hat{j}$:

$$\begin{aligned}\vec{F} &= \left(1.602\times10^{-19}\text{ C}\right)\left[\left\{\left(4.0\text{ km/s}\right)\hat{i} + \left(3.0\text{ km/s}\right)\hat{j}\right\}\times\left(1.48\text{ T}\right)\hat{k}\right]\\ &= \boxed{\left(7.1\times10^{-16}\text{ N}\right)\hat{i} - \left(9.5\times10^{-16}\text{ N}\right)\hat{j}}\end{aligned}$$

17 • A straight wire segment that is 2.0 m long makes an angle of 30º with a uniform 0.37-T magnetic field. Find the magnitude of the force on the wire if the wire carries a current of 2.6 A.

Picture the Problem The magnitude of the magnetic force acting on a segment of wire carrying a current I is given by $F = I\ell B\sin\theta$ where ℓ is the length of the segment of wire, B is the magnetic field, and θ is the angle between direction of the current in the segment of wire and the direction of the magnetic field.

Express the magnitude of the magnetic force acting on the segment of wire:

$$F = I\ell B\sin\theta$$

Substitute numerical values and evaluate F:

$$\begin{aligned}F &= \left(2.6\text{ A}\right)\left(2.0\text{ m}\right)\left(0.37\text{ T}\right)\sin30° \\ &= \boxed{0.96\text{ N}}\end{aligned}$$

18 • A straight segment of a current-carrying wire has a current element $I\vec{L}$, where I = 2.7 A and $\vec{L} = 3.0\text{ cm}\,\hat{i} + 4.0\text{ cm}\,\hat{j}$. The segment is in a region with a uniform magnetic field given by $1.3\text{ T}\hat{i}$. Find the force on the segment of wire.

Picture the Problem We can use $\vec{F} = I\vec{L}\times\vec{B}$ to find the force acting on the wire segment.

Express the force acting on the wire segment:

$$\vec{F} = I\vec{L}\times\vec{B}$$

Substitute numerical values and evaluate $\vec{F}$:

$$\begin{aligned}\vec{F} &= \left(2.7\text{ A}\right)\left[\left(3.0\text{ cm}\right)\hat{i} + \left(4.0\text{ cm}\right)\hat{j}\right]\times\left(1.3\text{ T}\right)\hat{i} \\ &= \boxed{-\left(0.14\text{ N}\right)\hat{k}}\end{aligned}$$

19 • What is the force on an electron that has a velocity equal to $2.0\times10^{6}\ \text{m/s}\,\hat{\boldsymbol{i}} - 3.0\times10^{6}\ \hat{\boldsymbol{j}}$ when it is in a region with a magnetic field given by $0.80\ \text{T}\,\hat{\boldsymbol{i}} + 0.60\ \text{T}\,\hat{\boldsymbol{k}}$?

Picture the Problem The magnetic force acting on the electron is given by $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$.

The magnetic force acting on the proton is given by:

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$$

Substitute numerical values and evaluate $\vec{\boldsymbol{F}}$:

$$\begin{aligned}\vec{\boldsymbol{F}} &= \left(-1.602\times10^{-19}\ \text{C}\right)\left[\left\{\left(2\ \text{Mm/s}\right)\hat{\boldsymbol{i}} - \left(3\ \text{Mm/s}\right)\hat{\boldsymbol{j}}\right\}\times\left(0.8\,\hat{\boldsymbol{i}} + 0.6\,\hat{\boldsymbol{j}} - 0.4\,\hat{\boldsymbol{k}}\right)\text{T}\right]\\ &= \left(-0.192\ \text{pN}\right)\hat{\boldsymbol{k}} + \left(-0.128\ \text{pN}\right)\hat{\boldsymbol{j}} + \left(-0.384\ \text{pN}\right)\hat{\boldsymbol{k}} + \left(-0.192\ \text{pN}\right)\hat{\boldsymbol{i}}\\ &= -\left(0.192\ \text{pN}\right)\hat{\boldsymbol{i}} - \left(0.128\ \text{pN}\right)\hat{\boldsymbol{j}} - \left(0.577\ \text{pN}\right)\hat{\boldsymbol{k}}\\ &= \boxed{-\left(0.19\ \text{pN}\right)\hat{\boldsymbol{i}} - \left(0.13\ \text{pN}\right)\hat{\boldsymbol{j}} - \left(0.58\ \text{pN}\right)\hat{\boldsymbol{k}}}\end{aligned}$$

20 •• The section of wire shown in Figure 26-32 carries a current equal to 1.8 A from *a* to *b*. The segment is in a region that has a magnetic field whose value is $1.2\ \text{T}\,\hat{\boldsymbol{k}}$. Find the total force on the wire and show that the total force is the same as if the wire were in the form of a straight wire directly from *a* to *b* and carrying the same current.

Picture the Problem We can use $\vec{\boldsymbol{F}} = I\vec{\ell}\times\vec{\boldsymbol{B}}$ to find the force acting on the segments of the wire as well as the magnetic force acting on the wire if it were a straight segment from *a* to *b*.

Express the magnetic force acting on the wire:

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_{3\,\text{cm}} + \vec{\boldsymbol{F}}_{4\,\text{cm}}$$

Evaluate $\vec{\boldsymbol{F}}_{3\,\text{cm}}$:

$$\begin{aligned}\vec{\boldsymbol{F}}_{3\,\text{cm}} &= \left(1.8\ \text{A}\right)\left[\left(3.0\ \text{cm}\right)\hat{\boldsymbol{i}}\times\left(1.2\ \text{T}\right)\hat{\boldsymbol{k}}\right]\\ &= \left(0.0648\ \text{N}\right)\left(-\hat{\boldsymbol{j}}\right)\\ &= -\left(0.0648\ \text{N}\right)\hat{\boldsymbol{j}}\end{aligned}$$

Evaluate $\vec{\boldsymbol{F}}_{4\,\text{cm}}$:

$$\begin{aligned}\vec{\boldsymbol{F}}_{4\,\text{cm}} &= \left(1.8\ \text{A}\right)\left[\left(4.0\ \text{cm}\right)\hat{\boldsymbol{j}}\times\left(1.2\ \text{T}\right)\hat{\boldsymbol{k}}\right]\\ &= \left(0.0864\ \text{N}\right)\hat{\boldsymbol{i}}\end{aligned}$$

Substitute to obtain:

$$\vec{F} = -(0.0648\,\text{N})\hat{j} + (0.0864\,\text{N})\hat{i} = \boxed{(86\,\text{mN})\hat{i} - (65\,\text{mN})\hat{j}}$$

If the wire were straight from *a* to *b*:

$$\vec{\ell} = (3.0\,\text{cm})\hat{i} + (4.0\,\text{cm})\hat{j}$$

The magnetic force acting on the wire is:

$$\vec{F} = (1.8\,\text{A})\left[(3.0\,\text{cm})\hat{i} + (4.0\,\text{cm})\hat{j}\right] \times (1.2\,\text{T})\hat{k} = -(0.0648\,\text{N})\hat{j} + (0.0864\,\text{N})\hat{i}$$
$$= \boxed{(86\,\text{mN})\hat{i} - (65\,\text{mN})\hat{j}}$$

in agreement with the result obtained above when we treated the two straight segments of the wire separately.

21 •• A straight, stiff, horizontal 25-cm-long wire that has a mass equal to 50 g is connected to a source of emf by light, flexible leads. A magnetic field of 1.33 T is horizontal and perpendicular to the wire. Find the current necessary to "float" the wire, that is, when it is released from rest it remains at rest.

Picture the Problem Because the magnetic field is horizontal and perpendicular to the wire, the force it exerts on the current-carrying wire will be vertical. Under equilibrium conditions, this upward magnetic force will be equal to the downward gravitational force acting on the wire.

Apply $\sum F_{\text{vertical}} = 0$ to the wire:

$$F_{\text{mag}} - w = 0$$

Express F_{mag}:

$$F_{\text{mag}} = I\ell B \text{ because } \theta = 90°.$$

Substitute for F_{mag} to obtain:

$$I\ell B - mg = 0 \Rightarrow I = \frac{mg}{\ell B}$$

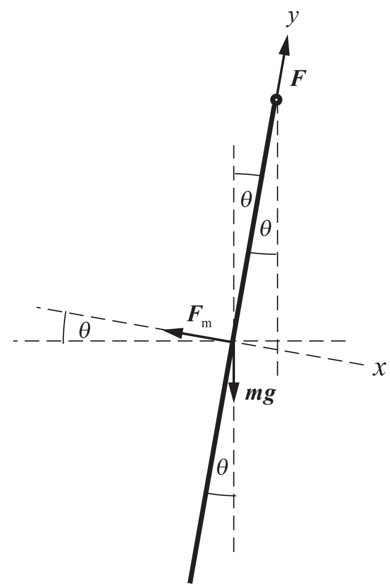
Substitute numerical values and evaluate I:

$$I = \frac{(50\,\text{g})(9.81\,\text{m/s}^2)}{(25\,\text{cm})(1.33\,\text{T})} = \boxed{1.5\,\text{A}}$$

22 •• In your physics laboratory class, you have constructed a simple *gaussmeter* for measuring the horizontal component of magnetic fields. The setup consists of a stiff 50-cm wire that hangs vertically from a conducting pivot so that its free end makes contact with a pool of mercury in a dish below (Figure 26-33). The mercury provides an electrical contact without constraining the movement of the wire. The wire has a mass of 5.0 g and conducts a current downward.
(*a*) What is the equilibrium angular displacement of the wire from vertical if the horizontal component of the magnetic field is 0.040 T and the current is 0.20 A?

(*b*) What is the sensitivity of this gaussmeter? That is, what is the ratio of the output to the input (in radians per tesla).

Picture the Problem The magnetic field is out of the page. The diagram shows the gaussmeter displaced from equilibrium under the influence of the gravitational force $m\vec{g}$, the magnetic force $\vec{F}_{\text{m}}$, and the force exerted by the conducting pivot $\vec{F}$. We can apply the condition for translational equilibrium in the *x* direction to find the equilibrium angular displacement of the wire from the vertical. In Part (*b*) we can solve the equation derived in Part (*a*) for *B* and evaluate this expression for the given data to find the horizontal magnetic field sensitivity of this gaussmeter.



(*a*) Apply $\sum F_x = 0$ to the wire to obtain:

$$\sum F_x = -F_{\text{m}} + mg\sin\theta = 0$$

The magnitude of the magnetic force acting on the wire is given by:

$F_{\text{m}} = I\ell B\sin\phi$

or, because $\phi = 90°$,

$F_{\text{m}} = I\ell B$

Substitute for F_{m} to obtain:

$$-I\ell B + mg\sin\theta = 0 \qquad (1)$$

Solving for θ yields:

$$\theta = \sin^{-1}\left[\frac{I\ell B}{mg}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{(0.20\,\text{A})(0.50\,\text{m})(0.040\,\text{T})}{(0.0050\,\text{kg})(9.81\,\text{m/s}^2)}\right] = 4.679° = \boxed{4.7°} = \boxed{82\,\text{mrad}}$$

(*b*) The sensitivity of this gaussmeter is the ratio of the output to the input:

$$\text{sensitivity} = \frac{\theta}{B}$$

Substitute numerical values and evaluate the sensitivity of the gaussmeter:

$$\text{sensitivity} = \frac{82\ \text{mrad}}{0.040\ \text{T}} = \boxed{2.0\ \text{rad/T}}$$

23 •• [SSM] A 10-cm long straight wire is parallel with the x axis and carries a current of 2.0 A in the $+x$ direction. The force on this wire due to the presence of a magnetic field $\vec{B}$ is $3.0\ \text{N}\hat{j} + 2.0\ \text{N}\hat{k}$. If this wire is rotated so that it is parallel with the y axis with the current in the $+y$ direction, the force on the wire becomes $-3.0\ \text{N}\hat{i} - 2.0\ \text{N}\hat{k}$. Determine the magnetic field $\vec{B}$.

Picture the Problem We can use the information given in the 1st and 2nd sentences to obtain an expression containing the components of the magnetic field $\vec{B}$. We can then use the information in the 1st and 3rd sentences to obtain a second equation in these components that we can solve simultaneously for the components of $\vec{B}$.

Express the magnetic field $\vec{B}$ in terms of its components:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \qquad (1)$$

Express $\vec{F}$ in terms of $\vec{B}$:

$$\begin{aligned}\vec{F} &= I\vec{\ell}\times\vec{B} = (2.0\,\text{A})\left[(0.10\,\text{m})\hat{i}\right]\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)\\ &= (0.20\,\text{A}\cdot\text{m})\hat{i}\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) = -(0.20\,\text{A}\cdot\text{m})B_z\hat{j} + (0.20\,\text{A}\cdot\text{m})B_y\hat{k}\end{aligned}$$

Equate the components of this expression for $\vec{F}$ with those given in the second sentence of the statement of the problem to obtain:

$$-(0.20\,\text{A}\cdot\text{m})B_z = 3.0\,\text{N}$$
and
$$(0.20\,\text{A}\cdot\text{m})B_y = 2.0\,\text{N}$$

Noting that B_x is undetermined, solve for B_z and B_y:

$$B_z = -15\,\text{T} \text{ and } B_y = 10\,\text{T}$$

When the wire is rotated so that the current flows in the positive y direction:

$$\begin{aligned}\vec{F} &= I\vec{\ell}\times\vec{B} = (2.0\,\text{A})\left[(0.10\,\text{m})\hat{j}\right]\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)\\ &= (0.20\,\text{A}\cdot\text{m})\hat{j}\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) = (0.20\,\text{A}\cdot\text{m})B_z\hat{i} - (0.20\,\text{A}\cdot\text{m})B_x\hat{k}\end{aligned}$$

Equate the components of this expression for $\vec{F}$ with those given in the third sentence of the problem statement to obtain:

$$(0.20\,\text{A}\cdot\text{m})B_x = -2.0\,\text{N}$$
and
$$-(0.20\,\text{A}\cdot\text{m})B_z = -3.0\,\text{N}$$

Solve for B_x and B_z to obtain:

$B_x = 10\,\text{T}$ and, in agreement with our results above, $B_z = -15\,\text{T}$

Substitute in equation (1) to obtain:

$$\vec{B} = \boxed{(10\,\text{T})\hat{\boldsymbol{i}} + (10\,\text{T})\hat{\boldsymbol{j}} - (15\,\text{T})\hat{\boldsymbol{k}}}$$

24 •• A 10-cm long straight wire is parallel with the z axis and carries a current of 4.0 A in the + z direction. The force on this wire due to a uniform magnetic field $\vec{B}$ is $-0.20\ \text{N}\,\hat{\boldsymbol{i}} + 0.20\ \text{N}\,\hat{\boldsymbol{j}}$. If this wire is rotated so that it is parallel with the x axis with the current is in the +x direction, the force on the wire becomes $0.20\,\hat{\boldsymbol{k}}\,\text{N}$. Find $\vec{B}$.

Picture the Problem We can use the information given in the 1st and 2nd sentences to obtain an expression containing the components of the magnetic field $\vec{B}$. We can then use the information in the 1st and 3rd sentences to obtain a second equation in these components that we can solve simultaneously for the components of $\vec{B}$.

Express the magnetic field $\vec{B}$ in terms of its components:

$$\vec{B} = B_x\hat{\boldsymbol{i}} + B_y\hat{\boldsymbol{j}} + B_z\hat{\boldsymbol{k}} \qquad (1)$$

Express $\vec{F}$ in terms of $\vec{B}$:

$$\begin{aligned}\vec{F} &= I\vec{\ell}\times\vec{B}\\ &= (4.0\,\text{A})\left[(0.1\,\text{m})\hat{\boldsymbol{k}}\right]\times\left(B_x\hat{\boldsymbol{i}} + B_y\hat{\boldsymbol{j}} + B_z\hat{\boldsymbol{k}}\right)\\ &= (0.40\,\text{A}\cdot\text{m})\hat{\boldsymbol{k}}\times\left(B_x\hat{\boldsymbol{i}} + B_y\hat{\boldsymbol{j}} + B_z\hat{\boldsymbol{k}}\right)\\ &= (0.40\,\text{A}\cdot\text{m})B_y\hat{\boldsymbol{j}} - (0.40\,\text{A}\cdot\text{m})B_y\hat{\boldsymbol{i}}\end{aligned}$$

Equate the components of this expression for $\vec{F}$ with those given in the second sentence of the statement of the problem to obtain:

$$(0.40\,\text{A}\cdot\text{m})B_y = 0.20\,\text{N}$$
and
$$(0.40\,\text{A}\cdot\text{m})B_x = 0.20\,\text{N}$$

Noting that B_z is undetermined, solve for B_x and B_y:

$B_x = 0.50\,\text{T}$ and $B_y = 0.50\,\text{T}$

When the wire is rotated so that the current flows in the positive x direction:

$$\vec{F} = I\vec{\ell} \times \vec{B} = (4.0\,\text{A})\left[(0.10\,\text{m})\hat{i}\right] \times \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$
$$= (0.40\,\text{A}\cdot\text{m})\hat{i} \times \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) = -(0.40\,\text{A}\cdot\text{m})B_z\hat{j} + (0.40\,\text{A}\cdot\text{m})B_y\hat{k}$$

Equate the components of this expression for $\vec{F}$ with those given in the third sentence of the problem statement to obtain:

$$-(0.40\,\text{A}\cdot\text{m})B_z = 0$$
and
$$(0.40\,\text{A}\cdot\text{m})B_y = 0.2\,\text{N}$$

Solve for B_z and B_y to obtain:

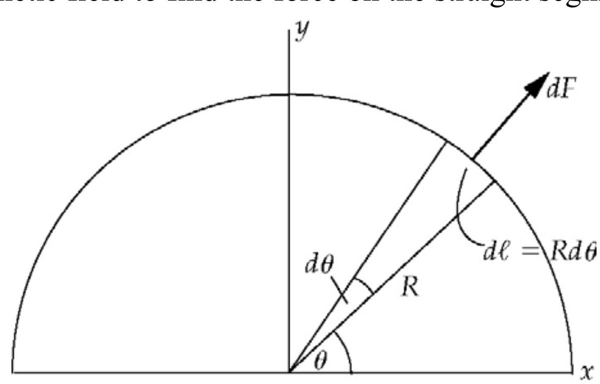
$$B_z = 0$$
and, in agreement with our results above,
$$B_y = 0.50\,\text{T}$$

Substitute in equation (1) to obtain:

$$\vec{B} = \boxed{(0.50\,\text{T})\hat{i} + (0.50\,\text{T})\hat{j}}$$

25 •• [SSM] A current-carrying wire is bent into a closed semicircular loop of radius R that lies in the xy plane (Figure 26-34). The wire is in a uniform magnetic field that is in the $+z$ direction, as shown. Verify that the force acting on the loop is zero.

Picture the Problem With the current in the direction indicated and the magnetic field in the z direction, pointing out of the plane of the page, the force is in the radial direction and we can integrate the element of force dF acting on an element of length $d\ell$ between $\theta = 0$ and π to find the force acting on the semicircular portion of the loop and use the expression for the force on a current-carrying wire in a uniform magnetic field to find the force on the straight segment of the loop.



Express the net force acting on the semicircular loop of wire:

$$\vec{F} = \vec{F}_{\substack{\text{semicircular}\\\text{loop}}} + \vec{F}_{\substack{\text{straight}\\\text{segment}}} \qquad (1)$$

Express the force acting on the straight segment of the loop:

$$\vec{F}_{\substack{\text{straight}\\ \text{segment}}} = I\vec{\ell} \times \vec{B} = 2RI\hat{i} \times B\hat{k} = -2RIB\hat{j}$$

Express the force dF acting on the element of the wire of length $d\ell$:

$$dF = Id\ell B = IRBd\theta$$

Express the x and y components of dF:

$$dF_x = dF\cos\theta \text{ and } dF_y = dF\sin\theta$$

Because, by symmetry, the x component of the force is zero, we can integrate the y component to find the force on the wire:

$$dF_y = IRB\sin\theta\, d\theta$$

and

$$\vec{F}_{\substack{\text{semicircular}\\ \text{loop}}} = F_y\hat{j} = \left(RIB\int_0^{\pi}\sin\theta\, d\theta\right)\hat{j}$$
$$= 2RIB\hat{j}$$

Substitute in equation (1) to obtain:

$$\vec{F} = 2RIB\hat{j} - 2RIB\hat{j} = \boxed{0}$$

26 ••• A wire bent in some arbitrary shape carries a current I. The wire is in a region with a uniform magnetic field $\vec{B}$. Show that the total force on the part of the wire from some arbitrary point on the wire (designated as a) to some other arbitrary point on the wire (designated as b) is $\vec{F} = I\vec{L} \times \vec{B}$, where $\vec{L}$ is the vector from point a to point b. In other words, show that the force on an arbitrary section of the bent wire is the same as the force would be on a straight section wire carrying the same current and connecting the two endpoints of the arbitrary section.

Picture the Problem We can integrate the expression for the force $d\vec{F}$ acting on an element of the wire of length $d\vec{L}$ from a to b to show that $\vec{F} = I\vec{L} \times \vec{B}$.

Express the force $d\vec{F}$ acting on the element of the wire of length $d\vec{L}$:

$$d\vec{F} = Id\vec{L} \times \vec{B}$$

Integrate this expression to obtain:

$$\vec{F} = \int_a^b Id\vec{L} \times \vec{B}$$

Because $\vec{B}$ and I are constant:

$$\vec{F} = I\left(\int_a^b d\vec{L}\right) \times \vec{B} = \boxed{I\vec{L} \times \vec{B}}$$

where $\vec{L}$ is the vector from a to b.

Motion of a Point Charge in a Magnetic Field

27 • [SSM] A proton moves in a 65-cm-radius circular orbit that is perpendicular to a uniform magnetic field of magnitude 0.75 T. (*a*) What is the orbital period for the motion? (*b*) What is the speed of the proton? (*c*) What is the kinetic energy of the proton?

Picture the Problem We can apply Newton's second law to the orbiting proton to relate its speed to its radius. We can then use $T = 2\pi r/v$ to find its period. In Part (*b*) we can use the relationship between T and v to determine v. In Part (*c*) we can use its definition to find the kinetic energy of the proton.

(*a*) Relate the period T of the motion of the proton to its orbital speed v:

$$T = \frac{2\pi r}{v} \quad (1)$$

Apply Newton's second law to the proton to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Substitute for r in equation (1) and simplify to obtain:

$$T = \frac{2\pi m}{qB}$$

Substitute numerical values and evaluate T:

$$T = \frac{2\pi\left(1.673\times10^{-27}\ \text{kg}\right)}{\left(1.602\times10^{-19}\ \text{C}\right)\left(0.75\,\text{T}\right)} = 87.4\,\text{ns}$$
$$= \boxed{87\,\text{ns}}$$

(*b*) From equation (1) we have:

$$v = \frac{2\pi r}{T}$$

Substitute numerical values and evaluate v:

$$v = \frac{2\pi\left(0.65\,\text{m}\right)}{87.4\,\text{ns}} = 4.67\times10^{7}\ \text{m/s}$$
$$= \boxed{4.7\times10^{7}\ \text{m/s}}$$

(*c*) Using its definition, express and evaluate the kinetic energy of the proton:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\left(1.673\times10^{-27}\ \text{kg}\right)\left(4.67\times10^{7}\ \text{m/s}\right)^2 = 1.82\times10^{-12}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}}$$
$$= \boxed{11\,\text{MeV}}$$

28 • A 4.5-keV electron (an electron that has a kinetic energy equal to 4.5 keV) moves in a circular orbit that is perpendicular to a magnetic field of 0.325 T. (*a*) Find the radius of the orbit. (*b*) Find the frequency and period of the orbital motion.

Picture the Problem (*a*) We can apply Newton's second law to the orbiting electron to obtain an expression for the radius of its orbit as a function of its mass *m*, charge *q*, speed *v*, and the magnitude of the magnetic field *B*. Using the definition of kinetic energy will allow us to express *r* in terms of *m*, *q*, *B*, and the electron's kinetic energy *K*. (*b*) The period of the orbital motion is given by $T = 2\pi r/v$. Substituting for *r* (or r/v) from Part (*a*) will eliminate the orbital speed of the electron and leave us with an expression for *T* that depends only on *m*, *q*, and *B*. The frequency of the orbital motion is the reciprocal of the period of the orbital motion.

(*a*) Apply Newton's second law to the orbiting electron to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Express the kinetic energy of the electron:

$$K = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}}$$

Substituting for *v* in the expression for *r* and simplifying yields:

$$r = \frac{m}{qB}\sqrt{\frac{2K}{m}} = \frac{\sqrt{2Km}}{qB}$$

Substitute numerical values and evaluate *r*:

$$r = \frac{\sqrt{2(4.5\,\text{keV})\left(9.109\times10^{-31}\,\text{kg}\right)\left(\dfrac{1.602\times10^{-19}\,\text{J}}{\text{eV}}\right)}}{\left(1.602\times10^{-19}\,\text{C}\right)(0.325\,\text{T})} = 0.696\,\text{mm} = \boxed{0.70\,\text{mm}}$$

(*b*) Relate the period of the electron's motion to the radius of its orbit and its orbital speed:

$$T = \frac{2\pi r}{v}$$

Because $r = \dfrac{mv}{qB}$:

$$T = \frac{2\pi\dfrac{mv}{qB}}{v} = \frac{2\pi\, m}{qB}$$

Substitute numerical values and evaluate T:

$$T = \frac{2\pi\left(9.109\times10^{-31}\ \text{kg}\right)}{\left(1.602\times10^{-19}\ \text{C}\right)\left(0.325\ \text{T}\right)} = 1.099\times10^{-10}\ \text{s} = \boxed{0.11\ \text{ns}}$$

The frequency of the motion is known as the cyclotron frequency and is the reciprocal of the period of the electron's motion:

$$f = \frac{1}{T} = \frac{1}{0.110\ \text{ns}} = \boxed{9.1\ \text{GHz}}$$

29 •• A proton, a deuteron and an alpha particle in a region with a uniform magnetic field each follow circular paths that have the same radius. The deuteron has a charge that is equal to the charge a proton has, and an alpha particle has a charge that is equal to twice the charge a proton has. Assume that $m_\alpha = 2m_\text{d} = 4m_\text{p}$. Compare (*a*) their speeds, (*b*) their kinetic energies, and (*c*) the magnitudes of their angular momenta about the centers of the orbits.

Picture the Problem We can apply Newton's second law to the orbiting particles to derive an expression for their orbital speeds as a function of their charge, their mass, the magnetic field in which they are moving, and the radii of their orbits. We can then compare their speeds by expressing their ratios. In Parts (*b*) and (*c*) we can proceed similarly starting with the definitions of kinetic energy and angular momentum.

(*a*) Apply Newton's second law to an orbiting particle to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow v = \frac{qBr}{m}$$

The speeds of the orbiting particles are given by:

$$v_\text{p} = \frac{q_\text{p}Br}{m_\text{p}}, \qquad (1)$$

$$v_\alpha = \frac{q_\alpha Br}{m_\alpha}, \text{ and} \qquad (2)$$

$$v_\text{d} = \frac{q_\text{d}Br}{m_\text{d}} \qquad (3)$$

Divide equation (2) by equation (1) and simplify to obtain:

$$\frac{v_\alpha}{v_\text{p}} = \frac{\dfrac{q_\alpha Br}{m_\alpha}}{\dfrac{q_\text{p}Br}{m_\text{p}}} = \frac{q_\alpha m_\text{p}}{q_\text{p}m_\alpha} = \frac{2em_\text{p}}{e\left(4m_\text{p}\right)} = \tfrac{1}{2}$$

or

$$2v_\alpha = v_\text{p}$$

Divide equation (3) by equation (1) and simplify to obtain:

$$\frac{v_\mathrm{d}}{v_\mathrm{p}}=\frac{\dfrac{q_\mathrm{d}Br}{m_\mathrm{d}}}{\dfrac{q_\mathrm{p}Br}{m_\mathrm{p}}}=\frac{q_\mathrm{d}m_\mathrm{p}}{q_\mathrm{p}m_\mathrm{d}}=\frac{em_\mathrm{p}}{e(2m_\mathrm{p})}=\tfrac{1}{2}$$

or

$$2v_\mathrm{d}=v_\mathrm{p}$$

Combining these results yields:

$$\boxed{2v_\alpha=2v_\mathrm{d}=v_\mathrm{p}}$$

(*b*) Using the expression for its orbital speed derived in (*a*), express the kinetic energy of an orbiting particle:

$$K=\tfrac{1}{2}mv^2=\tfrac{1}{2}m\left(\frac{qBr}{m}\right)^2=\frac{q^2B^2r^2}{2m}$$

The kinetic energies of the three particles are given by:

$$K_\mathrm{p}=\frac{q_\mathrm{p}^2B^2r^2}{2m_\mathrm{p}},\qquad(4)$$

$$K_\alpha=\frac{q_\alpha^2B^2r^2}{2m_\alpha},\ \text{and}\qquad(5)$$

$$K_\mathrm{d}=\frac{q_\mathrm{d}^2B^2r^2}{2m_\mathrm{d}}\qquad(6)$$

Dividing equation (7) by equation (6) and simplifying yields:

$$\frac{K_\alpha}{K_\mathrm{p}}=\frac{\tfrac{1}{2}\dfrac{q_\alpha^2B^2r^2}{m_\alpha}}{\tfrac{1}{2}\dfrac{q_\mathrm{p}^2B^2r^2}{m_\mathrm{p}}}=\frac{q_\alpha^2m_\mathrm{p}}{q_\mathrm{p}^2m_\alpha}=\frac{(2e)^2m_\mathrm{p}}{e^2(4m_\mathrm{p})}$$

$$=1\Rightarrow K_\alpha=K_\mathrm{p}$$

Divide equation (8) by equation (6) and simplify to obtain:

$$\frac{K_\mathrm{d}}{K_\mathrm{p}}=\frac{\dfrac{q_\mathrm{d}^2B^2r^2}{2m_\mathrm{d}}}{\dfrac{q_\mathrm{p}^2B^2r^2}{2m_\mathrm{p}}}=\frac{q_\mathrm{d}^2m_\mathrm{p}}{q_\mathrm{p}^2m_\mathrm{d}}=\frac{e^2m_\mathrm{p}}{e^2(2m_\mathrm{p})}$$

$$=\tfrac{1}{2}\Rightarrow K_\mathrm{p}=2K_\mathrm{d}$$

Combining these results yields:

$$\boxed{K_\alpha=2K_\mathrm{d}=K_\mathrm{p}}$$

(*c*) The angular momenta of the orbiting particles are given by:

$$L_p = m_p v_p r\,,$$
$$L_\alpha = m_\alpha v_\alpha r\,, \text{ and}$$
$$L_d = m_d v_d r$$

Express the ratio of L_α to L_p:

$$\frac{L_\alpha}{L_p} = \frac{m_\alpha v_\alpha r}{m_p v_p r} = \frac{(4m_p)\left(\frac{1}{2}v_p\right)}{m_p v_p} = 2$$
or
$$L_\alpha = 2L_p$$

Express the ratio of L_d to L_p:

$$\frac{L_d}{L_p} = \frac{m_d v_d r}{m_p v_p r} = \frac{(2m_p)\left(\frac{1}{2}v_p\right)}{m_p v_p} = 1$$
or
$$L_d = L_p$$

Combining these results yields:

$$\boxed{L_\alpha = 2L_d = 2L_p}$$

30 •• A particle has a charge q, a mass m, a linear momentum of magnitude p and a kinetic energy K. The particle moves in a circular orbit of radius R perpendicular to a uniform magnetic field $\vec{B}$. Show that (*a*) $p = BqR$ and (*b*) $K = \frac{1}{2}B^2q^2R^2/m$.

Picture the Problem We can use the definition of momentum to express p in terms of v and apply Newton's second law to the orbiting particle to express v in terms of q, B, R, and m. In Part (*b*) we can express the particle's kinetic energy in terms of its momentum and use our result from Part (*a*) to show that $K = \frac{1}{2}B^2q^2R^2/m.$

(*a*) Express the momentum of the particle:

$$p = mv \qquad (1)$$

Apply $\sum F_{\text{radial}} = ma_c$ to the orbiting particle to obtain:

$$qvB = m\frac{v^2}{R} \Rightarrow v = \frac{qBR}{m}$$

Substitute for v in equation (1) to obtain:

$$p = m\left(\frac{qBR}{m}\right) = \boxed{qBR}$$

(*b*) Express the kinetic energy of the orbiting particle as a function of its momentum:

$$K = \frac{p^2}{2m}$$

Substitute our result for *p* from Part (*a*) to obtain:

$$K = \frac{(qBR)^2}{2m} = \boxed{\frac{q^2B^2R^2}{2m}}$$

31 •• **[SSM]** A beam of particles with velocity $\vec{v}$ enters a region that has a uniform magnetic field $\vec{B}$ in the $+x$ direction. Show that when the x component of the displacement of one of the particles is $2\pi(m/qB)v\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{B}$, the velocity of the particle is in the same direction as it was when the particle entered the field.

Picture the Problem The particle's velocity has a component v_1 parallel to $\vec{B}$ and a component v_2 normal to $\vec{B}$. $v_1 = v\cos\theta$ and is constant, whereas $v_2 = v\sin\theta$, being normal to $\vec{B}$, will result in a magnetic force acting on the beam of particles and circular motion perpendicular to $\vec{B}$. We can use the relationship between distance, rate, and time and Newton's second law to express the distance the particle moves in the direction of the field during one period of the motion.

Express the distance moved in the direction of $\vec{B}$ by the particle during one period:

$$x = v_1T \qquad (1)$$

Express the period of the circular motion of the particles in the beam:

$$T = \frac{2\pi r}{v_2} \qquad (2)$$

Apply Newton's second law to a particle in the beam to obtain:

$$qv_2B = m\frac{v_2^2}{r} \Rightarrow v_2 = \frac{qBr}{m}$$

Substituting for v_2 in equation (2) and simplifying yields:

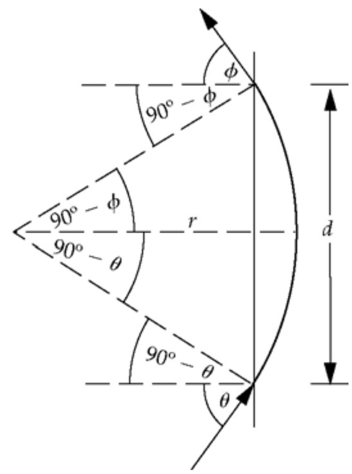
$$T = \frac{2\pi r}{\frac{qBr}{m}} = \frac{2\pi m}{qB}$$

Because $v_1 = v\cos\theta$, equation (1) becomes:

$$x = (v\cos\theta)\left(\frac{2\pi m}{qB}\right) = \boxed{2\pi\left(\frac{m}{qB}\right)v\cos\theta}$$

32 •• A proton that has a speed equal to 1.00×10^6 m/s enters a region that has a uniform magnetic field that has a magnitude of 0.800 T and points into the page, as shown in Figure 26-35. The proton enters the region at an angle θ = 60º. Find the exit angle ϕ and the distance d.

Picture the Problem The trajectory of the proton is shown to the right. We know that, because the proton enters the uniform field perpendicularly to the field, its trajectory while in the field will be circular. We can use symmetry considerations to determine ϕ. The application of Newton's second law to the proton while it is in the magnetic field and of trigonometry will allow us to conclude that $r = d$ and to determine the value of d.



From symmetry, it is evident that the angle θ in Figure 26-35 equals the angle ϕ:

$$\phi = \boxed{60°}$$

Apply $\sum F_{\text{radial}} = ma_c$ to the proton while it is in the magnetic field to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Use trigonometry to obtain:

$$\sin(90° - \theta) = \sin 30° = \frac{1}{2} = \frac{d/2}{r}$$

Solving for d yields:

$$r = d$$

Substitute for r to obtain:

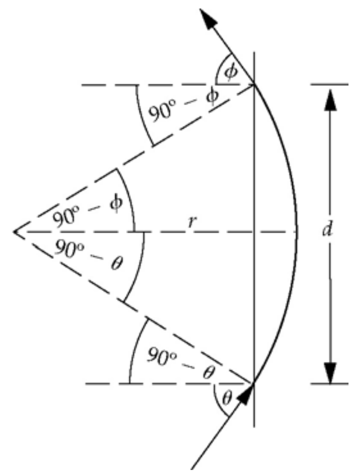
$$d = \frac{mv}{qB}$$

Substitute numerical values and evaluate d:

$$d = r = \frac{(1.673\times10^{-27}\text{ kg})(1.00\times10^{6}\text{ m/s})}{(1.602\times10^{-19}\text{ C})(0.800\text{ T})} = \boxed{13.1\text{ mm}}$$

33 •• [SSM] Suppose that in Figure 26-35, the magnetic field has a magnitude of 60 mT, the distance d is 40 cm, and θ is 24º. Find the speed v at which a particle enters the region and the exit angle ϕ if the particle is a (*a*) proton and (*b*) deuteron. Assume that $m_d = 2m_p$.

Picture the Problem The trajectory of the proton is shown to the right. We know that, because the proton enters the uniform field perpendicularly to the field, its trajectory while in the field will be circular. We can use symmetry considerations to determine ϕ. The application of Newton's second law to the proton and deuteron while they are in the uniform magnetic field will allow us to determine the values of v_p and v_d.



(*a*) From symmetry, it is evident that the angle θ in Figure 26-35 equals the angle ϕ:

$$\phi = \boxed{24°}$$

Apply $\sum F_{\text{radial}} = ma_c$ to the proton while it is in the magnetic field to obtain:

$$q_p v_p B = m_p \frac{v_p^2}{r_p} \Rightarrow v_p = \frac{q_p r_p B}{m_p} \quad (1)$$

Use trigonometry to obtain:

$$\sin(90° - \theta) = \sin 66° = \frac{d/2}{r}$$

Solving for r yields:

$$r = \frac{d}{2\sin 66°}$$

Substituting for r in equation (1) and simplifying yields:

$$v_p = \frac{q_p B d}{2m_p \sin 66°} \quad (2)$$

Substitute numerical values and evaluate v_p:

$$v_p = \frac{(1.602\times10^{-19}\ \text{C})(60\ \text{mT})(0.40\ \text{m})}{2(1.673\times10^{-27}\ \text{kg})\sin 66°} = \boxed{1.3\times10^{6}\ \text{m/s}}$$

(*b*) From symmetry, it is evident that the angle θ in Figure 26-35 equals the angle ϕ:

$\phi = \boxed{24°}$ independently of whether the particles are protons or deuterons.

For deuterons equation (2) becomes:

$$v_d = \frac{q_d Bd}{2m_d \sin 66°}$$

Because $m_d = 2m_p$ and $q_d = q_p$:

$$v_d \approx \frac{q_p Bd}{2(2m_p)\sin 66°} = \frac{q_p Bd}{4m_p \sin 66°}$$

Substitute numerical values and evaluate v_d:

$$v_d = \frac{(1.602\times10^{-19}\text{ C})(60\text{ mT})(0.40\text{ m})}{4(1.673\times10^{-27}\text{ kg})\sin 66°} = \boxed{6.3\times10^5\text{ m/s}}$$

34 •• The galactic magnetic field in some region of interstellar space has a magnitude of 1.00×10^{-9} T. A particle of interstellar dust has a mass of 10.0 μg and a total charge of 0.300 nC. How many years does it take for the particle to complete revolution of the circular orbit caused by its interaction with the magnetic field?

Picture the Problem We can apply Newton's second law of motion to express the orbital speed of the particle and then find the period of the dust particle from this orbital speed. Assume that the particle moves in a direction perpendicular to $\vec{B}$.

The period of the dust particle's motion is given by:

$$T = \frac{2\pi r}{v}$$

Apply $\sum F = ma_c$ to the particle:

$$qvB = m\frac{v^2}{r} \Rightarrow v = \frac{qBr}{m}$$

Substitute for v in the expression for T and simplify:

$$T = \frac{2\pi rm}{qBr} = \frac{2\pi m}{qB}$$

Substitute numerical values and evaluate T:

$$T = \frac{2\pi(10.0\times10^{-6}\text{ g}\times10^{-3}\text{ kg/g})}{(0.300\text{ nC})(1.00\times10^{-9}\text{ T})}$$

$$= 2.094\times10^{11}\text{ s}\times\frac{1\text{ y}}{31.56\text{ Ms}}$$

$$= \boxed{6.64\times10^3\text{ y}}$$

Applications of the Magnetic Force Acting on Charged Particles

35 • **[SSM]** A velocity selector has a magnetic field that has a magnitude equal to 0.28 T and is perpendicular to an electric field that has a magnitude equal

to 0.46 MV/m. (*a*) What must the speed of a particle be for that particle to pass through the velocity selector undeflected? What kinetic energy must (*b*) protons and (*c*) electrons have in order to pass through the velocity selector undeflected?

Picture the Problem Suppose that, for positively charged particles, their motion is from left to right through the velocity selector and the electric field is upward. Then the magnetic force must be downward and the magnetic field out of the page. We can apply the condition for translational equilibrium to relate v to E and B. In (*b*) and (*c*) we can use the definition of kinetic energy to find the energies of protons and electrons that pass through the velocity selector undeflected.

(*a*) Apply $\sum F_y = 0$ to the particle to obtain:

$$F_{\text{elec}} - F_{\text{mag}} = 0$$

or

$$qE - qvB = 0 \Rightarrow v = \frac{E}{B}$$

Substitute numerical values and evaluate v:

$$v = \frac{0.46\,\text{MV/m}}{0.28\,\text{T}} = 1.64\times10^{6}\ \text{m/s} = \boxed{1.6\times10^{6}\ \text{m/s}}$$

(*b*) The kinetic energy of protons passing through the velocity selector undeflected is:

$$K_{\text{p}} = \tfrac{1}{2}m_{\text{p}}v^2 = \tfrac{1}{2}\left(1.673\times10^{-27}\,\text{kg}\right)\left(1.64\times10^{6}\ \text{m/s}\right)^2 = 2.26\times10^{-15}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = \boxed{14\,\text{keV}}$$

(*c*) The kinetic energy of electrons passing through the velocity selector undeflected is:

$$K_{\text{e}} = \tfrac{1}{2}m_{\text{e}}v^2 = \tfrac{1}{2}\left(9.109\times10^{-31}\,\text{kg}\right)\left(1.64\times10^{6}\ \text{m/s}\right)^2 = 1.23\times10^{-18}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = \boxed{7.7\,\text{eV}}$$

36 •• A beam of protons is moving in the +*x* direction with a speed of 12.4 km/s through a region in which the electric field is perpendicular to the magnetic field. The beam is not deflected in this region. (*a*) If the magnetic field has a magnitude of 0.85 T and points in the + *y* direction, find the magnitude and direction of the electric field. (*b*) Would electrons that have the same velocity as the protons be deflected by these fields? If so, in what direction would they be deflected? If not, why not?

Picture the Problem Because the beam of protons is not deflected; we can conclude that the electric force acting on them is balanced by the magnetic force. Hence, we can find the magnetic force from the given data and use its definition to express the electric field.

(*a*) Because the beam of protons is not deflected:

$$\vec{F}_{\text{elec}} + \vec{F}_{\text{mag}} = 0$$

or

$$q\vec{E} + q\left(v\hat{\boldsymbol{i}} \times B\hat{\boldsymbol{j}}\right) = 0$$

Solving for $\vec{E}$ gives:

$$\vec{E} = -\left(v\hat{\boldsymbol{i}} \times B\hat{\boldsymbol{j}}\right) = -vB\hat{\boldsymbol{k}}$$

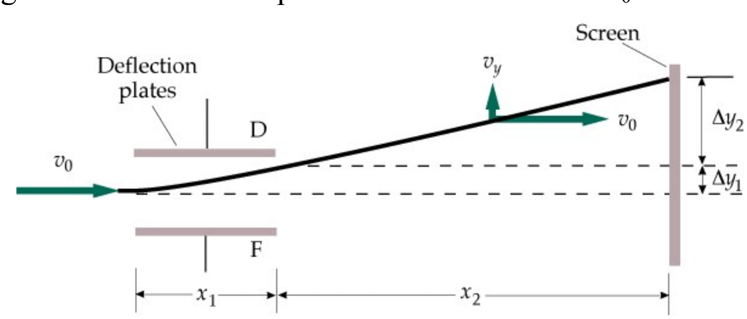
Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -(12.4\text{ km/s})(0.85\text{ T})\hat{\boldsymbol{k}}$$
$$= \boxed{-(11\text{kV/m})\hat{\boldsymbol{k}}}$$

(*b*) Because both $\vec{F}_{\text{mag}}$ and $\vec{F}_{\text{elec}}$ would be reversed, electrons are not deflected either.

37 •• The plates of a Thomson q/m apparatus are 6.00 cm long and are separated by 1.20 cm. The end of the plates is 30.0 cm from the tube screen. The kinetic energy of the electrons is 2.80 keV. If a potential difference of 25.0 V is applied across the deflection plates, by how much will the point where the beam strikes the screen displaced?

Picture the Problem Figure 26-18 is reproduced below. We can express the total deflection of the electron beam as the sum of the deflections while the beam is in the field between the plates and its deflection while it is in the field-free space. We can, in turn, use constant-acceleration equations to express each of these deflections. The resulting equation is in terms of v_0 and E. We can find v_0 from the kinetic energy of the beam and E from the potential difference across the plates and their separation. In Part (*b*) we can equate the electric and magnetic forces acting on an electron to express B in terms of E and v_0.



Express the total deflection Δy of the electrons:

$$\Delta y = \Delta y_1 + \Delta y_2 \qquad (1)$$

where Δy_1 is the deflection of the beam while it is in the electric field and Δy_2 is the deflection of the beam while it travels along a straight-line path outside the electric field.

Use a constant-acceleration equation to express Δy_1:

$$\Delta y_1 = \tfrac{1}{2} a_y (\Delta t)^2 \qquad (2)$$

where $\Delta t = x_1/v_0$ is the time an electron is in the electric field between the plates.

Apply Newton's second law to an electron between the plates to obtain:

$$qE = ma_y \Rightarrow a_y = \frac{qE}{m}$$

Substitute for a_y in equation (2) to obtain:

$$\Delta y_1 = \tfrac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{x_1}{v_0}\right)^2 \qquad (3)$$

Express the vertical deflection Δy_2 of the electrons once they are out of the electric field:

$$\Delta y_2 = v_y \Delta t_2 \qquad (4)$$

Use a constant-acceleration equation to find the vertical speed of an electron as it leaves the electric field:

$$\begin{aligned} v_y &= v_{0y} + a_y \Delta t_1 \\ &= 0 + \frac{qE}{m}\left(\frac{x_1}{v_0}\right) \end{aligned}$$

Substitute in equation (4) to obtain:

$$\Delta y_2 = \frac{qE}{m}\left(\frac{x_1}{v_0}\right)\left(\frac{x_2}{v_0}\right) = \frac{qEx_1x_2}{mv_0^2} \qquad (5)$$

Substitute equations (3) and (5) in equation (1) to obtain:

$$\Delta y = \tfrac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{x_1}{v_0}\right)^2 + \frac{qEx_1x_2}{mv_0^2}$$

or

$$\Delta y = \frac{qEx_1}{mv_0^2}\left(\frac{x_1}{2} + x_2\right) \qquad (6)$$

Use the definition of kinetic energy to express the square of the speed of the electrons:

$$K = \tfrac{1}{2}mv_0^2 \Rightarrow v_0^2 = \frac{2K}{m}$$

Express the electric field between the plates in terms of their potential difference:

$$E = \frac{V}{d}$$

Substituting for E and v_0^2 in equation (6) and simplifying yields:

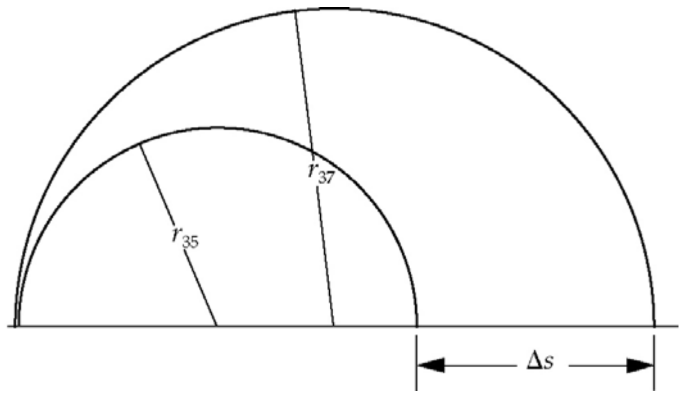
$$\Delta y = \frac{q\frac{V}{d}x_1}{m\frac{2K}{m}}\left(\frac{x_1}{2}+x_2\right) = \frac{qVx_1}{2dK}\left(\frac{x_1}{2}+x_2\right)$$

Substitute numerical values and evaluate Δy:

$$\Delta y = \frac{(1.602\times10^{-19}\ \text{C})(25.0\ \text{V})(6.00\,\text{cm})}{2(1.20\ \text{cm})(2.80\ \text{keV})}\left(\frac{6.00\,\text{cm}}{2}+30.0\,\text{cm}\right) = \boxed{7.37\,\text{mm}}$$

38 •• Chlorine has two stable isotopes, ^{35}Cl and ^{37}Cl. Chlorine gas which consists of singly-ionized ions is to be separated into its isotopic components using a mass spectrometer. The magnetic field strength in the spectrometer is 1.2 T. What is the minimum value of the potential difference through which these ions must be accelerated so that the separation between them, after they complete their semicircular path, is 1.4 cm?

Picture the Problem The diagram below represents the paths of the two ions entering the magnetic field at the left. The magnetic force acting on each causes them to travel in circular paths of differing radii due to their different masses. We can apply Newton's second law to an ion in the magnetic field to obtain an expression for its radius and then express their final separation in terms of these radii that, in turn, depend on the energy with which the ions enter the field. We can connect their energy to the potential through which they are accelerated using the work-kinetic energy theorem and relate their separation Δs to the accelerating potential difference ΔV.



Express the separation Δs of the chlorine ions:

$$\Delta s = 2(r_{37} - r_{35}) \quad (1)$$

Apply Newton's second law to an ion in the magnetic field of the mass spectrometer:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \quad (2)$$

Relate the speed of an ion as it enters the magnetic field to the potential difference through which it has been accelerated:

$$q\Delta V = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2q\Delta V}{m}}$$

Substitute for v in equation (2) to obtain:

$$r = \frac{m}{qB}\sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2m\Delta V}{qB^2}}$$

Use this equation to express the radii of the paths of the two chlorine isotopes to obtain:

$$r_{35} = \sqrt{\frac{2m_{35}\Delta V}{qB^2}} \text{ and } r_{37} = \sqrt{\frac{2m_{37}\Delta V}{qB^2}}$$

Substitute for r_{35} and r_{37} in equation (1) to obtain:

$$\Delta s = 2\left(\sqrt{\frac{2m_{35}\Delta V}{qB^2}} - \sqrt{\frac{2m_{35}\Delta V}{qB^2}}\right)$$
$$= 2\left(\frac{1}{B}\sqrt{\frac{2\Delta V}{q}}\left(\sqrt{m_{37}} - \sqrt{m_{35}}\right)\right)$$

Solving for ΔV yields:

$$\Delta V = \frac{qB^2\left(\frac{\Delta s}{2}\right)^2}{2\left(\sqrt{m_{37}} - \sqrt{m_{35}}\right)^2}$$

Substitute numerical values and evaluate ΔV:

$$\Delta V = \frac{(1.602\times10^{-19}\text{ C})(1.2\text{ T})^2\left(\frac{1.4\text{ cm}}{2}\right)^2}{2\left(\sqrt{37\text{ u}}-\sqrt{35\text{ u}}\right)^2}$$

$$= \frac{5.65\times10^{-24}\text{ C}\cdot\text{T}^2\cdot\text{m}^2}{\left(\sqrt{37}-\sqrt{35}\right)^2(1.66\times10^{-27}\text{ kg})}$$

$$= \boxed{0.12\text{ MV}}$$

39 •• **[SSM]** In a mass spectrometer, a singly ionized ^{24}Mg ion has a mass equal to 3.983×10^{-26} kg and is accelerated through a 2.50-kV potential difference. It then enters a region where it is deflected by a magnetic field of 557 G. (*a*) Find the radius of curvature of the ion's orbit. (*b*) What is the difference in the orbital radii of the ^{26}Mg and ^{24}Mg ions? Assume that their mass ratio is 26:24.

Picture the Problem We can apply Newton's second law to an ion in the magnetic field to obtain an expression for r as a function of m, v, q, and B and use the work-kinetic energy theorem to express the kinetic energy in terms of the potential difference through which the ion has been accelerated. Eliminating v between these equations will allow us to express r in terms of m, q, B, and ΔV.

Apply Newton's second law to an ion in the magnetic field of the mass spectrometer:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \qquad (1)$$

Apply the work-kinetic energy theorem to relate the speed of an ion as it enters the magnetic field to the potential difference through which it has been accelerated:

$$q\Delta V = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2q\Delta V}{m}}$$

Substitute for v in equation (1) and simplify to obtain:

$$r = \frac{m}{qB}\sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2m\Delta V}{qB^2}} \qquad (2)$$

(*a*) Substitute numerical values and evaluate equation (2) for ^{24}Mg :

$$r_{24} = \sqrt{\frac{2(3.983\times10^{-26}\text{ kg})(2.50\text{ kV})}{(1.602\times10^{-19}\text{ C})(557\times10^{-4}\text{ T})^2}}$$

$$= \boxed{63.3\text{ cm}}$$

(*b*) Express the difference in the radii for ^{24}Mg and ^{26}Mg:

$$\Delta r = r_{26} - r_{24}$$

Substituting for r_{26} and r_{24} and simplifying yields:

$$\Delta r = \sqrt{\frac{2m_{26}\Delta V}{qB^2}} - \sqrt{\frac{2m_{24}\Delta V}{qB^2}} = \frac{1}{B}\sqrt{\frac{2\Delta V}{q}}\left(\sqrt{m_{26}} - \sqrt{m_{24}}\right)$$
$$= \frac{1}{B}\sqrt{\frac{2\Delta V}{q}}\left(\sqrt{\frac{26}{24}m_{24}} - \sqrt{m_{24}}\right) = \frac{1}{B}\sqrt{\frac{2\Delta V m_{24}}{q}}\left(\sqrt{\frac{26}{24}} - 1\right)$$

Substitute numerical values and evaluate Δr:

$$\Delta r = \frac{1}{557\times10^{-4}\ \text{T}}\sqrt{\frac{2(2.50\ \text{kV})(3.983\times10^{-26}\ \text{kg})}{1.602\times10^{-19}\ \text{C}}}\left(\sqrt{\frac{26}{24}} - 1\right) = \boxed{2.58\ \text{cm}}$$

40 •• A beam of singly ionized ^{6}Li and ^{7}Li ions passes through a velocity selector and enters a region of uniform magnetic field with a velocity that is perpendicular to the direction of the field. If the diameter of the orbit of the ^{6}Li ions is 15 cm, what is the diameter of the orbit for ^{7}Li ions? Assume their mass ratio is 7:6.

Picture the Problem We can apply Newton's second law to an ion in the magnetic field of the spectrometer to relate the diameter of its orbit to its charge, mass, velocity, and the magnetic field. If we assume that the velocity is the same for the two ions, we can then express the ratio of the two diameters as the ratio of the masses of the ions and solve for the diameter of the orbit of ^{7}Li.

Apply Newton's second law to an ion in the field of the spectrometer:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Express the diameter of the orbit:

$$d = \frac{2mv}{qB}$$

The diameters of the orbits for ^{6}Li and ^{7}Li are:

$$d_6 = \frac{2m_6 v}{qB} \text{ and } d_7 = \frac{2m_7 v}{qB}$$

Assume that the velocities of the two ions are the same and divide the 2nd of these diameters by the first to obtain:

$$\frac{d_7}{d_6} = \frac{\frac{2m_7 v}{qB}}{\frac{2m_6 v}{qB}} = \frac{m_7}{m_6}$$

Solve for and evaluate d_7:

$$d_7 = \frac{m_7}{m_6} d_6 = \frac{7}{6}(15\,\text{cm}) = \boxed{18\,\text{cm}}$$

41 •• Using Example 26- 6, determine the time required for a ^{58}Ni ion and a ^{60}Ni ion to complete the semicircular path.

Picture the Problem The time required for each of the ions to complete its semicircular paths is half its period. We can apply Newton's second law to an ion in the magnetic field of the spectrometer to obtain an expression for r as a function of the charge and mass of the ion, its velocity, and the magnetic field.

Express the time for each ion to complete its semicircular path:

$$\Delta t = \tfrac{1}{2}T = \frac{\pi r}{v}$$

Apply Newton's second law to an ion in the field of the spectrometer:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Substitute for r to obtain:

$$\Delta t = \frac{\pi m}{qB}$$

Substitute numerical values and evaluate Δt_{58} and Δt_{60}:

$$\Delta t_{58} = \frac{58\pi(1.6606\times10^{-27}\,\text{kg})}{(1.602\times10^{-19}\,\text{C})(0.120\,\text{T})} = \boxed{15.7\,\mu\text{s}}$$

and

$$\Delta t_{60} = \frac{60\pi(1.6606\times10^{-27}\,\text{kg})}{(1.602\times10^{-19}\,\text{C})(0.120\,\text{T})} = \boxed{16.3\,\mu\text{s}}$$

42 •• Before entering a mass spectrometer, ions pass through a velocity selector consisting of parallel plates that are separated by 2.0 mm and have a potential difference of 160 V. The magnetic field strength is 0.42 T in the region between the plates. The magnetic field strength in the mass spectrometer is 1.2 T. Find (*a*) the speed of the ions entering the mass spectrometer and (*b*) the difference in the diameters of the orbits of singly ionized ^{238}U and ^{235}U. The mass of a ^{235}U ion is 3.903×10^{-25} kg.

Picture the Problem We can apply a condition for equilibrium to ions passing through the velocity selector to obtain an expression relating E, B, and v that we can solve for v. We can, in turn, express E in terms of the potential difference V between the plates of the selector and their separation d. In (*b*) we can apply

Newton's second law to an ion in the bending field of the spectrometer to relate its diameter to its mass, charge, velocity, and the magnetic field.

(*a*) Apply $\sum F_y = 0$ to the ions in the crossed fields of the velocity selector to obtain:

$$F_{\text{elec}} - F_{\text{mag}} = 0$$

or

$$qE - qvB = 0 \Rightarrow v = \frac{E}{B}$$

Express the electric field between the plates of the velocity selector in terms of their separation and the potential difference across them:

$$E = \frac{V}{d}$$

Substituting for *E* yields:

$$v = \frac{V}{dB}$$

Substitute numerical values and evaluate *v*:

$$v = \frac{160\,\text{V}}{(2.0\,\text{mm})(0.42\,\text{T})} = 1.905\times10^5\ \text{m/s}$$
$$= \boxed{1.9\times10^5\ \text{m/s}}$$

(*b*) Express the difference in the diameters of the orbits of singly ionized ^{238}U and ^{235}U:

$$\Delta d = d_{238} - d_{235} \qquad (1)$$

Apply $\sum F_{\text{radial}} = ma_{\text{c}}$ to an ion in the spectrometer's magnetic field:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Express the diameter of the orbit:

$$d = \frac{2mv}{qB}$$

The diameters of the orbits for ^{238}U and ^{235}U are:

$$d_{238} = \frac{2m_{238}v}{qB} \text{ and } d_{235} = \frac{2m_{235}v}{qB}$$

Substitute in equation (1) to obtain:

$$\Delta d = \frac{2m_{238}v}{qB} - \frac{2m_{235}v}{qB}$$
$$= \frac{2v}{qB}(m_{238} - m_{235})$$

Substitute numerical values and evaluate Δd:

$$\Delta d = \frac{2\left(1.905\times10^{5}\text{ m/s}\right)\left(238\text{u}-235\text{u}\right)\left(\dfrac{1.6606\times10^{-27}\text{ kg}}{\text{u}}\right)}{\left(1.602\times10^{-19}\text{ C}\right)\left(1.2\text{T}\right)} = \boxed{1\text{cm}}$$

43 •• [SSM] A cyclotron for accelerating protons has a magnetic field strength of 1.4 T and a radius of 0.70 m. (*a*) What is the cyclotron's frequency? (*b*) Find the kinetic energy of the protons when they emerge. (*c*) How will your answers change if deuterons are used instead of protons?

Picture the Problem We can express the cyclotron frequency in terms of the maximum orbital radius and speed of the protons/deuterons. By applying Newton's second law, we can relate the radius of the particle's orbit to its speed and, hence, express the cyclotron frequency as a function of the particle's mass and charge and the cyclotron's magnetic field. In Part (*b*) we can use the definition of kinetic energy and their maximum speed to find the maximum energy of the emerging protons.

(*a*) Express the cyclotron frequency in terms of the proton's orbital speed and radius:

$$f = \frac{1}{T} = \frac{1}{2\pi r/v} = \frac{v}{2\pi r} \qquad (1)$$

Apply Newton's second law to a proton in the magnetic field of the cyclotron:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \qquad (2)$$

Substitute for r in equation (1) and simplify to obtain:

$$f = \frac{qBv}{2\pi mv} = \frac{qB}{2\pi m} \qquad (3)$$

Substitute numerical values and evaluate f:

$$f = \frac{\left(1.602\times10^{-19}\text{ C}\right)\left(1.4\text{T}\right)}{2\pi\left(1.673\times10^{-27}\text{ kg}\right)} = 21.3\text{MHz}$$
$$= \boxed{21\text{MHz}}$$

(*b*) Express the maximum kinetic energy of a proton:

$$K_{\text{max}} = \tfrac{1}{2}mv_{\text{max}}^2$$

From equation (2), v_{max} is given by:

$$v_{\text{max}} = \frac{qBr_{\text{max}}}{m}$$

Substitute for v_{max} and simplify to obtain:

$$K_{max} = \tfrac{1}{2}m\left(\frac{qBr_{max}}{m}\right)^2 = \tfrac{1}{2}\left(\frac{q^2B^2}{m}\right)r_{max}^2$$

Substitute numerical values and evaluate K_{max}:

$$K_{max} = \tfrac{1}{2}\left(\frac{(1.602\times10^{-19}\text{ C})^2(1.4\text{ T})^2}{1.673\times10^{-27}\text{ kg}}\right)(0.7\text{ m})^2 = 7.37\times10^{-12}\text{ J}\times\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}}$$
$$= 46.0\text{ MeV} = \boxed{46\text{ MeV}}$$

(c) From equation (3) we see that doubling m halves f:

$$f_{\text{deuterons}} = \tfrac{1}{2}f_{\text{protons}} = \boxed{11\text{MHz}}$$

From our expression for K_{max} we see that doubling m halves K:

$$K_{\text{deuterons}} = \tfrac{1}{2}K_{\text{protons}} = \boxed{23\text{ MeV}}$$

44 •• A certain cyclotron that has a magnetic field whose magnitude is 1.8 T is designed to accelerate protons to a kinetic energy of 25 MeV. (a) What is the cyclotron frequency for this cyclotron? (b) What must the minimum radius of the magnet be to achieve this energy? (c) If the alternating potential difference applied to the dees has a maximum value of 50 kV, how many revolutions must the protons make before emerging with kinetic energies of 25 MeV?

Picture the Problem We can express the cyclotron frequency in terms of the maximum orbital radius and speed of the protons be accelerated in the cyclotron. By applying Newton's second law, we can relate the radius of the proton's orbit to its speed and, hence, express the cyclotron frequency as a function of the its mass and charge and the cyclotron's magnetic field. In Part (b) we can use the definition of kinetic energy express the minimum radius required to achieve the desired emergence energy. In Part (c) we can find the number of revolutions required to achieve this emergence energy from the energy acquired during each revolution.

(a) Express the cyclotron frequency in terms of the proton's orbital speed and radius:

$$f = \frac{1}{T} = \frac{1}{2\pi r/v} = \frac{v}{2\pi r}$$

Apply Newton's second law to a proton in the magnetic field of the cyclotron:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \qquad (1)$$

Substitute for v and simplify to obtain:

$$f=\frac{qBv}{2\pi mv}=\frac{qB}{2\pi m}$$

Substitute numerical values and evaluate f:

$$f=\frac{\left(1.602\times10^{-19}\text{ C}\right)\left(1.8\text{ T}\right)}{2\pi\left(1.673\times10^{-27}\text{ kg}\right)}$$
$$=\boxed{27\text{ MHz}}$$

(b) Using the definition of kinetic energy, relate emergence energy of the protons to their velocity:

$$K=\tfrac{1}{2}mv^2\Rightarrow v=\sqrt{\frac{2K}{m}}$$

Substitute for v in equation (1) and simplify to obtain:

$$r=\frac{m}{qB}\sqrt{\frac{2K}{m}}=\frac{\sqrt{2Km}}{qB}$$

Substitute numerical values and evaluate r_{min}:

$$r=\frac{\sqrt{2\left(25\text{ MeV}\right)\left(1.673\times10^{-27}\text{ kg}\right)}}{\left(1.602\times10^{-19}\text{ C}\right)\left(1.8\text{ T}\right)}$$
$$=\boxed{40\text{ cm}}$$

(c) Express the required number of revolutions N in terms of the energy gained per revolution:

$$N=\frac{25\text{ MeV}}{E_{\text{rev}}}$$

Because the beam is accelerated through a potential difference of 50 kV twice during each revolution:

$$E_{\text{rev}}=2q\Delta V=100\text{ keV}$$

Substitute the numerical value of E_{rev} and evaluate N:

$$N=\frac{25\text{ MeV}}{100\text{ keV/rev}}=\boxed{2.5\times10^{2}\text{ rev}}$$

45 •• Show that for a given cyclotron the cyclotron frequency for accelerating deuterons is the same as the frequency for accelerating alpha particles is half the frequency for accelerating protons in the same magnetic field. The deuteron has a charge that is equal to the charge a proton has, and an alpha particle has a charge that is equal to twice the charge a proton has. Assume that $m_\alpha = 2m_{\text{d}} = 4m_{\text{p}}$.

Picture the Problem We can express the cyclotron frequency in terms of the maximum orbital radius and speed of a particle being accelerated in the cyclotron. By applying Newton's second law, we can relate the radius of the particle's orbit

to its speed and, hence, express the cyclotron frequency as a function of its charge-to-mass ratio and the cyclotron's magnetic field. We can then use data for the relative charges and masses of deuterons, alpha particles, and protons to establish the ratios of their cyclotron frequencies.

Express the cyclotron frequency in terms of a particle's orbital speed and radius:

$$f = \frac{1}{T} = \frac{1}{2\pi r/v} = \frac{v}{2\pi r}$$

Apply Newton's second law to a particle in the magnetic field of the cyclotron:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Substitute for r to obtain:

$$f = \frac{qBv}{2\pi mv} = \frac{B}{2\pi}\frac{q}{m} \qquad (1)$$

Evaluate equation (1) for deuterons:

$$f_{\mathrm{d}} = \frac{B}{2\pi}\frac{q_{\mathrm{d}}}{m_{\mathrm{d}}} = \frac{B}{2\pi}\frac{e}{m_{\mathrm{d}}}$$

Evaluate equation (1) for alpha particles:

$$f_\alpha = \frac{B}{2\pi}\frac{q_\alpha}{m_\alpha} = \frac{B}{2\pi}\frac{2e}{2m_{\mathrm{d}}} = \frac{B}{2\pi}\frac{e}{m_{\mathrm{d}}}$$

and

$$\boxed{f_{\mathrm{d}} = f_\alpha}$$

Evaluate equation (1) for protons:

$$f_{\mathrm{p}} = \frac{B}{2\pi}\frac{q_{\mathrm{p}}}{m_{\mathrm{p}}} = \frac{B}{2\pi}\frac{e}{\frac{1}{2}m_{\mathrm{d}}} = 2\left(\frac{B}{2\pi}\frac{e}{m_{\mathrm{d}}}\right)$$

$$= 2f_{\mathrm{d}}$$

and

$$\tfrac{1}{2}f_{\mathrm{p}} = \boxed{f_{\mathrm{d}} = f_\alpha}$$

46 ••• Show that the radius of the orbit of a charged particle in a cyclotron is proportional to the square root of the number of orbits completed.

Picture the Problem We can apply Newton's second law to the orbiting charged particle to obtain an expression for its radius as a function of its particle's kinetic energy. Because the energy gain per revolution is constant, we can express this kinetic energy as the product of the number of orbits completed and the energy gained per revolution and, hence, show that the radius is proportional to the square root of the number of orbits completed.

Apply Newton's second law to a particle in the magnetic field of the cyclotron:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \qquad (1)$$

Express the kinetic energy of the particle in terms of its speed and solve for v:

$$K = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}} \qquad (2)$$

Noting that the energy gain per revolution is constant, express the kinetic energy in terms of the number of orbits N completed by the particle and energy E_{rev} gained by the particle each revolution:

$$K = NE_{\text{rev}} \qquad (3)$$

Substitute equations (2) and (3) in equation (1) to obtain:

$$r = \frac{m}{qB}\sqrt{\frac{2K}{m}} = \frac{1}{qB}\sqrt{2mK}$$
$$= \frac{1}{qB}\sqrt{2mNE_{\text{rev}}} = \frac{\sqrt{2mE_{\text{rev}}}}{qB}N^{1/2}$$
or $\boxed{r \propto N^{1/2}}$

Torques on Current Loops, Magnets, and Magnetic Moments

47 • [SSM] A small circular coil consisting of 20 turns of wire lies in a region with a uniform magnetic field whose magnitude is 0.50 T. The arrangement is such that the normal to the plane of the coil makes an angle of 60º with the direction of the magnetic field. The radius of the coil is 4.0 cm, and the wire carries a current of 3.0 A. (*a*) What is the magnitude of the magnetic moment of the coil? (*b*) What is the magnitude of the torque exerted on the coil?

Picture the Problem We can use the definition of the magnetic moment of a coil to evaluate μ and the expression for the torque exerted on the coil $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$ to find the magnitude of τ.

(*a*) Using its definition, express the magnetic moment of the coil:

$$\mu = NIA = NI\pi r^2$$

Substitute numerical values and evaluate μ:

$$\mu = (20)(3.0\,\text{A})\pi(0.040\,\text{m})^2$$
$$= 0.302\,\text{A}\cdot\text{m}^2 = \boxed{0.30\,\text{A}\cdot\text{m}^2}$$

(*b*) Express the magnitude of the torque exerted on the coil:

$$\tau = \mu B \sin\theta$$

Substitute numerical values and evaluate τ:

$$\tau = \left(0.302\,\mathrm{A\cdot m^2}\right)\left(0.50\,\mathrm{T}\right)\sin 60° = \boxed{0.13\,\mathrm{N\cdot m}}$$

48 • What is the maximum torque on a 400-turn circular coil of radius 0.75 cm that carries a current of 1.6 mA and is in a region with a uniform magnetic field of 0.25 T?

Picture the Problem The coil will experience the maximum torque when the plane of the coil makes an angle of 90° with the direction of $\vec{B}$. The magnitude of the maximum torque is then given by $\tau_{\max} = \mu B$.

The maximum torque acting on the coil is:

$$\tau_{\max} = \mu B$$

Use its definition to express the magnetic moment of the coil:

$$\mu = NIA = NI\pi r^2$$

Substitute to obtain:

$$\tau_{\max} = NI\pi r^2 B$$

Substitute numerical values and evaluate τ:

$$\tau_{\max} = (400)(1.6\,\mathrm{mA})\pi(0.75\,\mathrm{cm})^2(0.25\,\mathrm{T}) = \boxed{28\,\mu\mathrm{N\cdot m}}$$

49 • **[SSM]** A current-carrying wire is in the shape of a square of edge-length 6.0 cm. The square lies in the $z = 0$ plane. The wire carries a current of 2.5 A. What is the magnitude of the torque on the wire if it is in a region with a uniform magnetic field of magnitude 0.30 T that points in the (*a*) +*z* direction and (*b*) +*x* direction?

Picture the Problem We can use $\vec{\tau} = \vec{\mu}\times\vec{B}$ to find the torque on the coil in the two orientations of the magnetic field.

Express the torque acting on the coil:

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

Express the magnetic moment of the coil:

$$\vec{\mu} = \pm IA\hat{k} = \pm IL^2\hat{k}$$

(*a*) Evaluate $\vec{\tau}$ for $\vec{B}$ in the $+z$ direction:

$$\vec{\tau} = \pm IL^2\hat{k}\times B\hat{k} = \pm IL^2B\left(\hat{k}\times\hat{k}\right) = \boxed{0}$$

(*b*) Evaluate $\vec{\tau}$ for $\vec{B}$ in the $+x$ direction:

$$\vec{\tau} = \pm IL^2\hat{k}\times B\hat{i} = \pm IL^2B\left(\hat{k}\times\hat{i}\right)$$
$$= \pm(2.5\,\text{A})(0.060\,\text{m})^2(0.30\,\text{T})\hat{j}$$
$$= \pm(2.7\,\text{mN}\cdot\text{m})\hat{j}$$

and

$$\left|\vec{\tau}\right| = \boxed{2.7\times10^{-3}\,\text{N}\cdot\text{m}}$$

50 • A current-carrying wire is in the shape of an equilateral triangle of edge-length 8.0 cm. The triangle lies in the $z = 0$ plane. The wire carries a current of 2.5 A. What is the magnitude of the torque on the wire if it is in a region with a uniform magnetic field of magnitude 0.30 T that points in the (*a*) $+z$ direction and (*b*) $+x$ direction?

Picture the Problem We can use $\vec{\tau} = \vec{\mu}\times\vec{B}$ to find the torque on the equilateral triangle in the two orientations of the magnetic field.

Express the torque acting on the coil:

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

Express the magnetic moment of the coil:

$$\vec{\mu} = \pm IA\hat{k}$$

Relate the area of the equilateral triangle to the length of its side:

$$A = \tfrac{1}{2}\,\text{base}\times\text{altitude}$$
$$= \frac{1}{2}(L)\left(\frac{\sqrt{3}L}{2}\right) = \frac{\sqrt{3}}{4}L^2$$

Substitute to obtain:

$$\vec{\mu} = \pm\frac{\sqrt{3}L^2I}{4}\hat{k}$$

(*a*) Evaluate $\vec{\tau}$ for $\vec{B}$ in the $+z$ direction:

$$\vec{\tau} = \pm\frac{\sqrt{3}L^2I}{4}\hat{k}\times B\hat{k}$$
$$= \pm\frac{\sqrt{3}L^2IB}{4}\left(\hat{k}\times\hat{k}\right) = \boxed{0}$$

(*b*) Evaluate $\vec{\boldsymbol{\tau}}$ for $\vec{\boldsymbol{B}}$ in the $+x$ direction:

$$\vec{\boldsymbol{\tau}} = \pm\frac{\sqrt{3}L^2 I}{4}\hat{\boldsymbol{k}}\times B\hat{\boldsymbol{i}} = \pm\frac{\sqrt{3}L^2 IB}{4}\left(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{i}}\right)$$

$$= \pm\frac{\sqrt{3}(0.080\,\text{m})^2(2.5\,\text{A})(0.30\,\text{T})}{4}\hat{\boldsymbol{j}}$$

$$= \pm\left(2.1\times10^{-3}\ \text{N}\cdot\text{m}\right)\hat{\boldsymbol{j}}$$

and

$$\left|\vec{\boldsymbol{\tau}}\right| = \boxed{2.1\times10^{-3}\ \text{N}\cdot\text{m}}$$

51 •• A rigid wire is in the shape of a square of edge-length L. The square has mass m and the wire carries current I. The square lies on flat horizontal surface in a region where there is a magnetic field of magnitude B that is parallel to two edges of the square. What is the minimum value of B so that one edge of the square will lift off the surface?

Picture the Problem One edge of the square will lift off the surface when the magnitude of the magnetic torque acting on it equals the magnitude of the gravitational torque acting on it.

The condition for liftoff is that the magnitudes of the torques must be equal:

$$\tau_{\text{mag}} = \tau_{\text{grav}} \qquad (1)$$

Express the magnetic torque acting on the square:

$$\tau_{\text{mag}} = \mu B = IL^2 B$$

Express the gravitational torque acting on one edge of the square:

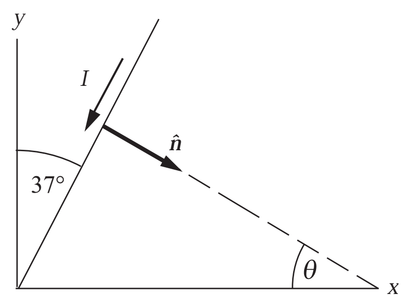
$$\tau_{\text{grav}} = mgL$$

Substituting in equation (1) yields:

$$IL^2 B_{\text{min}} = mgL \Rightarrow B_{\text{min}} = \boxed{\frac{mg}{IL}}$$

52 •• A rectangular current-carrying 50-turn coil, as shown in Figure 26-36, is pivoted about the z axis. (*a*) If the wires in the $z = 0$ plane make an angle $\theta = 37°$ with the y axis, what angle does the magnetic moment of the coil make with the unit vector $\hat{\boldsymbol{i}}$? (*b*) Write an expression for $\hat{\boldsymbol{n}}$ in terms of the unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, where $\hat{\boldsymbol{n}}$ is a unit vector in the direction of the magnetic moment. (*c*) What is the magnetic moment of the coil? (*d*) Find the torque on the coil when there is a uniform magnetic field $\vec{\boldsymbol{B}} = 1.5\ \text{T}\hat{\boldsymbol{j}}$ in the region occupied by the coil. (*e*) Find the potential energy of the coil in this field. (The potential energy is zero when $\theta = 0$.)

Picture the Problem The diagram shows the coil as it would appear from along the positive z axis. The right-hand rule for determining the direction of $\hat{\boldsymbol{n}}$ has been used to establish $\hat{\boldsymbol{n}}$ as shown. We can use the geometry of this figure to determine θ and to express the unit normal vector $\hat{\boldsymbol{n}}$. The magnetic moment of the coil is given by $\vec{\boldsymbol{\mu}} = NIA\hat{\boldsymbol{n}}$ and the torque exerted on the coil by $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$. Finally, we can find the potential energy of the coil in this field from $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}}$.



(*a*) Noting that θ and the angle whose measure is 37° have their right and left sides mutually perpendicular, we can conclude that:

$$\theta = \boxed{37°}$$

(*b*) Use the components of $\hat{\boldsymbol{n}}$ to express $\hat{\boldsymbol{n}}$ in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$:

$$\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{i}} + n_y\hat{\boldsymbol{j}} = \cos 37°\hat{\boldsymbol{i}} - \sin 37°\hat{\boldsymbol{j}}$$
$$= 0.799\hat{\boldsymbol{i}} - 0.602\hat{\mathbf{j}}$$
$$= \boxed{0.80\hat{\boldsymbol{i}} - 0.60\hat{\boldsymbol{j}}}$$

(*c*) Express the magnetic moment of the coil:

$$\vec{\boldsymbol{\mu}} = NIA\hat{\boldsymbol{n}}$$

Substitute numerical values and evaluate $\vec{\boldsymbol{\mu}}$:

$$\vec{\boldsymbol{\mu}} = (50)(1.75\,\text{A})(48.0\,\text{cm}^2)(0.799\hat{\boldsymbol{i}} - 0.602\hat{\boldsymbol{j}}) = (0.335\,\text{A}\cdot\text{m}^2)\,\hat{\boldsymbol{i}} - (0.253\,\text{A}\cdot\text{m}^2)\hat{\boldsymbol{j}}$$
$$= \boxed{(0.34\,\text{A}\cdot\text{m}^2)\,\hat{\boldsymbol{i}} - (0.25\,\text{A}\cdot\text{m}^2)\hat{\boldsymbol{j}}}$$

(*d*) Express the torque exerted on the coil:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$$

Substitute for $\vec{\mu}$ and $\vec{B}$ to obtain:

$$\begin{aligned}\vec{\tau} &= \left\{\left(0.335\,\text{A}\cdot\text{m}^2\right)\hat{i} - \left(0.253\,\text{A}\cdot\text{m}^2\right)\hat{j}\right\}\times\left(1.5\,\text{T}\right)\hat{j} \\ &= \left(0.503\,\text{N}\cdot\text{m}\right)\left(\hat{i}\times\hat{j}\right) - \left(0.379\,\text{N}\cdot\text{m}\right)\left(\hat{j}\times\hat{j}\right) = \boxed{\left(0.50\,\text{N}\cdot\text{m}\right)\hat{k}}\end{aligned}$$

(*e*) Express the potential energy of the coil in terms of its magnetic moment and the magnetic field:

$$U = -\vec{\mu}\cdot\vec{B}$$

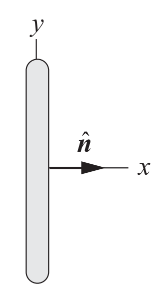
Substitute for $\vec{\mu}$ and $\vec{B}$ and evaluate U:

$$\begin{aligned}U &= -\left\{\left(0.335\,\text{A}\cdot\text{m}^2\right)\hat{i} - \left(0.253\,\text{A}\cdot\text{m}^2\right)\hat{j}\right\}\cdot\left(1.5\,\text{T}\right)\hat{j} \\ &= -\left(0.503\,\text{N}\cdot\text{m}\right)\left(\hat{i}\cdot\hat{j}\right) + \left(0.379\,\text{N}\cdot\text{m}\right)\left(\hat{j}\cdot\hat{j}\right) = \boxed{0.38\,\text{J}}\end{aligned}$$

53 •• **[SSM]** For the coil in Problem 52 the magnetic field is now $\vec{B} = 2.0\text{ T}\hat{j}$. Find the torque exerted on the coil when $\hat{n}$ is equal to (*a*) $\hat{i}$, (*b*) $\hat{j}$, (*c*) $-\hat{j}$, and (*d*) $\dfrac{\hat{i}}{\sqrt{2}} + \dfrac{\hat{j}}{\sqrt{2}}$.

Picture the Problem We can use the right-hand rule for determining the direction of $\hat{n}$ to establish the orientation of the coil for value of $\hat{n}$ and $\vec{\tau} = \vec{\mu}\times\vec{B}$ to find the torque exerted on the coil in each orientation.

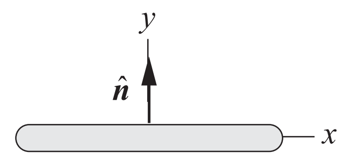
(*a*) The orientation of the coil is shown to the right:



Evaluate $\vec{\tau}$ for $\vec{B} = 2.0\text{ T}\hat{j}$ and $\hat{n} = \hat{i}$:

$$\begin{aligned}\vec{\tau} &= \vec{\mu}\times\vec{B} = NIA\hat{n}\times\vec{B} \\ &= (50)(1.75\,\text{A})\left(48.0\,\text{cm}^2\right)\hat{i}\times(2.0\,\text{T})\hat{j} \\ &= \left(0.840\,\text{N}\cdot\text{m}\right)\left(\hat{i}\times\hat{j}\right) = \left(0.840\,\text{N}\cdot\text{m}\right)\hat{k} \\ &= \boxed{\left(0.84\,\text{N}\cdot\text{m}\right)\hat{k}}\end{aligned}$$

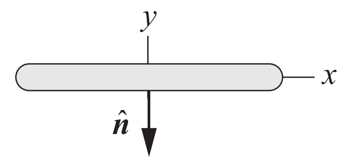
(b) The orientation of the coil is shown to the right:



Evaluate $\vec{\tau}$ for $\vec{B} = 2.0\text{ T}\,\hat{j}$ and $\hat{n} = \hat{j}$:

$$\begin{aligned}\vec{\tau} &= \vec{\mu}\times\vec{B} = NIA\hat{n}\times\vec{B}\\ &= (50)(1.75\text{ A})(48.0\text{ cm}^2)\hat{j}\times(2.0\text{ T})\hat{j}\\ &= (0.840\text{ N}\cdot\text{m})(\hat{j}\times\hat{j})\\ &= \boxed{0}\end{aligned}$$

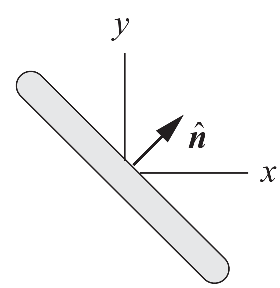
(c) The orientation of the coil is shown to the right:



Evaluate $\vec{\tau}$ for $\vec{B} = 2.0\text{ T}\,\hat{j}$ and $\hat{n} = -\hat{j}$:

$$\begin{aligned}\vec{\tau} &= \vec{\mu}\times\vec{B} = NIA\hat{n}\times\vec{B}\\ &= -(50)(1.75\text{ A})(48.0\text{ cm}^2)\hat{j}\times(2.0\text{ T})\hat{j}\\ &= (-0.840\text{ N}\cdot\text{m})(\hat{j}\times\hat{j})\\ &= \boxed{0}\end{aligned}$$

(d) The orientation of the coil is shown to the right:



Evaluate $\vec{\tau}$ for $\vec{B} = 2.0\text{ T}\,\hat{j}$ and $\hat{n} = (\hat{i}+\hat{j})/\sqrt{2}$:

$$\begin{aligned}\vec{\tau} &= \vec{\mu}\times\vec{B} = NIA\hat{n}\times\vec{B}\\ &= \frac{(50)(1.75\text{ A})(48.0\text{ cm}^2)}{\sqrt{2}}(\hat{i}+\hat{j})\times(2.0\text{ T})\hat{j}\\ &= (0.594\text{ N}\cdot\text{m})(\hat{i}\times\hat{j})\\ &\quad+(0.594\text{ N}\cdot\text{m})(\hat{j}\times\hat{j})\\ &= \boxed{(0.59\text{ N}\cdot\text{m})\hat{k}}\end{aligned}$$

54 •• A small bar magnet has a length equal to 6.8 cm and its magnetic moment is aligned with a uniform magnetic field of magnitude 0.040 T. The bar magnet is then rotated through an angle of 60 about an axis perpendicular to its length The observed torque on the bar magnet has a magnitude of 0.10 N·m.

(*a*) Find the magnetic moment of the magnet. (*b*) Find the potential energy of the magnet.

Picture the Problem Because the small magnet can be modeled as a magnetic dipole; we can use the equation for the torque on a current loop to find its magnetic moment.

(*a*) Express the magnitude of the torque acting on the magnet:

$$\tau = \mu B \sin\theta$$

Solve for μ to obtain:

$$\mu = \frac{\tau}{B\sin\theta}$$

Substitute numerical values and evaluate μ:

$$\mu = \frac{0.10\,\text{N}\cdot\text{m}}{(0.040\,\text{T})\sin 60^\circ} = \boxed{2.9\,\text{A}\cdot\text{m}^2}$$

(*b*) The potential energy of the magnet is given by:

$$U = -\vec{\boldsymbol{\mu}}\cdot\vec{\boldsymbol{B}} = -\mu B\cos\theta$$

Substitute numerical values and evaluate U:

$$U = -\left(2.887\,\text{A}\cdot\text{m}^2\right)(0.040\,\text{T})\cos 60^\circ = \boxed{-58\,\text{mJ}}$$

55 •• A wire loop consists of two semicircles connected by straight segments (Figure 26-37). The inner and outer radii are 0.30 m and 0.50 m, respectively. A current of 1.5 A is in this wire and the current in the outer semicircle is in the clockwise direction. What is the magnetic moment of this current loop?

Picture the Problem We can use the definition of the magnetic moment to find the magnetic moment of the given current loop and a right-hand rule to find its direction.

Using its definition, express the magnetic moment of the current loop:

$$\mu = IA$$

Express the area bounded by the loop:

$$A = \tfrac{1}{2}\left(\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2\right) = \frac{\pi}{2}\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$$

Substitute for A to obtain:

$$\mu = \frac{\pi I}{2}\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$$

Substitute numerical values and evaluate μ:

$$\mu = \frac{\pi(1.5\,\text{A})}{2}\left[(0.50\,\text{m})^2 - (0.30\,\text{m})^2\right] = \boxed{0.38\,\text{A}\cdot\text{m}^2}$$

Apply the right-hand rule for determining the direction of the unit normal vector (the direction of μ) to conclude that $\vec{\boldsymbol{\mu}}$ points into the page.

56 •• A wire of length L is wound into a circular coil that has N turns. Show that when the wire carries a current I, the magnetic moment of the coil has a magnitude given by $IL^2/(4\pi N)$.

Picture the Problem We can use the definition of the magnetic moment of a coil to find the magnetic moment of a wire of length L that is wound into a circular coil of N loops. We can find the area of the coil from its radius R and we can find R by dividing the length of the wire by the number of turns.

Use its definition to express the magnetic moment of the coil:

$$\mu = NIA \qquad (1)$$

Express the circumference of each loop:

$$\frac{L}{N} = 2\pi R \Rightarrow R = \frac{L}{2\pi N}$$

where R is the radius of a loop.

The area of the coil is given by:

$$A = \pi R^2$$

Substituting for A and simplifying yields:

$$A = \pi\left(\frac{L}{2\pi N}\right)^2 = \frac{L^2}{4\pi N^2}$$

Substitute for A in equation (1) and simplify to obtain:

$$\mu = NI\left(\frac{L^2}{4\pi N^2}\right) = \boxed{\frac{IL^2}{4\pi N}}$$

57 •• **[SSM]** A particle that has a charge q and a mass m moves with angular velocity ω in a circular path of radius r. (a) Show that the average current created by this moving particle is $\omega q/(2\pi)$ and that the magnetic moment of its orbit has a magnitude of $\frac{1}{2}q\omega r^2$. (b) Show that the angular momentum of this particle has the magnitude of $mr^2\omega$ and that the magnetic moment and angular momentum vectors are related by $\vec{\boldsymbol{\mu}} = \left(\frac{q}{2m}\right)\vec{\boldsymbol{L}}$, where $\vec{\boldsymbol{L}}$ is the angular momentum about the center of the circle.

Picture the Problem We can use the definition of current and the relationship between the frequency of the motion and its period to show that $I = q\omega/2\pi$. We can use the definition of angular momentum and the moment of inertia of a point particle to show that the magnetic moment has the magnitude $\mu = \frac{1}{2}q\omega r^2$. Finally, we can express the ratio of μ to L and the fact that $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{L}}$ are both parallel to $\vec{\boldsymbol{\omega}}$ to conclude that $\vec{\boldsymbol{\mu}} = (q/2m)\vec{\boldsymbol{L}}$.

(*a*) Using its definition, relate the average current to the charge passing a point on the circumference of the circle in a given period of time:

$$I = \frac{\Delta q}{\Delta t} = \frac{q}{T} = qf$$

Relate the frequency of the motion to the angular frequency of the particle:

$$f = \frac{\omega}{2\pi}$$

Substitute for f to obtain:

$$I = \boxed{\frac{q\omega}{2\pi}}$$

From the definition of the magnetic moment we have:

$$\mu = IA = \left(\frac{q\omega}{2\pi}\right)\left(\pi r^2\right) = \boxed{\tfrac{1}{2}q\omega r^2}$$

(*b*) Express the angular momentum of the particle:

$$L = I\omega$$

The moment of inertia of the particle is:

$$I = mr^2$$

Substituting for I yields:

$$L = \left(mr^2\right)\omega = \boxed{mr^2\omega}$$

Express the ratio of μ to L and simplify to obtain:

$$\frac{\mu}{L} = \frac{\frac{1}{2}q\omega r^2}{mr^2\omega} = \frac{q}{2m} \Rightarrow \mu = \frac{q}{2m}L$$

Because $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{L}}$ are both parallel to $\vec{\boldsymbol{\omega}}$:

$$\vec{\boldsymbol{\mu}} = \boxed{\left(\frac{q}{2m}\right)\vec{\boldsymbol{L}}}$$

58 ••• A hollow non-conducting cylinder has length L and inner and outer radii R_i A uniformly charged non-conducting cylindrical shell (Figure 26-38) has length L, inner and outer radii R_i and R_o, respectively, a charge density ρ and an

angular velocity ω about its axis. Derive an expression for the magnetic moment of the cylinder.

Picture the Problem We can express the magnetic moment of an element of charge dq in a cylinder of length L, radius r, and thickness dr, relate this charge to the length, radius, and thickness of the cylinder, express the current due to this rotating charge, substitute for A and dI in our expression for μ and then integrate to complete our derivation for the magnetic moment of the rotating cylinder as a function of its angular velocity.

Express the magnetic moment of an element of charge dq in a cylinder of length L, radius r, and thickness dr:

$$d\mu = AdI = \pi r^2 dI \qquad (1)$$

Relate the charge dq in the cylinder to the length of the cylinder, its radius, and thickness:

$$dq = 2\pi L\rho r dr$$

The current due to this rotating charge is given by:

$$dI = \frac{\omega}{2\pi}dq = \frac{\omega}{2\pi}(2\pi L\rho r dr) = L\rho r\omega dr$$

Substitute for dI in equation (1) and simplify to obtain:

$$d\mu = \pi r^2(L\rho\omega r dr) = L\rho\pi\omega r^3 dr$$

Integrate r from R_i to R_0 to obtain:

$$\mu = L\rho\pi\omega\int_{R_\text{i}}^{R_0} r^3 dr = \tfrac{1}{4}L\rho\pi\omega\left(R_0^4 - R_\text{i}^4\right)$$

Because $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{\omega}}$ are parallel:

$$\vec{\boldsymbol{\mu}} = \boxed{\tfrac{1}{4}L\rho\pi\left(R_0^4 - R_\text{i}^4\right)\vec{\boldsymbol{\omega}}}$$

59 ••• [SSM] A uniform non-conducting thin rod of mass m and length L has a uniform charge per unit length λ and rotates with angular speed ω about an axis through one end and perpendicular to the rod. (*a*) Consider a small segment of the rod of length dx and charge $dq = \lambda dr$ at a distance r from the pivot (Figure 26-40). Show that the average current created by this moving segment is $\omega dq/(2\pi)$ and show that the magnetic moment of this segment is $\frac{1}{2}\lambda\omega r^2 dx$. (*b*) Use this to show that the magnitude of the magnetic moment of the rod is $\frac{1}{6}\lambda\omega L^3$. (*c*) Show that the magnetic moment $\vec{\boldsymbol{\mu}}$ and angular momentum $\vec{\boldsymbol{L}}$ are related by $\vec{\boldsymbol{\mu}} = \left(\dfrac{Q}{2m}\right)\vec{\boldsymbol{L}}$, where Q is the total charge on the rod.

Picture the Problem We can follow the step-by-step outline provided in the problem statement to establish the given results.

(*a*) Express the magnetic moment of the rotating element of charge:

$$d\mu = A\,dI \qquad (1)$$

The area enclosed by the rotating element of charge is:

$$A = \pi x^2$$

Express dI in terms of dq and Δt:

$$dI = \frac{dq}{\Delta t} = \frac{\lambda dx}{\Delta t}$$ where Δt is the time required for one revolution.

The time Δt required for one revolution is:

$$\Delta t = \frac{1}{f} = \frac{2\pi}{\omega}$$

Substitute for Δt and simplify to obtain:

$$dI = \frac{\lambda\omega}{2\pi}dx$$

Substituting for dI in equation (1) and simplifying yields:

$$d\mu = \left(\pi x^2\right)\left(\frac{\lambda\omega}{2\pi}dx\right) = \boxed{\tfrac{1}{2}\lambda\omega x^2 dx}$$

(*b*) Integrate $d\mu$ from $x = 0$ to $x = L$ to obtain:

$$\mu = \tfrac{1}{2}\lambda\omega\int_0^L x^2 dx = \boxed{\tfrac{1}{6}\lambda\omega L^3}$$

(*c*) Express the angular momentum of the rod:

$$L = I\omega$$

where L is the angular momentum of the rod and I is the moment of inertia of the rod with respect to the point about which it is rotating.

Express the moment of inertia of the rod with respect to an axis through its end:

$$I = \tfrac{1}{3}mL^2$$

where L is now the length of the rod.

Substitute for I to obtain:

$$L = \tfrac{1}{3}mL^2\omega$$

Divide the expression for μ by L to obtain:

$$\frac{\mu}{L}=\frac{\frac{1}{6}\lambda\omega L^3}{\frac{1}{3}mL^2\omega}=\frac{\lambda L}{2m}$$

or, because $Q = \lambda L$,

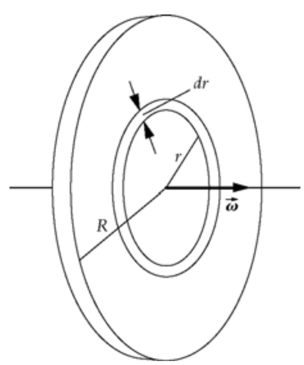
$$\mu=\frac{Q}{2m}L$$

Because $\vec{\omega}$ and $\vec{L} = I\vec{\omega}$ are parallel:

$$\vec{\mu}=\boxed{\left(\frac{Q}{2M}\right)\vec{L}}$$

60 ••• A non-uniform, non-conducting thin disk of mass m, radius R, and total charge Q has a charge per unit area σ that varies as $\sigma_0 r/R$ and a mass per unit area σ_m that is given by $(m/Q)\,\sigma$. The disk rotates with angular speed ω about its central axis. (*a*) Show that the magnetic moment of the disk has a magnitude $\frac{1}{5}\pi\omega\sigma_0 R^4$ which can be alternatively rewritten as $\frac{3}{10}\omega QR^2$. (*b*) Show that the magnetic moment $\vec{\mu}$ and angular momentum $\vec{L}$ are related by $\vec{\mu}=\frac{Q}{2M}\vec{L}$.

Picture the Problem We can express the magnetic moment of an element of current dI due to a ring of radius r, and thickness dr with charge dq. Integrating this expression from $r = 0$ to $r = R$ will give us the magnetic moment of the disk. We can integrate the charge on the ring between these same limits to find the total charge on the disk and divide μ by Q to establish the relationship between them. In Part (*b*) we can find the angular momentum of the disk by first finding the moment of inertia of the disk by integrating r^2dm between the same limits used above.



(*a*) Express the magnetic moment of an element of the disk:

$$d\mu = AdI$$

The area enclosed by the rotating element of charge is:

$$A = \pi x^2$$

Express the element of current dI:

$$dI = \frac{dq}{\Delta t} = \frac{\sigma dA}{\Delta t} = f\sigma dA$$
$$= \frac{\omega}{2\pi}\left(\sigma_0 \frac{r}{R}\right)(2\pi\, r dr) = \frac{\sigma_0 \omega}{R} r^2 dr$$

Substitute for A and dI and simplify to obtain:

$$d\mu = \pi r^2 \frac{\sigma_0 \omega}{R} r^2 dr = \frac{\sigma_0 \pi \omega}{R} r^4 dr$$

Integrate $d\mu$ from $r = 0$ to $r = R$ to obtain:

$$\mu = \frac{\sigma_0 \pi \omega}{R} \int_0^R r^4 dr = \boxed{\tfrac{1}{5}\sigma_0 \pi \omega R^4} \qquad (1)$$

The charge dq within a distance r of the center of the disk is given by:

$$dq = 2\pi r \sigma dr = 2\pi r\left(\sigma_0 \frac{r}{R}\right)dr$$
$$= \frac{2\pi\sigma_0}{R} r^2 dr$$

Integrate dq from $r = 0$ to $r = R$ to obtain:

$$Q = \frac{2\pi\sigma_0}{R} \int_0^R r^2 dr = \tfrac{2}{3}\pi\sigma_0 R^2 \qquad (2)$$

Divide equation (1) by Q to obtain:

$$\frac{\mu}{Q} = \frac{\tfrac{1}{5}\sigma_0 \pi \omega R^4}{\tfrac{2}{3}\pi\sigma_0 R^2} = \frac{3\omega R^2}{10}$$

and

$$\mu = \boxed{\tfrac{3}{10} Q\omega R^2} \qquad (3)$$

(b) Express the moment of inertia of an element of mass dm of the disk:

$$dI = r^2 dm = r^2 \sigma_{\mathrm{m}} dA$$
$$= r^2\left(\frac{m}{Q}\sigma\right)(2\pi r dr)$$
$$= \frac{2\pi m\left(\frac{r}{R}\sigma_0\right)}{Q} r^3 dr$$
$$= \frac{2\pi m \sigma_0}{QR} r^4 dr$$

Integrate dI from $r = 0$ to $r = R$ to obtain:

$$I = \frac{2\pi m \sigma_0}{QR} \int_0^R r^4 dr = \frac{2\pi m \sigma_0}{5Q} R^4$$

Divide I by equation (2) and simplify to obtain:

$$\frac{I}{Q}=\frac{\frac{2\pi m\sigma_0}{5Q}R^4}{\frac{2}{3}\pi\sigma_0 R^2}=\frac{3m}{5Q}R^2$$

and

$$I=\frac{3m}{5}R^2$$

Express the angular momentum of the disk:

$$L=I\omega=\tfrac{3}{5}mR^2\omega$$

Divide equation (3) by L and simplify to obtain:

$$\frac{\mu}{L}=\frac{\frac{3}{10}Q\omega R^2}{\frac{3}{5}mR^2\omega}=\frac{Q}{2m}\Rightarrow\mu=\frac{Q}{2m}L$$

Because $\vec{\boldsymbol{\mu}}$ is in the same direction as $\vec{\boldsymbol{\omega}}$:

$$\vec{\boldsymbol{\mu}}=\boxed{\frac{Q}{2m}\vec{\boldsymbol{L}}}$$

61 ••• [SSM] A spherical shell of radius R carries a constant surface charge density σ. The shell rotates about its diameter with angular speed ω. Find the magnitude of the magnetic moment of the rotating shell.

Picture the Problem We can use the result of Problem 57 to express μ as a function of Q, M, and L. We can then use the definitions of surface charge density and angular momentum to substitute for Q and L to obtain the magnetic moment of the rotating shell.

Express the magnetic moment of the spherical shell in terms of its mass, charge, and angular momentum:

$$\mu=\frac{Q}{2M}L$$

Use the definition of surface charge density to express the charge on the spherical shell:

$$Q=\sigma A=4\pi\sigma R^2$$

Express the angular momentum of the spherical shell:

$$L=I\omega=\tfrac{2}{3}MR^2\omega$$

Substitute for L and simplify to obtain:

$$\mu=\left(\frac{4\pi\sigma R^2}{2M}\right)\left(\frac{2}{3}MR^2\omega\right)=\boxed{\tfrac{4}{3}\pi\sigma R^4\omega}$$

62 ••• A uniform solid uniformly charged sphere of radius R has a volume charge density ρ. The sphere rotates about an axis through its center with angular speed ω. Find the magnitude of the magnetic moment of this rotating sphere.

Picture the Problem We can use the result of Problem 57 to express μ as a function of Q, M, and L. We can then use the definitions of volume charge density and angular momentum to substitute for Q and L to obtain the magnetic moment of the rotating sphere.

Express the magnetic moment of the solid sphere in terms of its mass, charge, and angular momentum:

$$\mu = \frac{Q}{2M}L$$

Use the definition of volume charge density to express the charge of the sphere:

$$Q = \rho V = \tfrac{4}{3}\pi\rho R^3$$

Express the angular momentum of the solid sphere:

$$L = I\omega = \tfrac{2}{5}MR^2\omega$$

Substitute for Q and L and simplify to obtain:

$$\mu = \left(\frac{\tfrac{4}{3}\pi\rho R^3}{2M}\right)\left(\frac{2}{5}MR^2\omega\right) = \boxed{\tfrac{4}{15}\pi\rho R^5\omega}$$

63 ••• A uniform thin uniformly charged disk of mass m, radius R, and uniform surface charge density σ rotates with angular speed ω about an axis through its center and perpendicular to the disk (Figure 26-40). The disk in in a region with a uniform magnetic field $\vec{B}$ that makes an angle θ with the rotation axis. Calculate (*a*) the magnitude of the torque exerted on the disk by the magnetic field and (*b*) the precession frequency of the disk in the magnetic field.

Picture the Problem We can use its definition to express the torque acting on the disk, Example 26-11 to express the magnetic moment of the disk, and the definition of the precession frequency to find the precession frequency of the disk.

(*a*) The magnitude of the net torque acting on the disk is:

$$\tau = \mu B\sin\theta$$

where μ is the magnetic moment of the disk.

From Example 26-11:

$$\mu = \tfrac{1}{4}\pi\sigma r^4\omega$$

Substitute for μ in the expression for τ to obtain:

$$\tau = \boxed{\tfrac{1}{4}\pi\sigma r^4\omega B\sin\theta}$$

(*b*) The precession frequency Ω is equal to the ratio of the torque divided by the spin angular momentum:

$$\Omega = \frac{\tau}{I\omega}$$

For a solid disk, the moment of inertia is given by:

$$I = \tfrac{1}{2}mr^2$$

Substitute for τ and I to obtain:

$$\Omega = \frac{\frac{1}{4}\pi\sigma r^4 \omega B \sin\theta}{\frac{1}{2}mr^2\omega} = \boxed{\frac{\pi\sigma r^2 B}{2m}\sin\theta}$$

Remarks: Note that the precession frequency is independent of ω.

The Hall Effect

64 • A metal strip that is 2.00-cm wide and 0.100-cm thick carries a current of 20.0 A in region with a uniform magnetic field of 2.00 T, as shown in Figure 26-41. The Hall voltage is measured to be 4.27 μV. (*a*) Calculate the drift speed of the free electrons in the strip. (*b*) Find the number density of the free electrons in the strip. (*c*) Is point *a* or point *b* at the higher potential? Explain your answer.

Picture the Problem We can use the Hall effect equation to find the drift speed of the electrons and the relationship between the current and the number density of charge carriers to find *n*. In (*c*) we can use a right-hand rule to decide whether *a* or *b* is at the higher potential.

(*a*) Express the Hall voltage as a function of the drift speed of the electrons in the strip:

$$V_{\mathrm{H}} = v_{\mathrm{d}}Bw \Rightarrow v_{\mathrm{d}} = \frac{V_{\mathrm{H}}}{Bw}$$

Substitute numerical values and evaluate v_{d}:

$$v_{\mathrm{d}} = \frac{4.27\,\mu\mathrm{V}}{(2.00\,\mathrm{T})(2.00\,\mathrm{cm})} = 0.1068\,\mathrm{mm/s} = \boxed{0.107\,\mathrm{mm/s}}$$

(*b*) Express the current as a function of the number density of charge carriers:

$$I = nAqv_{\mathrm{d}} \Rightarrow n = \frac{I}{Aqv_{\mathrm{d}}}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{20.0\,\mathrm{A}}{(2.00\,\mathrm{cm})(0.100\,\mathrm{cm})(1.602\times10^{-19}\,\mathrm{C})(0.1068\,\mathrm{mm/s})} = \boxed{5.85\times10^{28}\,\mathrm{m^{-3}}}$$

(c) Apply a right-hand rule to $I\vec{\ell}$ and $\vec{\boldsymbol{B}}$ to conclude that positive charge will accumulate at a and negative charge at b and therefore $\boxed{V_a > V_b}$. The Hall effect electric field is directed from a toward b.

65 •• **[SSM]** The number density of free electrons in copper is 8.47×10^{22} electrons per cubic centimeter. If the metal strip in Figure 26-41 is copper and the current is 10.0 A, find (a) the drift speed v_d and (b) the potential difference $V_a - V_b$. Assume that the magnetic field strength is 2.00 T.

Picture the Problem We can use $I = nqv_d A$ to find the drift speed and $V_H = v_d Bw$ to find the potential difference $V_a - V_b$.

(a) Express the current in the metal strip in terms of the drift speed of the electrons:

$$I = nqv_d A \Rightarrow v_d = \frac{I}{nqA}$$

Substitute numerical values and evaluate v_d:

$$v_d = \frac{10.0\,\text{A}}{\left(8.47\times10^{22}\,\text{cm}^{-3}\right)\left(1.602\times10^{-19}\,\text{C}\right)\left(2.00\,\text{cm}\right)\left(0.100\,\text{cm}\right)}$$
$$= 3.685\times10^{-5}\,\text{m/s} = \boxed{3.68\times10^{-5}\,\text{m/s}}$$

(b) The potential difference $V_a - V_b$ is the Hall voltage and is given by:

$$V_a - V_b = V_H = v_d Bw$$

Substitute numerical values and evaluate $V_a - V_b$:

$$V_a - V_b = \left(3.685\times10^{-5}\,\text{m/s}\right)\left(2.00\,\text{T}\right)\left(2.00\,\text{cm}\right) = \boxed{1.47\,\mu\text{V}}$$

66 •• A copper strip has 8.47×10^{22} free electrons per cubic centimeter is 2.00-cm wide, is 0.100-cm thick, and is used to measure the magnitudes of unknown magnetic fields that are perpendicular to it. Find the magnitude of B when the current is 20.0 A and the Hall voltage is (a) 2.00 μV, (b) 5.25 μV, and (c) 8.00 μV.

Picture the Problem We can use $V_H = v_d Bw$ to express B in terms of V_H and $I = nqv_d A$ to eliminate the drift velocity v_d and derive an expression for B in terms of V_H, n, and t.

Relate the Hall voltage to the drift velocity and the magnetic field:

$$V_H = v_d B w \Rightarrow B = \frac{V_H}{v_d w}$$

Express the current in the metal strip in terms of the drift velocity of the electrons:

$$I = nqv_d A \Rightarrow v_d = \frac{I}{nqA}$$

Substitute for v_d and simplify to obtain:

$$B = \frac{V_H}{\frac{I}{nqA}w} = \frac{nqAV_H}{Iw} = \frac{nqwtV_H}{Iw} = \frac{nqt}{I}V_H$$

Substitute numerical values and simplify to obtain:

$$B = \frac{(8.47\times10^{22}\ \text{cm}^{-3})(1.602\times10^{-19}\ \text{C})(0.100\,\text{cm})V_H}{20.0\,\text{A}} = (6.7845\times10^{5}\ \text{s/m}^2)V_H$$

(*a*) Evaluate B for $V_H = 2.00\ \mu\text{V}$:

$$B = (6.7845\times10^{5}\ \text{s/m}^2)(2.00\,\mu\text{V}) = \boxed{1.36\,\text{T}}$$

(*b*) Evaluate B for $V_H = 5.25\ \mu\text{V}$:

$$B = (6.7845\times10^{5}\ \text{s/m}^2)(5.25\,\mu\text{V}) = \boxed{3.56\,\text{T}}$$

(*c*) Evaluate B for $V_H = 8.00\ \mu\text{V}$:

$$B = (6.7845\times10^{5}\ \text{s/m}^2)(8.00\,\mu\text{V}) = \boxed{5.43\,\text{T}}$$

67 •• Because blood contains ions, moving blood in the presence of a magnetic field develops a Hall voltage across the diameter of an artery. A large artery that has a diameter of 0.85 cm can have blood flowing through it with a maximum speed of 0.60 m/s. If a section of this artery is in a magnetic field of 0.20 T, what is the maximum potential difference across the diameter of the artery?

Picture the Problem We can use $V_H = v_d B w$ to find the Hall voltage developed across the diameter of the artery.

Relate the Hall voltage to the flow speed of the blood v_d, the diameter of the artery w, and the magnetic field B:

$$V_H = v_d B w$$

Substitute numerical values and evaluate V_H:

$$V_H = (0.60\,\text{m/s})(0.20\,\text{T})(0.85\,\text{cm}) = \boxed{1.0\,\text{mV}}$$

68 •• The Hall coefficient R_H is a property of conducting material (just as resistivity is). It is defined as $R_H = E_y/(J_xB_z)$, where J_x is x component of the current density in the material, B_z is the z component of the magnetic field, and E_y is the y component resulting Hall electric field. Show that the Hall coefficient is equal to $1/(nq)$, where q is the charge of the charge carriers ($-e$ if they are electrons). (The Hall coefficients of monovalent metals, such as copper, silver, and sodium are therefore negative.)

Picture the Problem Let the width of the slab be w and its thickness t. We can use the definition of the Hall electric field in the slab, the expression for the Hall voltage across it, and the definition of current density to show that the Hall coefficient is also given by $1/(nq)$.

The Hall coefficient is:

$$R = \frac{E_y}{J_x B_z}$$

Using its definition, express the Hall electric field in the slab:

$$E_y = \frac{V_H}{w}$$

The current density in the slab is:

$$J_x = \frac{I}{wt} = nqv_d$$

Substitute for E_y and J_x and simplify to obtain:

$$R = \frac{\frac{V_H}{w}}{nqv_d B_z} = \frac{V_H}{nqv_d w B_z}$$

Express the Hall voltage in terms of v_d, B, and w:

$$V_H = v_d B_z w$$

Substitute for V_H and simplify to obtain:

$$R = \frac{v_d B_z w}{nqv_d w B_z} = \boxed{\frac{1}{nq}}$$

69 •• [SSM] Aluminum has a density of 2.7×10^3 kg/m^3 and a molar mass of 27 g/mol. The Hall coefficient of aluminum is $R = -0.30 \times 10^{-10}$ m^3/C. (See Problem 68 for the definition of R.) What is the number of conduction electrons per aluminum atom?

Picture the Problem We can determine the number of conduction electrons per atom from the quotient of the number density of charge carriers and the number of charge carriers per unit volume. Let the width of a slab of aluminum be w and its thickness t. We can use the definition of the Hall electric field in the slab, the expression for the Hall voltage across it, and the definition of current density to find n in terms of R and q and $n_a = \rho N_A / M$, to express n_a.

Express the number of electrons per atom N:

$$N = \frac{n}{n_a} \qquad (1)$$

where n is the number density of charge carriers and n_a is the number of atoms per unit volume.

From the definition of the Hall coefficient we have:

$$R = \frac{E_y}{J_x B_z}$$

Express the Hall electric field in the slab:

$$E_y = \frac{V_H}{w}$$

The current density in the slab is:

$$J_x = \frac{I}{wt} = nqv_d$$

Substitute for E_y and J_x in the expression for R to obtain:

$$R = \frac{\frac{V_H}{w}}{nqv_d B_z} = \frac{V_H}{nqv_d w B_z}$$

Express the Hall voltage in terms of v_d, B, and w:

$$V_H = v_d B_z w$$

Substitute for V_H and simplify to obtain:

$$R = \frac{v_d B_z w}{nqv_d w B_z} = \frac{1}{nq} \Rightarrow n = \frac{1}{Rq} \qquad (2)$$

Express the number of atoms n_a per unit volume:

$$n_a = \rho \frac{N_A}{M} \qquad (3)$$

Substitute equations (2) and (3) in equation (1) to obtain:

$$N=\frac{M}{qR\rho N_{\text{A}}}$$

Substitute numerical values and evaluate N:

$$N=\frac{27\,\frac{\text{g}}{\text{mol}}}{\left(-1.602\times10^{-19}\,\text{C}\right)\left(-0.30\times10^{-10}\,\frac{\text{m}^3}{\text{C}}\right)\left(2.7\times10^{3}\,\frac{\text{kg}}{\text{m}^3}\right)\left(6.022\times10^{23}\,\frac{\text{atoms}}{\text{mol}}\right)}$$
$$\approx \boxed{4}$$

General Problems

70 • A long wire parallel to the x axis carries a current of 6.50 A in the $+x$ direction. The wire occupies a region that has a uniform magnetic field $\vec{B}=1.35\text{ T}\,\hat{\boldsymbol{j}}$. Find the magnetic force per unit length on the wire.

Picture the Problem We can use the expression for the magnetic force acting on a wire ($\vec{F}=I\vec{\ell}\times\vec{B}$) to find the force per unit length on the wire.

Express the magnetic force on the wire:

$$\vec{F}=I\vec{\ell}\times\vec{B}$$

Substitute for $I\vec{\ell}$ and $\vec{B}$ to obtain:

$$\vec{F}=(6.50\,\text{A})\ell\hat{\boldsymbol{i}}\times(1.35\,\text{T})\hat{\boldsymbol{j}}$$

and

$$\frac{\vec{F}}{\ell}=(6.50\,\text{A})\hat{\boldsymbol{i}}\times(1.35\,\text{T})\hat{\boldsymbol{j}}$$

Simplify to obtain:

$$\frac{\vec{F}}{\ell}=(8.78\,\text{N/m})\left(\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}\right)=\boxed{(8.78\,\text{N/m})\hat{\boldsymbol{k}}}$$

71 • An alpha particle (charge $+2e$) travels in a circular path of radius 0.50 m in a region with a magnetic field whose magnitude is 0.10 T. Find (*a*) the period, (*b*) the speed, and (*c*) the kinetic energy (in electron volts) of the alpha particle. (The mass of an alpha particle is 6.65×10^{-27} kg.)

Picture the Problem We can express the period of the alpha particle's motion in terms of its orbital speed and use Newton's second law to express its orbital speed in terms of known quantities. Knowing the particle's period and the radius of its motion we can find its speed and kinetic energy.

(*a*) Relate the period of the alpha particle's motion to its orbital speed:

$$T = \frac{2\pi r}{v} \qquad (1)$$

Apply Newton's second law to the alpha particle to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow v = \frac{qBr}{m}$$

Substitute for v in equation (1) and simplify to obtain:

$$T = \frac{2\pi r}{\frac{qBr}{m}} = \frac{2\pi m}{qB}$$

Substitute numerical values and evaluate T:

$$T = \frac{2\pi\left(6.65\times10^{-27}\text{ kg}\right)}{2\left(1.602\times10^{-19}\text{ C}\right)\left(0.10\text{ T}\right)} = 1.30\ \mu\text{s}$$
$$= \boxed{1.3\ \mu\text{s}}$$

(*b*) Solve equation (1) for v:

$$v = \frac{2\pi r}{T}$$

Substitute numerical values and evaluate v:

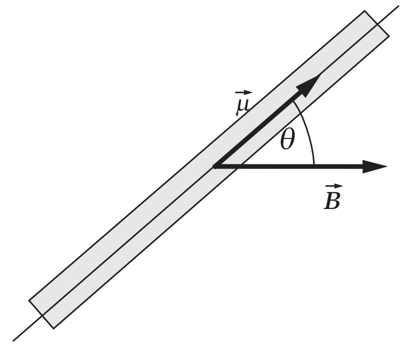
$$v = \frac{2\pi\left(0.50\text{ m}\right)}{1.30\ \mu\text{s}} = 2.409\times10^{6}\text{ m/s}$$
$$= \boxed{2.4\times10^{6}\text{ m/s}}$$

(*c*) The kinetic energy of the alpha particle is:

$$K = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}\left(6.65\times10^{-27}\text{ kg}\right)\left(2.409\times10^{6}\text{ m/s}\right)^2$$
$$= 1.930\times10^{-14}\text{ J}\times\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}}$$
$$= \boxed{0.12\text{ MeV}}$$

72 •• The pole strength q_m of a bar magnet is defined by $\vec{\boldsymbol{\mu}} = q_m\vec{\ell}$, where $\vec{\boldsymbol{\mu}}$ is the magnetic moment of the magnet and $\vec{\ell}$ is the position of the north-pole end of the magnet relative to the south-pole end. Show that the torque exerted on a bar magnet in a uniform magnetic field $\vec{\boldsymbol{B}}$ is the same as if a force $+q_m\vec{\boldsymbol{B}}$ is exerted on the north-pole of the magnetic and a force $-q_m\vec{\boldsymbol{B}}$ is exerted on the south-pole.

Picture the Problem The configuration of the magnet and field are shown in the figure. We'll assume that a force $+q_{\text{m}}\vec{\boldsymbol{B}}$ is exerted on the north-pole end and a force $-q_{\text{m}}\vec{\boldsymbol{B}}$ is exerted on the south-pole end and show that this assumption leads to the familiar expression for the torque acting on a magnetic dipole.



Assuming that a force $+q_{\text{m}}\vec{\boldsymbol{B}}$ is exerted on the north-pole end and a force $-q_{\text{m}}\vec{\boldsymbol{B}}$ is exerted on the south-pole end, express the net torque acting on the bar magnet:

$$\tau = \frac{Bq_{\text{m}}\ell}{2}\sin\theta - \frac{-Bq_{\text{m}}\ell}{2}\sin\theta$$
$$= Bq_{\text{m}}\ell\sin\theta$$

Substitute for q_{m} to obtain:

$$\tau = B\frac{|\vec{\mu}|}{\ell}\ell\sin\theta = \mu B\sin\theta$$

or

$$\vec{\boldsymbol{\tau}} = \boxed{\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{B}}}$$

73 •• [SSM] A particle of mass m and charge q enters a region where there is a uniform magnetic field $\vec{\boldsymbol{B}}$ parallel with the x axis. The initial velocity of the particle is $\vec{\boldsymbol{v}} = v_{0x}\hat{\boldsymbol{i}} + v_{0y}\hat{\boldsymbol{j}}$, so the particle moves in a helix. (*a*) Show that the radius of the helix is $r = mv_{0y}/qB$. (*b*) Show that the particle takes a time $\Delta t = 2\pi m/qB$ to complete each turn of the helix. (*c*) What is the x component of the displacement of the particle during time given in Part (*b*)?

Picture the Problem We can use $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$ to show that motion of the particle in the x direction is not affected by the magnetic field. The application of Newton's second law to motion of the particle in yz plane will lead us to the result that $r = mv_{0y}/qB$. By expressing the period of the motion in terms of v_{0y} we can show that the time for one complete orbit around the helix is $t = 2\pi m/qB$.

(*a*) Express the magnetic force acting on the particle:

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$$

Substitute for $\vec{v}$ and $\vec{B}$ and simplify to obtain:

$$\begin{aligned}\vec{F} &= q\left(v_{0x}\hat{i} + v_{0y}\hat{j}\right)\times B\hat{i}\\ &= qv_{0x}B\left(\hat{i}\times\hat{i}\right) + qv_{0y}B\left(\hat{j}\times\hat{i}\right)\\ &= 0 - qv_{0y}B\hat{k} = -qv_{0y}B\hat{k}\end{aligned}$$

that is, the motion in the direction of the magnetic field (the x direction) is not affected by the field.

Apply Newton's second law to the particle in the plane perpendicular to $\hat{i}$ (i.e., the yz plane):

$$qv_{0y}B = m\frac{v_{0y}^2}{r} \qquad (1)$$

Solving for r yields:

$$r = \boxed{\frac{mv_{0y}}{qB}}$$

(b) Relate the time for one orbit around the helix to the particle's orbital speed:

$$\Delta t = \frac{2\pi r}{v_{0y}}$$

Solve equation (1) for v_{0y}:

$$v_{0y} = \frac{qBr}{m}$$

Substitute for v_{0y} and simplify to obtain:

$$\Delta t = \frac{2\pi r}{\frac{qBr}{m}} = \boxed{\frac{2\pi m}{qB}}$$

(c) Because, as was shown in Part (a), the motion in the direction of the magnetic field (the x direction) is not affected by the field, the x component of the displacement of the particle as a function of t is:

$$x(t) = v_{\mathrm{ox}}t$$

For $t = \Delta t$:

$$x(\Delta t) = v_{\mathrm{ox}}\left(\frac{2\pi m}{qB}\right) = \boxed{\frac{2\pi mv_{\mathrm{ox}}}{qB}}$$

74 •• A metal crossbar of mass m rides on a parallel pair of long horizontal conducting rails separated by a distance L and connected to a device that supplies constant current I to the circuit, as shown in Figure 26-42. The circuit is in a region with a uniform magnetic field $\vec{B}$ whose direction is vertically downward.

There is no friction and the bar starts from rest at $t = 0$. (*a*) In which direction will the bar start to move? (*b*) Show that at time *t* the bar has a speed of $(BIL/m)t$.

Picture the Problem We can use a constant-acceleration equation to relate the velocity of the crossbar to its acceleration and Newton's second law to express the acceleration of the crossbar in terms of the magnetic force acting on it. We can determine the direction of motion of the crossbar using a right-hand rule or, equivalently, by applying $\vec{\boldsymbol{F}} = I\vec{\ell} \times \vec{\boldsymbol{B}}$.

(*a*) Because the magnetic force is to the right and the crossbar starts from rest, the motion of the crossbar will also be toward the right.

(*b*) Using a constant-acceleration equation, express the velocity of the bar as a function of its acceleration and the time it has been in motion:

$$v = v_0 + at$$

or, because $v_0 = 0$,

$$v = at$$

Use Newton's second law to express the acceleration of the rail:

$$a = \frac{F}{m}$$

where F is the magnitude of the magnetic force acting in the direction of the crossbar's motion.

Substitute for a to obtain:

$$v = \frac{F}{m}t$$

Express the magnetic force acting on the current-carrying crossbar:

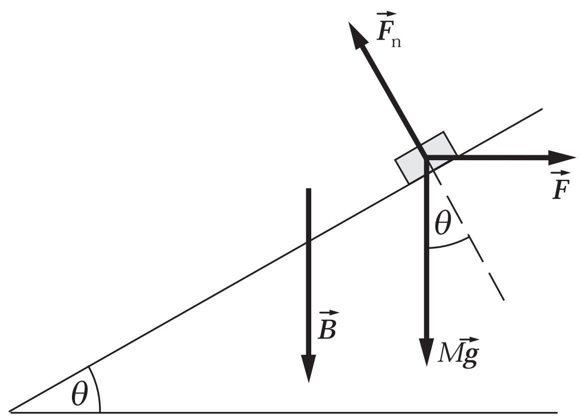
$$F = ILB$$

Substitute to obtain:

$$v = \boxed{\frac{ILB}{m}t}$$

75 •• **[SSM]** Assume that the rails Problem 74 are frictionless but tilted upward so that they make an angle θ with the horizontal, and with the current source attached to the low end of the rails. The magnetic field is still directed vertically downward. (*a*) What minimum value of B is needed to keep the bar from sliding down the rails? (*b*) What is the acceleration of the bar if B is twice the value found in Part (*a*)?

Picture the Problem Note that with the rails tilted, $\vec{F}$ still points horizontally to the right (I, and hence $\vec{\ell}$, is out of the page). Choose a coordinate system in which down the incline is the $+x$ direction. Then we can apply a condition for translational equilibrium to find the vertical magnetic field $\vec{B}$ needed to keep the bar from sliding down the rails. In Part (b) we can apply Newton's second law to find the acceleration of the crossbar when B is twice its value found in (a).



(a) Apply $\sum F_x = 0$ to the crossbar to obtain:

$$mg\sin\theta - I\ell B\cos\theta = 0$$

Solving for B yields:

$$B = \frac{mg}{I\ell}\tan\theta \text{ and } \vec{B} = \boxed{-\frac{mg}{I\ell}\tan\theta\,\hat{\boldsymbol{u}}_\text{v}}$$

where $\hat{\boldsymbol{u}}_\text{v}$ is a unit vector in the vertical direction.

(b) Apply Newton's second law to the crossbar to obtain:

$$I\ell B'\cos\theta - mg\sin\theta = ma$$

Solving for a yields:

$$a = \frac{I\ell B'}{m}\cos\theta - g\sin\theta$$

Substitute $B' = 2B$ and simplify to obtain:

$$a = \frac{2I\ell\frac{mg}{I\ell}\tan\theta}{m}\cos\theta - g\sin\theta$$

$$= 2g\sin\theta - g\sin\theta = \boxed{g\sin\theta}$$

Note that the direction of the acceleration is up the incline.

76 •• A long, narrow bar magnet that has magnetic moment $\vec{\boldsymbol{\mu}}$ parallel to its long axis is suspended at its center as a frictionless compass needle. When placed in region with a horizontal magnetic field $\vec{\boldsymbol{B}}$, the needle lines up with the field. If it is displaced by a small angle θ, show that the needle will oscillate about its equilibrium position with frequency $f=\dfrac{1}{2\pi}\sqrt{\dfrac{\mu B}{I}}$, where I is the moment of inertia of the needle about the point of suspension.

Picture the Problem We're being asked to show that, for small displacements from equilibrium, the bar magnet executes simple harmonic motion. To show its motion is SHM we need to show that the bar magnet experiences a linear restoring torque when displaced from equilibrium. We can accomplish this by applying Newton's second law in rotational form and using a small angle approximation to obtain the differential equation for simple harmonic motion. Once we have the differential equation of motion we can identify ω and express f.

Apply Newton's second law to the bar magnet:

$$-\mu B\sin\theta = I\frac{d^2\theta}{dt^2}$$

where the minus sign indicates that the torque acts in such a manner as to align the magnet with the magnetic field and I is the moment of inertia of the magnet.

For small displacements from equilibrium, $\theta << 1$ and:

$$\sin\theta\approx\theta$$

Hence our differential equation of motion becomes:

$$I\frac{d^2\theta}{dt^2}=-\mu B\theta$$

Thus for small displacements from equilibrium we see that the differential equation describing the motion of the bar magnet is the differential equation of simple harmonic motion. Solve this equation for $d^2\theta/dt^2$ to obtain:

$$\frac{d^2\theta}{dt^2}=-\frac{\mu B}{I}\theta=-\omega^2\theta$$

where $\omega=\sqrt{\dfrac{\mu B}{I}}$

Relate f to ω to obtain:

$$f=\frac{\omega}{2\pi}=\boxed{\frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}}$$

77 •• A straight conducting wire whose length is 20 m is parallel to the *y* axis and is moving in the +*x* direction with a speed of 20 m/s in a region with a magnetic field given by $0.50\,\text{T}\,\hat{\boldsymbol{k}}$. (*a*) Because of the magnetic force, electrons move to one end of the wire leaving the other end positively charged, until the electric field due to this charge separation exerts a force on the conduction electrons that balances the magnetic force. Find the magnitude and direction of this electric field in the steady state situation. (*b*) Which end of the wire is positively charged and which end is negatively charged? (*c*) Suppose the moving wire is 2.0-m long. What is the potential difference between its two ends due to this electric field?

Picture the Problem (*a*) We can use a condition for translational equilibrium to relate $\vec{E}$ to $\vec{F}$. In Part (*c*) we can apply the definition of electric field in terms of potential difference to evaluate the difference in potential between the ends of the moving wire.

(*a*) Sum the forces acting on an electron under steady-state conditions to obtain:

$$q\vec{E} + \vec{F} = 0 \Rightarrow \vec{E} = -\frac{\vec{F}}{q}$$

The magnetic force on an electron in the conductor is given by:

$$\vec{F} = q\vec{v} \times \vec{B} = qv\hat{\boldsymbol{i}} \times B\hat{\boldsymbol{k}} = qvB\left(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}}\right) = -qvB\hat{\boldsymbol{j}}$$

Substituting for $\vec{F}$ and simplifying yields:

$$\vec{E} = -\frac{-qvB\hat{\boldsymbol{j}}}{q} = vB\hat{\boldsymbol{j}}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = (20\,\text{m/s})(0.50\,\text{T})\hat{\boldsymbol{j}} = \boxed{(10\,\text{V/m})\hat{\boldsymbol{j}}}$$

(*b*) Because the magnetic force acting on the conduction electrons is in the +*y* direction, the end of the wire that is in the +*y* direction becomes negatively charged and the end of the wire that is in the −*y* direction becomes positively charged. The positive end has the lesser *y* coordinate.

(*c*) The potential difference between the ends of the wire is:

$$\Delta V = E\Delta y = (10.0\,\text{V/m})(2.0\,\text{m}) = \boxed{20\,\text{V}}$$

78 ••• A circular loop of wire that has a mass *m* and carries a constant current *I* is in a region with a uniform magnetic field. It is initially in equilibrium and its magnetic moment is aligned with the magnetic field. The loop is given a small angular displacement about an axis through it center and perpendicular to the magnetic field and then released. What is the period of the subsequent motion?

(Assume that the only torque exerted on the loop is due to the magnetic field and that there are no other forces acting on the loop.)

Picture the Problem We're being asked to show that, for small displacements from equilibrium, the circular loop executes simple harmonic motion. To show its motion is SHM we must show that the loop experiences a linear restoring torque when displaced from equilibrium. We can accomplish this by applying Newton's second law in rotational form and using a small angle approximation to obtain the differential equation for simple harmonic motion. Once we have the differential equation we can identify ω and express the period T of the motion.

Apply Newton's second law to the loop:

$$-IAB\sin\theta = I_{\text{inertia}}\frac{d^2\theta}{dt^2}$$

where the minus sign indicates that the torque acts in such a manner as to align the loop with the magnetic field and I_{inertia} is the moment of inertia of the loop.

For small displacements from equilibrium, $\theta << 1$ and:

$$\sin\theta \approx \theta$$

Hence, our differential equation of motion becomes:

$$I_{\text{inertia}}\frac{d^2\theta}{dt^2} = -IAB\theta$$

Thus for small displacements from equilibrium we see that the differential equation describing the motion of the current loop is the differential equation of simple harmonic motion. Solve this equation for $d^2\theta/dt^2$ to obtain:

$$\frac{d^2\theta}{dt^2} = -\frac{IAB}{I_{\text{inertia}}}\theta$$

.

Noting that the moment of inertia of a hoop about its diameter is $\frac{1}{2}mR^2$, substitute for I_{inertia} and simplify to obtain:

$$\frac{d^2\theta}{dt^2} = -\frac{I\pi R^2 B}{\frac{1}{2}mR^2}\theta = -\frac{2I\pi B}{m}\theta = -\omega^2\theta$$

where $\omega = \sqrt{\frac{2\pi IB}{m}}$

The period T of the motion is related to the angular frequency ω:

$$T = \frac{2\pi}{\omega}$$

Substituting for ω and simplifying yields:

$$T = \boxed{\sqrt{\frac{2\pi m}{IB}}}$$

79 ••• A small bar magnet has a magnetic moment $\vec{\boldsymbol{\mu}}$ that makes an angle θ with the x axis. The magnet is in a region that has a *non-uniform* magnetic field given by $\vec{\boldsymbol{B}} = B_x(x)\hat{\boldsymbol{i}} + B_y(y)\hat{\boldsymbol{j}}$. Using $F_x = -\partial U/\partial x$, $F_y = -\partial U/\partial y$ and $F_z = -\partial U/\partial z$, show that there is a net magnetic force on the magnet that is given by $\vec{\boldsymbol{F}} = \mu_x \frac{\partial B_x}{\partial x}\hat{\boldsymbol{i}} + \mu_y \frac{\partial B_y}{\partial y}\hat{\boldsymbol{j}}$.

Picture the Problem We can express $\vec{\boldsymbol{\mu}}$ in terms of its components and calculate U from $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{B}}$ using $U = -\vec{\boldsymbol{\mu}}\cdot\vec{\boldsymbol{B}}$. Knowing U we can calculate the components of $\vec{\boldsymbol{F}}$ using $F_x = -dU/dx$ and $F_y = -dU/dy$.

Express the net force acting on the magnet in terms of its components:

$$\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} \qquad (1)$$

Express $\vec{\boldsymbol{\mu}}$ in terms of its components:

$$\vec{\boldsymbol{\mu}} = \mu_x\hat{\boldsymbol{i}} + \mu_y\hat{\boldsymbol{j}} + \mu_z\hat{\boldsymbol{k}}$$

Express the potential energy of the bar magnetic in the nonuniform magnetic field:

$$\begin{aligned} U &= -\vec{\boldsymbol{\mu}}\cdot\vec{\boldsymbol{B}} \\ &= -\left(\mu_x\hat{\boldsymbol{i}} + \mu_y\hat{\boldsymbol{j}} + \mu_z\hat{\boldsymbol{k}}\right)\cdot\left(B_x(x)\hat{\boldsymbol{i}} + B_y(y)\hat{\boldsymbol{j}}\right) \\ &= -\mu_x B_x(x) - \mu_y B_y(y) \end{aligned}$$

Because $\vec{\boldsymbol{\mu}}$ is constant but $\vec{\boldsymbol{B}}$ depends on x and y:

$$F_x = -\frac{dU}{dx} = \mu_x\left(\frac{\partial B_x}{\partial x}\right)$$

and

$$F_y = -\frac{dU}{dy} = \mu_y\left(\frac{\partial B_y}{\partial y}\right)$$

Substitute in equation (1) to obtain:

$$\vec{\boldsymbol{F}} = \boxed{\mu_x \frac{\partial B_x}{\partial x}\hat{\boldsymbol{i}} + \mu_y \frac{\partial B_y}{\partial y}\hat{\boldsymbol{j}}}$$

80 •• A proton, a deuteron and an alpha particle all have the same kinetic energy. They are moving in a region with a uniform magnetic field that is perpendicular to each of their velocities. Let R_p, R_d, and R_α be the radii of their circular orbits, respectively. The deuteron has a charge that is equal to the charge a proton has, and an alpha particle has a charge that is equal to twice the charge a proton has. Find the ratios R_d/R_p and R_α/R_p. Assume that $m_\alpha = 2m_\text{d} = 4m_\text{p}$.

Picture the Problem We can apply Newton's second law to an orbiting particle to obtain an expression for the radius of its orbit R as a function of its mass m, charge q, speed v, and the magnitude of the magnetic field B.

Apply Newton's second law to an orbiting particle to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

Express the kinetic energy of the particle:

$$K = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}}$$

Substitute for v in the expression for r and simplify to obtain:

$$r = \frac{m}{qB}\sqrt{\frac{2K}{m}} = \frac{1}{qB}\sqrt{2Km} \qquad (1)$$

Using equation (1), express the ratio $R_\mathrm{d}/R_\mathrm{p}$:

$$\frac{R_\mathrm{d}}{R_\mathrm{p}} = \frac{\frac{1}{q_\mathrm{d}B}\sqrt{2Km_\mathrm{d}}}{\frac{1}{q_\mathrm{p}B}\sqrt{2Km_\mathrm{p}}} = \frac{q_\mathrm{p}}{q_\mathrm{d}}\sqrt{\frac{m_\mathrm{d}}{m_\mathrm{p}}} = \frac{e}{e}\sqrt{\frac{2m_\mathrm{p}}{m_\mathrm{p}}} = \boxed{\sqrt{2}}$$

Using equation (1), express the ratio R_α/R_p:

$$\frac{R_\alpha}{R_\mathrm{p}} = \frac{\frac{1}{q_\alpha B}\sqrt{2Km_\alpha}}{\frac{1}{q_\mathrm{p}B}\sqrt{2Km_\mathrm{p}}} = \frac{q_\mathrm{p}}{q_\alpha}\sqrt{\frac{m_\alpha}{m_\mathrm{p}}} = \frac{e}{2e}\sqrt{\frac{4m_\mathrm{p}}{m_\mathrm{p}}} = \boxed{1}$$

81 ••• Your forensic chemistry group, working closely with the local law enforcement agencies, has acquired a mass spectrometer similar to that discussed in the text. It employs a uniform magnetic field that has a magnitude of 0.75 T. To calibrate the mass spectrometer, you decide to measure the masses of various carbon isotopes by measuring the position of impact of the various singly ionized carbon ions that have entered the spectrometer with a kinetic energy of 25 keV. A wire chamber with position sensitivity of 0.50 mm is part of the apparatus. What will be the limit on its mass resolution (in kg) for ions in this mass range, that is those whose mass is on the order of that of a carbon atom?

Picture the Problem We can apply Newton's second law, with the force on a moving charged particle in a magnetic field as the net force, to an ion in the spectrometer to obtain an expression for the radius of its trajectory as a function of its momentum. We can then use the definition of kinetic energy to eliminate

the speed of the ion from the expression for the radius of its trajectory. Differentiating the expression for the range (twice the radius of curvature) of the ions with respect to their mass will yield the mass resolution for ions whose masses are roughly 19.9×10^{-27} kg. We'll assume that the carbon atoms are singly ionized.

Apply Newton's second law to an ion in the spectrometer to obtain:

$$qvB = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} \qquad (1)$$

where q is the charge of the ion, m is its mass, and r is the radius of curvature of its path.

From the definition of kinetic energy we have:

$$E = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2E}{m}}$$

Substituting for v in equation (1) and simplifying yields:

$$r = \frac{m\sqrt{\frac{2E}{m}}}{qB} = \frac{\sqrt{2mE}}{qB} \qquad (2)$$

The range R of the ions is twice their radius of curvature:

$$R = \frac{2\sqrt{2mE}}{qB} = \frac{\sqrt{8mE}}{qB} \qquad (3)$$

Differentiate R with respect to m to obtain:

$$\frac{dR}{dm} = \frac{d}{dm}\left(\frac{\sqrt{8mE}}{qB}\right) = \frac{\sqrt{8E}}{qB}\frac{d}{dm}\left(\sqrt{m}\right)$$

$$= \frac{\sqrt{8E}}{qB}\frac{1}{2\sqrt{m}} = \frac{\sqrt{\frac{2E}{m}}}{qB} = \sqrt{\frac{2E}{mq^2B^2}}$$

Solving for dm yields:

$$dm = \frac{dR}{\sqrt{\frac{2E}{mq^2B^2}}} = dR\sqrt{\frac{mq^2B^2}{2E}}$$

Substitute numerical values and evaluate dm:

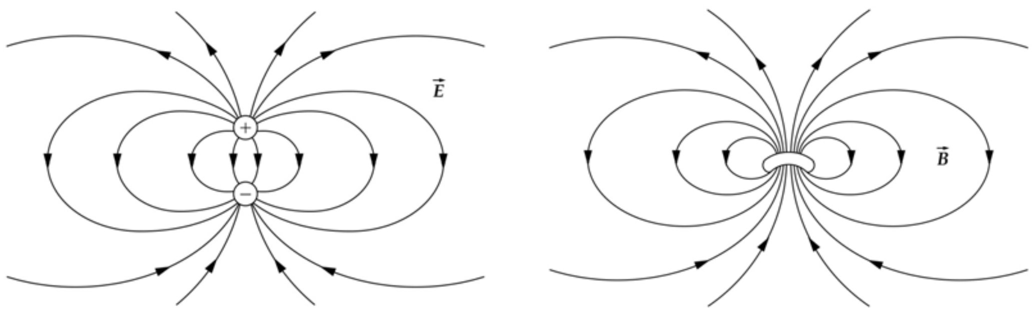
$$dm = (0.50\,\text{mm})\sqrt{\frac{\left(19.9\times10^{-27}\,\text{kg}\right)\left(1.602\times10^{-19}\,\text{C}\right)^2\left(0.80\,\text{T}\right)^2}{2\left(25\,\text{keV}\times\frac{1.602\times10^{-19}\,\text{C}}{\text{eV}}\right)}} = \boxed{1.0\times10^{-28}\,\text{kg}}$$

Chapter 27
Sources of the Magnetic Field

Conceptual Problems

1 • Sketch the field lines for the electric dipole and the magnetic dipole shown in Figure 27-47. How do the field lines differ in appearance close to the center of each dipole?

Picture the Problem Note that, while the two far fields (the fields far from the dipoles) are the same, the two near fields (the fields near the dipoles) are not. At the center of the electric dipole, the electric field is antiparallel to the direction of the far field above and below the dipole, and at the center of the magnetic dipole, the magnetic field is parallel to the direction of the far field above and below the dipole. It is especially important to note that while the electric field lines begin and terminate on electric charges, the magnetic field lines are continuous, that is, they form closed loops.



2 • Two wires lie in the plane of the page and carry equal currents in opposite directions, as shown in Figure 27- 48. At a point midway between the wires, the magnetic field is (*a*) zero, (*b*) into the page, (*c*) out of the page, (*d*) toward the top or bottom of the page, (*e*) toward one of the two wires.

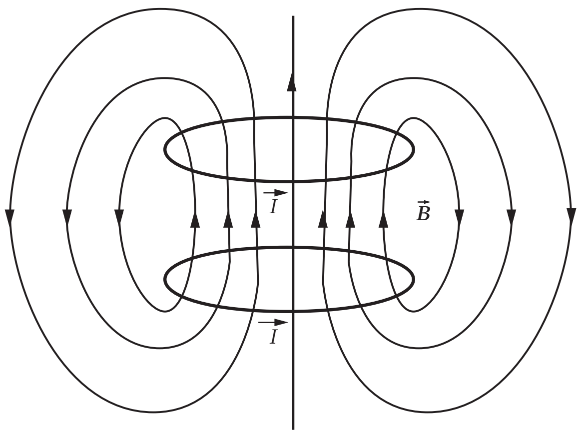
Determine the Concept Applying the right-hand rule to the wire to the left we see that the magnetic field due to its current is out of the page at the midpoint. Applying the right-hand rule to the wire to the right we see that the magnetic field due to its current is also out of the page at the midpoint. Hence, the sum of the magnetic fields is out of the page as well. 1is correct.

3 • Parallel wires 1 and 2 carry currents I_1 and I_2, respectively, where $I_2 = 2I_1$. The two currents are in the same direction. The magnitudes of the magnetic force by current 1 on wire 2 and by current 2 on wire 1 are F_{12} and F_{21}, respectively. These magnitudes are related by (*a*) $F_{21} = F_{12}$, (*b*) $F_{21} = 2F_{12}$, (*c*) $2F_{21} = F_{12}$, (*d*) $F_{21} = 4F_{12}$, (*e*) $4F_{21} = F_{12}$.

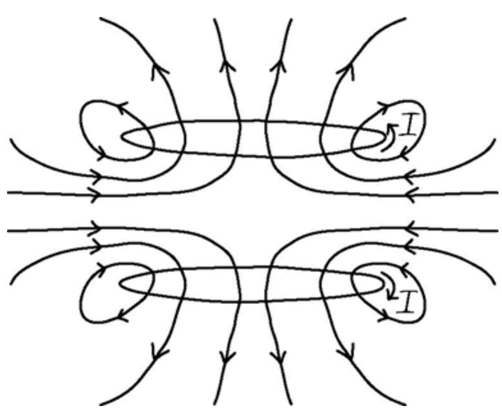
Determine the Concept While we could express the force wire 1 exerts on wire 2 and compare it to the force wire 2 exerts on wire 1 to show that they are the same, it is simpler to recognize that these are action and reaction forces. $\boxed{(a)}$ is correct.

4 • Make a field-line sketch of the magnetic field due to the currents in the pair of identical coaxial coils shown in Figure 27-49. Consider two cases: (*a*) the currents in the coils have the same magnitude and have the same direction and (*b*) the currents in the coils have the same magnitude and have the opposite directions.

Picture the Problem (*a*) The field-line sketch follows. An assumed direction for the current in the coils is shown in the diagram. Note that the field is stronger in the region between the coaxial coils and that the field lines have neither beginning nor ending points as do electric-field lines. Because there are an uncountable infinity of lines, only a representative few have been shown.



(*b*) The field-line sketch is shown below. An assumed direction for the current in the coils is shown in the diagram. Note that the field lines never begin or end and that they do not touch or cross each other. Because there are an uncountable infinity of lines, only a representative few have been shown.



5 • [SSM] Discuss the differences and similarities between Gauss's law for magnetism and Gauss's law for electricity.

Determine the Concept Both tell you about the respective fluxes through closed surfaces. In the electrical case, the flux is proportional to the net charge enclosed. In the magnetic case, the flux is always zero because there is no such thing as magnetic charge (a magnetic monopole). The source of the magnetic field is NOT the equivalent of electric charge; that is, it is NOT a thing called magnetic charge, but rather it is moving electric charges.

6 • Explain how you would modify Gauss's law if scientists discovered that single, isolated magnetic poles actually existed.

Determine the Concept Gauss' law for magnetism now reads: ″The flux of the magnetic field through any closed surface is equal to zero.″ Just like each electric pole has an electric pole strength (an amount of electric charge), each magnetic pole would have a magnetic pole strength (an amount of magnetic charge). Gauss' law for magnetism would read: ″The flux of the magnetic field through any closed surface is proportional to the total amount of magnetic charge inside.″

7 • [SSM] You are facing directly into one end of a long solenoid and the magnetic field inside of the solenoid points away from you. From your perspective, is the direction of the current in the solenoid coils clockwise or counterclockwise? Explain your answer.

Determine the Concept Application of the right-hand rule leads one to conclude that the current is clockwise.

8 • Opposite ends of a helical metal spring are connected to the terminals of a battery. Do the spacings between the coils of the spring tend to increase, decrease, or remain the same when the battery is connected? Explain your answer.

Determine the Concept Decrease. The coils attract each other and tend to move closer together when there is current in the spring.

9 • The current density is constant and uniform in a long straight wire that has a circular cross section. True or false:

(*a*) The magnitude of the magnetic field produced by this wire is greatest at the surface of the wire.
(*b*) The magnetic field strength in the region surrounding the wire varies inversely with the square of the distance from the wire's central axis.
(*c*) The magnetic field is zero at all points on the wire's central axis.
(*d*) The magnitude of the magnetic field inside the wire increases linearly with the distance from the wire's central axis.

(*a*) True. The magnetic field due to an infinitely long, straight wire is given by $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$, where R is the perpendicular distance to the field point. Because the magnetic field decreases linearly as the distance from the wire's central axis, the maximum field produced by this current is at the surface of the wire.

(*b*) False. Because the magnetic field due to an infinitely long, straight wire is given by $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$, where R is the perpendicular distance to the field point, the magnetic field outside the wire is inversely proportional to the distance from the wire's central axis.

(*c*) True. Because $I_C = 0$ at the center of the wire, Ampere's law ($\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\ell} = \mu_0 I_C$) tells us that the magnetic field is zero at the center of the wire.

(*d*) True. Application of Ampere's law shows that, inside the wire, $B = \frac{\mu_0}{2\pi}\frac{I}{R^2}r$, where R is the radius of the wire and r is the distance from the center of the wire.

10 • If the magnetic susceptibility of a material is positive,

(*a*) paramagnetic effects or ferromagnetic effects must be greater than diamagnetic effects,
(*b*) diamagnetic effects must be greater than paramagnetic effects,
(*c*) diamagnetic effects must be greater than ferromagnetic effects,
(*d*) ferromagnetic effects must be greater than paramagnetic effects,
(*e*) paramagnetic effects must be greater than ferromagnetic effects.

Determine the Concept The magnetic susceptibility χ_m is defined by the equation $\vec{\boldsymbol{M}} = \chi_m \frac{\vec{\boldsymbol{B}}_{app}}{\mu_0}$, where $\vec{\boldsymbol{M}}$ is the magnetization vector and $\vec{\boldsymbol{B}}_{app}$ is the applied magnetic field. For paramagnetic materials, χ_m is a small positive number that depends on temperature, whereas for diamagnetic materials, it is a small negative constant independent of temperature. $\boxed{(a)}$ is correct.

11 • **[SSM]** Of the four gases listed in Table 27-1, which are diamagnetic and which are paramagnetic?

Determine the Concept H_2, CO_2, and N_2 are diamagnetic ($\chi_m < 0$); O_2 is paramagnetic ($\chi_m > 0$).

12 • When a current is passed through the wire in Figure 27- 50, will the wire tend to bunch up or will it tend to form a circle? Explain your answer.

Determine the Concept It will tend to form a circle. Oppositely directed current elements will repel each other, and so opposite sides of the loop will repel.

The Magnetic Field of Moving Point Charges

13 • **[SSM]** At time t = 0, a particle has a charge of 12 μC, is located in the z = 0 plane at x = 0, y = 2.0 m, and has a velocity equal to $30\ \text{m/s}\ \hat{\boldsymbol{i}}$. Find the magnetic field in the z = 0 plane at (a) the origin, (b) x = 0, y = 1.0 m, (c) x = 0, y = 3.0 m, and (d) x = 0, y = 4.0 m.

Picture the Problem We can substitute for $\vec{v}$ and q in the equation describing the magnetic field of the moving charged particle ($\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{q\vec{v}\times\hat{r}}{r^2}$), evaluate r and $\hat{r}$ for each of the given points of interest, and then find $\vec{B}$.

The magnetic field of the moving charged particle is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

Substitute numerical values and simplify to obtain:

$$\vec{B} = \left(10^{-7}\ \text{N/A}^2\right)(12\ \mu\text{C})\frac{(30\ \text{m/s})\hat{i}\times\hat{r}}{r^2} = \left(36.0\ \text{pT}\cdot\text{m}^2\right)\frac{\hat{i}\times\hat{r}}{r^2}$$

(a) Find r and $\hat{r}$ for the particle at (0, 2.0 m) and the point of interest at the origin:

$\vec{r} = -(2.0\ \text{m})\hat{j}$, $r = 2.0\ \text{m}$, and $\hat{r} = -\hat{j}$

Evaluating $\vec{B}(0,0)$ yields:

$$\vec{B}(0,0) = \left(36.0\ \text{pT}\cdot\text{m}^2\right)\frac{\hat{i}\times\left(-\hat{j}\right)}{(2.0\ \text{m})^2} = \boxed{-(9.0\ \text{pT})\hat{k}}$$

(b) Find r and $\hat{r}$ for the particle at (0, 2.0 m) and the point of interest at (0, 1.0 m):

$\vec{r} = -(1.0\ \text{m})\hat{j}$, $r = 1.0\ \text{m}$, and $\hat{r} = -\hat{j}$

Evaluate $\vec{B}(0,1.0\,\text{m})$ to obtain:

$$\vec{B}(0,1.0\,\text{m}) = (36.0\,\text{pT}\cdot\text{m}^2)\frac{\hat{i}\times(-\hat{j})}{(1.0\,\text{m})^2} = \boxed{-(36\,\text{pT})\hat{k}}$$

(*c*) Find *r* and $\hat{r}$ for the particle at (0, 2.0 m) and the point of interest at (0, 3.0 m):

$\vec{r} = (1.0\,\text{m})\hat{j}$, $r = 1.0\,\text{m}$, and $\hat{r} = \hat{j}$

Evaluating $\vec{B}(0,3.0\,\text{m})$ yields:

$$\vec{B}(0,3.0\,\text{m}) = (36.0\,\text{pT}\cdot\text{m}^2)\frac{\hat{i}\times\hat{j}}{(1.0\,\text{m})^2} = \boxed{(36\,\text{pT})\hat{k}}$$

(*d*) Find *r* and $\hat{r}$ for the particle at (0, 2.0 m) and the point of interest at (0, 4.0 m):

$\vec{r} = (2.0\,\text{m})\hat{j}$, $r = 2.0\,\text{m}$, and $\hat{r} = \hat{j}$

Evaluate $\vec{B}(0,4.0\,\text{m})$ to obtain:

$$\vec{B}(0,4.0\,\text{m}) = (36.0\,\text{pT}\cdot\text{m}^2)\frac{\hat{i}\times\hat{j}}{(2.0\,\text{m})^2} = \boxed{(9.0\,\text{pT})\hat{k}}$$

14 • At time $t = 0$, a particle has a charge of 12 μC, is located in the $z = 0$ plane at $x = 0$, $y = 2.0$ m, and has a velocity equal to $30\ \text{m/s}\,\hat{i}$. Find the magnetic field in the $z = 0$ plane at (*a*) $x = 1.0$ m, $y = 3.0$ m, (*b*) $x = 2.0$ m, $y = 2.0$ m, and (*c*) $x = 2.0$ m, $y = 3.0$ m.

Picture the Problem We can substitute for $\vec{v}$ and *q* in the equation describing the magnetic field of the moving charged particle ($\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$), evaluate *r* and $\hat{r}$ for each of the given points of interest, and substitute to find $\vec{B}$.

The magnetic field of the moving charged particle is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

Substitute numerical values and simplify to obtain:

$$\vec{B} = (10^{-7}\ \text{N/A}^2)(12\,\mu\text{C})\frac{(30\,\text{m/s})\hat{i}\times\hat{r}}{r^2} = (36.0\,\text{pT}\cdot\text{m}^2)\frac{\hat{i}\times\hat{r}}{r^2}$$

(*a*) Find r and $\hat{\boldsymbol{r}}$ for the particle at (0, 2.0 m) and the point of interest at (1.0 m, 3.0 m):

$\vec{r} = (1.0\,\mathrm{m})\hat{\boldsymbol{i}} + (1.0\,\mathrm{m})\hat{\boldsymbol{j}}$, $r = \sqrt{2}\,\mathrm{m}$, and $\hat{\boldsymbol{r}} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{i}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{j}}$

Substitute for $\hat{\boldsymbol{r}}$ and r in $\vec{\boldsymbol{B}} = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\hat{\boldsymbol{r}}}{r^2}$ and evaluate $\vec{\boldsymbol{B}}(1.0\,\mathrm{m},3.0\,\mathrm{m})$:

$$\vec{\boldsymbol{B}}(1.0\,\mathrm{m},3.0\,\mathrm{m}) = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\left(\frac{1}{\sqrt{2}}\hat{\boldsymbol{i}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{j}}\right)}{\left(\sqrt{2}\,\mathrm{m}\right)^2} = \frac{(36.0\,\mathrm{pT\cdot m^2})}{\sqrt{2}}\frac{\hat{\boldsymbol{k}}}{\left(\sqrt{2}\,\mathrm{m}\right)^2}$$

$$= \boxed{(13\,\mathrm{pT})\hat{\boldsymbol{k}}}$$

(*b*) Find r and $\hat{\boldsymbol{r}}$ for the particle at (0, 2.0 m) and the point of interest at (2.0 m, 2.0 m):

$\vec{r} = (2.0\,\mathrm{m})\hat{\boldsymbol{i}}$, $r = 2.0\,\mathrm{m}$, and $\hat{\boldsymbol{r}} = \hat{\boldsymbol{i}}$

Substitute for $\hat{\boldsymbol{r}}$ and r in $\vec{\boldsymbol{B}} = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\hat{\boldsymbol{r}}}{r^2}$ and evaluate $\vec{\boldsymbol{B}}(2.0\,\mathrm{m},2.0\,\mathrm{m})$:

$$\vec{\boldsymbol{B}}(2.0\,\mathrm{m},2.0\,\mathrm{m}) = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}}}{(2.0\,\mathrm{m})^2} = \boxed{0}$$

(*c*) Find r and $\hat{\boldsymbol{r}}$ for the particle at (0, 2.0 m) and the point of interest at (2.0 m, 3.0 m):

$\vec{r} = (2.0\,\mathrm{m})\hat{\boldsymbol{i}} + (1.0\,\mathrm{m})\hat{\boldsymbol{j}}$, $r = \sqrt{5}\,\mathrm{m}$, and $\hat{\boldsymbol{r}} = \frac{2}{\sqrt{5}}\hat{\boldsymbol{i}} + \frac{1}{\sqrt{5}}\hat{\boldsymbol{j}}$

Substitute for $\hat{\boldsymbol{r}}$ and r in $\vec{\boldsymbol{B}} = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\hat{\boldsymbol{r}}}{r^2}$ and evaluate $\vec{\boldsymbol{B}}(2.0\,\mathrm{m},3.0\,\mathrm{m})$:

$$\vec{\boldsymbol{B}}(2.0\,\mathrm{m},3.0\,\mathrm{m}) = (36.0\,\mathrm{pT\cdot m^2})\frac{\hat{\boldsymbol{i}}\times\left(\frac{2}{\sqrt{5}}\hat{\boldsymbol{i}} + \frac{1}{\sqrt{5}}\hat{\boldsymbol{j}}\right)}{\left(\sqrt{5}\,\mathrm{m}\right)^2} = \boxed{(3.2\,\mathrm{pT})\hat{\boldsymbol{k}}}$$

15 • A proton has a velocity of 1.0×10^2 m/s $\hat{\boldsymbol{i}} + 2.0\times10^2$ m/s $\hat{\boldsymbol{j}}$ and is located in the z = 0 plane at x = 3.0 m, y = 4.0 m at some time t. Find the magnetic field in the z = 0 plane at (*a*) x = 2.0 m, y = 2.0 m, (*b*) x = 6.0 m, y = 4.0 m, and (*c*) x = 3.0 m, y = 6.0 m.

Picture the Problem We can substitute for $\vec{v}$ and q in the equation describing the magnetic field of the moving proton ($\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$), evaluate r and $\hat{r}$ for each of the given points of interest, and substitute to find $\vec{B}$.

The magnetic field of the moving proton is given by:

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

Substituting numerical values yields:

$$\begin{aligned}\vec{B}&=\left(10^{-7}\ \text{N/A}^2\right)\left(1.602\times10^{-19}\,\text{C}\right)\frac{\left[\left(1.0\times10^2\ \text{m/s}\right)\hat{i}+\left(2.0\times10^2\ \text{m/s}\right)\hat{j}\right]\times\hat{r}}{r^2}\\&=\left(1.60\times10^{-24}\ \text{T}\cdot\text{m}^2\right)\frac{\left(1.0\,\hat{i}+2.0\hat{j}\right)\times\hat{r}}{r^2}\end{aligned}$$

(*a*) Find r and $\hat{r}$ for the proton at (3.0 m, 4.0 m) and the point of interest at (2.0 m, 2.0 m):

$\vec{r}=-(1.0\,\text{m})\hat{i}-(2.0\,\text{m})\hat{j}$, $r=\sqrt{5}\,\text{m}$, and $\hat{r}=-\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}$

Substitute for $\hat{r}$ and r in $\vec{B}=\left(1.60\times10^{-24}\ \text{T}\cdot\text{m}^2\right)\frac{\left(1.0\,\hat{i}+2.0\hat{j}\right)\times\hat{r}}{r^2}$ and evaluate $\vec{B}(2.0\,\text{m},2.0\,\text{m})$:

$$\begin{aligned}\vec{B}(2.0\,\text{m},2.0\,\text{m})&=\left(1.60\times10^{-24}\ \text{T}\cdot\text{m}^2\right)\frac{\left(1.0\hat{i}+2.0\hat{j}\right)\times\left(-\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}\right)}{r^2}\\&=\frac{\left(1.60\times10^{-24}\ \text{T}\cdot\text{m}^2\right)}{\sqrt{5}}\left[\frac{-2.0\hat{k}+2.0\hat{k}}{\left(\sqrt{5}\,\text{m}\right)^2}\right]=\boxed{0}\end{aligned}$$

(*b*) Find r and $\hat{r}$ for the proton at (3.0 m, 4.0 m) and the point of interest at (6.0 m, 4.0 m):

$\vec{r}=(3.0\,\text{m})\hat{i}$, $r=3.0\,\text{m}$, and $\hat{r}=\hat{i}$

Substitute for $\hat{r}$ and r in $\vec{B}=\left(1.60\times10^{-24}\ \text{T}\cdot\text{m}^2\right)\frac{\left(1.0\,\hat{i}+2.0\hat{j}\right)\times\hat{r}}{r^2}$ and evaluate $\vec{B}(6.0\,\text{m},4.0\,\text{m})$:

$$\vec{B}(6.0\,\text{m},4.0\,\text{m}) = \left(1.60\times10^{-24}\,\text{T}\cdot\text{m}^2\right)\frac{\left(1.0\,\hat{i}+2.0\,\hat{j}\right)\times\hat{i}}{(3.0\,\text{m})^2}$$

$$= \left(1.60\times10^{-24}\,\text{T}\cdot\text{m}^2\right)\left(\frac{-2.0\,\hat{k}}{9.0\,\text{m}^2}\right) = \boxed{-\left(3.6\times10^{-25}\,\text{T}\right)\hat{k}}$$

(*c*) Find *r* and $\hat{r}$ for the proton at (3.0 m, 4.0 m) and the point of interest at the (3.0 m, 6.0 m):

$\vec{r} = (2.0\,\text{m})\hat{j}$, $r = 2.0\,\text{m}$, and $\hat{r} = \hat{j}$

Substitute for $\hat{r}$ and *r* in $\vec{B} = \left(1.60\times10^{-24}\,\text{T}\cdot\text{m}^2\right)\dfrac{\left(1.0\,\hat{i}+2.0\,\hat{j}\right)\times\hat{r}}{r^2}$ and evaluate $\vec{B}(3.0\,\text{m},6.0\,\text{m})$:

$$\vec{B}(3.0\,\text{m},6.0\,\text{m}) = \left(1.60\times10^{-24}\,\text{T}\cdot\text{m}^2\right)\frac{\left(1.0\,\hat{i}+2.0\,\hat{j}\right)\times\hat{j}}{(2.0\,\text{m})^2}$$

$$= \left(1.60\times10^{-24}\,\text{T}\cdot\text{m}^2\right)\left(\frac{\hat{k}}{4.0\,\text{m}^2}\right) = \boxed{\left(4.0\times10^{-25}\,\text{T}\right)\hat{k}}$$

16 •• In a pre–quantum-mechanical model of the hydrogen atom, an electron orbits a proton at a radius of 5.29×10^{-11} m. According to this model, what is the magnitude of the magnetic field at the proton due to the orbital motion of the electron? Neglect any motion of the proton.

Picture the Problem The centripetal force acting on the orbiting electron is the Coulomb force between the electron and the proton. We can apply Newton's second law to the electron to find its orbital speed and then use the expression for the magnetic field of a moving charge to find *B*.

Express the magnetic field due to the motion of the electron:

$$B = \frac{\mu_0}{4\pi}\frac{ev}{r^2}$$

Apply $\sum F_{\text{radial}} = ma_{\text{c}}$ to the electron:

$$\frac{ke^2}{r^2} = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{ke^2}{mr}}$$

Substitute for *v* in the expression for *B* and simplify to obtain:

$$B = \frac{\mu_0}{4\pi}\frac{e}{r^2}\sqrt{\frac{ke^2}{mr}} = \frac{\mu_0 e^2}{4\pi r^2}\sqrt{\frac{k}{mr}}$$

Substitute numerical values and evaluate B:

$$B = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{4\pi\left(5.29\times10^{-11}\ \text{m}\right)^2}\sqrt{\frac{8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2}{\left(9.109\times10^{-31}\ \text{kg}\right)\left(5.29\times10^{-11}\ \text{m}\right)}}$$
$$= \boxed{12.5\,\text{T}}$$

17 •• Two equal point charges q are, at some instant, located at (0, 0, 0) and at (0, b, 0). They are both moving with speed v in the $+x$ direction (assume $v \ll c$). Find the ratio of the magnitude of the magnetic force to the magnitude of the electric force on each charge.

Picture the Problem We can find the ratio of the magnitudes of the magnetic and electrostatic forces by using the expression for the magnetic field of a moving charge and Coulomb's law. Note that $\vec{v}$ and $\vec{r}$, where $\vec{r}$ is the vector from one charge to the other, are at right angles. The field $\vec{B}$ due to the charge at the origin at the location (0, b, 0) is perpendicular to $\vec{v}$ and $\vec{r}$.

Express the magnitude of the magnetic force on the moving charge at (0, b, 0):

$$F_B = qvB = \frac{\mu_0}{4\pi}\frac{q^2v^2}{b^2}$$

and, applying the right hand rule, we find that the direction of the force is toward the charge at the origin; i.e., the magnetic force between the two moving charges is attractive.

Express the magnitude of the repulsive electrostatic interaction between the two charges:

$$F_E = \frac{1}{4\pi\in_0}\frac{q^2}{b^2}$$

Express the ratio of F_B to F_E and simplify to obtain:

$$\frac{F_B}{F_E} = \frac{\dfrac{\mu_0}{4\pi}\dfrac{q^2v^2}{b^2}}{\dfrac{1}{4\pi\in_0}\dfrac{q^2}{b^2}} = \boxed{\in_0\mu_0v^2}$$

The Magnetic Field Using the Biot–Savart Law

18 • A small current element at the origin has a length of 2.0 mm and carries a current of 2.0 A in the $+z$ direction. Find the magnetic field due to the current element: (a) on the x axis at x = 3.0 m, (b) on the x axis at x = –6.0 m, (c) on the z axis at z = 3.0 m, and (d) on the y axis at y = 3.0 m.

Picture the Problem We can substitute for I and $d\vec{\ell} \approx \Delta\vec{\ell}$ in the Biot-Savart relationship ($d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\hat{r}}{r^2}$), evaluate r and $\hat{r}$ for each of the points of interest, and substitute to find $d\vec{B}$.

Express the Biot-Savart law for the given current element:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{\ell}\times\hat{r}}{r^2} = \left(10^{-7}\ \text{N/A}^2\right)\frac{(2.0\,\text{A})(2.0\,\text{mm})\hat{k}\times\hat{r}}{r^2} = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\hat{r}}{r^2}$$

(*a*) Find r and $\hat{r}$ for the point whose coordinates are (3.0 m, 0, 0):

$\vec{r} = (3.0\,\text{m})\hat{i}$, $r = 3.0\,\text{m}$, and $\hat{r} = \hat{i}$

Evaluate $\vec{B}$ at (3.0 m, 0, 0):

$$\vec{B}(3.0\,\text{m},0,0) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\hat{i}}{(3.0\,\text{m})^2} = \boxed{(44\,\text{pT})\hat{j}}$$

(*b*) Find r and $\hat{r}$ for the point whose coordinates are (–6.0 m, 0, 0):

$\vec{r} = -(6.0\,\text{m})\hat{i}$, $r = 6.0\,\text{m}$, and $\hat{r} = -\hat{i}$

Evaluate $\vec{B}$ at (–6.0 m, 0, 0):

$$\vec{B}(-6.0\,\text{m},0,0) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\cdot\frac{\hat{k}\times\left(-\hat{i}\right)}{(6.0\,\text{m})^2} = \boxed{-(11\,\text{pT})\hat{j}}$$

(*c*) Find r and $\hat{r}$ for the point whose coordinates are (0, 0, 3.0 m):

$\vec{r} = (3.0\,\text{m})\hat{k}$, $r = 3.0\,\text{m}$, and $\hat{r} = \hat{k}$

Evaluate $\vec{B}$ at (0, 0, 3.0 m):

$$\vec{B}(0,0,3.0\,\text{m}) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\hat{k}}{(3.0\,\text{m})^2} = \boxed{0}$$

(*d*) Find r and $\hat{r}$ for the point whose coordinates are (0, 3.0 m, 0):

$\vec{r} = (3.0\,\text{m})\hat{j}$, $r = 3.0\,\text{m}$, and $\hat{r} = \hat{j}$

Evaluate $\vec{B}$ at (0, 3.0 m, 0):

$$\vec{B}(0,3.0\,\text{m},0) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\hat{j}}{(3.0\,\text{m})^2} = \boxed{-(44\,\text{pT})\hat{i}}$$

19 • **[SSM]** A small current element at the origin has a length of 2.0 mm and carries a current of 2.0 A in the +z direction. Find the magnitude and direction of the magnetic field due to the current element at the point (0, 3.0 m, 4.0 m).

Picture the Problem We can substitute for I and $d\vec{\ell} \approx \Delta\vec{\ell}$ in the Biot-Savart law ($d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\hat{r}}{r^2}$), evaluate r and $\hat{r}$ for (0, 3.0 m, 4.0 m), and substitute to find $d\vec{B}$.

The Biot-Savart law for the given current element is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{\ell}\times\hat{r}}{r^2}$$

Substituting numerical values yields:

$$\vec{B} = \left(1.0\times10^{-7}\ \text{N/A}^2\right)\frac{(2.0\,\text{A})(2.0\,\text{mm})\hat{k}\times\hat{r}}{r^2} = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\hat{r}}{r^2}$$

Find r and $\hat{r}$ for the point whose coordinates are (0, 3.0 m, 4.0 m):

$$\vec{r} = (3.0\,\text{m})\hat{j} + (4.0\,\text{m})\hat{k},$$

$$r = 5.0\,\text{m},\ \text{and}\ \hat{r} = \frac{3}{5}\hat{j} + \frac{4}{5}\hat{k}$$

Evaluate $\vec{B}$ at (0, 3.0 m, 4.0 m):

$$\vec{B}(0,3.0\,\text{m},4.0\,\text{m}) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k}\times\left(\frac{3}{5}\hat{j}+\frac{4}{5}\hat{k}\right)}{(5.0\,\text{m})^2} = \boxed{-(9.6\,\text{pT})\hat{i}}$$

20 • A small current element at the origin has a length of 2.0 mm and carries a current of 2.0 A in the +z direction. Find the magnitude of the magnetic field due to this element and indicate its direction on a diagram at (*a*) x = 2.0 m, y = 4.0 m, z = 0 and (*b*) x = 2.0 m, y = 0, z = 4.0 m.

Picture the Problem We can substitute for I and $d\vec{\ell} \approx \Delta\vec{\ell}$ in the Biot-Savart law ($d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\hat{r}}{r^2}$), evaluate r and $\hat{r}$ for the given points, and substitute to find $d\vec{B}$.

Apply the Biot-Savart law to the given current element to obtain:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{\ell} \times \hat{r}}{r^2} = \left(10^{-7}\ \text{N/A}^2\right)\frac{(2.0\,\text{A})(2.0\,\text{mm})\hat{k} \times \hat{r}}{r^2} = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k} \times \hat{r}}{r^2}$$

(*a*) Find r and $\hat{r}$ for the point whose coordinates are (2.0 m, 4.0 m, 0):

$$\vec{r} = (2.0\,\text{m})\hat{i} + (4.0\,\text{m})\hat{j},$$

$$r = 2.0\sqrt{5}\,\text{m},$$

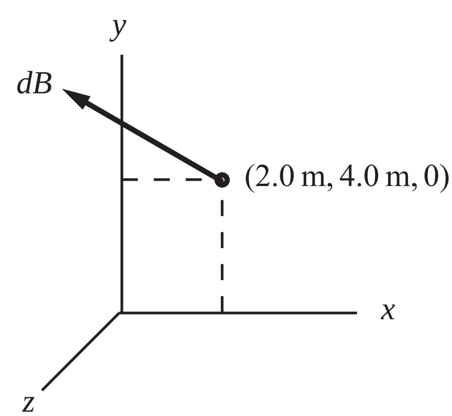
and

$$\hat{r} = \frac{2.0}{2\sqrt{5}}\hat{i} + \frac{4.0}{2\sqrt{5}}\hat{j} = \frac{1.0}{\sqrt{5}}\hat{i} + \frac{2.0}{\sqrt{5}}\hat{j}$$

Evaluate $d\vec{B}$ at (2.0 m, 4.0 m, 0):

$$d\vec{B}(2.0\,\text{m}, 4.0\,\text{m}, 0) = \left(0.400\,\text{nT}\cdot\text{m}^2\right)\frac{\hat{k} \times \left(\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}\right)}{\left(2\sqrt{5}\,\text{m}\right)^2} = \boxed{-(18\,\text{pT})\hat{i} + (8.9\,\text{pT})\hat{j}}$$

The diagram is shown to the right:



(*b*) Find r and $\hat{\boldsymbol{r}}$ for the point whose coordinates are (2.0 m, 0, 4.0 m):

$$\vec{\boldsymbol{r}} = (2.0\,\text{m})\hat{\boldsymbol{i}} + (4.0\,\text{m})\hat{\boldsymbol{k}}\,,$$

$$r = 2.0\sqrt{5}\,\text{m}\,,$$

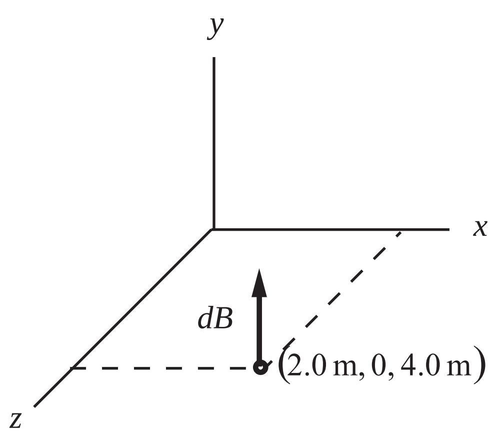
and

$$\hat{\boldsymbol{r}} = \frac{2.0}{2.0\sqrt{5}}\hat{\boldsymbol{i}} + \frac{4.0}{2.0\sqrt{5}}\hat{\boldsymbol{k}} = \frac{1.0}{\sqrt{5}}\hat{\boldsymbol{i}} + \frac{2.0}{\sqrt{5}}\hat{\boldsymbol{k}}$$

Evaluate $d\vec{\boldsymbol{B}}$ at (2.0 m, 0, 4.0 m):

$$d\vec{\boldsymbol{B}}(2.0\,\text{m},0,4.0\,\text{m}) = (0.400\,\text{nT}\cdot\text{m}^2)\frac{\hat{\boldsymbol{k}}\times\left(\frac{1.0}{\sqrt{5}}\hat{\boldsymbol{i}} + \frac{2.0}{\sqrt{5}}\hat{\boldsymbol{k}}\right)}{\left(2.0\sqrt{5}\,\text{m}\right)^2} = \boxed{(8.9\,\text{pT})\hat{\boldsymbol{j}}}$$

The diagram is shown to the right:



The Magnetic Field Due to Current Loops and Coils

21 • A single conducting loop has a radius equal to 3.0 cm and carries a current equal to 2.6 A. What is the magnitude of the magnetic field on the line through the center of the loop and perpendicular to the plane of the loop (*a*) the center of the loop, (*b*) 1.0 cm from the center, (*c*) 2.0 cm from the center, and (*d*) 35 cm from the center?

Picture the Problem We can use $B_x = \frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{\left(x^2 + R^2\right)^{3/2}}$ to find B on the axis of the current loop.

B on the axis of a current loop is given by:

$$B_x = \frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{\left(x^2 + R^2\right)^{3/2}}$$

Substitute numerical values to obtain:

$$B_x = \left(10^{-7}\ \text{N/A}^2\right)\frac{2\pi(0.030\,\text{m})^2(2.6\,\text{A})}{\left(x^2+(0.030\,\text{m})^2\right)^{3/2}} = \frac{1.470\times10^{-9}\ \text{T}\cdot\text{m}^3}{\left(x^2+(0.030\,\text{m})^2\right)^{3/2}}$$

(*a*) Evaluate *B* at the center of the loop:

$$B(0) = \frac{1.470\times10^{-9}\ \text{T}\cdot\text{m}^3}{\left(0+(0.030\,\text{m})^2\right)^{3/2}} = \boxed{54\,\mu\text{T}}$$

(*b*) Evaluate *B* at $x = 1.0$ cm:

$$B(0.010\,\text{m}) = \frac{1.470\times10^{-9}\ \text{T}\cdot\text{m}^3}{\left((0.010\,\text{m})^2+(0.030\,\text{m})^2\right)^{3/2}} = \boxed{46\,\mu\text{T}}$$

(*c*) Evaluate *B* at $x = 2.0$ cm:

$$B(0.020\,\text{m}) = \frac{1.470\times10^{-9}\ \text{T}\cdot\text{m}^3}{\left((0.020\,\text{m})^2+(0.030\,\text{m})^2\right)^{3/2}} = \boxed{31\,\mu\text{T}}$$

(*d*) Evaluate *B* at $x = 35$ cm:

$$B(0.35\,\text{m}) = \frac{1.470\times10^{-9}\ \text{T}\cdot\text{m}^3}{\left((0.35\,\text{m})^2+(0.030\,\text{m})^2\right)^{3/2}} = \boxed{34\,\text{nT}}$$

22 ••• A pair of identical coils, each having a radius of 30 cm, are separated by a distance equal to their radii, that is 30 cm. Called *Helmholtz coils*, they are coaxial and carry equal currents in directions such that their axial fields are in the same direction. A feature of Helmholtz coils is that the resultant magnetic field in the region between the coils is very uniform. Assume the current in each is 15 A and there are 250 turns for each coil. Using a **spreadsheet** program calculate and graph the magnetic field as a function of *z,* the distance from the center of the coils along the common axis, for -30 cm $< z < +30$ cm. Over what range of *z* does the field vary by less than 20%?

Picture the Problem Let the origin be midway between the coils so that one of them is centered at $z = -R/2$ and the other is centered at $z = R/2$, where *R* is the radius of the coils. Let the numeral 1 denote the coil centered at $z = -R/2$ and the numeral 2 the coil centered at $z = R/2$. We can express the magnetic field in the region between the coils as the sum of the magnetic fields B_1 and B_2 due to the two coils.

The magnetic field on the z is the sum of the magnetic fields due to the currents in coils 1 and 2:

$$B_z = B_1(z) + B_2(z) \quad (1)$$

Express the magnetic field on the z axis due to the coil centered at $z = -R/2$:

$$B_1(z) = \frac{\mu_0 NR^2 I}{2\left[z + \left(\frac{1}{2}R\right)^2 + R^2\right]^{3/2}}$$

where N is the number of turns.

Express the magnetic field on the z axis due to the coil centered at $z = R/2$:

$$B_2(z) = \frac{\mu_0 NR^2 I}{2\left[\left(z - \frac{1}{2}R\right)^2 + R^2\right]^{3/2}}$$

Substitute for $B_1(z)$ and $B_2(z)$ in equation (1) to express the total magnetic field along the z axis:

$$\begin{aligned} B_z &= \frac{\mu_0 NR^2 I}{2\left[\left(z + \frac{1}{2}R\right)^2 + R^2\right]^{3/2}} + \frac{\mu_0 NR^2 I}{2\left[\left(z - \frac{1}{2}R\right)^2 + R^2\right]^{3/2}} \\ &= \frac{\mu_0 NR^2 I}{2}\left(\left[\left(z + \frac{1}{2}R\right)^2 + R^2\right]^{-3/2} + \left[\left(z - \frac{1}{2}R\right)^2 + R^2\right]^{-3/2}\right) \end{aligned}$$

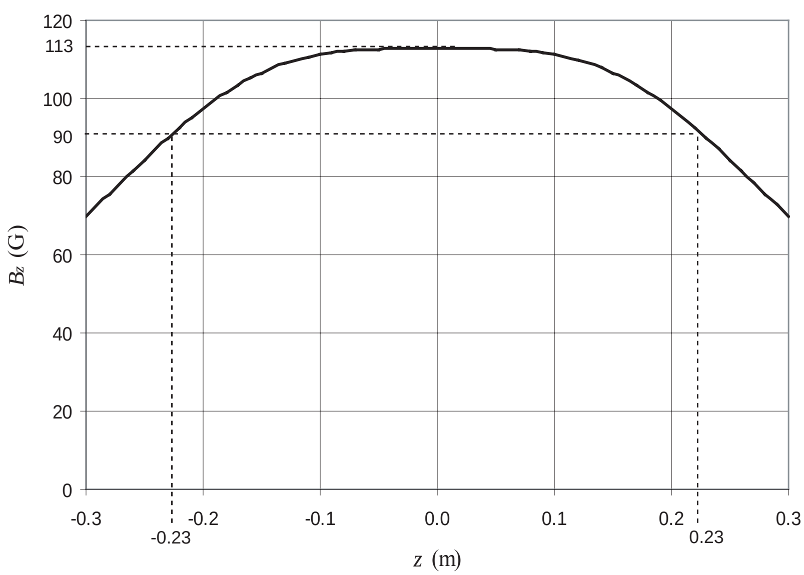
The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	1.13×10^{-7}	μ_0
B2	0.30	R
B3	250	N
B3	15	I
B5	0.5*$B$1*$B$3*($B$2^2)*$B$4	Coeff $= \frac{\mu_0 NR^2 I}{2}$
A8	−0.30	$-R$
B8	$B$5*(($B$2/2+A8)^2+$B$2^2)^(−3/2)	$\frac{\mu_0 NR^2 I}{2}\left[\left(z - \frac{1}{2}R\right)^2 + R^2\right]^{-3/2}$
C8	$B$5* (($B$2/2−A8)^2+$B$2^2)^(−3/2)	$\frac{\mu_0 NR^2 I}{2}\left[\left(z + \frac{1}{2}R\right)^2 + R^2\right]^{-3/2}$
D8	10^4(B8+C8)	$B_z = 10^4\left(B_1(z) + B_2(z)\right)$

	A	B	C	D
1	μ_0=	1.26E-06	N/A^2	
2	R=	0.30	m	
3	N=	250	turns	
4	I=	15	A	

5	Coeff=	2.13E–04		
6				
7	z	B_1	B_2	$B(z)$
8	–0.30	5.63E–03	1.34E–03	70
9	–0.29	5.86E–03	1.41E–03	73
10	–0.28	6.08E–03	1.48E–03	76
11	–0.27	6.30E–03	1.55E–03	78
12	–0.26	6.52E–03	1.62E–03	81
13	–0.25	6.72E–03	1.70E–03	84
14	–0.24	6.92E–03	1.78E–03	87
15	–0.23	7.10E–03	1.87E–03	90
61	0.23	1.87E–03	7.10E–03	90
62	0.24	1.78E–03	6.92E–03	87
63	0.25	1.70E–03	6.72E–03	84
64	0.26	1.62E–03	6.52E–03	81
65	0.27	1.55E–03	6.30E–03	78
66	0.28	1.48E–03	6.08E–03	76
67	0.29	1.41E–03	5.86E–03	73
68	0.30	1.34E–03	5.63E–03	70

The following graph of B_z as a function of z was plotted using the data in the above table.



The maximum value of B_z is 113 G. Eighty percent of this maximum value is 90 G. We see that the field is within 20 percent of 113 G in the interval

$$\boxed{-0.23\,\text{m} < z < 0.23\,\text{m}.}$$

23 ••• A pair of Helmholtz coils that have radii R have their axes along the z axis (see Problem 22). One coil is in the $z=-\frac{1}{2}R$ plane and the second coil is in the $z=\frac{1}{2}R$ plane. Show that on the z axis at $z = 0$ $dB_z/dz = 0$, $d^2B_z/dz^2 = 0$, and $d^3B_z/dz^3 = 0$. (*Note:* These results show that the magnitude and direction of the magnetic field in the region to either side of the midpoint is approximately equal to the magnitude and direction of the magnetic field at the midpoint.)

Picture the Problem Let the numeral 1 denote the coil centered at $z=-\frac{1}{2}R$ and the numeral 2 the coil centered at $z=\frac{1}{2}R$. We can express the magnetic field strength in the region between the coils as the sum of the magnetic field strengths due to the two coils and then evaluate the derivatives of this function to show that $dB_z/dz = 0$, $d^2B_z/dz^2 = 0$, and $d^3B_z/dz^3 = 0$ at $z = 0$.

Express the magnetic field strength on the z axis due to the coil centered at $z=-\frac{1}{2}R$:

$$B_1(z)=\frac{\mu_0 NR^2 I}{2z_1^3}$$

where $z_1=\sqrt{\left(z+\frac{1}{2}R\right)^2+R^2}$ and N is the number of turns and.

Express the magnetic field strength on the z axis due to the coil centered at $z=\frac{1}{2}R$:

$$B_2(x)=\frac{\mu_0 NR^2 I}{2z_2^3}$$

where $z_2=\sqrt{\left(z-\frac{1}{2}R\right)^2+R^2}$

Add these equations to express the total magnetic field along the x axis:

$$B_z(z)=B_1(z)+B_2(z)=\frac{\mu_0 NR^2 I}{2z_1^3}+\frac{\mu_0 NR^2 I}{2z_2^3}=\frac{\mu_0 NR^2 I}{2}\left(z_1^{-3}+z_2^{-3}\right)$$

Differentiate B_z with respect to z to obtain:

$$\frac{dB_z}{dz}=\frac{\mu_0 NR^2 I}{2}\frac{d}{dz}\left(z_1^{-3}+z_2^{-3}\right)$$
$$=\frac{\mu_0 NR^2 I}{2}\left(-3z_1^{-4}\frac{dz_1}{dz}-3z_2^{-4}\frac{dz_2}{dz}\right)$$

Because $z_1=\sqrt{\left(z+\frac{1}{2}R\right)^2+R^2}$:

$$\frac{dz_1}{dz}=\frac{1}{2}\left[\left(z+\frac{1}{2}R\right)^2+R^2\right]^{-1/2}\left[2\left(z+\frac{1}{2}R\right)\right]$$
$$=\frac{z+\frac{1}{2}R}{z_1}$$

Because $z_2 = \sqrt{\left(z - \frac{1}{2}R\right)^2 + R^2}$:

$$\frac{dz_2}{dz} = \tfrac{1}{2}\left[\left(z - \tfrac{1}{2}R\right)^2 + R^2\right]^{-1/2}\left[2\left(z - \tfrac{1}{2}R\right)\right] = \frac{z - \frac{1}{2}R}{z_2}$$

Evaluating $z_1(0)$ and $z_2(0)$ yields:

$$z_1(0) = \sqrt{\left(\tfrac{1}{2}R\right)^2 + R^2} = \left(\tfrac{5}{4}R^2\right)^{1/2}$$

and

$$z_2(0) = \sqrt{\left(\tfrac{1}{2}R\right)^2 + R^2} = \left(\tfrac{5}{4}R^2\right)^{1/2}$$

Substituting for $\frac{dz_1}{dz}$ and $\frac{dz_2}{dz}$ in the expression for $\frac{dB_z}{dz}$ and simplifying yields:

$$\frac{dB_z}{dz} = \frac{\mu_0 NR^2 I}{2}\left[\left(-3z_1^{-4}\left(\frac{z + \frac{1}{2}R}{z_1}\right) + -3z_2^{-4}\left(\frac{z - \frac{1}{2}R}{z_2}\right)\right)\right] = \frac{\mu_0 NR^2 I}{2}\left[\frac{-3\left(z + \frac{1}{2}R\right)}{z_1^5} + \frac{-3\left(z - \frac{1}{2}R\right)}{z_2^5}\right]$$

Substitute for $z_1(0)$ and $z_2(0)$ and evaluate $\frac{dB_z}{dz}$ at $z = 0$:

$$\left.\frac{dB_z}{dz}\right|_{z=0} = \frac{\mu_0 NR^2 I}{2}\left(\frac{-3\left(\frac{1}{2}R\right)}{\left(\frac{5}{4}R^2\right)^{5/2}} + \frac{-3\left(-\frac{1}{2}R\right)}{\left(\frac{5}{4}R^2\right)^{5/2}}\right) = \boxed{0}$$

Differentiate $\frac{dB_z}{dz}$ with respect to z to obtain:

$$\frac{d^2B_z}{dz^2} = \frac{\mu_0 NR^2 I}{2}\frac{d}{dz}\left[\frac{-3\left(z + \frac{1}{2}R\right)}{z_1^5} + \frac{-3\left(z - \frac{1}{2}R\right)}{z_2^5}\right] = \frac{\mu_0 NR^2 I}{2}(-3)\left[\frac{1}{z_1^5} - \frac{5\left(z + \frac{1}{2}R\right)^2}{z_1^7} + \frac{1}{z_2^5} - \frac{5\left(z - \frac{1}{2}R\right)^2}{z_2^7}\right]$$

Evaluate $\frac{d^2B_z}{dz^2}$ at $z = 0$ to obtain:

$$\left.\frac{d^2B_z}{dz^2}\right|_{z=0} = \frac{\mu_0 NR^2 I}{2}(-3)\left[\frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{5\left(\frac{1}{2}R\right)^2}{\left(\frac{5}{4}R\right)^{7/2}} + \frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{5\left(-\frac{1}{2}R\right)^2}{\left(\frac{5}{4}R\right)^{7/2}}\right]$$

$$= \frac{\mu_0 NR^2 I}{2}(-3)\left[\frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{\frac{5}{4}R^2}{\left(\frac{5}{4}R\right)^{7/2}} + \frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{\frac{5}{4}R^2}{\left(\frac{5}{4}R\right)^{7/2}}\right]$$

$$= \frac{\mu_0 NR^2 I}{2}(-3)\left[\frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{1}{\left(\frac{5}{4}R\right)^{5/2}} + \frac{1}{\left(\frac{5}{4}R\right)^{5/2}} - \frac{1}{\left(\frac{5}{4}R\right)^{5/2}}\right]$$

$$= \boxed{0}$$

Differentiate $\frac{d^2B_z}{dz^2}$ with respect to z to obtain:

$$\frac{d^3B_z}{dz^3} = \frac{\mu_0 NR^2 I}{2}(-3)\frac{d}{dz}\left[\frac{1}{z_1^5} - \frac{5\left(z+\frac{1}{2}R\right)^2}{z_1^7} + \frac{1}{z_2^5} - \frac{5\left(z-\frac{1}{2}R\right)^2}{z_2^7}\right]$$

$$= \frac{\mu_0 NR^2 I}{2}(-3)\left(\frac{35\left(z+\frac{1}{2}R\right)^3}{z_1^9} - \frac{15\left(z+\frac{1}{2}R\right)}{z_1^7} - \frac{15\left(z-\frac{1}{2}R\right)}{z_2^7} + \frac{35\left(z-\frac{1}{2}R\right)^3}{z_2^9}\right)$$

Evaluate $\frac{d^3B_z}{dz^3}$ at $z = 0$ to obtain:

$$\left.\frac{d^3B_z}{dz^3}\right|_{z=0} = \frac{\mu_0 NR^2 I}{2}(-3)\left(\frac{35\left(\frac{1}{2}R\right)^3}{z_1^9} - \frac{15\left(\frac{1}{2}R\right)}{z_1^7} - \frac{15\left(-\frac{1}{2}R\right)}{z_2^7} + \frac{35\left(-\frac{1}{2}R\right)^3}{z_2^9}\right)$$

$$= \frac{\mu_0 NR^2 I}{2}(-3)\left(\frac{35\left(\frac{1}{2}R\right)^3}{\left(\frac{5}{4}R^2\right)^{9/2}} - \frac{15\left(\frac{1}{2}R\right)}{\left(\frac{5}{4}R^2\right)^{7/2}} + \frac{15\left(\frac{1}{2}R\right)}{\left(\frac{5}{4}R^2\right)^{7/2}} - \frac{35\left(\frac{1}{2}R\right)^3}{\left(\frac{5}{4}R^2\right)^{9/2}}\right)$$

$$= \boxed{0}$$

24 ••• *Anti-Helmholtz* coils are used in many physics applications, such as laser cooling and trapping, where a field with a uniform gradient is desired. These coils have the same construction as a Helmholtz coil, except that the currents have opposite directions, so that the axial fields are in opposite directions, and the coil separation is $\sqrt{3}R$ rather than R. Graph the magnetic field as a function of z, the axial distance from the center of the coils, for an anti-Helmholtz coil using the same parameters as in Problem 22. Over what interval of

the z axis is dB_z/dz within one percent of its value at the midpoint between the coils?

Picture the Problem Let the origin be midway between the coils so that one of them is centered at $z=-\frac{\sqrt{3}}{2}R$ and the other is centered at $z=\frac{\sqrt{3}}{2}R$. Let the numeral 1 denote the coil centered at $z=-\frac{\sqrt{3}}{2}R$ and the numeral 2 the coil centered at $z=\frac{\sqrt{3}}{2}R$. We can express the magnetic field in the region between the coils as the difference of the magnetic fields B_1 and B_2 due to the two coils and use the two-point formula to approximate the slope of the graph of B_z as a function of z.

The magnetic field on the z is the difference between the magnetic fields due to the currents in coils 1 and 2:

$$B_z = B_1(z) - B_2(z) \qquad (1)$$

Express the magnetic field on the z axis due to the coil centered at $z=-R\sqrt{3}/2$:

$$B_1(z)=\frac{\mu_0 NR^2 I}{2\left[\left(z+\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{3/2}}$$

where N is the number of turns.

Express the magnetic field on the z axis due to the coil centered at $z=R\sqrt{3}/2$:

$$B_2(z)=\frac{\mu_0 NR^2 I}{2\left[\left(z-\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{3/2}}$$

Substitute for $B_1(z)$ and $B_2(z)$ in equation (1) to express the total magnetic field along the z axis:

$$\begin{aligned} B_z &= \frac{\mu_0 NR^2 I}{2\left[\left(z+\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{3/2}} - \frac{\mu_0 NR^2 I}{2\left[\left(z-\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{3/2}} \\ &= \frac{\mu_0 NR^2 I}{2}\left(\left[\left(z+\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{-3/2} - \left[\left(z-\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{-3/2}\right) \end{aligned} \qquad (1)$$

The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

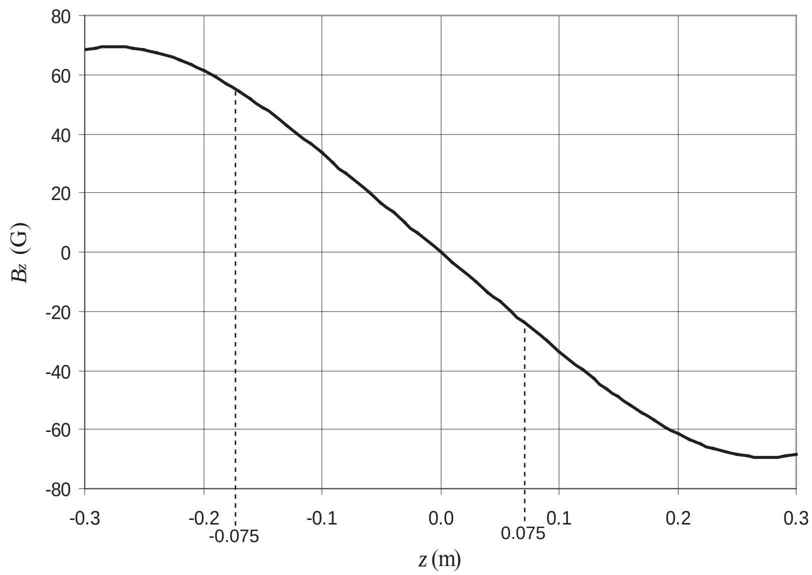
Cell	Formula/Content	Algebraic Form
B1	1.26×10^{-6}	μ_0
B2	0.30	R
B3	250	N
B3	15	I

B5	0.5*B$1*B$3*(B$2^2)*B$4	$\text{Coeff} = \dfrac{\mu_0 N R^2 I}{2}$
A8	–0.30	$-R$
B8	B$5*((B$2*SQRT(3)/2+A8)^2 +B$2^2)^(–3/2)	$\dfrac{\mu_0 N R^2 I}{2}\left[\left(z+\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{-3/2}$
C8	B$5* ((B$2*SQRT(3)/2–A8)^2 +B$2^2)^(–3/2)	$\dfrac{\mu_0 N R^2 I}{2}\left[\left(z-\frac{\sqrt{3}}{2}R\right)^2+R^2\right]^{-3/2}$
D8	10^4*(B8–C8)	$B_x = B_1(z) - B_2(z)$
E9	(D10 – D8)/(A10 – A8)	$\Delta B_z/\Delta z$
F9	ABS(100*(E9 – E$38)/E$38)	% diff

	A	B	C	D	E	F
1	μ_0=	1.26E–06	N/A^2			
2	R=	0.30	m			
3	N=	250	turns			
4	I=	15	A			
5	Coeff=	2.13E–04				
6						
7	z	B_1	B_2	$B(z)$	slope	% diff
8	–0.30	7.67E–03	8.30E-04	68.4		
9	–0.29	7.76E–03	8.65E-04	68.9	41	112.1
10	–0.28	7.82E–03	9.03E-04	69.2	14	104.1
11	–0.27	7.86E–03	9.42E-04	69.2	–14	95.9
12	–0.26	7.87E–03	9.84E-04	68.9	–42	87.5
30	–0.08	4.97E–03	2.28E–03	26.9	–332	1.3
31	–0.07	4.75E–03	2.40E–03	23.5	–334	0.8
32	–0.06	4.54E–03	2.52E–03	20.2	–335	0.4
33	–0.05	4.33E–03	2.65E–03	16.8	–336	0.2
34	–0.04	4.13E–03	2.79E–03	13.5	–336	0.1
35	–0.03	3.94E–03	2.93E–03	10.1	–337	0.0
36	–0.02	3.75E–03	3.08E–03	6.7	–337	0.0
37	–0.01	3.57E–03	3.24E–03	3.4	–337	0.0
38	0.00	3.40E–03	3.40E–03	0.0	–337	0.0
39	0.01	3.24E–03	3.57E–03	–3.4	–337	0.0
40	0.02	3.08E–03	3.75E–03	–6.7	–337	0.0
41	0.03	2.93E–03	3.94E–03	–10.1	–337	0.0
42	0.04	2.79E–03	4.13E–03	–13.5	–336	0.1
43	0.05	2.65E–03	4.33E–03	–16.8	–336	0.2
44	0.06	2.52E–03	4.54E–03	–20.2	–335	0.4
45	0.07	2.40E–03	4.75E–03	–23.5	–334	0.8
46	0.08	2.28E–03	4.97E–03	–26.9	–332	1.3

64	0.26	9.84E−04	7.87E−03	−68.9	−42	87.5
65	0.27	9.42E−04	7.86E−03	−69.2	−14	95.9
66	0.28	9.03E−04	7.82E−03	−69.2	14	104.1
67	0.29	8.65E−04	7.76E−03	−68.9	41	112.1
68	0.30	8.30E−04	7.67E−03	−68.4		

The following graph of B_z as a function of z was plotted using the data in the above table.



Inspection of the table reveals that the slope of the graph of B_z, evaluated at $z = 0$, is −337 G. 1% of this value corresponds approximately to $-0.075\text{ m} < z < 0.075\text{ m}$ or $\boxed{-0.25R < z < 0.25R}$.

The Magnetic Field Due to Straight-Line Currents

25 •• **[SSM]** If the currents are both in the $-x$ direction, find the magnetic field at the following points on the y axis: (a) y = –3.0 cm, (b) y = 0, (c) y = +3.0 cm, and (d) y = +9.0 cm.

Picture the Problem Let + denote the wire (and current) at y = +6.0 cm and − the wire (and current) at y = −6.0 cm. We can use $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ to find the magnetic field due to each of the current-carrying wires and superimpose the magnetic fields due to the currents in these wires to find B at the given points on the y axis. We can apply the right-hand rule to find the direction of each of the fields and, hence, of $\vec{B}$.

(*a*) Express the resultant magnetic field at $y = -3.0$ cm:

$$\vec{B}(-3.0\,\text{cm}) = \vec{B}_+(-3.0\,\text{cm}) + \vec{B}_-(-3.0\,\text{cm}) \qquad (1)$$

Find the magnitudes of the magnetic fields at $y = -3.0$ cm due to each wire:

$$B_+(-3.0\,\text{cm}) = \left(10^{-7}\,\text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.090\,\text{m}} = 44.4\,\mu\text{T}$$

and

$$B_-(-3.0\,\text{cm}) = \left(10^{-7}\,\text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.030\,\text{m}} = 133\,\mu\text{T}$$

Apply the right-hand rule to find the directions of $\vec{B}_+$ and $\vec{B}_-$:

$$\vec{B}_+(-3.0\,\text{cm}) = (44.4\,\mu\text{T})\hat{k}$$

and

$$\vec{B}_-(-3.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k}$$

Substituting in equation (1) yields:

$$\vec{B}(-3.0\,\text{cm}) = (44.4\,\mu\text{T})\hat{k} - (133\,\mu\text{T})\hat{k} = \boxed{-(89\,\mu\text{T})\hat{k}}$$

(*b*) Express the resultant magnetic field at $y = 0$:

$$\vec{B}(0) = \vec{B}_+(0) + \vec{B}_-(0)$$

Because $\vec{B}_+(0) = -\vec{B}_-(0)$:

$$\vec{B}(0) = \boxed{0}$$

(*c*) Proceed as in (*a*) to obtain:

$$\vec{B}_+(3.0\,\text{cm}) = (133\,\mu\text{T})\hat{k}\,,$$

$$\vec{B}_-(3.0\,\text{cm}) = -(44.4\,\mu\text{T})\hat{k}\,,$$

and

$$\vec{B}(3.0\,\text{cm}) = (133\,\mu\text{T})\hat{k} - (44.4\,\mu\text{T})\hat{k} = \boxed{(89\,\mu\text{T})\hat{k}}$$

(*d*) Proceed as in (*a*) with $y = 9.0$ cm to obtain:

$$\vec{B}_+(9.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k}\,,$$

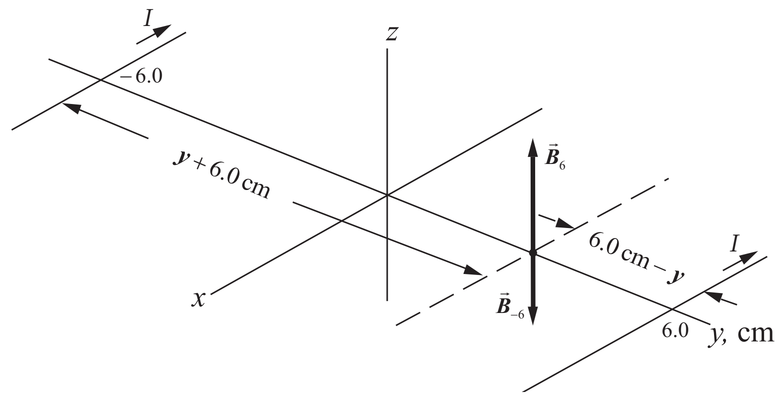
$$\vec{B}_-(9.0\,\text{cm}) = -(26.7\,\mu\text{T})\hat{k}\,,$$

and

$$\vec{B}(9.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k} - (26.7\,\mu\text{T})\hat{k} = \boxed{-(160\,\mu\text{T})\hat{k}}$$

26 •• Using a **spreadsheet** program or graphing calculator, graph B_z versus y when both currents are in the $-x$ direction.

Picture the Problem The diagram shows the two wires with the currents flowing in the negative x direction. We can use the expression for B due to a long, straight wire to express the difference of the fields due to the two currents. We'll denote each field by the subscript identifying the position of each wire.



The field due to the current in the wire located at $y = 6.0$ cm is:

$$B_6 = \frac{\mu_0}{4\pi}\frac{2I}{0.060\,\text{m} - y}$$

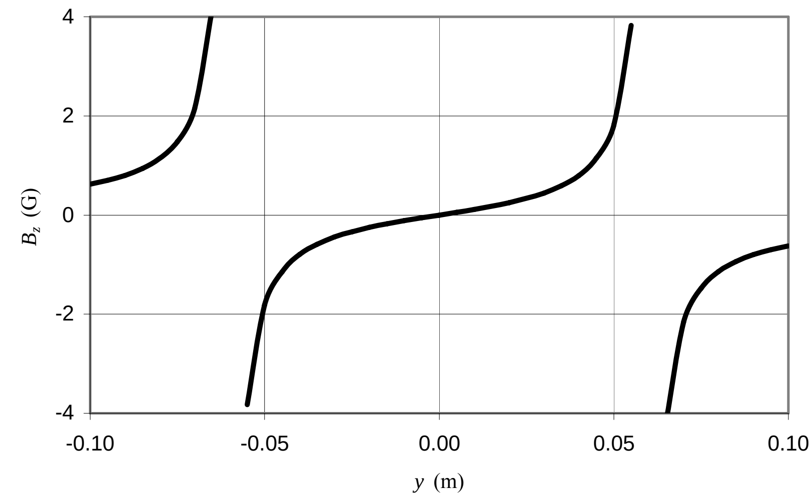
The field due to the current in the wire located at $y = -6.0$ cm is:

$$B_{-6} = \frac{\mu_0}{4\pi}\frac{2I}{0.060\,\text{m} + y}$$

The resultant field B_z is the difference between B_6 and B_{-6}:

$$\begin{aligned} B_z = B_6 - B_{-6} &= \frac{\mu_0}{4\pi}\frac{I}{0.060\,\text{m} - y} - \frac{\mu_0}{4\pi}\frac{I}{0.060\,\text{m} - y} \\ &= \frac{\mu_0 I}{4\pi}\left(\frac{1}{0.060\,\text{m} - y} - \frac{1}{0.060\,\text{m} + y}\right) \end{aligned}$$

The following graph of B_z as a function of y was plotted using a spreadsheet program:



27 •• The current in the wire at y = –6.0 cm is in the –x direction and the current in the wire at y = +6.0 cm is in the +x direction. Find the magnetic field at the following points on the y axis: (a) y = –3.0 cm, (b) y = 0, (c) y = +3.0 cm, and (d) y = +9.0 cm.

Picture the Problem Let + denote the wire (and current) at y = +6 cm and − the wire (and current) at y = –6.0 cm. We can use $B=\frac{\mu_0}{4\pi}\frac{2I}{R}$ to find the magnetic field due to each of the current carrying wires and superimpose the magnetic fields due to the currents in the wires to find B at the given points on the y axis. We can apply the right-hand rule to find the direction of each of the fields and, hence, of $\vec{B}$.

(a) Express the resultant magnetic field at $y = -3.0$ cm:

$$\vec{B}(-3.0\,\text{cm}) = \vec{B}_+(-3.0\,\text{cm}) + \vec{B}_-(-3.0\,\text{cm}) \quad (1)$$

Find the magnitudes of the magnetic fields at $y = -3.0$ cm due to each wire:

$$B_+(-3.0\,\text{cm}) = \left(10^{-7}\,\text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.090\,\text{m}} = 44.4\,\mu\text{T}$$

and

$$B_-(-3.0\,\text{cm}) = \left(10^{-7}\,\text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.030\,\text{m}} = 133\,\mu\text{T}$$

Apply the right-hand rule to find the directions of $\vec{B}_+$ and $\vec{B}_-$:

$$\vec{B}_+(-3.0\,\text{cm}) = -(44.4\,\mu\text{T})\hat{k}$$
and
$$\vec{B}_-(-3.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k}$$

Substituting in equation (1) yields:

$$\vec{B}(-3.0\,\text{cm}) = -(44.4\,\mu\text{T})\hat{k} - (133\,\mu\text{T})\hat{k} = \boxed{-(0.18\,\text{mT})\hat{k}}$$

(*b*) Express the resultant magnetic field at $y = 0$:

$$\vec{B}(0) = \vec{B}_+(0) + \vec{B}_-(0)$$

Find the magnitudes of the magnetic fields at $y = 0$ cm due to each wire:

$$B_+(0) = \left(10^{-7}\ \text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.060\,\text{m}} = 66.7\,\mu\text{T}$$
and
$$B_-(0) = \left(10^{-7}\ \text{T}\cdot\text{m/A}\right)\frac{2(20\,\text{A})}{0.060\,\text{m}} = 66.7\,\mu\text{T}$$

Apply the right-hand rule to find the directions of $\vec{B}_+$ and $\vec{B}_-$:

$$\vec{B}_+(0) = -(66.7\,\mu\text{T})\hat{k}$$
and
$$\vec{B}_-(0) = -(66.7\,\mu\text{T})\hat{k}$$

Substitute to obtain:

$$\vec{B}(0) = -(66.7\,\mu\text{T})\hat{k} - (66.7\,\mu\text{T})\hat{k} = \boxed{-(0.13\,\text{mT})\hat{k}}$$

(*c*) Proceed as in (*a*) with $y = +3.0$ cm to obtain:

$$\vec{B}_+(3.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k},$$
$$\vec{B}_-(3.0\,\text{cm}) = -(44.4\,\mu\text{T})\hat{k},$$
and
$$\vec{B}(3.0\,\text{cm}) = -(133\,\mu\text{T})\hat{k} - (44.4\,\mu\text{T})\hat{k} = \boxed{-(0.18\,\text{mT})\hat{k}}$$

(*d*) Proceed as in (*a*) with $y = +9.0$ m to obtain:

$$\vec{B}_+(9.0\,\text{cm}) = (133\,\mu\text{T})\hat{k}\,,$$

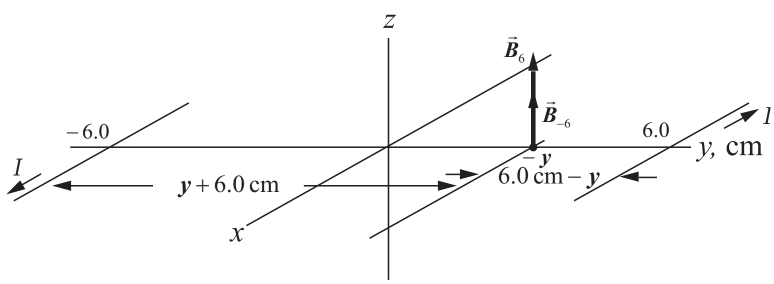
$$\vec{B}_-(9.0\,\text{cm}) = -(26.7\,\mu\text{T})\hat{k}\,,$$

and

$$\vec{B}(9.0\,\text{cm}) = (133\,\mu\text{T})\hat{k} - (26.7\,\mu\text{T})\hat{k} = \boxed{(0.11\,\text{mT})\hat{k}}$$

28 •• The current in the wire at $y = -6.0$ cm is in the $+x$ direction and the current in the wire at $y = +6.0$ cm is in the $-x$ direction. Using a **spreadsheet** program or graphing calculator, graph B_z versus y.

Picture the Problem The diagram shows the two wires with the currents flowing in the negative *x* direction. We can use the expression for *B* due to a long, straight wire to express the difference of the fields due to the two currents. We'll denote each field by the subscript identifying the position of each wire.



The field due to the current in the wire located at $y = 6.0$ cm is:

$$B_6 = \frac{\mu_0}{4\pi}\frac{2I}{0.060\,\text{m} - y}$$

The field due to the current in the wire located at $y = -6.0$ cm is:

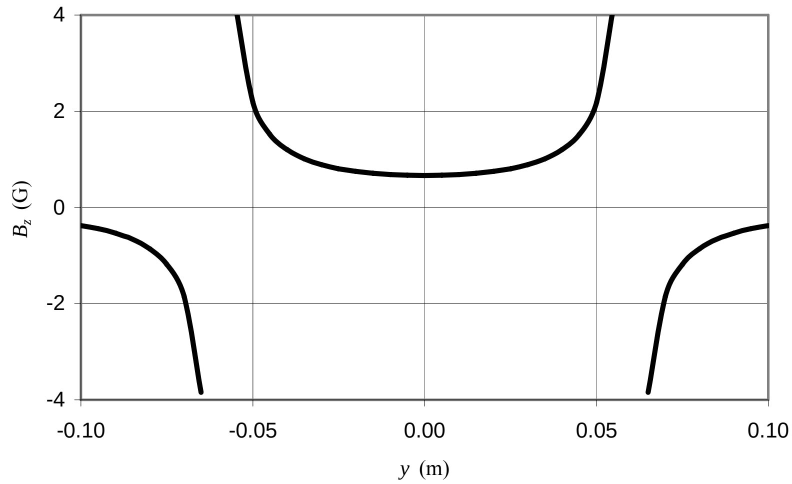
$$B_{-6} = \frac{\mu_0}{4\pi}\frac{2I}{0.060\,\text{m} + y}$$

The resultant field B_z is the sum of B_6 and B_{-6}:

$$B_z = B_6 + B_{-6} = \frac{\mu_0}{4\pi}\frac{I}{0.060\,\text{m} - y} + \frac{\mu_0}{4\pi}\frac{I}{0.060\,\text{m} - y}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{0.060\,\text{m} - y} + \frac{1}{0.060\,\text{m} + y}\right)$$

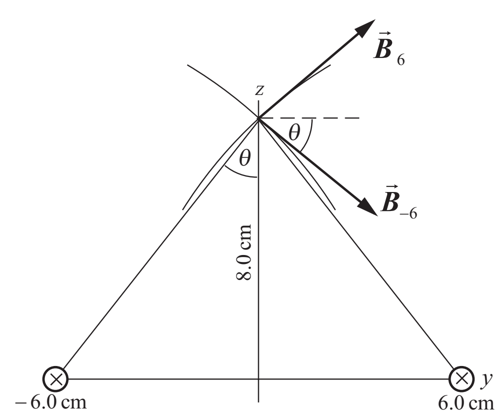
The following graph of B_z as a function of y was plotted using a spreadsheet program:



29 • Find the magnetic field on the z axis at z = +8.0 cm if (a) the currents are both in the $-x$ direction, and (b) the current in the wire at y = –6.0 cm is in the $-x$ direction and the current in the wire at y = +6.0 cm is in the $+x$ direction.

Picture the Problem Let + denote the wire (and current) at y = +6.0 cm and – the wire (and current) at y = –6.0 cm. We can use $B=\frac{\mu_0}{4\pi}\frac{2I}{R}$ to find the magnetic field due to each of the current carrying wires and superimpose the magnetic fields due to the currents in the wires to find B at the given points on the z axis.

(a) Apply the right-hand rule to show that, for the currents parallel and in the negative x direction, the directions of the fields are as shown to the right:



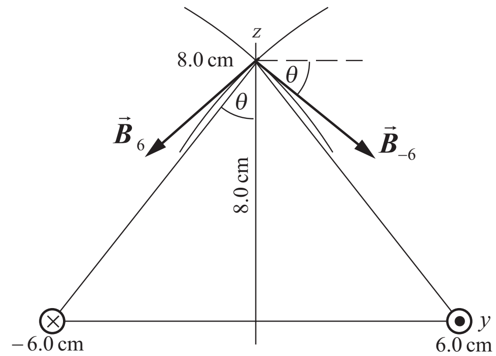
Express the magnitudes of the magnetic fields at z = +8.0 cm due to the current-carrying wires at y = –6.0 cm and y = +6.0 cm:

$$B_{z-}=B_{z+}=\left(10^{-7}\ \mathrm{T\cdot m/A}\right)\times\frac{2(20\ \mathrm{A})}{\sqrt{(0.060\ \mathrm{m})^2+(0.080\ \mathrm{m})^2}}$$
$$=40.0\ \mu\mathrm{T}$$

Noting that the z components add to zero, express the resultant magnetic field at $z = +8.0$ cm:

$$\vec{B}(8.0\,\text{cm}) = 2(40.0\,\mu\text{T})\cos\theta\,\hat{j} = 2(40.0\,\mu\text{T})(0.80)\,\hat{j} = \boxed{(64\,\mu\text{T})\,\hat{j}}$$

(b) Apply the right-hand rule to show that, for the currents antiparallel with the current in the wire at $y = -6.0$ cm in the negative x direction, the directions of the fields are as shown to the right:



Noting that the y components add to zero, express the resultant magnetic field at $z = +8.0$ cm:

$$\vec{B}(8.0\,\text{cm}) = -2(40.0\,\mu\text{T})\sin\theta\,\hat{k} = -2(40.0\,\mu\text{T})(0.60)\,\hat{k} = \boxed{-(48\,\mu\text{T})\,\hat{k}}$$

30 • Find the magnitude of the force per unit length exerted by one wire on the other.

Picture the Problem Let + denote the wire (and current) at $y = +6.0$ cm and − the wire (and current) at $y = -6.0$ cm. The forces per unit length the wires exert on each other are action and reaction forces and hence are equal in magnitude. We can use $F = I\ell B$ to express the force on either wire and $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ to express the magnetic field at the location of either wire due to the current in the other.

Express the force exerted on either wire:

$$F = I\ell B$$

Express the magnetic field at either location due to the current in the wire at the other location:

$$B = \frac{\mu_0}{4\pi}\frac{2I}{R}$$

Substitute for B to obtain:

$$F = I\ell\left(\frac{\mu_0}{4\pi}\frac{2I}{R}\right) = \frac{2\ell\mu_0}{4\pi}\frac{I^2}{R}$$

Divide both sides of the equation by ℓ to obtain:

$$\frac{F}{\ell}=\frac{2\mu_0}{4\pi}\frac{I^2}{R}$$

Substitute numerical values and evaluate F/ℓ:

$$\frac{F}{\ell}=\frac{2\left(10^{-7}\ \mathrm{T\cdot m/A}\right)\left(20\ \mathrm{A}\right)^2}{0.12\ \mathrm{m}}=\boxed{0.67\ \mathrm{mN/m}}$$

31 • Two long, straight parallel wires 8.6 cm apart carry equal currents. The wires repel each other with a force per unit length of 3.6 nN/m. (*a*) Are the currents parallel or anti-parallel? Explain your answer. (*b*) Determine the current in each wire.

Picture the Problem We can use $\frac{F}{\ell}=\frac{2\mu_0}{4\pi}\frac{I^2}{R}$ to relate the force per unit length each current-carrying wire exerts on the other to their common current.

(*a*) Because the currents repel, they are antiparallel.

(*b*)The force per unit length experienced by each wire is given by:

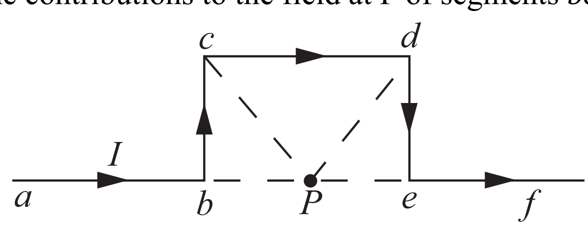
$$\frac{F}{\ell}=\frac{2\mu_0}{4\pi}\frac{I^2}{R}\Rightarrow I=\sqrt{\frac{4\pi R}{2\mu_0}\frac{F}{\ell}}$$

Substitute numerical values and evaluate I:

$$I=\sqrt{\frac{\left(8.6\ \mathrm{cm}\right)}{2\left(10^{-7}\ \mathrm{T\cdot m/A}\right)}\left(3.6\ \mathrm{nN/m}\right)}=\boxed{39\ \mathrm{mA}}$$

32 •• The current in the wire shown in Figure 27-52 is 8.0 A. Find the magnetic field at point *P*.

Picture the Problem Note that the current segments *a*-*b* and *e*-*f* do not contribute to the magnetic field at point *P*. The current in the segments *b*-*c*, *c*-*d*, and *d*-*e* result in a magnetic field at *P* that points into the plane of the paper. Note that the angles *bPc* and *ePd* are 45° and use the expression for *B* due to a straight wire segment to find the contributions to the field at P of segments *bc*, *cd*, and *de*.



Express the resultant magnetic field at *P*:

$$B = B_{bc} + B_{cd} + B_{de}$$

Express the magnetic field due to a straight line segment:

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin\theta_1 + \sin\theta_2\right) \qquad (1)$$

Use equation (1) to express B_{bc} and B_{de}:

$$B_{bc} = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin 45° + \sin 0°\right) = \frac{\mu_0}{4\pi}\frac{I}{R}\sin 45°$$

Use equation (1) to express B_{cd}:

$$B_{cd} = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin 45° + \sin 45°\right) = 2\frac{\mu_0}{4\pi}\frac{I}{R}\sin 45°$$

Substitute to obtain:

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}\sin 45° + 2\frac{\mu_0}{4\pi}\frac{I}{R}\sin 45° + \frac{\mu_0}{4\pi}\frac{I}{R}\sin 45° = 4\frac{\mu_0}{4\pi}\frac{I}{R}\sin 45°$$

Substitute numerical values and evaluate *B*:

$$B = 4\left(10^{-7}\ \text{T}\cdot\text{m/A}\right)\frac{8.0\,\text{A}}{0.010\,\text{m}}\sin 45° = \boxed{0.23\,\text{mT into the page}}$$

33 •• [SSM] As a student technician, you are preparing a lecture demonstration on "magnetic suspension." You have a 16-cm long straight rigid wire that will be suspended by flexible conductive lightweight leads above a long straight wire. Currents that are equal but are in opposite directions will be established in the two wires so the 16-cm wire "floats" a distance *h* above the long wire with no tension in its suspension leads. If the mass of the 16-cm wire is 14 g and if *h* (the distance between the central axes of the two wires) is 1.5 mm, what should their common current be?

Picture the Problem The forces acting on the wire are the upward magnetic force F_B and the downward gravitational force *mg*, where *m* is the mass of the wire. We can use a condition for translational equilibrium and the expression for the force per unit length between parallel current-carrying wires to relate the required current to the mass of the wire, its length, and the separation of the two wires.

Apply $\sum F_y = 0$ to the floating wire to obtain:

$$F_B - mg = 0$$

Express the repulsive force acting on the upper wire:

$$F_B = 2\frac{\mu_0}{4\pi}\frac{I^2\ell}{R}$$

Substitute to obtain:

$$2\frac{\mu_0}{4\pi}\frac{I^2\ell}{R} - mg = 0 \Rightarrow I = \sqrt{\frac{4\pi mgR}{2\mu_0\ell}}$$

Substitute numerical values and evaluate I:

$$I = \sqrt{\frac{(14\times10^{-3}\text{ kg})(9.81\text{m/s}^2)(1.5\times10^{-3}\text{ m})}{2(10^{-7}\text{ T}\cdot\text{m/A})(0.16\text{m})}} = \boxed{80\text{A}}$$

34 •• Three long, parallel straight wires pass through the vertices of an equilateral triangle that has sides equal to 10 cm, as shown in Figure 27-53. The dot indicates that the direction of the current is out of the page and a cross indicates that the direction of the current is into the page. If each current is 15 A, find (*a*) the magnetic field at the location of the upper wire due to the currents in the two lower wires and (*b*) the force per unit length on the upper wire.

Picture the Problem (*a*) Choose a coordinate system in which the +*x* direction is to the right and the +*y* direction is upward. We can use the right-hand rule to determine the directions of the magnetic fields at the upper wire due to the currents in the two lower wires and use $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ to find the magnitude of the resultant field due to these currents. (*b*) Note that the forces on the upper wire are away from and directed along the lines to the lower wire and that their horizontal components cancel. We can use the expression for the magnetic force on a current-carrying wire to find the force per unit length on the upper wire.

(*a*) Noting, from the geometry of the wires, the magnetic field vectors both are at an angle of 30° with the horizontal and that their *y* components cancel, express the resultant magnetic field:

$$\vec{B} = 2\left(\frac{\mu_0}{4\pi}\frac{2I}{R}\cos 30°\right)\hat{i}$$

Substitute numerical values and evaluate B:

$$\vec{B} = 2\left(10^{-7}\ \text{T}\cdot\text{m/A}\right)\frac{2(15\,\text{A})}{0.10\,\text{m}}\cos 30°\hat{i}$$

$$= \boxed{(52\,\mu\text{T})\hat{i}}$$

That is, the magnitude of the magnetic field is 52 μT and its direction is to the right.

(*b*) Express the force per unit length each of the lower wires exerts on the upper wire:

$$\vec{F} = I\left(\vec{\ell}\times\vec{B}\right) = I\left(\ell\hat{k}\times B\hat{i}\right) = I\ell B\sin 90°\hat{j}$$

and

$$\frac{\vec{F}}{\ell} = IB\sin 90°\hat{j}$$

Substitute numerical values and evaluate $\frac{\vec{F}}{\ell}$:

$$\frac{\vec{F}}{\ell} = (15\,\text{A})(52\,\mu\text{T})\hat{j}$$

$$= \boxed{\left(7.8\times10^{-4}\ \text{N/m}\right)\hat{j}}$$

That is, the magnitude of the force per unit length on the upper wire is 7.8×10^{-4} N/m and the direction of this force is up the page.

35 •• Rework Problem 34 with the current in the lower right corner of Figure 27- 53 reversed.

Picture the Problem (*a*) Choose a coordinate system in which the +*x* direction is to the right and the +*y* direction is upward. We can use the right-hand rule to determine the directions of the magnetic fields at the upper wire due to the currents in the two lower wires and use $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ to find the magnitude of the resultant field due to these currents. (*b*) Note that the forces on the upper wire are away from and directed along the lines to the lower wire and that their horizontal components cancel. We can use the expression for the magnetic force on a current-carrying wire to find the force per unit length on the upper wire.

(*a*) Noting, from the geometry of the wires, that the magnetic field vectors both are at an angle of 30° with the horizontal and that their *x* components cancel, express the resultant magnetic field:

$$\vec{B} = -2\left(\frac{\mu_0}{4\pi}\frac{2I}{R}\sin 30°\right)\hat{j}$$

Substitute numerical values and evaluate B:

$$\vec{B} = -2\left[\left(10^{-7}\ \text{T}\cdot\text{m/A}\right)\frac{2(15\ \text{A})}{0.10\ \text{m}}\sin 30°\right]\hat{j}$$

$$= \boxed{-(30\ \mu\text{T})\,\hat{j}}$$

That is, the magnitude of the magnetic field is 30 μT and its direction is down the page.

(*b*) Express the force per unit length each of the lower wires exerts on the upper wire:

$$\vec{F} = I\left(\vec{\ell}\times\vec{B}\right) = I\left(\ell\hat{k}\times -B\hat{j}\right) = I\ell B\sin 90°\,\hat{i}$$

and

$$\frac{\vec{F}}{\ell} = IB\sin 90°\,\hat{i}$$

Substitute numerical values and evaluate $\frac{\vec{F}}{\ell}$:

$$\frac{\vec{F}}{\ell} = (15\ \text{A})(30\ \mu\text{T})\hat{i}$$

$$= \boxed{\left(4.5\times10^{-4}\ \text{N/m}\right)\hat{i}}$$

That is, the magnitude of the force per unit length on the upper wire is 4.5×10^{-4} N/m and the direction of this force is to the right.

36 •• An infinitely long wire lies along the x axis, and carries current I in the $+x$ direction. A second infinitely long wire lies along the y axis, and carries current I in the $+y$ direction. At what points in the $z = 0$ plane is the resultant magnetic field zero?

Picture the Problem Let the numeral 1 denote the current flowing in the $+x$ direction and the magnetic field resulting from it and the numeral 2 denote the current flowing in the $+y$ direction and the magnetic field resulting from it. We can express the magnetic field anywhere in the xy plane using $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ and the right-hand rule and then impose the condition that $\vec{B} = 0$ to determine the set of points that satisfy this condition.

Express the resultant magnetic field due to the two current-carrying wires:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 \qquad (1)$$

Express the magnetic field due to the current flowing in the $+x$ direction:

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{2I_1}{y}\hat{k}$$

Express the magnetic field due to the current flowing in the +y direction:

$$\vec{B}_2 = -\frac{\mu_0}{4\pi}\frac{2I_2}{x}\hat{k}$$

Substitute for $\vec{B}_1$ and $\vec{B}_2$ in equation (1) and simplify to obtain:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{y}\hat{k} - \frac{\mu_0}{4\pi}\frac{2I}{x}\hat{k} = \left(\frac{\mu_0}{4\pi}\frac{2I}{y} - \frac{\mu_0}{4\pi}\frac{2I}{x}\right)\hat{k}$$

because $I = I_1 = I_2$.

For $\vec{B} = 0$:

$$\frac{\mu_0}{4\pi}\frac{2I}{y} - \frac{\mu_0}{4\pi}\frac{2I}{x} = 0 \Rightarrow x = y.$$

$\vec{B} = 0$ everywhere on the plane that contains both the z axis and the line $y = x$ in the $z = 0$ plane.

37 •• [SSM] An infinitely long wire lies along the z axis and carries a current of 20 A in the +z direction. A second infinitely long wire that is parallel to the z and intersects the x axis at x = 10.0 cm. (a) Find the current in the second wire if the magnetic field is zero at (2.0 cm, 0, 0) is zero. (b) What is the magnetic field at (5.0 cm, 0, 0)?

Picture the Problem Let the numeral 1 denote the current flowing along the positive z axis and the magnetic field resulting from it and the numeral 2 denote the current flowing in the wire located at x = 10 cm and the magnetic field resulting from it. We can express the magnetic field anywhere in the xy plane using $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ and the right-hand rule and then impose the condition that $\vec{B} = 0$ to determine the current that satisfies this condition.

(a) Express the resultant magnetic field due to the two current-carrying wires:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 \qquad (1)$$

Express the magnetic field at x = 2.0 cm due to the current flowing in the +z direction:

$$\vec{B}_1(2.0\,\text{cm}) = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{2.0\,\text{cm}}\right)\hat{j}$$

Express the magnetic field at x = 2.0 cm due to the current flowing in the wire at x = 10.0 cm:

$$\vec{B}_2(2.0\,\text{cm}) = -\frac{\mu_0}{4\pi}\left(\frac{2I_2}{8.0\,\text{cm}}\right)\hat{j}$$

Substitute for $\vec{B}_1$ and $\vec{B}_2$ in equation (1) and simplify to obtain:

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{2.0\,\text{cm}}\right)\hat{j} - \frac{\mu_0}{4\pi}\left(\frac{2I_2}{8.0\,\text{cm}}\right)\hat{j} = \left(\frac{\mu_0}{4\pi}\frac{2I_1}{2.0\,\text{cm}} - \frac{\mu_0}{4\pi}\frac{2I_2}{8.0\,\text{cm}}\right)\hat{j}$$

For $\vec{B} = 0$:

$$\frac{\mu_0}{4\pi}\left(\frac{2I_1}{2.0\,\text{cm}}\right) - \frac{\mu_0}{4\pi}\left(\frac{2I_2}{8.0\,\text{cm}}\right) = 0$$

or

$$\frac{I_1}{2.0} - \frac{I_2}{8.0} = 0 \Rightarrow I_2 = 4I_1$$

Substitute numerical values and evaluate I_2:

$$I_2 = 4(20\,\text{A}) = \boxed{80\,\text{A}}$$

(b) Express the magnetic field at $x = 5.0$ cm:

$$\vec{B}(5.0\,\text{cm}) = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{5.0\,\text{cm}}\right)\hat{j} - \frac{\mu_0}{4\pi}\left(\frac{2I_2}{5.0\,\text{cm}}\right)\hat{j} = \frac{2\mu_0}{4\pi(5.0\,\text{cm})}(I_1 - I_2)\hat{j}$$

Substitute numerical values and evaluate $\vec{B}(5.0\,\text{cm})$:

$$\vec{B}(5.0\,\text{cm}) = \frac{2(10^{-7}\,\text{T}\cdot\text{m/A})}{5.0\,\text{cm}}(20\,\text{A} - 80\,\text{A})\hat{j} = \boxed{-(0.24\,\text{mT})\hat{j}}$$

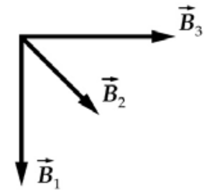
38 •• Three long parallel wires are at the corners of a square, as shown in Figure 27- 54. The wires each carry a current I. Find the magnetic field at the unoccupied corner of the square when (a) all the currents are into the page, (b) I_1 and I_3 are into the page and I_2 is out, and (c) I_1 and I_2 are into the page and I_3 is out. Your answers should be in terms of I and L.

Picture the Problem Choose a coordinate system with its origin at the lower left-hand corner of the square, the +x axis to the right and the +y axis upward. We can use $B = \frac{\mu_0}{4\pi}\frac{2I}{R}$ and the right-hand rule to find the magnitude and direction of the magnetic field at the unoccupied corner due to each of the currents, and superimpose these fields to find the resultant field.

(a) Express the resultant magnetic field at the unoccupied corner:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 \qquad (1)$$

When all the currents are into the paper their magnetic fields at the unoccupied corner are as shown to the right:



Express the magnetic field at the unoccupied corner due to the current I_1:

$$\vec{B}_1 = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{j}$$

Express the magnetic field at the unoccupied corner due to the current I_2:

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{2I}{L\sqrt{2}}\cos 45°\left(\hat{i}-\hat{j}\right) = \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(\hat{i}-\hat{j}\right)$$

Express the magnetic field at the unoccupied corner due to the current I_3:

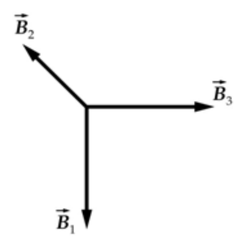
$$\vec{B}_3 = \frac{\mu_0}{4\pi}\frac{2I}{L}\hat{i}$$

Substitute in equation (1) and simplify to obtain:

$$\vec{B} = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{j} + \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(\hat{i}-\hat{j}\right) + \frac{\mu_0}{4\pi}\frac{2I}{L}\hat{i} = \frac{\mu_0}{4\pi}\frac{2I}{L}\left(-\hat{j}+\frac{1}{2}\left(\hat{i}-\hat{j}\right)+\hat{i}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{2I}{L}\left[\left(1+\frac{1}{2}\right)\hat{i}+\left(-1-\frac{1}{2}\right)\hat{j}\right] = \boxed{\frac{3\mu_0 I}{4\pi L}\left[\hat{i}-\hat{j}\right]}$$

(*b*) When I_2 is out of the paper the magnetic fields at the unoccupied corner are as shown to the right:



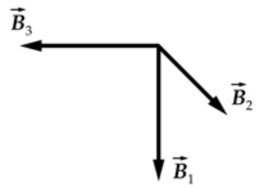
Express the magnetic field at the unoccupied corner due to the current I_2:

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{2I}{L\sqrt{2}}\cos 45°\left(-\hat{i}+\hat{j}\right) = \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(-\hat{i}+\hat{j}\right)$$

Substitute in equation (1) and simplify to obtain:

$$\vec{B} = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{j} + \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(-\hat{i}+\hat{j}\right) + \frac{\mu_0}{4\pi}\frac{2I}{L}\hat{i} = \frac{\mu_0}{4\pi}\frac{2I}{L}\left(-\hat{j}+\frac{1}{2}\left(-\hat{i}+\hat{j}\right)+\hat{i}\right)$$
$$= \frac{\mu_0}{4\pi}\frac{2I}{L}\left[\left(1-\frac{1}{2}\right)\hat{i}+\left(-1+\frac{1}{2}\right)\hat{j}\right] = \frac{\mu_0}{4\pi}\frac{2I}{L}\left[\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}\right] = \boxed{\frac{\mu_0 I}{4\pi L}\left[\hat{i}-\hat{j}\right]}$$

(*c*) When I_1 and I_2 are in and I_3 is out of the paper the magnetic fields at the unoccupied corner are as shown to the right:



From (*a*) or (*b*) we have:

$$\vec{B}_1 = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{j}$$

From (*a*) we have:

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{2I}{L\sqrt{2}}\cos 45°\left(\hat{i}-\hat{j}\right)$$
$$= \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(\hat{i}-\hat{j}\right)$$

Express the magnetic field at the unoccupied corner due to the current I_3:

$$\vec{B}_3 = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{i}$$

Substitute in equation (1) and simplify to obtain:

$$\vec{B} = -\frac{\mu_0}{4\pi}\frac{2I}{L}\hat{j} + \frac{\mu_0}{4\pi}\frac{2I}{2L}\left(\hat{i}-\hat{j}\right) - \frac{\mu_0}{4\pi}\frac{2I}{L}\hat{i} = \frac{\mu_0}{4\pi}\frac{2I}{L}\left(-\hat{j}+\frac{1}{2}\left(\hat{i}-\hat{j}\right)-\hat{i}\right)$$
$$= \frac{\mu_0}{4\pi}\frac{2I}{L}\left[\left(-1+\frac{1}{2}\right)\hat{i}+\left(-1-\frac{1}{2}\right)\hat{j}\right] = \boxed{\frac{\mu_0 I}{4\pi L}\left[-\hat{i}-3\hat{j}\right]}$$

39 •• [SSM] Four long, straight parallel wires each carry current *I*. In a plane perpendicular to the wires, the wires are at the corners of a square of side length *a*. Find the magnitude of the force per unit length on one of the wires if (*a*) all the currents are in the same direction and (*b*) the currents in the wires at adjacent corners are oppositely directed.

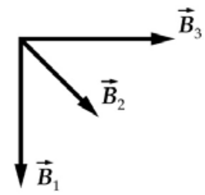
Picture the Problem Choose a coordinate system with its origin at the lower left-hand corner of the square, the +*x* axis to the right and the +*y* axis upward. Let the numeral 1 denote the wire and current in the upper left-hand corner of the

square, the numeral 2 the wire and current in the lower left-hand corner (at the origin) of the square, and the numeral 3 the wire and current in the lower right-hand corner of the square. We can use $B=\frac{\mu_0}{4\pi}\frac{2I}{R}$ and the right-hand rule to find the magnitude and direction of the magnetic field at, say, the upper right-hand corner due to each of the currents, superimpose these fields to find the resultant field, and then use $F = I\ell B$ to find the force per unit length on the wire.

(*a*) Express the resultant magnetic field at the upper right-hand corner:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 \quad (1)$$

When all the currents are into the paper their magnetic fields at the upper right-hand corner are as shown to the right:



Express the magnetic field due to the current I_1:

$$\vec{B}_1 = -\frac{\mu_0}{4\pi}\frac{2I}{a}\hat{j}$$

Express the magnetic field due to the current I_2:

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{2I}{a\sqrt{2}}\cos 45°\left(\hat{i}-\hat{j}\right) = \frac{\mu_0}{4\pi}\frac{2I}{2a}\left(\hat{i}-\hat{j}\right)$$

Express the magnetic field due to the current I_3:

$$\vec{B}_3 = \frac{\mu_0}{4\pi}\frac{2I}{a}\hat{i}$$

Substitute in equation (1) and simplify to obtain:

$$\vec{B} = -\frac{\mu_0}{4\pi}\frac{2I}{a}\hat{j} + \frac{\mu_0}{4\pi}\frac{2I}{2a}\left(\hat{i}-\hat{j}\right) + \frac{\mu_0}{4\pi}\frac{2I}{a}\hat{i} = \frac{\mu_0}{4\pi}\frac{2I}{a}\left(-\hat{j}+\frac{1}{2}\left(\hat{i}-\hat{j}\right)+\hat{i}\right)$$
$$= \frac{\mu_0}{4\pi}\frac{2I}{a}\left[\left(1+\frac{1}{2}\right)\hat{i}+\left(-1-\frac{1}{2}\right)\hat{j}\right] = \frac{3\mu_0 I}{4\pi a}\left[\hat{i}-\hat{j}\right]$$

Using the expression for the magnetic force on a current-carrying wire, express the force per unit length on the wire at the upper right-hand corner:

$$\frac{F}{\ell} = BI \quad (2)$$

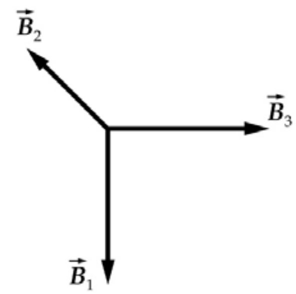
Substitute to obtain:

$$\frac{\vec{F}}{\ell}=\frac{3\mu_0 I^2}{4\pi a}\left[\hat{i}-\hat{j}\right]$$

and

$$\frac{F}{\ell}=\sqrt{\left(\frac{3\mu_0 I^2}{4\pi a}\right)^2+\left(\frac{3\mu_0 I^2}{4\pi a}\right)^2}=\boxed{\frac{3\sqrt{2}\mu_0 I^2}{4\pi a}}$$

(*b*) When the current in the upper right-hand corner of the square is out of the page, and the currents in the wires at adjacent corners are oppositely directed, the magnetic fields at the upper right-hand are as shown to the right:



Express the magnetic field at the upper right-hand corner due to the current I_2:

$$\vec{B}_2=\frac{\mu_0}{4\pi}\frac{2I}{a\sqrt{2}}\cos 45°\left(-\hat{i}+\hat{j}\right)=\frac{\mu_0}{4\pi}\frac{2I}{2a}\left(-\hat{i}+\hat{j}\right)$$

Using $\vec{B}_1$ and $\vec{B}_3$ from (*a*), substitute in equation (1) and simplify to obtain:

$$\vec{B}=-\frac{\mu_0}{4\pi}\frac{2I}{a}\hat{j}+\frac{\mu_0}{4\pi}\frac{2I}{2a}\left(-\hat{i}+\hat{j}\right)+\frac{\mu_0}{4\pi}\frac{2I}{a}\hat{i}=\frac{\mu_0}{4\pi}\frac{2I}{a}\left(-\hat{j}+\frac{1}{2}\left(-\hat{i}+\hat{j}\right)+\hat{i}\right)$$

$$=\frac{\mu_0}{4\pi}\frac{2I}{a}\left[\left(1-\frac{1}{2}\right)\hat{i}+\left(-1+\frac{1}{2}\right)\hat{j}\right]=\frac{\mu_0}{4\pi}\frac{2I}{a}\left[\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}\right]=\frac{\mu_0 I}{4\pi a}\left[\hat{i}-\hat{j}\right]$$

Substitute in equation (2) to obtain:

$$\frac{\vec{F}}{\ell}=\frac{\mu_0 I^2}{4\pi a}\left[\hat{i}-\hat{j}\right]$$

and

$$\frac{F}{\ell}=\sqrt{\left(\frac{\mu_0 I^2}{4\pi a}\right)^2+\left(\frac{\mu_0 I^2}{4\pi a}\right)^2}=\boxed{\frac{\sqrt{2}\mu_0 I^2}{4\pi a}}$$

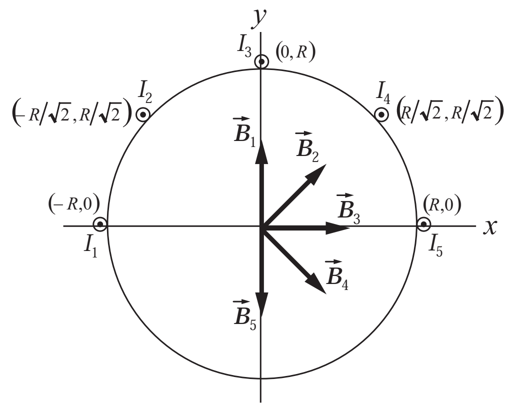
40 •• Five long straight current-carrying wires are parallel to the *z* axis, and each carries a current *I* in the +*z* direction. The wires each are a distance *R* from the *z* axis. Two of the wires intersect the *x* axis, one at $x = R$ and the other at $x = –R$. Another wire intersects the *y* axis at $y = R$. One of the remaining wires intersects the $z = 0$ plane at the point $\left(R/\sqrt{2}, R/\sqrt{2}\right)$ and the last remaining wire

intersects the $z = 0$ plane at the point $\left(-R/\sqrt{2}, R/\sqrt{2}\right)$. Find the magnetic field on the z axis.

Picture the Problem The configuration is shown in the adjacent figure. Here the $+z$ axis is out of the plane of the paper, the $+x$ axis is to the right, and the $+y$ axis is upward. We can use

$$B = \frac{\mu_0}{4\pi}\frac{2I}{R}$$

and the right-hand rule to find the magnetic field due to the current in each wire and add these magnetic fields vectorially to find the resultant field.



Express the resultant magnetic field on the z axis:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 + \vec{B}_5 \quad (1)$$

$\vec{B}_1$ is given by: $\vec{B}_1 = B\hat{j}$

$\vec{B}_2$ is given by: $\vec{B}_2 = (B\cos 45°)\hat{i} + (B\sin 45°)\hat{j}$

$\vec{B}_3$ is given by: $\vec{B}_3 = B\hat{i}$

$\vec{B}_4$ is given by: $\vec{B}_4 = (B\cos 45°)\hat{i} - (B\sin 45°)\hat{j}$

$\vec{B}_5$ is given by: $\vec{B}_5 = -B\hat{j}$

Substitute for $\vec{B}_1$, $\vec{B}_2$, $\vec{B}_3$, $\vec{B}_4$, and $\vec{B}_5$ in equation (1) and simplify to obtain:

$$\begin{aligned}\vec{B} &= B\hat{j} + (B\cos 45°)\hat{i} + (B\sin 45°)\hat{j} + B\hat{i} + (B\cos 45°)\hat{i} - (B\sin 45°)\hat{j} - B\hat{j} \\ &= (B\cos 45°)\hat{i} + B\hat{i} + (B\cos 45°)\hat{i} = (B + 2B\cos 45°)\hat{i} = \left(1+\sqrt{2}\right)B\hat{i}\end{aligned}$$

Express B due to each current at $z = 0$:

$$B = \frac{\mu_0}{4\pi}\frac{2I}{R}$$

Substitute for B to obtain:

$$\vec{B} = \boxed{\left(1+\sqrt{2}\right)\frac{\mu_0 I}{2\pi R}\,\hat{i}}$$

Magnetic Field Due to a Current-carrying Solenoid

41 •• **[SSM]** A solenoid that has length 30 cm, radius 1.2 cm, and 300 turns carries a current of 2.6 A. Find the magnitude of the magnetic field on the axis of the solenoid (*a*) at the center of the solenoid, (*b*) at one end of the solenoid.

Picture the Problem We can use $B_x = \frac{1}{2}\mu_0 nI\left(\frac{b}{\sqrt{b^2+R^2}} + \frac{a}{\sqrt{a^2+R^2}}\right)$ to find B at any point on the axis of the solenoid. Note that the number of turns per unit length for this solenoid is 300 turns/0.30 m = 1000 turns/m.

Express the magnetic field at any point on the axis of the solenoid:

$$B_x = \frac{1}{2}\mu_0 nI\left(\frac{b}{\sqrt{b^2+R^2}} + \frac{a}{\sqrt{a^2+R^2}}\right)$$

Substitute numerical values to obtain:

$$B_x = \frac{1}{2}\left(4\pi\times10^{-7}\ \text{T}\cdot\text{m/A}\right)(1000)(2.6\ \text{A})\left(\frac{b}{\sqrt{b^2+(0.012\,\text{m})^2}} + \frac{a}{\sqrt{a^2+(0.012\,\text{m})^2}}\right)$$
$$= (1.634\,\text{mT})\left(\frac{b}{\sqrt{b^2+(0.012\,\text{m})^2}} + \frac{a}{\sqrt{a^2+(0.012\,\text{m})^2}}\right)$$

(*a*) Evaluate B_x for $a = b = 0.15$ m:

$$B_x(0.15\,\text{m}) = (1.634\,\text{mT})\left(\frac{0.15\,\text{m}}{\sqrt{(0.15\,\text{m})^2+(0.012\,\text{m})^2}} + \frac{0.15\,\text{m}}{\sqrt{(0.15\,\text{m})^2+(0.012\,\text{m})^2}}\right)$$
$$= \boxed{3.3\,\text{mT}}$$

(*b*) Evaluate B_x (= B_{end}) for $a = 0$ and $b = 0.30$ m:

$$B_x(0.30\,\text{m}) = (1.634\,\text{mT})\left(\frac{0.30\,\text{m}}{\sqrt{(0.30\,\text{m})^2+(0.012\,\text{m})^2}}\right) = \boxed{1.6\,\text{mT}}$$

Note that $B_{\text{end}} \approx \frac{1}{2}B_{\text{center}}$.

42 • A solenoid is 2.7-m long, has a radius of 0.85 cm, and has 600 turns. It carries a current I of 2.5 A. What is the magnitude of the magnetic field B inside the solenoid and far from either end?

Picture the Problem We can use $B = \mu_0 nI$ to find the magnetic field inside the solenoid and far from either end.

The magnetic field inside the solenoid and far from either end is given by:

$$B = \mu_0 nI$$

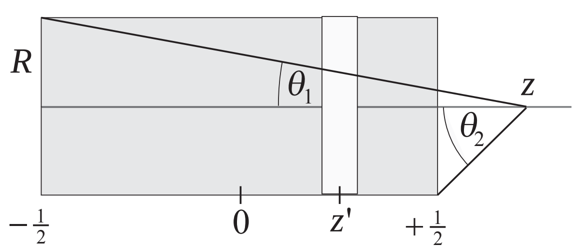
Substitute numerical values and evaluate B:

$$B = \left(4\pi \times 10^{-7}\ \text{N/A}^2\right)\left(\frac{600}{2.7\,\text{m}}\right)(2.5\,\text{A})$$

$$= \boxed{0.70\,\text{mT}}$$

43 •• A solenoid has n turns per unit length, has a radius R, and carries a current I. Its axis coincides with the z axis with one end at $z = -\frac{1}{2}\ell$ and the other end at $z = +\frac{1}{2}\ell$. Show that the magnitude of the magnetic field at a point on the z axis in the interval $z > \frac{1}{2}\ell$ is given by $B = \frac{1}{2}\mu_0 nI(\cos\theta_1 - \cos\theta_2)$, where the angles are related to the geometry by: $\cos\theta_1 = \left(z + \frac{1}{2}\ell\right)\Big/\sqrt{\left(z + \frac{1}{2}\ell\right)^2 + R^2}$ and $\cos\theta_2 = \left(z - \frac{1}{2}\ell\right)\Big/\sqrt{\left(z - \frac{1}{2}\ell\right)^2 + R^2}$.

Picture the Problem The solenoid, extending from $z = -\ell/2$ to $z = \ell/2$, with the origin at its center, is shown in the following diagram. To find the field at the point whose coordinate is z outside the solenoid we can determine the field at z due to an infinitesimal segment of the solenoid of width dz' at z', and then integrate from $z = -\ell/2$ to $z = \ell/2$. We'll treat the segment as a coil of thickness ndz' carrying a current I.



Express the field dB at the axial point whose coordinate is z:

$$dB = \frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{\left[(z - z')^2 + R^2\right]^{3/2}}dz'$$

Integrate dB from $z=-\ell/2$ to $z=\ell/2$ to obtain:

$$B=\frac{\mu_0 nIR^2}{2}\int_{-\ell/2}^{\ell/2}\frac{dz'}{\left[(z-z')^2+R^2\right]^{3/2}}=\frac{\mu_0 nI}{2}\left(\frac{z+\ell/2}{\sqrt{(z+\ell/2)^2+R^2}}-\frac{z-\ell/2}{\sqrt{(z-\ell/2)^2+R^2}}\right)$$

Refer to the diagram to express $\cos\theta_1$ and $\cos\theta_2$:

$$\cos\theta_1=\frac{z+\frac{1}{2}\ell}{\left[R^2+\left(z+\frac{1}{2}\ell\right)^2\right]^{1/2}}$$

and

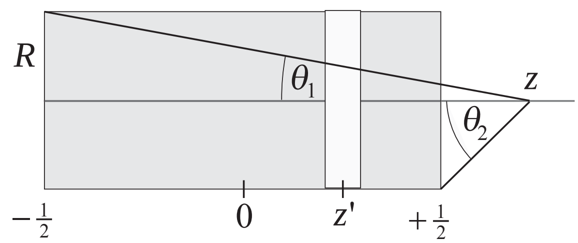
$$\cos\theta_2=\frac{z-\frac{1}{2}\ell}{\left[R^2+\left(z-\frac{1}{2}\ell\right)^2\right]^{1/2}}$$

Substitute for the terms in parentheses in the expression for B to obtain:

$$B=\boxed{\tfrac{1}{2}\mu_0 nI\left(\cos\theta_1-\cos\theta_2\right)}$$

44 ••• In Problem 43, an expression for the magnitude of the magnetic field along the axis of a solenoid is given. For $z \gg \ell$ and $\ell \gg R$, the angles θ_1 and θ_2 are very small, so the small-angle approximations $\cos\theta\approx 1-\frac{1}{2}\theta^2$ and $\sin\theta\approx\tan\theta\approx\theta$ are highly accurate. (*a*) Draw a diagram and use it to show that, for these conditions, the angles can be approximated as $\theta_1\approx R/\left(z+\frac{1}{2}\ell\right)$ and $\theta_2\approx R/\left(z-\frac{1}{2}\ell\right)$. (*b*) Using these approximations, show that the magnetic field at a points on the z axis where $z \gg \ell$ can be written as $B=\dfrac{\mu_0}{4\pi}\left(\dfrac{q_n}{r_2^2}-\dfrac{q_n}{r_1^2}\right)$ where $r_2=z-\frac{1}{2}\ell$ is the distance to the near end of the solenoid, $r_1=z+\frac{1}{2}\ell$ is the distance to the far end, and the quantity q_m is defined by $q_m=nI\pi R^2=\mu/\ell$, where $\mu=NI\pi R^2$ is the magnitude of the magnetic moment of the solenoid.

Picture the Problem (*a*) We can use the results of Problem 43, together with the small angle approximation for the cosine and tangent functions, to show that θ_1 and θ_2 are as given in the problem statement and that (*b*) $B=\dfrac{\mu_0}{4\pi}\left(\dfrac{q_m}{r_2^2}-\dfrac{q_m}{r_1^2}\right)$. The angles θ_1 and θ_2 are shown in the diagram. Note that $\tan\theta_1=R/\left(z+\ell/2\right)$ and $\tan\theta_2=R/\left(z-\ell/2\right)$.



(*a*) Apply the small angle approximation $\tan\theta \approx \theta$ to obtain:

$$\theta_1 \approx \boxed{\frac{R}{z+\frac{1}{2}\ell}} \text{ and } \theta_2 \approx \boxed{\frac{R}{z-\frac{1}{2}\ell}}$$

(*b*) Express the magnetic field outside the solenoid:

$$B = \tfrac{1}{2}\mu_0 nI\left(\cos\theta_1 - \cos\theta_2\right)$$

Apply the small angle approximation for the cosine function to obtain:

$$\cos\theta_1 = 1 - \tfrac{1}{2}\left(\frac{R}{z+\frac{1}{2}\ell}\right)^2$$

and

$$\cos\theta_2 = 1 - \tfrac{1}{2}\left(\frac{R}{z-\frac{1}{2}\ell}\right)^2$$

Substitute and simplify to obtain:

$$B = \tfrac{1}{2}\mu_0 nI\left[1 - \tfrac{1}{2}\left(\frac{R}{z+\frac{1}{2}\ell}\right)^2 - 1 + \tfrac{1}{2}\left(\frac{R}{z-\frac{1}{2}\ell}\right)^2\right] = \tfrac{1}{4}\mu_0 nIR^2\left[\frac{1}{\left(z-\frac{1}{2}\ell\right)^2} - \frac{1}{\left(z+\frac{1}{2}\ell\right)^2}\right]$$

Let $r_1 = z + \frac{1}{2}\ell$ be the distance to the near end of the solenoid, $r_2 = z - \frac{1}{2}\ell$ the distance to the far end, and $q_m = nI\pi R^2 = \mu/\ell$, where $\mu = nI\pi R^2$ is the magnetic moment of the solenoid to obtain:

$$B = \boxed{\frac{\mu_0}{4\pi}\left(\frac{q_\mathrm{m}}{r_2^2} - \frac{q_\mathrm{m}}{r_1^2}\right)}$$

Using Ampère's Law

45 • **[SSM]** A long, straight, thin-walled cylindrical shell of radius *R* carries a current *I* parallel to the central axis of the shell. Find the magnetic field (including direction) both inside and outside the shell.

Picture the Problem We can apply Ampère's law to a circle centered on the axis of the cylinder and evaluate this expression for $r < R$ and $r > R$ to find B inside and outside the cylinder. We can use the right-hand rule to determine the direction of the magnetic fields.

Apply Ampère's law to a circle centered on the axis of the cylinder:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$$

Note that, by symmetry, the field is the same everywhere on this circle.

Evaluate this expression for $r < R$:

$$\oint_C \vec{B}_{\text{inside}} \cdot d\vec{\ell} = \mu_0(0) = 0$$

Solve for B_{inside} to obtain:

$$B_{\text{inside}} = \boxed{0}$$

Evaluate this expression for $r > R$:

$$\oint_C \vec{B}_{\text{outside}} \cdot d\vec{\ell} = B(2\pi R) = \mu_0 I$$

Solve for B_{outside} to obtain:

$$B_{\text{outside}} = \boxed{\frac{\mu_0 I}{2\pi R}}$$

The direction of the magnetic field is in the direction of the curled fingers of your right hand when you grab the cylinder with your right thumb in the direction of the current.

46 • In Figure 27-55, one current is 8.0 A into the page, the other current is 8.0 A out of the page, and each curve is a circular path. (*a*) Find $\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}}$ for each path, assuming that each integral is to be evaluated in the counterclockwise direction. (*b*) Which path, if any, can be used to find the combined magnetic field of these currents?

Picture the Problem We can use Ampère's law, $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to find the line integral $\oint_C \vec{B} \cdot d\vec{\ell}$ for each of the three paths.

(*a*) Noting that the angle between $\vec{B}$ and $d\vec{\ell}$ is 180°, evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$ for C_1:

$$\oint_{C_1} \vec{B}\cdot d\vec{\ell} = \boxed{-\mu_0(8.0\,\mathrm{A})}$$

The positive tangential direction on C_1 is counterclockwise. Therefore, in accord with convention (a right-hand rule), the positive normal direction for the flat surface bounded by C_1 is out of the page. $\oint_C \vec{B}\cdot d\vec{\ell}$ is negative because the current through the surface is in the negative direction (into the page).

Noting that the net current bounded by C_2 is zero, evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$:

$$\oint_{C_2} \vec{B}\cdot d\vec{\ell} = \mu_0(8.0\,\mathrm{A} - 8.0\,\mathrm{A}) = \boxed{0}$$

Noting that the angle between $\vec{B}$ and $d\vec{\ell}$ is 0°, Evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$ for C_3:

$$\oint_{C_3} \vec{B}\cdot d\vec{\ell} = \boxed{+\mu_0(8.0\,\mathrm{A})}$$

(*b*) None of the paths can be used to find $\vec{B}$ because the current configuration does not have cylindrical symmetry, which means that $\vec{B}$ cannot be factored out of the integral.

47 •• [SSM] Show that a uniform magnetic field that has no fringing field, such as that shown in Figure 27- 56 is impossible because it violates Ampère's law. Do this calculation by applying Ampère's law to the rectangular curve shown by the dashed lines.

Determine the Concept The contour integral consists of four portions, two horizontal portions for which $\oint_C \vec{B}\cdot d\vec{\ell} = 0$, and two vertical portions. The portion within the magnetic field gives a nonvanishing contribution, whereas the portion outside the field gives no contribution to the contour integral. Hence, the contour integral has a finite value. However, it encloses no current; thus, it appears that Ampère's law is violated. What this demonstrates is that there must be a fringing field so that the contour integral does vanish.

48 •• A coaxial cable consists of a solid conducting cylinder that has a radius equal to 1.00 mm and a conducting cylindrical shell that has an inner radius equal to 2.00 mm and an outer radius equal to 3.00 mm. The solid cylinder carries a current of 15.0 A parallel to the central axis. The cylindrical shell carries and current of 15.0 A in the opposite direction. Assume that the current densities are uniformly distributed in both conductors. (*a*) Using a **spreadsheet** program or graphing calculator, graph the magnitude of the magnetic field as a function of the

radial distance r from the central axis for $0 < R < 3.00$ mm. (*b*) What is the magnitude of the field for $R > 3.00$ mm?

Picture the Problem Let r_1 = 1.00 mm, r_2 = 2.00 mm, and r_3 = 3.00 mm and apply Ampère's law in each of the three regions to obtain expressions for B in each part of the coaxial cable and outside the coaxial cable.

(*a*) Apply Ampère's law to a circular path of radius $r < r_1$ to obtain:

$$B_{r<r_1}(2\pi r) = \mu_0 I_C$$

Because the current is uniformly distributed over the cross section of the inner wire:

$$\frac{I_C}{\pi r^2} = \frac{I}{\pi r_1^2} \Rightarrow I_C = \frac{r^2}{r_1^2} I$$

Substitute for I_C to obtain:

$$B_{r<r_1}(2\pi r) = \mu_0 \frac{r^2}{r_1^2} I$$

Solving for $B_{r<r_1}$ yields:

$$B_{r<r_1} = \frac{2\mu_0 I}{4\pi} \frac{r}{r_1^2} \qquad (1)$$

Apply Ampère's law to a circular path of radius $r_1 < r < r_2$ to obtain:

$$B_{r_1<r<r_2}(2\pi r) = \mu_0 I$$

Solving for $B_{r_1<r<r_2}$ yields:

$$B_{r_1<r<r_2} = \frac{2\mu_0 I}{4\pi} \frac{1}{r} \qquad (2)$$

Apply Ampère's law to a circular path of radius $r_2 < r < r_3$ to obtain:

$$B_{r_2<r<r_3}(2\pi r) = \mu_0 I_C = \mu_0 (I - I')$$

where I' is the current in the outer conductor at a distance less than r from the center of the inner conductor.

Because the current is uniformly distributed over the cross section of the outer conductor:

$$\frac{I'}{\pi r^2 - \pi r_2^2} = \frac{I}{\pi r_3^2 - \pi r_2^2}$$

Solving for I' yields:

$$I' = \frac{r^2 - r_2^2}{r_3^2 - r_2^2} I$$

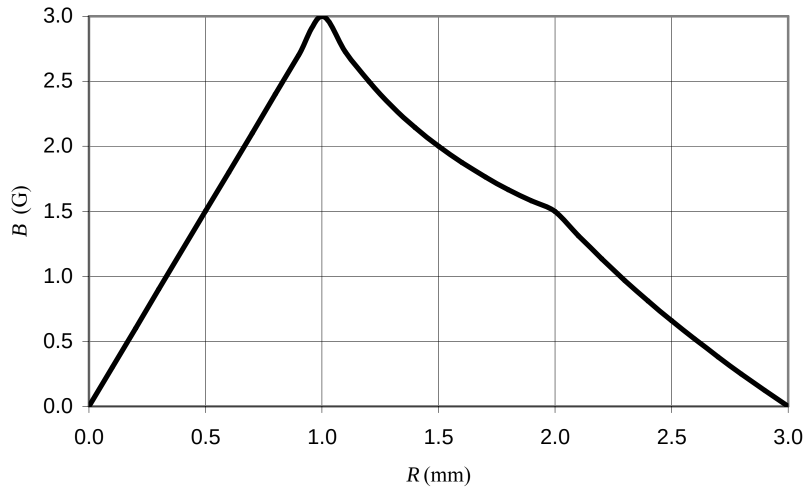
Substitute for I' to obtain:

$$B_{r_2<r<r_3}(2\pi r) = \mu_0 \left(I - \frac{r^2 - r_2^2}{r_3^2 - r_2^2} I \right)$$

Solving for $B_{r_1<r<r_2}$ yields:

$$B_{r_2<r<r_3} = \frac{2\mu_0 I}{4\pi r}\left(1-\frac{r^2-r_2^2}{r_3^2-r_2^2}\right) \qquad (3)$$

A spreadsheet program was used to plot the following graph of equations (1), (2), and (3).



(*b*) Apply Ampère's law to a circular path of radius $r > r_3$ to obtain:

$$B_{r>r_3}(2\pi r) = \mu_0 I_C = \mu_0(I - I) = 0$$

and $B_{r>r_3} = \boxed{0}$

49 •• **[SSM]** A long cylindrical shell has an inner radius a and an outer radius b and carries a current I parallel to the central axis. Assume that within the material of the shell the current density is uniformly distributed. Find an expression for the magnitude of the magnetic field for (*a*) $0 < R < a$, (*b*) $a < R < b$, and (*c*) $R > b$.

Picture the Problem We can use Ampère's law to calculate B because of the high degree of symmetry. The current through C depends on whether R is less than the inner radius a, greater than the inner radius a but less than the outer radius b, or greater than the outer radius b.

(*a*) Apply Ampère's law to a circular path of radius $R < a$ to obtain:

$$\oint_C \vec{B}_{r<a} \cdot d\vec{\ell} = \mu_0 I_C = \mu_0(0) = 0$$

and $B_{r<a} = \boxed{0}$

(*b*) Use the uniformity of the current over the cross-section of the conductor to express the current I' enclosed by a circular path whose radius satisfies the condition $a < R < b$:

$$\frac{I'}{\pi\left(R^2 - a^2\right)} = \frac{I}{\pi\left(b^2 - a^2\right)}$$

Solving for $I_C = I'$ yields:

$$I_C = I' = I\,\frac{R^2 - a^2}{b^2 - a^2}$$

Substitute in Ampère's law to obtain:

$$\oint_C \vec{\boldsymbol{B}}_{a<R<b} \cdot d\vec{\ell} = B_{a<r<b}\left(2\pi R\right)$$
$$= \mu_0 I' = \mu_0 I\,\frac{R^2 - a^2}{b^2 - a^2}$$

Solving for $B_{a<r<b}$ yields:

$$B_{a<r<b} = \boxed{\frac{\mu_0 I}{2\pi r}\,\frac{R^2 - a^2}{b^2 - a^2}}$$

(*c*) Express I_C for $R > b$:

$$I_C = I$$

Substituting in Ampère's law yields:

$$\oint_C \vec{\boldsymbol{B}}_{R>b} \cdot d\vec{\ell} = B_{r>b}\left(2\pi R\right) = \mu_0 I$$

Solve for $B_{R>b}$ to obtain:

$$B_{R>b} = \boxed{\frac{\mu_0 I}{2\pi R}}$$

50 •• Figure 27-57 shows a solenoid that has *n* turns per unit length and carries a current *I*. Apply Ampère's law to the rectangular curve shown in the figure to derive an expression for the magnitude of the magnetic field. Assume that inside the solenoid the magnetic field is uniform and parallel with the central axis, and that outside the solenoid there is no magnetic field.

Picture the Problem The number of turns enclosed within the rectangular area is *na*. Denote the corners of the rectangle, starting in the lower left-hand corner and proceeding counterclockwise, as 1, 2, 3, and 4. We can apply Ampère's law to each side of this rectangle in order to evaluate $\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\ell}$.

Express the integral around the closed path C as the sum of the integrals along the sides of the rectangle:

$$\oint_C \vec{B}\cdot d\vec{\ell} = \int_{1\to 2}\vec{B}\cdot d\vec{\ell} + \int_{2\to 3}\vec{B}\cdot d\vec{\ell} + \int_{3\to 4}\vec{B}\cdot d\vec{\ell} + \int_{4\to 1}\vec{B}\cdot d\vec{\ell}$$

Evaluate $\int_{1\to 2}\vec{B}\cdot d\vec{\ell}$:

$$\int_{1\to 2}\vec{B}\cdot d\vec{\ell} = aB$$

For the paths $2 \to 3$ and $4 \to 1$, $\vec{B}$ is either zero (outside the solenoid) or is perpendicular to $d\vec{\ell}$ and so:

$$\int_{2\to 3}\vec{B}\cdot d\vec{\ell} = \int_{4\to 1}\vec{B}\cdot d\vec{\ell} = 0$$

For the path $3 \to 4$, $\vec{B}$ =0 and:

$$\int_{3\to 4}\vec{B}\cdot d\vec{\ell} = 0$$

Substitute in Ampère's law to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = aB + 0 + 0 + 0 = aB = \mu_0 I_C = \mu_0 naI$$

Solving for B yields:

$$B = \boxed{\mu_0 nI}$$

51 •• [SSM] A tightly wound 1000-turn toroid has an inner radius 1.00 cm and an outer radius 2.00 cm, and carries a current of 1.50 A. The toroid is centered at the origin with the centers of the individual turns in the $z = 0$ plane. In the $z = 0$ plane: (*a*) What is the magnetic field strength at a distance of 1.10 cm from the origin? (*b*) What is the magnetic field strength at a distance of 1.50 cm from the origin?

Picture the Problem The magnetic field inside a tightly wound toroid is given by $B = \mu_0 NI/(2\pi r)$, where $a < r < b$ and a and b are the inner and outer radii of the toroid.

Express the magnetic field of a toroid:

$$B = \frac{\mu_0 NI}{2\pi r}$$

(*a*) Substitute numerical values and evaluate B(1.10 cm):

$$B(1.10\,\text{cm}) = \frac{(4\pi\times 10^{-7}\ \text{N/A}^2)(1000)(1.50\,\text{A})}{2\pi(1.10\,\text{cm})} = \boxed{27.3\,\text{mT}}$$

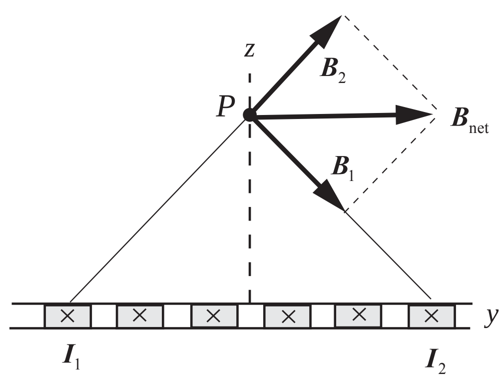
(*b*) Substitute numerical values and evaluate B(1.50 cm):

$$B(1.50\,\text{cm}) = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(1000)(1.50\,\text{A})}{2\pi(1.50\,\text{cm})} = \boxed{20.0\,\text{mT}}$$

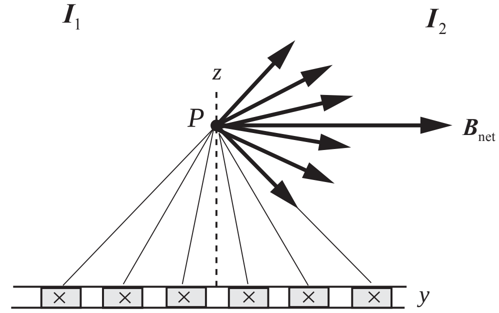
52 ••• A thin conducting sheet in the $z = 0$ plane carries current in the $-x$ direction (Figure 27-88*a*). The sheet extends indefinitely in all directions and the current is uniformly distributed throughout the sheet. To find the direction of the magnetic field at point P consider the field due only to the currents I_1 and I_2 in the two narrow strips shown. The strips are identical, so $I_1 = I_2$. (*a*) What is the direction of the magnetic field at point P due to just I_1 and I_2? Explain your answer using a sketch. (*b*) What is the direction of the magnetic field at point P due to the *entire* sheet? Explain your answer. (*c*) What is the direction of the field at a point to the right of point P (where $y \neq 0$)? Explain your answer. (*d*) What is the direction of the field at a point below the sheet (where $z < 0$)? Explain your answer using a sketch. (*e*) Apply Ampere's law to the rectangular curve (Figure 27-88*b*) to show that the magnetic field strength at point P is given by $B = \frac{1}{2}\mu_0\lambda$, where $\lambda = dI/dy$ is the current per unit length along the y axis.

Picture the Problem In Parts (*a*), (*b*), and (*c*) we can use a right-hand rule to determine the direction of the magnetic field at points above and below the infinite sheet of current. In Part (*d*) we can evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$ around the specified path and equate it to $\mu_0 I_C$ and solve for B.

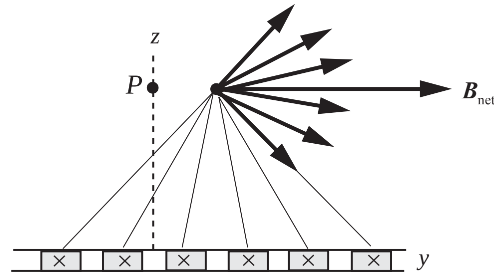
(*a*) Because its vertical components cancel at P, the magnetic field points to the right (i.e., in the $+y$ direction).



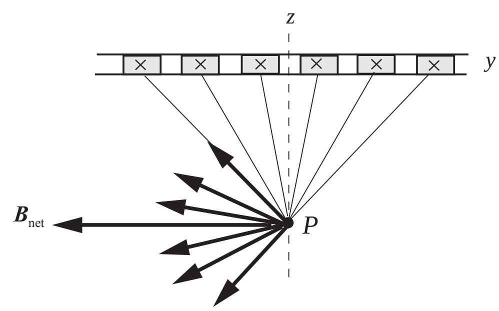
(*b*) The vertical components of the field cancel in pairs. The magnetic field is in the $+y$ direction.



(*c*) The magnetic field is in the +*y* direction. This result follows from the same arguments that were used in (*a*) and (*b*).



(*d*) Below the sheet the magnetic field points to the left; i.e., in the −*y* direction. The vertical components cancel in pairs.



(*e*) Express $\oint_C \vec{B}\cdot d\vec{\ell}$, in the counterclockwise direction, for the given path:

$$\oint_C \vec{B}\cdot d\vec{\ell} = 2\int_{\text{parallel}} \vec{B}\cdot d\vec{\ell} + 2\int_{\perp} \vec{B}\cdot d\vec{\ell} \quad (1)$$

For the paths perpendicular to the sheet, $\vec{B}$ and $d\vec{\ell}$ are perpendicular to each other and:

$$\int_{\perp} \vec{B}\cdot d\vec{\ell} = 0$$

For the paths parallel to the sheet, $\vec{B}$ and $d\vec{\ell}$ are in the same direction and:

$$\int_{\text{parallel}} \vec{B}\cdot d\vec{\ell} = Bw$$

Substituting in equation (1) and simplifying yields:

$$\oint_C \vec{B}\cdot d\vec{\ell} = 2\int_{\text{parallel}} \vec{B}\cdot d\vec{\ell} = 2Bw$$
$$= \mu_0 I_C = \mu_0(\lambda w)$$

Solving for *B* yields:

$$B = \boxed{\tfrac{1}{2}\mu_0\lambda}$$

Magnetization and Magnetic Susceptibility

53 • [SSM] A tightly wound solenoid is 20.0-cm long, has 400 turns, and carries a current of 4.00 A so that its axial field is in the +z direction. Find B and B_{app} at the center when (*a*) there is no core in the solenoid, and (*b*) there is a soft iron core that has a magnetization of 1.2×10^6 A/m.

Picture the Problem We can use $B = B_{\text{app}} = \mu_0 nI$ to find B and B_{app} at the center when there is no core in the solenoid and $B = B_{\text{app}} + \mu_0 M$ when there is an iron core with a magnetization $M = 1.2 \times 10^6$ A/m.

(*a*) Express the magnetic field, in the absence of a core, in the solenoid :

$$B = B_{\text{app}} = \mu_0 nI$$

Substitute numerical values and evaluate B and B_{app}:

$$B = B_{\text{app}} = \left(4\pi \times 10^{-7}\ \text{N/A}^2\right)\left(\frac{400}{0.200\,\text{m}}\right)(4.00\,\text{A}) = \boxed{10.1\,\text{mT}}$$

(*b*) With an iron core with a magnetization $M = 1.2 \times 10^6$ A/m present:

$$B_{\text{app}} = \boxed{10.1\,\text{mT}}$$

and

$$B = B_{\text{app}} + \mu_0 M = 10.1\,\text{mT} + \left(4\pi \times 10^{-7}\ \text{N/A}^2\right)\left(1.2 \times 10^6\ \text{A/m}\right) = \boxed{1.5\,\text{T}}$$

54 • A long tungsten-core solenoid carries a current. (*a*) If the core is removed while the current is held constant, does the magnetic field strength in the region inside the solenoid decrease or increase? (*b*) By what percentage does the magnetic field strength in the region inside the solenoid decrease or increase?

Picture the Problem We can use $B = B_{\text{app}}(1 + \chi_{\text{m}})$ to relate B and B_{app} to the magnetic susceptibility of tungsten. Dividing both sides of this equation by B_{app} and examining the value of $\chi_{\text{m, tungsten}}$ will allow us to decide whether the field inside the solenoid decreases or increases when the core is removed.

Express the magnetic field inside the solenoid with the tungsten core present B in terms of B_{app} and χ_{m}:

$$B = B_{\text{app}}(1 + \chi_{\text{m}})$$

where B_{app} is the magnetic field in the absence of the tungsten core.

Express the ratio of B to B_{app} :

$$\frac{B}{B_{\text{app}}} = 1 + \chi_{\text{m}} \qquad (1)$$

(*a*) Because $\chi_{\text{m, tungsten}} > 0$:

$B > B_{\text{app}}$ and B will decrease when the tungsten core is removed.

(*b*) From equation (1), the fractional change is:

$$\chi_m = 6.8\times10^{-5} = \boxed{6.8\times10^{-3}\%}$$

55 • As a liquid fills the interior volume of a solenoid that carries a constant current, the magnetic field inside the solenoid *decreases* by 0.0040 percent. Determine the magnetic susceptibility of the liquid.

Picture the Problem We can use $B = B_{app}(1+\chi_m)$ to relate B and B_{app} to the magnetic susceptibility of liquid sample.

Express the magnetic field inside the solenoid with the liquid sample present B in terms of B_{app} and $\chi_{m,\,sample}$:

$$B = B_{app}(1+\chi_{m,\,sample})$$

where B_{app} is the magnetic field in the absence of the liquid sample.

The fractional change in the magnetic field in the core is:

$$\frac{\Delta B}{B_{app}} = \chi_{m,\,sample}$$

Substitute numerical values and evaluate $\chi_{m,\,sample}$:

$$\chi_{m,\,sample} = \frac{\Delta B}{B_{app}} = -0.0040\%$$

$$= \boxed{-4.0\times10^{-5}}$$

56 • A long thin solenoid carrying a current of 10 A has 50 turns per centimeter of length. What is the magnetic field strength in the region occupied by the interior of the solenoid when the interior is (*a*) a vacuum, (*b*) filled with aluminum, and (*c*) filled with silver?

Picture the Problem We can use $B = B_{app} = \mu_0 nI$ to find B and B_{app} at the center when there is no core in the solenoid and $B = B_{app}(1+\chi_m)$ when there is an aluminum or silver core.

(*a*) Express the magnetic field, in the absence of a core, in the solenoid:

$$B = B_{app} = \mu_0 nI$$

Substitute numerical values and evaluate B and B_{app}:

$$B = B_{app} = \left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(\frac{50}{\text{cm}}\right)(10\ \text{A}) = \boxed{63\,\text{mT}}$$

(*b*) With an aluminum core:

$$B = B_{app}(1+\chi_m)$$

Use Table 27-1 to find the value of χ_m for aluminum:

$$\chi_{\mathrm{m,\,Al}} = 2.3\times10^{-5}$$
and
$$1+\chi_{\mathrm{m,\,Al}} = 1+2.3\times10^{-5} \approx 1$$

Substitute numerical values and evaluate B and B_{app}:

$$B = B_{\mathrm{app}} = \left(4\pi\times10^{-7}\ \mathrm{N/A^2}\right)\left(\frac{50}{\mathrm{cm}}\right)\left(10\,\mathrm{A}\right) = \boxed{63\,\mathrm{mT}}$$

(c) With a silver core:

$$B = B_{\mathrm{app}}\left(1+\chi_{\mathrm{m}}\right)$$

Use Table 27-1 to find the value of χ_m for silver:

$$\chi_{\mathrm{m,\,Ag}} = -2.6\times10^{-5}$$
and
$$1+\chi_{\mathrm{m,\,Ag}} = 1-2.6\times10^{-5} \approx 1$$

Substitute numerical values and evaluate B and B_{app}:

$$B = B_{\mathrm{app}} = \left(4\pi\times10^{-7}\ \mathrm{N/A^2}\right)\left(\frac{50}{\mathrm{cm}}\right)\left(10\,\mathrm{A}\right) = \boxed{63\,\mathrm{mT}}$$

57 •• [SSM] A cylinder of iron, initially unmagnetized, is cooled to 4.00 K. What is the magnetization of the cylinder at that temperature due to the influence of Earth's magnetic field of 0.300 G? Assume a magnetic moment of 2.00 Bohr magnetons per atom.

Picture the Problem We can use Curie's law to relate the magnetization M of the cylinder to its saturation magnetization M_S. The saturation magnetization is the product of the number of atoms n in the cylinder and the magnetic moment of each molecule. We can find n using the proportion $\dfrac{n}{N_A} = \dfrac{\rho_{\mathrm{Fe}}}{M_{\mathrm{Fe}}}$ where M_{Fe} is the molar mass of iron.

The magnetization of the cylinder is given by Curie's law:

$$M = \frac{1}{3}\frac{\mu B_{\mathrm{app}}}{kT}M_{\mathrm{S}}$$

Assuming a magnetic moment of 2.00 Bohr magnetons per atom:

$$M = \frac{1}{3}\frac{\left(2.00\mu_{\mathrm{B}}\right)B_{\mathrm{Earth}}}{kT}M_{\mathrm{S}} \qquad (1)$$

The saturation magnetization is given by:

$$M_S = n\mu = n(2.00\mu_B)$$

where n is the number of atoms and μ is the magnetic moment of each molecule.

The number of atoms of iron per unit volume n can be found from the molar mass M_{Fe} of iron, the density ρ_{Fe} of iron, and Avogadro's number N_A:

$$n = \frac{\rho_{Fe}}{M_{Fe}} N_A \Rightarrow M_S = \frac{2.00\rho_{Fe} N_A \mu_B}{M_{Fe}}$$

Substitute numerical values and evaluate M_S:

$$M_S = \frac{2.00\left(7.96\times10^3 \frac{\text{kg}}{\text{m}^3}\right)\left(6.022\times10^{23} \frac{\text{atoms}}{\text{mol}}\right)\left(9.27\times10^{-24}\ \text{A}\cdot\text{m}^2\right)}{55.85\frac{\text{g}}{\text{mol}}\times\frac{1\,\text{kg}}{10^3\ \text{g}}}$$

$$= 1.591\times10^6\ \frac{\text{A}}{\text{m}}$$

Substitute numerical values in equation (1) and evaluate M:

$$M = \frac{2.00\left(5.788\times10^{-5}\ \frac{\text{eV}}{\text{T}}\right)\left(0.300\ \text{G}\times\frac{1\,\text{T}}{10^4\ \text{G}}\right)}{3\left(8.617\times10^{-5}\ \frac{\text{eV}}{\text{K}}\right)(4.00\ \text{K})}\left(1.591\times10^6\ \frac{\text{A}}{\text{m}}\right) = \boxed{5.34\frac{\text{A}}{\text{m}}}$$

58 •• A cylinder of silver at a temperature of 77 K has a magnetization equal to 0.075% of its saturation magnetization. Assume a magnetic moment of one Bohr magneton per atom. The density of silver is 1.05×10^4 kg/m^3. (*a*) What value of applied magnetic field parallel to the central axis of the cylinder is required to reach this magnetization? (*b*) What is the magnetic field strength at the center of the cylinder?

Picture the Problem We can use Curie's law to relate the magnetization M of the cylinder to its saturation magnetization M_S. The saturation magnetization is the product of the number of atoms n in the cylinder and the magnetic moment of each atom. We can find n using the proportion $\frac{n}{N_A} = \frac{\rho_{Ag}}{\text{M}_{Ag}}$ where M_{Ag} is the molar mass of silver. In Part (*b*), we can use Equation 27-22 to find the magnetic field at the center of the cylinder, on the axis defined by the magnetic field.

(*a*) The magnetization of the cylinder is given by Curie's law:

$$M = \frac{1}{3}\frac{\mu B_{\text{app}}}{kT} M_{\text{S}}$$

Assuming a magnetic moment of 1Bohr magnetons per atom:

$$M = \frac{(\mu_{\text{B}})B_{\text{app}}}{3kT} M_{\text{S}} \Rightarrow B_{\text{app}} = \frac{3kT}{\mu_{\text{B}}}\left(\frac{M}{M_{\text{S}}}\right)$$

Substitute numerical values and evaluate B_{app}:

$$B_{\text{app}} = \frac{3\left(8.617\times10^{-5}\ \frac{\text{eV}}{\text{K}}\right)(77\ \text{K})}{5.788\times10^{-5}\ \frac{\text{eV}}{\text{T}}}(0.00075) = \boxed{0.26\ \text{T}}$$

(*b*) The magnetic field at the center of the cylinder on the axis defined by the magnetic field is:

$$B = B_{\text{app}} + \mu_0 M_{\text{S}} \qquad (1)$$

The saturation magnetization is given by:

$$M_{\text{S}} = n\mu = n(\mu_{\text{B}})$$

where n is the number of atoms and μ is the magnetic moment of each molecule.

The number of atoms of silver per unit volume n can be found from the molar mass M_{Ag} of silver, the density ρ_{Ag} of silver, and Avogadro's number N_{A}:

$$n = \frac{\rho_{\text{Ag}}}{M_{\text{Ag}}} N_{\text{A}} \Rightarrow M_{\text{S}} = \frac{\rho_{\text{Ag}} N_{\text{A}} \mu_{\text{B}}}{M_{\text{Ag}}}$$

Substitute numerical values and evaluate M_{S}:

$$M_{\text{S}} = \frac{\left(10.5\times10^3\ \frac{\text{kg}}{\text{m}^3}\right)\left(6.022\times10^{23}\ \frac{\text{atoms}}{\text{mol}}\right)\left(9.27\times10^{-24}\ \text{A}\cdot\text{m}^2\right)}{107.870\frac{\text{g}}{\text{mol}}\times\frac{1\,\text{kg}}{10^3\ \text{g}}} = 5.434\times10^5\ \frac{\text{A}}{\text{m}}$$

Substitute numerical values in equation (1) and evaluate B:

$$B = 0.26\ \text{T} + \left(4\pi\times10^{-7}\ \text{T}\cdot\text{A}\right)\left(5.434\times10^5\ \frac{\text{A}}{\text{m}}\right) = \boxed{0.94\ \text{T}}$$

59 •• During a solid-state physics lab, you are handed a cylindrically shaped sample of unknown magnetic material. You and your lab partners place the sample in a long solenoid that has n turns per unit length and a current I. The values for magnetic field B within the material versus nI, where B_{app} is the field due to the current I and K_m is the relative permeability of the sample, are given below. Use these values to plot B versus B_{app} and K_m versus nI.

nI, A/m	0	50	100	150	200	500	1000	10 000
B, T	0	0.04	0.67	1.00	1.2	1.4	1.6	1.7

Picture the Problem We can use the data in the table and $B_{app} = \mu_0 nI$ to plot B versus B_{app}. We can find K_m using $B = K_m B_{app}$.

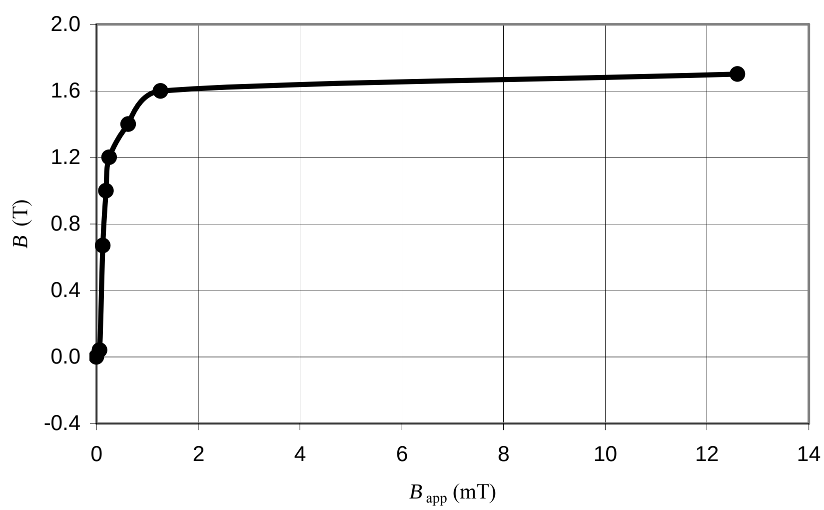
We can find the applied field B_{app} for a long solenoid using:

$$B_{app} = \mu_0 nI$$

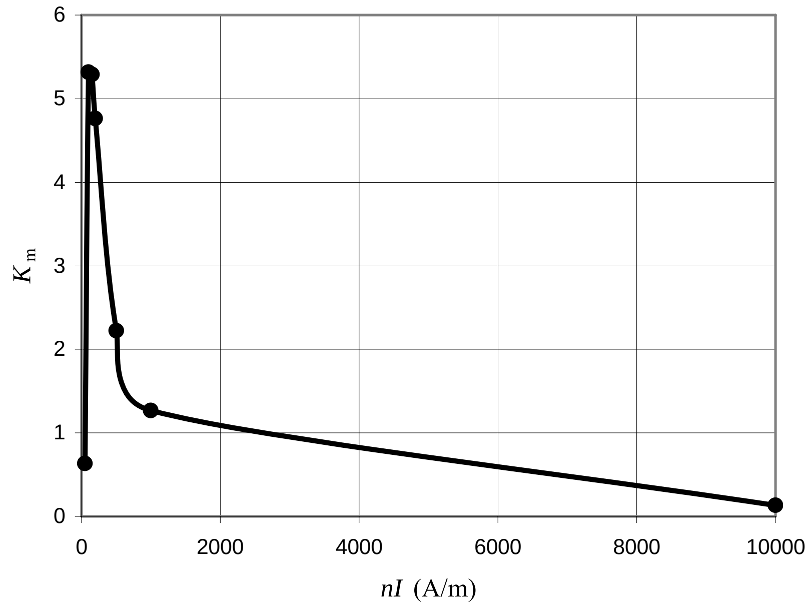
K_m can be found from B_{app} and B using:

$$K_m = \frac{B}{B_{app}}$$

The following graph was plotted using a spreadsheet program. The abscissa values for the graph were obtained by multiplying nI by μ_0. B initially rises rapidly, and then becomes nearly flat. This is characteristic of a ferromagnetic material.



The following graph of K_m versus nI was also plotted using a spreadsheet program. Note that K_m becomes quite large for small values of nI but then diminishes. A more revealing graph would be to plot $B/(nI)$, which would be quite large for small values of nI and then drop to nearly zero at nI = 10,000 A/m, corresponding to saturation of the magnetization.



Atomic Magnetic Moments

60 •• Nickel has a density of 8.70 g/cm^3 and a molar mass of 58.7 g/mol. Nickel's saturation magnetization is 0.610 T. Calculate the magnetic moment of a nickel atom in Bohr magnetons.

Picture the Problem We can find the magnetic moment of a nickel atom μ from its relationship to the saturation magnetization M_S using $M_S = n\mu$ where n is the number of molecules per unit volume. n, in turn, can be found from Avogadro's number, the density of nickel, and its molar mass using $n = \frac{N_A\rho}{M}$.

Express the saturation magnetic field in terms of the number of molecules per unit volume and the magnetic moment of each molecule:

$$M_S = n\mu \Rightarrow \mu = \frac{M_S}{n}$$

Express the number of molecules per unit volume in terms of Avogadro's number N_A, the molecular mass M, and the density ρ:

$$n = \frac{N_A \rho}{M}$$

Substitute for n in the equation for μ and simplify to obtain:

$$\mu = \frac{M_S}{\frac{N_A \rho}{M}} = \frac{\mu_0 M_S}{\frac{\mu_0 N_A \rho}{M}} = \frac{\mu_0 M_S M}{\mu_0 N_A \rho}$$

Substitute numerical values and evaluate μ:

$$\mu = \frac{(0.610\,T)(58.7\times10^{-3}\,\text{kg/mol})}{(4\pi\times10^{-7}\,\text{N/A}^2)(6.022\times10^{23}\,\text{atoms/mol})(8.70\,\text{g/cm}^3)} = 5.439\times10^{-24}\,\text{A}\cdot\text{m}^2$$

Divide μ by $\mu_B = 9.27\times10^{-24}\,\text{A}\cdot\text{m}^2$ to obtain:

$$\frac{\mu}{\mu_B} = \frac{5.439\times10^{-24}\,\text{A}\cdot\text{m}^2}{9.27\times10^{-24}\,\text{A}\cdot\text{m}^2} = 0.587$$

or $\mu = \boxed{0.587\mu_B}$

61 •• Repeat Problem 60 for cobalt, which has a density of 8.90 g/cm^3, a molar mass of 58.9 g/mol, and a saturation magnetization of 1.79 T.

Picture the Problem We can find the magnetic moment of a cobalt atom μ from its relationship to the saturation magnetization M_S using $M_S = n\mu$, where n is the number of molecules per unit volume. n, in turn, can be found from Avogadro's number, the density of cobalt, and its molar mass using $n = \frac{N_A \rho}{M}$.

Express the saturation magnetic field in terms of the number of molecules per unit volume and the magnetic moment of each molecule:

$$M_S = n\mu \Rightarrow \mu = \frac{M_S}{n}$$

Express the number of molecules per unit volume in terms of Avogadro's number N_A, the molar mass M, and the density ρ:

$$n = \frac{N_A \rho}{M}$$

Substitute for n in the equation for μ and simplify to obtain:

$$\mu = \frac{M_S}{\frac{N_A \rho}{M}} = \frac{\mu_0 M_S}{\frac{\mu_0 N_A \rho}{M}} = \frac{\mu_0 M_S M}{\mu_0 N_A \rho}$$

Substitute numerical values and evaluate μ:

$$\mu = \frac{(1.79\,\text{T})(58.9\times10^{-3}\,\text{kg/mol})}{(4\pi\times10^{-7}\,\text{N/A}^2)(6.022\times10^{23}\,\text{atoms/mol})(8.90\,\text{g/cm}^3)} = 1.57\times10^{-23}\,\text{A}\cdot\text{m}^2$$

Divide μ by $\mu_B = 9.27\times10^{-24}\,\text{A}\cdot\text{m}^2$ to obtain:

$$\frac{\mu}{\mu_B} = \frac{1.57\times10^{-23}\,\text{A}\cdot\text{m}^2}{9.27\times10^{-24}\,\text{A}\cdot\text{m}^2} = 1.69$$

or $\mu = \boxed{1.69\mu_B}$

Paramagnetism

62 • Show that Curie's law predicts that the magnetic susceptibility of a paramagnetic substance is given by $\chi_m = \mu\mu_0 M_s/3kT$.

Picture the Problem We can show that $\chi_m = \mu\mu_0 M_s/3kT$ by equating Curie's law and the equation that defines χ_m ($M = \chi_m \frac{B_{app}}{\mu_0}$) and solving for χ_m.

Express Curie's law:

$$M = \frac{1}{3}\frac{\mu B_{app}}{kT} M_S$$

where M_S is the saturation value.

Express the magnetization of the substance in terms of its magnetic susceptibility χ_m:

$$M = \chi_m \frac{B_{app}}{\mu_0}$$

Equate these expressions to obtain:

$$\chi_m \frac{B_{app}}{\mu_0} = \frac{1}{3}\frac{\mu B_{app}}{kT} M_S$$

and

$$\frac{\chi_m}{\mu_0} = \frac{1}{3}\frac{\mu}{kT} M_S \Rightarrow \chi_m = \boxed{\frac{\mu_0 \mu M_S}{3kT}}$$

63 •• In a simple model of paramagnetism, we can consider that some fraction f of the molecules have their magnetic moments aligned with the external magnetic field and that the rest of the molecules are randomly oriented and therefore do not contribute to the magnetic field. (*a*) Use this model and Curie's law to show that at temperature T and external magnetic field B, the fraction of aligned molecules f is given by $\mu B/(3kT)$. (*b*) Calculate this fraction for a sample temperature of 300 K, an external field of 1.00 T. Assume that μ has a value of 1.00 Bohr magneton.

Picture the Problem We can use the assumption that $M = fM_S$ and Curie's law to solve these equations simultaneously for the fraction *f* of the molecules have their magnetic moments aligned with the external magnetic field.

(*a*) Assume that some fraction *f* of the molecules have their magnetic moments aligned with the external magnetic field and that the rest of the molecules are randomly oriented and so do not contribute to the magnetic field:

$$M = fM_S$$

From Curie's law we have:

$$M = \frac{1}{3}\frac{\mu B_{app}}{kT}M_S$$

Equating these expressions yields:

$$fM_S = \frac{1}{3}\frac{\mu B_{app}}{kT}M_S \Rightarrow f = \boxed{\frac{\mu B}{3kT}}$$

because B given in the problem statement is the external magnetic field B_{app} .

(*b*) Substitute numerical values and evaluate *f*:

$$f = \frac{(9.27\times10^{-24}\ \text{A}\cdot\text{m}^2)(1.00\,\text{T})}{3(1.381\times10^{-23}\ \text{J/K})(300\,\text{K})} = \boxed{7.46\times10^{-4}}$$

64 •• Assume that the magnetic moment of an aluminum atom is 1.00 Bohr magneton. The density of aluminum is 2.70 g/cm^3 and its molar mass is 27.0 g/mol. (*a*) Calculate the value of the saturation magnetization and the saturation magnetic field for aluminum. (*b*) Use the result of Problem 62 to calculate the magnetic susceptibility at 300 K. (*c*) Explain why the result for Part (*b*) is larger than the value listed in Table 27-1.

Picture the Problem In (*a*) we can express the saturation magnetic field in terms of the number of molecules per unit volume and the magnetic moment of each molecule and use $n = N_A\rho/\text{M}$ to express the number of molecules per unit volume in terms of Avogadro's number N_A, the molecular mass M, and the density ρ. (*b*) We can use $\chi_m = \mu_0\mu M_S/3kT$ from Problem 62 to calculate χ_m.

(*a*) Express the saturation magnetic field in terms of the number of molecules per unit volume and the magnetic moment of each molecule:

$$M_S = n\mu_B$$

Express the number of molecules per unit volume in terms of Avogadro's number N_A, the molecular mass M, and the density ρ:

$$n = \frac{N_A \rho}{\text{M}}$$

Substitute for *n* to obtain:

$$M_S = \frac{N_A \rho}{\text{M}} \mu_B$$

Substitute numerical values and evaluate M_S:

$$M_S = \frac{(6.022 \times 10^{23}\ \text{atoms/mol})(2.70 \times 10^3\ \text{kg/m}^3)(9.27 \times 10^{-24}\ \text{A} \cdot \text{m}^2)}{27.0\,\text{g/mol}}$$
$$= 5.582 \times 10^5\ \text{A/m} = \boxed{5.58 \times 10^5\ \text{A/m}}$$

and

$$B_S = \mu_0 M_S = (4\pi \times 10^{-7}\ \text{N/A}^2)(5.582 \times 10^5\ \text{A/m}) = \boxed{0.702\,\text{T}}$$

(*b*) From Problem 62 we have:

$$\chi_m = \frac{\mu_0 \mu M_S}{3kT}$$

Substitute numerical values and evaluate χ_m:

$$\chi_m = \frac{(4\pi \times 10^{-7}\ \text{N/A}^2)(9.27 \times 10^{-24}\ \text{A} \cdot \text{m}^2)(5.582 \times 10^5\ \text{A/m})}{3(1.381 \times 10^{-23}\ \text{J/K})(300\,\text{K})} = \boxed{5.23 \times 10^{-4}}$$

(*c*) In calculating χ_m in Part (*b*) we neglected any diamagnetic effects.

65 •• [SSM] A toroid has *N* turns, carries a current *I*, has a mean radius *R*, and has a cross-sectional radius *r*, where $r \ll R$ (Figure 27-59). When the toroid is filled with material, it is called a *Rowland ring*. Find B_{app} and *B* in such a ring, assuming a magnetization that is everywhere parallel to $\vec{B}_{app}$.

Picture the Problem We can use $B_{app} = \dfrac{\mu_0 NI}{2\pi a}$ to express B_{app} and $B = B_{app} + \mu_0 M$ express *B* in terms of B_{app} and *M*.

Express B_{app} inside a tightly wound toroid:

$$B_{\text{app}} = \boxed{\frac{\mu_0 NI}{2\pi a}} \text{ for } R - r < a < R + r$$

The resultant field B in the ring is the sum of B_{app} and $\mu_0 M$:

$$B = B_{\text{app}} + \mu_0 M = \boxed{\frac{\mu_0 NI}{2\pi a} + \mu_0 M}$$

66 •• A toroid is filled with liquid oxygen that has a magnetic susceptibility of 4.00×10^{-3}. The toroid has 2000 turns and carries a current of 15.0 A. Its mean radius is 20.0 cm, and the radius of its cross section is 8.00 mm. (*a*) What is the magnetization? (*b*) What is the magnetic field? (*c*) What is the percentage change in magnetic field produced by the liquid oxygen?

Picture the Problem We can find the magnetization using $M = \chi_{\text{m}} B_{\text{app}} / \mu_0$ and the magnetic field using $B = B_{\text{app}}(1 + \chi_{\text{m}})$.

(*a*) The magnetization M in terms of χ_{m} and B_{app} is given by:

$$M = \chi_{\text{m}} \frac{B_{\text{app}}}{\mu_0}$$

Express B_{app} inside a tightly wound toroid:

$$B_{\text{app}} = \frac{\mu_0 NI}{2\pi r_{\text{mean}}}$$

Substitute for B_{app} to obtain:

$$M = \chi_{\text{m}} \frac{\dfrac{\mu_0 NI}{2\pi r_{\text{mean}}}}{\mu_0} = \chi_{\text{m}} \frac{NI}{2\pi r_{\text{mean}}}$$

Substitute numerical values and evaluate M:

$$M = \frac{(4.00 \times 10^{-3})(2000)(15.0\,\text{A})}{2\pi(0.200\,\text{m})} = \boxed{95.5\,\text{A/m}}$$

(*b*) Express B in terms of B_{app} and χ_{m}:

$$B = B_{\text{app}}(1 + \chi_{\text{m}})$$

Substitute for B_{app} to obtain:

$$B = \frac{\mu_0 NI}{2\pi r_{\text{mean}}}(1 + \chi_{\text{m}})$$

Substitute numerical values and evaluate B:

$$B = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(2000)(15.0\,\text{A})}{2\pi(0.200\,\text{m})}\left(1+4.00\times10^{-3}\right) = \boxed{30.1\,\text{mT}}$$

(c) Express the fractional increase in B produced by the liquid oxygen:

$$\frac{\Delta B}{B} = \frac{B - B_{\text{app}}}{B} = \frac{B_{\text{app}}(1+\chi_{\text{m}}) - B_{\text{app}}}{B} = \frac{\chi_{\text{m}} B_{\text{app}}}{B} = \frac{\chi_{\text{m}}}{1+\chi_{\text{m}}} = \frac{1}{\dfrac{1}{\chi_{\text{m}}}+1}$$

Substitute numerical values and evaluate $\Delta B/B$:

$$\frac{\Delta B}{B} = \frac{1}{\dfrac{1}{4.00\times10^{-3}}+1} = 3.98\times10^{-3} = \boxed{0.398\%}$$

67 •• The centers of the turns of a toroid form a circle with a radius of 14.0 cm. The cross-sectional area of each turn is 3.00 cm^2. It is wound with 5278 turns of fine wire, and the wire carries a current of 4.00 A. The core is filled with a paramagnetic material of magnetic susceptibility 2.90×10^{-4}. (a) What is the magnitude of the magnetic field within the substance? (b) What is the magnitude of the magnetization? (c) What would the magnitude of the magnetic field be if there were no paramagnetic core present?

Picture the Problem We can use $B = B_{\text{app}}(1+\chi_{\text{m}})$ and $B_{\text{app}} = \dfrac{\mu_0 NI}{2\pi r_{\text{mean}}}$ to find B within the substance and $M = \chi_{\text{m}}\dfrac{B_{\text{app}}}{\mu_0}$ to find the magnitude of the magnetization.

(a) Express the magnetic field B within the substance in terms of B_{app} and χ_{m}:

$$B = B_{\text{app}}(1+\chi_{\text{m}})$$

Express B_{app} inside the toroid:

$$B_{\text{app}} = \frac{\mu_0 NI}{2\pi r_{\text{mean}}}$$

Substitute to obtain:

$$B = \frac{\mu_0 NI(1+\chi_m)}{2\pi r_{mean}}$$

Substitute numerical values and evaluate B:

$$B = \frac{(4\pi \times 10^{-7}\ \mathrm{N/A^2})(5278)(4.00\ \mathrm{A})(1+2.90\times 10^{-4})}{2\pi(14.0\ \mathrm{cm})} = \boxed{30.2\ \mathrm{mT}}$$

(*b*) Express the magnetization M in terms of χ_m and B_{app}:

$$M = \chi_m \frac{B_{app}}{\mu_0}$$

Substitute for B_{app} and simplify to obtain:

$$M = \frac{\chi_m NI}{2\pi r_{mean}}$$

Substitute numerical values and evaluate M:

$$M = \frac{(2.90\times 10^{-4})(5278)(4.00\ \mathrm{A})}{2\pi(14.0\ \mathrm{cm})} = \boxed{6.96\ \mathrm{A/m}}$$

(*c*) If there were no paramagnetic core present:

$$B = B_{app} = \boxed{30.2\ \mathrm{mT}}$$

Ferromagnetism

68 • For annealed iron, the relative permeability K_m has its maximum value of approximately 5500 at $B_{app} = 1.57 \times 10^{-4}$ T. Find the magnitude of the magnetization and magnetic field in annealed iron when K_m is maximum.

Picture the Problem We can use $B = K_m B_{app}$ to find B and $M = (K_m - 1)B_{app}/\mu_0$ to find M.

Express B in terms of M and K_m:

$$B = K_m B_{app}$$

Substitute numerical values and evaluate B:

$$B = (5500)(1.57\times 10^{-4}\ \mathrm{T}) = \boxed{0.86\ \mathrm{T}}$$

Relate M to K_m and B_{app}:

$$M = (K_m - 1)\frac{B_{app}}{\mu_0} \approx \frac{K_m B_{app}}{\mu_0}$$

Substitute numerical values and evaluate M:

$$M = \frac{(5500)(1.57\times10^{-4}\ \text{T})}{4\pi\times10^{-7}\ \text{N/A}^2} = \boxed{6.9\times10^{5}\ \text{A/m}}$$

69 •• **[SSM]** The saturation magnetization for annealed iron occurs when B_{app} = 0.201 T. Find the permeability and the relative permeability of annealed iron at saturation. (See Table 27-2)

Picture the Problem We can relate the permeability μ of annealed iron to χ_{m} using $\mu = (1+\chi_{\text{m}})\mu_0$, find χ_{m} using $M = \chi_{\text{m}}\dfrac{B_{\text{app}}}{\mu_0}$, and use its definition ($K_{\text{m}} = 1+\chi_{\text{m}}$) to evaluate K_{m}.

Express the permeability μ of annealed iron in terms of its magnetic susceptibility χ_{m}:

$$\mu = (1+\chi_{\text{m}})\mu_0 \qquad (1)$$

The magnetization M in terms of χ_{m} and B_{app} is given by:

$$M = \chi_{\text{m}}\frac{B_{\text{app}}}{\mu_0}$$

Solve for and evaluate χ_{m} (see Table 27-2 for the product of μ_0 and M):

$$\chi_{\text{m}} = \frac{\mu_0 M}{B_{\text{app}}} = \frac{2.16\,\text{T}}{0.201\,\text{T}} = 10.75$$

Use its definition to express and evaluate the relative permeability K_{m}:

$$K_{\text{m}} = 1+\chi_{\text{m}} = 1+10.75 = 11.746 = \boxed{11.7}$$

Substitute numerical values in equation (1) and evaluate μ:

$$\mu = (1+10.746)(4\pi\times10^{-7}\ \text{N/A}^2) = \boxed{1.48\times10^{-5}\,\text{N/A}^2}$$

70 •• The *coercive force* (which is a misnomer because it is really a magnetic field value) is defined as the applied magnetic field needed to bring the magnetic field back to zero along the hysteresis curve (which is point *c* in Figure 27- 38). For a certain permanent bar magnet, the coercive force is known to be 5.53×10^{-2} T. The bar magnet is to be demagnetized by placing it inside a 15.0-cm-long solenoid that has 600 turns. What minimum current is needed in the solenoid to demagnetize the magnet?

Picture the Problem We can use the relationship between the magnetic field on the axis of a solenoid and the current in the solenoid to find the minimum current is needed in the solenoid to demagnetize the magnet.

Relate the magnetic field on the axis of a solenoid to the current in the solenoid:

$$B_x = \mu_0 nI \Rightarrow I = \frac{B_x}{\mu_0 n}$$

Let $B_{\text{app}} = B_x$ to obtain:

$$I = \frac{B_{\text{app}}}{\mu_0 n}$$

Substitute numerical values and evaluate I:

$$I = \frac{5.53\times10^{-2}\ \text{T}}{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(\dfrac{600}{0.150\,\text{m}}\right)} = \boxed{11.0\,\text{A}}$$

71 •• A long thin solenoid has 50 turns/cm and carries a current of 2.00 A. The solenoid is filled with iron and the magnetic field is measured to be 1.72 T. (*a*) Neglecting end effects, what is the magnitude of the applied magnetic field? (*b*) What is the magnetization? (*c*) What is the relative permeability?

Picture the Problem We can use the equation describing the magnetic field on the axis of a solenoid, as a function of the current in the solenoid, to find B_{app}. We can then use $B = B_{\text{app}} + \mu_0 M$ to find M and $B = K_{\text{m}} B_{\text{app}}$ to evaluate K_{m}.

(*a*) Relate the magnetic field on the axis of a solenoid to the current in the solenoid:

$$B_x = \mu_0 nI$$

Substitute numerical values and evaluate B_{app}:

$$B_{\text{app}} = \left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(50\,\text{cm}^{-1}\right)\left(2.00\,\text{A}\right) = \boxed{12.6\,\text{mT}}$$

(*b*) Relate M to B and B_{app}:

$$B = B_{\text{app}} + \mu_0 M \Rightarrow M = \frac{B - B_{\text{app}}}{\mu_0}$$

Substitute numerical values and evaluate M:

$$M = \frac{1.72\,\text{T} - 12.6\,\text{mT}}{4\pi\times10^{-7}\ \text{N/A}^2} = \boxed{1.36\times10^{6}\ \text{A/m}}$$

(*c*) Express B in terms of K_m and B_{app}:

$$B = K_m B_{app} \Rightarrow K_m = \frac{B}{B_{app}}$$

Substitute numerical values and evaluate K_m:

$$K_m = \frac{1.72\,\text{T}}{12.6\,\text{mT}} = \boxed{137}$$

72 •• When the current in Problem 71 is 0.200 A, the magnetic field is measured to be 1.58 T. (*a*) Neglecting end effects, what is the applied magnetic field? (*b*) What is the magnetization? (*c*) What is the relative permeability?

Picture the Problem We can use the equation describing the magnetic field on the axis of a solenoid, as a function of the current in the solenoid, to find B_{app}. We can then use $B = B_{app} + \mu_0 M$ to find M and $B = K_m B_{app}$ to evaluate K_m.

(*a*) Relate the magnetic field on the axis of the solenoid to the current in the solenoid:

$$B_x = \mu_0 nI$$

Substitute numerical values and evaluate B_{app}:

$$B_{app} = \left(4\pi \times 10^{-7}\,\frac{\text{N}}{\text{A}^2}\right)\left(50\,\text{cm}^{-1}\right)\left(0.200\,\text{A}\right) = 1.257\,\text{mT} = \boxed{1.26\,\text{mT}}$$

(*b*) Relate M to B and B_{app}:

$$B = B_{app} + \mu_0 M \Rightarrow M = \frac{B - B_{app}}{\mu_0}$$

Substitute numerical values and evaluate M:

$$M = \frac{1.58\,\text{T} - 1.257\,\text{mT}}{4\pi \times 10^{-7}\,\text{N/A}^2} = \boxed{1.26 \times 10^6\,\text{A/m}}$$

(*c*) Express B in terms of K_m and B_{app}:

$$B = K_m B_{app} \Rightarrow K_m = \frac{B}{B_{app}}$$

Substitute numerical values and evaluate K_m:

$$K_m = \frac{1.58\,\text{T}}{1.257\,\text{mT}} = \boxed{1.26 \times 10^3}$$

73 •• **[SSM]** A toroid has N turns, carries a current I, has a mean radius R, and has a cross-sectional radius r, where $r << R$ (Figure 27-53). The core of the toroid of is filled with iron. When the current is 10.0 A, the magnetic field in the region where the iron is has a magnitude of 1.80 T. (*a*) What is the

magnetization? (*b*) Find the values for the relative permeability, the permeability, and magnetic susceptibility for this iron sample.

Picture the Problem We can use $B = B_{app} + \mu_0 M$ and the expression for the magnetic field inside a tightly wound toroid to find the magnetization *M*. We can find K_m from its definition, $\mu = K_m \mu_0$ to find μ, and $K_m = 1 + \chi_m$ to find χ_m for the iron sample.

(*a*) Relate the magnetization to *B* and B_{app}:

$$B = B_{app} + \mu_0 M \Rightarrow M = \frac{B - B_{app}}{\mu_0}$$

Express the magnetic field inside a tightly wound toroid:

$$B_{app} = \frac{\mu_0 NI}{2\pi r}$$

Substitute for B_{app} and simplify to obtain:

$$M = \frac{B - \dfrac{\mu_0 NI}{2\pi r}}{\mu_0} = \frac{B}{\mu_0} - \frac{NI}{2\pi r}$$

Substitute numerical values and evaluate *M*:

$$M = \frac{1.80\,\text{T}}{4\pi \times 10^{-7}\ \text{N/A}^2} - \frac{2000(10.0\,\text{A})}{2\pi(0.200\,\text{m})} = \boxed{1.42 \times 10^6\ \text{A/m}}$$

(*b*) Use its definition to express K_m:

$$K_m = \frac{B}{B_{app}} = \frac{B}{\dfrac{\mu_0 NI}{2\pi r}} = \frac{2\pi r B}{\mu_0 NI}$$

Substitute numerical values and evaluate K_m:

$$K_m = \frac{2\pi(0.200\,\text{m})(1.80\,\text{T})}{(4\pi \times 10^{-7}\ \text{N/A}^2)(2000)(10.0\,\text{A})} = \boxed{90.0}$$

Now that we know K_m we can find μ using:

$$\mu = K_m \mu_0 = 90(4\pi \times 10^{-7}\ \text{N/A}^2) = \boxed{1.13 \times 10^{-4}\ \text{T} \cdot \text{m/A}}$$

Relate χ_m to K_m:

$$K_m = 1 + \chi_m \Rightarrow \chi_m = K_m - 1$$

Substitute the numerical value of K_m and evaluate χ_m:

$$\chi_m = 90 - 1 = \boxed{89}$$

74 •• The centers of the turns of a toroid form a circle with a radius of 14.0 cm. The cross-sectional area of each turn is 3.00 cm^2. It is wound with 5278 turns of fine wire, and the wire carries a current of 0.200 A. The core is filled with soft iron, which has a relative permeability of 500. What is the magnetic field strength in the core?

Picture the Problem We can substitute the expression for applied magnetic field ($B_{\text{app}} = \frac{\mu_0 NI}{2\pi r}$) in the defining equation for K_m ($B = K_m B_{\text{app}}$) to obtain an expression for the magnetic field B in the toroid.

Relate the magnetic field in the toroid to the relative permeability of its core:

$$B = K_m B_{\text{app}}$$

Express the applied magnetic field in the toroid in terms of the current in its winding:

$$B_{\text{app}} = \frac{\mu_0 NI}{2\pi r}$$

Substitute for B_{app} to obtain:

$$B = \frac{K_m \mu_0 NI}{2\pi r}$$

Substitute numerical values and evaluate B:

$$B = \frac{500\left(4\pi \times 10^{-7}\ \text{N/A}^2\right)(5278)(0.200\ \text{A})}{2\pi(14.0\ \text{cm})} = \boxed{0.754\ \text{T}}$$

75 ••• A long straight wire that has a radius of 1.00 mm is coated with an insulating ferromagnetic material that has a thickness of 3.00 mm and a relative magnetic permeability of 400. The coated wire is in air and the wire itself is nonmagnetic. The wire carries a current of 40.0 A. (*a*) Find the magnetic field in the region occupied by the inside of the wire as a function of the perpendicular distance, *r*, from the central axis of the wire. (*b*) Find the magnetic field in the region occupied by the inside of the ferromagnetic material as a function of the perpendicular distance, *r*, from the central axis of the wire. (*c*) Find the magnetic field in the region surrounding the wire and coating as a function of the perpendicular distance, *r*, from the central axis of the wire. (*d*)What must the magnitudes and directions of the Amperian currents be on the surfaces of the ferromagnetic material to account for the magnetic fields observed?

Picture the Problem We can use Ampère's law to obtain expressions for the magnetic field inside the wire, inside the ferromagnetic material, and in the region surrounding the wire and coating.

(*a*) Apply Ampère's law to a circle of radius r < 1.00 mm and concentric with the center of the wire:

$$\oint_C \vec{B}\cdot d\vec{\ell} = B(2\pi r) = \mu_0 I_C \qquad (1)$$

Assuming that the current is distributed uniformly over the cross-sectional area of the wire (uniform current density), express I_C in terms of the total current I:

$$\frac{I_C}{\pi r^2} = \frac{I}{\pi R^2} \Rightarrow I_C = \frac{r^2}{R^2} I$$

Substitute for I_C in equation (1) to obtain:

$$B(2\pi r) = \frac{\mu_0 I r^2}{R^2} \Rightarrow B = \frac{\mu_0 I}{2\pi R^2} r$$

Substitute numerical values and evaluate B:

$$B = \frac{(4\pi\times10^{-7}\ \mathrm{N/A^2})(40.0\ \mathrm{A})}{2\pi(1.00\ \mathrm{mm})^2} r = \boxed{(8.00\ \mathrm{T/m})r}$$

(*b*) Relate the magnetic field inside the ferromagnetic material to the magnetic field due to the current in the wire:

$$B = K_\mathrm{m} B_\mathrm{app} \qquad (1)$$

Apply Ampère's law to a circle of radius 1.00 mm < r < 4.00 mm and concentric with the center of the wire:

$$\int_C \vec{B}\cdot d\vec{\ell} = B_\mathrm{app}(2\pi r) = \mu_0 I_C = \mu_0 I$$

Solving for B_app yields:

$$B_\mathrm{app} = \frac{\mu_0 I}{2\pi r}$$

Substitute for B_app in equation (1) to obtain:

$$B = \frac{K_\mathrm{m}\mu_0 I}{2\pi r}$$

Substitute numerical values and evaluate B:

$$B = \frac{400(4\pi\times10^{-7}\ \mathrm{N/A^2})(40.0\ \mathrm{A})}{2\pi r} = \boxed{(3.20\times10^{-3}\ \mathrm{T\cdot m})\frac{1}{r}}$$

(*c*) Apply Ampère's law to a circle of radius $r > 4.00$ mm and concentric with the center of the wire:

$$\oint_C \vec{B}\cdot d\vec{\ell} = B(2\pi r) = \mu_0 I_C = \mu_0 I$$

Solving for B yields:

$$B = \frac{\mu_0 I}{2\pi r}$$

Substitute numerical values and evaluate B:

$$B = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(40.0\,\text{A})}{2\pi r}$$
$$= \boxed{\left(8.00\times10^{-6}\ \text{T}\cdot\text{m}\right)\frac{1}{r}}$$

(*d*) Note that the field in the ferromagnetic region is that which would be produced in a nonmagnetic region by a current of $400I$ = 1600 A. The ampèrian current on the inside of the surface of the ferromagnetic material must therefore be (1600 − 40) A = 1560 A in the direction of I. On the outside surface there must then be an ampèrian current of 1560 A in the opposite direction.

General Problems

76 • Find the magnetic field at point P in Figure 27-60.

Picture the Problem Because point P is on the line connecting the straight segments of the conductor, these segments do not contribute to the magnetic field at P. Hence, we can use the expression for the magnetic field at the center of a current loop to find B_P.

Express the magnetic field at the center of a current loop:

$$B = \frac{\mu_0 I}{2R}$$

where R is the radius of the loop.

Express the magnetic field at the center of half a current loop:

$$B = \frac{1}{2}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R}$$

Substitute numerical values and evaluate B:

$$B = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(15\,\text{A})}{4(0.20\,\text{m})}$$
$$= \boxed{24\,\mu\text{T out of the page}}$$

77 • [SSM] Using Figure 27-61, find the magnetic field (in terms of the parameters given in the figure) at point P, the common center of the two arcs.

Picture the Problem Let out of the page be the $+x$ direction. Because point P is on the line connecting the straight segments of the conductor, these segments do not contribute to the magnetic field at P. Hence, the resultant magnetic field at P will be the sum of the magnetic fields due to the current in the two semicircles, and we can use the expression for the magnetic field at the center of a current loop to find $\vec{B}_P$.

Express the resultant magnetic field at P:

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2 \qquad (1)$$

Express the magnetic field at the center of a current loop:

$$B = \frac{\mu_0 I}{2R}$$

where R is the radius of the loop.

Express the magnetic field at the center of half a current loop:

$$B = \frac{1}{2}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R}$$

Express $\vec{B}_1$ and $\vec{B}_2$:

$$\vec{B}_1 = \frac{\mu_0 I}{4R_1}\hat{\boldsymbol{i}} \text{ and } \vec{B}_2 = -\frac{\mu_0 I}{4R_2}\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\vec{B}_P = \frac{\mu_0 I}{4R_1}\hat{\boldsymbol{i}} - \frac{\mu_0 I}{4R_2}\hat{\boldsymbol{i}}$$

$$= \boxed{\frac{\mu_0 I}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\hat{\boldsymbol{i}}}$$

78 •• A wire of length ℓ, is wound into a circular coil of N turns, and carries a current I. Show that the magnetic field strength in the region occupied by the center of the coil is given by $\mu_0 \pi N^2 I/\ell$.

Picture the Problem We can express the magnetic field strength B as a function of N, I, and R using $B = \dfrac{\mu_0 NI}{2R}$ and eliminate R by relating ℓ to R.

Express the magnetic field at the center of a coil of N turns and radius R:

$$B = \frac{\mu_0 NI}{2R}$$

Relate ℓ to the number of turns N:

$$\ell = 2\pi RN \Rightarrow R = \frac{\ell}{2\pi N}$$

Substitute for R in the expression for B and simplify to obtain:

$$B = \frac{\mu_0 NI}{2\left(\frac{\ell}{2\pi N}\right)} = \boxed{\frac{\mu_0 \pi N^2 I}{\ell}}$$

79 •• A very long wire carrying a current I is bent into the shape shown in Figure 27-62. Find the magnetic field at point P.

Picture the Problem The magnetic field at P (which is out of the page) is the sum of the magnetic fields due to the three parts of the wire. Let the numerals 1, 2, and 3 denote the left-hand, center (short), and right-hand wires. Let out of the page be the $+x$ direction. We can then use the expression for B due to a straight wire segment to find each of these fields and their sum.

Express the resultant magnetic field at point P:

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$

Because $\vec{B}_1 = \vec{B}_3$

$$\vec{B}_P = 2\vec{B}_1 + \vec{B}_2 \qquad (1)$$

Express the magnetic field strength due to a straight wire segment:

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin\theta_1 + \sin\theta_2\right)$$

For wires 1 and 3 (the long wires), $\theta_1 = 90°$ and $\theta_2 = 45°$:

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{I}{a}\left(\sin 90° + \sin 45°\right)\hat{\boldsymbol{i}} = \frac{\mu_0}{4\pi}\frac{I}{a}\left(1 + \frac{1}{\sqrt{2}}\right)\hat{\boldsymbol{i}}$$

For wire 2, $\theta_1 = \theta_2 = 45°$:

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{I}{a}\left(\sin 45° + \sin 45°\right)\hat{\boldsymbol{i}} = \frac{\mu_0}{4\pi}\frac{I}{a}\left(\frac{2}{\sqrt{2}}\right)\hat{\boldsymbol{i}}$$

Substitute for $\vec{B}_1$ and $\vec{B}_2$ in equation (1) and simplify to obtain:

$$\vec{B}_P = 2\left[\frac{\mu_0}{4\pi}\frac{I}{a}\left(1+\frac{1}{\sqrt{2}}\right)\right]\hat{i} + \frac{\mu_0}{4\pi}\frac{I}{a}\left(\frac{2}{\sqrt{2}}\right)\hat{i} = \frac{\mu_0}{2\pi}\frac{I}{a}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\hat{i}$$

$$= \boxed{\frac{\mu_0}{2\pi}\frac{I}{a}\left(1+\sqrt{2}\right)\hat{i}}$$

80 •• A power cable carrying 50 A is 2.0 m below Earth's surface, but the cable's direction and precise position are unknown. Explain how you could locate the cable using a compass. Assume that you are at the equator, where Earth's magnetic field is horizontal and 0.700 G due north.

Picture the Problem Depending on the direction of the wire, the magnetic field due to its current (provided this field is a large enough fraction of Earth's magnetic field) will either add to or subtract from Earth's field and moving the compass over the ground in the vicinity of the wire will indicate the direction of the current.

Apply Ampère's law to a circle of radius r and concentric with the center of the wire:

$$\oint_C \vec{B}\cdot d\vec{\ell} = B_{\text{wire}}(2\pi r) = \mu_0 I_C = \mu_0 I$$

Solve for B to obtain:

$$B_{\text{wire}} = \frac{\mu_0 I}{2\pi r}$$

Substitute numerical values and evaluate B_{wire}:

$$B_{\text{wire}} = \frac{(4\pi\times10^{-7}\ \text{N/A}^2)(50\,\text{A})}{2\pi(2.0\,\text{m})} = 0.0500\,\text{G}$$

Express the ratio of B_{wire} to B_{Earth}:

$$\frac{B_{\text{wire}}}{B_{\text{Earth}}} = \frac{0.05\,\text{G}}{0.7\,\text{G}} \approx 7\%$$

Thus, the field of the current-carrying wire should be detectable with a good compass.

If the cable runs in a direction other than east-west, its magnetic field is in a direction different than that of Earth's, and by rotating the compass one should cause a change in the direction of the compass needle.

If the cable runs east-west, its magnetic field is in the north-south direction and thus either adds to or subtracts from Earth's field, depending on the current direction and location of the compass. If the magnetic field is toward the north, the two fields add and the resultant field is stronger. If perturbed, the compass needle will oscillate about its equilibrium position. The stronger the field, the higher the frequency of oscillation. By moving from place to place and systematically perturbing the needle one should be able to detect a change frequency, and thus a change in magnetic field strength.

81 •• [SSM] A long straight wire carries a current of 20.0 A, as shown in Figure 27-63. A rectangular coil that has two sides parallel to the straight wire has sides that are 5.00-cm long and 10.0-cm long. The side nearest to the wire is 2.00 cm from the wire. The coil carries a current of 5.00 A. (*a*) Find the force on each segment of the rectangular coil due to the current in the long straight wire. (*b*) What is the net force on the coil?

Picture the Problem Let I_1 and I_2 represent the currents of 20 A and 5.0 A, $\vec{F}_{\text{top}}$, $\vec{F}_{\text{left side}}$, $\vec{F}_{\text{bottom}}$, and $\vec{F}_{\text{right side}}$ the forces that act on the four segments of the rectangular current loop, and $\vec{B}_1$, $\vec{B}_2$, $\vec{B}_3$, and $\vec{B}_4$ the magnetic fields at these wire segments due to I_1. We'll need to take into account the fact that $\vec{B}_1$ and $\vec{B}_3$ are not constant over the segments 1 and 3 of the rectangular coil. Let the +*x* direction be to the right and the +*y* direction be upward. Then the +*z* direction is toward you (i.e., out of the page). Note that only the components of $\vec{B}_1$, $\vec{B}_2$, $\vec{B}_3$, and $\vec{B}_4$ into or out of the page contribute to the forces acting on the rectangular coil. The +*x* and +*y* directions are up the page and to the right.

(*a*) Express the force $d\vec{F}_1$ acting on a current element $I_2 d\vec{\ell}$ in the top segment of wire:

$$d\vec{F}_{\text{top}} = I_2 d\vec{\ell} \times \vec{B}_1$$

Because $I_2 d\vec{\ell} = I_2 d\ell\left(-\hat{i}\right)$ in this segment of the coil and the magnetic field due to I_1 is given by $\vec{B}_1 = \frac{\mu_0 I_1}{2\pi\,\ell}\left(-\hat{k}\right)$:

$$\vec{F}_{\text{top}} = I_2 d\ell\left(-\hat{i}\right)\times\frac{\mu_0 I_1}{2\pi\,\ell}\left(-\hat{k}\right) = -\frac{\mu_0 I_1 I_2}{2\pi}\frac{d\ell}{\ell}\hat{j}$$

Integrate $d\vec{F}_{\text{top}}$ to obtain:

$$\vec{F}_{\text{top}} = -\frac{\mu_0 I_1 I_2}{2\pi}\int_{2.0\,\text{cm}}^{7.0\,\text{cm}}\frac{d\ell}{\ell}\hat{j} = -\frac{\mu_0 I_1 I_2}{2\pi}\ln\left(\frac{7.0\text{ cm}}{2.0\text{ cm}}\right)\hat{j}$$

Substitute numerical values and evaluate $\vec{F}_{\text{top}}$:

$$\vec{F}_{\text{top}} = -\frac{\left(4\pi\times10^{-7}\ \frac{\text{N}}{\text{A}^2}\right)(20\text{ A})(5.0\text{ A})}{2\pi}\ln\left(\frac{7.0\text{ cm}}{2.0\text{ cm}}\right)\hat{j} = \boxed{-\left(2.5\times10^{-5}\text{ N}\right)\hat{j}}$$

Express the force $d\vec{F}_{\text{bottom}}$ acting on a current element $I_2 d\vec{\ell}$ in the horizontal segment of wire at the bottom of the coil:

$$d\vec{F}_{\text{bottom}} = I_2 d\vec{\ell}\times\vec{B}_3$$

Because $I_2 d\vec{\ell} = I_2 d\ell\left(\hat{i}\right)$ in this segment of the coil and the magnetic field due to I_1 is given by $\vec{B}_1 = \frac{\mu_0 I_1}{2\pi\,\ell}\left(-\hat{k}\right)$:

$$d\vec{F}_{\text{bottom}} = I_2 d\ell\hat{i}\times\frac{\mu_0 I_1}{2\pi\ell}\left(-\hat{k}\right) = \frac{\mu_0 I_1 I_2}{2\pi}\frac{d\ell}{\ell}\hat{j}$$

Integrate $d\vec{F}_{\text{bottom}}$ to obtain:

$$d\vec{F}_{\text{bottom}} = \frac{\mu_0 I_1 I_2}{2\pi}\int_{2.0\,\text{cm}}^{7.0\,\text{cm}}\frac{d\ell}{\ell}\hat{j} = \frac{\mu_0 I_1 I_2}{2\pi}\ln\left(\frac{7.0\text{ cm}}{2.0\text{ cm}}\right)\hat{j}$$

Substitute numerical values and evaluate $\vec{F}_{\text{bottom}}$:

$$\vec{F}_{\text{bottom}} = \frac{\left(4\pi\times10^{-7}\ \frac{\text{N}}{\text{A}^2}\right)(20\text{ A})(5.0\text{ A})}{2\pi}\ln\left(\frac{7.0\text{ cm}}{2.0\text{ cm}}\right)\hat{j} = \boxed{\left(2.5\times10^{-5}\text{ N}\right)\hat{j}}$$

Express the forces $\vec{F}_{\text{left side}}$ and $\vec{F}_{\text{right side}}$ in terms of I_2 and $\vec{B}_2$ and $\vec{B}_4$:

$$\vec{F}_{\text{left side}} = I_2\vec{\ell}_2\times\vec{B}_2$$

and

$$\vec{F}_{\text{right side}} = I_2\vec{\ell}_4\times\vec{B}_4$$

Express $\vec{B}_2$ and $\vec{B}_4$:

$$\vec{B}_2 = -\frac{\mu_0}{4\pi}\frac{2I_1}{R_1}\hat{k} \text{ and } \vec{B}_4 = -\frac{\mu_0}{4\pi}\frac{2I_1}{R_4}\hat{k}$$

Substitute for $\vec{B}_2$ and $\vec{B}_4$ to obtain:

$$\vec{F}_{\text{left side}} = -I_2\ell_2\hat{j}\times\left(-\frac{\mu_0}{4\pi}\frac{2I_1}{R_1}\hat{k}\right) = \frac{\mu_0\ell_2 I_1 I_2}{2\pi R_2}\hat{i}$$

and

$$\vec{F}_{\text{right side}} = I_2\ell_4\hat{j}\times\left(-\frac{\mu_0}{4\pi}\frac{2I_1}{R_4}\hat{k}\right) = -\frac{\mu_0\ell_4 I_1 I_2}{2\pi R_4}\hat{i}$$

Substitute numerical values and evaluate $\vec{F}_{\text{left side}}$ and $\vec{F}_{\text{right side}}$:

$$\vec{F}_{\text{left side}} = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(0.100\,\text{m})(20.0\,\text{A})(5.00\,\text{A})}{2\pi(0.0200\,\text{m})}\hat{i} = \boxed{\left(1.0\times10^{-4}\ \text{N}\right)\hat{i}}$$

and

$$\vec{F}_{\text{right side}} = -\frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(0.100\,\text{m})(20.0\,\text{A})(5.00\,\text{A})}{2\pi(0.0700\,\text{m})}\hat{i} = \boxed{\left(-0.29\times10^{-4}\ \text{N}\right)\hat{i}}$$

(*b*) Express the net force acting on the coil:

$$\vec{F}_{\text{net}} = \vec{F}_{\text{top}} + \vec{F}_{\text{left side}} + \vec{F}_{\text{bottom}} + \vec{F}_{\text{right side}}$$

Substitute for $\vec{F}_{\text{top}}$, $\vec{F}_{\text{left side}}$, $\vec{F}_{\text{bottom}}$, and $\vec{F}_{\text{right side}}$ and simplify to obtain:

$$\begin{aligned}\vec{F}_{\text{net}} &= \left(-2.5\times10^{-5}\ \text{N}\right)\hat{j} + \left(1.0\times10^{-4}\ \text{N}\right)\hat{i} + \left(2.5\times10^{-5}\ \text{N}\right)\hat{j} + \left(-0.29\times10^{-4}\ \text{N}\right)\hat{i}\\ &= \boxed{\left(0.71\times10^{-4}\ \text{N}\right)\hat{i}}\end{aligned}$$

82 •• The closed loop shown in Figure 27-64 carries a current of 8.0 A in the counterclockwise direction. The radius of the outer arc is 0.60 m and that of the inner arc is 0.40 m. Find the magnetic field at point *P*.

Picture the Problem Let out of the page be the positive *x* direction and the numerals 40 and 60 refer to the circular arcs whose radii are 40 cm and 60 cm. Because point *P* is on the line connecting the straight segments of the conductor, these segments do not contribute to the magnetic field at *P*. Hence the resultant magnetic field at *P* will be the sum of the magnetic fields due to the current in the two circular arcs and we can use the expression for the magnetic field at the center of a current loop to find $\vec{B}_P$.

Express the resultant magnetic field at P:

$$\vec{B}_P = \vec{B}_{40} + \vec{B}_{60} \qquad (1)$$

Express the magnetic field strength at the center of a current loop:

$$B = \frac{\mu_0 I}{2R}$$

where R is the radius of the loop.

Express the magnetic field strength at the center of one-sixth of a current loop:

$$B = \frac{1}{6}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{12R}$$

Express $\vec{B}_{40}$ and $\vec{B}_{60}$:

$$\vec{B}_{40} = -\frac{\mu_0 I}{12R_{40}}\hat{\boldsymbol{i}} \text{ and } \vec{B}_{60} = \frac{\mu_0 I}{12R_{60}}\hat{\boldsymbol{i}}$$

Substitute for $\vec{B}_{40}$ and $\vec{B}_{60}$ in equation (1) and simplify to obtain:

$$\vec{B}_P = -\frac{\mu_0 I}{12R_{40}}\hat{\boldsymbol{i}} + \frac{\mu_0 I}{12R_{60}}\hat{\boldsymbol{i}} = \frac{\mu_0 I}{12}\left(\frac{1}{R_{60}} - \frac{1}{R_{40}}\right)\hat{\boldsymbol{i}}$$

Substitute numerical values and evaluate $\vec{B}_P$:

$$\vec{B}_P = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(8.0\ \text{A})}{12}\left(\frac{1}{0.60\ \text{m}} - \frac{1}{0.40\ \text{m}}\right)\hat{\boldsymbol{i}} = (-0.70\ \mu\text{T})\hat{\boldsymbol{i}}$$

or

$$B_P = \boxed{0.70\ \mu\text{T into the page}}$$

83 •• A closed circuit consists of two semicircles of radii 40 cm and 20 cm that are connected by straight segments, as shown in Figure 27-65. A current of 3.0 A exists in this circuit and has a clockwise direction. Find the magnetic field at point P.

Picture the Problem Let the $+x$ direction be into the page and the numerals 20 and 40 refer to the circular arcs whose radii are 20 cm and 40 cm. Because point P is on the line connecting the straight segments of the conductor, these segments do not contribute to the magnetic field at P and the resultant field at P is the sum of the fields due to the two semicircular current loops.

Express the resultant magnetic field at P:

$$\vec{B}_P = \vec{B}_{20} + \vec{B}_{40} \qquad (1)$$

Express the magnetic field strength at the center of a circular current loop:

$$B = \frac{\mu_0 I}{2R}$$

where R is the radius of the loop.

Express the magnetic field strength at the center of half a circular current loop:

$$B = \frac{1}{2}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R}$$

Express $\vec{B}_{20}$ and $\vec{B}_{40}$:

$$\vec{B}_{20} = \frac{\mu_0 I}{4R_{20}}\hat{i} \text{ and } \vec{B}_{40} = \frac{\mu_0 I}{4R_{40}}\hat{i}$$

Substitute for $\vec{B}_{20}$ and $\vec{B}_{40}$ in equation (1) and simplify to obtain:

$$\vec{B}_P = \frac{\mu_0 I}{4R_{20}}\hat{i} + \frac{\mu_0 I}{4R_{40}}\hat{i} = \frac{\mu_0 I}{4}\left(\frac{1}{R_{20}} + \frac{1}{R_{40}}\right)\hat{i}$$

Substitute numerical values and evaluate B_P:

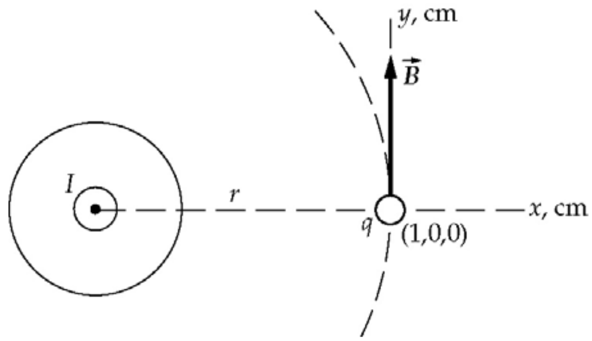
$$\vec{B}_P = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)(3.0\ \text{A})}{4}\left(\frac{1}{0.20\ \text{m}} + \frac{1}{0.40\ \text{m}}\right)\hat{i} = \boxed{(7.1\ \mu\text{T})\hat{i}}$$

or

$$B_P = \boxed{7.1\ \mu\text{T into the page}}$$

84 •• A very long straight wire carries a current of 20.0 A. An electron outside the wire is 1.00 cm from the central axis of the wire is moving with a speed of 5.00×10^6 m/s. Find the force on the electron when it moves (*a*) directly away from the wire, (*b*) parallel to the wire in the direction of the current, and (*c*) perpendicular to the central axis of wire and tangent to a circle that is coaxial with the wire.

Picture the Problem Chose the coordinate system shown below. Then the current is in the +z direction. Assume that the electron is at (1.00 cm, 0, 0). We can use $\vec{F} = q\vec{v}\times\vec{B}$ to relate the magnetic force on the electron to $\vec{v}$ and $\vec{B}$ and $\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{r}\hat{j}$ to express the magnetic field at the location of the electron. We'll need to express $\vec{v}$ for each of the three situations described in the problem in order to evaluate $\vec{F} = q\vec{v}\times\vec{B}$.



Express the magnetic force acting on the electron:

$$\vec{F} = q\vec{v} \times \vec{B}$$

Express the magnetic field due to the current in the wire as a function of distance from the wire:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{r}\hat{j}$$

Substitute for $\vec{B}$ and simplifying yields:

$$\vec{F} = q\vec{v} \times \frac{\mu_0}{4\pi}\frac{2I}{r}\hat{j} = \frac{2q\mu_0 I}{4\pi r}\left(\vec{v} \times \hat{j}\right) \quad (1)$$

(*a*) Express the velocity of the electron when it moves directly away from the wire:

$$\vec{v} = v\hat{i}$$

Substitute for $\vec{v}$ in equation (1) and simplify to obtain:

$$\vec{F} = \frac{2q\mu_0 I}{4\pi r}\left(v\hat{i} \times \hat{j}\right) = \frac{2q\mu_0 Iv}{4\pi r}\hat{k}$$

Substitute numerical values and evaluate $\vec{F}$:

$$\vec{F} = \frac{2\left(4\pi \times 10^{-7}\ \text{N/A}^2\right)\left(-1.602 \times 10^{-19}\ \text{C}\right)\left(5.00 \times 10^{6}\ \text{m/s}\right)\left(20.0\ \text{A}\right)\hat{k}}{4\pi\left(0.0100\ \text{m}\right)}$$
$$= \left(-3.20 \times 10^{-16}\ \text{N}\right)\hat{k}$$

and

$$F = \boxed{3.20 \times 10^{-16}\ \text{N into the page}}$$

(*b*) Express $\vec{v}$ when the electron is traveling parallel to the wire in the direction of the current:

$$\vec{v} = v\hat{k}$$

Substitute in equation (1) to obtain:

$$\vec{F} = \frac{2q\mu_0 I}{4\pi r}\left(v\hat{k} \times \hat{j}\right) = -\frac{2q\mu_0 Iv}{4\pi r}\hat{i}$$

Substitute numerical values and evaluate $\vec{F}$:

$$\vec{F} = -\frac{2(4\pi\times10^{-7}\ \mathrm{N/A^2})(-1.602\times10^{-19}\ \mathrm{C})(5.00\times10^{6}\ \mathrm{m/s})(20.0\,\mathrm{A})\hat{\boldsymbol{i}}}{4\pi(0.0100\,\mathrm{m})}$$
$$= (3.20\times10^{-16}\ \mathrm{N})\hat{\boldsymbol{i}}$$

and

$$F = \boxed{3.20\times10^{-16}\ \text{N toward the right}}$$

(*c*) Express $\vec{v}$ when the electron is traveling perpendicular to the wire and tangent to a circle around the wire:

$$\vec{v} = v\hat{\boldsymbol{j}}$$

Substitute for $\vec{v}$ in equation (1) and simplify to obtain:

$$\vec{F} = \frac{2q\mu_0 I}{4\pi r}\left(v\hat{\boldsymbol{j}}\times\hat{\boldsymbol{j}}\right) = \boxed{0}$$

85 •• A current of 5.00 A is uniformly distributed over the cross section of a long straight wire of radius $R_o = 2.55$ mm. Using a **spreadsheet** program, graph the magnetic field strength as a function of R, the distance from the central axis of the wire, for $0 \le R \le 10R_0$.

Picture the Problem We can apply Ampère's law to derive expressions for the magnetic field strength as a function of the distance from the center of the wire.

Apply Ampère's law to a closed circular path of radius $r \le R_0$ to obtain:

$$B_{r\le R_0}(2\pi r) = \mu_0 I_C$$

Because the current is uniformly distributed over the cross section of the wire:

$$\frac{I_C}{\pi r^2} = \frac{I}{\pi R_0^2} \Rightarrow I_C = \frac{r^2}{R_0^2}I$$

Substitute for I_C to obtain:

$$B_{r\le R_0}(2\pi r) = \frac{\mu_0 r^2 I}{R_0^2}$$

Solving for $B_{r\le R_0}$ yields:

$$B_{r\le R_0} = \frac{\mu_0 r I}{2\pi R_0^2} = \frac{\mu_0}{4\pi}\frac{2I}{R_0^2}r \qquad (1)$$

Apply Ampère's law to a closed circular path of radius $r > R_0$ to obtain:

$$B_{r>R_0}(2\pi r) = \mu_0 I_C = \mu_0 I$$

Solving for $B_{r>R_0}$ yields:

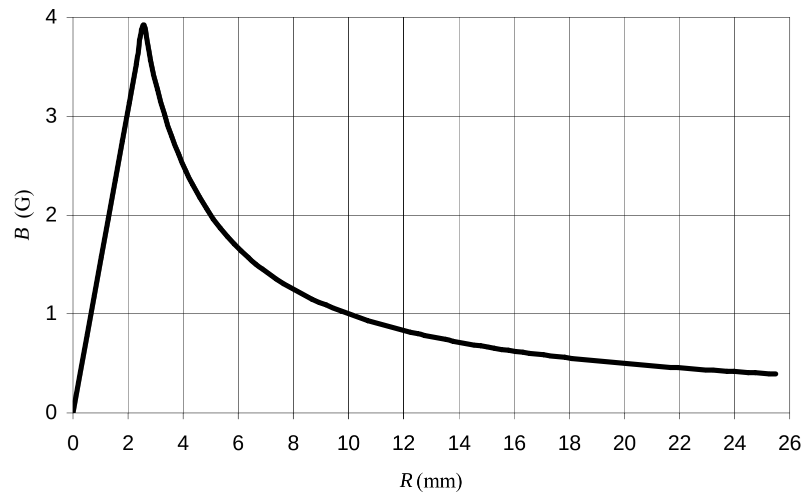
$$B_{r>r_0} = \frac{\mu_0}{4\pi}\frac{2I}{r} \tag{2}$$

The spreadsheet program to calculate B as a function of r in the interval $0 \le r \le 10R_0$ follows. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	1.00E−07	$\frac{\mu_0}{4\pi}$
B2	5.00	I
B3	2.55E−03	R_0
C6	10^4*$B$1*2*$B$2*A6/$B$3^2	$\frac{\mu_0}{4\pi}\frac{2I}{R_0^2}r$
C17	10^4*$B$1*2*$B$2*A6/A17	$\frac{\mu_0}{4\pi}\frac{2I}{r}$

	A	B	C
1	$\mu/4\pi$=	1.00E−07	N/A^2
2	I=	5	A
3	R_0=	2.55E−03	m
4			
5	R (m)	R (mm)	B (T)
6	0.00E+00	0.00E+00	0.00E+00
7	2.55E−04	2.55E−01	3.92E−01
8	5.10E−04	5.10E−01	7.84E−01
9	7.65E−04	7.65E−01	1.18E+00
10	1.02E−03	1.02E+00	1.57E+00
102	2.45E−02	2.45E+01	4.08E−01
103	2.47E−02	2.47E+01	4.04E−01
104	2.50E−02	2.50E+01	4.00E−01
105	2.52E−02	2.52E+01	3.96E−01
106	2.55E−02	2.55E+01	3.92E−01

A graph of B as a function of r follows.



86 •• A 50-turn coil of radius 10.0 cm carries a current of 4.00 A and a concentric 20-turn coil of radius 0.500 cm carries a current of 1.00 A. The planes of the two coils are perpendicular. Find the magnitude of the torque exerted by the large coil on the small coil. (Neglect any variation in magnetic field due to the current in the large coil over the region occupied by the small coil.)

Picture the Problem We can use $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$ to find the torque exerted on the small coil (magnetic moment = $\vec{\boldsymbol{\mu}}$) by the magnetic field $\vec{\boldsymbol{B}}$ due to the current in the large coil.

Relate the torque exerted by the large coil on the small coil to the magnetic moment $\vec{\boldsymbol{\mu}}$ of the small coil and the magnetic field $\vec{\boldsymbol{B}}$ due to the current in the large coil:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$$

or, because the planes of the two coils are perpendicular, $\tau = \mu B$

Express the magnetic moment of the small coil:

$$\mu = NIA$$

where I is the current in the coil, N is the number of turns in the coil, and A is the cross-sectional area of the coil.

Express the magnetic field at the center of the large coil:

$B = \dfrac{N'\mu_0 I'}{2R}$ where I' is the current in the large coil, N' is the number of turns in the coil, and R is its radius.

Substitute for B and μ in the expression for τ to obtain:

$$\tau = \frac{NN'II'A\mu_0}{2R}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{(50)(20)(4.00\,\text{A})(1.00\,\text{A})\pi(0.500\,\text{cm})^2\left(4\pi\times10^{-7}\ \text{N/A}^2\right)}{2(10.0\,\text{cm})} = \boxed{1.97\,\mu\text{N}\cdot\text{m}}$$

87 •• The magnetic needle of a compass is a uniform rod with a length of 3.00 cm, a radius of 0.850 mm, and a density of 7.96×10^3 kg/m^3. The needle is free to rotate in a horizontal plane, where the horizontal component of Earth's magnetic field is 0.600 G. When disturbed slightly, the compass executes simple harmonic motion about its midpoint with a frequency of 1.40 Hz. (*a*) What is the magnetic dipole moment of the needle? (*b*) What is the magnetization of the needle? (*c*) What is the amperian current on the surface of the needle?

Picture the Problem (*a*) We can solve the equation for the frequency f of the compass needle for the magnetic dipole moment of the needle. In Parts (*b*) and (*c*) we can use their definitions to find the magnetization M and the amperian current I_{amperian}.

(*a*) The frequency of the compass needle is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{\mu B}{I}} \Rightarrow \mu = \frac{4\pi^2 f^2 I}{B}$$

where I is the moment of inertia of the needle.

The moment of inertia of the needle is:

$$I = \tfrac{1}{12}mL^2 = \tfrac{1}{12}\rho VL^2 = \tfrac{1}{12}\rho\pi r^2 L^3$$

Substitute for I to obtain:

$$\mu = \frac{\pi^3 f^2 \rho r^2 L^3}{3B}$$

Substitute numerical values and evaluate μ:

$$\mu = \frac{\pi^3\left(1.40\,\text{s}^{-1}\right)^2\left(7.96\times10^3\ \text{kg/m}^3\right)\left(0.850\times10^{-3}\ \text{m}\right)^2(0.0300\,\text{m})^3}{3\left(0.600\times10^{-4}\ \text{T}\right)} = \boxed{5.24\times10^{-2}\ \text{A}\cdot\text{m}^2}$$

(*b*) Use its definition to express the magnetization M:

$$M = \frac{\mu}{V}$$

Substitute to obtain:

$$M=\frac{\mu}{V}=\frac{\pi^3 f^2\rho r^2L^3}{3BV}=\frac{\pi^2 f^2\rho L^2}{3B}$$

Substitute numerical values and evaluate M:

$$M=\frac{\pi^2\left(1.40\,\text{s}^{-1}\right)^2\left(7.96\times10^3\ \text{kg/m}^3\right)\left(0.0300\,\text{m}\right)^2}{3\left(0.600\times10^{-4}\ \text{T}\right)}=\boxed{7.70\times10^5\ \text{A/m}}$$

(*c*) The amperian current on the surface of the needle is:

$$I_{\text{amperian}}=ML=\left(7.70\times10^5\ \text{A/m}\right)\left(0.0300\,\text{m}\right)=\boxed{23.1\text{kA}}$$

88 •• A relatively inexpensive ammeter, called a *tangent galvanometer*, can be made using Earth's magnetic field. A plane circular coil that has N turns and a radius R is oriented so the magnetic field B_c it produces in the center of the coil is either east or west. A compass is placed at the center of the coil. When there is no current in the coil, assume the compass needle points due north. When there is a current in the coil (I), the compass needle points in the direction of the resultant magnetic field at an angle θ to the north. Show that the current I is related to θ and to the horizontal component of Earth's magnetic field B_e by $I=\dfrac{2RB_e}{\mu_0 N}\tan\theta$.

Picture the Problem Note that B_e and B_c are perpendicular to each other and that the resultant magnetic field is at an angle θ with north. We can use trigonometry to relate B_c and B_e and express B_c in terms of the geometry of the coil and the current flowing in it.

Express B_c in terms of B_e:

$$B_c=B_e\tan\theta$$

where θ is the angle of the resultant field from north.

Express the field B_c due to the current in the coil:

$$B_c=\frac{N\mu_0 I}{2R}$$

where N is the number of turns.

Substitute for B_c to obtain:

$$\frac{N\mu_0 I}{2R}=B_e\tan\theta\Rightarrow I=\boxed{\frac{2RB_e}{\mu_0 N}\tan\theta}$$

89 •• Earth's magnetic field is about 0.600 G at the magnetic poles, and is pointed vertically downward at the magnetic pole in the northern hemisphere. If the magnetic field were due to an electric current circulating in a loop at the radius of the inner iron core of Earth (approximately 1300 km), (*a*) what would be the magnitude of the current required? (*b*) What direction would this current have—the same as Earth's spin, or opposite? Explain your answer.

Picture the Problem The current required can be found by solving the equation for the magnetic field on the axis of a current loop for the current in the loop. We can use the right-hand rule to determine the direction of this current.

(*a*) The magnetic field on the axis of a current loop is given by:

$$B_x = \frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{\left(x^2 + R^2\right)^{3/2}}$$

Solving for I yields:

$$I = \frac{4\pi\left(x^2 + R^2\right)^{3/2} B_x}{2\pi\mu_0 R^2}$$

Substitute numerical values and evaluate I:

$$I = \frac{4\pi\left((6370\,\text{km})^2 + (1300\,\text{km})^2\right)^{3/2}\left(0.600\,\text{G}\times\frac{1\,\text{T}}{10^4\,\text{G}}\right)}{2\pi\left(4\pi\times10^{-7}\,\frac{\text{N}}{\text{A}^2}\right)(1300\,\text{km})^2} = \boxed{15.5\,\text{GA}}$$

(*b*) Because Earth's magnetic field points down at the north pole, application of the right-hand rule indicates that the current is opposite the spin direction of Earth.

90 •• A long, narrow bar magnet has its magnetic moment $\vec{\boldsymbol{\mu}}$ parallel to its long axis and is suspended at its center—in essence becoming a frictionless compass needle. When the magnet is placed in a magnetic field $\vec{\boldsymbol{B}}$, it lines up with the field. If it is displaced by a small angle and released, show that the magnet will oscillate about its equilibrium position with frequency given by $\frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}$, where I is the moment of inertia about the point of suspension.

Picture the Problem We can apply Newton's second law for rotational motion to obtain the differential equation of motion of the bar magnet. While this equation is not linear, we can use a small-angle approximation to render it linear and obtain an expression for the square of the angular frequency that we can solve for the frequency f of the motion.

Apply Newton's second law to the bar magnet to obtain the differential equation of motion for the magnet:

$$-\mu B\sin\theta = I\frac{d^2\theta}{dt^2}$$

where I is the moment of inertia of the magnet about an axis through its point of suspension.

For small displacements from equilibrium ($\theta << 1$):

$$-\mu B\theta \approx I\frac{d^2\theta}{dt^2}$$

Rewrite the differential equation as:

$$I\frac{d^2\theta}{dt^2} + \mu B\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{\mu B}{I}\theta = 0$$

Because the coefficient of the linear term is the square of the angular frequency, we have:

$$\omega^2 = \frac{\mu B}{I} \Rightarrow \omega = \sqrt{\frac{\mu B}{I}}$$

Because $\omega = 2\pi f$:

$$2\pi f = \sqrt{\frac{\mu B}{I}} \Rightarrow f = \boxed{\frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}}$$

91 •• An infinitely long straight wire is bent, as shown in Figure 27-66. The circular portion has a radius of 10.0 cm and its center a distance r from the straight part. Find r so that the magnetic field at the region occupied by the center of the circular portion is zero.

Picture the Problem Let the +x direction be out of the page. We can use the expressions for the magnetic fields due to an infinite straight line and a circular loop to express the net magnetic field at the center of the circular loop. We can set this net field to zero and solve for r.

Express the net magnetic field at the center of circular loop:

$$\vec{B} = \vec{B}_{\text{loop}} + \vec{B}_{\text{line}} \qquad (1)$$

Letting R represent the radius of the loop, express $\vec{B}_{\text{loop}}$:

$$\vec{B}_{\text{loop}} = -\frac{\mu_0 I}{2R}\hat{i}$$

Express the magnetic field due to the current in the infinite straight line:

$$\vec{B}_{\text{line}} = \frac{\mu_0 I}{2\pi r}\hat{i}$$

Substitute for $\vec{B}_{\text{loop}}$ and $\vec{B}_{\text{line}}$ in equation (1) and simplify to obtain:

$$\vec{B} = -\frac{\mu_0 I}{2R}\hat{i} + \frac{\mu_0 I}{2\pi r}\hat{i} = \left(-\frac{\mu_0 I}{2R} + \frac{\mu_0 I}{2\pi r}\right)\hat{i}$$

If $\vec{B} = 0$, then:

$$-\frac{\mu_0 I}{2R} + \frac{\mu_0 I}{2\pi r} = 0 \Rightarrow -\frac{1}{R} + \frac{1}{\pi r} = 0$$

Solving for r yields:

$$r = \frac{10.0\,\text{cm}}{\pi} = \boxed{3.18\,\text{cm}}$$

92 •• (*a*) Find the magnetic field strength point *P* on the perpendicular bisector of a wire segment carrying current *I*, as shown in Figure 27-67. (*b*) Use your result from Part (*a*) to find the magnetic field strength at the center of a regular polygon of *N* sides. (*c*) Show that when *N* is very large, your result approaches that for the magnetic field strength at the center of a circle.

Picture the Problem (*a*) We can use the expression for *B* due to a straight wire segment, to find the magnetic field strength at *P*. Note that the current in the wires whose lines contain point *P* do not contribute to the magnetic field strength at point *P*. In Part (*b*) we can use our result from (*a*), together with the value for θ when the polygon has *N* sides, to obtain an expression for *B* at the center of a polygon of *N* sides. (*c*) Letting *N* grow without bound will yield the equation for the magnetic field strength at the center of a circle.

(*a*) Express the magnetic field strength at *P* due to the straight wire segment:

$$B_P = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin\theta_1 + \sin\theta_2\right)$$

Because $\theta_1 = \theta_2 = \theta$:

$$B_P = \frac{\mu_0}{4\pi}\frac{I}{R}\left(2\sin\theta\right) = \left(\frac{\mu_0}{2\pi}\frac{I}{R}\right)\sin\theta$$

Refer to the figure to obtain:

$$\sin\theta = \frac{a}{\sqrt{a^2 + R^2}}$$

Substituting for $\sin\theta$ in the expression for B_P yields:

$$B_P = \boxed{\frac{\mu_0 a I}{2\pi R\sqrt{a^2 + R^2}}}$$

(*b*) θ for an *N*-sided polygon is given by:

$$\theta = \frac{\pi}{N}$$

Because each side of the polygon contributes to B an amount equal to that obtained in (a):

$$B = \boxed{\left(\frac{N\mu_0 I}{2\pi R}\right)\sin\left(\frac{\pi}{N}\right)},\ N = 3, 4, \ldots K$$

(c) For large N, π/N is small, so $\sin\left(\frac{\pi}{N}\right) \approx \frac{\pi}{N}$. Hence:

$$B_\infty = \text{Limit}_{N\to\infty} N\frac{\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{N}\right)$$

$$= \frac{\mu_0 I}{2\pi R}\text{Limit}_{N\to\infty} N\frac{\pi}{N} = \boxed{\frac{\mu_0 I}{2R}}$$

the expression for the magnetic field strength at the center of a current-carrying circular loop.

93 •• The current in a long cylindrical conductor of radius 10 cm varies with distance from the axis of the cylinder according to the relation $I(r) = (50\ \text{A/m})r$. Find the magnetic field at the following perpendicular distances from the wire's central axis (a) 5.0 cm, (b) 10 cm, and (c) 20 cm.

Picture the Problem We can use Ampère's law to derive expressions for $B(r)$ for $r \le R$ and $r > R$ that we can evaluate for the given distances from the center of the cylindrical conductor.

Apply Ampère's law to a closed circular path a distance $r \le R$ from the center of the cylindrical conductor to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = B(r)(2\pi r) = \mu_0 I_C = \mu_0 I(r)$$

Solve for $B(r)$ to obtain:

$$B(r) = \frac{\mu_0 I(r)}{2\pi r}$$

Substitute for $I(r)$:

$$B(r) = \frac{\mu_0(50\ \text{A/m})r}{2\pi r} = \frac{\mu_0(50\ \text{A/m})}{2\pi}$$

(a) and (b) Noting that B is independent of r, substitute numerical values and evaluate B(5.0 cm) and B(10 cm):

$$B(5.0\,\text{cm}) = B(10\,\text{cm}) = \frac{(4\pi\times10^{-7}\ \text{N/A}^2)(50\,\text{A/m})}{2\pi} = \boxed{10\,\mu\text{T}}$$

(*c*) Apply Ampère's law to a closed circular path a distance $r > R$ from the center of the cylindrical conductor to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = B(r)(2\pi r) = \mu_0 I_C = \mu_0 I(R)$$

Solving for $B(r)$ yields:

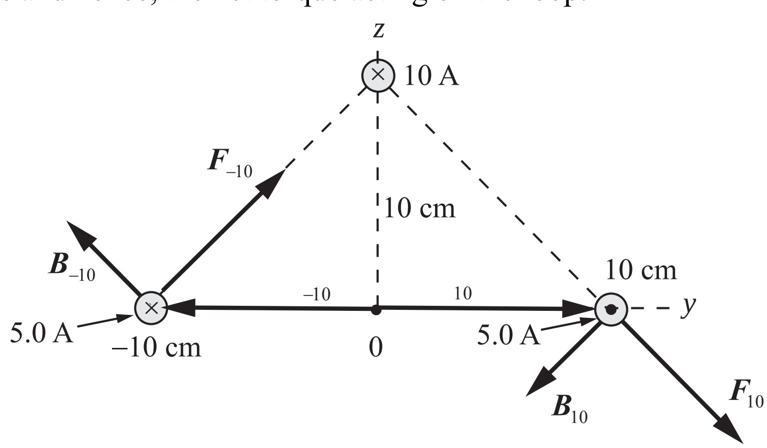
$$B(r) = \frac{\mu_0 I(R)}{2\pi r}$$

Substitute numerical values and evaluate $B(20\text{ cm})$:

$$B(20\,\text{cm}) = \frac{(4\pi\times10^{-7}\ \text{N/A}^2)(50\,\text{A/m})(0.10\,\text{m})}{2\pi(0.20\,\text{m})} = \boxed{5.0\,\mu\text{T}}$$

94 •• Figure 27- 68 shows a square loop that has 20-cm long sides and is in the $z = 0$ plane with its center at the origin. The loop carries a current of 5.0 A. An infinitely long wire that is parallel to the *x* axis and carries a current of 10 A intersects the *z* axis at $z = 10$ cm. The directions of the currents are shown in the figure. (*a*) Find the net torque on the loop. (*b*) Find the net force on the loop.

Picture the Problem The field $\vec{B}$ due to the 10-A current is in the *yz* plane. The net force on the wires of the square in the *y* direction cancel and do not contribute to a net torque or force. We can use $\vec{\tau} = \vec{\ell}\times\vec{F}$, $\vec{F} = I\vec{\ell}\times\vec{B}$, and the expression for the magnetic field due to a long straight wire to express the torque acting on each of the wires and hence, the net torque acting on the loop.



(*a*) The net torque about the *x* axis is the sum of the torques due to the forces $\vec{F}_{10}$ and $\vec{F}_{-10}$:

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{10} + \vec{\tau}_{-10}$$

Substituting for $\vec{\boldsymbol{\tau}}_{10}$ and $\vec{\boldsymbol{\tau}}_{-10}$ yields:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\ell}_{10} \times \vec{\boldsymbol{F}}_{10} + \vec{\ell}_{-10} \times \vec{\boldsymbol{F}}_{-10}$$

where the subscripts refer to the positions of the current-carrying wires.

The forces acting on the wires are given by:

$$\vec{\boldsymbol{F}}_{10} = \left(I\vec{\ell}\right)_{10} \times \vec{\boldsymbol{B}}_{10}$$

and

$$\vec{\boldsymbol{F}}_{-10} = \left(I\vec{\ell}\right)_{-10} \times \vec{\boldsymbol{B}}_{-10}$$

Substitute for $\vec{\boldsymbol{F}}_{10}$ and $\vec{\boldsymbol{F}}_{-10}$ to obtain:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\ell}_{10} \times \left[\left(I\vec{\ell}\right)_{10} \times \vec{\boldsymbol{B}}_{10}\right] + \vec{\ell}_{-10} \times \left[\left(I\vec{\ell}\right)_{-10} \times \vec{\boldsymbol{B}}_{-10}\right] \qquad (1)$$

The lever arms for the forces acting on the wires at $y = 10$ cm and $y = -10$ cm are:

$$\vec{\ell}_{10} = (0.10\text{ m})\hat{j} \text{ and } \vec{\ell}_{-10} = -(0.10\text{ m})\hat{j}$$

The magnetic field at the wire at $y = 10$ cm is given by:

$$\vec{\boldsymbol{B}}_{10} = \frac{\mu_0}{4\pi}\frac{2I}{R}\frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$$

where

$$R = \sqrt{(0.10\text{m})^2 + (0.10\text{m})^2} = 0.141\text{m}.$$

Substitute numerical values and evaluate $\vec{\boldsymbol{B}}_{10}$:

$$\vec{\boldsymbol{B}}_{10} = \frac{4\pi \times 10^{-7}\text{ N/A}^2}{4\pi\sqrt{2}}\frac{2(10\text{ A})}{0.141\text{m}}\left(-\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right) = (10.0\,\mu\text{T})\left(-\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$$

Proceed similarly to obtain:

$$\vec{\boldsymbol{B}}_{-10} = (10.0\,\mu\text{T})\left(-\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$$

Substitute in equation (1) and simplify to obtain:

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{net}} &= (0.10\text{ m})\hat{\boldsymbol{j}} \times \left[(5.0\text{ A})(0.20\text{ m})\hat{\boldsymbol{i}} \times (10.0\,\mu\text{T})\left(-\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)\right] \\ &\quad - (0.10\text{ m})\hat{\boldsymbol{j}} \times \left[(5.0\text{ A})(0.20\text{ m})\left(-\hat{\boldsymbol{i}}\right) \times (10.0\,\mu\text{T})\left(-\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)\right] \\ &= \boxed{-(2.0\,\mu\text{N}\cdot\text{m})\hat{\boldsymbol{i}}}\end{aligned}$$

(*b*) The net force acting on the loop is the sum of the forces acting on its four sides:

$$\vec{\boldsymbol{F}}_{\text{net}} = \vec{\boldsymbol{F}}_{10} + \vec{\boldsymbol{F}}_{-10} \qquad (2)$$

Evaluate $\vec{F}_{10}$ to obtain:

$$\begin{aligned}\vec{F}_{10} &= \left(I\vec{\ell}\right)_{10} \times \vec{B}_{10} = (5.0\,\text{A})(0.20\,\text{m})\hat{i} \times (10.0\,\mu\text{T})\left(-\hat{j} - \hat{k}\right) \\ &= (10\,\mu\text{N})\left[\hat{i} \times \left(-\hat{j} - \hat{k}\right)\right] \\ &= (10\,\mu\text{N})\left(-\hat{k} + \hat{j}\right)\end{aligned}$$

Evaluating $\vec{F}_{-10}$ yields:

$$\begin{aligned}\vec{F}_{-10} &= \left(I\vec{\ell}\right)_{-10} \times \vec{B}_{-10} = (5.0\,\text{A})(-0.20\,\text{m})\hat{i} \times (10.0\,\mu\text{T})\left(-\hat{j} + \hat{k}\right) \\ &= (-10\,\mu\text{N})\left[\hat{i} \times \left(-\hat{j} + \hat{k}\right)\right] \\ &= (10\,\mu\text{N})\left(\hat{k} + \hat{j}\right)\end{aligned}$$

Substitute for $\vec{F}_{10}$ and $\vec{F}_{-10}$ in equation (2) and simplify to obtain:

$$\vec{F}_{\text{net}} = (10\,\mu\text{N})\left(-\hat{k} + \hat{j}\right) + (10\,\mu\text{N})\left(\hat{k} + \hat{j}\right) = \boxed{(20\,\mu\text{N})\hat{j}}$$

95 •• [SSM] A current balance is constructed in the following way: A straight 10.0-cm-long section of wire is placed on top of the pan of an electronic balance (Figure 27-69). This section of wire is connected in series with a power supply and a long straight horizontal section of wire that is parallel to it and positioned directly above it. The distance between the central axes of the two wires is 2.00 cm. The power supply provides a current in the wires. When the power supply is switched on, the reading on the balance increases by 5.00 mg. What is the current in the wire?

Picture the Problem The force acting on the lower wire is given by $F_{\text{lower wire}} = I\ell B$, where I is the current in the lower wire, ℓ is the length of the wire on the balance, and B is the magnetic field strength at the location of the lower wire due to the current in the upper wire. We can apply Ampere's law to find B at the location of the wire on the pan of the balance.

The force experienced by the lower wire is given by:

$$F_{\text{lower wire}} = I\ell B$$

Apply Ampere's law to a closed circular path of radius r centered on the upper wire to obtain:

$$B(2\pi r) = \mu_0 I_C = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Substituting for B in the expression for the force on the lower wire and simplifying yields:

$$F_{\text{lower wire}} = I\ell\left(\frac{\mu_0 I}{2\pi r}\right) = \frac{\mu_0 \ell I^2}{2\pi r}$$

Solve for I to obtain:

$$I=\sqrt{\frac{2\pi r F_{\text{lower wire}}}{\mu_0\ell}}$$

Note that the force on the lower wire is the increase in the reading of the balance. Substitute numerical values and evaluate I:

$$I=\sqrt{\frac{2\pi(2.00\text{ cm})(5.00\times10^{-6}\text{ kg})}{(4\pi\times10^{-7}\text{ N/A}^2)(10.0\text{ cm})}}$$
$$=2.236\text{ A}=\boxed{2.24\text{ A}}$$

96 •• Consider the current balance of Problem 95. If the sensitivity of the balance is 0.100 mg, what is the minimum current detectable using this current balance?

Picture the Problem We can use a proportion to relate minimum current detectible using this balance to its sensitivity and to the current and change in balance reading from Problem 95.

The minimum current I_{min} detectible is to the sensitivity of the balance as the current in Problem 95 is to the change in the balance reading in Problem 95:

$$\frac{I_{\text{min}}}{0.100\,\text{mg}}=\frac{2.236\,\text{A}}{5.00\,\text{mg}}$$

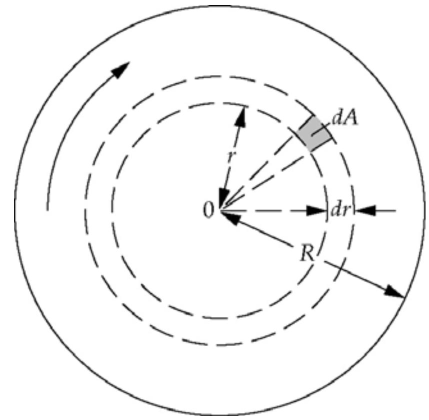
Solving for I_{min} yields:

$$I_{\text{min}}=(0.100\,\text{mg})\left(\frac{2.236\,\text{A}}{5.00\,\text{mg}}\right)$$
$$=\boxed{44.7\,\text{mA}}$$

The ″standard″ current balance can be made very sensitive by increasing the length (i.e., moment arm) of the wire balance, which one cannot do with this kind; however, this is compensated somewhat by the high sensitivity of the electronic balance.

97 ••• **[SSM]** A non-conducting disk that has radius R, carries a uniform surface charge density σ, and rotates with angular speed ω. (*a*) Consider an annular strip that has a radius r, a width dr, and a charge dq. Show that the current (dI) produced by this rotating strip is given by $\omega\sigma r dr$. (b) Use your result from Part (*a*) to show that the magnetic field strength at the center of the disk is given by the expression $\frac{1}{2}\mu_0\sigma\omega R$. (*c*) Use your result from Part (*a*) to find an expression for the magnetic field strength at a point on the central axis of the disk a distance z from its center.

Picture the Problem The diagram shows the rotating disk and the circular strip of radius *r* and width *dr* with charge *dq*. We can use the definition of surface charge density to express *dq* in terms of *r* and *dr* and the definition of current to show that $dI = \omega\sigma r\, dr$. We can then use this current and expression for the magnetic field on the axis of a current loop to obtain the results called for in Parts (*b*) and (*c*).



(*a*) Express the total charge *dq* that passes a given point on the circular strip once each period:

$$dq = \sigma dA = 2\pi\sigma r dr$$

Letting *q* be the total charge that passes along a radial section of the disk in a period of time *T*, express the current in the element of width *dr*:

$$dI = \frac{dq}{dt} = \frac{2\pi\sigma r dr}{\frac{2\pi}{\omega}} = \boxed{\omega\sigma r dr}$$

(*c*) Express the magnetic field dB_x at a distance *z* along the axis of the disk due to the current loop of radius *r* and width *dr*:

$$dB_x = \frac{\mu_0}{4\pi}\frac{2\pi r^2 dI}{\left(z^2 + r^2\right)^{3/2}} = \frac{\mu_0\omega\sigma r^3}{2\left(z^2 + r^2\right)^{3/2}}dr$$

Integrate from $r = 0$ to $r = R$ to obtain:

$$B_x = \frac{\mu_0\omega\sigma}{2}\int_0^R \frac{r^3}{\left(z^2 + r^2\right)^{3/2}}dr = \boxed{\frac{\mu_0\omega\sigma}{2}\left(\frac{R^2 + 2z^2}{\sqrt{R^2 + z^2}} - 2x\right)}$$

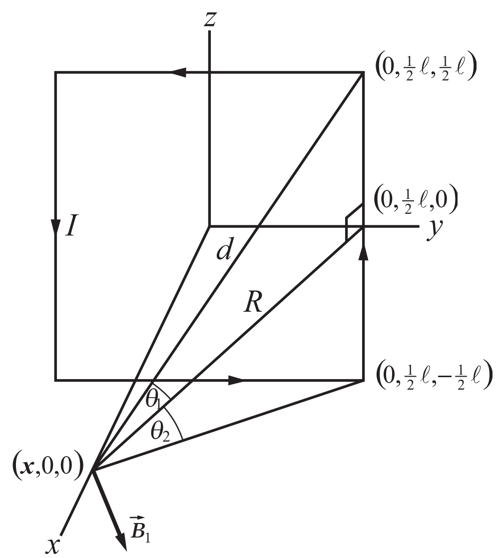
(*b*) Evaluate B_x for $x = 0$:

$$B_x(0) = \frac{\mu_0\omega\sigma}{2}\left(\frac{R^2}{\sqrt{R^2}}\right) = \boxed{\tfrac{1}{2}\mu_0\sigma\omega R}$$

98 ••• A square loop that has sides of length ℓ lies in the $z = 0$ plane with its center at the origin. The loop carries a current *I*. (*a*) Derive an expression for the magnetic field strength at any point on the *z* axis. (*b*) Show that for *z* much larger

than ℓ, your result from Part (*a*) becomes $B \approx \mu\mu_0/(2\pi z^3)$, where μ is the magnitude of the magnetic moment of the loop.

Picture the Problem From the symmetry of the system it is evident that the magnetic fields due to each segment have the same magnitude. We can express the magnetic field at (x,0,0) due to one side (segment) of the square, find its component in the x direction, and then multiply by four to find the resultant field.



(*a*) B due to a straight wire segment is given by:

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}(\sin\theta_1 + \sin\theta_2)$$

where R is the perpendicular distance from the wire segment to the field point.

Use $\theta_1 = \theta_2$ and $R = \sqrt{x^2 + \ell^2/4}$ to express B due to one side at (x,0,0):

$$B_1(x,0,0) = \frac{\mu_0}{4\pi}\frac{I}{\sqrt{x^2 + \frac{\ell^2}{4}}}(2\sin\theta_1)$$

$$= \frac{\mu_0}{2\pi}\frac{I}{\sqrt{x^2 + \frac{\ell^2}{4}}}(\sin\theta_1)$$

Referring to the diagram, express $\sin\theta_1$:

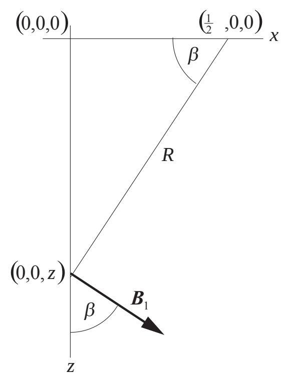
$$\sin\theta_1 = \frac{\frac{\ell}{2}}{d} = \frac{\frac{\ell}{2}}{\sqrt{x^2 + \frac{\ell^2}{2}}}$$

Substituting for $\sin\theta_1$ and simplifying yields:

$$B_1(x,0,0) = \frac{\mu_0}{2\pi}\frac{I}{\sqrt{x^2+\frac{\ell^2}{4}}}\frac{\frac{\ell}{2}}{\sqrt{x^2+\frac{\ell^2}{2}}}$$

$$= \frac{\mu_0 I}{4\pi\sqrt{x^2+\frac{\ell^2}{4}}}\frac{\ell}{\sqrt{x^2+\frac{\ell^2}{2}}}$$

By symmetry, the sum of the y and z components of the fields due to the four wire segments must vanish, whereas the x components will add. The diagram to the right is a view of the xy plane showing the relationship between $\vec{B}_1$ and the angle β it makes with the x axis.



Express B_{1x}:

$$B_{1x} = B_1\cos\beta$$

Substituting for $\cos\beta$ and simplifying yields:

$$B_{1x} = \frac{\mu_0 I}{4\pi\sqrt{x^2+\frac{\ell^2}{4}}}\frac{\ell}{\sqrt{x^2+\frac{\ell^2}{2}}}\frac{\frac{\ell}{2}}{\sqrt{x^2+\frac{\ell^2}{4}}}$$

$$= \frac{\mu_0 I\ell^2}{8\pi\left(x^2+\frac{\ell^2}{4}\right)\sqrt{x^2+\frac{\ell^2}{2}}}$$

The resultant magnetic field is the sum of the fields due to the four wire segments (sides of the square):

$$\vec{B} = 4B_{1x}\hat{\boldsymbol{i}}$$

$$= \boxed{\frac{\mu_0 I\ell^2}{2\pi\left(x^2+\frac{\ell^2}{4}\right)\sqrt{x^2+\frac{\ell^2}{2}}}\hat{\boldsymbol{i}}}$$

(*b*) Factor x^2 from the two factors in the denominator to obtain:

$$\vec{B} = \frac{\mu_0 I\ell^2}{2\pi x^2\left(1+\frac{\ell^2}{4x^2}\right)\sqrt{x^2\left(1+\frac{\ell^2}{2x^2}\right)}}\hat{i}$$

$$= \frac{\mu_0 I\ell^2}{2\pi x^3\left(1+\frac{\ell^2}{4x^2}\right)\sqrt{1+\frac{\ell^2}{2x^2}}}\hat{i}$$

For $x >> \ell$:

$$\vec{B} \approx \frac{\mu_0 I\ell^2}{2\pi x^3}\hat{i} = \boxed{\frac{\mu_0 \mu}{2\pi x^3}\hat{i}}$$

where $\mu = I\ell^2$.

Chapter 28
Magnetic Induction

Conceptual Problems

1 • [SSM] (*a*) The magnetic equator is a line on the surface of Earth on which Earth's magnetic field is horizontal. At the magnetic equator, how would you orient a flat sheet of paper so as to create the maximum magnitude of magnetic flux through it? (*b*) How about the minimum magnitude of magnetic flux?

Determine the Concept
(*a*) Orient the sheet so the normal to the sheet is both horizontal and perpendicular to the local tangent to the magnetic equator.

(*b*) Orient the sheet of paper so the normal to the sheet is perpendicular to the direction of the normal described in the answer to Part (*a*).

2 • At one of Earth's magnetic poles, how would you orient a flat sheet of paper so as to create the maximum magnitude of magnetic flux through it?

Determine the Concept
(*a*) Orient the sheet so the normal to the sheet is vertical.

(*b*) Any orientation as long as the paper's plane is perpendicular to Earth's surface at that location.

3 • [SSM] Show that the following combination of SI units is equivalent to the volt: $\mathrm{T\cdot m^2/s}$.

Determine the Concept Because a volt is a joule per coulomb, we can show that the SI units $\dfrac{\mathrm{T\cdot m^2}}{\mathrm{s}}$ are equivalent to a volt by making a series of substitutions and simplifications that reduces these units to a joule per coulomb.

The units of a tesla are $\dfrac{\mathrm{N}}{\mathrm{A\cdot m}}$:

$$\frac{\mathrm{T\cdot m^2}}{\mathrm{s}}=\frac{\dfrac{\mathrm{N}}{\mathrm{A\cdot m}}\cdot\mathrm{m^2}}{\mathrm{s}}=\frac{\dfrac{\mathrm{N\cdot m}}{\mathrm{A}}}{\mathrm{s}}$$

Substitute the units of an ampere (C/s), replace $\mathrm{N\cdot m}$ with J, and simplify to obtain:

$$\frac{\mathrm{T\cdot m^2}}{\mathrm{s}}=\frac{\dfrac{\mathrm{J}}{\dfrac{\mathrm{C}}{\mathrm{s}}}}{\mathrm{s}}=\frac{\mathrm{J}}{\mathrm{C}}$$

Finally, because a joule per coulomb is a volt:

$$\frac{\mathrm{T\cdot m^2}}{\mathrm{s}}=\boxed{\mathrm{V}}$$

4 • Show that the following combination of SI units is equivalent to the ohm: $\dfrac{\mathrm{Wb}}{\mathrm{A\cdot s}}$.

Determine the Concept Because a weber is a newton·meter per ampere, we can show that the SI units $\dfrac{\mathrm{Wb}}{\mathrm{A\cdot s}}$ are equivalent to an ohm by making a series of substitutions and simplifications that reduces these units to a volt per ampere.

Because a weber is a $\dfrac{\mathrm{N\cdot m}}{\mathrm{A}}$:

$$\frac{\mathrm{Wb}}{\mathrm{A\cdot s}}=\frac{\dfrac{\mathrm{N\cdot m}}{\mathrm{A}}}{\mathrm{A\cdot s}}=\frac{\mathrm{J}}{\mathrm{A^2\cdot s}}$$

Substitute the units of an ampere and simplify to obtain:

$$\frac{\mathrm{Wb}}{\mathrm{A\cdot s}}=\frac{\mathrm{J}}{\mathrm{A\cdot\dfrac{C}{s}\cdot s}}=\frac{\mathrm{J}}{\mathrm{A\cdot C}}$$

Finally, because a joule per coulomb is a volt:

$$\frac{\mathrm{Wb}}{\mathrm{A\cdot s}}=\frac{\mathrm{V}}{\mathrm{A}}=\boxed{\Omega}$$

5 • **[SSM]** A current is induced in a conducting loop that lies in a horizontal plane and the induced current is clockwise when viewed from above. Which of the following statements could be true? (*a*) A constant magnetic field is directed vertically downward. (*b*) A constant magnetic field is directed vertically upward. (*c*) A magnetic field whose magnitude is increasing is directed vertically downward. (*d*) A magnetic field whose magnitude is decreasing is directed vertically downward. (*e*) A magnetic field whose magnitude is decreasing is directed vertically upward.

Determine the Concept We know that the magnetic flux (in this case the magnetic field because the area of the conducting loop is constant and its orientation is fixed) must be changing so the only issues are whether the field is increasing or decreasing and in which direction. Because the direction of the magnetic field associated with the clockwise current is vertically downward, the changing field that is responsible for it must be either increasing vertically upward (not included in the list of possible answers) or a decreasing field directed into the page. $\boxed{(d)}$ is correct.

6 • Give the direction of the induced current in the circuit, shown on the right in Figure 28- 37, when the resistance in the circuit on the left is suddenly (*a*) increased and (*b*) decreased. Explain your answer.

Determine the Concept The induced emf and induced current in the circuit on the right are in such a direction as to oppose the change that produces them (Lenz's Law). We can determine the direction of the induced current in the circuit. Note that when R is constant, $\vec{B}$ in the circuit to the right points out of the paper.

(*a*) If R increases, I decreases and B in the circuit to the right decreases. Lenz's law tells us that the induced current is counterclockwise.

(*b*) If R decreases, I increases and B in the circuit to the right increases. Lenz's law tells us that the induced current is clockwise.

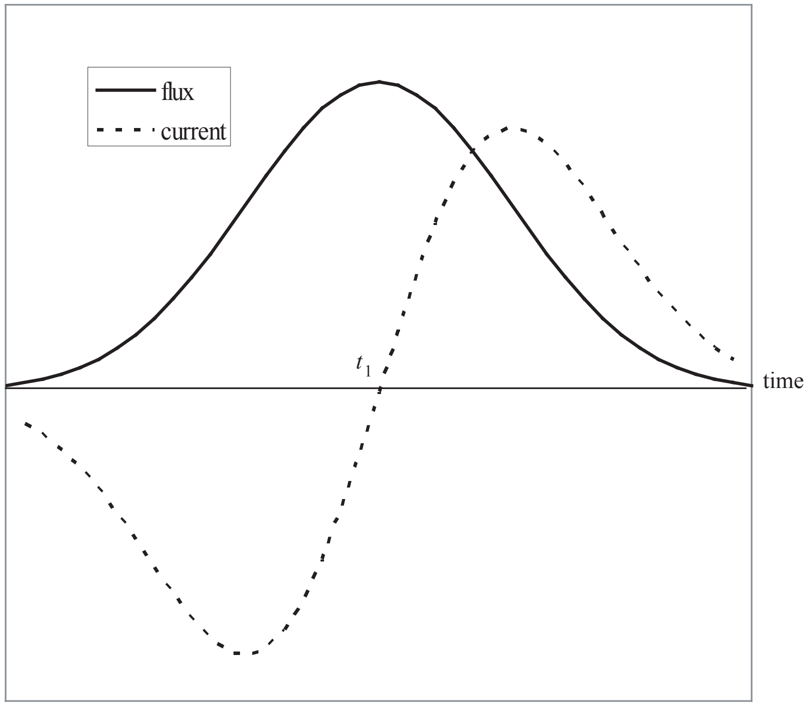
7 • **[SSM]** The planes of the two circular loops in Figure 28-38, are parallel. As viewed from the left, a counterclockwise current exists in loop A. If the magnitude of the current in loop A is increasing, what is the direction of the current induced in loop B? Do the loops attract or repel each other? Explain your answer.

Determine the Concept Clockwise as viewed from the left. The loops repel each other.

8 • A bar magnet moves with constant velocity along the axis of a loop, as shown in Figure 28-39, (*a*) Make a graph of the magnetic flux through the loop as a function of time. Indicate on the graph when the magnet is halfway through the loop by designating this time t_1. Choose the direction of the normal to the flat surface bounded by the flat surface to be to the right. (*b*) Make a graph of the induced current in the loop as a function of time. Choose the positive direction for the current to be clockwise as viewed from the left.

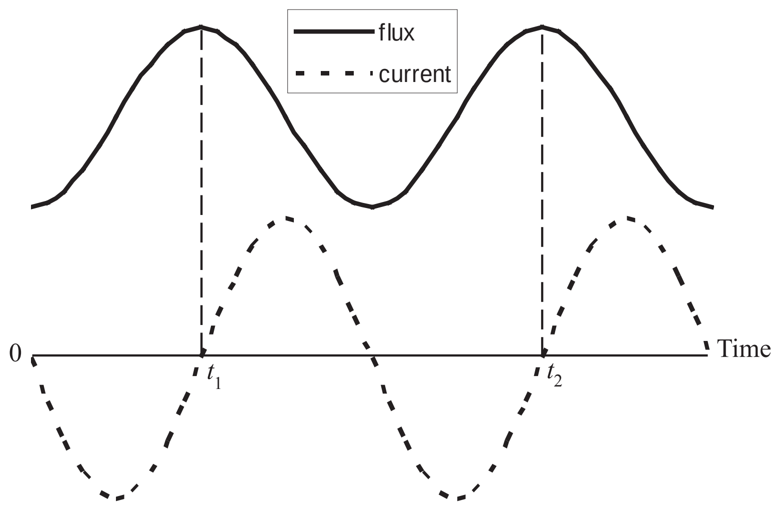
Determine the Concept We know that, as the magnet moves to the right, the flux through the loop first increases until the magnet is half way through the loop and then decreases. Because the flux first increases and then decreases, the current will change directions, having its maximum values when the flux is changing most rapidly.

(*a*) and (*b*) The following graph shows the flux and the induced current as a function of time as the bar magnet passes through the coil. When the center of the magnet passes through the plane of the coil $d\phi_m/dt = 0$ and the current is zero.



9 • A bar magnet is mounted on the end of a coiled spring and is oscillating in simple harmonic motion along the axis of a loop, as shown in Figure 28-40. The magnet is in its equilibrium position when its midpoint is in the plane of the loop. (*a*) Make a graph of the magnetic flux through the loop as a function of time. Indicate when the magnet is halfway through the loop by designating these times t_1 and t_2. (*b*) Make a graph of the induced current in the loop as a function of time, choosing the current to be positive when it is clockwise as viewed from above.

Determine the Concept Because the magnet moves with simple harmonic motion, the flux and the induced current will vary sinusoidally. The current will be a maximum wherever the flux is changing most rapidly and will be zero wherever the flux is momentarily constant. (*a*), (*b*) The following graph shows the flux, ϕ_m, and the induced current (proportional to $-d\phi_m/dt$) in the loop as a function of time.



10 • A pendulum is fabricated from a thin, flat piece of aluminum. At the bottom of its arc, it passes between the poles of a strong permanent magnet. In Figure 28-41*a*, the metal sheet is continuous, whereas in Figure 28-41*b*, there are slots in the sheet. When released from the same angle, the pendulum that has slots swings back and forth many times, but the pendulum that does not have slots stops swinging after no more than one complete oscillation. Explain why.

Determine the Concept In the configuration shown in (*a*), energy is dissipated by eddy currents from the emf induced by the pendulum movement. In the configuration shown in (*b*), the slits inhibit the eddy currents and the braking effect is greatly reduced.

11 • A bar magnet is dropped inside a long vertical tube. If the tube is made of metal, the magnet quickly approaches a terminal speed, but if the tube is made of cardboard, the magnet falls with constant acceleration. Explain why the magnet falls differently in the metal tube than it does in the cardboard tube.

Determine the Concept The magnetic field of the falling magnet sets up eddy currents in the metal tube. The eddy currents establish a magnetic field that exerts a force on the magnet opposing its motion; thus the magnet is slowed down. If the tube is made of a nonconducting material, there are no eddy currents.

12 • A small square wire loop lies in the plane of this page, and a constant magnetic field is directed into the page. The loop is moving to the right, which is the $+x$ direction. Find the direction of the induced current, if any, in the loop if (*a*) the magnetic field is uniform, (*b*) the magnetic field strength increases as x increases, and (*c*) the magnetic field strength decreases as x increases.

Determine the Concept The direction of the induced current is in such a direction as to oppose, or tend to oppose, the change that produces it (Lenz's Law).

(*a*) Because the applied field is constant and uniform, there is no change in flux through the loop and, in accord with Faraday's law, no induced current in the loop.

(*b*) Let the positive normal direction on the flat surface bounded by the loop be into the page. Because the strength of the applied field increases to the right, the flux through the loop increases as it moves to the right. In accord with Lenz's law, the direction of the induced current will be such that the flux through the loop due to its magnetic field will be opposite in sign the change in flux of the applied field. Thus, on the flat surface bounded by the loop the magnetic field due to the induced current is out of the page. Using the right-hand rule, the induced current must be counterclockwise.

(*c*) Let the positive normal direction on the flat surface bounded by the loop be into the page. Because the strength of the applied field increases to the right, the flux through the loop decreases as it moves to the right. In accord with Lenz's law, the direction of the induced current will be such that the flux through the loop due to its magnetic field will be opposite in sign the change in flux of the applied field. Thus, on the flat surface bounded by the loop the magnetic field due to the induced current is into the page. Using the right-hand rule, the induced current must be clockwise.

13 • If the current in an inductor doubles, the energy stored in the inductor will (*a*) remain the same, (*b*) double, (*c*) quadruple, (*d*) halve.

Determine the Concept The magnetic energy stored in an inductor is given by $U_m = \frac{1}{2}LI^2$. Doubling I quadruples U_m. $\boxed{(c)}$ is correct.

14 • Two solenoids are equal in length and radius, and the cores of both are identical cylinders of iron. However, solenoid A has three times the number of turns per unit length as solenoid B. (*a*) Which solenoid has the larger self-inductance? (*b*) What is the ratio of the self-inductance of solenoid A to the self-inductance of solenoid B?

Determine the Concept The self-inductance of a coil is given by $L = \mu_0 n^2 A\ell$, where n is the number of turns per unit length and ℓ is the length of the coil.

(*a*) Because the two solenoids are equal in length and radius and have identical cores, their self-inductances are proportional to the square of their number of turns per unit length. Hence A has the larger self-inductance.

(*b*) The self-inductances of the two coils are given by:

$$L_A = \mu_0 n_A^2 A_A \ell_A$$

and

$$L_B = \mu_0 n_B^2 A_B \ell_B$$

Divide the self-inductance of coil A by the self-inductance of coil B and simplify to obtain:

$$\frac{L_A}{L_B} = \frac{\mu_0 n_A^2 A_A \ell_A}{\mu_0 n_B^2 A_B \ell_B} = \frac{n_A^2 A_A \ell_A}{n_B^2 A_B \ell_B}$$

or, because the coils have the same lengths and radii (hence, the same cross-sectional areas),

$$\frac{L_A}{L_B} = \frac{n_A^2}{n_B^2} = \left(\frac{n_A}{n_B}\right)^2$$

If n increases by a factor of 3, ℓ will decrease by the same factor, because the inductors are made from the same length of wire. Hence:

$$\frac{L_A}{L_B} = \left(\frac{3n_B}{n_B}\right)^2 = \boxed{9}$$

15 • [SSM] True or false:

(*a*) The induced emf in a circuit is equal to the negative of the magnetic flux through the circuit.
(*b*) There can be a non-zero induced emf at an instant when the flux through the circuit is equal to zero.
(*c*) The self inductance of a solenoid is proportional to the rate of change of the current in the solenoid.
(*d*) The magnetic energy density at some point in space is proportional to the square of the magnitude of the magnetic field at that point.
(*e*) The inductance of a solenoid is proportional to the current in it.

(*a*) False. The induced emf in a circuit is equal to *the rate of change of* the magnetic flux through the circuit.

(*b*) True. The rate of change of the magnetic flux can be non-zero when the flux through the circuit is momentarily zero

(*c*) False. The self inductance of a solenoid is determined by its length, cross-sectional area, number of turns per unit length, and the permeability of the matter in its core.

(*d*) True. The magnetic energy density at some point in space is given by Equation 28-20: $u_m = \dfrac{B^2}{2\mu_0}$.

(*e*) False. The inductance of a solenoid is determined by its length, cross-sectional area, number of turns per unit length, and the permeability of the matter in its core.

Estimation and Approximation

16 • Your baseball teammates, having just studied this chapter, are concerned about generating enough voltage to shock them while swinging aluminum bats at fast balls. Estimate the maximum possible motional emf measured between the ends of an aluminum baseball bat during a swing. Do you think your team should switch to wooden bats to avoid electrocution?

Picture the Problem The bat is swung in Earth's magnetic field. We'll assume that the batter swings such that the maximum linear velocity of the bat occurs at an angle such that it is moving perpendicular to Earth's field (i.e. when the bat is aligned north-south and moving east-west). The induced emf in the bat is given by $\mathcal{E} = vB\ell$. A bat is roughly 1 m long, and at most its center is probably moving at 75 mph, or about 33 m/s. Earth's magnetic field is about 0.3 G.

The emf induced in the bat is given by:

$$\mathcal{E} = vB\ell$$

Under the conditions resulting in a maximum induced emf outlined above:

$$\mathcal{E} = (33\ \text{m/s})\left(0.3\ \text{G} \times \frac{1\ \text{T}}{10^4\ \text{G}}\right)(1\ \text{m}) \approx \boxed{1\ \text{mV}}$$

Because 1 mV is so low, there is no danger of being shocked and no reason to switch to wooden bats.

17 • Compare the energy density stored in Earth's electric field near its surface to that stored in Earth's magnetic field near its surface.

Picture the Problem We can compare the energy density stored in Earth's electric field to that of Earth's magnetic field by finding their ratio. We'll take Earth's magnetic field to be 0.3 G and its electric field to be 100 V/m.

The energy density in an electric field E is given by:

$$u_e = \tfrac{1}{2}\epsilon_0 E^2$$

The energy density in a magnetic field B is given by:

$$u_{\mathrm{m}} = \frac{B^2}{2\mu_0}$$

Express the ratio of u_{m} to u_{e} to obtain:

$$\frac{u_{\mathrm{m}}}{u_{\mathrm{e}}} = \frac{\dfrac{B^2}{2\mu_0}}{\frac{1}{2}\epsilon_0 E^2} = \frac{B^2}{\mu_0 \epsilon_0 E^2}$$

Substitute numerical values and evaluate $u_{\mathrm{m}}/ u_{\mathrm{e}}$:

$$\frac{u_{\mathrm{m}}}{u_{\mathrm{e}}} = \frac{\left(0.3\,\mathrm{G} \times \dfrac{1\,\mathrm{T}}{10^4\,\mathrm{G}}\right)^2}{\left(4\pi \times 10^{-7}\,\mathrm{N/A^2}\right)\left(8.854 \times 10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(100\,\mathrm{V/m}\right)^2} = 8.09 \times 10^3$$

or

$$u_{\mathrm{m}} \approx \boxed{\left(8 \times 10^3\right) u_{\mathrm{e}}}$$

18 •• A physics teacher does the following emf demonstration. She has two students hold a long wire connected to a voltmeter. The wire is held slack, so that it sags with a large arc in it. When she says ″start, ″the students begin rotating the wire as if they were playing jump rope. The students stand 3.0 m apart, and the sag in the wire is about 1.5 m. The motional emf from the ″jump rope″ is then measured on the voltmeter. (*a*) Estimate a reasonable value for the maximum angular speed that the students can rotate the wire. (*b*) From this, estimate the maximum motional emf in the wire. *HINT: What field is involved in creating the induced emf?*

Picture the Problem We can use Faraday's law to relate the motional emf in the wire to the angular speed with which the students turn the jump rope. Assume that Earth's magnetic field is 0.3 G.

(*a*) It seems unlikely that the students could turn the ″jump rope″ wire faster than 5.0 rev/s.

(*b*) The magnetic flux ϕ_{m} through the rotating circular loop of wire varies sinusoidally with time according to:

$$\phi_{\mathrm{m}} = BA\sin\omega t \Rightarrow \frac{d\phi_{\mathrm{m}}}{dt} = BA\omega\cos\omega t$$

Because the average value of the cosine function, over one revolution, is ½, the average rate at which the flux changes through the circular loop is:

$$\left.\frac{d\phi_{\mathrm{m}}}{dt}\right|_{\mathrm{av}} = \tfrac{1}{2}BA\omega = \tfrac{1}{2}\pi r^2 B\omega$$

From Faraday's law, the magnitude of the average motional emf in the loop is:

$$\mathcal{E} = \frac{d\phi_m}{dt} = \tfrac{1}{2}\pi r^2 B\omega$$

Substitute numerical values and evaluate $\mathcal{E}$:

$$\mathcal{E} = \tfrac{1}{2}\pi\left(\frac{1.5\,\text{m}}{2}\right)^2\left(0.3\,\text{G}\times\frac{1\,\text{T}}{10^4\,\text{G}}\right)(31.4\,\text{rad/s}) \approx \boxed{0.8\,\text{mV}}$$

19 •• (*a*) Estimate the maximum possible motional emf between the wingtips of a typical commercial airliner in flight. (*b*) Estimate the magnitude of the electric field between the wingtips.

Picture the Problem The motional emf between the wingtips of an airliner is given by $\mathcal{E} = vB\ell$. Assume a speed, relative to Earth's magnetic field, of 500 mi/h or about 220 m/s and a wingspan of 70 m. Assume that Earth's magnetic field is 0.3 G.

(*a*) The motional emf between the wingtips is given by:

$$\mathcal{E} = vB\ell$$

Substitute numerical values and evaluate $\mathcal{E}$:

$$\mathcal{E} = (220\,\text{m/s})\left(0.3\,\text{G}\times\frac{1\,\text{T}}{10^4\,\text{G}}\right)(70\,\text{m}) \approx \boxed{0.5\,\text{V}}$$

(*b*) The magnitude of the electric field between the wingtips is the ratio of the potential difference between them and their separation:

$$E = \frac{V}{d} = \frac{0.5\,\text{V}}{70\,\text{m}} \approx \boxed{7\,\text{mV/m}}$$

Magnetic Flux

20 • A uniform magnetic field of magnitude 0.200 T is in the +*x* direction. A square coil that has 5.00-cm long sides has a single turn and makes an angle θ with the z axis, as shown in Figure 28-42. Find the magnetic flux through the coil when θ is (*a*) 0º, (*b*) 30º, (*c*) 60º, and (*d*) 90º.

Picture the Problem Because the surface is a plane with area A and $\vec{B}$ is constant in magnitude and direction over the surface and makes an angle θ with the unit normal vector, we can use $\phi_m = BA\cos\theta$ to find the magnetic flux through the coil.

The magnetic flux through the coil is given by: $\phi_m = BA\cos\theta$

Substitute for B and A to obtain:

$$\phi_m = \left(2000\,\text{G}\cdot\frac{1\,\text{T}}{10^4\,\text{G}}\right)\left(5.00\times10^{-2}\,\text{m}\right)^2\cos\theta = \left(5.00\times10^{-4}\,\text{Wb}\right)\cos\theta$$

(*a*) For $\theta = 0°$:
$$\phi_m = \left(5.00\times10^{-4}\,\text{Wb}\right)\cos 0° = 5.00\times10^{-4}\,\text{Wb} = \boxed{0.50\,\text{mWb}}$$

(*b*) For $\theta = 30°$:
$$\phi_m = \left(5.00\times10^{-4}\,\text{Wb}\right)\cos 30° = 4.33\times10^{-4}\,\text{Wb} = \boxed{0.43\,\text{mWb}}$$

(*c*) For $\theta = 60°$:
$$\phi_m = \left(5.00\times10^{-4}\,\text{Wb}\right)\cos 60° = 2.50\times10^{-4}\,\text{Wb} = \boxed{0.25\,\text{mWb}}$$

(*d*) For $\theta = 90°$:
$$\phi_m = \left(5.00\times10^{-4}\,\text{Wb}\right)\cos 90° = \boxed{0}$$

21 • [SSM] A circular coil has 25 turns and a radius of 5.0 cm. It is at the equator, where Earth's magnetic field is 0.70 G, north. The axis of the coil is the line that passes through the center of the coil and is perpendicular to the plane of the coil. Find the magnetic flux through the coil when the axis of the coil is (*a*) vertical, (*b*) horizontal with the axis pointing north, (*c*) horizontal with the axis pointing east, and (*d*) horizontal with the axis making an angle of 30º with north.

Picture the Problem Because the coil defines a plane with area A and $\vec{B}$ is constant in magnitude and direction over the surface and makes an angle θ with the unit normal vector, we can use $\phi_m = NBA\cos\theta$ to find the magnetic flux through the coil.

The magnetic flux through the coil is given by: $\phi_m = NBA\cos\theta = NB\pi r^2\cos\theta$

Substitute for numerical values to obtain:

$$\phi_m = 25\left(0.70\,\text{G}\cdot\frac{1\,\text{T}}{10^4\,\text{G}}\right)\pi\left(5.0\times10^{-2}\,\text{m}\right)^2\cos\theta = \left(13.7\,\mu\text{Wb}\right)\cos\theta$$

(*a*) When the plane of the coil is horizontal, $\theta = 90°$:

$$\phi_{\mathrm{m}} = (13.7\,\mu\mathrm{Wb})\cos 90° = \boxed{0}$$

(*b*) When the plane of the coil is vertical with its axis pointing north, $\theta = 0°$:

$$\phi_{\mathrm{m}} = (13.7\,\mu\mathrm{Wb})\cos 0° = \boxed{14\,\mu\mathrm{Wb}}$$

(*c*) When the plane of the coil is vertical with its axis pointing east, $\theta = 90°$:

$$\phi_{\mathrm{m}} = (13.7\,\mu\mathrm{Wb})\cos 90° = \boxed{0}$$

(*d*) When the plane of the coil is vertical with its axis making an angle of 30° with north, $\theta = 30°$:

$$\phi_{\mathrm{m}} = (13.7\,\mu\mathrm{Wb})\cos 30° = \boxed{12\,\mu\mathrm{Wb}}$$

22 • A magnetic field of 1.2 T is perpendicular to the plane of a14 turn square coil with sides 5.0-cm long. (*a*) Find the magnetic flux through the coil. (*b*) Find the magnetic flux through the coil if the magnetic field makes an angle of 60º with the normal to the plane of the coil.

Picture the Problem Because the square coil defines a plane with area A and $\vec{B}$ is constant in magnitude and direction over the surface and makes an angle θ with the unit normal vector, we can use $\phi_{\mathrm{m}} = NBA\cos\theta$ to find the magnetic flux through the coil.

The magnetic flux through the coil is given by:

$$\phi_{\mathrm{m}} = NBA\cos\theta$$

Substitute numerical values for N, B, and A to obtain:

$$\begin{aligned}\phi_{\mathrm{m}} &= 14(1.2\,\mathrm{T})\left(5.0\times10^{-2}\,\mathrm{m}\right)^2\cos\theta \\ &= (42.0\,\mathrm{mWb})\cos\theta\end{aligned}$$

(*a*) For $\theta = 0°$:

$$\phi_{\mathrm{m}} = (42.0\,\mathrm{mWb})\cos 0° = \boxed{42\,\mathrm{mWb}}$$

(*b*) For $\theta = 60°$:

$$\phi_{\mathrm{m}} = (42.0\,\mathrm{mWb})\cos 60° = \boxed{21\,\mathrm{mWb}}$$

23 • A uniform magnetic field $\vec{B}$ is perpendicular to the base of a hemisphere of radius R. Calculate the magnetic flux (in terms of B and R) through the spherical surface of the hemisphere.

Picture the Problem Noting that the flux through the base must also penetrate the spherical surface (the ± in the answer below), we can apply its definition to express ϕ_m.

Apply the definition of magnetic flux to obtain:

$$\phi_m = \pm AB = \boxed{\pm\pi R^2 B}$$

24 • Find the magnetic flux through a 400-turn solenoid that has a length equal to 25.0 cm, has a radius equal to 1.00 cm, and carries a current of 3.00 A.

Picture the Problem We can use $\phi_m = NBA\cos\theta$ to express the magnetic flux through the solenoid and $B = \mu_0 nI$ to relate the magnetic field in the solenoid to the current in its coils. Assume that the magnetic field in the solenoid is constant.

Express the magnetic flux through a coil with N turns:

$$\phi_m = NBA\cos\theta$$

Express the magnetic field inside a long solenoid:

$$B = \mu_0 nI$$

where n is the number of turns per unit length.

Substitute to obtain:

$$\phi_m = N\mu_0 nIA\cos\theta$$

or, because $n = N/L$ and $\theta = 0°$,

$$\phi_m = \frac{N^2\mu_0 IA}{L} = \frac{N^2\mu_0 I\pi r^2}{L}$$

Substitute numerical values and evaluate ϕ_m:

$$\phi_m = \frac{(400)^2(4\pi\times10^{-7}\ \text{N/A}^2)(3.00\ \text{A})\pi(0.0100\ \text{m})^2}{0.250\ \text{m}} = \boxed{758\ \mu\text{Wb}}$$

25 • Find the magnetic flux through a 800-turn solenoid that has a length equal to 30.0 cm, has a radius equal to 2.00 cm, and carries a current of 2.00 A.

Picture the Problem We can use $\phi_m = NBA\cos\theta$ to express the magnetic flux through the solenoid and $B = \mu_0 nI$ to relate the magnetic field in the solenoid to the current in its coils. Assume that the magnetic field in the solenoid is constant.

Express the magnetic flux through a coil with N turns:

$$\phi_m = NBA\cos\theta$$

Express the magnetic field inside a long solenoid:

$$B = \mu_0 nI$$

where n is the number of turns per unit length.

Substitute for B to obtain:

$$\phi_m = N\mu_0 nIA\cos\theta$$

or, because $n = N/L$ and $\theta = 0°$,

$$\phi_m = \frac{N^2\mu_0 IA}{L} = \frac{N^2\mu_0 I\pi r^2}{L}$$

Substitute numerical values and evaluate ϕ_m:

$$\phi_m = \frac{(800)^2(4\pi\times10^{-7}\ \text{N/A}^2)(2.00\ \text{A})\pi(0.0200\ \text{m})^2}{0.300\ \text{m}} = \boxed{6.74\ \text{mWb}}$$

26 •• A circular coil has 15.0 turns, has a radius 4.00 cm, and is in a uniform magnetic field of 4.00 kG in the $+x$ direction. Find the flux through the coil when the unit normal to the plane of the coil is $(a)\,\hat{\boldsymbol{i}}$, $(b)\,\hat{\boldsymbol{j}}$, $(c)\,(\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}})/\sqrt{2}$, $(d)\,\hat{\boldsymbol{k}}$, and $(e)\,0.60\hat{\boldsymbol{i}}+0.80\hat{\boldsymbol{j}}$.

Picture the Problem We can apply the definitions of magnet flux and of the dot product to find the flux for the given unit vectors.

Apply the definition of magnetic flux to the coil to obtain:

$$\phi_m = N\int_S \vec{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}}\,dA$$

Because $\vec{\boldsymbol{B}}$ is constant:

$$\phi_m = N\vec{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}}\int_S dA = N(\vec{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})A = N(\vec{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})\pi r^2$$

Evaluate $\vec{\boldsymbol{B}}$:

$$\vec{\boldsymbol{B}} = (0.400\,T)\hat{\boldsymbol{i}}$$

Substitute numerical values and simplify to obtain:

$$\phi_m = (15.0)[(0.400\ \text{T})]\pi(0.0400\ \text{m})^2 = (0.03016\ \text{T}\cdot\text{m}^2)\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{n}}$$

(a) Evaluate ϕ_m for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{i}}$:

$$\phi_m = (0.03016\ \text{T}\cdot\text{m}^2)\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}} = \boxed{30.2\ \text{mWb}}$$

(b) Evaluate ϕ_m for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{j}}$:

$$\phi_m = (0.03016\ \text{T}\cdot\text{m}^2)\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}} = \boxed{0}$$

(*c*) Evaluate ϕ_m for $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})/\sqrt{2}$:

$$\phi_m = (0.03016\,\mathrm{T\cdot m^2})\hat{\boldsymbol{i}} \cdot \frac{(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})}{\sqrt{2}} = \frac{0.03016\,\mathrm{T\cdot m^2}}{\sqrt{2}} = \boxed{21.3\,\mathrm{mWb}}$$

(*d*) Evaluate ϕ_m for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$:

$$\phi_m = (0.03016\,\mathrm{T\cdot m^2})\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}} = \boxed{0}$$

(*e*) Evaluate ϕ_m for $\hat{\boldsymbol{n}} = 0.60\hat{\boldsymbol{i}} + 0.80\hat{\boldsymbol{j}}$:

$$\phi_m = (0.03016\,\mathrm{T\cdot m^2})\hat{\boldsymbol{i}} \cdot (0.60\hat{\boldsymbol{i}} + 0.80\hat{\boldsymbol{j}}) = \boxed{18\,\mathrm{mWb}}$$

27 •• **[SSM]** A long solenoid has n turns per unit length, has a radius R_1, and carries a current I. A circular coil with radius R_2 and with N total turns is coaxial with the solenoid and equidistant from its ends. (*a*) Find the magnetic flux through the coil if $R_2 > R_1$. (*b*) Find the magnetic flux through the coil if $R_2 < R_1$.

Picture the Problem The magnetic field outside the solenoid is, to a good approximation, zero. Hence, the flux through the coil is the flux in the core of the solenoid. The magnetic field inside the solenoid is uniform. Hence, the flux through the circular coil is given by the same expression with R_2 replacing R_1:

(*a*) The flux through the large circular loop outside the solenoid is given by:

$$\phi_m = NBA$$

Substituting for B and A and simplifying yields:

$$\phi_m = N(\mu_0 nI)(\pi R_1^2) = \boxed{\mu_0 nIN\pi R_1^2}$$

(*b*) The flux through the coil when $R_2 < R_1$ is given by:

$$\phi_m = N(\mu_0 nI)(\pi R_2^2) = \boxed{\mu_0 nIN\pi R_2^2}$$

28 ••• (*a*) Compute the magnetic flux through the rectangular loop shown in Figure 28-45. (*b*) Evaluate your answer for $a = 5.0$ cm, $b = 10$ cm, $d = 2.0$ cm, and $I = 20$ A.

Picture the Problem We can use the hint to set up the element of area dA and express the flux $d\phi_m$ through it and then carry out the details of the integration to express ϕ_m.

(*a*) The flux through the strip of area dA is given by:

$$d\phi_m = BdA$$

where $dA = bdx$.

Express B at a distance x from a long, straight wire:

$$B = \frac{\mu_0}{4\pi}\frac{2I}{x} = \frac{\mu_0}{2\pi}\frac{I}{x}$$

Substitute to obtain:

$$d\phi_m = \frac{\mu_0}{2\pi}\frac{I}{x}b\,dx = \frac{\mu_0 Ib}{2\pi}\frac{dx}{x}$$

Integrate from $x = d$ to $x = d + a$:

$$\phi_m = \frac{\mu_0 Ib}{2\pi}\int_d^{d+a}\frac{dx}{x} = \boxed{\frac{\mu_0 Ib}{2\pi}\ln\left(\frac{d+a}{d}\right)}$$

(*b*) Substitute numerical values and evaluate ϕ_m:

$$\phi_m = \frac{\left(4\pi\times10^{-7}\ \mathrm{N/A^2}\right)(20\ \mathrm{A})(0.10\ \mathrm{m})}{2\pi}\ln\left(\frac{7.0\ \mathrm{cm}}{2.0\ \mathrm{cm}}\right) = \boxed{0.50\ \mu\mathrm{Wb}}$$

29 ••• A long cylindrical conductor with a radius R and a length L carries a current I. Find the magnetic flux per unit length through the area indicated in Figure 28-44.

Picture the Problem Consider an element of area $dA = Ldr$ where $r \le R$. We can use its definition to express $d\phi_m$ through this area in terms of B and Ampere's law to express B as a function of I. The fact that the current is uniformly distributed over the cross-sectional area of the conductor allows us to set up a proportion from which we can obtain I as a function of r. With these substitutions in place we can integrate $d\phi_m$ to obtain ϕ_m/L.

Noting that $\vec{B}$ is parallel to $\hat{n}$ over the entire area, express the flux $d\phi_m$ through an area Ldr:

$$d\phi_m = BdA = BLdr \qquad (1)$$

Apply Ampere's law to the current contained inside a cylindrical region of radius $r < R$:

$$\oint_C \vec{\boldsymbol{B}}\cdot d\vec{\ell} = 2\pi rB = \mu_0 I_C$$

Solving for B yields:

$$B = \frac{\mu_0 I_C}{2\pi r} \qquad (2)$$

Using the fact that the current I is uniformly distributed over the cross-sectional area of the conductor, express its variation with distance r from the center of the conductor:

$$\frac{I(r)}{I} = \frac{\pi r^2}{\pi R^2} \Rightarrow I(r) = I_C = I\frac{r^2}{R^2}$$

Substitute for I_C in equation (2) and simplify to obtain:

$$B = \frac{\mu_0 I}{2\pi r}\frac{r^2}{R^2} = \frac{\mu_0 I}{2\pi R^2}r$$

Substituting for B in equation (1) yields:

$$d\phi_{\mathrm{m}} = \frac{\mu_0 LI}{2\pi R^2}rdr$$

Integrate $d\phi_{\mathrm{m}}$ from $r = 0$ to $r = R$ to obtain:

$$\phi_{\mathrm{m}} = \frac{\mu_0 LI}{2\pi R^2}\int_0^R rdr = \frac{\mu_0 LI}{4\pi}$$

Divide both sides of this equation by L to express the magnetic flux per unit length:

$$\frac{\phi_{\mathrm{m}}}{L} = \boxed{\frac{\mu_0 I}{4\pi}}$$

Induced EMF and Faraday's Law

30 • The flux through a loop is given by $\phi_{\mathrm{m}} = 0.10t^2 - 0.40t$, where ϕ_{m} is in webers and t is in seconds. (a) Find the induced emf as a function of time. (b) Find both ϕ_{m} and $\mathcal{E}$ at $t = 0$, $t = 2.0$ s, $t = 4.0$ s, and $t = 6.0$ s.

Picture the Problem Given ϕ_{m} as a function of time, we can use Faraday's law to express $\mathcal{E}$ as a function of time.

(a) Apply Faraday's law to express the induced emf in the loop in terms of the rate of change of the magnetic flux:

$$\mathcal{E}(t) = -\frac{d\phi_{\mathrm{m}}}{dt} = -\frac{d}{dt}\left[0.10\left(t^2 - 0.40t\right)\right]$$
$$= -(0.20t - 0.40)\frac{\mathrm{Wb}}{\mathrm{s}}$$
$$= \boxed{-(0.20t - 0.40)\,\mathrm{V}}$$

where $\mathcal{E}$ is in volts and t is in seconds.

(b) Evaluate ϕ_{m} at $t = 0$:

$$\phi_{\mathrm{m}}(0\,\mathrm{s}) = 0.10(0)^2 - (0.40)(0) = \boxed{0}$$

Evaluate $\mathcal{E}$ at $t = 0$:

$$\mathcal{E}(0\,\mathrm{s}) = -[0.20(0) - 0.40]\mathrm{V} = \boxed{0.40\,\mathrm{V}}$$

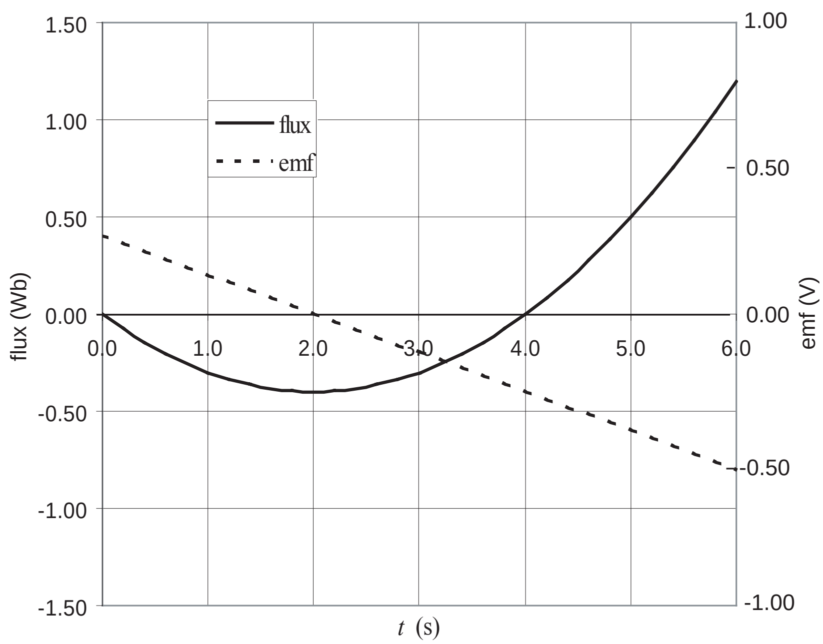
Proceed as above to complete the table to the right:

t	ϕ_{m}	$\mathcal{E}$
(s)	(Wb)	(V)
0	0	0.40
2.0	–0.40	0.00
4.0	0.0	–0.40
6.0	1.2	–0.80

31 • The flux through a loop is given by $\phi_m = 0.10t^2 - 0.40t$, where ϕ_m is in webers and t is in seconds. (*a*) Sketch graphs of magnetic flux and induced emf as a function of time. (*b*) At what time(s) is the flux minimum? What is the induced emf at that (those) time(s)? (*c*) At what time(s) is the flux zero? What is (are) the induced emf(s) at those time(s)?

Picture the Problem We can find the time at which the flux is a minimum by looking for the lowest point on the graph of $\mathcal{E}$ versus t and the emf at this time by determining the value of V at this time from the graph. We can interpret the graphs to find the times at which the flux is zero and the corresponding values of the emf.

(*a*) The flux, ϕ_m, and the induced emf, $\mathcal{E}$, are shown as functions of t in the following graph. The solid curve represents ϕ_m, the dashed curve represents $\mathcal{E}$.



(*b*) Referring to the graph, we see that the flux is a minimum when $t = 2.0$ s and that $V(2.0\text{ s}) = 0$.

(*c*) The flux is zero when $t = 0$ and $t = 4.0$ s. $\mathcal{E}(0) = 0.40$ V and

$\mathcal{E}(4.0\text{ s}) = -0.40$ V.

32 • A solenoid that has a length equal to 25.0 cm, a radius equal to 0.800 cm, and 400 turns is in a region where a magnetic field of 600 G exists and makes an angle of 50º with the axis of the solenoid. (*a*) Find the magnetic flux

through the solenoid. (*b*) Find the magnitude of the average emf induced in the solenoid if the magnetic field is reduced to zero in 1.40 s.

Picture the Problem We can use its definition to find the magnetic flux through the solenoid and Faraday's law to find the emf induced in the solenoid when the external field is reduced to zero in 1.4 s.

(*a*) Express the magnetic flux through the solenoid in terms of N, B, A, and θ:

$$\phi_{\mathrm{m}} = NBA\cos\theta = NB\pi R^2\cos\theta$$

Substitute numerical values and evaluate ϕ_{m}:

$$\phi_{\mathrm{m}} = (400)(60.0\,\mathrm{mT})\pi(0.00800\,\mathrm{m})^2\cos 50° = 3.10\,\mathrm{mWb} = \boxed{3.1\,\mathrm{mWb}}$$

(*b*) Apply Faraday's law to obtain:

$$\mathcal{E} = -\frac{\Delta\phi_{\mathrm{m}}}{\Delta t} = -\frac{0 - 3.10\,\mathrm{mWb}}{1.40\,\mathrm{s}} = \boxed{2.2\,\mathrm{mV}}$$

33 •• [SSM] A 100-turn circular coil has a diameter of 2.00 cm, a resistance of 50.0 Ω, and the two ends of the coil are connected together. The plane of the coil is perpendicular to a uniform magnetic field of magnitude 1.00 T. The direction of the field is reversed. (*a*) Find the total charge that passes through a cross section of the wire. If the reversal takes 0.100 s, find (*b*) the average current and (*c*) the average emf during the reversal.

Picture the Problem We can use the definition of average current to express the total charge passing through the coil as a function of I_{av}. Because the induced current is proportional to the induced emf and the induced emf, in turn, is given by Faraday's law, we can express ΔQ as a function of the number of turns of the coil, the magnetic field, the resistance of the coil, and the area of the coil. Knowing the reversal time, we can find the average current from its definition and the average emf from Ohm's law.

(*a*) Express the total charge that passes through the coil in terms of the induced current:

$$\Delta Q = I_{\mathrm{av}}\Delta t \qquad (1)$$

Relate the induced current to the induced emf:

$$I = I_{\mathrm{av}} = \frac{\mathcal{E}}{R}$$

Using Faraday's law, express the induced emf in terms of ϕ_m:

$$\mathcal{E} = -\frac{\Delta\phi_m}{\Delta t}$$

Substitute in equation (1) and simplify to obtain:

$$\Delta Q = \frac{|\mathcal{E}|}{R}\Delta t = \frac{\dfrac{\Delta\phi_m}{\Delta t}}{R}\Delta t = \frac{2\phi_m}{R}$$

$$= \frac{2NBA}{R} = \frac{2NB\left(\dfrac{\pi}{4}d^2\right)}{R}$$

$$= \frac{NB\pi d^2}{2R}$$

where d is the diameter of the coil.

Substitute numerical values and evaluate ΔQ:

$$\Delta Q = \frac{(100)(1.00\,\text{T})\pi(0.0200\,\text{m})^2}{2(50.0\,\Omega)}$$

$$= 1.257\,\text{mC} = \boxed{1.26\,\text{mC}}$$

(*b*) Apply the definition of average current to obtain:

$$I_{av} = \frac{\Delta Q}{\Delta t} = \frac{1.257\,\text{mC}}{0.100\,\text{s}} = 12.57\,\text{mA}$$

$$= \boxed{12.6\,\text{mA}}$$

(*c*) Using Ohm's law, relate the average emf in the coil to the average current:

$$\mathcal{E}_{av} = I_{av}R = (12.57\,\text{mA})(50.0\,\Omega)$$

$$= \boxed{628\,\text{mV}}$$

34 •• At the equator, a 1000-turn coil that has a cross-sectional area of 300 cm^2 and a resistance of 15.0 Ω is aligned so that its plane is perpendicular to Earth's magnetic field of 0.700 G. (*a*) If the coil is flipped over in 0.350 s, what is the average induced current in it during the 0.350 s? (*b*) How much charge flows through a cross section of the coil wire during the 0.350 s?

Picture the Problem (*a*) Because the average induced current is proportional to the induced emf and the induced emf, in turn, is given by Faraday's law, we can find I_{av} from the change in the magnetic flux through the coil, the resistance of the coil, and the time required for the flipping of the coil. (*b*) Knowing the average current, we can use its definition to find the charge flowing in the coil.

(*a*) The average induced current is given by:

$$I_{av} = \frac{\mathcal{E}}{R}$$

The induced emf in the coil is the rate at which the magnetic flux is changing:

$$\mathcal{E}=\frac{\Delta\phi_{\mathrm{m}}}{\Delta t}=\frac{2\phi_{\mathrm{m}}}{\Delta t}=\frac{2NBA}{\Delta t}$$

Substituting for $\mathcal{E}$ yields:

$$I_{\mathrm{av}}=\frac{2NBA}{R\Delta t}$$

Substitute numerical values and evaluate I_{av}:

$$I_{\mathrm{av}}=\frac{2(1000)\left(0.700\ \mathrm{G}\times\frac{1\ \mathrm{T}}{10^4\ \mathrm{G}}\right)\left(300\ \mathrm{cm}^2\times\left(\frac{1\ \mathrm{m}}{10^2\ \mathrm{cm}}\right)^2\right)}{(15.0\ \Omega)(0.350\ \mathrm{s})}=\boxed{800\ \mu\mathrm{A}}$$

(*b*) The average current is also given by:

$$I_{\mathrm{av}}=\frac{\Delta Q}{\Delta t}\Rightarrow\Delta Q=I_{\mathrm{av}}\Delta t$$

Substitute numerical values and evaluate ΔQ:

$$\Delta Q=(0.800\ \mathrm{mA})(0.350\ \mathrm{s})=\boxed{280\ \mu\mathrm{C}}$$

35 •• A *current integrator* measures the current as a function of time and integrates (adds) the current to find the total charge passing through it. (Because $I = dq/dt$, the integrator calculates the integral of the current or $Q=\int I dt$.) A circular coil that has 300 turns and a radius equal to 5.00 cm is connected to such an instrument. The total resistance of the circuit is 20.0 Ω. The plane of the coil is originally aligned perpendicular to Earth's magnetic field at some point. When the coil is rotated through 90º about an axis that is in the plane of the coil, a charge of 9.40 μC passes through the current integrator is measured to be 9.40 μC. Calculate the magnitude of Earth's magnetic field at that point.

Picture the Problem We can use Faraday's law to express Earth's magnetic field at this location in terms of the induced emf and Ohm's law to relate the induced emf to the charge that passes through the current integrator.

Using Faraday's law, express the induced emf in terms of the change in the magnetic flux as the coil is rotated through 90°:

$$\mathcal{E}=\left|-\frac{\Delta\phi_{\mathrm{m}}}{\Delta t}\right|=\frac{NBA}{\Delta t}=\frac{NB\pi r^2}{\Delta t}$$

Solving for B yields:

$$B=\frac{\mathcal{E}\Delta t}{N\pi r^2}$$

Using Ohm's law, relate the induced emf to the induced current:

$$\mathcal{E} = IR = \frac{\Delta Q}{\Delta t}R$$

where ΔQ is the charge that passes through the current integrator.

Substitute for $\mathcal{E}$ and simplify to obtain:

$$B = \frac{\frac{\Delta Q}{\Delta t}R\Delta t}{N\pi r^2} = \frac{\Delta QR}{N\pi r^2}$$

Substitute numerical values and evaluate B:

$$B = \frac{(9.40\,\mu\text{C})(20.0\,\Omega)}{(300)\pi(0.0500\,\text{m})^2} = \boxed{79.8\,\mu\text{T}}$$

Motional EMF

36 •• A 30.0-cm long rod moves steadily at 8.00 m/s in a plane that is perpendicular to a magnetic field of 500 G. The velocity of the rod is perpendicular to its length. Find (*a*) the magnetic force on an electron in the rod, (*b*) the electrostatic field in the rod, and (*c*) the potential difference between the ends of the rod.

Picture the Problem We can apply the equation for the force on a charged particle moving in a magnetic field to find the magnetic force acting on an electron in the rod. We can use $\vec{E} = \vec{v} \times \vec{B}$ to find E and $V = E\ell$, where ℓ is the length of the rod, to find the potential difference between its ends.

(*a*) Relate the magnetic force on an electron in the rod to the speed of the rod, the electronic charge, and the magnetic field in which the rod is moving:

$$\vec{F} = q\vec{v} \times \vec{B}$$

and

$$F = qvB\sin\theta$$

Substitute numerical values and evaluate F:

$$F = (1.602\times10^{-19}\,\text{C})(8.00\,\text{m/s})\times(0.0500\,\text{T})\sin 90° = \boxed{6.4\times10^{-20}\,\text{N}}$$

(*b*) Express the electrostatic field $\vec{E}$ in the rod in terms of the magnetic field $\vec{B}$:

$\vec{E} = \vec{v} \times \vec{B}$ and $E = vB\sin\theta$ where θ is the angle between $\vec{v}$ and $\vec{B}$.

Substitute numerical values and evaluate B:

$$E = (8.00\,\text{m/s})(0.0500\,\text{T})\sin 90° = 0.400\,\text{V/m} = \boxed{0.40\,\text{V/m}}$$

(c) Relate the potential difference between the ends of the rod to its length ℓ and the electric field E:

$$V = E\ell$$

Substitute numerical values and evaluate V:

$$V = (0.400\,\text{V/m})(0.300\,\text{m}) = \boxed{0.12\,\text{V}}$$

37 •• A 30.0-cm long rod moves in a plane that is perpendicular to a magnetic field of 500 G. The velocity of the rod is perpendicular to its length. Find the speed of the rod if the potential difference between the ends is 6.00 V.

Picture the Problem We can use $\vec{E} = \vec{v} \times \vec{B}$ to relate the speed of the rod to the electric field in the rod and magnetic field in which it is moving and $V = E\ell$ to relate the electric field in the rod to the potential difference between its ends.

Express the electrostatic field $\vec{E}$ in the rod in terms of the magnetic field $\vec{B}$ and solve for v:

$\vec{E} = \vec{v} \times \vec{B}$ and $v = \dfrac{E}{B\sin\theta}$ where θ is the angle between $\vec{v}$ and $\vec{B}$.

Relate the potential difference between the ends of the rod to its length ℓ and the electric field E and solve for E:

$$V = E\ell \Rightarrow E = \frac{V}{\ell}$$

Substitute for E to obtain:

$$v = \frac{V}{B\ell\sin\theta}$$

Substitute numerical values and evaluate v:

$$v = \frac{6.00\,\text{V}}{(0.0500\,\text{T})(0.300\,\text{m})\sin 90°} = \boxed{400\,\text{m/s}}$$

38 •• In Figure 28-45, let the magnetic field strength be 0.80 T, the rod speed be 10 m/s, the rod length be 20 cm, and the resistance of the resistor be 2.0 Ω. (The resistance of the rod and rails are negligible.) Find (a) the induced emf in the circuit, (b) the induced current in the circuit (including direction), and (c) the force needed to move the rod with constant speed (assuming negligible friction). Find (d) the power delivered by the force found in Part (c) and (e) the rate of Joule heating in the resistor.

Picture the Problem Because the speed of the rod is constant, an external force must act on the rod to counter the magnetic force acting on the induced current. We can use the motional-emf equation $\mathcal{E} = vB\ell$ to evaluate the induced emf, Ohm's law to find the current in the circuit, Newton's second law to find the force needed to move the rod with constant speed, and $P = Fv$ to find the power input by the force.

(*a*) Relate the induced emf in the circuit to the speed of the rod, the magnetic field, and the length of the rod:

$$\mathcal{E} = vB\ell = (10\,\text{m/s})(0.80\,\text{T})(0.20\,\text{m}) = 1.60\,\text{V} = \boxed{1.6\,\text{V}}$$

(*b*) Using Ohm's law, relate the current in the circuit to the induced emf and the resistance of the circuit:

$$I = \frac{\mathcal{E}}{R} = \frac{1.60\,\text{V}}{2.0\,\Omega} = \boxed{0.80\,\text{A}}$$

Note that, because the rod is moving to the right, the flux in the region defined by the rod, the rails, and the resistor is increasing. Hence, in accord with Lenz's law, the current must be counterclockwise if its magnetic field is to counter this increase in flux.

(*c*) Because the rod is moving with constant speed in a straight line, the net force acting on it must be zero. Apply Newton's second law to relate F to the magnetic force F_{m}:

$$\sum F_x = F - F_{\text{m}} = 0$$

Solving for F and substituting for F_{m} yields:

$$F = F_{\text{m}} = BI\ell$$

Substitute numerical values and evaluate F:

$$F = (0.80\,\text{T})(0.80\,\text{A})(0.20\,\text{m}) = 0.128\,\text{N} = \boxed{0.13\,\text{N}}$$

(*d*) Express the power input by the force in terms of the force and the velocity of the rod:

$$P = Fv = (0.128\,\text{N})(10\,\text{m/s}) = \boxed{1.3\,\text{W}}$$

(*e*) The rate of Joule heat production is given by:

$$P = I^2R = (0.80\,\text{A})^2(2.0\,\Omega) = \boxed{1.3\,\text{W}}$$

39 •• A 10-cm by 5.0-cm rectangular loop (Figure 28-46) that has a resistance equal to 2.5 Ω moves at a constant speed of 2.4 cm/s through a region that has a uniform 1.7-T magnetic field directed out of the page as shown. The front of the loop enters the field region at time $t = 0$. (*a*) Graph the flux through the loop as a function of time. (*b*) Graph the induced emf and the current in the loop as functions of time. Neglect any self-inductance of the loop and construct your graphs to include the interval $0 \le t \le 16$ s.

Picture the Problem We'll need to determine how long it takes for the loop to completely enter the region in which there is a magnetic field, how long it is in the region, and how long it takes to leave the region. Once we know these times, we can use its definition to express the magnetic flux as a function of time. We can use Faraday's law to find the induced emf as a function of time.

(*a*) Find the time required for the loop to enter the region where there is a uniform magnetic field:

$$t = \frac{\ell_{\text{side of loop}}}{v} = \frac{10\,\text{cm}}{2.4\,\text{cm/s}} = 4.17\,\text{s}$$

Letting w represent the width of the loop, express and evaluate ϕ_m for $0 < t < 4.17\,\text{s}$:

$$\begin{aligned}\phi_m &= NBA = NBwvt\\ &= (1.7\,\text{T})(0.050\,\text{m})(0.024\,\text{m/s})t\\ &= (2.04\,\text{mWb/s})t\end{aligned}$$

Find the time during which the loop is fully in the region where there is a uniform magnetic field:

$$t = \frac{\ell_{\text{side of loop}}}{v} = \frac{10\,\text{cm}}{2.4\,\text{cm/s}} = 4.17\,\text{s}$$

that is, the loop will begin to exit the region when $t = 8.33$ s.

Express ϕ_m for $4.17\,\text{s} < t < 8.33\,\text{s}$:

$$\begin{aligned}\phi_m &= NBA = NB\ell w\\ &= (1.7\,\text{T})(0.10\,\text{m})(0.050\,\text{m})\\ &= 8.50\,\text{mWb}\end{aligned}$$

The left-end of the loop will exit the field when $t = 12.5$ s. Express ϕ_m for $8.33\,\text{s} < t < 12.5\,\text{s}$:

$$\phi_m = mt + b$$

where m is the slope of the line and b is the ϕ_m-intercept.

For $t = 8.33$ s and $\phi_m = 8.50$ mWb:

$$8.50\,\text{mWb} = m(8.33\,\text{s}) + b \qquad (1)$$

For $t = 12.5$ s and $\phi_m = 0$:

$$0 = m(12.5\,\text{s}) + b \qquad (2)$$

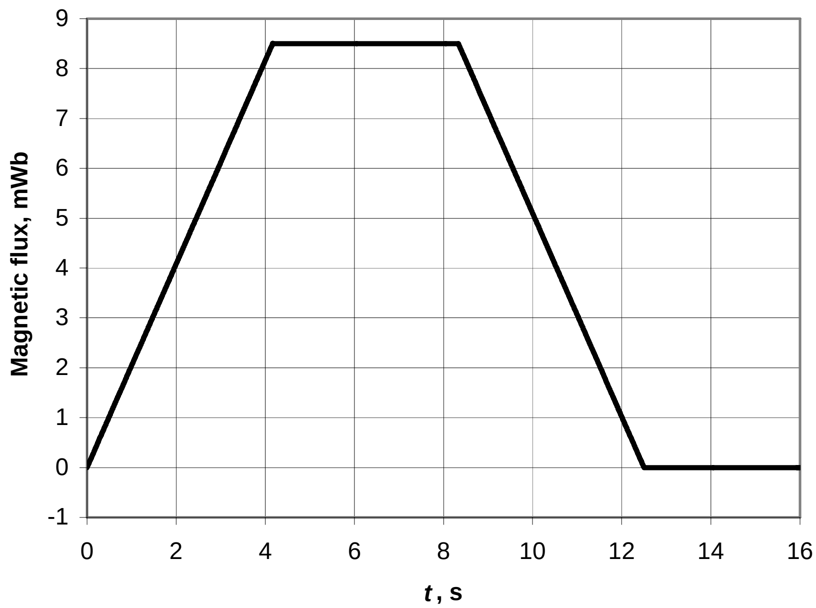
Solve equations (1) and (2) simultaneously to obtain:

$$\phi_m = -(2.04\,\text{mWb/s})t + 25.5\,\text{mWb}$$

The loop will be completely out of the magnetic field when $t > 12.5$ s and:

$$\phi_{\mathrm{m}} = 0$$

The following graph of $\phi_{\mathrm{m}}(t)$ was plotted using a spreadsheet program.



(*b*) Using Faraday's law, relate the induced emf to the magnetic flux:

$$\mathcal{E} = -\frac{d\phi_{\mathrm{m}}}{dt}$$

During the interval $0 < t < 4.17\,\mathrm{s}$:

$$\mathcal{E} = -\frac{d}{dt}\left[(2.04\,\mathrm{mWb/s})t\right] = -2.04\,\mathrm{mV}$$

During the interval $4.17\,\mathrm{s} < t < 8.33\,\mathrm{s}$:

$$\mathcal{E} = -\frac{d}{dt}\left[8.50\,\mathrm{mWb}\right] = 0$$

During the interval $8.33\,\mathrm{s} < t < 12.5\,\mathrm{s}$:

$$\mathcal{E} = -\frac{d}{dt}\left[(-2.04\,\mathrm{mWb/s})t + 25.5\,\mathrm{mWb}\right] = 2.04\,\mathrm{mV}$$

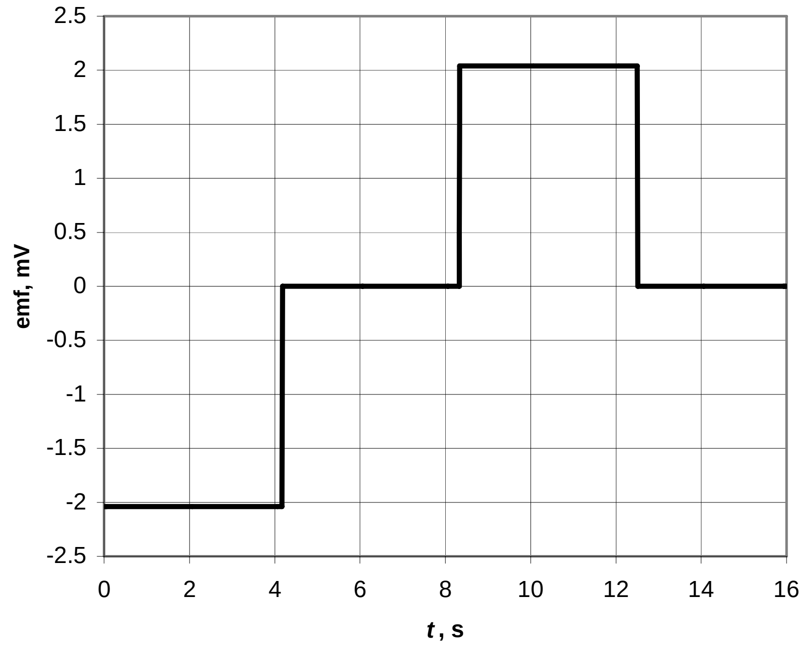
For $t > 12.5$ s:

$$\mathcal{E} = 0$$

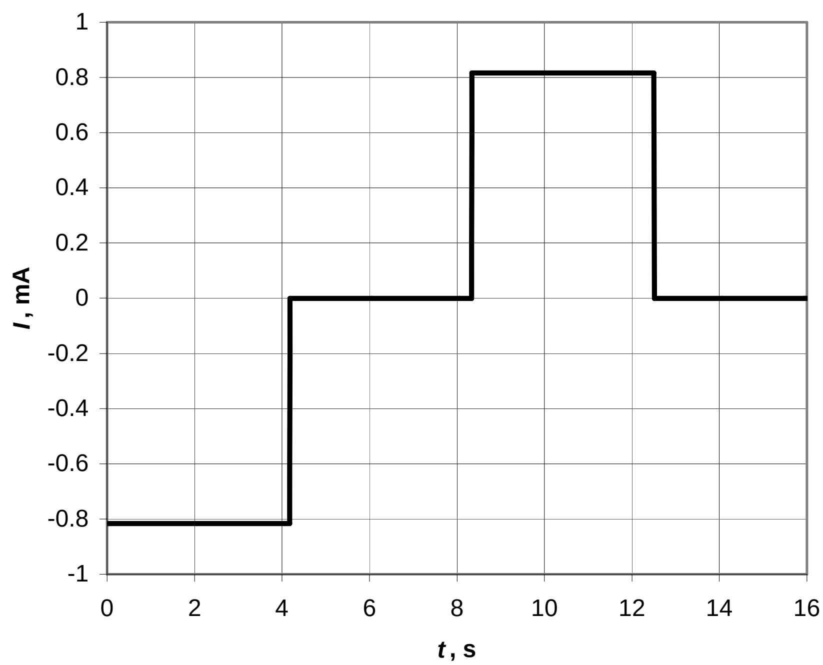
The current in each of these intervals is given by Ohm's law:

$$I = \frac{\mathcal{E}}{R}$$

The following graph of $\mathcal{E}(t)$ was plotted using a spreadsheet program.



The following graph of $I(t)$ was plotted using a spreadsheet program.



40 •• A uniform 1.2-T magnetic field is in the +*z* direction. A conducting rod of length 15 cm lies parallel to the *y* axis and oscillates in the *x* direction with displacement given by $x = (2.0\text{ cm})\cos(120\pi t)$, where 120π has units of rad/s . (*a*) Find an expression for the potential difference between the ends the rod as a function of time? (*b*) What is the maximum potential difference between the ends the rod?

Picture the Problem The rod is executing simple harmonic motion in the *xy* plane, i.e., in a plane perpendicular to the magnetic field. The emf induced in the rod is a consequence of its motion in this magnetic field and is given by $|\mathcal{E}| = vB\ell$. Because we're given the position of the oscillator as a function of time, we can differentiate this expression to obtain *v*.

(*a*) The potential difference between the ends of the rod is given by:

$$\mathcal{E} = vB\ell = B\ell\frac{dx}{dt}$$

Evaluate *dx*/*dt*:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left[(2.0\,\text{cm})\cos 120\pi t\right]\\ &= -(2.0\,\text{cm})\left(120\,\text{s}^{-1}\right)\pi\sin 120\pi t\\ &= -(7.54\,\text{m/s})\sin 120\pi t\end{aligned}$$

Substitute numerical values and evaluate $\mathcal{E}$:

$$\mathcal{E} = -(1.2\,\text{T})(0.15\,\text{m})(7.54\,\text{m/s})\sin 120\pi t = \boxed{-(1.4\,\text{V})\sin 120\pi t}$$

(*b*) The maximum potential difference between the ends the rod is the amplitude of the expression for $\mathcal{E}$ derived in Part (*a*):

$$\mathcal{E}_{\text{max}} = \boxed{1.4\,\text{V}}$$

41 •• **[SSM]** In Figure 28-47, the rod has a mass *m* and a resistance *R*. The rails are horizontal, frictionless and have negligible resistances. The distance between the rails is ℓ. An ideal battery that has an emf $\mathcal{E}$ is connected between points *a* and *b* so that the current in the rod is downward. The rod released from rest at $t = 0$. (*a*) Derive an expression for the force on the rod as a function of the speed. (*b*) Show that the speed of the rod approaches a terminal speed and find an expression for the terminal speed. (*c*) What is the current when the rod is moving at its terminal speed?

Picture the Problem (*a*) The net force acting on the rod is the magnetic force it experiences as a consequence of carrying a current and being in a magnetic field. The net emf that drives *I* in this circuit is the difference between the emf of the

battery and the emf induced in the rod as a result of its motion. Applying a right-hand rule to the rod reveals that the direction of this magnetic force is to the right. Hence the rod will accelerate to the right when it is released. (*b*) We can obtain the equation of motion of the rod by applying Newton's second law to relate its acceleration to $\mathcal{E}$, B, I, R and ℓ. (*c*) Letting $v = v_t$ in the equation for the current in the circuit will yield current when the rod is at its terminal speed.

(*a*) Express the magnetic force on the current-carrying rod:

$$F_m = I\ell B$$

The current in the rod is given by:

$$I = \frac{\mathcal{E} - B\ell v}{R} \qquad (1)$$

Substituting for I yields:

$$F_m = \left(\frac{\mathcal{E} - B\ell v}{R}\right)\ell B = \boxed{\frac{B\ell}{R}\left(\mathcal{E} - B\ell v\right)}$$

(*b*) Letting the direction of motion of the rod be the positive x direction, apply $\sum F_x = ma_x$ to the rod:

$$\frac{B\ell}{R}\left(\mathcal{E} - B\ell v\right) = m\frac{dv}{dt}$$

Solving for dv/dt yields:

$$\frac{dv}{dt} = \frac{B\ell}{mR}\left(\mathcal{E} - B\ell v\right)$$

Note that as v increases, $\mathcal{E} - B\ell v \to 0$, $dv/dt \to 0$ and the rod approaches its terminal speed v_t. Set $dv/dt = 0$ to obtain:

$$\frac{B\ell}{mR}\left(\mathcal{E} - B\ell v_t\right) = 0 \Rightarrow v_t = \boxed{\frac{\mathcal{E}}{B\ell}}$$

(*c*) Substitute v_t for v in equation (1) to obtain:

$$I = \frac{\mathcal{E} - B\ell\dfrac{\mathcal{E}}{B\ell}}{R} = \boxed{0}$$

42 • A uniform magnetic field is established perpendicular to the plane of a loop that has a radius equal to 5.00 cm and a resistance equal to 0.400 Ω. The magnitude of the field is increasing at a rate of 40.0 mT/s. Find (*a*) the magnitude of the induced emf in the loop, (*b*) the induced current in the loop, and (*c*) the rate of Joule heating in the loop.

Picture the Problem (*a*) We can find the magnitude of the induced emf by applying Faraday's law to the loop. (*b*) and (*c*) The application of Ohm's law will yield the induced current in the loop and we can find the rate of Joule heating using $P = I^2R$.

(*a*) Apply Faraday's law to express the induced emf in the loop in terms of the rate of change of the magnetic field:

$$|\mathcal{E}| = \frac{d\phi_{\mathrm{m}}}{dt} = \frac{d}{dt}(AB) = A\frac{dB}{dt} = \pi R^2\frac{dB}{dt}$$

Substitute numerical values and evaluate $|\mathcal{E}|$:

$$|\mathcal{E}| = \pi(0.0500\,\mathrm{m})^2(40.0\,\mathrm{mT/s}) = 0.3142\,\mathrm{mV} = \boxed{0.314\,\mathrm{mV}}$$

(*b*) Using Ohm's law, relate the induced current to the induced voltage and the resistance of the loop and evaluate *I*:

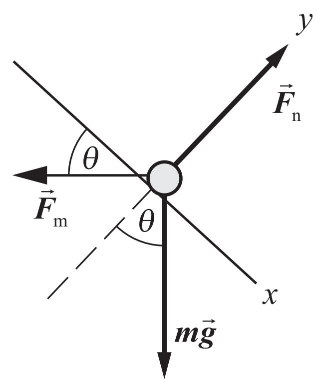
$$I = \frac{\mathcal{E}}{R} = \frac{0.3142\,\mathrm{mV}}{0.400\,\Omega} = 0.7854\,\mathrm{mA} = \boxed{0.785\,\mathrm{mA}}$$

(*c*) Express the rate at which power is dissipated in a conductor in terms of the induced current and the resistance of the loop and evaluate *P*:

$$P = I^2R = (0.7854\,\mathrm{mA})^2(0.400\,\Omega) = \boxed{0.247\,\mu\mathrm{W}}$$

43 •• In Figure 28-48, a conducting rod that has a mass *m* and a negligible resistance is free to slide without friction along two parallel frictionless rails that have negligible resistances separated by a distance ℓ and connected by a resistance *R*. The rails are attached to a long inclined plane that makes an angle θ with the horizontal. There is a magnetic field directed upward as shown. (*a*) Show that there is a retarding force directed up the incline given by $F = \left(B^2\ell^2 v\cos^2\theta\right)/R$. (*b*) Show that the terminal speed of the rod is $v_{\mathrm{t}} = (mgR\sin\theta)/\left(B^2\ell^2\cos^2\theta\right)$.

Picture the Problem The free-body diagram shows the forces acting on the rod as it slides down the inclined plane. The retarding force is the component of F_{m} acting up the incline, i.e., in the $-x$ direction. We can express F_{m} using the expression for the force acting on a conductor moving in a magnetic field. Recognizing that only the horizontal component of the rod's velocity $\vec{v}$ produces an induced emf, we can apply the expression for a motional emf in conjunction with Ohm's law to find the induced current in the rod. In Part (*b*) we can apply Newton's second law to obtain an expression for dv/dt and set this expression equal to zero to obtain v_{t}.



(*a*) Express the retarding force acting on the rod:

$$F = F_m \cos\theta \qquad (1)$$

where

$$F_m = I\ell B$$

and I is the current induced in the rod as a consequence of its motion in the magnetic field.

Express the induced emf due to the motion of the rod in the magnetic field:

$$\mathcal{E} = B\ell v\cos\theta$$

Using Ohm's law, relate the current I in the circuit to the induced emf:

$$I = \frac{\mathcal{E}}{R} = \frac{B\ell v\cos\theta}{R}$$

Substitute in equation (1) to obtain:

$$F = \left(\frac{B\ell v\cos\theta}{R}\right)\ell B\cos\theta = \boxed{\left(B^2\ell^2 v\cos^2\theta\right)/R}$$

(*b*) Apply $\sum F_x = ma_x$ to the rod:

$$mg\sin\theta - \frac{B^2\ell^2 v}{R}\cos^2\theta = m\frac{dv}{dt}$$

and

$$\frac{dv}{dt} = g\sin\theta - \frac{B^2\ell^2 v}{mR}\cos^2\theta$$

When the rod reaches its terminal speed v_t, $dv/dt = 0$:

$$0 = g\sin\theta - \frac{B^2\ell^2 v_t}{mR}\cos^2\theta$$

Solve for v_t to obtain:

$$v_t = \boxed{(mgR\sin\theta)/\left(B^2\ell^2\cos^2\theta\right)}$$

44 ••• A conducting rod of length ℓ rotates at constant angular speed ω about one end, in a plane perpendicular to a uniform magnetic field B (Figure 28-49). (*a*) Show that the potential difference between the ends of the rod is $\frac{1}{2}\ell^2\theta$. (*b*) Let the angle θ between the rotating rod and the dashed line be given by $\theta = \omega t$. Show that the area of the pie-shaped region swept out by the rod during time t is $\frac{1}{2}\ell^2\theta$. (*c*) Compute the flux ϕ_m through this area, and apply $\mathcal{E} = -d\phi_m/dt$ (Faraday's law) to show that the motional emf is given by $\frac{1}{2}B\omega\ell^2$.

Picture the Problem We can use $\vec{F} = q\vec{v} \times \vec{B}$ to express the magnetic force acting on the moving charged body. Expressing the emf induced in a segment of the rod of length dr and integrating this expression over the length of the rod will lead us to an expression for the induced emf.

(*a*) Use the motional emf equation to express the emf induced in a segment of the rod of length dr and at a distance r from the pivot:

$$d\mathcal{E} = Brdv = Br\omega dr$$

Integrate this expression from $r = 0$ to $r = \ell$ to obtain:

$$\int_0^{\mathcal{E}} d\mathcal{E} = B\omega\int_0^{\ell} rdr \Rightarrow \mathcal{E} = \boxed{\tfrac{1}{2}B\omega\ell^2}$$

(*b*) Using Faraday's law, relate the induced emf to the rate at which the flux changes:

$$|\mathcal{E}| = \frac{d\phi_m}{dt}$$

Express the area dA, for any value of θ, between r and $r + dr$:

$$dA = r\theta dr$$

Integrate from $r = 0$ to $r = \ell$ to obtain:

$$A = \theta\int_0^{\ell} rdr = \boxed{\tfrac{1}{2}\theta\ell^2}$$

(*c*) Using its definition, express the magnetic flux through this area:

$$\phi_m = BA = \boxed{\tfrac{1}{2}B\ell^2\theta}$$

Differentiate ϕ_m with respect to time to obtain:

$$|\mathcal{E}| = \frac{d}{dt}\left[\tfrac{1}{2}B\ell^2\theta\right] = \tfrac{1}{2}B\ell^2\frac{d\theta}{dt} = \boxed{\tfrac{1}{2}B\ell^2\omega}$$

45 • **[SSM]** A 2.00-cm by 1.50-cm rectangular coil has 300 turns and rotates in a region that has a magnetic field of 0.400 T. (*a*) What is the maximum emf generated when the coil rotates at 60 rev/s? (*b*) What must its angular speed be to generate a maximum emf of 110 V?

Picture the Problem We can use the relationship $\mathcal{E}_{max} = 2\pi NBAf$ to relate the maximum emf generated to the area of the coil, the number of turns of the coil, the magnetic field in which the coil is rotating, and the angular speed at which it rotates.

(*a*) Relate the induced emf to the magnetic field in which the coil is rotating:

$$\mathcal{E}_{max} = NBA\omega = 2\pi NBAf \qquad (1)$$

Substitute numerical values and evaluate $\mathcal{E}_{max}$:

$$\mathcal{E}_{max} = 2\pi(300)(0.400\,\mathrm{T})(2.00\times10^{-2}\,\mathrm{m})(1.50\times10^{-2}\,\mathrm{m})(60\,\mathrm{s}^{-1}) = \boxed{14\,\mathrm{V}}$$

(*b*) Solve equation (1) for *f*:

$$f = \frac{\mathcal{E}_{max}}{2\pi NBA}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{110\,\mathrm{V}}{2\pi(300)(0.400\,\mathrm{T})(2.00\times10^{-2}\,\mathrm{m})(1.50\times10^{-2}\,\mathrm{m})} = \boxed{486\,\mathrm{rev/s}}$$

46 • The coil of Problem 45 rotates at 60 rev/s in a magnetic field. If the maximum emf generated by the coil is 24 V, what is the magnitude of the magnetic field?

Picture the Problem We can use the relationship $\mathcal{E}_{max} = NBA\omega$ to relate the maximum emf generated to the area of the coil, the number of turns of the coil, the magnitude of the magnetic field in which the coil is rotating, and the angular speed at which it rotates.

Relate the induced emf to the magnetic field in which the coil is rotating:

$$\mathcal{E}_{max} = NBA\omega \Rightarrow B = \frac{\mathcal{E}_{max}}{NA\omega}$$

Substitute numerical values and evaluate B:

$$B=\frac{24\,\text{V}}{2\pi(300)\left(2.00\times10^{-2}\text{ m}\right)\left(1.50\times10^{-2}\text{ m}\right)(60\text{ rev/s})}=\boxed{0.71\,\text{T}}$$

Inductance

47 • When the current in an 8.00-H coil is equal to 3.00 A and is increasing at 200 A/s, find (*a*) the magnetic flux through the coil and (*b*) the induced emf in the coil.

Picture the Problem We can use $\phi_\text{m}=LI$ to find the magnetic flux through the coil. We can apply Faraday's law to find the induced emf in the coil.

(*a*) The magnetic flux through the coil is the product of the self-inductance of the coil and the current it is carrying:

$$\phi_\text{m}=LI$$

When the current is 3.00 A:

$$\phi_\text{m}=(8.00\,\text{H})(3.00\,\text{A})=\boxed{24.0\,\text{Wb}}$$

(*b*) Use Faraday's law to relate $\mathcal{E}$, L, and dI/dt:

$$\mathcal{E}=-L\frac{dI}{dt}$$

Substitute numerical values and evaluate $\mathcal{E}$:

$$\mathcal{E}=-(8.00\,\text{H})(200\,\text{A/s})=\boxed{-1.60\,\text{kV}}$$

48 •• A 300-turn solenoid has a radius equal to 2.00 cm; a length equal to 25.0 cm, and a 1000-turn solenoid has a radius equal to 5.00 cm and is also 25.0-cm long. The two solenoids are coaxial, with one nested completely inside the other. What is their mutual inductance?

Picture the Problem We can find the mutual inductance of the two coaxial solenoids using $M_{2,1}=\dfrac{\phi_{\text{m}2}}{I_1}=\mu_0 n_2 n_1 \ell \pi r_1^2$.

Substitute numerical values and evaluate $M_{2,1}$:

$$M_{2,1}=\left(4\pi\times10^{-7}\text{ N/A}^2\right)\left(\frac{300}{0.250\,\text{m}}\right)\left(\frac{1000}{0.250\,\text{m}}\right)(0.250\,\text{m})\pi(0.0200\,\text{m})^2=\boxed{1.89\,\text{mH}}$$

49 •• **[SSM]** An insulated wire that has a resistance of 18.0 Ω/m and a length of 9.00 m will be used to construct a resistor. First, the wire is bent in half and then the doubled wire is wound on a cylindrical form (Figure 28-50) to create a 25.0-cm-long helix that has a diameter equal to 2.00 cm. Find both the resistance and the inductance of this wire-wound resistor.

Picture the Problem Note that the current in the two parts of the wire is in opposite directions. Consequently, the total flux in the coil is zero. We can find the resistance of the wire-wound resistor from the length of wire used and the resistance per unit length.

Because the total flux in the coil is zero:

$$L = \boxed{0}$$

Express the total resistance of the wire:

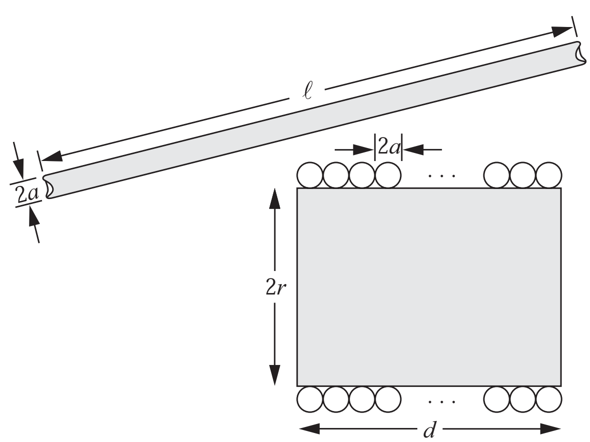
$$R = \left(18.0\frac{\Omega}{\text{m}}\right)L$$

Substitute numerical values and evaluate R:

$$R = \left(18.0\frac{\Omega}{\text{m}}\right)(9.00\,\text{m}) = \boxed{162\,\Omega}$$

50 •• You are given a length ℓ of wire that has radius a and are told to wind it into an inductor in the shape of a helix that has a circular cross section of radius r. The windings are to be as close together as possible without overlapping. Show that the self-inductance of this inductor is $L = \frac{1}{4}\mu_0 r\ell/a$.

Picture the Problem The wire of length ℓ and radius a is shown in the diagram, as is the inductor constructed with this wire and whose inductance L is to be found. We can use the equation for the self-inductance of a cylindrical inductor to derive an expression for L.



The self-inductance of an inductor with length d, cross-sectional area A, and number of turns per unit length n is:

$$L = \mu_0 n^2 A d \qquad (1)$$

The number of turns N is given by:

$$N = \frac{d}{2a} \Rightarrow n = \frac{N}{d} = \frac{1}{2a}$$

Assuming that $a << r$, the length of the wire ℓ is related to N and r:

$$\ell = N(2\pi r) = \left(\frac{d}{2a}\right) 2\pi r = \frac{\pi r}{a} d$$

Solving for d yields:

$$d = \frac{a\ell}{\pi r}$$

Substitute for d, A, and n in equation (1) to obtain:

$$L = \mu_0 \left(\frac{1}{2a}\right)^2 \left(\pi r^2\right)\left(\frac{a\ell}{\pi r}\right) = \boxed{\tfrac{1}{4}\mu_0 r\ell/a}$$

51 • Using the result of Problem 50, calculate the self-inductance of an inductor wound from 10 cm of wire that has a diameter of 1.0 mm into a coil that has a radius of 0.25 cm.

Picture the Problem We can substitute numerical values in the expression derived in Problem 50to find the self-inductance of the inductor.

From Problem 50 we have:

$$L = \frac{\mu_0 r d}{4a}$$

Substitute numerical values and evaluate L:

$$L = \frac{\left(4\pi \times 10^{-7}\ \text{N/A}^2\right)(0.25\,\text{cm})(10\,\text{cm})}{4(0.50\,\text{mm})} = \boxed{0.16\,\mu\text{H}}$$

52 ••• In Figure 28-51, circuit 2 has a total resistance of 300 Ω. After switch S is closed, the current in circuit 1 increases—reaching a value of 5.00 A after a long time. A charge of 200 μC passes through the galvanometer in circuit 2 during the time that the current in circuit 1 is increasing. What is the mutual inductance between the two coils?

Picture the Problem We can apply Kirchhoff's loop rule to the galvanometer circuit to relate the potential difference across L_2 to the potential difference across R_2. Integration of this equation over time will yield an equation that relates the mutual inductance between the two coils to the steady-state current in circuit 1 and the charge that flows through the galvanometer.

Apply Kirchhoff's loop rule to the galvanometer circuit:

$$M\frac{dI_1}{dt}+L_2\frac{dI_2}{dt}-R_2I_2=0$$

or

$$MdI_1+L_2dI_2-R_2I_2dt=0$$

Integrate each term from $t = 0$ to $t = \infty$:

$$M\int_0^\infty dI_1+L_2\int_0^\infty dI_2-R_2\int_0^\infty I_2dt=0$$

and

$$MI_{1\infty}+L_2I_{2\infty}-R_2Q=0$$

Because $I_{2\infty} = 0$:

$$MI_{1\infty}-R_2Q=0\Rightarrow M=\frac{R_2Q}{I_{1\infty}}$$

Substitute numerical values and evaluate M:

$$M=\frac{(300\,\Omega)(2.00\times10^{-4}\text{ C})}{5.00\text{ A}}=\boxed{12.0\text{ mH}}$$

53 ••• **[SSM]** Show that the inductance of a toroid of rectangular cross section, as shown in Figure 28-52 is given by $L=\frac{\mu_0N^2H\ln(b/a)}{2\pi}$ where N is the total number of turns, a is the inside radius, b is the outside radius, and H is the height of the toroid.

Picture the Problem We can use Ampere's law to express the magnetic field inside the rectangular toroid and the definition of magnetic flux to express ϕ_m through the toroid. We can then use the definition of self-inductance of a solenoid to express L.

Using the definition of the self-inductance of a solenoid, express L in terms of ϕ_m, N, and I:

$$L=\frac{N\phi_m}{I} \quad (1)$$

Apply Ampere's law to a closed path of radius $a < r < b$:

$$\oint_C \vec{B}\cdot d\vec{\ell}=B2\pi r=\mu_0I_C$$

or, because $I_C = NI$,

$$B2\pi r=\mu_0NI\Rightarrow B=\frac{\mu_0NI}{2\pi r}$$

Express the flux in a strip of height H and width dr:

$$d\phi_m=BHdr$$

Substituting for B yields:

$$d\phi_{\mathrm{m}} = \frac{\mu_0 NIH}{2\pi r}dr$$

Integrate $d\phi_{\mathrm{m}}$ from $r = a$ to $r = b$ to obtain:

$$\phi_{\mathrm{m}} = \frac{\mu_0 NIH}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 NIH}{2\pi}\ln\left(\frac{b}{a}\right)$$

Substitute for ϕ_{m} in equation (1) and simplify to obtain:

$$L = \boxed{\frac{\mu_0 N^2 H}{2\pi}\ln\left(\frac{b}{a}\right)}$$

Magnetic Energy

54 • A coil that has a self-inductance of 2.00 H and a resistance of 12.0 Ω is connected to an ideal 24.0-V battery. (*a*) What is the steady-state current? (*b*) How much energy is stored in the inductor when the steady-state current is established?

Picture the Problem The current in an *LR* circuit, as a function of time, is given by $I = I_{\mathrm{f}}\left(1 - e^{-t/\tau}\right)$, where $I_{\mathrm{f}} = \varepsilon_0/R$ and $\tau = L/R$. The energy stored in the inductor under steady-state conditions is stored in its magnetic field and is given by $U_{\mathrm{m}} = \frac{1}{2}LI_{\mathrm{f}}^2$.

(*a*) The final current is the quotient of the emf of the battery and the resistance of the coil:

$$I_{\mathrm{f}} = \frac{\mathcal{E}_0}{R} = \frac{24.0\,\mathrm{V}}{12.0\,\Omega} = \boxed{2.00\,\mathrm{A}}$$

(*b*) The energy stored in the inductor is:

$$U_{\mathrm{m}} = \tfrac{1}{2}LI_{\mathrm{f}}^2 = \tfrac{1}{2}(2.00\,\mathrm{H})(2.00\,\mathrm{A})^2 = \boxed{4.00\,\mathrm{J}}$$

55 • **[SSM]** In a plane electromagnetic wave, the magnitudes of the electric fields and magnetic fields are related by $E = cB$, where $c = 1\big/\sqrt{\epsilon_0\,\mu_0}$ is the speed of light. Show that when $E = cB$ the electric and the magnetic energy densities are equal.

Picture the Problem We can examine the ratio of u_{m} to u_{E} with $E = cB$ and $c = 1\big/\sqrt{\epsilon_0\,\mu_0}$ to show that the electric and magnetic energy densities are equal.

Express the ratio of the energy density in the magnetic field to the energy density in the electric field:

$$\frac{u_{\mathrm{m}}}{u_{\mathrm{E}}} = \frac{\dfrac{B^2}{2\mu_0}}{\frac{1}{2}\epsilon_0 E^2} = \frac{B^2}{\mu_0\,\epsilon_0\,E^2}$$

Because $E = cB$:

$$\frac{u_m}{u_E} = \frac{B^2}{\mu_0 \in_0 c^2 B^2} = \frac{1}{\mu_0 \in_0 c^2}$$

Substituting for c^2 and simplifying yields:

$$\frac{u_m}{u_E} = \frac{\mu_0 \in_0}{\mu_0 \in_0} = 1 \Rightarrow u_m = \boxed{u_E}$$

56 •• A 2000-turn solenoid has a cross-sectional area equal to 4.0 cm^2 and length equal to 30 cm. The solenoid carries a current of 4.0 A. (*a*) Calculate the magnetic energy stored in the solenoid using $U = \frac{1}{2}LI^2$, where $L = \mu_0 n^2 A\ell$. (*b*) Divide your answer in Part (*a*) by the volume of the region inside the solenoid to find the magnetic energy per unit volume in the solenoid. (*c*) Check your Part (*b*) result by computing the magnetic energy density from $\mu_m = B^2/2\mu_0$ where $B = \mu_0 nI$.

Picture the Problem We can use $L = \mu_0 n^2 A\ell$ to find the inductance of the solenoid and $B = \mu_0 nI$ to find the magnetic field inside it.

(*a*) Express the magnetic energy stored in the solenoid:

$$U_m = \tfrac{1}{2}LI^2$$

Relate the inductance of the solenoid to its dimensions and properties:

$$L = \mu_0 n^2 A\ell$$

Substitute for L to obtain:

$$U_m = \tfrac{1}{2}\mu_0 n^2 A\ell I^2$$

Substitute numerical values and evaluate U_m:

$$U_m = \tfrac{1}{2}\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(\frac{2000}{0.30\,\text{m}}\right)^2 \times\left(4.0\times10^{-4}\ \text{m}^2\right)(0.30\,\text{m})(4.0\,\text{A})^2$$

$$= 53.6\,\text{mJ} = \boxed{54\,\text{mJ}}$$

(*b*) The magnetic energy per unit volume in the solenoid is:

$$\frac{U_m}{V} = \frac{U_m}{A\ell} = \frac{53.6\,\text{mJ}}{\left(4.0\times10^{-4}\ \text{m}^2\right)(0.30\,\text{m})} = \boxed{0.45\,\text{kJ/m}^3}$$

(*c*) The magnetic energy density in the solenoid is given by:

$$u_m = \frac{B^2}{2\mu_0}$$

Substituting for B and simplifying yields:

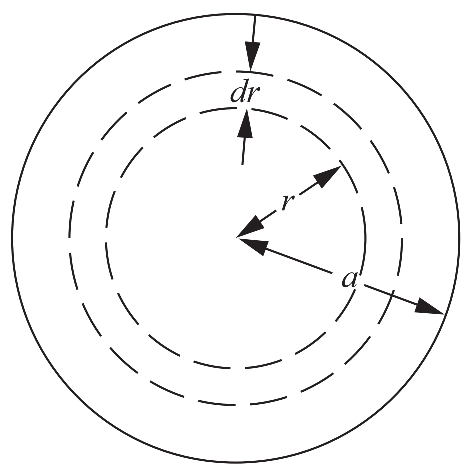
$$u_{\mathrm{m}} = \frac{(\mu_0 nI)^2}{2\mu_0} = \frac{\left(\mu_0 \frac{N}{\ell} I\right)^2}{2\mu_0} = \frac{\mu_0 N^2 I^2}{2\ell^2}$$

Substitute numerical values and evaluate u_{m}:

$$u_{\mathrm{m}} = \frac{(4\pi \times 10^{-7}\ \mathrm{N/A^2})(2000)^2(4.0\ \mathrm{A})^2}{2(0.30\ \mathrm{m})^2} = \boxed{0.45\ \mathrm{kJ/m^3}}$$

57 •• A long cylindrical wire has a radius equal to 2.0 cm and carries a current of 80 A uniformly distributed over its cross-sectional area. Find the magnetic energy per unit length within the wire.

Picture the Problem Consider a cylindrical annulus of thickness dr at a radius $r < a$. We can use its definition to express the total magnetic energy dU_{m} inside the cylindrical annulus and divide both sides of this expression by the length of the wire to express the magnetic energy per unit length dU'_m. Integration of this expression will give us the magnetic energy per unit length within the wire.



Express the magnetic energy within the cylindrical annulus:

$$dU_{\mathrm{m}} = \frac{B^2}{2\mu_0} V_{\mathrm{annulus}} = \frac{B^2}{2\mu_0} 2\pi r \ell dr = \frac{B^2}{\mu_0} \pi r \ell dr$$

Divide both sides of the equation by ℓ to express the magnetic energy per unit length dU'_m:

$$dU'_m = \frac{B^2}{\mu_0}\pi r dr \qquad (1)$$

Use Ampere's law to express the magnetic field inside the wire at a distance $r < a$ from its center:

$$2\pi rB = \mu_0 I_C \Rightarrow B = \frac{\mu_0 I_C}{2\pi r}$$

where I_C is the current inside the cylinder of radius r.

Because the current is uniformly distributed over the cross-sectional area of the wire:

$$\frac{I_C}{I} = \frac{\pi r^2}{\pi a^2} \Rightarrow I_C = \frac{r^2}{a^2}I$$

Substitute for I_C to obtain:

$$B = \frac{\mu_0 rI}{2\pi a^2}$$

Substituting for B in equation (1) and simplifying yields:

$$dU'_m = \frac{\left(\frac{\mu_0 rI}{2\pi a^2}\right)^2}{\mu_0}\pi r dr = \frac{\mu_0 I^2}{4\pi a^4}r^3 dr$$

Integrate dU'_m from $r = 0$ to $r = a$ to obtain:

$$U'_m = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{4\pi a^4}\cdot\frac{a^4}{4} = \frac{\mu_0 I^2}{16\pi}$$

Substitute numerical values and evaluate U'_m:

$$U'_m = \frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(80\ \text{A}\right)^2}{16\pi} = \boxed{0.16\ \text{mJ/m}}$$

Remarks: Note that the magnetic energy per unit length is independent of the radius of the cylinder and depends only on the total current.

58 •• A toroid that has a mean radius equal to 25.0 cm and a circular loops with radii equal to 2.00 cm is wound with a superconducting wire. The wire has a length equal to 1000 m and carries a current of 400 A. (*a*) What is the number of turns of the wire? (*b*) What is the magnetic field strength and magnetic energy density at the mean radius? (*c*) Estimate the total energy stored in this toroid by assuming that the energy density is uniformly distributed in the region inside the toroid.

Picture the Problem We can find the number of turns on the coil from the length of the superconducting wire and the cross-sectional radius of the coil. We can use $B=(\mu_0 NI)/(2\pi r_{\text{mean}})$ to find the magnetic field at the mean radius. We can find the energy density in the magnetic field from $u_{\text{m}}=B^2/(2\mu_0)$ and the total energy stored in the toroid by multiplying u_{m} by the volume of the toroid.

(*a*) Express the number of turns in terms of the length of the wire L and length required per turn $2\pi r$:

$$N=\frac{L}{2\pi r}$$

Substitute numerical values and evaluate N:

$$N=\frac{1000\,\text{m}}{2\pi(0.0200\,\text{m})}=7958$$
$$=\boxed{7.96\times10^3}$$

(*b*) B inside a tightly wound toroid or radius r is given by:

$$B=\frac{\mu_0 NI}{2\pi r}$$

Substitute numerical values and evaluate the magnetic field at the mean radius:

$$B=\frac{(4\pi\times10^{-7}\,\text{N/A}^2)(7958)(400\,\text{A})}{2\pi(0.250\,\text{m})}$$
$$=2.547\,\text{T}=\boxed{2.55\,\text{T}}$$

The energy density in the magnetic field is given by:

$$u_{\text{m}}=\frac{B^2}{2\mu_0}$$

Substitute numerical values and evaluate u_{m}:

$$u_{\text{m}}=\frac{(2.547\,\text{T})^2}{2(4\pi\times10^{-7}\,\text{N/A}^2)}=2.580\,\text{MJ/m}^3$$
$$=\boxed{2.58\,\text{MJ/m}^3}$$

(*c*) Relate the total energy stored in the toroid to the energy density in its magnetic field and the volume of the toroid:

$$U_{\text{m}}=u_{\text{m}}V_{\text{toroid}}$$

Think of the toroid as a cylinder of radius r and height $2\pi r_{\text{mean}}$ to obtain:

$$V_{\text{toroid}}=\pi r^2(2\pi r_{\text{mean}})=2\pi^2 r^2 r_{\text{mean}}$$

Substitute for V_{toroid} to obtain:

$$U_{\text{m}}=2\pi^2 r^2 r_{\text{mean}} u_{\text{m}}$$

Substitute numerical values and evaluate U_m:

$$U_m = 2\pi^2(0.0200\,\text{m})^2(0.250\,\text{m})(2.580\,\text{MJ/m}^3) = \boxed{5.09\,\text{kJ}}$$

RL Circuits

59 • [SSM] A circuit consists of a coil that has a resistance equal to 8.00 Ω and a self-inductance equal to 4.00 mH, an open switch and an ideal 100-V battery—all connected in series. At $t = 0$ the switch is closed. Find the current and its rate of change at times (*a*) $t = 0$, (*b*) $t = 0.100$ ms, (*c*) $t = 0.500$ ms, and (*d*) $t = 1.00$ ms.

Picture the Problem We can find the current using $I = I_f\left(1 - e^{-t/\tau}\right)$ where $I_f = \mathcal{E}_0/R$ and $\tau = L/R$ and its rate of change by differentiating this expression with respect to time.

Express the dependence of the current on I_f and τ:

$$I = I_f\left(1 - e^{-t/\tau}\right)$$

Evaluating I_f and τ yields:

$$I_f = \frac{\mathcal{E}_0}{R} = \frac{100\,\text{V}}{8.00\,\Omega} = 12.5\,\text{A}$$

and

$$\tau = \frac{L}{R} = \frac{4.00\,\text{mH}}{8.00\,\Omega} = 0.500\,\text{ms}$$

Substitute for I_f and τ to obtain:

$$I = (12.5\,\text{A})\left(1 - e^{-t/0.500\,\text{ms}}\right)$$

Express dI/dt:

$$\frac{dI}{dt} = (12.5\,\text{A})\left(-e^{-t/0.500\,\text{ms}}\right)\left(-2000\,\text{s}^{-1}\right)$$
$$= (25.0\,\text{kA/s})e^{-t/0.500\,\text{ms}}$$

(*a*) Evaluate I and dI/dt at $t = 0$:

$$I(0) = (12.5\,\text{A})\left(1 - e^0\right) = \boxed{0}$$

and

$$\left.\frac{dI}{dt}\right|_{t=0} = (25.0\,\text{kA/s})e^0 = \boxed{25.0\,\text{kA/s}}$$

(*b*) Evaluating I and dI/dt at $t = 0.100$ ms yields:

$$I(0.100\,\text{ms}) = (12.5\,\text{A})\left(1 - e^{-0.100\,\text{ms}/0.500\,\text{ms}}\right)$$
$$= \boxed{2.27\,\text{A}}$$

and

$$\left.\frac{dI}{dt}\right|_{t=0.500\text{ ms}} = (25.0\,\text{kA/s})e^{-0.100\text{ ms}/0.500\text{ ms}}$$
$$= \boxed{20.5\,\text{kA/s}}$$

(*c*) Evaluate I and dI/dt at $t = 0.500$ ms to obtain:

$$I(0.500\text{ ms}) = (12.5\,\text{A})\left(1 - e^{-0.500\text{ ms}/0.500\text{ ms}}\right)$$
$$= \boxed{7.90\,\text{A}}$$

and

$$\left.\frac{dI}{dt}\right|_{t=0.500\text{ ms}} = (25.0\,\text{kA/s})e^{-0.500\text{ ms}/0.500\text{ ms}}$$
$$= \boxed{9.20\,\text{kA/s}}$$

(*d*) Evaluating I and dI/dt at $t = 1.00$ ms yields:

$$I(1.00\text{ ms}) = (12.5\,\text{A})\left(1 - e^{-1.00\text{ ms}/0.500\text{ ms}}\right)$$
$$= \boxed{10.8\,\text{A}}$$

and

$$\left.\frac{dI}{dt}\right|_{t=1.00\text{ ms}} = (25.0\,\text{kA/s})e^{-1.00\text{ ms}/0.500\text{ ms}}$$
$$= \boxed{3.38\,\text{kA/s}}$$

60 • In the circuit shown in Figure 28-53, the throw of the make-before-break switch has been at contact *a* for a long time and the current in the 1.00 mH coil is equal to 2.00 A. At $t = 0$ the throw is quickly moved to contact *b*. The total resistance $R + r$ of the coil and the resistor is 10.0 Ω. Find the current when (*a*) $t = 0.500$ ms, and (*b*) $t = 10.0$ ms.

Picture the Problem We can find the current using $I(t) = I_0 e^{-t/\tau}$, where I_0 is the current at time $t = 0$ and $\tau = L/R$.

Express the current as a function of time:

$$I(t) = I_0 e^{-t/\tau} = (2.00\,\text{A})e^{-t/\tau}$$

Evaluating τ yields:

$$\tau = \frac{L}{R} = \frac{1.00\,\text{mH}}{10.0\,\Omega} = 0.100\,\text{ms}$$

Substitute for τ to obtain:

$$I(t) = (2.00\,\text{A})e^{-0.100\text{ ms}t}$$

(*a*) When $t = 0.500$ ms:

$$I(0.500\text{ ms}) = (2.00\,\text{A})e^{-0.500\text{ ms}/0.100\text{ ms}}$$
$$= \boxed{13.5\,\text{mA}}$$

(*b*) When $t = 10.0$ ms:

$$I(10.0\text{ ms}) = (2\text{ A})e^{-10.0\text{ ms}/0.100\text{ ms}} = (2.00\text{ A})e^{-100} = 7.44\times10^{-44}\text{ A} \approx \boxed{0}$$

61 •• [SSM] In the circuit shown in Figure 28-54, let $\mathcal{E}_0 = 12.0$ V, $R = 3.00\ \Omega$, and $L = 0.600$ H. The switch, which was initially open, is closed at time $t = 0$. At time $t = 0.500$ s, find (*a*) the rate at which the battery supplies energy, (*b*) the rate of Joule heating in the resistor, and (*c*) the rate at which energy is being stored in the inductor.

Picture the Problem We can find the current using $I = I_\text{f}\left(1-e^{-t/\tau}\right)$, where $I_\text{f} = \mathcal{E}_0/R,$ and $\tau = L/R,$ and its rate of change by differentiating this expression with respect to time.

Express the dependence of the current on I_f and τ:

$$I(t) = I_\text{f}\left(1-e^{-t/\tau}\right)$$

Evaluating I_f and τ yields:

$$I_\text{f} = \frac{\mathcal{E}_0}{R} = \frac{12.0\text{ V}}{3.00\ \Omega} = 4.00\text{ A}$$

and

$$\tau = \frac{L}{R} = \frac{0.600\text{ H}}{3.00\ \Omega} = 0.200\text{ s}$$

Substitute for I_f and τ to obtain:

$$I(t) = (4.00\text{ A})\left(1-e^{-t/0.200\text{ s}}\right)$$

Express dI/dt:

$$\frac{dI}{dt} = (4.00\text{ A})\left(-e^{-t/0.200\text{ s}}\right)\left(-5.00\text{ s}^{-1}\right) = (20.0\text{ A/s})e^{-t/0.200\text{ s}}$$

(*a*) The rate at which the battery supplies energy is given by:

$$P = I\mathcal{E}_0$$

Substituting for I and $\mathcal{E}_0$ yields:

$$P(t) = (4.00\text{ A})\left(1-e^{-t/0.200\text{s}}\right)(12.0\text{ V}) = (48.0\text{ W})\left(1-e^{-t/0.200\text{s}}\right)$$

The rate at which the battery supplies energy at $t = 0.500$ s is:

$$P(0.500\text{ s}) = (48.0\text{ W})\left(1-e^{-0.500/0.200\text{s}}\right) = \boxed{44.1\text{ W}}$$

(*b*) The rate of Joule heating is:

$$P_J = I^2 R$$

Substitute for *I* and *R* and simplify to obtain:

$$P_J = \left[(4.00\,\text{A})\left(1 - e^{-t/0.200\,\text{s}}\right)\right]^2 (3.00\,\Omega) = (48.0\,\text{W})\left(1 - e^{-t/0.200\,\text{s}}\right)^2$$

The rate of Joule heating at $t = 0.500$ s is:

$$P_J(0.500\,\text{s}) = (48.0\,\text{W})\left(1 - e^{-0.500\,\text{s}/0.200\,\text{s}}\right)^2 = \boxed{40.4\ \text{W}}$$

(*c*) Use the expression for the magnetic energy stored in an inductor to express the rate at which energy is being stored:

$$\frac{dU_L}{dt} = \frac{d}{dt}\left[\tfrac{1}{2} L I^2\right] = LI\frac{dI}{dt}$$

Substitute for *L, I,* and *dI*/*dt* to obtain:

$$\frac{dU_L}{dt} = (0.600\,\text{H})(4.00\,\text{A})\left(1 - e^{-t/0.200\,\text{s}}\right)(20.0\,\text{A/s})e^{-t/0.200\,\text{s}} = (48.0\,\text{W})\left(1 - e^{-t/0.200\,\text{s}}\right)e^{-t/0.200\,\text{s}}$$

Evaluate this expression for $t = 0.500$ s:

$$\left.\frac{dU_L}{dt}\right]_{t=0.500\,\text{s}} = (48.0\,\text{W})\left(1 - e^{-0.500\,\text{s}/0.200\,\text{s}}\right)e^{-0.500\,\text{s}/0.200\,\text{s}} = \boxed{3.62\ \text{W}}$$

Remarks: Note that, to a good approximation, $dU_L/dt = P - P_J$.

62 •• How many time constants must elapse before the current in an *RL* circuit (Figure 28-54) that is initially zero reaches (*a*) 90 percent, (*b*) 99 percent, and (*c*) 99.9 percent of its steady-state value?

Picture the Problem If the current is initially zero in an *LR* circuit, its value at some later time *t* is given by $I = I_f\left(1 - e^{-t/\tau}\right)$, where $I_f = \mathcal{E}_0/R$ and $\tau = L/R$ is the time constant for the circuit. We can find the number of time constants that must elapse before the current reaches any given fraction of its final value by solving this equation for t/τ.

Express the fraction of its final value to which the current has risen as a function of time:

$$\frac{I}{I_f} = 1 - e^{-t/\tau} \Rightarrow \frac{t}{\tau} = -\ln\left(1 - \frac{I}{I_f}\right)$$

(*a*) Evaluate t/τ for $I/I_f = 0.90$:

$$\left.\frac{t}{\tau}\right|_{90\%} = -\ln(1-0.90) = \boxed{2.3}$$

(*b*) Evaluate t/τ for $I/I_f = 0.99$:

$$\left.\frac{t}{\tau}\right|_{99\%} = -\ln(1-0.99) = \boxed{4.6}$$

(*c*) Evaluate t/τ for $I/I_f = 0.999$:

$$\left.\frac{t}{\tau}\right|_{99.9\%} = -\ln(1-0.999) = \boxed{6.91}$$

63 •• [SSM] A circuit consists of a 4.00-mH coil, a 150-Ω resistor, a 12.0-V ideal battery and an open switch—all connected in series. After the switch is closed: (*a*) What is the initial rate of increase of the current? (*b*) What is the rate of increase of the current when the current is equal to half its steady-state value? (*c*) What is the steady-state value of the current? (*d*) How long does it take for the current to reach 99 percent of its steady state value?

Picture the Problem If the current is initially zero in an *LR* circuit, its value at some later time *t* is given by $I = I_f\left(1-e^{-t/\tau}\right)$, where $I_f = \mathcal{E}_0/R$ and $\tau = L/R$ is the time constant for the circuit. We can find the rate of increase of the current by differentiating *I* with respect to time and the time for the current to reach any given fraction of its initial value by solving for *t*.

(*a*) Express the current in the circuit as a function of time:

$$I = \frac{\mathcal{E}_0}{R}\left(1-e^{-t/\tau}\right)$$

Express the initial rate of increase of the current by differentiating this expression with respect to time:

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{R}\frac{d}{dt}\left(1-e^{-t/\tau}\right) = \frac{\mathcal{E}_0}{R}\left(-e^{-t/\tau}\right)\left(-\frac{1}{\tau}\right) = \frac{\mathcal{E}_0}{\tau R}e^{-\frac{R}{L}t} = \frac{\mathcal{E}_0}{L}e^{-\frac{R}{L}t}$$

Evaluate dI/dt at $t = 0$ to obtain:

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{\mathcal{E}_0}{L}e^0 = \frac{12.0\,\text{V}}{4.00\,\text{mH}} = \boxed{3.00\,\text{kA/s}}$$

(*b*) When $I = 0.5I_f$:

$$0.5 = 1-e^{-t/\tau} \Rightarrow e^{-t/\tau} = 0.5$$

Evaluate dI/dt with $e^{-t/\tau}=0.5$ to obtain:

$$\left.\frac{dI}{dt}\right|_{e^{-t/\tau}=0.5}=0.5\frac{\mathcal{E}_0}{L}=0.5\left(\frac{12.0\,\text{V}}{4.00\,\text{mH}}\right)=\boxed{1.50\,\text{kA/s}}$$

(c) Calculate I_f from $\mathcal{E}_0$ and R:

$$I_\text{f}=\frac{\mathcal{E}_0}{R}=\frac{12.0\,\text{V}}{150\,\Omega}=\boxed{80.0\,\text{mA}}$$

(d) When $I = 0.99I_\text{f}$:

$$0.99=1-e^{-t/\tau}\Rightarrow e^{-t/\tau}=0.01$$

Solving for t and substituting for τ yields:

$$t=-\tau\ln(0.01)=-\frac{L}{R}\ln(0.01)$$

Substitute numerical values and evaluate t:

$$t=-\frac{4.00\,\text{mH}}{150\,\Omega}\ln(0.01)=\boxed{0.123\,\text{ms}}$$

64 •• A circuit consists of a large electromagnet that has an inductance of 50.0 H and a resistance of 8.00 Ω, a dc 250-V power source and an open switch—all connected in series. How long after the switch is closed is the current equal to (a) 10 A, and (b) 30 A.

Picture the Problem If the current is initially zero in an LR circuit, its value at some later time t is given by $I=I_\text{f}\left(1-e^{-t/\tau}\right)$, where $I_\text{f} = \mathcal{E}_0/R$ and $\tau = L/R$ is the time constant for the circuit. We can find the time for the current to reach any given value by solving this equation for t.

Evaluate I_f and τ:

$$I_\text{f}=\frac{\mathcal{E}_0}{R}=\frac{250\,\text{V}}{8.00\,\Omega}=31.25\,\text{A}$$

and

$$\tau=\frac{L}{R}=\frac{50.0\,\text{H}}{8.00\,\Omega}=6.25\,\text{s}$$

Solve $I=I_\text{f}\left(1-e^{-t/\tau}\right)$ for t:

$$t=-\tau\ln\left(1-\frac{I}{I_\text{f}}\right)$$

Substituting for τ and I_f yields:

$$t=-(6.25\,\text{s})\ln\left(1-\frac{I}{31.25\,\text{A}}\right)$$

(*a*) Evaluate *t* for *I* = 10 A:

$$t\big|_{10\,\text{A}} = -(6.25\,\text{s})\ln\left(1-\frac{10\,\text{A}}{31.25\,\text{A}}\right) = \boxed{2.4\,\text{s}}$$

(*b*) Evaluate *t* for *I* = 30 A:

$$t\big|_{30\,\text{A}} = -(6.25\,\text{s})\ln\left(1-\frac{30\,\text{A}}{31.25\,\text{A}}\right) = \boxed{20\,\text{s}}$$

65 •• [SSM] Given the circuit shown in Figure 28-55, assume that the inductor has negligible internal resistance and that the switch S has been closed for a long time so that a steady current exists in the inductor. (*a*) Find the battery current, the current in the 100 Ω resistor, and the current in the inductor. (*b*) Find the potential drop across the inductor immediately after the switch S is opened. (*c*) Using a **spreadsheet** program, make graphs of the current in the inductor and the potential drop across the inductor as functions of time for the period during which the switch is open.

Picture the Problem The self-induced emf in the inductor is proportional to the rate at which the current through it is changing. Under steady-state conditions, $dI/dt = 0$ and so the self-induced emf in the inductor is zero. We can use Kirchhoff's loop rule to obtain the current through and the voltage across the inductor as a function of time.

(*a*) Because, under steady-state conditions, the self-induced emf in the inductor is zero and because the inductor has negligible resistance, we can apply Kirchhoff's loop rule to the loop that includes the source, the 10-Ω resistor, and the 2-H inductor to find the current drawn from the battery and flowing through the inductor and the 10-Ω resistor:

$$10\,\text{V} - (10\,\Omega)I_{10-\Omega} = 0$$

Solving for $I_{10-\Omega}$ yields:

$$I_{10-\Omega} = I_{2-\text{H}} = \boxed{1.0\,\text{A}}$$

By applying Kirchhoff's junction rule at the junction between the resistors, we can conclude that:

$$I_{100\text{-}\Omega} = I_{\text{battery}} - I_{2\text{-H}} = \boxed{0}$$

(*b*) When the switch is opened, the current cannot immediately go to zero in the circuit because of the inductor. For a time, a current will circulate in the circuit loop between the inductor and the 100-Ω resistor. Because the current flowing through this circuit is initially 1 A, the voltage drop across the 100-Ω resistor is initially $\boxed{100\,\text{V.}}$ Conservation of energy (Kirchhoff's loop rule) requires that the voltage drop across the 2-H inductor is $V_{\text{2-H}} = \boxed{100\,\text{V.}}$

(*c*) Apply Kirchhoff's loop rule to the *RL* circuit to obtain:

$$L\frac{dI}{dt} + IR = 0$$

The solution to this differential equation is:

$$I(t) = I_0 e^{-\frac{R}{L}t} = I_0 e^{-\frac{t}{\tau}}$$

$$\text{where } \tau = \frac{L}{R} = \frac{2.0\,\text{H}}{100\,\Omega} = 0.020\,\text{s}$$

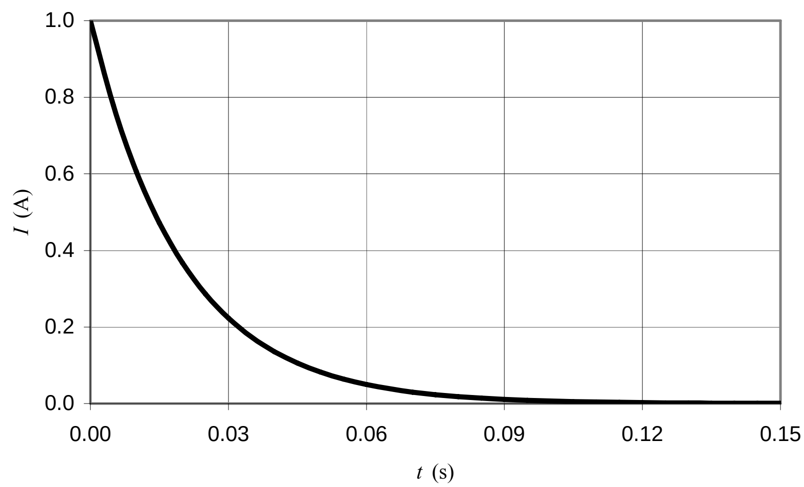
A spreadsheet program to generate the data for graphs of the current and the voltage across the inductor as functions of time is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	2.0	L
B2	100	R
B3	1	I_0
A6	0	t_0
B6	$B$3*EXP((–$B$2/$B$1)*A6)	$I_0 e^{-\frac{R}{L}t}$

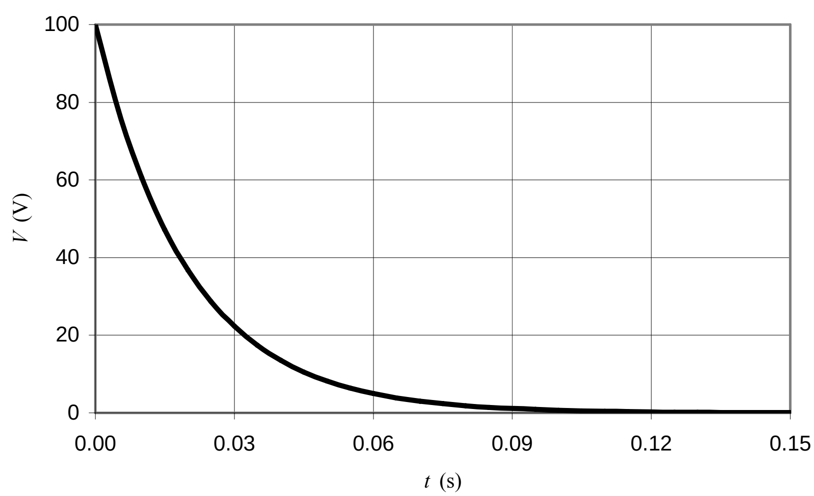
	A	B	C
1	L=	2	H
2	R=	100	ohms
3	I_0=	1	A
4			
5	t	$I(t)$	$V(t)$
6	0.000	1.00E+00	100.00
7	0.005	7.79E–01	77.88
8	0.010	6.07E–01	60.65
9	0.015	4.72E–01	47.24
10	0.020	3.68E–01	36.79
11	0.025	2.87E–01	28.65
12	0.030	2.23E–01	22.31

32	0.130	1.50E−03	0.15
33	0.135	1.17E−03	0.12
34	0.140	9.12E−04	0.09
35	0.145	7.10E−04	0.07
36	0.150	5.53E−04	0.06

The following graph of the current in the inductor as a function of time was plotted using the data in columns A and B of the spreadsheet program.



The following graph of the voltage across the inductor as a function of time was plotted using the data in columns A and C of the spreadsheet program.



66 •• Given the circuit shown in Figure 28-56, the inductor has negligible internal resistance and the switch S has been open for a long time. The switch is then closed. (*a*) Find the current in the battery, the current in the 100-Ω resistor, and the current in the inductor immediately after the switch is closed. (*b*) Find the

current in the battery, the current in the 100-Ω resistor, and the current in the inductor a long time after the switch is closed. After being closed for a long time the switch is now opened. (*c*) Find the current in the battery, the current in the 100-Ω resistor, and the current in the inductor immediately after the switch is opened. (*d*) Find the current in the battery, the current in the 100-Ω resistor, and the current in the inductor after the switch is opened for a long time.

Picture the Problem Let the current supplied by the battery be I_{battery}, the current through the inductor be I_L, and the current in the 100-Ω resistor be $I_{100\text{-}\Omega}$. (*a*) We simplify our calculations by using the fact that the current in an inductor cannot change abruptly. Thus, the current in the inductor must be zero just after the switch is closed, because the current is zero before the switch is closed. (*b*) When the current reaches its final value, *dI*/*dt* equals zero, and there is no potential drop across the inductor. The inductor thus acts like a short circuit; that is, the inductor acts like a wire with zero resistance. (*c*) Immediately after the switch is opened, the current in the inductor is the same as it was before. (*d*) A long time after the switch is opened, all the currents must be zero.

(*a*) The switch is just opened. The current through the inductor is zero, just as it was before the switch was closed. Apply the junction rule to relate I_{battery} and $I_{100\text{-}\Omega}$:

$$I_L = \boxed{0}$$

$$I_{\text{battery}} = I_{100-\Omega} + I_L \Rightarrow I_{\text{battery}} = I_{100-\Omega}$$

Apply Kirchhoff's loop rule to the loop on the right to obtain:

$$10.0\text{ V} - I_{\text{battery}}(10.0\,\Omega) - I_{100\text{-}\Omega}(100\,\Omega) = 0$$

or, because $I_{\text{battery}} = I_{100-\Omega}$,

$$10.0\text{ V} - I_{\text{battery}}(10.0\,\Omega) - I_{\text{battery}}(100\,\Omega) = 0$$

Solving for I_{battery} yields:

$$I_{\text{battery}} = I_{100\text{-}\Omega} = \frac{10.0\text{ V}}{10.0\,\Omega + 100\,\Omega} = \boxed{90.9\text{ mA}}$$

(*b*) After a long time, the currents are steady and the inductor acts like a short circuit, so the potential drop across the 100-Ω resistor is zero. Apply the loop rule to the loop to the right to obtain:

$$-(2.00\text{ H})\frac{dI_{\text{L}}}{dt} + I_{100-\Omega}(100\,\Omega) = 0$$

Because $\frac{dI_{\text{L}}}{dt}=0$:

$$I_{100-\Omega}(100\ \Omega)=0$$

and

$$I_{100-\Omega}=\boxed{0}$$

Apply the loop rule to the loop to the right to obtain:

$$10.0\ \text{V}-I_{\text{battery}}(10.0\ \text{V})-I_{100-\Omega}(100\ \Omega)=0$$

Because $I_{100-\Omega}=0$:

$$10.0\ \text{V}-I_{\text{battery}}(10.0\ \text{V})=0$$

and

$$I_{\text{battery}}=\boxed{1.00\ \text{A}}$$

Apply the junction rule to the three currents to obtain:

$$I_{\text{battery}}=I_L+I_{100-\Omega}\Rightarrow 1.00\ \text{A}=I_L+0$$

and so $I_L=\boxed{1.00\ \text{A}}$

(*c*) When the switch is reopened, I_1 *instantly* becomes zero. The current I_L in the inductor changes continuously, so at that instant I_L = 1.00 A. Apply the junction rule to obtain:

$$I_L=\boxed{1.00\ \text{A}},$$

$$I_{\text{battery}}=I_{100-\Omega}+I_L,$$

and

$$I_{100-\Omega}=I_{\text{battery}}-I_L$$

Substitute numerical values for I_{battery} and I_{L} and evaluate $I_{100\text{-}\Omega}$:

$$I_{100-\Omega}=0-1.00\ \text{A}=\boxed{-1.00\ \text{A}}$$

(*d*) A long time after the switch is opened, all the currents must be zero:

$$I_{\text{battery}}=I_L=I_{100-\Omega}=\boxed{0}$$

67 •• An inductor, two resistors, a make-before-break switch an a battery are connected as shown in Figure 28-57. The switch throw has been at contact *e* for a long time and the current in the inductor is 2.5 A. Then, at t = 0, the throw is quickly moved to contact *f*. During the next 45 ms the current in the inductor drops to 1.5 A. (*a*) What is the time constant for this circuit? (*b*) If the resistance *R* is equal to 0.40 Ω, what is the value of the inductance *L*?

Picture the Problem The current in an initially energized but source-free *RL* circuit is given by $I=I_0e^{-t/\tau}$. We can find τ from this equation and then use its definition to evaluate *L*.

(*a*) Express the current in the *RL* circuit as a function of time:

$$I = I_0 e^{-t/\tau} \Rightarrow \tau = -\frac{t}{\ln\left(\frac{I}{I_0}\right)}$$

Substitute numerical values and evaluate τ:

$$\tau = -\frac{45\,\text{ms}}{\ln\left(\frac{1.5\,\text{A}}{2.5\,\text{A}}\right)} = 88.1\,\text{ms} = \boxed{88\,\text{ms}}$$

(*b*) Using the definition of the inductive time constant, relate *L* to *R*:

$$L = \tau R$$

Substitute numerical values and evaluate *L*:

$$L = (88.1\,\text{ms})(0.40\,\Omega) = \boxed{35\,\text{mH}}$$

68 •• A circuit consists of a coil that has a self-inductance equal to 5.00 mH and an internal resistance equal to 15.0 Ω, an ideal 12.0-V battery and an open switch—all connected in series (Figure 28-58), At $t = 0$ the switch is closed. Find the time at which the rate at which energy is dissipated in the coil equals the rate at which magnetic energy is stored in the coil.

Picture the Problem If the current is initially zero in an *LR* circuit, its value at some later time *t* is given by $I = I_f\left(1 - e^{-t/\tau}\right)$, where $I_f = \mathcal{E}_0/R$ and $\tau = L/R$ is the time constant for the circuit. We can find the time at which the power dissipation in the resistor equals the rate at which magnetic energy is stored in the inductor by equating expressions for these rates and using the expression for *I* and its rate of change.

Express the rate at which magnetic energy is stored in the inductor:

$$\frac{dU_L}{dt} = \frac{d}{dt}\left[\tfrac{1}{2}LI^2\right] = LI\frac{dI}{dt}$$

Express the rate at which power is dissipated in the resistor:

$$P = I^2 R$$

Equate these expressions to obtain:

$$I^2 R = LI\frac{dI}{dt} \Rightarrow I = \tau\frac{dI}{dt} \qquad (1)$$

Express the current and its rate of change:

$$I = I_{\text{f}}\left(1 - e^{-t/\tau}\right)$$

and

$$\frac{dI}{dt} = I_{\text{f}}\frac{d}{dt}\left(1 - e^{-t/\tau}\right) = -I_{\text{f}}e^{-t/\tau}\left(-\frac{1}{\tau}\right)$$

$$= \frac{I_{\text{f}}}{\tau}e^{-t/\tau}$$

Substitute for dI/dt in equation (1) and simplify to obtain:

$$I_{\text{f}}\left(1 - e^{-t/\tau}\right) = \tau\left(\frac{I_{\text{f}}}{\tau}e^{-t/\tau}\right)$$

or

$$1 - e^{-t/\tau} = e^{-t/\tau} \Rightarrow 1 = 2e^{-t/\tau}$$

Solving for t and substituting for τ yields:

$$t = -\tau\ln\left(\tfrac{1}{2}\right) = -\frac{L}{R}\ln\left(\tfrac{1}{2}\right)$$

Substitute numerical values and evaluate t:

$$t = -\frac{5.00\text{ mH}}{15.0\,\Omega}\ln\left(\tfrac{1}{2}\right) = \boxed{231\,\mu\text{s}}$$

69 ••• [SSM] In the circuit shown in Figure 28-54, let $\mathcal{E}_0$ = 12.0 V, R = 3.00 Ω, and L = 0.600 H. The switch is closed at time t = 0. During the time from t = 0 to t = L/R, find (*a*) the amount of energy supplied by the battery, (*b*) the amount of energy dissipated in the resistor, and (*c*) the amount of energy delivered to the inductor. *Hint: Find the energy transfer rates as functions of time and integrate.*

Picture the Problem We can integrate $dE/dt = \mathcal{E}_0 I,$ where $I = I_{\text{f}}\left(1 - e^{-t/\tau}\right)$, to find the energy supplied by the battery, $dE_{\text{J}}/dt = I^2R$ to find the energy dissipated in the resistor, and $U_L(\tau) = \frac{1}{2}L(I(\tau))^2$ to express the energy that has been stored in the inductor when $t = L/R$.

(*a*) Express the rate at which energy is supplied by the battery:

$$\frac{dE}{dt} = \mathcal{E}_0 I$$

Express the current in the circuit as a function of time:

$$I = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right)$$

Substitute for I to obtain:

$$\frac{dE}{dt} = \frac{\mathcal{E}_0^2}{R}\left(1 - e^{-t/\tau}\right)$$

Separate variables and integrate from $t = 0$ to $t = \tau$ to obtain:

$$E = \frac{\mathcal{E}_0^2}{R}\int_0^{\tau}\left(1 - e^{-t/\tau}\right)dt = \frac{\mathcal{E}_0^2}{R}\left[\tau - \left(-\tau e^{-1} + \tau\right)\right] = \frac{\mathcal{E}_0^2}{R}\frac{\tau}{e} = \frac{\mathcal{E}_0^2 L}{R^2 e}$$

Substitute numerical values and evaluate E:

$$E = \frac{(12.0\,\text{V})^2(0.600\,\text{H})}{(3.00\,\Omega)^2 e} = \boxed{3.53\,\text{J}}$$

(b) Express the rate at which energy is being dissipated in the resistor:

$$\frac{dE_\text{J}}{dt} = I^2 R = \left[\frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right)\right]^2 R = \frac{\mathcal{E}_0^2}{R}\left(1 - 2e^{-t/\tau} + e^{-2t/\tau}\right)$$

Separate variables and integrate from $t = 0$ to $t = L/R$ to obtain:

$$E_\text{J} = \frac{\mathcal{E}_0^2}{R}\int_0^{L/R}\left(1 - 2e^{-t/\tau} + e^{-2t/\tau}\right)dt = \frac{\mathcal{E}_0^2}{R}\left(\frac{2\frac{L}{R}}{e} - \frac{\frac{L}{R}}{2} - \frac{\frac{L}{R}}{2e^2}\right) = \frac{\mathcal{E}_0^2 L}{R^2}\left(\frac{2}{e} - \frac{1}{2} - \frac{1}{2e^2}\right)$$

Substitute numerical values and evaluate E_J:

$$E_\text{J} = \frac{(12.0\,\text{V})^2(0.600\,\text{H})}{(3.00\,\Omega)^2}\left(\frac{2}{e} - \frac{1}{2} - \frac{1}{2e^2}\right) = \boxed{1.61\,\text{J}}$$

(c) Express the energy stored in the inductor when $t = \dfrac{L}{R}$:

$$U_L\left(\frac{L}{R}\right) = \tfrac{1}{2}L\left(I\left(\frac{L}{R}\right)\right) = \tfrac{1}{2}L\left(\frac{\mathcal{E}_0}{R}\left(1 - e^{-1}\right)\right)^2 = \frac{L\mathcal{E}_0^2}{2R^2}\left(1 - e^{-1}\right)^2$$

Substitute numerical values and evaluate U_L:

$$U_L\left(\frac{L}{R}\right)=\frac{(0.600\,\text{H})(12.0\,\text{V})^2}{2(3.00\,\Omega)^2}\left(1-e^{-1}\right)^2$$
$$=\boxed{1.92\,\text{J}}$$

Remarks: Note that, as we would expect from energy conservation, $\boldsymbol{E = E_J + E_L}$.

General Problems

70 • A 100-turn coil has a radius of 4.00 cm and a resistance of 25.0 Ω. (*a*) The coil is in a uniform magnetic field that is perpendicular to the plane of the coil. What rate of change of the magnetic field strength will induce a current of 4.00 A in the coil? (*b*) What rate of change of the magnetic field strength is required if the magnetic field makes an angle of 20° with the normal to the plane of the coil?

Picture the Problem We can apply Faraday's and Ohm's laws to obtain expressions for the induced emf that we can equate and solve for the rate at which the perpendicular magnetic field must change to induce a current of 4.00 A in the coil.

(*a*) Using Faraday's law, relate the induced emf in the coil to the changing magnetic flux:

$$|\mathcal{E}|=\frac{d\phi_{\text{m}}}{dt}=NA\frac{dB}{dt}$$

Using Ohm's law, relate the induced emf to the resistance of the coil and the current in it:

$$|\mathcal{E}|=IR$$

Equate these expressions and solve for dB/dt:

$$NA\frac{dB}{dt}=IR \Rightarrow \frac{dB}{dt}=\frac{IR}{NA}=\frac{IR}{N\pi r^2}$$

Substitute numerical values and evaluate dB/dt:

$$\frac{dB}{dt}=\frac{(4.00\,\text{A})(25.0\,\Omega)}{(100)\pi(0.0400\,\text{m})^2}=\boxed{199\,\text{T/s}}$$

(*b*) Using Faraday's law, relate the induced emf in the coil to the changing magnetic flux when the field makes an angle θ with respect to the normal to the coil area:

$$|\mathcal{E}|=\frac{d\phi_{\text{m}}}{dt}=\frac{d}{dt}\left(NA\vec{B}\cdot\hat{n}\right)$$
$$=NA\cos\theta\frac{dB}{dt}$$

Proceed as in (*a*) to obtain:

$$\frac{dB}{dt}=\frac{IR}{N\pi r^2\cos\theta}$$

Substitute numerical values and evaluate dB/dt:

$$\frac{dB}{dt}=\frac{(4.00\,\text{A})(25.0\,\Omega)}{(100)\pi(0.0400\,\text{m})^2\cos 20°}=\boxed{212\,\text{T/s}}$$

71 •• [SSM] Figure 28-59 shows a schematic drawing of an *ac generator*. The basic generator consists of a rectangular loop of dimensions *a* and *b* and has *N* turns connected to *slip rings*. The loop rotates (driven by a gasoline engine) at an angular speed of ω in a uniform magnetic field $\vec{B}$. (*a*) Show that the induced potential difference between the two slip rings is given by $\mathcal{E} = NBab\omega\sin\omega t$. (*b*) If $a = 2.00$ cm, $b = 4.00$ cm, $N = 250$, and $B = 0.200$ T, at what angular frequency ω must the coil rotate to generate an emf whose maximum value is 100V?

Picture the Problem (*a*) We can apply Faraday's law and the definition of magnetic flux to derive an expression for the induced emf in the coil (potential difference between the slip rings). In Part (*b*) we can solve the equation derived in Part (*a*) for ω and evaluate this expression under the given conditions.

(*a*) Use Faraday's law to express the induced emf:

$$\mathcal{E}=-\frac{d\phi_\text{m}(t)}{dt}$$

Using the definition of magnetic flux, relate the magnetic flux through the loop to its angular velocity:

$$\phi_\text{m}(t)=NBA\cos\omega t$$

Substitute for $\phi_\text{m}(t)$ to obtain:

$$\mathcal{E}=-\frac{d}{dt}[NBA\cos\omega t]=-NBab\omega(-\sin\omega t)=\boxed{NBab\omega\sin\omega t}$$

(*b*) Express the condition under which $\mathcal{E}=\mathcal{E}_\text{max}$:

$$\sin\omega t=1$$

and

$$\mathcal{E}_\text{max}=NBab\omega\Rightarrow\omega=\frac{\mathcal{E}_\text{max}}{NBab}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{100\,\text{V}}{(250)(0.200\,\text{T})(0.0200\,\text{m})(0.0400\,\text{m})} = \boxed{2.50\,\text{krad/s}}$$

72 •• Prior to 1960, magnetic field strengths were usually measured by a *rotating coil gaussmeter*. This device used a small multi-turn coil rotating at a high speed on an axis perpendicular to the magnetic field. This coil was connected to an ac voltmeter by means of slip rings, like those shown in Figure 28-61. In one specific design, the rotating coil has 400 turns and an area of 1.40 cm^2. The coil rotates at 180 rev/min. If the magnetic field strength is 0.450 T, find the maximum induced emf in the coil and the orientation of the normal to the plane of the coil relative to the field for which this maximum induced emf occurs.

Picture the Problem We can apply Faraday's law and the definition of magnetic flux to derive an expression for the induced emf in the rotating coil gaussmeter.

Use Faraday's law to express the induced emf:

$$\mathcal{E} = -\frac{d\phi_\text{m}}{dt}$$

Using the definition of magnetic flux, relate the magnetic flux through the loop to its angular velocity:

$$\phi_\text{m}(t) = NBA\cos\omega t$$

Substitute for $\phi_\text{m}(t)$ and simplify to obtain:

$$\begin{aligned}\mathcal{E} &= -\frac{d}{dt}\left[NBA\cos\omega t\right] \\ &= -NBA\omega(-\sin\omega t) \\ &= NBA\omega\sin\omega t = \mathcal{E}_\text{max}\sin\omega t\end{aligned}$$

where

$$\mathcal{E}_\text{max} = NBA\omega$$

Substitute numerical values and evaluate $\mathcal{E}_\text{max}$:

$$\mathcal{E}_\text{max} = (400)(0.450\,\text{T})(1.40\times10^{-4}\,\text{m}^2)\left(180\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right) = \boxed{0.475\,\text{V}}$$

The maximum induced emf occurs at the instant the normal to the plane of the coil is perpendicular to the magnetic field $\vec{B}$. At this instant, ϕ_m is zero, but $\mathcal{E}$ is a maximum.

73 •• Show that the equivalent self-inductance for two inductors that have self-inductances L_1 and L_2, and are connected in series is given by $L_{eq} = L_1 + L_2$ if there is no flux linkage between the two inductors. (Saying there is no flux linkage between them is equivalent to saying that the mutual inductance between them is zero.)

Picture the Problem We can use the equality of the currents in the inductors connected in series and the additive nature of the total induced emf across the inductors to show that the self-inductances are additive.

Relate the total induced emf $\mathcal{E}$ to the effective self-inductance L_{eq} and the rate at which the current is changing in the inductors:

$$\mathcal{E} = L_{eq}\frac{dI}{dt}$$

Because the inductors L_1 and L_2 are in series:

$$I_1 = I_2 = I \Rightarrow \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dI}{dt}$$

Express the total induced emf:

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = L_1\frac{dI}{dt} + L_2\frac{dI}{dt} = (L_1 + L_2)\frac{dI}{dt}$$

Substitute in equation (1) and simplify to obtain:

$$L_{eq} = \boxed{L_1 + L_2}$$

74 •• Show that the equivalent self-inductance for two inductors that have self-inductances L_1 and L_2, and are connected in parallel is given by $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$ if there is no flux linkage between the two inductors. (Saying there is no flux linkage between them is equivalent to saying that the mutual inductance between them is equal to zero.)

Picture the Problem We can use the common potential difference across the parallel combination of inductors and the fact that the current into the parallel combination is the sum of the currents through each inductor to find an expression of the equivalent self-inductance.

Define L_{eq} by:

$$L_{eq} = \frac{\mathcal{E}}{dI/dt} \Rightarrow \frac{dI}{dt} = \mathcal{E}\frac{1}{L_{eq}} \quad (1)$$

Relate the common potential difference across the inductors to their self-inductances and the rate at which the current is changing in each:

$$\mathcal{E}_1 = L_1 \frac{dI_1}{dt} \qquad (2)$$

and

$$\mathcal{E}_2 = L_2 \frac{dI_2}{dt} \qquad (3)$$

Because the current divides at the parallel junction:

$$I = I_1 + I_2 \Rightarrow \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

Solve equations (2) and (3) for dI_1/dt and dI_2/dt and substitute to obtain:

$$\frac{dI}{dt} = \frac{\mathcal{E}_1}{L_1} + \frac{\mathcal{E}_2}{L_2}$$

Express the relationship between an emf $\mathcal{E}$ applied across the parallel combination of inductors and the emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ across the individual inductors:

$$\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2$$

Substituting yields:

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L_1} + \frac{\mathcal{E}}{L_2} = \mathcal{E}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

Substitute in equation (1) and solve for $1/L_{\text{eq}}$:

$$\frac{1}{L_{\text{eq}}} = \boxed{\frac{1}{L_1} + \frac{1}{L_2}}$$

75 •• A circuit consists of a 12V battery, a switch, and a light bulb—all connected in series. It is known that the light bulb requires a minimum current of 0.10 A in order to produce a visible glow. In this circuit, this particular bulb draws 2.0 W when the switch has been closed for a long time. Now, an inductor is put in series with the bulb and the rest of the circuit. If the light bulb begins to glow 3.5 ms after the switch is closed, how large is the self-inductance of the inductor? Ignore any heating time of the filament and assume the glow is observed as soon as the current in the filament reaches the 0.10 A threshold.

Picture the Problem We can use Equation 28-25 to express the current in the circuit as a function of time and the expression $P = \mathcal{E}^2/R$ for the rate at which energy is dissipated in the light bulb to express the resistance of the circuit.

Use Equation 28-25 to express the current in the RL circuit:

$$I(t) = I_0\left(1 - e^{-t/\tau}\right) = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right)$$

The resistance of the light bulb is related to the rate at which it dissipates energy:

$$P = \frac{\mathcal{E}^2}{R} \Rightarrow R = \frac{\mathcal{E}^2}{P}$$

Substitute for R and simplify to obtain:

$$I(t) = \frac{\mathcal{E}}{\frac{\mathcal{E}^2}{P}}\left(1 - e^{-\frac{V^2}{P}t/L}\right) = \frac{P}{\mathcal{E}}\left(1 - e^{-V^2t/PL}\right)$$

or, for $I(t) = I_{\min}$,

$$I_{\min} = \frac{P}{\mathcal{E}}\left(1 - e^{-V^2t/PL}\right)$$

Solving for L yields:

$$L = \frac{-\mathcal{E}^2 t}{P\ln\left(1 - \frac{\mathcal{E}I_{\min}}{P}\right)}$$

Substitute numerical values and evaluate L:

$$L = \frac{-(12\text{ V})^2(3.5\text{ ms})}{(2.0\text{ W})\ln\left(1 - \frac{(12\text{ V})(0.10\text{ A})}{2.0\text{ W}}\right)}$$

$$= \boxed{0.28\text{ H}}$$

76 •• Your friend decides to generate electrical power by rotating a 100 000-turn coil of wire around an axis in the plane of the coil and through its center The coil is perpendicular to Earth' magnetic field in a region where the field strength is equal to 0.300 G. The loops of the coil have a radius 25.0 cm, and the coil has negligible resistance. (*a*) If your friend turns the coil at a rate of 150 rev/s, what peak current will exist in a 1500-Ω resistor that is connected across the terminals the coil? (*b*) The average of the square of the current will equal half of the square of the peak current. What will be the average power delivered to the resistor? Is this an economical way to generate power? *HINT: Energy has to be expended to keep the coil rotating.*

Picture the Problem We can use Ohm's law to express the peak current in terms of the peak induced emf in the coil and the resistance of the resistor attached to the coil and Faraday's law to find the peak induced emf in the coil.

(*a*) Apply Ohm's law to the coil and external resistor to obtain:

$$I_{\text{peak}} = \frac{\mathcal{E}_{\max}}{R}$$

Appling Faraday's law yields:

$$\mathcal{E} = -\frac{d\phi_{\text{m}}}{dt} = -\frac{d}{dt}(NBA\cos\omega t)$$

$$= NBA\omega\sin\omega t = \mathcal{E}_{\max}\sin\omega t$$

where $\mathcal{E}_{\max} = NBA\omega$.

Substitute for $\mathcal{E}_{\max}$ to obtain:

$$I_{\text{peak}} = \frac{NBA\omega}{R} = \frac{NB\pi r^2(2\pi f)}{R} = \frac{2\pi^2 NBr^2 f}{R}$$

Substitute numerical values and evaluate I_{peak}:

$$I_{\text{peak}} = \frac{2\pi^2(10^5)\left(0.300\ \text{G}\times\frac{1\ \text{T}}{10^4\ \text{G}}\right)(0.250\ \text{m})^2\left(150\ \frac{\text{rev}}{\text{s}}\right)}{1500\ \Omega} = 370.11\ \text{mA}$$
$$= \boxed{370\ \text{mA}}$$

(*b*) The average power supplied is the power dissipated in Joule heat in the resistor:

$$P_{\text{av}} = \left(I^2\right)_{\text{av}} R = \tfrac{1}{2} I_{\text{peak}}^2 R$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = \tfrac{1}{2}(370.11\ \text{mA})^2(1500\ \Omega) = \boxed{103\ \text{W}}$$

A power of 103 W is equal to 0.137 hp. Your friend is likely to tire. This is probably not an efficient way to generate power.

77 •• [SSM] Figure 28-60*a* shows an experiment designed to measure the acceleration due to gravity. A large plastic tube is encircled by a wire, which is arranged in single loops separated by a distance of 10 cm. A strong magnet is dropped through the top of the loop. As the magnet falls through each loop the voltage rises and then the voltage rapidly falls through zero to a large negative value and then returns to zero. The shape of the voltage signal is shown in Figure 28-62*b*. (*a*) Explain the basic physics behind the generation of this voltage pulse. (*b*) Explain why the tube cannot be made of a conductive material.
(*c*) Qualitatively explain the *shape* of the voltage signal in Figure 28-60*b*. (*d*) The times at which the voltage crosses zero as the magnet falls through each loop in succession are given in the table in the next column. Use these data to calculate a value for *g*.

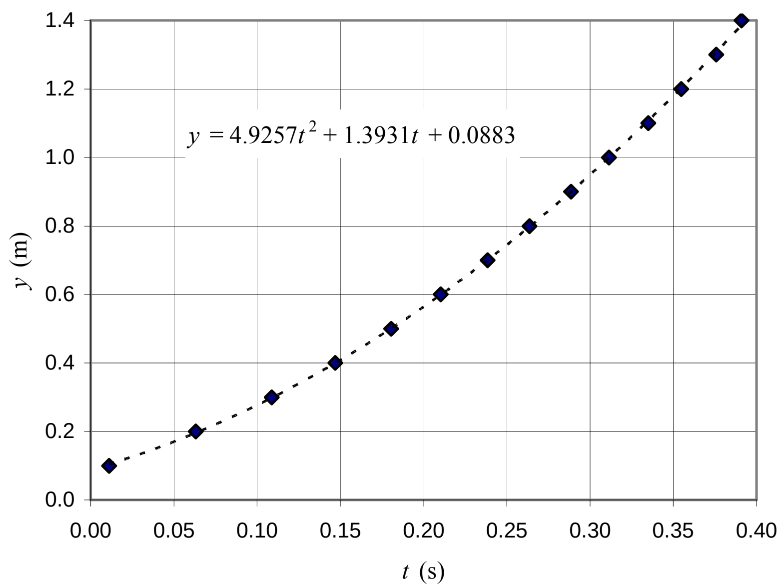
Picture the Problem

(*a*) As the magnet passes through a loop it induces an emf because of the changing flux through the loop. This allows the coil to ″sense″ when the magnet is passing through it.

(*b*) One cannot use a cylinder made of conductive material because eddy currents induced in it by a falling magnet would slow the magnet.

(*c*) As the magnet approaches the loop the flux increases, resulting in the negative voltage signal of increasing magnitude. When the magnet is passing a loop, the flux reaches a maximum value and then decreases, so the induced emf becomes zero and then positive. The instant at which the induced emf is zero is the instant at which the magnet is at the center of the loop.

(*d*) Each time represents a point when the distance has increased by 10 cm. The following graph of distance versus time was plotted using a spreadsheet program. The regression curve, obtained using Excel's ″Add Trendline″ feature, is shown as a dashed line.



The coefficient of the second-degree term is $\frac{1}{2}g$. Consequently,

$$g = 2\left(4.9257\,\text{m/s}^2\right) = \boxed{9.85\,\text{m/s}^2}$$

78 •• The rectangular coil shown in Figure 28-61 has 80 turns, is 25 cm wide, is 30 cm long, and is located in a magnetic field of 0.14 T directed out of the page, as shown. Only half of the coil is in the region of the magnetic field. The resistance of the coil is 24 Ω. Find the magnitude and the direction of the induced current if the coil is moved with a speed of 2.0 m/s (*a*) to the right, (*b*) up the page, (*c*) to the left, and (*d*) down the page.

Picture the Problem The current equals the induced emf divided by the resistance. We can calculate the emf induced in the circuit as the coil moves by calculating the rate of change of the flux through the coil. The flux is proportional to the area of the coil in the magnetic field. We can find the direction of the current from Lenz's law.

(*a*) and (*c*) Express the magnitude of the induced current:

$$I = \frac{|\mathcal{E}|}{R} \quad (1)$$

Using Faraday's law, express the magnitude of the induced emf:

$$|\mathcal{E}| = \frac{d\phi_m}{dt}$$

When the coil is moving to the right (or to the left), the flux does not change (until the coil leaves the region of magnetic field). Thus:

$$|\mathcal{E}| = \frac{d\phi_m}{dt} = 0 \Rightarrow I = \frac{|\mathcal{E}|}{R} = \boxed{0}$$

(*b*) Letting *x* represent the length of the side of the rectangular coil that is in the magnetic field, express the magnetic flux through the coil:

$$\phi_m = NBwx$$

Compute the rate of change of the flux when the coil is moving up or down:

$$\begin{aligned}\frac{d\phi_m}{dt} &= NBw\frac{dx}{dt}\\ &= (80)(0.14\,\text{T})(0.25\,\text{m})(2.0\,\text{m/s})\\ &= 5.60\,\text{V}\end{aligned}$$

Substitute in equation (1) to obtain:

$$I = \frac{5.60\,\text{V}}{24\,\Omega} = \boxed{0.23\,\text{A clockwise}}$$

(*d*) When the coil is moving downward, the outward flux decreases and the induced current will be in such a direction as to produce outward flux. The magnitude of the current is the same as in Part (*b*) and

$$I = \boxed{0.23\,\text{A counterclockwise}}\,.$$

79 •• [SSM] A long solenoid has *n* turns per unit length and carries a current that varies with time according to $I = I_0 \sin \omega t$. The solenoid has a circular cross section of radius *R*. Find the induced electric field, at points near the plane equidistant from the ends of the solenoid, as a function of both the time *t* and the perpendicular distance *r* from the axis of the solenoid for (*a*) $r < R$ and (*b*) $r > R$.

Picture the Problem We can apply Faraday's law to relate the induced electric field *E* to the rates at which the magnetic flux is changing at distances $r < R$ and $r > R$ from the axis of the solenoid.

(*a*) Apply Faraday's law to relate the induced electric field to the magnetic flux in the solenoid within a cylindrical region of radius $r < R$:

$$\int_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt}$$

or

$$E(2\pi r) = -\frac{d\phi_m}{dt} \qquad (1)$$

Express the field within the solenoid:

$$B = \mu_0 nI$$

Express the magnetic flux through an area for which $r < R$:

$$\phi_m = BA = \pi r^2 \mu_0 nI$$

Substitute in equation (1) to obtain:

$$E(2\pi r) = -\frac{d}{dt}\left[\pi r^2 \mu_0 nI\right] = -\pi r^2 \mu_0 n \frac{dI}{dt}$$

Because $I = I_0 \sin \omega t$:

$$E_{r<R} = -\tfrac{1}{2} r\mu_0 n \frac{d}{dt}\left[I_0 \sin \omega t\right] = \boxed{-\tfrac{1}{2} r\mu_0 nI_0 \omega \cos \omega t}$$

(*b*) Proceed as in (*a*) with $r > R$ to obtain:

$$E(2\pi r) = -\frac{d}{dt}\left[\pi R^2 \mu_0 nI\right] = -\pi R^2 \mu_0 n \frac{dI}{dt} = -\pi R^2 \mu_0 nI_0 \omega \cos \omega t$$

Solving for $E_{r>R}$ yields:

$$E_{r>R} = \boxed{-\frac{\mu_0 nR^2 I_0 \omega}{2r} \cos \omega t}$$

80 ••• A coaxial cable consists of two very thin-walled conducting cylinders of radii r_1 and r_2 (Figure 28-62). The currents in the inner and outer cylinders are equal in magnitude but opposite in direction. (*a*) Use Ampère's law to find the magnetic field as a function of the perpendicular distance r from the central axis of the cable for (1) $0 < r < r_1$ (2) $r_1 < r < r_2$ and (3) $r > r_2$. (*b*) Show that the magnetic energy density in the region between the cylinders is given by $\mu_m = \frac{1}{2}(\mu_0/4\pi)I^2/(\pi r^2)$. (*c*) Show that the total magnetic energy in a cable volume of length ℓ is given by $U_m = (\mu_0/4\pi)I^2\ell \ln(r_2/r_1)$. (*d*) Use the result in Part (*c*) and the relationship between magnetic energy, current and inductance to show that the self-inductance per unit length of this cable arrangement is given by $L/\ell = (\mu_0/2\pi)\ln(r_2/r_1)$.

Picture the Problem The system exhibits cylindrical symmetry, so one can use Ampère's law to determine B inside the inner cylinder, between the cylinders, and outside the outer cylinder. We can use $u_m = B^2/2\mu_0$ and the expression for B from Part (a) to express the magnetic energy density in the region between the cylinders. We can integrate this expression for u_m over the volume between the cylinders to find the total magnetic energy in a volume of length ℓ. Finally, we can use our result in Part (c) and $U_m = \frac{1}{2}LI^2$ to find the self-inductance of the cylinders per unit length.

(a) For $r < r_1$ and for $r > r_2$ the net enclosed current is zero; consequently, in these regions:

$$B(r < r_1) = \boxed{0}$$

and

$$B(r > r_2) = \boxed{0}$$

For $r_1 < r < r_2$:

$$2\pi rB = \mu_0 I_C \Rightarrow B(r_1 < r < r_2) = \boxed{\frac{\mu_0 I}{2\pi r}}$$

(b) Express the magnetic energy density in the region between the cylinders:

$$u_m = \frac{B^2}{2\mu_0}$$

Substitute for B and simplify to obtain:

$$u_m = \frac{\left(\frac{\mu_0 I}{2\pi r}\right)^2}{2\mu_0} = \boxed{\frac{\mu_0 I^2}{8\pi^2 r^2}}$$

(c) Express the magnetic energy dU_m in the cylindrical element of volume dV:

$$dU_m = u_m dV = \frac{\mu_0 I^2}{8\pi^2 r^2}(\ell 2\pi r dr)$$
$$= \frac{\mu_0 I^2 \ell}{4\pi} \cdot \frac{dr}{r}$$

Integrate this expression from $r = r_1$ to $r = r_2$ to obtain:

$$U_m = \frac{\mu_0 I^2 \ell}{4\pi}\int_{r_1}^{r_2} \frac{dr}{r} = \boxed{\frac{\mu_0}{4\pi} I^2 \ell \ln\left(\frac{r_2}{r_1}\right)}$$

(d) Express the energy in the magnetic field in terms of L and I:

$$U_m = \tfrac{1}{2}LI^2 \Rightarrow L = \frac{2U_m}{I^2}$$

From our result in (c):

$$\frac{U_m}{I^2} = \frac{\mu_0}{4\pi}\ell \ln\left(\frac{r_2}{r_1}\right)$$

Substitute to obtain:

$$L = 2\left[\frac{\mu_0}{4\pi}\ell \ln\left(\frac{r_2}{r_1}\right)\right] = \frac{\mu_0}{2\pi}\ell \ln\left(\frac{r_2}{r_1}\right)$$

Express the ratio L/ℓ:

$$\frac{L}{\ell} = \boxed{\frac{\mu_0}{2\pi}\ln\left(\frac{r_2}{r_1}\right)}$$

81 ••• A coaxial cable consists of two very thin-walled conducting cylinders of radii r_1 and r_2 (Figure 28-63). The currents in the inner and outer cylinders are equal in magnitude but opposite in direction. Compute the flux through a rectangular area of sides ℓ and $r_2 - r_1$ between the conductors shown in Figure 28-P95. Use the relationship between flux and current ($\phi_m = LI$) to show the self-inductance per unit length of the cable by is given by $L/\ell = (\mu_0/2\pi)\ln(r_2/r_1)$.

Picture the Problem We can use its definition to express the magnetic flux through a rectangular element of area dA and then integrate from $r = r_1$ to $r = r_2$ to express the total flux through the region. Substituting in $L = \phi_m/I$ will yield the same result found in Part (*d*) of Problem 80.

Use the definition of self-inductance to relate the magnetic flux through the region of interest to the current I:

$$L = \frac{\phi_m}{I} \qquad (1)$$

Consider a strip of unit length ℓ and width dr at a distance r from the axis. The flux through this area is given by:

$$d\phi_m = BdA = B\ell dr$$

Apply Ampere's law to express the magnetic field at a distance r from the axis:

$$2\pi rB = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Substituting for B yields:

$$d\phi_m = \frac{\mu_0 I\ell}{2\pi}\frac{dr}{r}$$

Integrate from $r = r_1$ to $r = r_2$ to obtain:

$$\phi_m = \frac{\mu_0 I\ell}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r} \Rightarrow \phi_m = \frac{\mu_0 I\ell}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$$

Substituting for ϕ_m in equation (1) and dividing both sides by ℓ yields:

$$\frac{L}{\ell} = \boxed{\frac{\mu_0}{2\pi}\ln\left(\frac{r_2}{r_1}\right)}$$

82 ••• Figure 28-64 shows a rectangular loop of wire that is 0.300 m wide, is 1.50 m long, and lies in the vertical plane which is perpendicular to a region that has a uniform magnetic field. The magnitude of the uniform magnetic field is 0.400 T and the direction of the magnetic field is into the page. The portion of the loop not in the magnetic field is 0.100 m long. The resistance of the loop is 0.200 Ω and its mass is 50.0 g. The loop is released from rest at $t = 0$. (*a*) What is the magnitude and direction of the induced current when the loop has a downward speed v? (*b*) What is the force that acts on the loop as a result of this current? (*c*) What is the net force acting on the loop? (*d*) Write out Newton's second law for the loop. (*e*) Obtain an expression for the speed of the loop as a function of time. (*f*) Integrate the expression obtained in Part (*e*) to find the distance the loop falls as a function of time. (*g*) Using a **spreadsheet** program, make a graph of the position of the loop as a function of time (letting $t = 0$ at the start) for values of y between 0 and 1.40 m (i.e., when the loop leaves the magnetic field). (*h*) At what time does the loop completely leave the field region? Compare this to the time it would have taken if there were no field.

Picture the Problem We can use $I = \mathcal{E}/R$ and $\mathcal{E} = Bv\ell$ to find the current induced in the loop and Lenz's law to determine its direction. We can apply the equation for the force on a current-carrying wire to find the net magnetic force acting on the loop and then sum the forces to find the net force on the loop. Separating the variables in the differential equation and integrating will lead us to an expression for $v(t)$ and a second integration to an expression for $y(t)$. We can solve the latter equation for $y = 1.40$ m to find the time it takes the loop to exit the magnetic field and our expression for $v(t)$ to find its exit speed. Finally, we can use a constant-acceleration equation to find its exit speed in the absence of the magnetic field.

(*a*) Relate the magnitude of the induced current to the induced emf and the resistance of the loop:

$$I = \frac{\mathcal{E}}{R}$$

Relate the induced emf $\mathcal{E}$ to the motion of the loop:

$$\mathcal{E} = Bv\ell$$

where ℓ is the length of the horizontal portion of the loop.

Substitute for $\mathcal{E}$ to obtain:

$$I = \boxed{\frac{B\ell}{R}v}$$

As the loop falls, the flux into it (the loop) decreases. The direction of the induced current is such that its magnetic field opposes this decrease; i.e., clockwise.

(*b*) Express the velocity-dependent force that acts on the loop in terms of the current in the loop:

$$F_v = BI\ell$$

Substitute for I to obtain:

$$F_v = B\left(\frac{B\ell}{R}\right)v\ell = \boxed{\frac{B^2\ell^2}{R}v}$$

Apply $d\vec{F} = Id\vec{\ell}\times\vec{B}$ to the horizontal portion of the loop that is in the magnetic field to conclude that the net magnetic force is upward. Note that the magnetic force on the left side of the loop is to the left and the magnetic force on the right side of the loop is to the right.

(*c*) The net force acting on the loop is the difference between the downward gravitational force and the upward magnetic force:

$$F_{\text{net}} = mg - F_v = \boxed{mg - \frac{B^2\ell^2}{R}v}$$

(*d*) Apply Newton's second law of motion to the loop to obtain its equation of motion:

$$mg - \frac{B^2\ell^2}{R}v = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{B^2\ell^2}{mR}v$$

Factor g to obtain an alternate form of the equation of motion:

$$\frac{dv}{dt} = g\left(1 - \frac{B^2\ell^2}{mgR}v\right) = \boxed{g\left(1 - \frac{v}{v_{\text{t}}}\right)}$$

where $v_{\text{t}} = \dfrac{mgR}{B^2\ell^2}$

(*e*) Separating the variables yields:

$$\frac{dv}{g - \dfrac{B^2\ell^2}{mR}v} = dt \text{ or } \frac{dv}{a - bv} = dt$$

where $a = g$ and $b = \dfrac{B^2\ell^2}{mR}$

Integrate v from 0 to v and t from 0 to t:

$$\int_0^v \frac{dv}{a - bv} = \int_0^t dt \Rightarrow -\frac{1}{b}\ln\left(\frac{a - bv}{a}\right) = t$$

Transforming from logarithmic to exponential form and solving for v yields:

$$v(t) = \frac{a}{b}\left(1 - e^{-bt}\right)$$

Noting that $v_t = \dfrac{a}{b}$, we have:

$$v(t) = \boxed{v_t\left(1 - e^{-t/\tau}\right)}$$

where $v_t = \dfrac{mgR}{B^2\ell^2}$ and $\tau = \dfrac{v_t}{a} = \dfrac{v_t}{g}$.

(*f*) Write v as dy/dt and separate variables to obtain:

$$dy = v_t\left(1 - e^{-t/\tau}\right)dt$$

Integrate y from 0 to y and t from 0 to t:

$$\int_0^y dy = v_t \int_0^t \left(1 - e^{-t/\tau}\right)dt$$

and

$$y(t) = \boxed{v_t\left[t - \tau\left(1 - e^{-t/\tau}\right)\right]}$$

(*g*) A spreadsheet program to generate the data for graphs of position y as a function of time t is shown below. The formulas used to calculate the quantities in the columns are as follows:

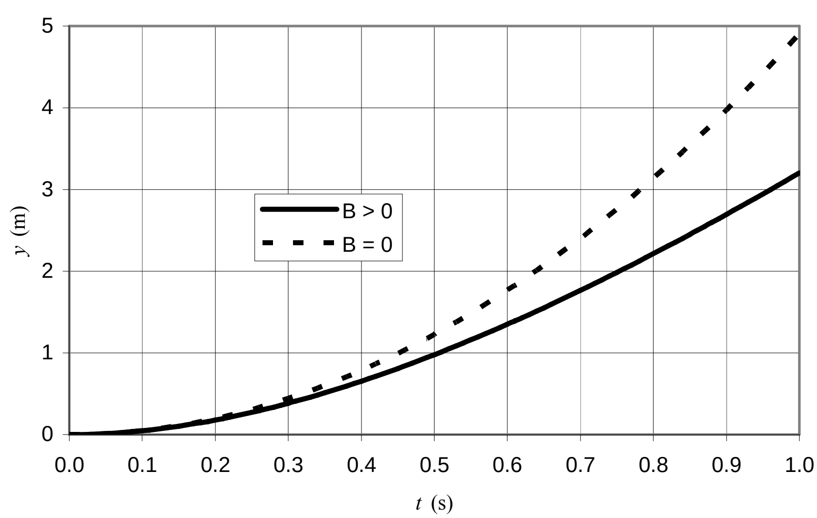
Cell	Formula/Content	Algebraic Form
B1	0.0500	m
B2	0.200	R
B3	0.400	B
B4	0.300	L
B5	$B$1*$B$7*$B$2/($B$3^2*$B$4^2)	v_t
B6	$B$5/$B$7	τ
B7	9.81	g
A10	0.00	t
B10	$B$5*(A10−$B$6*(1−EXP(−A10/$B$6)))	y
C10	0.5*$B$7*A10^2	$\frac{1}{2}gt^2$

	A	B	C
1	m=	0.0500	kg
2	R=	0.200	ohms
3	B=	0.400	T
4	L=	0.300	m
5	v_t=	6.813	m/s
6	τ=	0.694	s
7	g=	9.81	m/s^2
8			
9	t	y	y (no B)
10	0.00	0.000	0.000
11	0.01	0.000	0.000
12	0.02	0.002	0.002

13	0.03	0.004	0.004
14	0.04	0.008	0.008
15	0.05	0.012	0.012
62	0.52	1.049	1.326
63	0.53	1.085	1.378
64	0.54	1.122	1.430
65	0.55	1.159	1.484
66	0.56	1.196	1.538
67	0.57	1.234	1.594
68	0.58	1.273	1.650
69	0.59	1.311	1.707
70	0.60	1.351	1.766
71	0.61	1.390	1.825
72	0.62	1.430	1.885
73	0.63	1.471	1.947
74	0.64	1.511	2.009

Examination of either the table or the following graph shows that, when the loop falls in the magnetic field, $y = 1.4$ m when $t \approx \boxed{0.61\text{s.}}$ In the absence of the magnetic field, $y = 1.4$ m when $t \approx \boxed{0.53\text{s.}}$

The following graph shows y as a function of t for $B \neq 0$ (solid curve) and $B = 0$ (dashed curve).



In the absence of the magnetic field, the loop fall a distance of 1.40 m in about 0.08 s less time than it takes to fall the same distance in the presence of the magnetic field.

83 ••• A coil of N turns and area A suspended from the ceiling by a wire that provides a linear restoring torque with torsion constant κ. The two ends of the coil are connected to each other, the coil has resistance R, and the moment of inertia of the coil is I. The plane of the coil is vertical, and parallel to a uniform horizontal magnetic field $\vec{B}$ when the wire is not twisted (i.e., $\theta = 0$). The coil is rotated about a vertical axis through its center by a small angle θ_0 and released. The coil then undergoes damped harmonic oscillation. Show that its angle with its equilibrium position will vary with time according to $\theta(t) = \theta_0 e^{-t/2\tau}\cos\omega' t$, where $\tau = RI/(NBA)^2$, $\omega_0 = \sqrt{\kappa/I}$ and $\omega' = \omega_0\sqrt{1-(2\omega_0\tau)^{-2}}$.

Picture the Problem Picture the Problem If the coil is rotated through an angle θ, the wire exerts a restoring torque equal to $-\kappa\theta$ acts on it returning it to its equilibrium position. However, when it rotates with angular speed $d\theta/dt$, there will be an emf induced in the coil. The direction of the current resulting from this induced emf will be such that its magnetic field will oppose the change in flux resulting from the rotation of the coil. The net effect is that the magnetic field exerts a torque on the coil in a direction opposite to the direction of the angular velocity of the coil. We can show that θ will vary with time according to $\theta(t) = \theta_0 e^{-t/2\tau}\cos\omega' t$,where $\tau = RI/(NBA)^2$, $\omega_0 = \sqrt{\kappa/I}$ and $\omega' = \omega_0\sqrt{1-(2\omega_0\tau)^{-2}}$ by demonstrating that $\theta(t) = \theta_0 e^{-t/2\tau}\cos\omega' t$ satisfies the differential equation obtained in our solution for Part (a).

Apply $\sum\tau = I\alpha$ to the rotating coil to obtain:

$$\tau_{\text{restoring}} - \tau_{\text{retarding}} = I\frac{d^2\theta}{dt^2}$$

The magnitude of the retarding (damping) torque is given by:

$$\tau_{\text{retarding}} = NiBA\cos\theta$$

where i is the current induced in the coil whose cross-sectional area is A.

Substitute for $\tau_{\text{restoring}}$ and $\tau_{\text{retarding}}$ to obtain:

$$-\kappa\theta - NiBA\cos\theta = I\frac{d^2\theta}{dt^2} \qquad (1)$$

Apply Faraday's law to express the emf induced in the coil:

$$\mathcal{E} = -\frac{d}{dt}(NBA\sin\theta) = -(NBA\cos\theta)\frac{d\theta}{dt}$$

From Ohm's law, the magnitude of the induced current i in the coil is:

$$i=\frac{\varepsilon}{R}=\frac{NBA\cos\theta}{R}\frac{d\theta}{dt}$$

Substitute for the induced current i in equation (1) to obtain:

$$-\kappa\theta-\frac{(NBA)^2\cos^2\theta}{R}\frac{d\theta}{dt}=I\frac{d^2\theta}{dt^2}$$

For small displacements from equilibrium, $\cos\theta\approx 1$ and:

$$-\kappa\theta-\frac{(NBA)^2}{R}\frac{d\theta}{dt}\approx I\frac{d^2\theta}{dt^2}$$

Rearrange to express the differential equation in standard form, then substitute using $\omega_0=\sqrt{\kappa/I}$ and $\tau=RI/(NBA)^2$:

$$\frac{d^2\theta}{dt^2}+\frac{(NBA)^2}{RI}\frac{d\theta}{dt}+\frac{\kappa}{I}\theta\approx 0$$

or

$$\frac{d^2\theta}{dt^2}+\frac{1}{\tau}\frac{d\theta}{dt}+\omega_0^2\theta=0 \qquad (2)$$

Assume that the solution to equation (2) is given by $\theta(t)=\theta_0e^{-t/2\tau}\cos\omega't$ and evaluate its first and second derivatives with respect to time:

$$\begin{aligned}\frac{d\theta}{dt}&=\theta_0\frac{d}{dt}\left[e^{-t/2\tau}\cos\omega't\right]=\theta_0\left[e^{-t/2\tau}\frac{d}{dt}(\cos\omega't)+\cos\omega't\frac{d}{dt}\left(e^{-t/2\tau}\right)\right]\\&=\theta_0\left(-\omega'e^{-t/2\tau}\sin\omega't-\frac{1}{2\tau}e^{-t/2\tau}\theta_0\cos\omega't\right)\\&=-\theta_0e^{-t/2\tau}\left(\omega'\sin\omega't+\frac{1}{2\tau}\cos\omega't\right)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\theta}{dt^2}&=\frac{d}{dt}\left[-\theta_0\omega'e^{-t/2\tau}\sin\omega't-\theta_0\frac{1}{2\tau}e^{-t/2\tau}\cos\omega't\right]\\&=-\theta_0e^{-t/2\tau}\frac{d}{dt}\left(\omega'\sin\omega't+\frac{1}{2\tau}\cos\omega't\right)\\&\qquad-\theta_0\left(\omega'\sin\omega't+\frac{1}{2\tau}\cos\omega't\right)\frac{d}{dt}e^{-t/2\tau}\\&=-\theta_0e^{-t/2\tau}\left(\omega'^2\cos\omega't-\frac{\omega'}{2\tau}\sin\omega't\right)\\&\qquad+\theta_0\frac{1}{2\tau}\left(\omega'\sin\omega't+\frac{1}{2\tau}\cos\omega't\right)e^{-t/2\tau}\\&=-\theta_0e^{-t/2\tau}\left(\omega'^2\cos\omega't-\frac{\omega'}{\tau}\sin\omega't-\frac{1}{4\tau^2}\cos\omega't\right)\end{aligned}$$

Substitute these derivatives in equation (2) to obtain:

$$-\theta_0 e^{-t/2\tau}\left(\omega'^2\cos\omega' t-\frac{\omega'}{\tau}\sin\omega' t-\frac{1}{4\tau^2}\cos\omega' t\right)$$
$$-\frac{1}{\tau}\theta_0 e^{-t/2\tau}\left(\omega'\sin\omega' t+\frac{1}{2\tau}\cos\omega' t\right)+\omega_0^2\theta_0 e^{-t/2\tau}\cos\omega' t=0$$

Because θ_0 and $e^{-t/\tau}$ are never zero, we can divide them out of the equation and simplify to obtain:

$$-\omega'^2\cos\omega' t-\frac{1}{4\tau^2}\cos\omega' t+\omega_0^2\cos\omega' t=0$$

or

$$\left(-\omega'^2-\frac{1}{4\tau^2}+\omega_0^2\right)\cos\omega' t=0$$

This equation is satisfied provided: $$-\omega'^2-\frac{1}{4\tau^2}+\omega_0^2=0$$

Solving for ω' yields: $$\omega'=\omega_0\sqrt{1-(2\omega_0\tau)^{-2}}$$

Thus we've shown that the angular position of the oscillating coil, relative to its equilibrium position, varies with time according to $\theta(t)=\theta_0 e^{-t/2\tau}\cos\omega' t$, where $\tau=RI/(NBA)^2$, $\omega_0=\sqrt{\kappa/I}$ and $\omega'=\omega_0\sqrt{1-(2\omega_0\tau)^{-2}}$.

Chapter 29
Alternating-Current Circuits

Conceptual Problems

1 • A coil in an ac generator rotates at 60 Hz. How much time elapses between successive peak emf values of the coil?

Determine the Concept Successive peaks are one-half period apart. Hence the elapsed time between the peaks is $\frac{1}{2}T = \frac{1}{2f} = \frac{1}{2(60\ \text{s}^{-1})} = \boxed{8.33\ \text{ms}}$.

2 • If the rms voltage in an ac circuit is doubled, the peak voltage is (*a*) doubled, (*b*) halved, (*c*) increased by a factor of $\sqrt{2}$, (*d*) not changed.

Picture the Problem We can use the relationship between V and V_{peak} to decide the effect of doubling the rms voltage on the peak voltage.

Express the initial rms voltage in terms of the peak voltage:

$$V_{\text{rms}} = \frac{V_{\text{peak}}}{\sqrt{2}}$$

Express the doubled rms voltage in terms of the new peak voltage V'_{peak}:

$$2V_{\text{rms}} = \frac{V'_{\text{peak}}}{\sqrt{2}}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{2V_{\text{rms}}}{V_{\text{rms}}} = \frac{\frac{V'_{\text{peak}}}{\sqrt{2}}}{\frac{V_{\text{peak}}}{\sqrt{2}}} \Rightarrow 2 = \frac{V'_{\text{peak}}}{V_{\text{peak}}}$$

Solving for V'_{peak} yields:

$$V'_{\text{peak}} = 2V_{\text{peak}} \Rightarrow \boxed{(a)} \text{ is correct.}$$

3 • **[SSM]** If the frequency in the circuit shown in Figure 29-27 is doubled, the inductance of the inductor will (*a*) double, (*b*) not change, (*c*) halve, (*d*) quadruple.

Determine the Concept The inductance of an inductor is determined by the details of its construction and is independent of the frequency of the circuit. The inductive reactance, on the other hand, is frequency dependent. $\boxed{(b)}$ is correct.

4 • If the frequency in the circuit shown in Figure 29-27 is doubled, the inductive reactance of the inductor will (*a*) double, (*b*) not change, (*c*) halve, (*d*) quadruple.

Determine the Concept The inductive reactance of an inductor varies with the frequency according to $X_L = \omega L$. Hence, doubling ω will double X_L. $\boxed{(a)}$ is correct.

5 • If the frequency in the circuit in Figure 29-28 is doubled, the capacitive reactance of the circuit will (*a*) double, (*b*) not change, (*c*) halve, (*d*) quadruple.

Determine the Concept The capacitive reactance of an capacitor varies with the frequency according to $X_C = 1/\omega C$. Hence, doubling ω will halve X_C. $\boxed{(c)}$ is correct.

6 • (*a*) In a circuit consisting solely of a ac generator and an ideal inductor, are there any time intervals when the inductor receives energy from the generator? If so, when? Explain your answer. (*b*) Are there any time intervals when the inductor supplies energy back to the generator? If so when? Explain your answer.

Determine the Concept Yes to both questions. (*a*) While the magnitude of the current in the inductor is increasing, the inductor absorbs power from the generator. (*b*) When the magnitude of the current in the inductor decreases, the inductor supplies power to the generator.

7 • **[SSM]** (*a*) In a circuit consisting of a generator and a capacitor, are there any time intervals when the capacitor receives energy from the generator? If so, when? Explain your answer. (*b*) Are there any time intervals when the capacitor supplies power to the generator? If so, when? Explain your answer.

Determine the Concept Yes to both questions. (*a*) While the magnitude of the charge is accumulating on either plate of the capacitor, the capacitor absorbs power from the generator. (*b*) When the magnitude of the charge is on either plate of the capacitor is decreasing, it supplies power to the generator.

8 • (*a*) Show that the SI unit of inductance multiplied by the SI unit of capacitance is equivalent to seconds squared. (*b*) Show that the SI unit of inductance divided by the SI unit of resistance is equivalent to seconds.

Determine the Concept

(*a*) Substitute the SI units of inductance and capacitance and simplify to obtain:

$$\frac{\text{V}\cdot\text{s}}{\text{A}}\cdot\frac{\text{C}}{\text{V}} = \frac{\text{s}}{\frac{\text{C}}{\text{s}}}\cdot\text{C} = \boxed{\text{s}^2}$$

(*b*) Substitute the SI units of inductance divided by resistance and simplify to obtain:

$$\frac{\frac{\text{V}\cdot\text{s}}{\text{A}}}{\Omega}=\frac{\frac{\text{V}\cdot\text{s}}{\text{A}}}{\frac{\text{V}}{\text{A}}}=\boxed{\text{s}}$$

9 • **[SSM]** Suppose you increase the rotation rate of the coil in the generator shown in the simple ac circuit in Figure 29-29. Then the rms current (*a*) increases, (*b*) does not change, (*c*) may increase or decrease depending on the magnitude of the original frequency, (*d*) may increase or decrease depending on the magnitude of the resistance, (*e*) decreases.

Determine the Concept Because the rms current through the resistor is given by $I_{\text{rms}}=\frac{\mathcal{E}_{\text{rms}}}{R}=\frac{\mathcal{E}_{\text{peak}}}{\sqrt{2}}=\frac{NBA}{\sqrt{2}}\omega$, I_{rms} is directly proportional to ω. $\boxed{(a)}$ is correct.

10 • If the inductance value is tripled in a circuit consisting solely of a variable inductor and a variable capacitor, how would you have to change the capacitance so that the natural frequency of the circuit is unchanged? (*a*) triple the capacitance, (*b*) decrease the capacitance to one-third of its original value, (*c*) You should not change the capacitance.(*d*) You cannot determine how to change the capacitance from the data given.

Determine the Concept The natural frequency of an *LC* circuit is given by $f_0=1/2\pi\sqrt{LC}$.

Express the natural frequencies of the circuit before and after the inductance is tripled:

$$f_0=\frac{1}{2\pi\sqrt{LC}}\text{ and }f_0^{'}=\frac{1}{2\pi\sqrt{L'C'}}$$

Divide the second of the these equations by the first and simplify to obtain:

$$\frac{f_0^{'}}{f_0}=\frac{\frac{1}{2\pi\sqrt{L'C'}}}{\frac{1}{2\pi\sqrt{LC}}}=\sqrt{\frac{LC}{L'C'}}$$

Because the natural frequency is unchanged:

$$1=\sqrt{\frac{LC}{L'C'}}\Rightarrow\frac{LC}{L'C'}=1\Rightarrow C'=\frac{L}{L'}C$$

When the inductance is tripled:

$$C'=\frac{L}{3L}C=\frac{1}{3}C\Rightarrow\boxed{(b)}\text{ is correct.}$$

11 • **[SSM]** Consider a circuit consisting solely of an ideal inductor and an ideal capacitor. How does the maximum energy stored in the capacitor compare to the maximum value stored in the inductor? (*a*) They are the same and each equal to the total energy stored in the circuit. (*b*) They are the same and each

equal to half of the total energy stored in the circuit. (*c*) The maximum energy stored in the capacitor is larger than the maximum energy stored in the inductor. (*d*) The maximum energy stored in the inductor is larger than the maximum energy stored in the capacitor. (*e*) You cannot compare the maximum energies based on the data given because the ratio of the maximum energies depends on the actual capacitance and inductance values.

Determine the Concept The maximum energy stored in the electric field of the capacitor is given by $U_e = \frac{1}{2}\frac{Q^2}{C}$ and the maximum energy stored in the magnetic field of the inductor is given by $U_m = \frac{1}{2}LI^2$. Because energy is conserved in an LC circuit and oscillates between the inductor and the capacitor, $U_e = U_m = U_{total}$. $\boxed{(a)}$ is correct.

12 • True or false:

(*a*) A driven series *RLC* circuit that has a high *Q* factor has a narrow resonance curve.
(*b*) A circuit consists solely of a resistor, an inductor and a capacitor, all connected in series. If the resistance of the resistor is doubled, the natural frequency of the circuit remains the same.
(*c*) At resonance, the impedance of a driven series *RLC* combination equals the resistance *R*.
(*d*) At resonance, the current in a driven series *RLC* circuit is in phase with the voltage applied to the combination.

(*a*) True. The *Q* factor and the width of the resonance curve at half power are related according to $Q = \omega_0/\Delta\omega$; i.e., they are inversely proportional to each other.

(*b*) True. The natural frequency of the circuit depends only on the inductance *L* of the inductor and the capacitance *C* of the capacitor and is given by $\omega = 1/\sqrt{LC}$.

(*c*) True. The impedance of an *RLC* circuit is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. At resonance $X_L = X_C$ and so $Z = R$.

(*d*) True. The phase angle δ is related to X_L and X_C according to $\delta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$. At resonance $X_L = X_C$ and so $\delta = 0$.

13 • True or false (all questions related to a driven series *RLC* circuit):

(*a*) Near resonance, the power factor of a driven series *RLC* circuit is close to zero.
(*b*) The power factor of a driven series *RLC* circuit does not depend on the value of the resistance.
(*c*) The resonance frequency of a driven series *RLC* circuit does not depend on the value of the resistance.
(*d*) At resonance, the peak current of a driven series *RLC* circuit does not depend on the capacitance or the inductance.
(*e*) For frequencies below the resonant frequency, the capacitive reactance of a driven series *RLC* circuit is larger than the inductive reactance.
(*f*) For frequencies below the resonant frequency of a driven series *RLC* circuit, the phase of the current leads (is ahead of) the phase of the applied voltage.

(*a*) False. Near resonance, the power factor, given by $\cos\delta = \dfrac{R}{\sqrt{(X_L - X_C)^2 + R^2}}$, is close to 1.

(*b*) False. The power factor is given by $\cos\delta = \dfrac{R}{\sqrt{(X_L - X_C)^2 + R^2}}$.

(*c*) True. The resonance frequency for a driven series *RLC* circuit depends only on *L* and *C* and is given by $\omega_{\text{res}} = 1/\sqrt{LC}$

(*d*) True. At resonance $X_L - X_C = 0$ and so $Z = R$ and the peak current is given by $I_{\text{peak}} = V_{\text{app, peak}}/R$.

(*e*) True. Because the capacitive reactance varies inversely with the driving frequency and the inductive reactance varies directly with the driving frequency, at frequencies well below the resonance frequency the capacitive reactance is larger than the inductive reactance.

(*f*) True. For frequencies below the resonant frequency, the circuit is more capacitive than inductive and the phase constant ϕ is negative. This means that the current leads the applied voltage.

14 • You may have noticed that sometimes two radio stations can be heard when your receiver is tuned to a specific frequency. This situation often occurs when you are driving and are between two cities. Explain how this situation can occur.

Determine the Concept Because the power curves received by your radio from two stations have width, you could have two frequencies overlapping as a result of receiving signals from both stations.

15 • True or false (all questions related to a driven series *RLC* circuit):

(*a*) At frequencies much higher than or much lower than the resonant frequency of a driven series *RLC* circuit, the power factor is close to zero.
(*b*) The larger the resonance width of a driven series *RLC* circuit is, the larger the *Q* factor for the circuit is.
(*c*) The larger the resistance of a driven series *RLC* circuit is, the larger the resonance width for the circuit is.

(*a*) True. Because the power factor is given by $\cos\delta = R\Big/\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}$, for values of ω that are much higher or much lower than the resonant frequency, the term in parentheses becomes very large and $\cos\delta$ approaches zero.

(*b*) False. When the resonance curve is reasonably narrow, the *Q* factor can be approximated by $Q = \omega_0/\Delta\omega$. Hence a large value for *Q* corresponds to a narrow resonance curve.

(*c*) True. See Figure 29-21.

16 • An ideal transformer has N_1 turns on its primary and N_2 turns on its secondary. The average power delivered to a load resistance *R* connected across the secondary is P_2 when the primary rms voltage is V_1. The rms current in the primary windings can then be expressed as (*a*) P_2/V_1, (*b*) $(N_1/N_2)(P_2/V_1)$, (*c*) $(N_2/N_1)(P_2/V_1)$, (*d*) $(N_2/N_1)2(P_2/V_1)$.

Picture the Problem Let the subscript 1 denote the primary and the subscript 2 the secondary. Assuming no loss of power in the transformer, we can equate the power in the primary circuit to the power in the secondary circuit and solve for the rms current in the primary windings.

Assuming no loss of power in the transformer: $P_1 = P_2$

Substitute for P_1 and P_2 to obtain: $I_{1,\,\text{rms}}V_{1,\,\text{rms}} = I_{2,\,\text{rms}}V_{2,\,\text{rms}}$

Solving for $I_{1,\mathrm{rms}}$ and simplifying yields:

$$I_{1,\,\mathrm{rms}} = \frac{I_{2,\,\mathrm{rms}} V_{2,\,\mathrm{rms}}}{V_{1,\,\mathrm{rms}}} = \frac{P_2}{V_{1,\,\mathrm{rms}}}$$

$\boxed{(a)}$ is correct.

17 • **[SSM]** True or false:

(*a*) A transformer is used to change frequency.
(*b*) A transformer is used to change voltage.
(*c*) If a transformer steps up the current, it must step down the voltage.
(*d*) A step-up transformer, steps down the current.
(*e*) The standard household wall-outlet voltage in Europe is 220 V, about twice that used in the United States. If a European traveler wants her hair dryer to work properly in the United States, she should use a transformer that has more windings in its secondary coil than in its primary coil.
(*f*) The standard household wall-outlet voltage in Europe is 220 V, about twice that used in the United States. If an American traveler wants his electric razor to work properly in Europe, he should use a transformer that steps up the current.

(*a*) False. A transformer is a device used to raise or lower the voltage in a circuit.

(*b*) True. A transformer is a device used to raise or lower the voltage in a circuit.

(*c*) True. If energy is to be conserved, the product of the current and voltage must be constant.

(*d*) True. Because the product of current and voltage in the primary and secondary circuits is the same, increasing the current in the secondary results in a lowering (or stepping down) of the voltage.

(*e*) True. Because electrical energy is provided at a higher voltage in Europe, the visitor would want to step-up the voltage in order to make her hair dryer work properly.

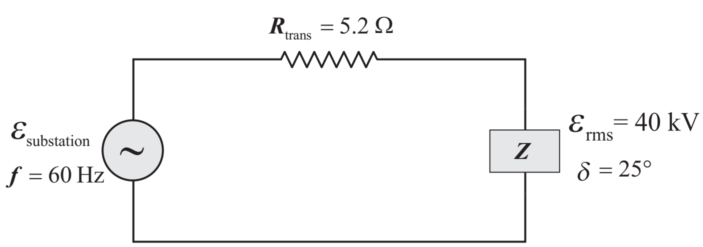
(*f*) True. Because electrical energy is provided at a higher voltage in Europe, the visitor would want to step-up the current (and decrease the voltage) in order to make his razor work properly.

Estimation and Approximation

18 •• The impedances of motors, transformers, and electromagnets include both resistance and inductive reactance. Suppose that phase of the current to a

large industrial plant lags the phase of the applied voltage by 25° when the plant is under full operation and using 2.3 MW of power. The power is supplied to the plant from a substation 4.5 km from the plant; the 60 Hz rms line voltage at the plant is 40 kV. The resistance of the transmission line from the substation to the plant is 5.2 Ω. The cost per kilowatt-hour to the company that owns the plant is $0.14, and the plant pays only for the actual energy used. (*a*) Estimate the resistance and inductive reactance of the plant's total load. (*b*) Estimate the rms current in the power lines and the rms voltage at the substation. (*c*) How much power is lost in transmission? (*d*) Suppose that the phase that the current lags the phase of the applied voltage is reduced to 18º by adding a bank of capacitors in series with the load. How much money would be saved by the electric utility during one month of operation, assuming the plant operates at full capacity for 16 h each day? (*e*) What must be the capacitance of this bank of capacitors to achieve this change in phase angle?

Picture the Problem We can find the resistance and inductive reactance of the plant's total load from the impedance of the load and the phase constant. The current in the power lines can be found from the total impedance of the load the potential difference across it and the rms voltage at the substation by applying Kirchhoff's loop rule to the substation-transmission wires-load circuit. The power lost in transmission can be found from $P_{\text{trans}} = I_{\text{rms}}^2 R_{\text{trans}}$. We can find the cost savings by finding the difference in the power lost in transmission when the phase angle is reduced to 18°. Finally, we can find the capacitance that is required to reduce the phase angle to 18° by first finding the capacitive reactance using the definition of tanδ and then applying the definition of capacitive reactance to find *C*.



(*a*) Relate the resistance and inductive reactance of the plant's total load to *Z* and δ:

$$R = Z\cos\delta$$
and
$$X_L = Z\sin\delta$$

Express *Z* in terms of the rms current I_{rms} in the power lines and the rms voltage $\mathcal{E}_{\text{rms}}$ at the plant:

$$Z = \frac{\mathcal{E}_{\text{rms}}}{I_{\text{rms}}}$$

Express the power delivered to the plant in terms of $\mathcal{E}_{rms}$, I_{rms}, and δ and solve for I_{rms}:

$$P_{av} = \mathcal{E}_{rms} I_{rms} \cos\delta$$

and

$$I_{rms} = \frac{P_{av}}{\mathcal{E}_{rms}\cos\delta} \qquad (1)$$

Substitute to obtain:

$$Z = \frac{\varepsilon_{rms}^2 \cos\delta}{P_{av}}$$

Substitute numerical values and evaluate Z:

$$Z = \frac{(40\,\text{kV})^2 \cos 25°}{2.3\,\text{MW}} = 630\,\Omega$$

Substitute numerical values and evaluate R and X_L:

$$R = (630\,\Omega)\cos 25° = 571\,\Omega = \boxed{0.57\,\text{k}\Omega}$$

and

$$X_L = (630\,\Omega)\sin 25° = 266\,\Omega = \boxed{0.27\,\text{k}\Omega}$$

(*b*) Use equation (1) to find the current in the power lines:

$$I_{rms} = \frac{2.3\,\text{MW}}{(40\,\text{kV})\cos 25°} = 63.4\,\text{A} = \boxed{63\,\text{A}}$$

Apply Kirchhoff's loop rule to the circuit:

$$\mathcal{E}_{sub} - I_{rms}R_{trans} - I_{rms}Z_{tot} = 0$$

Solve for $\mathcal{E}_{sub}$:

$$\mathcal{E}_{sub} = I_{rms}(R_{trans} + Z_{tot})$$

Substitute numerical values and evaluate $\mathcal{E}_{sub}$:

$$\mathcal{E}_{sub} = (63.4\,\text{A})(5.2\,\Omega + 630\,\Omega) = \boxed{40.3\,\text{kV}}$$

(*c*) The power lost in transmission is:

$$P_{trans} = I_{rms}^2 R_{trans} = (63.4\,\text{A})^2(5.2\,\Omega) = 20.9\,\text{kW} = \boxed{21\,\text{kW}}$$

(*d*) Express the cost savings ΔC in terms of the difference in energy consumption $(P_{25°} - P_{18°})\Delta t$ and the per-unit cost u of the energy:

$$\Delta C = (P_{25°} - P_{18°})\Delta t u$$

Express the power lost in transmission when $\delta = 18°$:

$$P_{18°} = I_{18°}^2 R_{\text{trans}}$$

Find the current in the transmission lines when $\delta = 18°$:

$$I_{18°} = \frac{2.3\,\text{MW}}{(40\,\text{kV})\cos 18°} = 60.5\,\text{A}$$

Evaluate $P_{18°}$:

$$P_{18°} = (60.5\,\text{A})^2(5.2\,\Omega) = 19.0\,\text{kW}$$

Substitute numerical values and evaluate ΔC:

$$\Delta C = (20.9\,\text{kW} - 19.0\,\text{kW})\left(16\frac{\text{h}}{\text{d}}\right)\left(30\frac{\text{d}}{\text{month}}\right)\left(\frac{\$0.14}{\text{kW}\cdot\text{h}}\right) = \boxed{\$128}$$

(*e*) The required capacitance is given by:

$$C = \frac{1}{2\pi f X_C}$$

Relate the new phase angle δ to the inductive reactance X_L, the reactance due to the added capacitance X_C, and the resistance of the load R:

$$\tan\delta = \frac{X_L - X_C}{R} \Rightarrow X_C = X_L - R\tan\delta$$

Substituting for X_C yields:

$$C = \frac{1}{2\pi f\,(X_L - R\tan\delta)}$$

Substitute numerical values and evaluate C:

$$C = \frac{1}{2\pi(60\,\text{s}^{-1})(266\,\Omega - (571\,\Omega)\tan 18°)} = \boxed{33\,\mu\text{F}}$$

Alternating Current in Resistors, Inductors, and Capacitors

19 • [SSM] A 100-W light bulb is screwed into a standard 120-V-rms socket. Find (*a*) the rms current, (*b*) the peak current, and (*c*) the peak power.

Picture the Problem We can use $P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}}$ to find I_{rms}, $I_{\text{peak}} = \sqrt{2} I_{\text{rms}}$ to find I_{peak}, and $P_{\text{peak}} = I_{\text{peak}}\mathcal{E}_{\text{peak}}$ to find P_{peak}.

(*a*) Relate the average power delivered by the source to the rms voltage across the bulb and the rms current through it:

$$P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \Rightarrow I_{\text{rms}} = \frac{P_{\text{av}}}{\mathcal{E}_{\text{rms}}}$$

Substitute numerical values and evaluate I_{rms}:

$$I_{\text{rms}} = \frac{100\,\text{W}}{120\,\text{V}} = 0.8333\text{ A} = \boxed{0.833\,\text{A}}$$

(*b*) Express I_{peak} in terms of I_{rms}:

$$I_{\text{peak}} = \sqrt{2}I_{\text{rms}}$$

Substitute for I_{rms} and evaluate I_{peak}:

$$I_{\text{peak}} = \sqrt{2}(0.8333\,\text{A}) = 1.1785\text{ A} = \boxed{1.18\,\text{A}}$$

(*c*) Express the maximum power in terms of the maximum voltage and maximum current:

$$P_{\text{peak}} = I_{\text{peak}}\mathcal{E}_{\text{peak}}$$

Substitute numerical values and evaluate P_{peak}:

$$P_{\text{peak}} = (1.1785\,\text{A})\sqrt{2}(120\,\text{V}) = \boxed{200\,\text{W}}$$

20 • A circuit breaker is rated for a current of 15 A rms at a voltage of 120 V rms. (*a*) What is the largest value of the peak current that the breaker can carry? (*b*) What is the maximum average power that can be supplied by this circuit?

Picture the Problem We can $I_{\text{peak}} = \sqrt{2}I_{\text{rms}}$ to find the largest peak current the breaker can carry and $P_{\text{av}} = I_{\text{rms}}V_{\text{rms}}$ to find the average power supplied by this circuit.

(*a*) Express I_{peak} in terms of I_{rms}:

$$I_{\text{peak}} = \sqrt{2}I_{\text{rms}} = \sqrt{2}(15\,\text{A}) = \boxed{21\,\text{A}}$$

(*b*) Relate the average power to the rms current and voltage:

$$P_{\text{av}} = I_{\text{rms}}V_{\text{rms}} = (15\,\text{A})(120\,\text{V}) = \boxed{1.8\,\text{kW}}$$

21 • **[SSM]** What is the reactance of a 1.00-μH inductor at (*a*) 60 Hz, (*b*) 600 Hz, and (*c*) 6.00 kHz?

Picture the Problem We can use $X_L = \omega L$ to find the reactance of the inductor at any frequency.

Express the inductive reactance as a function of *f*:

$$X_L = \omega L = 2\pi f L$$

(*a*) At *f* = 60 Hz:

$$X_L = 2\pi\left(60\,\text{s}^{-1}\right)(1.00\,\text{mH}) = \boxed{0.38\,\Omega}$$

(*b*) At $f = 600$ Hz: $X_L = 2\pi\left(600\,\text{s}^{-1}\right)\left(1.00\,\text{mH}\right) = \boxed{3.77\,\Omega}$

(*c*) At $f = 6.00$ kHz: $X_L = 2\pi\left(6.00\,\text{kHz}\right)\left(1.00\,\text{mH}\right) = \boxed{37.7\,\Omega}$

22 • An inductor has a reactance of 100 Ω at 80 Hz. (*a*) What is its inductance? (*b*) What is its reactance at 160 Hz?

Picture the Problem We can use $X_L = \omega L$ to find the inductance of the inductor at any frequency.

(*a*) Relate the reactance of the inductor to its inductance:

$$X_L = \omega L = 2\pi f L \Rightarrow L = \frac{X_L}{2\pi f}$$

Solve for and evaluate *L*:

$$L = \frac{100\,\Omega}{2\pi\left(80\,\text{s}^{-1}\right)} = 0.199\,\text{H} = \boxed{0.20\,\text{H}}$$

(*b*) At 160 Hz:

$$X_L = 2\pi\left(160\,\text{s}^{-1}\right)\left(0.199\,\text{H}\right) = \boxed{0.20\,\text{k}\Omega}$$

23 • At what frequency would the reactance of a 10-μF capacitor equal the reactance of a 1.0-μH inductor?

Picture the Problem We can equate the reactances of the capacitor and the inductor and then solve for the frequency.

Express the reactance of the inductor:

$$X_L = \omega L = 2\pi f L$$

Express the reactance of the capacitor:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

Equate these reactances to obtain:

$$2\pi f L = \frac{1}{2\pi f C} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{\left(10\,\mu\text{F}\right)\left(1.0\,\mu\text{H}\right)}} = \boxed{50\,\text{kHz}}$$

24 • What is the reactance of a 1.00-nF capacitor at (*a*) 60.0 Hz, (*b*) 6.00 kHz, and (*c*) 6.00 MHz?

Picture the Problem We can use $X_C = 1/\omega C$ to find the reactance of the capacitor at any frequency.

Express the capacitive reactance as a function of *f*:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

(*a*) At *f* = 60.0 Hz:

$$X_C = \frac{1}{2\pi\left(60.0\,\mathrm{s}^{-1}\right)\left(1.00\,\mathrm{nF}\right)} = \boxed{2.65\,\mathrm{M\Omega}}$$

(*b*) At *f* = 6.00 kHz:

$$X_C = \frac{1}{2\pi\left(6.00\,\mathrm{kHz}\right)\left(1.00\,\mathrm{nF}\right)} = \boxed{26.5\,\mathrm{k\Omega}}$$

(*c*) At *f* = 6.00 MHz:

$$X_C = \frac{1}{2\pi\left(6.00\,\mathrm{MHz}\right)\left(1.00\,\mathrm{nF}\right)} = \boxed{26.5\,\Omega}$$

25 • [SSM] A 20-Hz ac generator that produces a peak emf of 10 V is connected to a 20-μF capacitor. Find (*a*) the peak current and (*b*) the rms current.

Picture the Problem We can use $I_{\mathrm{peak}} = \mathcal{E}_{\mathrm{peak}}/X_C$ and $X_C = 1/\omega C$ to express I_{peak} as a function of $\mathcal{E}_{\mathrm{peak}}$, *f*, and *C*. Once we've evaluate I_{peak}, we can use $I_{\mathrm{rms}} = I_{\mathrm{peak}}/\sqrt{2}$ to find I_{rms}.

Express I_{peak} in terms of $\mathcal{E}_{\mathrm{peak}}$ and X_C:

$$I_{\mathrm{peak}} = \frac{\mathcal{E}_{\mathrm{peak}}}{X_C}$$

Express the capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

Substitute for X_C and simplify to obtain:

$$I_{\mathrm{peak}} = 2\pi f C\mathcal{E}_{\mathrm{peak}}$$

(*a*) Substitute numerical values and evaluate I_{peak}:

$$I_{\mathrm{peak}} = 2\pi\left(20\,\mathrm{s}^{-1}\right)\left(20\,\mu\mathrm{F}\right)\left(10\,\mathrm{V}\right) = 25.1\,\mathrm{mA} = \boxed{25\,\mathrm{mA}}$$

(*b*) Express I_{rms} in terms of I_{peak}:

$$I_{\mathrm{rms}} = \frac{I_{\mathrm{peak}}}{\sqrt{2}} = \frac{25.1\,\mathrm{mA}}{\sqrt{2}} = \boxed{18\,\mathrm{mA}}$$

26 • At what frequency is the reactance of a 10-μF capacitor (*a*) 1.00 Ω, (*b*) 100 Ω, and (*c*) 10.0 mΩ?

Picture the Problem We can use $X_C = 1/\omega C = 1/2\pi fC$ to relate the reactance of the capacitor to the frequency.

The reactance of the capacitor is given by:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC} \Rightarrow f = \frac{1}{2\pi CX_C}$$

(*a*) Find f when $X_C = 1.00\ \Omega$:

$$f = \frac{1}{2\pi(10\ \mu\text{F})(1.00\ \Omega)} = \boxed{16\,\text{kHz}}$$

(*b*) Find f when $X_C = 100\ \Omega$:

$$f = \frac{1}{2\pi(10\ \mu\text{F})(100\ \Omega)} = \boxed{0.16\,\text{kHz}}$$

(*c*) Find f when $X_C = 10.0\ \text{m}\Omega$:

$$f = \frac{1}{2\pi(10\ \mu\text{F})(10.0\ \text{m}\Omega)} = \boxed{1.6\,\text{MHz}}$$

27 •• A circuit consists of two ideal ac generators and a 25-Ω resistor, all connected in series. The potential difference across the terminals of one of the generators is given by $V_1 = (5.0\ \text{V})\cos(\omega t - \alpha)$, and the potential difference across the terminals of the other generator is given by $V_2 = (5.0\ \text{V})\cos(\omega t + \alpha)$, where $\alpha = \pi/6$. (*a*) Use Kirchhoff's loop rule and a trigonometric identity to find the peak current in the circuit. (*b*) Use a phasor diagram to find the peak current in the circuit. (*c*) Find the current in the resistor if $\alpha = \pi/4$ and the amplitude of V_2 is increased from 5.0 V to 7.0 V.

Picture the Problem We can use the trigonometric identity

$$\cos\theta + \cos\phi = 2\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)$$

to find the sum of the phasors V_1 and V_2 and then use this sum to express I as a function of time. In (*b*) we'll use a phasor diagram to obtain the same result and in (*c*) we'll use the phasor diagram appropriate to the given voltages to express the current as a function of time.

(*a*) Applying Kirchhoff's loop rule to the circuit yields:

$$V_1 + V_2 - IR = 0$$

Solve for I to obtain:

$$I = \frac{V_1 + V_2}{R}$$

Use the trigonometric identity $\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$ to find $V_1 + V_2$:

$$V_1 + V_2 = (5.0\,\text{V})[\cos(\omega t - \alpha) + \cos(\omega t + \alpha)] = (5\,\text{V})[2\cos\tfrac{1}{2}(2\omega t)\cos\tfrac{1}{2}(-2\alpha)]$$
$$= (10\,\text{V})\cos\frac{\pi}{6}\cos\omega t = (8.66\,\text{V})\cos\omega t$$

Substitute for $V_1 + V_2$ and R to obtain:

$$I = \frac{(8.66\,\text{V})\cos\omega t}{25\,\Omega} = (0.346\,\text{A})\cos\omega t$$
$$= (0.35\,\text{A})\cos\omega t$$

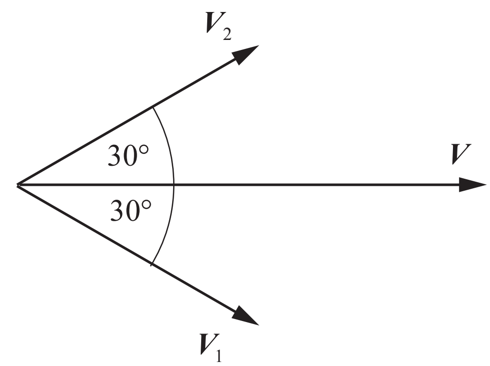
where

$$I_{\text{peak}} = \boxed{0.35\,\text{A}}$$

(b) Express the magnitude of the current in R:

$$|I| = \frac{|\vec{V}|}{R}$$

The phasor diagram for the voltages is shown to the right.



Use vector addition to find $|\vec{V}|$:

$$|\vec{V}| = 2|\vec{V}_1|\cos 30° = 2(5.0\,\text{V})\cos 30°$$
$$= 8.66\,\text{V}$$

Substitute for $|\vec{V}|$ and R to obtain:

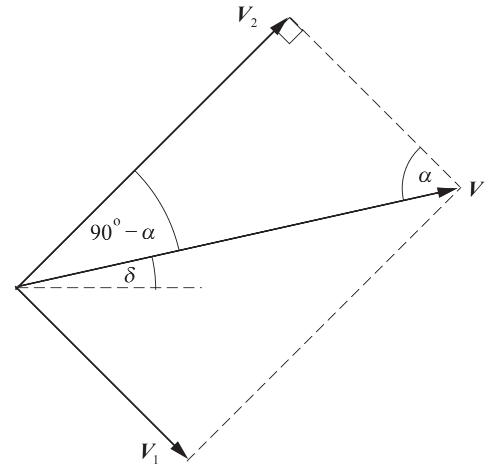
$$|I| = \frac{8.66\,\text{V}}{25\,\Omega} = 0.346\,\text{A}$$

and $I = (0.35\,\text{A})\cos\omega t$

where

$$I_{\text{peak}} = \boxed{0.35\,\text{A}}$$

(*c*) The phasor diagram is shown to the right. Note that the phase angle between $\vec{V}_1$ and $\vec{V}_2$ is now 90°.



Use the Pythagorean theorem to find $|\vec{V}|$:

$$|\vec{V}| = \sqrt{|\vec{V}_1|^2 + |\vec{V}_2|^2} = \sqrt{(5.0\text{ V})^2 + (7.0\text{ V})^2}$$
$$= 8.60\text{ V}$$

Express I as a function of t:

$$I = \frac{|\vec{V}|}{R}\cos(\omega t + \delta)$$

where

$$\delta = 45° - (90° - \alpha) = \alpha - 45°$$
$$= \tan^{-1}\left(\frac{7.0\text{ V}}{5.0\text{ V}}\right) - 45°$$
$$= 9.462° = 0.165\text{ rad}$$

Substitute numerical values and evaluate I:

$$I = \frac{8.60\text{ V}}{25\,\Omega}\cos(\omega t + 0.165\text{ rad})$$
$$= \boxed{(0.34\text{ A})\cos(\omega t + 0.17\text{ rad})}$$

Undriven Circuits Containing Capacitors, Resistors and Inductors

28 • (*a*) Show that $1/\sqrt{LC}$ has units of inverse seconds by substituting SI units for inductance and capacitance into the expression. (*b*) Show that $\omega_0 L/R$ (the expression for the *Q*-factor) is dimensionless by substituting SI units for angular frequency, inductance, and resistance into the expression.

Picture the Problem We can substitute the units of the various physical quantitities in $1/\sqrt{LC}$ and $Q = \omega_0 L/R$ to establish their units.

(*a*) Substitute the units for *L* and *C* in the expression $1/\sqrt{LC}$ and simplify to obtain:

$$\frac{1}{\sqrt{\text{H}\cdot\text{F}}}=\frac{1}{\sqrt{(\Omega\cdot\text{s})\left(\frac{\text{s}}{\Omega}\right)}}=\frac{1}{\sqrt{\text{s}^2}}=\boxed{\text{s}^{-1}}$$

(*b*) Substitute the units for ω_0, *L*, and *R* in the expression $Q=\omega_0 L/R$ and simplify to obtain:

$$\frac{\frac{1}{\text{s}}\cdot\frac{\text{V}\cdot\text{s}}{\text{A}}}{\frac{\text{V}}{\text{A}}}=\frac{\frac{1}{\text{s}}\cdot\frac{\text{V}\cdot\text{s}}{\text{A}}}{\frac{\text{V}}{\text{A}}}=1\Rightarrow\boxed{\text{no units}}$$

29 • **[SSM]** (*a*) What is the period of oscillation of an *LC* circuit consisting of an ideal 2.0-mH inductor and a 20-μF capacitor? (*b*) A circuit that oscillates consists solely of an 80-μF capacitor and a variable ideal inductor. What inductance is needed in order to tune this circuit to oscillate at 60 Hz?

Picture the Problem We can use $T=2\pi/\omega$ and $\omega=1/\sqrt{LC}$ to relate *T* (and hence *f*) to *L* and *C*.

(*a*) Express the period of oscillation of the *LC* circuit:

$$T=\frac{2\pi}{\omega}$$

For an *LC* circuit:

$$\omega=\frac{1}{\sqrt{LC}}$$

Substitute for ω to obtain:

$$T=2\pi\sqrt{LC}\qquad(1)$$

Substitute numerical values and evaluate *T*:

$$T=2\pi\sqrt{(2.0\,\text{mH})(20\,\mu\text{F})}=\boxed{1.3\,\text{ms}}$$

(*b*) Solve equation (1) for *L* to obtain:

$$L=\frac{T^2}{4\pi^2C}=\frac{1}{4\pi^2f^2C}$$

Substitute numerical values and evaluate *L*:

$$L=\frac{1}{4\pi^2\left(60\,\text{s}^{-1}\right)^2(80\,\mu\text{F})}=\boxed{88\,\text{mH}}$$

30 • An *LC* circuit has capacitance C_0 and inductance *L*. A second *LC* circuit has capacitance $\frac{1}{2}C_0$ and inductance 2*L*, and a third *LC* circuit has capacitance $2C_0$ and inductance $\frac{1}{2}L$. (*a*) Show that each circuit oscillates with the same frequency. (*b*) In which circuit would the peak current be greatest if the peak voltage across the capacitor in each circuit was the same?

Picture the Problem We can use the expression $f_0 = 1/2\pi\sqrt{LC}$ for the resonance frequency of an *LC* circuit to show that each circuit oscillates with the same frequency. In (*b*) we can use $I_{peak} = \omega Q_0$, where Q_0 is the charge of the capacitor at time zero, and the definition of capacitance $Q_0 = CV$ to express I_{peak} in terms of ω, *C* and *V*.

Express the resonance frequency for an *LC* circuit:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

(*a*) Express the product of *L* and C_0 for each circuit:

Circuit 1: $L_1C_1 = L_1C_0$,
Circuit 2: $L_2C_2 = (2L_1)(\frac{1}{2}C_0) = L_1C_1$,
and
Circuit 3: $L_3C_3 = (\frac{1}{2}L_1)(2C_0) = L_1C_1$

Because $L_1C_1 = L_2C_2 = L_3C_3$, the resonance frequencies of the three circuits are the same.

(*b*) Express I_{peak} in terms of the charge stored in the capacitor:

$$I_{peak} = \omega Q_0$$

Express Q_0 in terms of the capacitance of the capacitor and the potential difference across the capacitor:

$$Q_0 = CV$$

Substituting for Q_0 yields:

$$I_{peak} = \omega CV$$

or, for ω and *V* constant, $I_{peak} \propto C$.

Hence, the circuit with capacitance $2C_0$ has the greatest peak current.

31 •• A 5.0-μF capacitor is charged to 30 V and is then connected across an ideal 10-mH inductor. (*a*) How much energy is stored in the system? (*b*) What is the frequency of oscillation of the circuit? (*c*) What is the peak current in the circuit?

Picture the Problem We can use $U = \frac{1}{2}CV^2$ to find the energy stored in the electric field of the capacitor, $\omega_0 = 2\pi f_0 = 1/\sqrt{LC}$ to find f_0, and $I_{peak} = \omega Q_0$ and $Q_0 = CV$ to find I_{peak}.

(*a*) Express the energy stored in the system as a function of *C* and *V*:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(5.0\,\mu\text{F})(30\,\text{V})^2 = \boxed{2.3\,\text{mJ}}$$

(*b*) Express the resonance frequency of the circuit in terms of *L* and *C*:

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Substitute numerical values and evaluate f_0:

$$f_0 = \frac{1}{2\pi\sqrt{(10\,\text{mH})(5.0\,\mu\text{F})}} = 712\,\text{Hz}$$
$$= \boxed{0.71\,\text{kHz}}$$

(*c*) Express I_{peak} in terms of the charge stored in the capacitor:

$$I_{\text{peak}} = \omega Q_0$$

Express Q_0 in terms of the capacitance of the capacitor and the potential difference across the capacitor:

$$Q_0 = CV$$

Substituting for Q_0 yields:

$$I_{\text{peak}} = \omega CV$$

Substitute numerical values and evaluate I_{peak}:

$$I_{\text{peak}} = 2\pi\left(712\,\text{s}^{-1}\right)(5.0\,\mu\text{F})(30\,\text{V})$$
$$= \boxed{0.67\,\text{A}}$$

32 •• A coil with internal resistance can be modeled as a resistor and an ideal inductor in series. Assume that the coil has an internal resistance of 1.00 Ω and an inductance of 400 mH. A 2.00-μF capacitor is charged to 24.0 V and is then connected across coil. (*a*) What is the initial voltage across the coil? (*b*) How much energy is dissipated in the circuit before the oscillations die out? (*c*) What is the frequency of oscillation the circuit? (Assume the internal resistance is sufficiently small that has no impact on the frequency of the circuit.) (*d*) What is the quality factor of the circuit?

Picture the Problem In Part (*a*) we can apply Kirchhoff's loop rule to find the initial voltage across the coil. (*b*) The total energy lost via joule heating is the total energy initially stored in the capacitor. (*c*) The natural frequency of the circuit is given by $f_0 = 1/2\pi\sqrt{LC}$. In Part (*d*) we can use its definition to find the quality factor of the circuit.

(*a*) Application of Kirchhoff's loop rule leads us to conclude that the initial voltage across the coil is $\boxed{24.0\text{ V}}$.

(*b*) Because the ideal inductor can not dissipate energy as heat, all of the energy initially stored in the capacitor will be dissipated as joule heat in the resistor:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(2.00\,\mu\text{F})(24.0\text{ V})^2 = \boxed{0.576\text{ mJ}}$$

(*c*) The natural frequency of the circuit is:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(400\text{ mH})(2.00\,\mu\text{F})}} = \boxed{178\text{ Hz}}$$

(*d*) The quality factor of the circuit is given by:

$$Q = \frac{\omega_0 L}{R}$$

Substituting for ω_0 and simplifying yields:

$$Q = \frac{\frac{1}{\sqrt{LC}}L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Substitute numerical values and evaluate Q:

$$Q = \frac{1}{1.00\,\Omega}\sqrt{\frac{400\text{ mH}}{2.00\,\mu\text{F}}} = \boxed{447}$$

33 ••• [SSM] An inductor and a capacitor are connected, as shown in Figure 29-30. Initially, the switch is open, the left plate of the capacitor has charge Q_0. The switch is then closed. (*a*) Plot both Q versus t and I versus t on the same graph, and explain how it can be seen from these two plots that the current leads the charge by 90º. (*b*) The expressions for the charge and for the current are given by Equations 29-38 and 29-39, respectively. Use trigonometry and algebra to show that the current leads the charge by 90º.

Picture the Problem Let Q represent the instantaneous charge on the capacitor and apply Kirchhoff's loop rule to obtain the differential equation for the circuit. We can then solve this equation to obtain an expression for the charge on the capacitor as a function of time and, by differentiating this expression with respect

to time, an expression for the current as a function of time. We'll use a spreadsheet program to plot the graphs.

Apply Kirchhoff's loop rule to a clockwise loop just after the switch is closed:

$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$

Because $I = dQ/dt$:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \text{ or } \frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

The solution to this equation is:

$$Q(t) = Q_0 \cos(\omega t - \delta)$$

where $\omega = \sqrt{\frac{1}{LC}}$

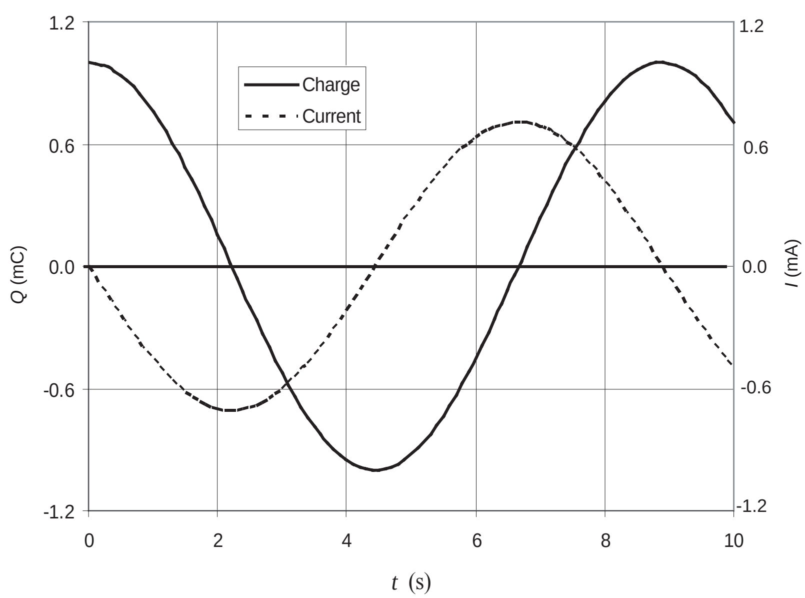
Because $Q(0) = Q_0$, $\delta = 0$ and:

$$Q(t) = Q_0 \cos \omega t$$

The current in the circuit is the derivative of Q with respect to t:

$$I = \frac{dQ}{dt} = \frac{d}{dt}\left[Q_0 \cos \omega t\right] = -\omega Q_0 \sin \omega t$$

(a) A spreadsheet program was used to plot the following graph showing both the charge on the capacitor and the current in the circuit as functions of time. L, C, and Q_0 were all arbitrarily set equal to one to obtain these graphs. Note that the current leads the charge by one-fourth of a cycle or 90°.



(b) The equation for the current is:

$$I = -\omega Q_0 \sin \omega t \qquad (1)$$

The sine and cosine functions are related through the identity:

$$-\sin\theta = \cos\left(\theta + \frac{\pi}{2}\right)$$

Use this identity to rewrite equation (1):

$$I = -\omega Q_0 \sin\omega t = \boxed{\omega Q_0 \cos\left(\omega t + \frac{\pi}{2}\right)}$$

Thus, the current leads the charge by 90°.

Driven *RL* Circuits

34 •• A circuit consists of a resistor, an ideal 1.4-H inductor and an ideal 60-Hz generator, all connected in series. The rms voltage across the resistor is 30 V and the rms voltage across the inductor is 40 V. (*a*) What is the resistance of the resistor? (*b*) What is the peak emf of the generator?

Picture the Problem We can express the ratio of V_R to V_L and solve this expression for the resistance *R* of the circuit. In (*b*) we can use the fact that, in an *LR* circuit, V_L leads V_R by 90° to find the ac input voltage.

(*a*) Express the potential differences across *R* and *L* in terms of the common current through these components:

$$V_L = IX_L = I\omega L$$

and

$$V_R = IR$$

Divide the second of these equations by the first to obtain:

$$\frac{V_R}{V_L} = \frac{IR}{I\omega L} = \frac{R}{\omega L} \Rightarrow R = \left(\frac{V_R}{V_L}\right)\omega L$$

Substitute numerical values and evaluate *R*:

$$R = \left(\frac{30\,\text{V}}{40\,\text{V}}\right)2\pi\left(60\,\text{s}^{-1}\right)(1.4\,\text{H}) = \boxed{0.40\,\text{k}\Omega}$$

(*b*) Because V_R leads V_L by 90° in an *LR* circuit:

$$V_{\text{peak}} = \sqrt{2}V_{\text{rms}} = \sqrt{2}\sqrt{V_R^2 + V_L^2}$$

Substitute numerical values and evaluate V_{peak}:

$$V_{\text{peak}} = \sqrt{2}\sqrt{(30\,\text{V})^2 + (40\,\text{V})^2} = \boxed{71\,\text{V}}$$

35 •• **[SSM]** A coil that has a resistance of 80.0 Ω has an impedance of 200 Ω when driven at a frequency of 1.00 kHz. What is the inductance of the coil?

Picture the Problem We can solve the expression for the impedance in an *LR* circuit for the inductive reactance and then use the definition of X_L to find *L*.

Express the impedance of the coil in terms of its resistance and inductive reactance:

$$Z=\sqrt{R^2+X_L^2}$$

Solve for X_L to obtain:

$$X_L=\sqrt{Z^2-R^2}$$

Express X_L in terms of L:

$$X_L=2\pi fL$$

Equate these two expressions to obtain:

$$2\pi fL=\sqrt{Z^2-R^2}\Rightarrow L=\frac{\sqrt{Z^2-R^2}}{2\pi f}$$

Substitute numerical values and evaluate L:

$$L=\frac{\sqrt{(200\,\Omega)^2-(80.0\,\Omega)^2}}{2\pi(1.00\,\text{kHz})}=\boxed{29.2\,\text{mH}}$$

36 •• A two conductor transmission line simultaneously carries a superposition of two voltage signals, so the potential difference between the two conductors is given by $V = V_1 + V_2$, where $V_1 = (10.0\text{ V})\cos(\omega_1 t)$ and $V_2 = (10.0\text{ V})\cos(\omega_2 t)$, where $\omega_1 = 100$ rad/s and $\omega_2 = 10\,000$ rad/s. A 1.00 H inductor and a 1.00 kΩ shunt resistor are inserted into the transmission line as shown in Figure 29-31. (Assume that the output is connected to a load that draws only an insignificant amount of current.) (*a*) What is the voltage (V_{out}) at the output of the transmission line? (*b*) What is the ratio of the low-frequency amplitude to the high-frequency amplitude at the output?

Picture the Problem We can express the two output voltage signals as the product of the current from each source and R = 1.00 kΩ. We can find the currents due to each source using the given voltage signals and the definition of the impedance for each of them.

(*a*) Express the voltage signals observed at the output side of the transmission line in terms of the potential difference across the resistor:

$$V_{1,\text{out}}=I_1R$$

and

$$V_{2,\text{out}}=I_2R$$

Evaluate I_1 and I_2:

$$I_1 = \frac{V_1}{Z_1} = \frac{(10.0\,\text{V})\cos 100t}{\sqrt{(1.00\,\text{k}\Omega)^2 + \left[(100\,\text{s}^{-1})(1.00\,\text{H})\right]^2}} = (9.95\,\text{mA})\cos 100t$$

and

$$I_2 = \frac{V_2}{Z_2} = \frac{(10.0\,\text{V})\cos 10^4 t}{\sqrt{(1.00\,\text{k}\Omega)^2 + \left[(10^4\,\text{s}^{-1})(1.00\,\text{H})\right]^2}} = (0.995\,\text{mA})\cos 10^4 t$$

Substitute for I_1 and I_2 to obtain:

$$V_{1,\,\text{out}} = (1.00\,\text{k}\Omega)(9.95\,\text{mA})\cos 100t = \boxed{(9.95\,\text{V})\cos 100t}$$

where $\omega_1 = 100$ rad/s and $\omega_2 = 10\,000$ rad/s.

and

$$V_{2,\,\text{out}} = (1.00\,\text{k}\Omega)(0.995\,\text{mA})\cos 10^4 t = \boxed{(0.995\,\text{V})\cos 10^4 t}$$

(*b*) Express the ratio of $V_{1,\text{out}}$ to $V_{2,\text{out}}$:

$$\frac{V_{1,\,\text{out}}}{V_{2,\,\text{out}}} = \frac{9.95\,\text{V}}{0.995\,\text{V}} = \boxed{10:1}$$

37 •• A coil is connected to a 120-V rms, 60-Hz line. The average power supplied to the coil is 60 W, and the rms current is 1.5 A. Find (*a*) the power factor, (*b*) the resistance of the coil, and (*c*) the inductance of the coil. (*d*) Does the current lag or lead the voltage? Explain your answer. (*e*) Support your answer to Part (*d*) by determining the phase angle.

Picture the Problem The average power supplied to coil is related to the power factor by $P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos\delta$. In (*b*) we can use $P_{\text{av}} = I_{\text{rms}}^2 R$ to find *R*. Because the inductance *L* is related to the resistance *R* and the phase angle δ according to $X_L = \omega L = R\tan\delta$, we can use this relationship to find the resistance of the coil. Finally, we can decide whether the current leads or lags the voltage by noting that the circuit is inductive.

(*a*) Express the average power supplied to the coil in terms of the power factor of the circuit:

$$P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos\delta \Rightarrow \cos\delta = \frac{P_{\text{av}}}{\mathcal{E}_{\text{rms}} I_{\text{rms}}}$$

Substitute numerical values and evaluate $\cos\delta$:

$$\cos\delta = \frac{60\,\text{W}}{(120\,\text{V})(1.5\,\text{A})} = 0.333 = \boxed{0.33}$$

(*b*) Express the power supplied by the source in terms of the resistance of the coil:

$$P_{av} = I_{rms}^2 R \Rightarrow R = \frac{P_{av}}{I_{rms}^2}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{60\ \text{W}}{(1.5\ \text{A})^2} = 26.7\ \Omega = \boxed{27\ \Omega}$$

(*c*) Relate the inductive reactance to the resistance and phase angle:

$$X_L = \omega L = R\tan\delta$$

Solving for *L* yields:

$$L = \frac{R\tan\delta}{\omega} = \frac{R\tan\left[\cos^{-1}(0.333)\right]}{2\pi f}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{(26.7\ \Omega)\tan(70.5°)}{2\pi(60\ \text{s}^{-1})} = \boxed{0.20\ \text{H}}$$

(*d*) Evaluate X_L:

$$X_L = (26.7\ \Omega)\tan(70.5°) = 75.4\ \Omega$$

Because the circuit is inductive, the current lags the voltage.

(*e*) From Part (*a*):

$$\delta = \cos^{-1}(0.333) = \boxed{71°}$$

38 •• A 36-mH inductor that has a resistance of 40 Ω is connected to an ideal ac voltage source whose output is given by $\mathcal{E}$ = (345 V) cos(150πt), where *t* is in seconds. Determine (*a*) the peak current in the circuit, (*b*) the peak and rms voltages across the inductor, (*c*) the average power dissipation, and (*d*) the peak and average magnetic energy stored in the inductor.

Picture the Problem (*a*) We can use $I_{peak} = \mathcal{E}_{peak} \Big/ \sqrt{R^2 + (\omega L)^2}$ and $V_{L,\,peak} = I_{peak} X_L = \omega L I_{peak}$ to find the peak current in the circuit and the peak voltage across the inductor. (*b*) Once we've found $V_{L,\,peak}$ we can find $V_{L,\,rms}$ using $V_{L,\,rms} = V_{L,\,peak} \big/ \sqrt{2}$. (*c*) We can use $P_{av} = \frac{1}{2} I_{rms}^2 R$ to find the average power dissipation, and (*d*) $U_{L,\,peak} = \frac{1}{2} L I_{peak}^2$ to find the peak and average magnetic energy stored in the inductor. The average energy stored in the magnetic field of the inductor can be found using $U_{L,av} = \int P_{av}\,dt$.

(*a*) Apply Kirchhoff's loop rule to the circuit to obtain:

$$\mathcal{E} - IZ = 0 \Rightarrow I = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}}{\sqrt{R^2 + (\omega L)^2}}$$

Substitute numerical values and evaluate *I*:

$$I = \frac{(345\,\text{V})\cos(150\pi t)}{\sqrt{(40\,\Omega)^2 + \left[(150\pi\,\text{s}^{-1})(36\,\text{mH})\right]^2}} = (7.94\,\text{A})\cos(150\pi t)$$

and $I_{\text{peak}} = \boxed{7.9\,\text{A}}$.

(*b*) Because $\mathcal{E}$ = (345 V) cos(150πt) :

$$V_{L,\,\text{peak}} = \boxed{345\,\text{V}}$$

Find $V_{L,\text{rms}}$ from $V_{L,\,\text{peak}}$:

$$V_{L,\text{rms}} = \frac{V_{L,\,\text{peak}}}{\sqrt{2}} = \frac{345\,\text{V}}{\sqrt{2}} = \boxed{244\,\text{V}}$$

(*c*) Relate the average power dissipation to I_{peak} and *R*:

$$P_{\text{av}} = I_{\text{rms}}^2 R = \left(\frac{I_{\text{peak}}}{\sqrt{2}}\right)^2 R = \tfrac{1}{2} I_{\text{peak}}^2 R$$

Substitute numerical values and evaluate P_{av} :

$$P_{\text{av}} = \tfrac{1}{2}(7.94\,\text{A})^2(40\,\Omega) = \boxed{1.3\,\text{kW}}$$

(*d*) The maximum energy stored in the magnetic field of the inductor is:

$$U_{L,\,\text{peak}} = \tfrac{1}{2} L I_{\text{peak}}^2 = \tfrac{1}{2}(36\,\text{mH})(7.94\,\text{A})^2 = \boxed{1.1\,\text{J}}$$

The definition of $U_{L,\text{av}}$ is:

$$U_{L,\text{av}} = \frac{1}{T}\int_0^T U(t)\,dt$$

U(*t*) is given by:

$$U(t) = \frac{1}{2} L[I(t)]^2$$

Substitute for *U*(*t*) to obtain:

$$U_{L,\text{av}} = \frac{L}{2T}\int_0^T [I(t)]^2\,dt$$

Evaluating the integral yields:

$$U_{L,\text{av}} = \frac{L}{2T}\left[\frac{1}{2} I_{\text{peak}}^2\right] T = \frac{1}{4} L I_{\text{peak}}^2$$

Substitute numerical values and evaluate $U_{L,\text{av}}$:

$$U_{L,\text{av}} = \frac{1}{4}(36\,\text{mH})(7.94\,\text{A})^2 = \boxed{0.57\,\text{J}}$$

39 •• **[SSM]** A coil that has a resistance R and an inductance L has a power factor equal to 0.866 when driven at a frequency of 60 Hz. What is the coil's power factor it is driven at 240 Hz?

Picture the Problem We can use the definition of the power factor to find the relationship between X_L and R when the coil is driven at a frequency of 60 Hz and then use the definition of X_L to relate the inductive reactance at 240 Hz to the inductive reactance at 60 Hz. We can then use the definition of the power factor to determine its value at 240 Hz.

Using the definition of the power factor, relate R and X_L:

$$\cos\delta = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_L^2}} \qquad (1)$$

Square both sides of the equation to obtain:

$$\cos^2\delta = \frac{R^2}{R^2 + X_L^2}$$

Solve for $X_L^2(60\,\text{Hz})$:

$$X_L^2(60\,\text{Hz}) = R^2\left(\frac{1}{\cos^2\delta} - 1\right)$$

Substitute for $\cos\delta$ and simplify to obtain:

$$X_L^2(60\,\text{Hz}) = R^2\left(\frac{1}{(0.866)^2} - 1\right) = \tfrac{1}{3}R^2$$

Use the definition of X_L to obtain:

$$X_L^2(f) = 4\pi f^2 L^2 \text{ and } X_L^2(f') = 4\pi f'^2 L^2$$

Dividing the second of these equations by the first and simplifying yields:

$$\frac{X_L^2(f')}{X_L^2(f)} = \frac{4\pi f'^2 L^2}{4\pi f^2 L^2} = \frac{f'^2}{f^2}$$

or

$$X_L^2(f') = \left(\frac{f'}{f}\right)^2 X_L^2(f)$$

Substitute numerical values to obtain:

$$X_L^2(240\,\text{Hz}) = \left(\frac{240\,\text{s}^{-1}}{60\,\text{s}^{-1}}\right)^2 X_L^2(60\,\text{Hz})$$

$$= 16\left(\frac{1}{3}R^2\right) = \frac{16}{3}R^2$$

Substitute in equation (1) to obtain:

$$(\cos\delta)_{240\,\text{Hz}} = \frac{R}{\sqrt{R^2 + \frac{16}{3}R^2}} = \sqrt{\frac{3}{19}} = \boxed{0.397}$$

40 •• A resistor and an inductor are connected in parallel across an ideal ac voltage source whose output is given by $\mathcal{E} = \mathcal{E}_{peak}\cos\omega t$ as shown in Figure 29-32. Show that (*a*) the current in the resistor is given by $I_R = \mathcal{E}_{peak}/R\cos\omega t$, (*b*) the current in the inductor is given by $I_L = \mathcal{E}_{peak}/X_L\cos(\omega t - 90°)$, and (*c*) the current in the voltage source is given by $I = I_R + I_L = I_{peak}\cos(\omega t - \delta)$, where $I_{peak} = \mathcal{E}_{max}/Z$.

Picture the Problem Because the resistor and the inductor are connected in parallel, the voltage drops across them are equal. Also, the total current is the sum of the current through the resistor and the current through the inductor. Because these two currents are not in phase, we'll need to use phasors to calculate their sum. The amplitudes of the applied voltage and the currents are equal to the magnitude of the phasors. That is $\left|\vec{\boldsymbol{\mathcal{E}}}\right| = \mathcal{E}_{peak}$, $\left|\vec{\boldsymbol{I}}\right| = I_{peak}$, $\left|\vec{\boldsymbol{I}}_R\right| = I_{R,\,peak}$, and $\left|\vec{\boldsymbol{I}}_L\right| = I_{L,\,peak}$.

(*a*) The ac source applies a voltage given by $\mathcal{E} = \mathcal{E}_{peak}\cos\omega t$. Thus, the voltage drop across both the load resistor and the inductor is:

$$\mathcal{E}_{peak}\cos\omega t = I_R R$$

The current in the resistor is in phase with the applied voltage:

$$I_R = I_{R,\,peak}\cos\omega t$$

Because $I_{R,\,peak} = \dfrac{\mathcal{E}_{peak}}{R}$:

$$I_R = \boxed{\frac{\mathcal{E}_{peak}}{R}\cos\omega t}$$

(*b*) The current in the inductor lags the applied voltage by 90°:

$$I_L = I_{L,\,peak}\cos(\omega t - 90°)$$

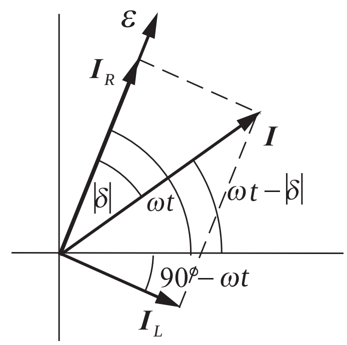
Because $I_{L,\,peak} = \dfrac{\mathcal{E}_{peak}}{X_L}$:

$$I_L = \boxed{\frac{\mathcal{E}_{peak}}{X_L}\cos(\omega t - 90°)}$$

(*c*) The net current I is the sum of the currents through the parallel branches:

$$I = I_R + I_L$$

Draw the phasor diagram for the circuit. The projections of the phasors onto the horizontal axis are the instantaneous values. The current in the resistor is in phase with the applied voltage, and the current in the inductor lags the applied voltage by 90°. The net current phasor is the sum of the branch current phasors $\left(\vec{I} = \vec{I}_L + \vec{I}_R\right)$.



The peak current through the parallel combination is equal to $\mathcal{E}_{\text{peak}}/Z$, where Z is the impedance of the combination:

$$I = I_{\text{peak}} \cos(\omega t - |\delta|),$$

where $I_{\text{peak}} = \dfrac{\mathcal{E}_{\text{peak}}}{Z}$

From the phasor diagram we have:

$$I_{\text{peak}}^2 = I_{R,\,\text{peak}}^2 + I_{L,\,\text{peak}}^2$$

$$= \left(\frac{\mathcal{E}_{\text{peak}}}{R}\right)^2 + \left(\frac{\mathcal{E}_{\text{peak}}}{X_{\text{L}}}\right)^2$$

$$= \mathcal{E}_{\text{peak}}^2\left(\frac{1}{R^2} + \frac{1}{X_{\text{L}}^2}\right) = \frac{\mathcal{E}_{\text{peak}}^2}{Z^2}$$

where $\dfrac{1}{Z^2} = \dfrac{1}{R^2} + \dfrac{1}{X_L^2}$

.

Solving for I_{peak} yields:

$$I_{\text{peak}} = \boxed{\frac{\mathcal{E}_{\text{peak}}}{Z}} \text{ where } Z^{-2} = R^{-2} + X_L^{-2}$$

From the phasor diagram:

$$I = \boxed{I_{\text{peak}} \cos(\omega t - |\delta|)}$$

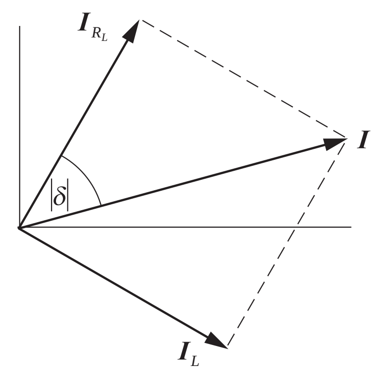
where

$$\tan|\delta| = \frac{I_{L,\,\text{peak}}}{I_{R,\,\text{peak}}} = \frac{\dfrac{\mathcal{E}_{\text{peak}}}{X_L}}{\dfrac{\mathcal{E}_{\text{peak}}}{R}} = \boxed{\frac{R}{X_L}}$$

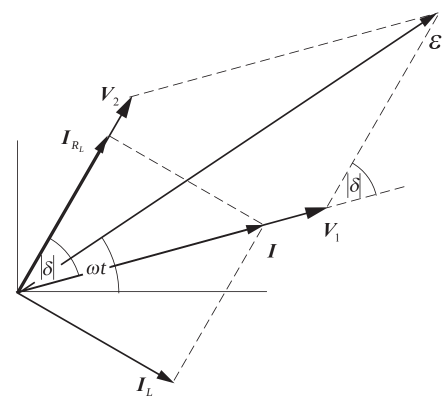
41 •• **[SSM]** Figure 29-33 shows a load resistor that has a resistance of $R_L = 20.0\ \Omega$ connected to a high-pass filter consisting of an inductor that has inductance $L = 3.20\text{-mH}$ and a resistor that has resistance $R = 4.00\text{-}\Omega$. The output of the ideal ac generator is given by $\mathcal{E} = (100\text{ V})\cos(2\pi ft)$. Find the rms currents in all three branches of the circuit if the driving frequency is (*a*) 500 Hz and (*b*) 2000 Hz. Find the fraction of the total average power supplied by the ac generator that is delivered to the load resistor if the frequency is (*c*) 500 Hz and (*d*) 2000 Hz.

Picture the Problem $\mathcal{E} = V_1 + V_2$, where V_1 is the voltage drop across R and V_2 is the voltage drop across the parallel combination of L and R_L. $\vec{\boldsymbol{\varepsilon}} = \vec{\boldsymbol{V}}_1 + \vec{\boldsymbol{V}}_2$ is the relation for the phasors. For the parallel combination $\vec{\boldsymbol{I}} = \vec{\boldsymbol{I}}_{R_L} + \vec{\boldsymbol{I}}_L$. Also, V_1 is in phase with I and V_2 is in phase with I_{R_L}. First draw the phasor diagram for the currents in the parallel combination, then add the phasors for the voltages to the diagram.

The phasor diagram for the currents in the circuit is:



Adding the voltage phasors to the diagram gives:



The maximum current in the inductor, $I_{2,\text{peak}}$, is given by:

$$I_{2,\text{peak}} = \frac{V_{2,\text{peak}}}{Z_2} \tag{1}$$

where $Z_2^{-2} = R_L^{-2} + X_L^{-2}$ (2)

$\tan|\delta|$ is given by:

$$\tan|\delta| = \frac{I_{L,\text{peak}}}{I_{R,\text{peak}}} = \frac{V_{2,\text{peak}}/X_\text{L}}{V_{2,\text{peak}}/R_\text{L}} = \frac{R_L}{X_L} = \frac{R_L}{\omega L} = \frac{R_L}{2\pi f L}$$

Solve for $|\delta|$ to obtain:

$$|\delta| = \tan^{-1}\left(\frac{R_\text{L}}{2\pi f L}\right) \tag{3}$$

Apply the law of cosines to the triangle formed by the voltage phasors to obtain:

$$\mathcal{E}_{\text{peak}}^2 = V_{1,\,\text{peak}}^2 + V_{2,\,\text{peak}}^2 + 2V_{1,\,\text{peak}}V_{2,\,\text{peak}}\cos|\delta|$$

or

$$I_{\text{peak}}^2 Z^2 = I_{\text{peak}}^2 R^2 + I_{\text{peak}}^2 Z_2^2 + 2I_{\text{peak}} R I_{\text{peak}} Z_2 \cos|\delta|$$

Dividing out the current squared yields:

$$Z^2 = R^2 + Z_2^2 + 2RZ_2\cos|\delta|$$

Solving for Z yields:

$$Z = \sqrt{R^2 + Z_2^2 + 2RZ_2\cos|\delta|} \quad (4)$$

The maximum current I_{peak} in the circuit is given by:

$$I_{\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{Z} \quad (5)$$

I_{rms} is related to I_{peak} according to:

$$I_{\text{rms}} = \frac{1}{\sqrt{2}} I_{\text{peak}} \quad (6)$$

(*a*) Substitute numerical values in equation (3) and evaluate $|\delta|$:

$$|\delta| = \tan^{-1}\left(\frac{20.0\,\Omega}{2\pi(500\,\text{Hz})(3.20\,\text{mH})}\right) = \tan^{-1}\left(\frac{20.0\,\Omega}{10.053\,\Omega}\right) = 63.31°$$

Solving equation (2) for Z_2 yields:

$$Z_2 = \frac{1}{\sqrt{R_L^{-2} + X_L^{-2}}}$$

Substitute numerical values and evaluate Z_2:

$$Z_2 = \frac{1}{\sqrt{(20.0\,\Omega)^{-2} + (10.053\,\Omega)^{-2}}} = 8.982\,\Omega$$

Substitute numerical values and evaluate Z:

$$Z = \sqrt{(4.00\,\Omega)^2 + (8.982\,\Omega)^2 + 2(4.00\,\Omega)(8.982\,\Omega)\cos 63.31°} = 11.36\,\Omega$$

Substitute numerical values in equation (5) and evaluate I_{peak}:

$$I_{\text{peak}} = \frac{100\,\text{V}}{11.36\,\Omega} = 8.806\,\text{A}$$

Substitute for I_{peak} in equation (6) and evaluate I_{rms}:

$$I_{\text{rms}} = \frac{1}{\sqrt{2}}(8.806\,\text{A}) = \boxed{6.23\,\text{A}}$$

The maximum and rms values of V_2 are given by:

$$V_{2,\,\text{peak}} = I_{\text{peak}} Z_2 = (8.806\,\text{A})(8.982\,\Omega) = 79.095\,\text{V}$$

and

$$V_{2,\text{rms}} = \frac{1}{\sqrt{2}} V_{2,\,\text{peak}} = \frac{1}{\sqrt{2}}(79.095\,\text{V}) = 55.929\,\text{V}$$

The rms values of $I_{R_L,\text{rms}}$ and $I_{L,\text{rms}}$ are:

$$I_{R_L,\text{rms}} = \frac{V_{2,\text{rms}}}{R_L} = \frac{55.929\,\text{V}}{20.0\,\Omega} = \boxed{2.80\,\text{A}}$$

and

$$I_{L,\text{rms}} = \frac{V_{2,\text{rms}}}{X_L} = \frac{55.929\,\text{V}}{10.053\,\Omega} = \boxed{5.56\,\text{A}}$$

(*b*) Proceed as in (*a*) with f = 2000 Hz to obtain:

$X_L = 40.2\,\Omega$, $|\delta| = 26.4°$, $Z_2 = 17.9\,\Omega$, $Z = 21.6\,\Omega$, $I_{\text{peak}} = 4.64\,\text{A}$, and $I_{\text{rms}} = \boxed{3.28\,\text{A}}$, $V_{2,\text{max}} = 83.0\,\text{V}$, $V_{2,\text{rms}} = 58.7\,\text{V}$, $I_{R_L,\text{rms}} = \boxed{2.94\,\text{A}}$, and $I_{L,\text{rms}} = \boxed{1.46\,\text{A}}$

(*c*) The power delivered by the ac source equals the sum of the power dissipated in the two resistors. The fraction of the total power delivered by the source that is dissipated in load resistor is given by:

$$\frac{P_{R_L}}{P_{R_L} + P_R} = \left(1 + \frac{P_R}{P_{R_L}}\right)^{-1} = \left(1 + \frac{I_{\text{rms}}^2 R}{I_{R_L,\text{rms}}^2 R_L}\right)^{-1}$$

Substitute numerical values for f = 500 Hz to obtain:

$$\left.\frac{P_{R_L}}{P_{R_L} + P_R}\right|_{f=500\,\text{Hz}} = \left(1 + \frac{(6.23\,\text{A})^2(4.00\,\Omega)}{(2.80\,\text{A})^2(20.0\,\Omega)}\right)^{-1} = 0.502 = \boxed{50.2\%}$$

(*d*) Substitute numerical values for f = 2000 Hz to obtain:

$$\left.\frac{P_{R_L}}{P_{R_L}+P_R}\right|_{f=2000\,\text{Hz}}=\left(1+\frac{(3.28\,\text{A})^2(4.00\,\Omega)}{(2.94\,\text{A})^2(20.0\,\Omega)}\right)^{-1}=0.800=\boxed{80.0\%}$$

42 •• An ideal ac voltage source whose emf $\mathcal{E}_1$ is given by (20 V) cos($2\pi ft$) and an ideal battery whose emf $\mathcal{E}_2$ is 16 V are connected to a combination of two resistors and an inductor (Figure 29-34), where R_1 = 10 Ω, R_2 = 8.0 Ω, and L = 6.0 mH. Find the average power delivered to each resistor if (*a*) the driving frequency is 100 Hz, (*b*) the driving frequency is 200 Hz, and (*c*) the driving frequency is 800 Hz.

Picture the Problem We can treat the ac and dc components separately. For the dc component, L acts like a short circuit. Let $\mathcal{E}_{1,\text{ peak}}$ denote the peak value of the voltage supplied by the ac voltage source. We can use $P=\mathcal{E}_2^2/R$ to find the power dissipated in the resistors by the current from the ideal battery. We'll apply Kirchhoff's loop rule to the loop including L, R_1, and R_2 to derive an expression for the average power delivered to each resistor by the ac voltage source.

(*a*) The total power delivered to R_1 and R_2 is:

$$P_1=P_{1,\text{ dc}}+P_{1,\text{ ac}} \qquad (1)$$

and

$$P_2=P_{2,\text{ dc}}+P_{2,\text{ ac}} \qquad (2)$$

The dc power delivered to the resistors whose resistances are R_1 and R_2 is:

$$P_{1,\text{dc}}=\frac{\mathcal{E}_2^2}{R_1}\text{ and }P_{2,\text{dc}}=\frac{\mathcal{E}_2^2}{R_2}$$

Express the average ac power delivered to R_1:

$$P_{1,\text{ac}}=\frac{\mathcal{E}_{1,\text{ rms}}^2}{R_1}=\frac{\mathcal{E}_{1,\text{ peak}}^2}{2R_1}$$

Apply Kirchhoff's loop rule to a clockwise loop that includes R_1, L, and R_2:

$$R_1I_1-Z_2I_2=0$$

Solving for I_2 yields:

$$I_2=\frac{R_1}{Z_2}I_1=\frac{R_1}{Z_2}\frac{\mathcal{E}_{1,\text{ peak}}}{R_1}=\frac{\mathcal{E}_{1,\text{ peak}}}{Z_2}$$

Express the average ac power delivered to R_2:

$$P_{2,\,\text{ac}} = \tfrac{1}{2} I_{2,\,\text{rms}}^2 R_2 = \tfrac{1}{2}\left(\frac{\mathcal{E}_{1,\,\text{peak}}}{Z_2}\right)^2 R_2 = \frac{\mathcal{E}_{1,\,\text{peak}}^2 R_2}{2Z_2^2}$$

Substituting in equations (1) and (2) yields:

$$P_1 = \frac{\mathcal{E}_2^2}{R_1} + \frac{\mathcal{E}_{1,\,\text{peak}}^2}{2R_1}$$

and

$$P_2 = \frac{\mathcal{E}_2^2}{R_2} + \frac{\mathcal{E}_{1,\,\text{peak}}^2 R_2}{2Z_2^2}$$

Substitute numerical values and evaluate P_1:

$$P_1 = \frac{(16\,\text{V})^2}{10\,\Omega} + \frac{(20\,\text{V})^2}{2(10\,\Omega)} = \boxed{46\ \text{W}}$$

Substitute numerical values and evaluate P_2:

$$P_2 = \frac{(16\,\text{V})^2}{8.0\,\Omega} + \frac{(20\,\text{V})^2(8.0\,\Omega)}{2\left[(8.0\,\Omega)^2 + \left(2\pi\{100\,\text{s}^{-1}\}\{6.0\,\text{mH}\}\right)^2\right]} = \boxed{52\ \text{W}}$$

(*b*) Proceed as in (*a*) to evaluate P_1 and P_2 with $f = 200$ Hz:

$$P_1 = 25.6\,\text{W} + 20.0\,\text{W} = \boxed{46\,\text{W}}$$

$$P_2 = 32.0\,\text{W} + 13.2\,\text{W} = \boxed{45\,\text{W}}$$

(*c*) Proceed as in (*a*) to evaluate P_1 and P_2 with $f = 800$ Hz:

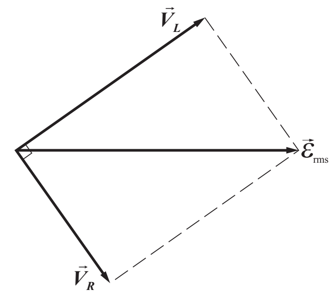
$$P_1 = 25.6\,\text{W} + 20.0\,\text{W} = \boxed{46\,\text{W}}$$

$$P_2 = 32.0\,\text{W} + 1.64\,\text{W} = \boxed{34\,\text{W}}$$

43 •• An ac circuit contains a resistor and an ideal inductor connected in series. The voltage rms drop across this series combination is 100-V and the rms voltage drop across the inductor alone is 80 V. What is the rms voltage drop across the resistor?

Picture the Problem We can use the phasor diagram for an *RL* circuit to find the voltage across the resistor.

The phasor diagram for the voltages in the circuit is shown to the right:



Use the Pythagorean theorem to express V_R:

$$V_R = \sqrt{\mathcal{E}_{\rm rms}^2 - V_L^2}$$

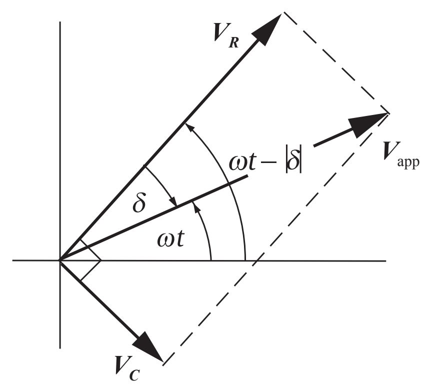
Substitute numerical values and evaluate V_R:

$$V_R = \sqrt{(100\,\text{V})^2 - (80\,\text{V})^2} = \boxed{60\,\text{V}}$$

Filters and Rectifiers

44 •• The circuit shown in Figure 29-35 is called an *RC high-pass filter* because it transmits input voltage signals that have high frequencies with greater amplitude than it transmits input voltage signals that have low frequencies. If the input voltage is given by $V_{\rm in} = V_{\rm in\,peak}\cos\omega t$, show that the output voltage is $V_{\rm out} = V_{\rm H}\cos(\omega t - \delta)$ where $V_{\rm H} = V_{\rm in\,peak}\big/\sqrt{1+(\omega RC)^{-2}}$. (Assume that the output is connected to a load that draws only an insignificant amount of current.) Show that this result justifies calling this circuit a high-pass filter.

Picture the Problem The phasor diagram for the *RC* high-pass filter is shown to the right. $\vec{V}_{\rm app}$ and $\vec{V}_R$ are the phasors for $V_{\rm in}$ and $V_{\rm out}$, respectively. Note that $\tan\delta = -X_C/R$. That δ is negative follows from the fact that $\vec{V}_{\rm app}$ lags $\vec{V}_R$ by $|\delta|$. The projection of $\vec{V}_{\rm app}$ onto the horizontal axis is $V_{\rm app} = V_{\rm in}$, and the projection of $\vec{V}_R$ onto the horizontal axis is $V_R = V_{\rm out}$.



Express $V_{\rm app}$:

$$V_{\rm app} = V_{\rm app,peak}\cos\omega t$$

where $V_{\rm app,peak} = V_{\rm peak} = I_{\rm peak}Z$

and $Z^2 = R^2 + X_C^2$ (1)

Because $\delta < 0$:

$$\omega t + |\delta| = \omega t - \delta$$

V_R is given by:

$$V_R = V_{R,\,\text{peak}} \cos(\omega t - \delta)$$
where $V_{R,\,\text{peak}} = V_{\text{H}} = I_{\text{peak}} R$

Solving equation (1) for Z and substituting for X_C yields:

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \qquad (2)$$

Because $V_{\text{out}} = V_R$:

$$\begin{aligned} V_{\text{out}} &= V_{R,\,\text{peak}} \cos(\omega t - \delta) \\ &= I_{\text{in peak}} R \cos(\omega t - \delta) \\ &= \frac{V_{\text{in peak}}}{Z} R \cos(\omega t - \delta) \end{aligned}$$

Using equation (2) to substitute for Z yields:

$$V_{\text{out}} = \frac{V_{\text{in peak}}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} R \cos(\omega t - \delta)$$

Simplify further to obtain:

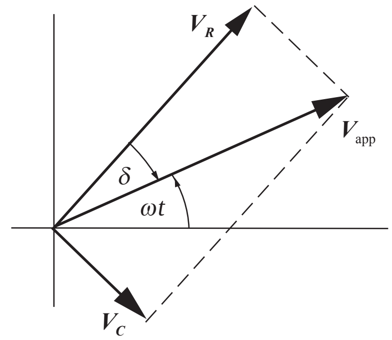
$$V_{\text{out}} = \frac{V_{\text{in peak}}}{\sqrt{1 + (\omega RC)^{-2}}} \cos(\omega t - \delta)$$
or
$$V_{\text{out}} = \boxed{V_{\text{H}} \cos(\omega t - \delta)}$$
where
$$V_{\text{H}} = \boxed{\frac{V_{\text{in peak}}}{\sqrt{1 + (\omega RC)^{-2}}}}$$

As $\omega \to \infty$:

$$V_{\text{H}} \to \frac{V_{\text{in peak}}}{\sqrt{1 + (0)^2}} = V_{\text{in peak}}$$ showing that the result is consistent with the high-pass name for this circuit.

45 •• (*a*) Find an expression for the phase constant δ in Problem 44 in terms of ω, R and C. (*b*) What is the value of δ in the limit that $\omega \to 0$? (*c*) What is the value of δ in the limit that $\omega \to \infty$? (*d*) Explain your answers to Parts (*b*) and (*c*).

Picture the Problem The phasor diagram for the *RC* high-pass filter is shown below. $\vec{V}_{\text{app}}$ and $\vec{V}_R$ are the phasors for V_{in} and V_{out}, respectively. The projection of $\vec{V}_{\text{app}}$ onto the horizontal axis is $V_{\text{app}} = V_{\text{in}}$, and the projection of $\vec{V}_{\text{R}}$ onto the horizontal axis is $V_R = V_{\text{out}}$.



(*a*) Because $\vec{V}_{\text{app}}$ lags $\vec{V}_{\text{R}}$ by $|\delta|$.

$$\tan\delta = -\frac{V_C}{V_R} = -\frac{IX_C}{IR} = -\frac{X_C}{R}$$

Use the definition of X_C to obtain:

$$\tan\delta = -\frac{\frac{1}{\omega C}}{R} = -\frac{1}{\omega RC}$$

Solving for δ yields:

$$\delta = \boxed{\tan^{-1}\left[-\frac{1}{\omega RC}\right]}$$

(*b*) As $\omega \to 0$:

$$\delta \to \boxed{-90°}$$

(*c*) As $\omega \to \infty$:

$$\delta \to \boxed{0}$$

(*d*) For very low driving frequencies, $X_C >> R$ and so $\vec{V}_C$ effectively lags $\vec{V}_{\text{in}}$ by 90°. For very high driving frequencies, $X_C << R$ and so $\vec{V}_R$ is effectively in phase with $\vec{V}_{\text{in}}$.

46 •• Assume that in Problem 44, $R = 20$ kΩ and $C = 15$ nF. (*a*) At what frequency is $V_{\text{H}} = \frac{1}{\sqrt{2}}V_{\text{in peak}}$? This particular frequency is known as the 3 dB frequency, or $f_{3\text{dB}}$ for the circuit. (*b*) Using a **spreadsheet** program, make a graph of $\log_{10}(V_{\text{H}})$ versus $\log_{10}(f)$, where f is the frequency. Make sure that the scale extends from at least 10% of the 3-dB frequency to ten times the 3-dB frequency. (*c*) Make a graph of δ versus $\log_{10}(f)$ for the same range of frequencies as in Part (*b*). What is the value of the phase constant when the frequency is equal to the 3-dB frequency?

Picture the Problem We can use the results obtained in Problems 44 and 45 to find $f_{3\text{ dB}}$ and to plot graphs of $\log(V_{\text{out}})$ versus $\log(f)$ and δ versus $\log(f)$.

(*a*) Use the result of Problem 44 to express the ratio $V_{out}/V_{in\ peak}$:

$$\frac{V_{out}}{V_{in\ peak}}=\frac{\dfrac{V_{in\ peak}}{\sqrt{1+(\omega RC)^{-2}}}}{V_{in\ peak}}=\frac{1}{\sqrt{1+(\omega RC)^{-2}}}$$

When $V_{out}=V_{in\ peak}/\sqrt{2}$:

$$\frac{1}{\sqrt{1+(\omega RC)^{-2}}}=\frac{1}{\sqrt{2}}$$

Square both sides of the equation and solve for ωRC to obtain:

$$\omega RC=1\Rightarrow\ \omega=\frac{1}{RC}\Rightarrow\ f_{3\,\mathrm{dB}}=\frac{1}{2\pi RC}$$

Substitute numerical values and evaluate $f_{3\,\mathrm{dB}}$:

$$f_{3\,\mathrm{dB}}=\frac{1}{2\pi(20\,\mathrm{k\Omega})(15\,\mathrm{nF})}=\boxed{0.53\,\mathrm{kHz}}$$

(*b*) From Problem 44 we have:

$$V_{out}=\frac{V_{in\ peak}}{\sqrt{1+(\omega RC)^{-2}}}$$

In Problem 45 it was shown that:

$$\delta=\tan^{-1}\left[-\frac{1}{\omega RC}\right]$$

Rewrite these expressions in terms of $f_{3\,\mathrm{dB}}$ to obtain:

$$V_{out}=\frac{V_{in\ peak}}{\sqrt{1+\left(\dfrac{1}{2\pi fRC}\right)^2}}=\frac{V_{peak}}{\sqrt{1+\left(\dfrac{f_{3\,\mathrm{dB}}}{f}\right)^2}}$$

and

$$\delta=\tan^{-1}\left[-\frac{1}{2\pi fRC}\right]=\tan^{-1}\left[-\frac{f_{3\,\mathrm{dB}}}{f}\right]$$

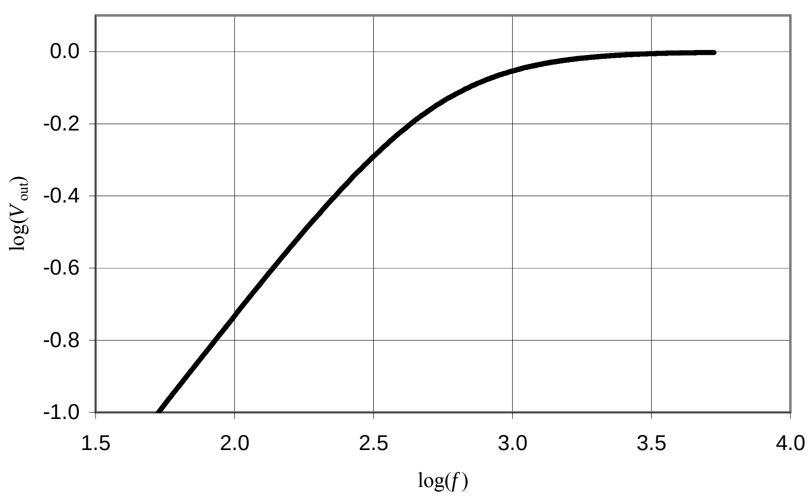
A spreadsheet program to generate the data for a graph of V_{out} versus f and δ versus f is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	2.00E+03	R
B2	1.50E−08	C
B3	1	$V_{in\ peak}$
B4	531	$f_{3\,\mathrm{dB}}$
A8	53	$0.1f_{3\,\mathrm{dB}}$
C8	$B$3/SQRT(1+(1($B$4/A8))^2)	$\dfrac{V_{in\ peak}}{\sqrt{1+\left(\dfrac{f_{3\,\mathrm{dB}}}{f}\right)^2}}$

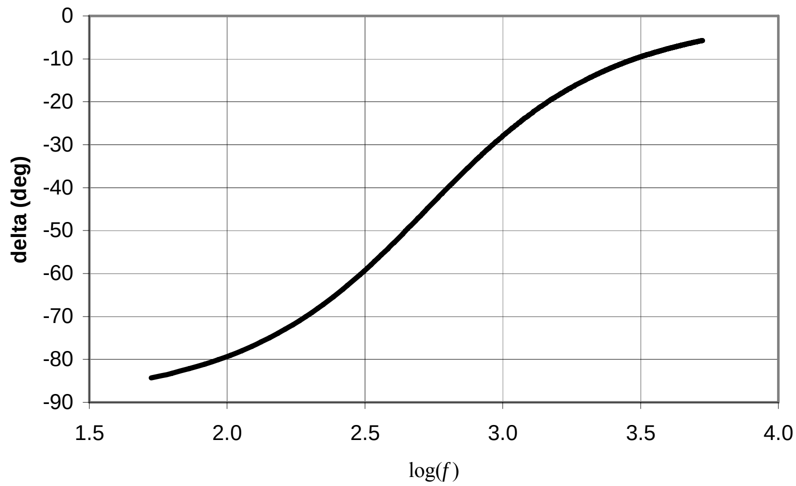
D8	LOG(C8)	$\log(V_{out})$
E8	ATAN(−$B$4/A8)	$\tan^{-1}\left[-\frac{f_{3\,dB}}{f}\right]$
F8	E8*180/PI()	δ in degrees

	A	B	C	D	E	F
1	R=	2.00E+04	ohms			
2	C=	1.50E−08	F			
3	$V_{in\ peak}$=	1	V			
4	$f_{3\ dB}$=	531	Hz			
5						
6						
7	f	log(f)	V_{out}	log(V_{out})	delta(rad)	delta(deg)
8	53	1.72	0.099	−1.003	−1.471	−84.3
9	63	1.80	0.118	−0.928	−1.453	−83.2
10	73	1.86	0.136	−0.865	−1.434	−82.2
11	83	1.92	0.155	−0.811	−1.416	−81.1
55	523	2.72	0.702	−0.154	−0.793	−45.4
56	533	2.73	0.709	−0.150	−0.783	−44.9
57	543	2.73	0.715	−0.146	−0.774	−44.3
531	5283	3.72	0.995	−0.002	−0.100	−5.7
532	5293	3.72	0.995	−0.002	−0.100	−5.7
533	5303	3.72	0.995	−0.002	−0.100	−5.7
534	5313	3.73	0.995	−0.002	−0.100	−5.7

The following graph of $\log(V_{out})$ versus $\log(f)$ was plotted for $V_{in\ peak} = 1$ V.



A graph of δ (in degrees) as a function of log(f) follows.



Referring to the spreadsheet program, we see that when $f = f_{3\text{ dB}}$, $\delta \approx \boxed{-44.9°.}$ This result is in good agreement with its calculated value of $-45.0°$.

47 •• [SSM] A slowly varying voltage signal $V(t)$ is applied to the input of the high-pass filter of Problem 44. Slowly varying means that during one time constant (equal to RC) there is no significant change in the voltage signal. Show that under these conditions the output voltage is proportional to the time derivative of $V(t)$. This situation is known as a *differentiation circuit.*

Picture the Problem We can use Kirchhoff's loop rule to obtain a differential equation relating the input, capacitor, and resistor voltages. Because the voltage drop across the resistor is small compared to the voltage drop across the capacitor, we can express the voltage drop across the capacitor in terms of the input voltage.

Apply Kirchhoff's loop rule to the input side of the filter to obtain:

$$V(t) - V_C - IR = 0$$

where V_C is the potential difference across the capacitor.

Substitute for $V(t)$ and I to obtain:

$$V_{\text{in peak}} \cos \omega t - V_c - R\frac{dQ}{dt} = 0$$

Because $Q = CV_C$:

$$\frac{dQ}{dt} = \frac{d}{dt}\left[CV_C\right] = C\frac{dV_C}{dt}$$

Substitute for dQ/dt to obtain:

$$V_{\text{peak}}\cos\omega t - V_C - RC\frac{dV_C}{dt} = 0$$

the differential equation describing the potential difference across the capacitor.

Because there is no significant change in the voltage signal during one time constant:

$$\frac{dV_C}{dt} = 0 \Rightarrow RC\frac{dV_C}{dt} = 0$$

Substituting for $RC\dfrac{dV_C}{dt}$ yields:

$$V_{\text{in peak}}\cos\omega t - V_C = 0$$

and

$$V_C = V_{\text{in peak}}\cos\omega t$$

Consequently, the potential difference across the resistor is given by:

$$V_R = RC\frac{dV_C}{dt} = \boxed{RC\frac{d}{dt}\left[V_{\text{in peak}}\cos\omega t\right]}$$

48 •• We can describe the output from the high-pass filter from Problem 44 using a decibel scale: $\beta = (20\text{ dB})\log_{10}\left(V_{\text{H}}/V_{\text{in peak}}\right)$, where β is the output in decibels. Show that for $V_{\text{H}} = \frac{1}{\sqrt{2}}V_{\text{in peak}}$, $\beta = -3.0$ dB. The frequency at which $V_{\text{H}} = \frac{1}{\sqrt{2}}V_{\text{in peak}}$ is known as $f_{3\text{dB}}$ (the 3-dB frequency). Show that for $f << f_{3\text{dB}}$, the output β drops by 6 dB if the frequency f is halved.

Picture the Problem We can use the expression for V_{H} from Problem 44 and the definition of β given in the problem to show that every time the frequency is halved, the output drops by 6 dB.

We're given that:

$$\beta = (20\text{ dB})\log_{10}\left(\frac{V_{\text{H}}}{V_{\text{in peak}}}\right)$$

For $V_{\text{H}} = \frac{1}{\sqrt{2}}V_{\text{in peak}}$:

$$\beta = (20\text{ dB})\log_{10}\left(\frac{\frac{1}{\sqrt{2}}V_{\text{in peak}}}{V_{\text{in peak}}}\right)$$

$$= (20\text{ dB})\log_{10}\left(\frac{1}{\sqrt{2}}\right) = \boxed{-3.0\text{ dB}}$$

From Problem 44:

$$V_H = \frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^{-2}}}$$

or

$$\frac{V_H}{V_{\text{in peak}}} = \frac{1}{\sqrt{1+(\omega RC)^{-2}}}$$

Express this ratio in terms of f and $f_{3\text{ dB}}$ and simplify to obtain:

$$\frac{V_H}{V_{\text{peak}}} = \frac{1}{\sqrt{1+\left(\frac{f_{3\text{ dB}}}{f}\right)^2}} = \frac{f}{\sqrt{f_{3\text{ dB}}^2\left(1+\frac{f^2}{f_{3\text{ dB}}^2}\right)}}$$

For $f << f_{3\text{dB}}$:

$$\frac{V_H}{V_{\text{peak}}} \approx \frac{f}{\sqrt{f_{3\text{ dB}}^2\left(1+\frac{f^2}{f_{3\text{ dB}}^2}\right)}} = \frac{f}{f_{3\text{ dB}}}$$

From the definition of β we have:

$$\beta = 20\log_{10}\left(\frac{V_H}{V_{\text{peak}}}\right)$$

Substitute for V_H/V_{peak} to obtain:

$$\beta = 20\log_{10}\left(\frac{f}{f_{3\text{ dB}}}\right)$$

Doubling the frequency yields:

$$\beta' = 20\log_{10}\left(\frac{\frac{1}{2}f}{f_{3\text{ dB}}}\right)$$

The change in decibel level is:

$$\begin{aligned}\Delta\beta &= \beta' - \beta \\ &= 20\log_{10}\left(\frac{\frac{1}{2}f}{f_{3\text{ dB}}}\right) - 20\log_{10}\left(\frac{f}{f_{3\text{ dB}}}\right) \\ &= 20\log_{10}\left(\tfrac{1}{2}\right) \approx \boxed{-6\,\text{dB}}\end{aligned}$$

49 •• [SSM] Show that the average power dissipated in the resistor of the high-pass filter of Problem 44 is given by $P_{\text{ave}} = \dfrac{V_{\text{in peak}}^2}{2R\left[1+(\omega RC)^{-2}\right]}$.

Picture the Problem We can express the instantaneous power dissipated in the resistor and then use the fact that the average value of the square of the cosine function over one cycle is ½ to establish the given result.

The instantaneous power $P(t)$ dissipated in the resistor is:

$$P(t)=\frac{V_{\text{out}}^2}{R}$$

The output voltage V_{out} is:

$$V_{\text{out}}=V_{\text{H}}\cos(\omega t-\delta)$$

From Problem 44:

$$V_{\text{H}}=\frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^{-2}}}$$

Substitute in the expression for $P(t)$ to obtain:

$$P(t)=\frac{V_{\text{H}}^2}{R}\cos^2(\omega t-\delta)$$
$$=\frac{V_{\text{in peak}}^2}{R\left[1+(\omega RC)^{-2}\right]}\cos^2(\omega t-\delta)$$

Because the average value of the square of the cosine function over one cycle is ½:

$$P_{\text{ave}}=\boxed{\frac{V_{\text{in peak}}^2}{2R\left[1+(\omega RC)^{-2}\right]}}$$

50 •• One application of the high-pass filter of Problem 44 is a noise filter for electronic circuits (a filter that blocks out low-frequency noise). Using a resistance value of 20 kΩ, find a value for the capacitance for the high-pass filter that attenuates a 60-Hz input voltage signal by a factor of 10. That is, so $V_{\text{H}}=\frac{1}{10}V_{\text{in peak}}$.

Picture the Problem We can solve the expression for V_{H} from Problem 44 for the required capacitance of the capacitor.

From Problem 44:

$$V_{\text{H}}=\frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^{-2}}}$$

We require that:

$$\frac{V_{\text{H}}}{V_{\text{in peak}}}=\frac{1}{\sqrt{1+(\omega RC)^{-2}}}=\frac{1}{10}$$

or

$$\sqrt{1+(\omega RC)^{-2}}=10$$

Solving for C yields:

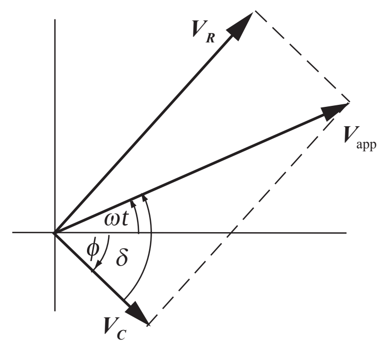
$$C=\frac{1}{\sqrt{99}\,\omega R}=\frac{1}{2\pi\sqrt{99}\,Rf}$$

Substitute numerical values and evaluate C:

$$C=\frac{1}{2\pi\sqrt{99}(20\,\text{k}\Omega)(60\,\text{Hz})}=\boxed{13\,\text{nF}}$$

51 •• [SSM] The circuit shown in Figure 29-36 is an example of a low-pass filter. (Assume that the output is connected to a load that draws only an insignificant amount of current.) (*a*) If the input voltage is given by $V_{in} = V_{in\,peak}\cos\omega t$, show that the output voltage is $V_{out} = V_L\cos(\omega t - \delta)$ where $V_L = V_{in\,peak}\Big/\sqrt{1+(\omega RC)^2}$. (*b*) Discuss the trend of the output voltage in the limiting cases $\omega \to 0$ and $\omega \to \infty$.

Picture the Problem In the phasor diagram for the *RC* low-pass filter, $\vec{V}_{app}$ and $\vec{V}_C$ are the phasors for V_{in} and V_{out}, respectively. The projection of $\vec{V}_{app}$ onto the horizontal axis is $V_{app} = V_{in}$, the projection of $\vec{V}_C$ onto the horizontal axis is $V_C = V_{out}$, $V_{peak} = \left|\vec{V}_{app}\right|$, and ϕ is the angle between $\vec{V}_C$ and the horizontal axis.



(*a*) Express V_{app}:

$$V_{app} = V_{in\,peak}\cos\omega t$$

where $V_{in\,peak} = I_{peak}Z$

and $Z^2 = R^2 + X_C^2$ (1)

$V_{out} = V_C$ is given by:

$$V_{out} = V_{C,peak}\cos\phi = I_{peak}X_C\cos\phi$$

If we define δ as shown in the phasor diagram, then:

$$V_{out} = I_{peak}X_C\cos(\omega t - \delta) = \frac{V_{in\,peak}}{Z}X_C\cos(\omega t - \delta)$$

Solving equation (1) for *Z* and substituting for X_C yields:

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \qquad (2)$$

Using equation (2) to substitute for *Z* and substituting for X_C yields:

$$V_{out} = \frac{V_{in\,peak}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\frac{1}{\omega C}\cos(\omega t - \delta)$$

Simplify further to obtain:

$$V_{out} = \frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^2}}\cos(\omega t - \delta)$$

or

$$V_{out} = \boxed{V_L \cos(\omega t - \delta)}$$

where

$$V_L = \boxed{\frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^2}}}$$

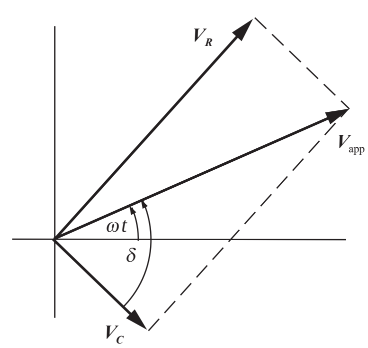
(*b*) Note that, as $\omega \to 0$, $V_L \to V_{peak}$. This makes sense physically in that, for low frequencies, X_C is large and, therefore, a larger peak input voltage will appear across it than appears across it for high frequencies.

Note further that, as $\omega \to \infty$, $V_L \to 0$. This makes sense physically in that, for high frequencies, X_C is small and, therefore, a smaller peak voltage will appear across it than appears across it for low frequencies.

Remarks: In Figures 29-19 and 29-20, δ is defined as the phase of the voltage drop across the combination relative to the voltage drop across the resistor.

52 •• (*a*) Find an expression for the phase angle δ for the low-pass filter of Problem 51 in terms of ω, R and C. (*b*) Find the value of δ in the limit that $\omega \to 0$ and in the limit that $\omega \to \infty$. Explain your answer.

Picture the Problem The phasor diagram for the *RC* low-pass filter is shown to the right. $\vec{V}_{app}$ and $\vec{V}_C$ are the phasors for V_{in} and V_{out}, respectively. The projection of $\vec{V}_{app}$ onto the horizontal axis is $V_{app} = V_{in}$ and the projection of $\vec{V}_C$ onto the horizontal axis is $V_C = V_{out}$. $V_{peak} = \left|\vec{V}_{app}\right|$.



(*a*) From the phasor diagram we have:

$$\tan\delta = \frac{V_R}{V_C} = \frac{I_{peak}R}{I_{peak}X_C} = \frac{R}{X_C}$$

Use the definition of X_C to obtain:

$$\tan\delta = \frac{R}{\frac{1}{\omega C}} = \omega RC$$

Solving for δ yields:

$$\delta = \boxed{\tan^{-1}(\omega RC)}$$

(b) As $\omega \to 0$, $\delta \to \boxed{0°}$. This behavior makes sense physically in that, at low frequencies, X_C is very large compared to R and, as a consequence, V_C is in phase with V_{in} .

As $\omega \to \infty$, $\delta \to \boxed{90°}$. This behavior makes sense physically in that, at high frequencies, X_C is very small compared to R and, as a consequence, V_C is out of phase with V_{in} .

Remarks: See the spreadsheet solution in the following problem for additional evidence that our answer for Part (*b*) is correct.

53 •• Using a **spreadsheet** program, make a graph of V_L versus input frequency f and a graph of phase angle δ versus input frequency for the low-pass filter of Problems 51 and 52. Use a resistance value of 10 kΩ and a capacitance value of 5.0 nF.

Picture the Problem We can use the expressions for V_L and δ derived in Problems 51 and 52 to plot the graphs of V_L versus f and δ versus f for the low-pass filter of Problem 51. We'll simplify the spreadsheet program by expressing both V_L and δ as functions of $f_{3\text{ dB}}$.

From Problems 51 and 52 we have:

$$V_L = \frac{V_{\text{in peak}}}{\sqrt{1+(\omega RC)^2}}$$

and

$$\delta = \tan^{-1}(\omega RC)$$

Rewrite each of these expressions in terms of f to obtain:

$$V_L = \frac{V_{\text{in peak}}}{\sqrt{1+(2\pi fRC)^2}}$$

and

$$\delta = \tan^{-1}(2\pi fRC)$$

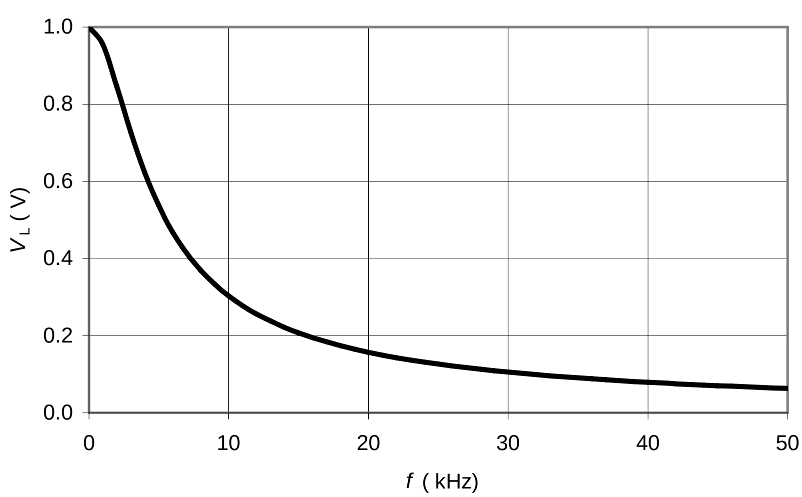
A spreadsheet program to generate the data for graphs of V_L versus f and δ versus f for the low-pass filter is shown below. Note that $V_{\text{in peak}}$ has been arbitrarily set equal to 1 V. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
B1	2.00E+03	R
B2	5.00E−09	C

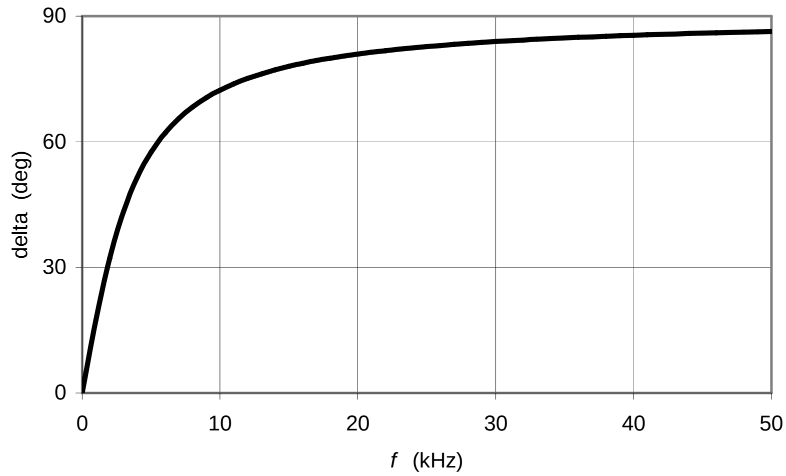
B3	1	$V_{\text{in peak}}$
B8	$B$3/SQRT(1+((2*PI()*A8*1000*$B$1*$B$2)^2))	$\dfrac{V_{\text{in peak}}}{\sqrt{1+(2\pi fRC)^2}}$
C8	ATAN(2*PI()*A8*1000*$B$1*$B$2)	$\tan^{-1}(2\pi fRC)$
D8	C8*180/PI()	δ in degrees

	A	B	C	D
1	R=	1.00E+04	ohms	
2	C=	5.00E−09	F	
3	$V_{\text{in peak}}$=	1	V	
4				
5				
6	f(kHz)	V_{out}	δ(rad)	δ(deg)
7	0	1.000	0.000	0.0
8	1	0.954	0.304	17.4
9	2	0.847	0.561	32.1
10	3	0.728	0.756	43.3
54	47	0.068	1.503	86.1
55	48	0.066	1.505	86.2
56	49	0.065	1.506	86.3
57	50	0.064	1.507	86.4

A graph of V_L as a function of f follows:



A graph of δ as a function of f follows:



54 ••• A rapidly varying voltage signal $V(t)$ is applied to the input of the low-pass filter of Problem 51. Rapidly varying means that during one time constant (equal to RC) there are significant changes in the voltage signal. Show that under these conditions the output voltage is proportional to the integral of $V(t)$ with respect to time. This situation is known as an *integration circuit*.

Picture the Problem We can use Kirchhoff's loop rule to obtain a differential equation relating the input, capacitor, and resistor voltages. We'll then assume a solution to this equation that is a linear combination of sine and cosine terms with coefficients that we can find by substitution in the differential equation. The solution to these simultaneous equations will yield the amplitude of the output voltage.

Apply Kirchhoff's loop rule to the input side of the filter to obtain:

$$V(t) - IR - V_C = 0$$

where V_C is the potential difference across the capacitor.

Substitute for $V(t)$ and I to obtain:

$$V_{\text{in peak}} \cos\omega t - R\frac{dQ}{dt} - V_C = 0$$

Because $Q = CV_C$:

$$\frac{dQ}{dt} = \frac{d}{dt}[CV_C] = C\frac{dV_C}{dt}$$

Substitute for dQ/dt to obtain:

$$V_{\text{in peak}} \cos\omega t - RC\frac{dV_C}{dt} - V_C = 0$$

the differential equation describing the potential difference across the capacitor.

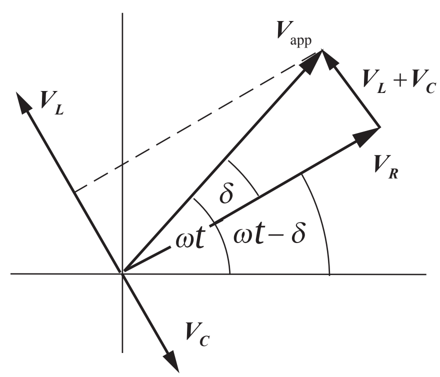
V_C is given by: $V_C = IX_c = \dfrac{I}{\omega C}$

The fact that $V(t)$ varies rapidly means that $\omega >> 1$ and so: $V_C \approx 0$
and
$V_{\text{peak}} \cos\omega t - RC\dfrac{dV_C}{dt} = 0$

Separating the variables in this differential equation and solving for V_C yields: $V_C = \boxed{\dfrac{1}{RC}\int V_{\text{peak}} \cos\omega t dt}$

55 ••• **[SSM]** The circuit shown in Figure 29-37 is a *trap filter*. (Assume that the output is connected to a load that draws only an insignificant amount of current.) (*a*) Show that the *trap filter* acts to reject signals in a band of frequencies centered at $\omega = 1/\sqrt{LC}$. (*b*) How does the width of the frequency band rejected depend on the resistance *R*?

Picture the Problem The phasor diagram for the *trap* filter is shown below. $\vec{V}_{\text{app}}$ and $\vec{V}_L + \vec{V}_C$ are the phasors for V_{in} and V_{out}, respectively. The projection of $\vec{V}_{\text{app}}$ onto the horizontal axis is $V_{\text{app}} = V_{\text{in}}$, and the projection of $\vec{V}_L + \vec{V}_C$ onto the horizontal axis is $V_L + V_C = V_{\text{out}}$. Requiring that the impedance of the trap be zero will yield the frequency at which the circuit rejects signals. Defining the bandwidth as $\Delta\omega = \left|\omega - \omega_{\text{trap}}\right|$ and requiring that $\left|Z_{\text{trap}}\right| = R$ will yield an expression for the bandwidth and reveal its dependence on *R*.



(*a*) Express V_{app}: $V_{\text{app}} = V_{\text{app, peak}} \cos\omega t$
where $V_{\text{app, peak}} = V_{\text{peak}} = I_{\text{peak}} Z$
and $Z^2 = R^2 + \left(X_L - X_C\right)^2$ (1)

V_{out} is given by:

$$V_{\text{out}} = V_{\text{out, peak}} \cos(\omega t - \delta)$$
where $V_{\text{out, peak}} = I_{\text{peak}} Z_{\text{trap}}$
and $Z_{\text{trap}} = X_L - X_C$

Solving equation (1) for Z yields:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad (2)$$

Because $V_{\text{out}} = V_{\text{L}} + V_C$:

$$\begin{aligned} V_{\text{out}} &= V_{\text{out, peak}} \cos(\omega t - \delta) \\ &= I_{\text{peak}} Z_{\text{trap}} \cos(\omega t - \delta) \\ &= \frac{V_{\text{peak}}}{Z} Z_{\text{trap}} \cos(\omega t - \delta) \end{aligned}$$

Using equation (2) to substitute for Z yields:

$$V_{\text{out}} = \frac{V_{\text{peak}}}{\sqrt{R^2 + Z_{\text{trap}}^2}} Z_{\text{trap}} \cos(\omega t - \delta)$$

Noting that $V_{\text{out}} = 0$ provided $Z_{\text{trap}} = 0$, set $Z_{\text{trap}} = 0$ to obtain:

$$Z_{\text{trap}} = X_L - X_C = 0$$

Substituting for X_L and X_C yields:

$$\omega L - \frac{1}{\omega C} = 0 \Rightarrow \omega = \boxed{\frac{1}{\sqrt{LC}}}$$

(*b*) Let the bandwidth $\Delta\omega$ be:

$$\Delta\omega = \left|\omega - \omega_{\text{trap}}\right| \qquad (3)$$

Let the frequency bandwidth be defined by the frequency at which $\left|Z_{\text{trap}}\right| = R$. Then:

$$\omega L - \frac{1}{\omega C} = R \Rightarrow \omega^2 LC - 1 = \omega RC$$

Because $\omega_{\text{trap}} = \dfrac{1}{\sqrt{LC}}$:

$$\left(\frac{\omega}{\omega_{\text{trap}}}\right)^2 - 1 = \omega RC$$

For $\omega \approx \omega_{\text{trap}}$:

$$\left(\frac{\omega^2 - \omega_{\text{trap}}^2}{\omega_{\text{trap}}}\right) \approx \omega_{\text{trap}} RC$$

Solve for $\omega^2 - \omega_{\text{trap}}^2$:

$$\omega^2 - \omega_{\text{trap}}^2 = (\omega - \omega_{\text{trap}})(\omega + \omega_{\text{trap}})$$

Because $\omega \approx \omega_{\text{trap}}$, $\omega + \omega_{\text{trap}} \approx 2\omega_{\text{trap}}$:

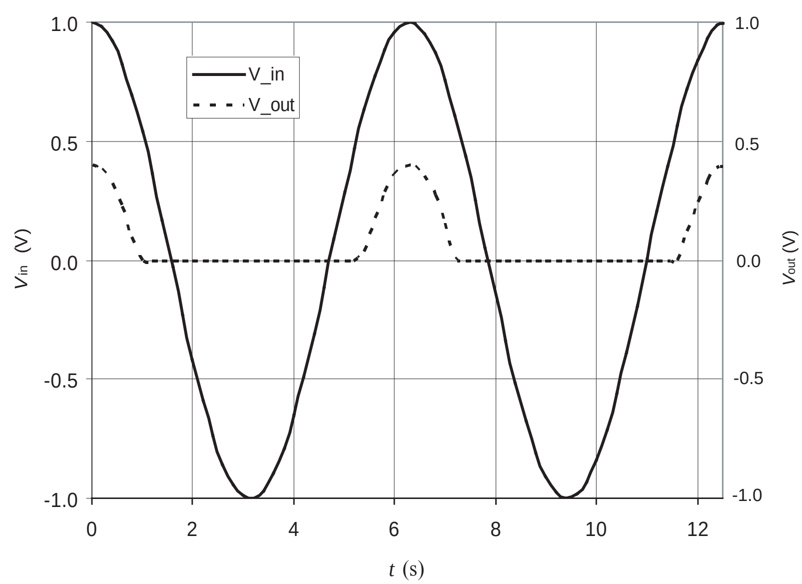
$$\omega^2 - \omega_{\text{trap}}^2 \approx 2\omega_{\text{trap}}(\omega - \omega_{\text{trap}})$$

Substitute in equation (3) to obtain:

$$\Delta\omega = \left|\omega - \omega_{\text{trap}}\right| = \frac{RC\omega_{\text{trap}}^2}{2} = \boxed{\frac{R}{2L}}$$

56 ••• A half-wave rectifier for transforming an ac voltage into a dc voltage is shown in Figure 29-38. The diode in the figure can be thought of as a one-way valve for current. It allows current to pass in the forward direction (the direction of the arrowhead) only when V_{in} is at a higher electric potential than V_{out} by 0.60 V (that is, whenever $V_{\text{in}} - V_{\text{out}} \geq +0.60\text{ V}$). The resistance of the diode is effectively infinite when $V_{\text{in}} - V_{\text{out}}$ is less than +0.60 V. Plot two cycles of both input and output voltages as a function of time, on the same graph, assuming the input voltage is given by $V_{\text{in}} = V_{\text{in peak}} \cos \omega t$.

Picture the Problem For voltages greater than 0.60 V, the output voltage will mirror the input voltage minus a 0.60 V drop. But when the voltage swings below 0.60 V, the output voltage will be 0. A spreadsheet program was used to plot the following graph. The angular frequency and the peak voltage were both arbitrarily set equal to one.



57 ••• The output of the rectifier of Problem 56, can be further filtered by putting its output through a low-pass filter as shown in Figure 29-39*a*. The resulting output is a dc voltage with a small ac component (ripple) shown in Figure 29-39*b*. If the input frequency is 60 Hz and the load resistance is 1.00 kΩ, find the value for the capacitance so that the output voltage varies by less than 50 percent of the mean value over one cycle.

Picture the Problem We can use the decay of the potential difference across the capacitor to relate the time constant for the *RC* circuit to the frequency of the input signal. Expanding the exponential factor in the expression for V_C will allow

us to find the approximate value for C that will limit the variation in the output voltage by less than 50 percent (or any other percentage).

The voltage across the capacitor is given by:

$$V_C = V_{\text{in}} e^{-t/RC}$$

Expand the exponential factor to obtain:

$$e^{-t/RC} \approx 1 - \frac{1}{RC}t$$

For a decay of less than 50 percent:

$$1 - \frac{1}{RC}t \leq 0.5 \Rightarrow C \leq \frac{2}{R}t$$

Because the voltage goes positive every cycle, $t = 1/60$ s and:

$$C \leq \frac{2}{1.00\,\text{k}\Omega}\left(\frac{1}{60}\text{s}\right) = \boxed{33\,\mu\text{F}}$$

Driven *LC* Circuits

58 •• The generator voltage in Figure 29-40, is given by $\mathcal{E} = (100\text{ V})\cos(2\pi ft)$. (*a*) For each branch, what is the peak current and what is the phase of the current relative to the phase of the generator voltage? (*b*) At the resonance frequency there is no current in the generator. What is the angular frequency at resonance? (*c*) At the resonance frequency, find the current in the inductor and what is the current in the capacitor. Express your results as functions of time. (*d*) Draw a phasor diagram showing the phasors for the applied voltage, the generator current, the capacitor current, and the inductor current for the case where frequency is higher than the resonance frequency.

Picture the Problem We know that the current leads the voltage across and capacitor and lags the voltage across an inductor. We can use $I_{L,\,\text{peak}} = \mathcal{E}_{\text{peak}}/X_L$ and $I_{C,\,\text{peak}} = \mathcal{E}_{\text{peak}}/X_C$ to find the amplitudes of these currents. The current in the generator will vanish under resonance conditions, i.e., when $|I_L| = |I_C|$. To find the currents in the inductor and capacitor at resonance, we can use the common potential difference across them and their reactances together with our knowledge of the phase relationships mentioned above.

(*a*) Express the amplitudes of the currents through the inductor and the capacitor:

$$I_{L,\,\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{X_L} = \frac{\mathcal{E}_{\text{peak}}}{2\pi fL}$$

and

$$I_{C,\,\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{X_C} = \frac{\mathcal{E}_{\text{peak}}}{\frac{1}{2\pi fC}} = 2\pi fC\mathcal{E}_{\text{peak}}$$

Substitute numerical values to obtain:

$$I_{L,\,\text{peak}} = \frac{100\,\text{V}}{(4.00\,\text{H})2\pi f} = \boxed{\frac{25.0\,\text{V/H}}{2\pi f},\ \text{lagging } \mathcal{E} \text{ by } 90°}$$

and

$$I_{C,\,\text{peak}} = (25.0\,\mu\text{F})(100\,\text{V})\omega = \boxed{(2.50\,\text{mV}\cdot\text{F})2\pi f\text{, leading } \mathcal{E} \text{ by } 90°}$$

(*b*) Express the condition that $I = 0$:

$$|I_L| = |I_C| \text{ or } \frac{\mathcal{E}}{\omega L} = \frac{\mathcal{E}}{\frac{1}{\omega C}} = \omega C\mathcal{E}$$

Solve for ω to obtain:

$$\omega = \frac{1}{\sqrt{LC}}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{1}{\sqrt{(4.00\,\text{H})(25.0\,\mu\text{F})}} = \boxed{100\,\text{rad/s}}$$

(*c*) Express the current in the inductor at $\omega = \omega_0$:

$$I_L = \left(\frac{25.0\,\text{V/H}}{100\,\text{s}^{-1}}\right)\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$= \boxed{(250\,\text{mA})\cos\left(\omega t - \frac{\pi}{2}\right)}$$

where $\omega = 100$ rad/s.

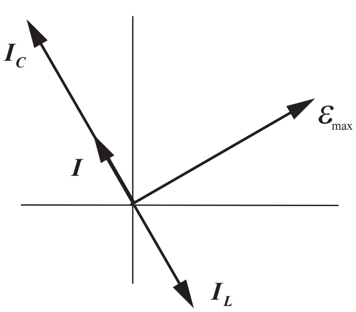
Express the current in the capacitor at $\omega = \omega_0$:

$$I_C = (2.50\,\text{mV}\cdot\text{F})(100\,\text{s}^{-1})\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$= \boxed{-(250\,\text{mA})\cos\left(\omega t + \frac{\pi}{2}\right)}$$

where $\omega = 100$ rad/s.

(*d*) The phasor diagram for the case where the inductive reactance is larger than the capacitive reactance is shown to the right.



59 •• A circuit consist of an ideal ac generator, a capacitor and an ideal inductor, all connected in series. The charge on the capacitor is given by $Q = (15\ \mu\text{C})\cos(\omega t + \frac{\pi}{4})$, where $\omega = 1250$ rad/s. (*a*) Find the current in the circuit as a function of time. (*b*) Find the capacitance if the inductance is 28 mH. (*c*) Write expressions for the electrical energy U_e, the magnetic energy U_m, and the total energy U as functions of time.

Picture the Problem We can differentiate Q with respect to time to find I as a function of time. In (*b*) we can find C by using $\omega = 1/\sqrt{LC}$. The energy stored in the magnetic field of the inductor is given by $U_m = \frac{1}{2}LI^2$ and the energy stored in the electric field of the capacitor by $U_e = \frac{1}{2}\frac{Q^2}{C}$.

(*a*) Differentiate the charge with respect to time to obtain the current:

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}\left[(15\ \mu\text{C})\cos\left(\omega t + \frac{\pi}{4}\right)\right] = -(15\ \mu\text{C})\left(1250\,\text{s}^{-1}\right)\sin\left(\omega t + \frac{\pi}{4}\right)$$

$$= -(18.75\,\text{mA})\sin\left(\omega t + \frac{\pi}{4}\right) = \boxed{-(19\,\text{mA})\sin\left(\omega t + \frac{\pi}{4}\right)}$$

where $\omega = 1250$ rad/s

(*b*) Relate C to L and ω:

$$\omega = \frac{1}{\sqrt{LC}}$$

Solve for C to obtain:

$$C = \frac{1}{\omega^2 L}$$

Substitute numerical values and evaluate C:

$$C = \frac{1}{\left(1250\,\text{s}^{-1}\right)^2(28\,\text{mH})} = 22.86\ \mu\text{F}$$

$$= \boxed{23\ \mu\text{F}}$$

(*c*) Express and evaluate the magnetic energy $U_m(t)$:

$$U_m(t) = \tfrac{1}{2}LI^2 = \tfrac{1}{2}(28\,\text{mH})(18.75\,\text{mA})^2\sin^2\left(\omega t + \frac{\pi}{4}\right) = \boxed{(4.9\ \mu\text{J})\sin^2\left(\omega t + \frac{\pi}{4}\right)}$$

where $\omega = 1250$ rad/s

Use $U_e = \frac{1}{2}\frac{Q^2}{C}$ to find the electrical energy stored in the capacitor as a function of time:

$$U_e(t) = \frac{1}{2}\frac{(15\,\mu\text{F})^2}{22.86\,\mu\text{F}}\cos^2\left(\omega t + \frac{\pi}{4}\right)$$
$$= (4.92\,\mu\text{J})\cos^2\left(\omega t + \frac{\pi}{4}\right)$$
$$= \boxed{(4.9\,\mu\text{J})\cos^2\left(\omega t + \frac{\pi}{4}\right)}$$

where $\omega = 1250$ rad/s

The total energy stored in the electric and magnetic fields is the sum of $U_m(t)$ and $U_e(t)$:

$$U = (4.92\,\mu\text{J})\sin^2\left(\omega t + \frac{\pi}{4}\right) + (4.92\,\mu\text{J})\cos^2\left(\omega t + \frac{\pi}{4}\right) = \boxed{4.9\,\mu\text{J}}$$

where $\omega = 1250$ rad/s.

60 ••• One method for determining the compressibility of a dielectric material uses a driven *LC* circuit that has a parallel-plate capacitor. The dielectric is inserted between the plates and the change in resonance frequency is determined as the capacitor plates are subjected to a compressive stress. In one such arrangement, the resonance frequency is 120 MHz when a dielectric of thickness 0.100 cm and dielectric constant $\kappa = 6.80$ is placed between the plates. Under a compressive stress of 800 atm, the resonance frequency decreases to 116 MHz. Find the Young's modulus of the dielectric material. (Assume that the dielectric constant does not change with pressure.)

Picture the Problem We can use the definition of the capacitance of a dielectric-filled capacitor and the expression for the resonance frequency of an *LC* circuit to derive an expression for the fractional change in the thickness of the dielectric in terms of the resonance frequency and the frequency of the circuit when the dielectric is under compression. We can then use this expression for $\Delta t/t$ to calculate the value of Young's modulus for the dielectric material.

Use its definition to express Young's modulus of the dielectric material:

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\Delta P}{\Delta t/t} \qquad (1)$$

Letting t be the initial thickness of the dielectric, express the initial capacitance of the capacitor:

$$C_0 = \frac{\kappa \in_0 A}{t}$$

Express the capacitance of the capacitor when it is under compression:

$$C_c = \frac{\kappa \in_0 A}{t - \Delta t}$$

Express the resonance frequency of the capacitor before the dielectric is compressed:

$$\omega_0 = \frac{1}{\sqrt{C_0 L}} = \frac{1}{\sqrt{\frac{\kappa \in_0 AL}{t}}}$$

When the dielectric is compressed:

$$\omega_c = \frac{1}{\sqrt{C_c L}} = \frac{1}{\sqrt{\frac{\kappa \in_0 AL}{t - \Delta t}}}$$

Express the ratio of ω_c to ω_0 and simplify to obtain:

$$\frac{\omega_c}{\omega_0} = \frac{\sqrt{\frac{\kappa \in_0 AL}{t}}}{\sqrt{\frac{\kappa \in_0 AL}{t - \Delta t}}} = \sqrt{1 - \frac{\Delta t}{t}}$$

Expand the radical binomially to obtain:

$$\frac{\omega_c}{\omega_0} = \left(1 - \frac{\Delta t}{t}\right)^{1/2} \approx 1 - \frac{\Delta t}{2t}$$

provided $\Delta t << t$.

Solve for $\Delta t/t$:

$$\frac{\Delta t}{t} = 2\left(1 - \frac{\omega_c}{\omega_0}\right)$$

Substitute in equation (1) to obtain:

$$Y = \frac{\Delta P}{2\left(1 - \frac{\omega_c}{\omega_0}\right)}$$

Substitute numerical values and evaluate Y:

$$Y = \frac{(800\,\text{atm})(101.325\,\text{kPa/atm})}{2\left(1 - \frac{116\,\text{MHz}}{120\,\text{MHz}}\right)} = \boxed{1.22 \times 10^9\ \text{N/m}^2}$$

61 ••• Figure 29-41 shows an inductor in series with a parallel plate capacitor. The capacitor has a width w of 20 cm and a gap of 2.0 mm. A dielectric that has a dielectric constant of 4.8 can be slid in and out of the gap. The inductor has an inductance of 2.0 mH. When half the dielectric is between the capacitor plates (when $x = \frac{1}{2}w$), the resonant frequency of this combination is 90 MHz.
(a) What is the capacitance of the capacitor without the dielectric?
(b) Find the resonance frequency as a function of x for $0 \le x \le w$.

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel. Let C_1 be the capacitance of the dielectric-filled capacitor and C_2 be the capacitance of the air-filled capacitor. We'll derive expressions for the capacitances of the parallel capacitors and add these expressions to obtain $C(x)$. We can then use the given resonance frequency when $x = w/2$ and the given value for L to evaluate C_0. In Part (b) we can use our result for $C(x)$ and the relationship between f, L, and $C(x)$ at resonance to express $f(x)$.

(a) Express the equivalent capacitance of the two capacitors in parallel:

$$C(x) = C_1 + C_2 = \frac{\kappa \in_0 A_1}{d} + \frac{\in_0 A_2}{d} \quad (1)$$

Express A_2 in terms of the total area of a capacitor plate A, w, and the distance x:

$$\frac{A_2}{A} = \frac{x}{w} \Rightarrow A_2 = A\frac{x}{w}$$

Express A_1 in terms of A and A_2:

$$A_1 = A - A_2 = A\left(1 - \frac{x}{w}\right)$$

Substitute in equation (1) and simplify to obtain:

$$C(x) = \frac{\kappa \in_0 A}{d}\left(1 - \frac{x}{w}\right) + \frac{\in_0 A}{d}\frac{x}{w}$$

$$= \frac{\in_0 A}{d}\left[\kappa\left(1 - \frac{x}{w}\right) + \frac{x}{w}\right]$$

$$= \kappa C_0\left[1 - \frac{\kappa - 1}{\kappa w}x\right]$$

where $C_0 = \dfrac{\in_0 A}{d}$

Find $C(w/2)$:

$$C\left(\frac{w}{2}\right) = \kappa C_0\left[1 - \frac{\kappa - 1}{\kappa w}\frac{w}{2}\right] = \kappa C_0\left[1 - \frac{\kappa - 1}{2\kappa}\right] = C_0\frac{\kappa + 1}{2}$$

Express the resonance frequency of the circuit in terms of L and $C(x)$:

$$f(x) = \frac{1}{2\pi\sqrt{LC(x)}} \qquad (2)$$

Evaluate $f(w/2)$:

$$f\left(\frac{w}{2}\right) = \frac{1}{2\pi\sqrt{LC_0\frac{\kappa + 1}{2}}} = \frac{1}{2\pi}\sqrt{\frac{2}{(\kappa + 1)LC_0}}$$

Solve for C_0 to obtain:

$$C_0 = \frac{1}{2\pi^2 f^2\left(\frac{w}{2}\right)L(\kappa + 1)}$$

Substitute numerical values and evaluate C_0:

$$C_0 = \frac{1}{2\pi^2(90\text{ MHz})^2\left(\frac{20\text{ cm}}{2}\right)(2.0\text{ mH})(4.8 + 1)} = \boxed{5.4\text{ fF}}$$

(*b*) Substitute for $C(x)$ in equation (2) to obtain:

$$f(x) = \frac{1}{2\pi\sqrt{L\kappa C_0\left[1 - \frac{\kappa - 1}{\kappa w}x\right]}}$$

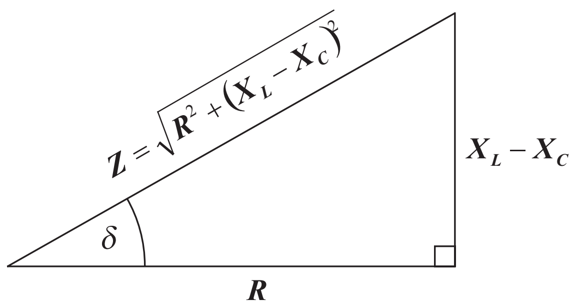
Substitute numerical values and evaluate $f(x)$:

$$f(x) = \frac{1}{2\pi\sqrt{(2.0\text{ mH})(4.8)(5.39\times10^{-16}\text{ F})\left[1 - \frac{4.8 - 1}{4.8(0.20\text{ m})}x\right]}} = \boxed{\frac{70\text{ MHz}}{\sqrt{1 - (4.0\text{ m}^{-1})x}}}$$

Driven *RLC* Circuits

62 • A circuit consists of an ideal ac generator, a 20-μF capacitor and an 80-Ω resistor, all connected in series. The output of the generator has a peak emf of 20-V, and the armature of the generator rotates at 400 rad/s. Find (*a*) the power factor, (*b*) the rms current, and (*c*) the average power supplied by the generator.

Picture the Problem The diagram shows the relationship between δ, X_L, X_C, and R. We can use this reference triangle to express the power factor for the given circuit. In (*b*) we can find the rms current from the rms potential difference and the impedance of the circuit. We can find the average power delivered by the source from the rms current and the resistance of the resistor.



(*a*) The power factor is defined to be:

$$\cos\delta = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$$

With no inductance in the circuit:

$$X_L = 0$$

and

$$\cos\delta = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{R}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}$$

Substitute numerical values and evaluate $\cos\delta$:

$$\cos\delta = \frac{80\,\Omega}{\sqrt{(80\,\Omega)^2 + \dfrac{1}{(400\,\text{s}^{-1})^2 (20\,\mu\text{F})^2}}} = \boxed{0.54}$$

(*b*) Express the rms current in the circuit:

$$I_{\text{rms}} = \frac{\mathcal{E}_{\text{rms}}}{Z} = \frac{\dfrac{\mathcal{E}_{\max}}{\sqrt{2}}}{\sqrt{R^2 + X_C^2}} = \frac{\mathcal{E}_{\max}}{\sqrt{2}\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}$$

Substitute numerical values and evaluate I_{rms}:

$$I_{\text{rms}} = \frac{20\,\text{V}}{\sqrt{2}\sqrt{(80\,\Omega)^2 + \dfrac{1}{(400\,\text{s}^{-1})^2(20\,\mu\text{F})^2}}} = 95.3\,\text{mA} = \boxed{95\,\text{mA}}$$

(*c*) The average power delivered by the generator is given by:

$$P_{\text{av}} = I_{\text{rms}}^2 R$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = (95.3\,\text{mA})^2(80\,\Omega) = \boxed{0.73\,\text{W}}$$

63 •• **[SSM]** Show that the expression $P_{\text{av}} = R\mathcal{E}_{\text{rms}}^2 / Z^2$ gives the correct result for a circuit containing only an ideal ac generator and (*a*) a resistor, (*b*) a capacitor, and (*c*) an inductor. In the expression $P_{\text{av}} = R\mathcal{E}_{\text{rms}}^2 / Z^2$, P_{av} is the average power supplied by the generator, $\mathcal{E}_{\text{rms}}$ is the root-mean-square of the emf of the generator, *R* is the resistance, *C* is the capacitance and *L* is the inductance. (In Part (*a*), $C = L = 0$, in Part (*b*), $R = L = 0$ and in Part (*c*), $R = C = 0$.

Picture the Problem The impedance of an ac circuit is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. We can evaluate the given expression for P_{av} first for $X_L = X_C = 0$ and then for $R = 0$.

(*a*) For $X = 0$, $Z = R$ and:

$$P_{\text{av}} = \frac{R\mathcal{E}_{\text{rms}}^2}{Z^2} = \frac{R\mathcal{E}_{\text{rms}}^2}{R^2} = \boxed{\frac{\mathcal{E}_{\text{rms}}^2}{R}}$$

(*b*) and (*c*) If $R = 0$, then:

$$P_{\text{av}} = \frac{R\mathcal{E}_{\text{rms}}^2}{Z^2} = \frac{(0)\mathcal{E}_{\text{rms}}^2}{(X_L - X_C)^2} = \boxed{0}$$

Remarks: Recall that there is no energy dissipation in an ideal inductor or capacitor.

64 •• A series *RLC* circuit that has an inductance of 10 mH, a capacitance of 2.0 μF, and a resistance of 5.0 Ω is driven by an ideal ac voltage source that has a peak emf of 100 V. Find (*a*) the resonant frequency and (*b*) the root-mean-square current at resonance. When the frequency is 8000 rad/s, find (*c*) the capacitive and inductive reactances, (*d*) the impedance, (*e*) the root-mean-square current, and (*f*) the phase angle.

Picture the Problem We can use $\omega_0 = 1/\sqrt{LC}$ to find the resonant frequency of the circuit, $I_{\text{rms}} = \mathcal{E}_{\text{rms}}/R$ to find the rms current at resonance, the definitions of X_C and X_L to find these reactances at ω = 8000 rad/s, the definitions of Z and I_{rms} to find the impedance and rms current at ω = 8000 rad/s, and the definition of the phase angle to find δ.

(*a*) Express the resonant frequency ω_0 in terms of L and C:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Substitute numerical values and evaluate ω_0:

$$\omega_0 = \frac{1}{\sqrt{(10\,\text{mH})(2.0\,\mu\text{F})}} = \boxed{7.1\times10^3\ \text{rad/s}}$$

(*b*) Relate the rms current at resonance to $\mathcal{E}_{\text{rms}}$ and the impedance of the circuit at resonance:

$$I_{\text{rms}} = \frac{\mathcal{E}_{\text{rms}}}{R} = \frac{\mathcal{E}_{\max}}{\sqrt{2}R} = \frac{100\,\text{V}}{\sqrt{2}(5.0\,\Omega)} = \boxed{14\,\text{A}}$$

(*c*) Express and evaluate X_C and X_L at ω = 8000 rad/s:

$$X_C = \frac{1}{\omega C} = \frac{1}{(8000\,\text{s}^{-1})(2.0\,\mu\text{F})} = 62.50\,\Omega = \boxed{63\,\Omega}$$

and

$$X_L = \omega L = (8000\,\text{s}^{-1})(10\,\text{mH}) = \boxed{80\,\Omega}$$

(*d*) Express the impedance in terms of the reactances, substitute the results from (*c*), and evaluate Z:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(5.0\,\Omega)^2 + (80\,\Omega - 62.5\,\Omega)^2} = 18.2\,\Omega = \boxed{18\,\Omega}$$

(*e*) Relate the rms current at ω = 8000 rad/s to $\mathcal{E}_{\text{rms}}$ and the impedance of the circuit at this frequency:

$$I_{\text{rms}} = \frac{\mathcal{E}_{\text{rms}}}{Z} = \frac{\mathcal{E}_{\max}}{\sqrt{2}Z} = \frac{100\,\text{V}}{\sqrt{2}(18.2\,\Omega)} = \boxed{3.9\,\text{A}}$$

(*f*) δ is given by:

$$\delta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Substitute numerical values and evaluate δ:

$$\delta = \tan^{-1}\left(\frac{80\,\Omega - 62.5\,\Omega}{5.0\,\Omega}\right) = \boxed{74°}$$

65 •• **[SSM]** Find (*a*) the *Q* factor and (*b*) the resonance width (in hertz) for the circuit in Problem 64. (*c*) What is the power factor when ω = 8000 rad/s?

Picture the Problem The *Q* factor of the circuit is given by $Q = \omega_0 L/R$, the resonance width by $\Delta f = f_0/Q = \omega_0/2\pi Q$, and the power factor by $\cos\delta = R/Z$. Because *Z* is frequency dependent, we'll need to find X_C and X_L at ω = 8000 rad/s in order to evaluate $\cos\delta$.

Using their definitions, express the *Q* factor and the resonance width of the circuit:

$$Q = \frac{\omega_0 L}{R} \qquad (1)$$

and

$$\Delta f = \frac{f_0}{Q} = \frac{\omega_0}{2\pi Q} \qquad (2)$$

(*a*) Express the resonance frequency for the circuit:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Substituting for ω_0 in equation (1) yields:

$$Q = \frac{L}{\sqrt{LC}\,R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Substitute numerical values and evaluate *Q*:

$$Q = \frac{1}{5.0\,\Omega}\sqrt{\frac{10\text{ mH}}{2.0\ \mu\text{F}}} = 14.1 = \boxed{14}$$

(*b*) Substitute numerical values in equation (2) and evaluate Δf:

$$\Delta f = \frac{7.07\times10^3\text{ rad/s}}{2\pi(14.1)} = \boxed{80\,\text{Hz}}$$

(*c*) The power factor of the circuit is given by:

$$\cos\delta = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

Substitute numerical values and evaluate $\cos\delta$:

$$\cos\delta = \frac{5.0\,\Omega}{\sqrt{(5.0\,\Omega)^2 + \left((8000\,\text{s}^{-1})(10\,\text{mH}) - \frac{1}{(8000\,\text{s}^{-1})(2.0\,\mu\text{F})}\right)^2}} = \boxed{0.27}$$

66 •• FM radio stations typically operate at frequencies separated by 0.20 MHz. Thus, when your radio is tuned to a station operating at a frequency of 100.1 MHz, the resonance width of the receiver circuit should be much smaller than 0.20 MHz, so that you do not receive a signal from stations operating at adjacent frequencies. Assume your receiving circuit has a resonance width of 0.050 MHz. When tuned in to this particular station, what is the *Q* factor of your circuit?

Picture the Problem We can use its definition, $Q = f_0/\Delta f$ to find the *Q* factor for the circuit.

The *Q* factor for the circuit is given by:

$$Q = \frac{f_0}{\Delta f}$$

Substitute numerical values and evaluate *Q*:

$$Q = \frac{100.1\,\text{MHz}}{0.050\,\text{MHz}} \approx \boxed{2.0\times10^3}$$

67 •• A coil is connected to a 60-Hz ac generator with a peak emf equal to 100 V. At this frequency, the coil has an impedance of 10 Ω and a reactance of 8.0 Ω. (*a*) What is the peak current in the coil? (*b*) What is the phase angle between the current and the applied voltage? (*c*) A capacitor is put in series with the coil and the generator. What capacitance is required so that the current is in phase with the generator emf? (*d*) What is the peak voltage measured across this capacitor?

Picture the Problem We can use $I_{\text{peak}} = \mathcal{E}_{\text{peak}}/Z$ to find the current in the coil and the definition of the phase angle to evaluate δ. We can equate X_L and X_C to find the capacitance required so that the current and the voltage are in phase. Finally, we can find the voltage measured across the capacitor by using $V_C = IX_C$.

(*a*) Express the current in the coil in terms of the potential difference across it and its impedance:

$$I_{\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{Z}$$

Substitute numerical values and evaluate I_{peak}:

$$I_{\text{peak}} = \frac{100\,\text{V}}{10\,\Omega} = \boxed{10\,\text{A}}$$

(*b*) The phase angle δ is given by:

$$\delta = \cos^{-1}\left(\frac{R}{Z}\right) = \sin^{-1}\left(\frac{X_L}{Z}\right)$$

Substitute numerical values and evaluate δ:

$$\delta = \sin^{-1}\left(\frac{8.0\,\Omega}{10\,\Omega}\right) = \boxed{53°}$$

(*c*) Express the condition on the reactances that must be satisfied if the current and voltage are to be in phase:

$$X_L = X_C = \frac{1}{\omega C} \Rightarrow C = \frac{1}{\omega X_L} = \frac{1}{2\pi f X_L}$$

Substitute numerical values and evaluate C:

$$C = \frac{1}{2\pi\left(60\,\text{s}^{-1}\right)\left(8.0\,\Omega\right)} = 332\,\mu\text{F}$$
$$= \boxed{0.33\,\text{mF}}$$

(*d*) Express the potential difference across the capacitor:

$$V_C = I_{\text{peak}} X_C$$

Relate the peak current in the circuit to the impedance of the circuit when $X_L = X_C$:

$$I_{\text{peak}} = \frac{V_{\text{peak}}}{R}$$

Substitute for I to obtain:

$$V_C = \frac{V_{\text{peak}} X_C}{R} = \frac{V_{\text{peak}}}{2\pi f C R}$$

Relate the impedance of the circuit to the resistance of the coil:

$$Z = \sqrt{R^2 + X^2} \Rightarrow R = \sqrt{Z^2 - X^2}$$

Substituting for R yields:

$$V_C = \frac{V_{\text{peak}}}{2\pi f C\sqrt{Z^2 - X^2}}$$

Substitute numerical values and evaluate V_C:

$$V_C = \frac{100\,\text{V}}{2\pi\left(60\,\text{s}^{-1}\right)\left(332\,\mu\text{F}\right)\sqrt{\left(10\,\Omega\right)^2 - \left(8.0\,\Omega\right)^2}} = \boxed{0.13\,\text{kV}}$$

68 •• An ideal 0.25-H inductor and a capacitor are connected in series with an ideal 60-Hz generator. An digital voltmeter is used to measure the rms voltages across the inductor and capacitor independently. The voltmeter reading across the capacitor is 75 V and that across the inductor is 50 V. (*a*) Find the capacitance and the rms current in the circuit. (*b*) What is the rms voltage across the series combination of the capacitor and the inductor?

Picture the Problem We can find C using $V_C = I_{\text{rms}}X_C$ and I_{rms} from the potential difference across the inductor. In the absence of resistance in the circuit, the measured rms voltage across both the capacitor and inductor is $V = |V_L - V_C|$.

(*a*) Relate the capacitance C to the potential difference across the capacitor:

$$V_C = I_{\text{rms}}X_C = \frac{I_{\text{rms}}}{2\pi fC} \Rightarrow C = \frac{I_{\text{rms}}}{2\pi fV_C}$$

Use the potential difference across the inductor to express the rms current in the circuit:

$$I_{\text{rms}} = \frac{V_L}{X_L} = \frac{V_L}{2\pi fL}$$

Substitute for I_{rms} to obtain:

$$C = \frac{V_L}{(2\pi f)^2 LV_C}$$

Substitute numerical values and evaluate C:

$$C = \frac{50\,\text{V}}{\left[2\pi\left(60\,\text{s}^{-1}\right)\right]^2(0.25\,\text{H})(75\,\text{V})} = \boxed{19\,\mu\text{F}}$$

(*b*) Express the measured rms voltage V across both the capacitor and the inductor when $R = 0$:

$$V = |V_L - V_C|$$

Substitute numerical values and evaluate V:

$$V = |50\,\text{V} - 75\,\text{V}| = \boxed{25\,\text{V}}$$

69 •• **[SSM]** In the circuit shown in Figure 29-42 the ideal generator produces an rms voltage of 115 V when operated at 60 Hz. What is the rms voltage between points (*a*) *A* and *B*, (*b*) *B* and *C*, (*c*) *C* and *D*, (*d*) *A* and *C*, and (*e*) *B* and *D*?

Picture the Problem We can find the rms current in the circuit and then use it to find the potential differences across each of the circuit elements. We can use phasor diagrams and our knowledge of the phase shifts between the voltages across the three circuit elements to find the voltage differences across their combinations.

(*a*) Express the potential difference between points *A* and *B* in terms of I_{rms} and X_L:

$$V_{AB} = I_{\text{rms}} X_L \qquad (1)$$

Express I_{rms} in terms of $\mathcal{E}$ and Z:

$$I_{\text{rms}} = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

Evaluate X_L and X_C to obtain:

$$X_L = 2\pi fL = 2\pi(60\,\text{s}^{-1})(137\,\text{mH}) = 51.648\,\Omega$$

and

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(60\,\text{s}^{-1})(25\,\mu\text{F})} = 106.10\,\Omega$$

Substitute numerical values and evaluate I_{rms}:

$$I_{\text{rms}} = \frac{115\,\text{V}}{\sqrt{(50\,\Omega)^2 + (51.648\,\Omega - 106.10\,\Omega)^2}} = 1.5556\,\text{A}$$

Substitute numerical values in equation (1) and evaluate V_{AB}:

$$V_{AB} = (1.5556\,\text{A})(51.648\,\Omega) = 80.344\text{ V} = \boxed{80\,\text{V}}$$

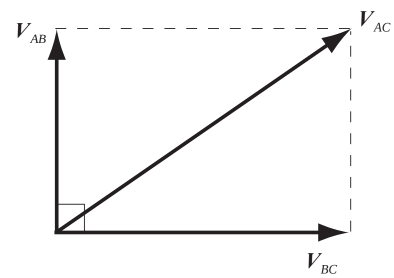
(*b*) Express the potential difference between points *B* and *C* in terms of I_{rms} and R:

$$V_{BC} = I_{\text{rms}} R = (1.5556\,\text{A})(50\,\Omega) = 77.780\text{ V} = \boxed{78\,\text{V}}$$

(*c*) Express the potential difference between points *C* and *D* in terms of I_{rms} and X_C:

$$V_{CD} = I_{\text{rms}} X_C = (1.5556\,\text{A})(106.10\,\Omega) = 165.05\,\text{V} = \boxed{0.17\,\text{kV}}$$

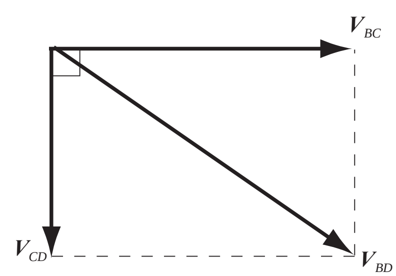
(d) The voltage across the inductor leads the voltage across the resistor as shown in the phasor diagram to the right:



Use the Pythagorean theorem to find V_{AC}:

$$V_{AC} = \sqrt{V_{AB}^2 + V_{BC}^2} = \sqrt{(80.0\,\text{V})^2 + (77.780\,\text{V})^2} = 111.58\,\text{V} = \boxed{0.11\,\text{kV}}$$

(e) The voltage across the capacitor lags the voltage across the resistor as shown in the phasor diagram to the right:



Use the Pythagorean theorem to find V_{BD}:

$$V_{BD} = \sqrt{V_{CD}^2 + V_{BC}^2} = \sqrt{(165.05\,\text{V})^2 + (77.780\,\text{V})^2} = 182.46\,\text{V} = \boxed{0.18\,\text{kV}}$$

70 •• When an *RLC* series circuit is connected to a 120-V rms, 60-Hz line, the rms current in the circuit is 11 A and this current leads the line voltage by 45º. (a) Find the average power supplied to the circuit. (b) What is the resistance in the circuit? (c) If the inductance in the circuit is 50 mH, find the capacitance in the circuit. (d) Without changing the inductance, by how much should you change the capacitance to make the power factor equal to 1? (e) Without changing the capacitance, by how much should you change the inductance to make the power factor equal to 1?

Picture the Problem We can use $P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos\delta$ to find the power supplied to the circuit and $P_{\text{av}} = I_{\text{rms}}^2 R$ to find the resistance. In Part (c) we can relate the capacitive reactance to the impedance, inductive reactance, and resistance of the circuit and solve for the capacitance C. We can use the condition on X_L and X_C at resonance to find the capacitance or inductance you would need to add to the circuit to make the power factor equal to 1.

(*a*) Express the power supplied to the circuit in terms of $\mathcal{E}_{\text{rms}}$, I_{rms}, and the power factor $\cos\delta$:

$$P_{\text{av}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos\delta$$

Substitute numerical values and evaluate P_{av}:

$$P_{\text{av}} = (120\,\text{V})(11\,\text{A})\cos 45° = 933\,\text{W} = \boxed{0.93\,\text{kW}}$$

(*b*) Relate the power dissipated in the circuit to the resistance of the resistor:

$$P_{\text{av}} = I_{\text{rms}}^2 R \Rightarrow R = \frac{P_{\text{av}}}{I_{\text{rms}}^2}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{933\,\text{W}}{(11\,\text{A})^2} = 7.71\,\Omega = \boxed{7.7\,\Omega}$$

(*c*) Express the capacitance of the capacitor in terms of its reactance:

$$C = \frac{1}{\omega X_C} = \frac{1}{2\pi f X_C} \qquad (1)$$

Relate the capacitive reactance to the impedance, inductive reactance, and resistance of the circuit:

$$Z^2 = R^2 + (X_L - X_C)^2$$

Express the impedance of the circuit in terms of the rms emf $\mathcal{E}$ and the rms current I_{rms}:

$$Z^2 = \frac{\mathcal{E}^2}{I_{\text{rms}}^2}$$

Equating these two expressions yields:

$$\frac{\mathcal{E}^2}{I_{\text{rms}}^2} = R^2 + (X_L - X_C)^2$$

Solve for $|X_L - X_C|$:

$$|X_L - X_C| = \sqrt{\frac{\mathcal{E}^2}{I_{\text{rms}}^2} - R^2}$$

Note that because *I* leads $\mathcal{E}$, the circuit is capacitive and $X_C > X_L$. Hence:

$$|X_L - X_C| = -(X_L - X_C)$$

and

$$X_C = X_L + \sqrt{\frac{\mathcal{E}^2}{I_{\text{rms}}^2} - R^2} = 2\pi f L + \sqrt{\frac{\mathcal{E}^2}{I_{\text{rms}}^2} - R^2}$$

Substitute numerical values and evaluate X_C:

$$X_C = 2\pi\left(60\,\text{s}^{-1}\right)\left(50\,\text{mH}\right) + \sqrt{\frac{\left(120\,\text{V}\right)^2}{\left(11\,\text{A}\right)^2} - \left(7.71\,\Omega\right)^2}$$
$$= 18.8\,\Omega + 7.7\,\Omega = 26.6\,\Omega$$

Substitute in equation (1) and evaluate C:

$$C = \frac{1}{2\pi\left(60\,\text{s}^{-1}\right)\left(26.6\,\Omega\right)} = 99.7\,\mu\text{F}$$
$$= \boxed{0.10\,\text{mF}}$$

(*d*) Let $C_{\text{pf}=1}$ represent the capacitance required to make $\cos\delta = 1$. The necessary change in capacitance is given by:

$$\Delta C = C_{\text{pf}=1} - C = C_{\text{pf}=1} - 99.9\,\mu\text{F}$$

Relate $C_{\text{pf}=1}$ to X_L:

$$X_L = X_C = \frac{1}{2\pi f C_{\text{pf}=1}}$$

Solving for $C_{\text{pf}=1}$ yields:

$$C_{\text{pf}=1} = \frac{1}{2\pi f X_L}$$

Substitute for $C_{\text{pf}=1}$ in the expression for ΔC to obtain:

$$\Delta C = \frac{1}{2\pi f X_L} - 99.9\,\mu\text{F}$$

Substitute numerical values and evaluate ΔC:

$$\Delta C = \frac{1}{2\pi\left(60\,\text{s}^{-1}\right)\left(18.8\,\Omega\right)} - 99.9\,\mu\text{F}$$
$$= \boxed{41\,\mu\text{F}}$$

(*e*) Let $L_{\text{pf}=1}$ represent the inductance required to make $\cos\delta = 1$. The necessary change in inductance is given by:

$$\Delta L = L_{\text{pf}=1} - L = L_{\text{pf}=1} - 50\,\text{mH}$$

Relate $L_{\text{pf}=1}$ to X_C:

$$X_L = X_C = 2\pi f L_{\text{pf}=1}$$

Solving for $L_{\text{pf}=1}$ yields:

$$L_{\text{pf}=1} = \frac{X_C}{2\pi f}$$

Substitute for $L_{\text{pf}=1}$ in the expression for ΔL to obtain:

$$\Delta L = \frac{X_C}{2\pi f} - 50\text{ mH}$$

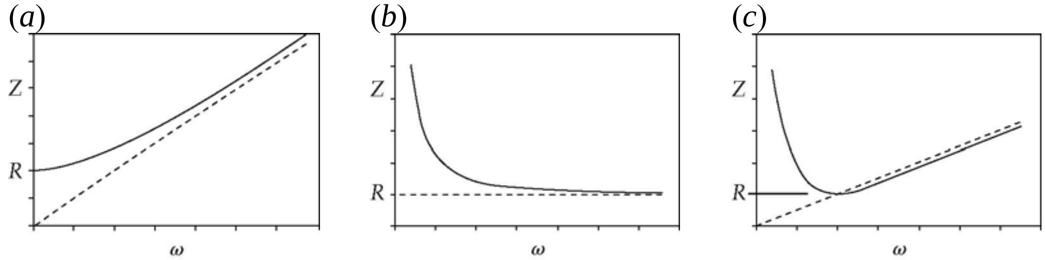
Substitute numerical values and evaluate ΔL:

$$\Delta L = \frac{26.6\,\Omega}{2\pi\left(60\text{ s}^{-1}\right)} - 50\text{ mH} = \boxed{21\text{mH}}$$

71 •• Plot the circuit impedance versus the angular frequency for each of the following circuits. (*a*) A driven series *LR* circuit, (*b*) a driven series *RC* circuit, and (*c*) a driven series *RLC* circuit.

Picture the Problem The impedance for the three circuits as functions of the angular frequency is shown in the three figures below. Also shown in each figure (dashed line) is the asymptotic approach for large angular frequencies.

(*a*) (*b*) (*c*)



72 •• In a driven series *RLC* circuit, the ideal generator has a peak emf equal to 200 V, the resistance is 60.0 Ω and the capacitance is 8.00 μF. The inductance can be varied from 8.00 mH to 40.0 mH by the insertion of an iron core in the solenoid. The angular frequency of the generator is 2500 rad/s. If the capacitor voltage is not to exceed 150 V, find (*a*) the peak current and (*b*) the range of inductances that is safe to use.

Picture the Problem We can find the maximum current in the circuit from the maximum voltage across the capacitor and the reactance of the capacitor. To find the range of inductance that is safe to use we can express Z^2 for the circuit in terms of $\mathcal{E}_{\text{peak}}^2$ and I_{peak}^2 and solve the resulting quadratic equation for *L*.

(*a*) Express the peak current in terms of the maximum potential difference across the capacitor and its reactance:

$$I_{\text{peak}} = \frac{V_{C,\text{peak}}}{X_C} = \omega C V_{C,\text{peak}}$$

Substitute numerical values and evaluate I_{peak}:

$$I_{\text{peak}} = (2500\text{ rad/s})(8.00\,\mu\text{F})(150\text{ V}) = \boxed{3.00\text{ A}}$$

(*b*) Relate the maximum current in the circuit to the emf of the source and the impedance of the circuit:

$$I_{\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{Z} \Rightarrow Z^2 = \frac{\mathcal{E}_{\text{peak}}^2}{I_{\text{peak}}^2}$$

Express Z^2 in terms of R, X_L, and X_C:

$$Z^2 = R^2 + (X_L - X_C)^2$$

Substitute to obtain:

$$\frac{\mathcal{E}_{\text{peak}}^2}{I_{\text{peak}}^2} = R^2 + (X_L - X_C)^2$$

Evaluating X_C yields:

$$X_C = \frac{1}{\omega C} = \frac{1}{(2500\,\text{rad/s})(8.00\,\mu\text{F})} = 50.0\,\Omega$$

Substitute numerical values to obtain:

$$\frac{(200\,\text{V})^2}{(3.00\,\text{A})^2} = (60.0\,\Omega)^2 + ((2500\,\text{rad/s})L - 50.0\,\Omega)^2$$

Solving for L yields:

$$L = \frac{50.0\,\Omega \pm \sqrt{844\,\Omega^2}}{2500\,\text{s}^{-1}}$$

Denoting the solutions as L_+ and L_-, find the values for the inductance:

$$L_+ = 31.6\,\text{mH and } L_- = 8.38\,\text{mH}$$

The ranges for L are:

$$\boxed{8.00\,\text{mH} < L < 8.38\,\text{mH}}$$

and

$$\boxed{31.6\,\text{mH} < L < 40.0\,\text{mH}}$$

73 •• A certain electrical device draws an rms current of 10 A at an average power of 720 W when connected to a 120-V rms, 60-Hz power line. (*a*) What is the impedance of the device? (*b*) What series combination of resistance and reactance would have the same impedance as this device? (*c*) If the current leads the emf, is the reactance inductive or capacitive?

Picture the Problem We can find the impedance of the circuit from the applied emf and the current drawn by the device. In Part (*b*) we can use $P_{\text{av}} = I_{\text{rms}}^2 R$ to find R and the definition of the impedance of a series *RLC* circuit to find $X = X_L - X_C$.

(*a*) Express the impedance of the device in terms of the current it draws and the emf provided by the power line:

$$Z = \frac{\mathcal{E}_{\text{rms}}}{I_{\text{rms}}}$$

Substitute numerical values and evaluate *Z*:

$$Z = \frac{120\,\text{V}}{10\,\text{A}} = \boxed{12\,\Omega}$$

(*b*) Use the relationship between the average power supplied to the device and the rms current it draws to express *R*:

$$P_{\text{av}} = I_{\text{rms}}^2 R \Rightarrow R = \frac{P_{\text{av}}}{I_{\text{rms}}^2}$$

Substitute numerical values and evaluate *R*:

$$R = \frac{720\,\text{W}}{(10\,\text{A})^2} = 7.20\,\Omega = \boxed{7.2\,\Omega}$$

The impedance of a series *RLC* circuit is given by:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

or

$$Z^2 = R^2 + (X_L - X_C)^2$$

Solving for $X_L - X_C$ yields:

$$X = X_L - X_C = \sqrt{Z^2 - R^2}$$

Substitute numerical values and evaluate *X*:

$$X = \sqrt{(12\,\Omega)^2 - (7.20\,\Omega)^2} = \boxed{9.6\,\Omega}$$

(*c*) If the current leads the emf, the reactance is capacitive.

74 •• A method for measuring inductance is to connect the inductor in series with a known capacitance, a known resistance, an ac ammeter, and a variable-frequency signal generator. The frequency of the signal generator is varied and the emf is kept constant until the current is maximum. (*a*) If the capacitance is 10 μF, the peak emf is 10 V, the resistance is 100 Ω, and the rms current in the circuit is maximum when the driving frequency is 5000 rad/s, what is the value of the inductance? (*b*) What is the maximum rms current?

Picture the Problem (*a*) We can use the fact that when the current is a maximum, $X_L = X_C$, to find the inductance of the circuit. In Part (*b*), we can find $I_{\text{rms, max}}$ from $\mathcal{E}_{\text{peak}}$ and the impedance of the circuit at resonance.

(*a*) Relate X_L and X_C at resonance:

$$X_L = X_C \text{ or } \omega_0 L = \frac{1}{\omega_0 C}$$

Solve for L to obtain:

$$L = \frac{1}{\omega_0^2 C}$$

Substitute numerical values and evaluate L:

$$L = \frac{1}{\left(5000\,\text{s}^{-1}\right)^2\left(10\,\mu\text{F}\right)} = \boxed{4.0\,\text{mH}}$$

(*b*) Noting that, at resonance, $X = 0$, express $I_{\text{rms, max}}$ in terms of the applied emf and the impedance of the circuit at resonance:

$$I_{\text{rms, max}} = \frac{\mathcal{E}_{\text{rms}}}{Z} = \frac{\mathcal{E}_{\text{max}}}{\sqrt{2}Z} = \frac{10\,\text{V}}{\sqrt{2}\left(100\,\Omega\right)} = \boxed{71\,\text{mA}}$$

75 •• A resistor and a capacitor are connected in parallel across an ac generator (Figure 29-43) that has an emf given by $\mathcal{E} = \mathcal{E}_{\text{peak}} \cos \omega t$. (*a*) Show that the current in the resistor is given by $I_R = (\mathcal{E}_{\text{peak}}/R) \cos \omega t$. (*b*) Show that the current in the capacitor branch is given by $I_C = (\mathcal{E}_{\text{peak}}/X_C) \cos(\omega t + 90^\circ)$. (*c*) Show that the total current is given by $I = I_{\text{peak}} \cos(\omega t + \delta)$, where $\tan \delta = R/X_C$ and $I_{\text{peak}} = \mathcal{E}_{\text{peak}}/Z$.

Picture the Problem Because the resistor and the capacitor are connected in parallel, the voltage drops across them are equal. Also, the total current is the sum of the current through the resistor and the current through the capacitor. Because these two currents are not in phase, we use phasors to calculate their sum. The amplitudes of the applied voltage and the currents are equal to the magnitude of the phasors. That is $\left|\vec{\boldsymbol{\mathcal{E}}}\right| = \mathcal{E}_{\text{peak}}$, $\left|\vec{\boldsymbol{I}}\right| = I_{\text{peak}}$, $\left|\vec{\boldsymbol{I}}_R\right| = I_{R,\text{peak}}$, and $\left|\vec{\boldsymbol{I}}_C\right| = I_{C,\text{peak}}$.

(*a*) The ac source applies a voltage given by $\mathcal{E} = \mathcal{E}_{\text{peak}} \cos \omega t$. Thus, the voltage drop across both the load resistor and the capacitor is:

$$\mathcal{E}_{\text{peak}} \cos \omega t = I_R R$$

The current in the resistor is in phase with the applied voltage:

$$I_R = I_{R,\text{peak}} \cos \omega t$$

Because $I_{R,\text{peak}} = \dfrac{\mathcal{E}_{\text{peak}}}{R}$:

$$I_R = \boxed{\frac{\mathcal{E}_{\text{peak}}}{R} \cos \omega t}$$

(*b*) The current in the capacitor leads the applied voltage by 90°:

$$I_C = I_{C,\text{peak}}\cos(\omega t + 90°)$$

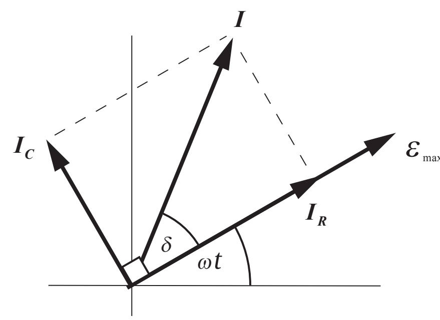
Because $I_{C,\text{peak}} = \dfrac{\mathcal{E}_{\text{peak}}}{X_C}$:

$$I_C = \boxed{\frac{\mathcal{E}_{\text{peak}}}{X_C}\cos(\omega t + 90°)}$$

(*c*) The net current *I* is the sum of the currents through the parallel branches:

$$I = I_R + I_C$$

Draw the phasor diagram for the circuit. The projections of the phasors onto the horizontal axis are the instantaneous values. The current in the resistor is in phase with the applied voltage, and the current in the capacitor leads the applied voltage by 90°. The net current phasor is the sum of the branch current phasors $\left(\vec{I} = \vec{I}_C + \vec{I}_R\right)$.



The peak current through the parallel combination is equal to $\mathcal{E}_{\text{peak}}/Z$, where *Z* is the impedance of the combination:

$$I = I_{\text{peak}}\cos(\omega t - |\delta|),$$

where $I_{\text{peak}} = \dfrac{\mathcal{E}_{\text{peak}}}{Z}$

From the phasor diagram we have:

$$I_{\text{peak}}^2 = I_{R,\text{peak}}^2 + I_{C,\text{peak}}^2$$

$$= \left(\frac{\mathcal{E}_{\text{peak}}}{R}\right)^2 + \left(\frac{\mathcal{E}_{\text{peak}}}{X_C}\right)^2$$

$$= \mathcal{E}_{\text{peak}}^2\left(\frac{1}{R^2} + \frac{1}{X_C^2}\right) = \frac{\mathcal{E}_{\text{peak}}^2}{Z^2}$$

where $\dfrac{1}{Z^2} = \dfrac{1}{R^2} + \dfrac{1}{X_C^2}$

Solving for I_{peak} yields:

$$I_{\text{peak}} = \boxed{\frac{\mathcal{E}_{\text{peak}}}{Z}} \text{ where } Z^{-2} = R^{-2} + X_C^{-2}$$

From the phasor diagram:

$$I = \boxed{I_{\text{peak}}\cos(\omega t + \delta)}$$

where

$$\tan\delta = \frac{I_C}{I_R} = \frac{\dfrac{\mathcal{E}_{\text{peak}}}{X_C}}{\dfrac{\mathcal{E}_{\text{peak}}}{R}} = \boxed{\frac{R}{X_C}}$$

76 ••• Figure 29-44 shows a plot of average power P_{av} versus generator frequency ω for a series *RLC* circuit driven by an ac generator. The average power P_{av} is given by Equation 29-56. The full width at half-maximum, $\Delta\omega$, is the width of the resonance curve between the two points, where P_{av} is one-half its maximum value. Show that for a sharply peaked resonance, $\Delta\omega \approx R/L$ and that $Q \approx \omega_0/\Delta\omega$ in this case (Equation 29-58). *Hint: The half-power points occur when the denominator of Equation 29-56 is equal to twice the value it has at resonance; that is, when* $L^2\left(\omega^2-\omega_0^2\right)+\omega^2R^2 \approx 2\omega_0^2R^2$. *Let* ω_1 *and* ω_2 *be the solutions of this equation. Then, show that* $\Delta\omega = \omega_2 - \omega_1 \approx R/L$.

Picture the Problem We can use the condition determining the half-power points to obtain a quadratic equation that we can solve for the frequencies corresponding to the half-power points. Letting ω_1 be the half-power frequency that is less than ω_0 and ω_2 be the half-power frequency that is greater than ω_0 will lead us to the result that $\Delta\omega = \omega_2 - \omega_1 \approx R/L$. We can then use the definition of Q to complete the proof that $Q \approx \omega_0/\Delta\omega$.

Equation 29-56 is:

$$P_{av} = \frac{V_{\text{app, rms}}^2 R\omega^2}{L^2\left(\omega^2-\omega_0^2\right)^2+\omega^2R^2}$$

The half-power points occur when the denominator of Equation 29-56 is twice the value near resonance; that is, when:

$$L^2\left(\omega^2-\omega_0^2\right)^2+\omega^2R^2 \approx 2\omega_0^2R^2$$

or

$$L^2\left[\left(\omega-\omega_0\right)\left(\omega+\omega_0\right)\right]^2+\omega^2R^2 \approx 2\omega_0^2R^2$$

For a sharply peaked resonance, $\omega+\omega_0 \approx 2\omega_0$. Hence:

$$L^2\left[\left(\omega-\omega_0\right)\left(2\omega_0\right)\right]^2+\omega^2R^2 \approx 2\omega_0^2R^2$$

or

$$4\omega_0^2L^2\left(\omega-\omega_0\right)^2+\omega^2R^2 \approx 2\omega_0^2R^2$$

Let ω_1 be a solution to this equation. Noting that, for a sharply peaked resonance, $\omega_1 \approx \omega_0$, it follows that:

$$4\omega_0^2L^2\left(\omega_1-\omega_0\right)^2+\omega_0^2R^2 \approx 2\omega_0^2R^2$$

or, simplifying,

$$\left(\omega_1-\omega_0\right)^2 \approx \frac{R^2}{4L^2}$$

Solving for ω_1 yields:

$$\omega_1 \approx \omega_0 - \frac{R}{2L}$$

where we've used the minus sign because $\omega_1 < \omega_0$.

Similarly for ω_2:

$$\omega_2 \approx \omega_0 + \frac{R}{2L}$$

where we've used the plus sign because $\omega_2 > \omega_0$.

Evaluating $\Delta\omega = \omega_2 - \omega_1$ yields:

$$\Delta\omega \approx \omega_0 + \frac{R}{2L} - \left(\omega_0 - \frac{R}{2L}\right) = \boxed{\frac{R}{L}}$$

From the definition of Q:

$$\frac{R}{L} = \frac{\omega_0}{Q}$$

Substitute in the expression for $\Delta\omega$ to obtain:

$$\Delta\omega \approx \frac{\omega_0}{Q} \Rightarrow Q \approx \boxed{\frac{\omega_0}{\Delta\omega}}$$

77 ••• Show by direct substitution that $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = 0$ (Equation 29-43*b*) is satisfied by $Q = Q_0e^{-t/\tau}\cos\omega' t$, where $\tau = 2L/R$, $\omega' = \sqrt{1/(LC) - 1/\tau^2}$, and Q_0 is the charge on the capacitor at $t = 0$.

Picture the Problem We'll differentiate $Q = Q_0e^{-t/\tau}\cos\omega' t$ twice and substitute this function and both its derivatives in the differential equation of the circuit. Rewriting the resulting equation in the form A$\cos\omega' t$ + B$\sin\omega' t$ = 0 will reveal that B vanishes. Requiring that A$\cos\omega' t = 0$ hold for all values of t will lead to the result that $\omega' = \sqrt{1/(LC) - 1/\tau^2}$.

Equation 29-43*b* is:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

Assume a solution of the form:

$$Q = Q_0e^{-t/\tau}\cos\omega' t$$

Differentiate $Q(t)$ twice to obtain:

$$\begin{aligned}\frac{dQ}{dt} &= Q_0\frac{d}{dt}\left[e^{-t/\tau}\cos\omega' t\right] \\ &= Q_0\left(e^{-t/\tau}\frac{d}{dt}\cos\omega' t + \cos\omega' t\frac{d}{dt}e^{-t/\tau}\right) \\ &= Q_0e^{-t/\tau}\left(-\omega'\sin\omega' t - \frac{1}{\tau}\cos\omega' t\right)\end{aligned}$$

and

$$\frac{d^2Q}{dt^2}=Q_0\frac{d}{dt}\left[e^{-t/\tau}\left(-\omega'\sin\omega't-\frac{1}{\tau}\cos\omega't\right)\right]$$
$$=Q_0e^{-t/\tau}\left[\left(\frac{1}{\tau^2}-\omega'^2\right)\cos\omega't+\frac{2\omega'}{\tau}\sin\omega't\right]$$

Substitute these derivatives in the differential equation and simplify to obtain:

$$LQ_0e^{-t/\tau}\left[\left(\frac{1}{\tau^2}-\omega'^2\right)\cos\omega't+\frac{2\omega'}{\tau}\sin\omega't\right]+RQ_0e^{-t/\tau}\left(-\omega'\sin\omega't-\frac{1}{\tau}\cos\omega't\right)$$
$$+\frac{1}{C}Q_0e^{-t/\tau}\cos\omega't=0$$

Because Q_0 and $e^{-t/\tau}$ are never zero, divide them out of the equation and simplify to obtain:

$$L\left(\frac{1}{\tau^2}-\omega'^2\right)\cos\omega't+\frac{2L\omega'}{\tau}\sin\omega't-\omega'R\sin\omega't-\frac{R}{\tau}\cos\omega't+\frac{1}{C}\cos\omega't=0$$

Rewriting this equation in the form A$\cos\omega't$ + B$\sin\omega't$ = 0 yields:

$$(R\omega'-R\omega')\sin\omega't+\left[L\left(\frac{1}{\tau^2}-\omega'^2\right)+\frac{1}{C}-\frac{R}{\tau}\right]\cos\omega't=0$$

or

$$\left[L\left(\frac{1}{\tau^2}-\omega'^2\right)+\frac{1}{C}-\frac{R}{\tau}\right]\cos\omega't=0$$

If this equation is to hold for all values of t, its coefficient must vanish:

$$L\left(\frac{1}{\tau^2}-\omega'^2\right)+\frac{1}{C}-\frac{R}{\tau}=0$$

Solving for ω' yields:

$$\omega'=\boxed{\sqrt{\frac{1}{LC}-\left(\frac{1}{2L}\right)^2}},$$

the condition that must be satisfied if $Q=Q_0e^{-t/\tau}\cos\omega't$ is the solution to Equation 29-43*b*.

78 ••• One method for measuring the magnetic susceptibility of a sample uses an *LC* circuit consisting of an air-core solenoid and a capacitor. The resonant frequency of the circuit without the sample is determined and then measured

again with the sample inserted in the solenoid. Suppose you have a solenoid that is 4.00 cm long, 3.00 mm in diameter, and has 400 turns of fine wire. You have a sample that is inserted in the solenoid and completely fills the air space. Neglect end effects. (*a*) Calculate the inductance of your empty solenoid. (*b*) What value for the capacitance of the capacitor should you choose that the resonance frequency of the circuit without a sample is exactly 6.0000 MHz? (*c*) When a sample is inserted in the solenoid, you determine that the resonance frequency drops to 5.9989 MHz. Use your data to determine the sample's susceptibility.

Picture the Problem We can use $L=\mu_0 n^2 A\ell$ to determine the inductance of the empty solenoid and the resonance condition to find the capacitance of the sample-free circuit when the resonance frequency of the circuit is 6.0000 MHz. By expressing L as a function of f_0 and then evaluating df_0/dL and approximating the derivative with $\Delta f_0/\Delta L$, we can evaluate χ from its definition.

(*a*) Express the inductance of an air-core solenoid:

$$L=\mu_0 n^2 A\ell$$

Substitute numerical values and evaluate L:

$$L=\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(\frac{400}{4.00\,\text{cm}}\right)^2\frac{\pi}{4}(3.00\,\text{cm})^2(4.00\,\text{cm})=3.553\,\text{mH}=\boxed{3.55\,\text{mH}}$$

(*b*) Express the condition for resonance in the LC circuit:

$$X_L=X_C\Rightarrow 2\pi f_0 L=\frac{1}{2\pi f_0 C}\qquad(1)$$

Solving for C yields:

$$C=\frac{1}{4\pi^2 f_0^2 L}$$

Substitute numerical values and evaluate C:

$$C=\frac{1}{4\pi^2(6.0000\,\text{MHz})^2(3.553\,\text{mH})}=1.9803\times10^{-13}\ \text{F}=\boxed{0.198\,\text{pF}}$$

(*c*) Express the sample's susceptibility in terms of L and ΔL:

$$\chi=\frac{\Delta L}{L}\qquad(2)$$

Solve equation (1) for f_0:

$$f_0=\frac{1}{2\pi\sqrt{LC}}$$

Differentiate f_0 with respect to L:

$$\frac{df_0}{dL}=\frac{1}{2\pi\sqrt{C}}\frac{d}{dL}L^{-1/2}=-\frac{1}{4\pi\sqrt{C}}L^{-3/2}$$

$$=-\frac{1}{4\pi L\sqrt{LC}}=-\frac{f_0}{2L}$$

Approximate df_0/dL by $\Delta f_0/\Delta L$:

$$\frac{\Delta f_0}{\Delta L}=-\frac{f_0}{2L}\ \text{or}\ \frac{\Delta f_0}{f_0}=-\frac{\Delta L}{2L}$$

Substitute in equation (2) to obtain:

$$\chi=-2\frac{\Delta f_0}{f_0}$$

Substitute numerical values and evaluate χ:

$$\chi=-2\left(\frac{5.9989\,\text{MHz}-6.0000\,\text{MHz}}{6.0000\,\text{MHz}_0}\right)$$

$$=\boxed{3.7\times10^{-4}}$$

The Transformer

79 • **[SSM]** A rms voltage of 24 V is required for a device whose impedance is 12 Ω. (*a*) What should the turns ratio of a transformer be, so that the device can be operated from a 120-V line? (*b*) Suppose the transformer is accidentally connected in reverse with the secondary winding across the 120-V-rms line and the 12-Ω load across the primary. How much rms current will then be in the primary winding?

Picture the Problem Let the subscript 1 denote the primary and the subscript 2 the secondary. We can use $V_2N_1=V_1N_2$ and $N_1I_1=N_2I_2$ to find the turns ratio and the primary current when the transformer connections are reversed.

(*a*) Relate the number of primary and secondary turns to the primary and secondary voltages:

$$V_{2,\,\text{rms}}N_1=V_{1,\,\text{rms}}N_2 \qquad (1)$$

Solve for and evaluate the ratio N_2/N_1:

$$\frac{N_2}{N_1}=\frac{V_{2,\,\text{rms}}}{V_{1,\,\text{rms}}}=\frac{24\,\text{V}}{120\,\text{V}}=\boxed{\frac{1}{5}}$$

(*b*) Relate the current in the primary to the current in the secondary and to the turns ratio:

$$I_{1,\,\text{rms}}=\frac{N_2}{N_1}I_{2,\,\text{rms}}$$

Express the current in the primary winding in terms of the voltage across it and its impedance:

$$I_{2,\,\mathrm{rms}} = \frac{V_{2,\,\mathrm{rms}}}{Z_2}$$

Substitute for $I_{2,\,\mathrm{rms}}$ to obtain:

$$I_{1,\,\mathrm{rms}} = \frac{N_2}{N_1}\frac{V_{2,\,\mathrm{rms}}}{Z_2}$$

Substitute numerical values and evaluate $I_{1,\,\mathrm{rms}}$:

$$I_1 = \left(\frac{5}{1}\right)\left(\frac{120\,\mathrm{V}}{12\,\Omega}\right) = \boxed{50\,\mathrm{A}}$$

80 • A transformer has 400 turns in the primary and 8 turns in the secondary. (*a*) Is this a step-up or a step-down transformer? (*b*) If the primary is connected to a 120 V rms voltage source, what is the open-circuit rms voltage across the secondary? (*c*) If the primary rms current is 0.100 A, what is the secondary rms current, assuming negligible magnetization current and no power loss?

Picture the Problem Let the subscript 1 denote the primary and the subscript 2 the secondary. We can decide whether the transformer is a step-up or step-down transformer by examining the ratio of the number of turns in the secondary to the number of terms in the primary. We can relate the open-circuit rms voltage in the secondary to the primary rms voltage and the turns ratio.

(*a*) Because there are fewer turns in the secondary than in the primary it is a step-down transformer.

(*b*) Relate the open-circuit rms voltage $V_{2,\,\mathrm{rms}}$ in the secondary to the rms voltage $V_{1,\,\mathrm{rms}}$ in the primary:

$$V_{2,\,\mathrm{rms}} = \frac{N_2}{N_1}V_{1,\,\mathrm{rms}}$$

Substitute numerical values and evaluate $V_{2,\,\mathrm{rms}}$:

$$V_{2,\,\mathrm{rms}} = \frac{8}{400}(120\,\mathrm{V}) = \boxed{2.40\,\mathrm{V}}$$

(*c*) Because there are no power losses:

$$V_{1,\,\mathrm{rms}}I_{1,\,\mathrm{rms}} = V_{2,\,\mathrm{rms}}I_{2,\,\mathrm{rms}}$$

and

$$I_{2,\,\mathrm{rms}} = \frac{V_{1,\,\mathrm{rms}}}{V_{2,\,\mathrm{rms}}}I_{1,\,\mathrm{rms}}$$

Substitute numerical values and evaluate $I_{2,\,\mathrm{rms}}$:

$$I_{2,\,\mathrm{rms}} = \frac{120\,\mathrm{V}}{2.40\,\mathrm{V}}(0.100\,\mathrm{A}) = \boxed{5.00\,\mathrm{A}}$$

81 • The primary of a step-down transformer has 250 turns and is connected to a 120-V rms line. The secondary is to supply 20 A rms at 9.0 V rms. Find (*a*) the rms current in the primary and (*b*) the number of turns in the secondary, assuming 100 percent efficiency.

Picture the Problem Let the subscript 1 denote the primary and the subscript 2 the secondary. We can use $I_{1,\,\text{rms}}V_{1,\,\text{rms}} = I_{2,\,\text{rms}}V_{2,\,\text{rms}}$ to find the current in the primary and $V_{2,\,\text{rms}}N_1 = V_{1,\,\text{rms}}N_2$ to find the number of turns in the secondary.

(*a*) Because we have 100 percent efficiency:

$$I_{1,\,\text{rms}}V_{1,\,\text{rms}} = I_{2,\,\text{rms}}V_{2,\,\text{rms}}$$

and

$$I_{1,\,\text{rms}} = I_{2,\,\text{rms}}\frac{V_{2,\,\text{rms}}}{V_{1,\,\text{rms}}}$$

Substitute numerical values and evaluate $I_{1,\,\text{rms}}$:

$$I_{1,\,\text{rms}} = (20\,\text{A})\frac{9.0\,\text{V}}{120\,\text{V}} = \boxed{1.5\,\text{A}}$$

(*b*) Relate the number of primary and secondary turns to the primary and secondary voltages:

$$V_{2,\,\text{rms}}N_1 = V_{1,\,\text{rms}}N_2 \Rightarrow N_2 = \frac{V_{2,\,\text{rms}}}{V_{1,\,\text{rms}}}N_1$$

Substitute numerical values and evaluate N_2/N_1:

$$N_2 = \frac{9.0\,\text{V}}{120\,\text{V}}(250) \approx \boxed{19}$$

82 •• An audio oscillator (ac source) that has an internal resistance of 2000 Ω and an open-circuit rms output voltage of 12.0 V is to be used to drive a loudspeaker coil that has a resistance of 8.00 Ω. (*a*) What should be the ratio of primary to secondary turns of a transformer, so that maximum average power is transferred to the speaker? (*b*) Suppose a second identical speaker is connected in parallel with the first speaker. How much average power is then supplied to the two speakers combined?

Picture the Problem Note: In a simple circuit maximum power transfer from source to load requires that the load resistance equals the internal resistance of the source. We can use Ohm's law and the relationship between the primary and secondary currents and the primary and secondary voltages and the turns ratio of the transformer to derive an expression for the turns ratio as a function of the effective resistance of the circuit and the resistance of the speaker(s).

(*a*) Express the effective loudspeaker resistance at the primary of the transformer:

$$R_{\text{eff}} = \frac{V_{1,\,\text{rms}}}{I_{1,\,\text{rms}}}$$

Relate $V_{1,\,\text{rms}}$ to $V_{2,\,\text{rms}}$, N_1, and N_2:

$$V_{1,\,\text{rms}} = V_{2,\,\text{rms}}\frac{N_1}{N_2}$$

Express $I_{1,\,\text{rms}}$ in terms of $I_{2,\,\text{rms}}$, N_1, and N_2:

$$I_{1,\,\text{rms}} = I_{2,\,\text{rms}}\frac{N_2}{N_1}$$

Substitute for $V_{1,\,\text{rms}}$ and $I_{1,\,\text{rms}}$ and simplify to obtain:

$$R_{\text{eff}} = \frac{V_{2,\,\text{rms}}\dfrac{N_1}{N_2}}{I_{2,\,\text{rms}}\dfrac{N_2}{N_1}} = \left(\frac{V_{2,\,\text{rms}}}{I_{2,\,\text{rms}}}\right)\left(\frac{N_1}{N_2}\right)^2$$

Solve for N_1/N_2:

$$\frac{N_1}{N_2} = \sqrt{\frac{I_{2,\,\text{rms}}R_{\text{eff}}}{V_{2,\,\text{rms}}}} = \sqrt{\frac{R_{\text{eff}}}{R_2}} \qquad (1)$$

Evaluate N_1/N_2 for $R_{\text{eff}} = R_{\text{coil}}$:

$$\frac{N_1}{N_2} = \sqrt{\frac{2000\,\Omega}{8.00\,\Omega}} = 15.811 = \boxed{15.8}$$

(*b*) Express the power delivered to the two speakers connected in parallel:

$$P_{\text{sp}} = I_{1,\,\text{rms}}^2 R_{\text{eff}} \qquad (2)$$

Find the equivalent resistance R_{sp} of the two 8.00-Ω speakers in parallel:

$$\frac{1}{R_{\text{sp}}} = \frac{1}{8.00\,\Omega} + \frac{1}{8.00\,\Omega} \Rightarrow R_{\text{sp}} = 4.00\,\Omega$$

Solve equation (1) for R_{eff} to obtain:

$$R_{\text{eff}} = R_2\left(\frac{N_1}{N_2}\right)^2$$

Substitute numerical values and evaluate R_{eff}:

$$R_{\text{eff}} = (4.00\,\Omega)(15.811)^2 = 1000\,\Omega$$

Find the current supplied by the source:

$$I_{1,\,\text{rms}} = \frac{V_{\text{rms}}}{R_{\text{tot}}} = \frac{12.0\,\text{V}}{2000\,\Omega + 1000\,\Omega} = 4.00\,\text{mA}$$

Substitute numerical values in equation (2) and evaluate the power delivered to the parallel speakers:

$$P_{\text{sp}} = (4.00\,\text{mA})^2(1000\,\Omega) = \boxed{16.0\,\text{mW}}$$

General Problems

83 • The distribution circuit of a residential power line is operated at 2000 V rms. This voltage must be reduced to 240 V rms for use within residences. If the secondary side of the transformer has 400 turns, how many turns are in the primary?

Picture the Problem We can relate the input and output voltages to the number of turns in the primary and secondary using $V_{2,\,\mathrm{rms}}N_1 = V_{1,\,\mathrm{rms}}N_2$.

Relate the output voltages $V_{2,\,\mathrm{rms}}$ to the input voltage $V_{1,\,\mathrm{rms}}$ and the number of turns in the primary N_1 and secondary N_2:

$$V_{2,\,\mathrm{rms}} = \frac{N_2}{N_1}V_{1,\,\mathrm{rms}} \Rightarrow N_1 = N_2\frac{V_{1,\,\mathrm{rms}}}{V_{2,\,\mathrm{rms}}}$$

Substitute numerical values and evaluate N_1:

$$N_1 = (400)\left(\frac{2000\,\mathrm{V}}{240\,\mathrm{V}}\right) = \boxed{3.33\times10^3}$$

84 •• A resistor that has a resistance R carries a current given by (5.0 A) sin $2\pi ft$ + (7.0 A) sin $4\pi ft$, where f = 60 Hz. (*a*) What is the rms current in the resistor? (*b*) If R =12 Ω, what is the average power delivered to the resistor? (*c*) What is the rms voltage across the resistor?

Picture the Problem We can use its definition, $I_{\mathrm{rms}} = \sqrt{\left(I^2\right)_{\mathrm{av}}}$ to relate the rms current to the current carried by the resistor and find $\left(I^2\right)_{\mathrm{av}}$ by integrating I^2.

(*a*) Express the rms current in terms of the $\left(I^2\right)_{\mathrm{av}}$:

$$I_{\mathrm{rms}} = \sqrt{\left(I^2\right)_{\mathrm{av}}}$$

Evaluate I^2:

$$\begin{aligned} I^2 &= \left[(5.0\,\mathrm{A})\sin 2\pi ft + (7.0\,\mathrm{A})\sin 4\pi ft\right]^2 \\ &= \left(25\,\mathrm{A}^2\right)\sin^2 2\pi ft + \left(70\,\mathrm{A}^2\right)\sin 2\pi ft\sin 4\pi ft + \left(49\,\mathrm{A}^2\right)\sin^2 4\pi ft \end{aligned}$$

Find $\left(I^2\right)_{\mathrm{av}}$ by integrating I^2 from t = 0 to $t = T = 2\pi/\omega$ and dividing by T:

$$\begin{aligned} \left(I^2\right)_{\mathrm{av}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} &\left\{\left(25\,\mathrm{A}^2\right)\sin^2 2\pi ft + \left(70\,\mathrm{A}^2\right)\sin 2\pi ft\sin 4\pi ft \right. \\ &\left. + \left(49\,\mathrm{A}^2\right)\sin^2 4\pi ft\right\}dt \end{aligned}$$

Use the trigonometric identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ to simplify and evaluate the 1st and 3rd integrals and recognize that the middle term is of the form sin*x*sin2*x* to obtain:

$$\left(I^2\right)_{\text{av}} = 12.5\,\text{A}^2 + 0 + 24.5\,\text{A}^2 = 37.0\,\text{A}^2$$

Substitute for $\left(I^2\right)_{\text{av}}$ and evaluate I_{rms}:

$$I_{\text{rms}} = \sqrt{37.0\,\text{A}^2} = \boxed{6.1\,\text{A}}$$

(*b*) Relate the power dissipated in the resistor to its resistance and the rms current in it:

$$P = I_{\text{rms}}^2 R$$

Substitute numerical values and evaluate *P*:

$$P = (6.08\,\text{A})^2(12\,\Omega) = \boxed{0.44\,\text{kW}}$$

(*c*) Express the rms voltage across the resistor in terms of *R* and I_{rms}:

$$V_{\text{rms}} = I_{\text{rms}} R$$

Substitute numerical values and evaluate V_{rms}:

$$V_{\text{rms}} = (6.08\,\text{A})(12\,\Omega) = \boxed{73\,\text{V}}$$

85 •• [SSM] Figure 29-45 shows the voltage versus time for a *square-wave* voltage source. If V_0 = 12 V, (*a*) what is the rms voltage of this source? (*b*) If this alternating waveform is rectified by eliminating the negative voltages, so that only the positive voltages remain, what is the new rms voltage?

Picture the Problem The average of any quantity over a time interval Δ*T* is the integral of the quantity over the interval divided by Δ*T*. We can use this definition to find both the average of the voltage squared, $\left(V^2\right)_{\text{av}}$ and then use the definition of the rms voltage.

(*a*) From the definition of V_{rms} we have:

$$V_{\text{rms}} = \sqrt{\left(V_0^2\right)_{\text{av}}}$$

Noting that $-V_0^2 = V_0^2$, evaluate V_{rms}:

$$V_{\text{rms}} = \sqrt{V_0^2} = V_0 = \boxed{12\,\text{V}}$$

(*b*) Noting that the voltage during the second half of each cycle is now zero, express the voltage during the first half cycle of the time interval $\frac{1}{2}\Delta T$:

$$V = V_0$$

Express the square of the voltage during this half cycle:

$$V^2 = V_0^2$$

Calculate $\left(V^2\right)_{\text{av}}$ by integrating V^2 from $t = 0$ to $t = \frac{1}{2}\Delta T$ and dividing by ΔT:

$$\left(V^2\right)_{\text{av}} = \frac{V_0^2}{\Delta T}\int_0^{\frac{1}{2}\Delta T} dt = \frac{V_0^2}{\Delta T}\left[t\right]_0^{\frac{1}{2}\Delta T} = \tfrac{1}{2}V_0^2$$

Substitute to obtain:

$$V_{\text{rms}} = \sqrt{\tfrac{1}{2}V_0^2} = \frac{V_0}{\sqrt{2}} = \frac{12\text{ V}}{\sqrt{2}} = \boxed{8.5\text{ V}}$$

86 •• What are the average values and rms values of current for the two current waveforms shown in Figure 29-46?

Picture the Problem The average of any quantity over a time interval ΔT is the integral of the quantity over the interval divided by ΔT. We can use this definition to find both the average current I_{av}, and the average of the current squared, $\left(I^2\right)_{\text{av}}$

From the definition of I_{av} and I_{rms} we have:

$$I_{\text{av}} = \frac{1}{\Delta T}\int_0^{\Delta T} I dt \text{ and } I_{\text{rms}} = \sqrt{\left(I^2\right)_{\text{av}}}$$

<u>Waveform (*a*)</u> Express the current during the first half cycle of time interval ΔT:

$$I_a = \frac{4\text{ A}}{\Delta T}t$$

where I is in A when t and T are in seconds.

Evaluate $I_{\text{av},\,a}$:

$$I_{\text{av},\,a} = \frac{1}{\Delta T}\int_0^{\Delta T}\frac{4.0\text{ A}}{\Delta T}tdt = \frac{4.0\text{ A}}{(\Delta T)^2}\int_0^{\Delta T} tdt$$

$$= \frac{4.0\text{ A}}{(\Delta T)^2}\left[\frac{t^2}{2}\right]_0^{\Delta T} = \boxed{2.0\text{ A}}$$

Express the square of the current during this half cycle:

$$I_a^2 = \frac{(4.0\text{ A})^2}{(\Delta T)^2}t^2$$

Noting that the average value of the squared current is the same for each time interval ΔT, calculate $(I_a^2)_{av}$ by integrating I_a^2 from $t = 0$ to $t = \Delta T$ and dividing by ΔT:

$$(I_a^2)_{av} = \frac{1}{\Delta T}\int_0^{\Delta T}\frac{(4.0\text{ A})^2}{(\Delta T)^2}t^2 dt = \frac{(4.0\text{ A})^2}{(\Delta T)^3}\left[\frac{t^3}{3}\right]_0^{\Delta T} = \frac{16}{3}\text{A}^2$$

Substitute in the expression for $I_{\text{rms},a}$ to obtain:

$$I_{\text{rms},a} = \sqrt{\frac{16}{3}\text{A}^2} = \boxed{2.3\text{ A}}$$

Waveform (*b*) Noting that the current during the second half of each cycle is zero, express the current during the first half cycle of the time interval $\frac{1}{2}\Delta T$:

$$I_b = 4.0\text{ A}$$

Evaluate $I_{\text{av},b}$:

$$I_{\text{av},b} = \frac{4.0\text{ A}}{\Delta T}\int_0^{\frac{1}{2}\Delta T} dt = \frac{4.0\text{ A}}{\Delta T}[t]_0^{\frac{1}{2}\Delta T} = \boxed{2.0\text{ A}}$$

Express the square of the current during this half cycle:

$$I_b^2 = (4.0\text{ A})^2$$

Calculate $(I_b^2)_{av}$ by integrating I_b^2 from $t = 0$ to $t = \frac{1}{2}\Delta T$ and dividing by ΔT:

$$(I_b^2)_{av} = \frac{(4.0\text{ A})^2}{\Delta T}\int_0^{\frac{1}{2}\Delta} dt = \frac{(4.0\text{ A})^2}{\Delta T}[t]_0^{\frac{1}{2}\Delta T} = 8.0\text{ A}^2$$

Substitute in the expression for $I_{\text{rms},b}$ to obtain:

$$I_{\text{rms},b} = \sqrt{8.0\text{ A}^2} = \boxed{2.8\text{ A}}$$

87 •• In the circuit shown in Figure 29-47, $\mathcal{E}_1 = (20\text{ V})\cos 2\pi ft$, where $f = 180$ Hz; $\mathcal{E}_2 = 18$ V, and $R = 36\ \Omega$. Find the maximum, minimum, average, and rms values of the current in the resistor.

Picture the Problem We can apply Kirchhoff's loop rule to express the current in the circuit in terms of the emfs of the sources and the resistance of the resistor. We can then find $I_{\max}$ and $I_{\min}$ by considering the conditions under which the time-dependent factor in I will be a maximum or a minimum. Finally, we can use

$I_{\text{rms}} = \sqrt{(I^2)_{\text{av}}}$ to derive an expression for I_{rms} that we can use to determine its value.

Apply Kirchhoff's loop rule to obtain: $\mathcal{E}_{1,\text{peak}} \cos\omega t + \mathcal{E}_2 - IR = 0$

Solving for I yields:
$$I = \frac{\mathcal{E}_{1,\text{peak}}}{R}\cos\omega t + \frac{\mathcal{E}_2}{R}$$
or
$$I = A_1 \cos\omega t + A_2$$
where $A_1 = \dfrac{\mathcal{E}_{1,\text{peak}}}{R}$ and $A_2 = \dfrac{\mathcal{E}_2}{R}$

Substitute numerical values to obtain:
$$\begin{aligned} I &= \left(\frac{20\,\text{V}}{36\,\Omega}\right)\cos\left(2\pi\left(180\,\text{s}^{-1}\right)t\right) + \frac{18\,\text{V}}{36\,\Omega} \\ &= (0.556\,\text{A})\cos\left(1131\,\text{s}^{-1}\right)t + 0.50\,\text{A} \end{aligned}$$

The current is a maximum when $\cos\left(1131\,\text{s}^{-1}\right)t = 1$. Hence:
$$I_{\max} = 0.50\,\text{A} + 0.556\,\text{A} = \boxed{1.06\,\text{A}}$$

Evaluate $I_{\min}$:
$$I_{\min} = 0.50\,\text{A} - 0.556\,\text{A} = \boxed{-0.06\,\text{A}}$$

Because the average value of $\cos\omega t = 0$:
$$I_{\text{av}} = \boxed{0.50\,\text{A}}$$

The rms current is the square root of the average of the squared current:
$$I_{\text{rms}} = \sqrt{\left[I^2\right]_{\text{av}}} \qquad (1)$$

$\left[I^2\right]_{\text{av}}$ is given by:
$$\begin{aligned} \left[I^2\right]_{\text{av}} &= \left[(A_1 \cos\omega t + A_2)^2\right]_{\text{av}} \\ &= \left[A_1^2 \cos^2 \omega t + 2A_1A_2 \cos\omega t + A_2^2\right]_{\text{av}} \\ &= \left[A_1^2 \cos^2 \omega t\right]_{\text{av}} + \left[2A_1A_2 \cos\omega t\right]_{\text{av}} + \left[A_2^2\right]_{\text{av}} \\ &= A_1^2\left[\cos^2 \omega t\right]_{\text{av}} + 2A_1A_2\left[\cos\omega t\right]_{\text{av}} + A_2^2 \end{aligned}$$

Because $\left[\cos^2 \omega t\right]_{\text{av}} = \frac{1}{2}$ and $\left[\cos\omega t\right]_{\text{av}} = 0$:
$$\left[I^2\right]_{\text{av}} = \tfrac{1}{2}A_1^2 + A_2^2$$

Substituting in equation (1) yields:
$$I_{\text{rms}} = \sqrt{\tfrac{1}{2}A_1^2 + A_2^2}$$

Substitute for A_1 and A_2 to obtain:

$$I_{\text{rms}} = \sqrt{\tfrac{1}{2}\left(\frac{\mathcal{E}_1}{R}\right)^2 + \left(\frac{\mathcal{E}_2}{R}\right)^2}$$

Substitute numerical values and evaluate I_{rms}:

$$I_{\text{rms}} = \sqrt{\tfrac{1}{2}\left(\frac{20\text{ V}}{36\,\Omega}\right)^2 + \left(\frac{18\text{ V}}{36\,\Omega}\right)^2} = \boxed{0.64\text{ A}}$$

88 •• Repeat Problem 87 if the resistor is replaced by a 2.0-μF capacitor.

Picture the Problem We can apply Kirchhoff's loop rule to obtain an expression for charge on the capacitor as a function of time. Differentiating this expression with respect to time will give us the current in the circuit. We can then find I_{max} and I_{min} by considering the conditions under which the time-dependent factor in I will be a maximum or a minimum. Finally, we can use $I_{\text{rms}} = \sqrt{\left(I^2\right)_{\text{av}}}$ to derive an expression for I_{rms} that we can use to determine its value.

Apply Kirchhoff's loop rule to obtain:

$$\mathcal{E}_{1\,\text{peak}}\cos\omega t + \mathcal{E}_2 - \frac{q(t)}{C} = 0$$

Solving for $q(t)$ yields:

$$q(t) = C\left(\mathcal{E}_{1\,\text{peak}}\cos\omega t + \mathcal{E}_2\right) = A_1\cos\omega t + A_2$$

where

$A_1 = C\mathcal{E}_{1\,\text{peak}}$ and $A_2 = C\mathcal{E}_2$

Differentiate this expression with respect to t to obtain the current as a function of time:

$$I = \frac{dq}{dt} = \frac{d}{dt}\left(A_1\cos\omega t + A_2\right) = -\omega A_1\sin\omega t$$

Substituting numerical values yields:

$$I = -2\pi(180\text{ Hz})(2.0\;\mu\text{F})\sin\left(2\pi(180\text{ Hz})t\right) = (-2.26\text{ mA})\sin\left(1131\text{ s}^{-1}\right)t$$

The current is a minimum when $\sin\left(1131\text{s}^{-1}\right)t = 1$. Hence:

$$I_{\text{min}} = \boxed{-2.3\text{mA}}$$

The current is a maximum when $\sin\left(1131\text{s}^{-1}\right)t = -1$. Hence:

$$I_{\text{max}} = \boxed{2.3\text{mA}}$$

Because the dc source sees the capacitor as an open circuit and the average value of the sine function over a period is zero:

$$I_{av} = \boxed{0}$$

The rms current is the square root of the average of the squared current:

$$I_{rms} = \sqrt{\left[I^2\right]_{av}} \qquad (1)$$

$\left[I^2\right]_{av}$ is given by:

$$\begin{aligned}\left[I^2\right]_{av} &= \left[(A_1 \cos\omega t + A_2)^2\right]_{av}\\ &= \left[A_1^2 \cos^2 \omega t + 2A_1A_2 \cos\omega t + A_2^2\right]_{av}\\ &= \left[A_1^2 \cos^2 \omega t\right]_{av} + \left[2A_1A_2 \cos\omega t\right]_{av} + \left[A_2^2\right]_{av}\\ &= A_1^2\left[\cos^2 \omega t\right]_{av} + 2A_1A_2\left[\cos\omega t\right]_{av} + A_2^2\end{aligned}$$

Because $\left[\cos^2 \omega t\right]_{av} = \frac{1}{2}$ and $\left[\cos\omega t\right]_{av} = 0$:

$$\left[I^2\right]_{av} = \tfrac{1}{2}A_1^2 + A_2^2$$

Substituting in equation (1) yields:

$$I_{rms} = \sqrt{\tfrac{1}{2}A_1^2 + A_2^2}$$

Substitute for A_1 and A_2 to obtain:

$$\begin{aligned}I_{rms} &= \sqrt{\tfrac{1}{2}(C\mathcal{E}_1)^2 + (C\mathcal{E}_2)^2}\\ &= C\sqrt{\tfrac{1}{2}(\mathcal{E}_1)^2 + (\mathcal{E}_2)^2}\end{aligned}$$

Substitute numerical values and evaluate I_{rms}:

$$\begin{aligned}I_{rms} &= (2.0\ \mu\text{F})\sqrt{\tfrac{1}{2}(20\ \text{V})^2 + (18\ \text{V})^2}\\ &= \boxed{46\ \mu\text{A}}\end{aligned}$$

89 ••• [SSM] A circuit consists of an ac generator, a capacitor and an ideal inductor—all connected in series. The emf of the generator is given by $\mathcal{E}_{peak} \cos\omega t$. (*a*) Show that the charge on the capacitor obeys the equation $L\dfrac{d^2Q}{dt^2} + \dfrac{Q}{C} = \mathcal{E}_{peak} \cos\omega t$. (*b*) Show by direct substitution that this equation is satisfied by $Q = Q_{peak} \cos\omega t$ where $Q_{peak} = -\dfrac{\mathcal{E}_{peak}}{L\left(\omega^2 - \omega_0^2\right)}$. (*c*) Show that the current can be written as $I = I_{peak} \cos(\omega t - \delta)$, where $I_{peak} = \dfrac{\omega\mathcal{E}_{peak}}{L\left|\omega^2 - \omega_0^2\right|} = \dfrac{\mathcal{E}_{peak}}{\left|X_L - X_C\right|}$ and $\delta = -90^\circ$ for $\omega < \omega_0$ and $\delta = 90^\circ$ for $\omega > \omega_0$, where ω_0 is the resonance frequency.

Picture the Problem In Part (a) we can apply Kirchhoff's loop rule to obtain the 2nd order differential equation relating the charge on the capacitor to the time. In Part (b) we'll assume a solution of the form $Q = Q_{\text{peak}} \cos \omega t$, differentiate it twice, and substitute for d^2Q/dt^2 and Q to show that the assumed solution satisfies the differential equation provided $Q_{\text{peak}} = -\dfrac{\mathcal{E}_{\text{peak}}}{L\left(\omega^2 - \omega_0^2\right)}$. In Part ($c$) we'll use our results from (a) and (b) to establish the result for I_{peak} given in the problem statement.

(a) Apply Kirchhoff's loop rule to obtain:

$$\mathcal{E} - \frac{Q}{C} - L\frac{dI}{dt} = 0$$

Substitute for $\mathcal{E}$ and rearrange the differential equation to obtain:

$$L\frac{dI}{dt} + \frac{Q}{C} = \mathcal{E}_{\text{max}} \cos \omega t$$

Because $I = dQ/dt$:

$$\boxed{L\frac{d^2Q}{dt^2} + \frac{Q}{C} = \mathcal{E}_{\text{max}} \cos \omega t}$$

(b) Assume that the solution is:

$$Q = Q_{\text{peak}} \cos \omega t$$

Differentiate the assumed solution twice to obtain:

$$\frac{dQ}{dt} = -\omega Q_{\text{peak}} \sin \omega t$$

and

$$\frac{d^2Q}{dt^2} = -\omega^2 Q_{\text{peak}} \cos \omega t$$

Substitute for $\dfrac{dQ}{dt}$ and $\dfrac{d^2Q}{dt^2}$ in the differential equation to obtain:

$$-\omega^2 L Q_{\text{peak}} \cos \omega t + \frac{Q_{\text{peak}}}{C} \cos \omega t = \mathcal{E}_{\text{peak}} \cos \omega t$$

Factor $\cos\omega t$ from the left-hand side of the equation:

$$\left(-\omega^2 L Q_{\text{peak}} + \frac{Q_{\text{peak}}}{C}\right) \cos \omega t = \mathcal{E}_{\text{peak}} \cos \omega t$$

If this equation is to hold for all values of t it must be true that:

$$-\omega^2 L Q_{\text{peak}} + \frac{Q_{\text{peak}}}{C} = \mathcal{E}_{\text{peak}}$$

Solving for Q_{peak} yields:

$$Q_{\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{-\omega^2 L + \dfrac{1}{C}}$$

Factor L from the denominator and substitute for $1/LC$ to obtain:

$$Q_{\text{peak}} = \frac{\mathcal{E}_{\text{peak}}}{L\left(-\omega^2 + \dfrac{1}{LC}\right)} = \boxed{-\frac{\mathcal{E}_{\text{peak}}}{L\left(\omega^2 - \omega_0^2\right)}}$$

(c) From (a) and (b) we have:

$$I = \frac{dQ}{dt} = -\omega Q_{\text{peak}} \sin \omega t$$

$$= \frac{\omega \mathcal{E}_{\text{peak}}}{L\left(\omega^2 - \omega_0^2\right)} \sin \omega t = I_{\text{peak}} \sin \omega t$$

$$= \boxed{I_{\text{peak}} \cos(\omega t - \delta)}$$

where

$$I_{\text{peak}} = \boxed{\frac{\omega \mathcal{E}_{\text{peak}}}{L\left|\omega^2 - \omega_0^2\right|}} = \frac{\mathcal{E}_{\text{peak}}}{\dfrac{L}{\omega}\left|\omega^2 - \omega_0^2\right|}$$

$$= \frac{\mathcal{E}_{\text{peak}}}{\left|\omega L - \dfrac{1}{\omega C}\right|} = \boxed{\frac{\mathcal{E}_{\text{peak}}}{\left|X_L - X_C\right|}}$$

If $\omega > \omega_0$, $X_L > X_C$ and the current lags the voltage by 90° ($\delta = 90°$).

If $\omega < \omega_0$, $X_L < X_C$ and the current leads the voltage by 90°($\delta = -90°$).

Chapter 30
Maxwell's Equations and Electromagnetic Waves

Conceptual Problems

1 • **[SSM]** True or false:

(*a*) The displacement current has different units than the conduction current.
(*b*) Displacement current only exists if the electric field in the region is changing with time.
(*c*) In an oscillating *LC* circuit, no displacement current exists between the capacitor plates when the capacitor is momentarily fully charged.
(*d*) In an oscillating *LC* circuit, no displacement current exists between the capacitor plates when the capacitor is momentarily uncharged.

(*a*) False. Like those of conduction current, the units of displacement current are C/s.

(*b*) True. Because displacement current is given by $I_d = \epsilon_0\, d\phi_e/dt$, I_d is zero if $d\phi_e/dt = 0$.

(*c*) True. When the capacitor is fully charged, the electric flux is momentarily a maximum (its rate of change is zero) and, consequently, the displacement current between the plates of the capacitor is zero.

(*d*) False. I_d is zero if $d\phi_e/dt = 0$. At the moment when the capacitor is momentarily uncharged, $dE/dt \neq 0$ and so $d\phi_e/dt \neq 0$.

2 • Using SI units, show that $I_d = \epsilon_0\, d\phi_e/dt$ has units of current.

Determine the Concept We need to show that $\epsilon_0\, d\phi_e/dt$ has units of amperes. We can accomplish this by substituting the SI units of ϵ_0 and $d\phi_e/dt$ and simplifying the resulting expression.

$$\frac{\text{C}^2}{\text{N}\cdot\text{m}}\cdot\frac{\frac{\text{N}}{\text{C}}\cdot\text{m}^2}{\text{s}} = \frac{\text{C}}{\text{s}} = \boxed{\text{A}}$$

3 • **[SSM]** True or false:

(*a*) Maxwell's equations apply only to electric and magnetic fields that are constant over time.
(*b*) The electromagnetic wave equation can be derived from Maxwell's equations.

(*c*) Electromagnetic waves are transverse waves.
(*d*) The electric and magnetic fields of an electromagnetic wave in free space are in phase.

(*a*) False. Maxwell's equations apply to both time-independent and time-dependent fields.

(*b*) True. One can use Faraday's law and the modified version of Ampere's law to derive the wave equation.

(*c*) True. Both the electric and magnetic fields of an electromagnetic wave oscillate at right angles to the direction of propagation of the wave.

(*d*) True.

4 • Theorists have speculated about the existence of *magnetic monopoles*, and several experimental searches for such monopoles have occurred. Suppose magnetic monopoles were found and that the magnetic field at a distance r from a monopole of strength q_m is given by $B = (\mu_0/4\pi)q_m/r^2$. Modify the Gauss's law for magnetism equation to be consistent with such a discovery.

Determine the Concept Gauss's law for magnetism would become $\boxed{\oint_S B_n dA = \mu_0 q_{\text{m, inside}}}$ where $q_{\text{m, inside}}$ is the total magnetic charge inside the Gaussian surface. Note that Gauss's law for electricity follows from the existence of electric monopoles (charges), and the electric field due to a point charge follows from the inverse-square nature of Coulomb's law.

5 • (*a*) For each of the following pairs of electromagnetic waves, which has the higher frequency: (1) visible light or X rays, (2) green light or red light, (3) infrared waves or red light. (*b*) For each of the following pairs of electromagnetic waves, which has the longer wavelength: (1) visible light or microwaves, (2) green light or ultraviolet light, (3) gamma rays or ultraviolet light.

Determine the Concept Refer to Table 30-1 to rank order the frequencies and wavelengths of the given electromagnetic radiation.

(*a*) (1) X rays (2) green light (3) red light

(*b*) (1) microwaves (2) green light (3) ultraviolet light

6 • The detection of radio waves can be accomplished with either an electric dipole antenna or a loop antenna. True or false:

(*a*) The electric dipole antenna works according to Faraday's law.
(*b*) If a linearly polarized radio wave is approaching you head on such that its electric field oscillates vertically, to best detect this wave the normal to a loop antenna's plane should be oriented so that it points either right or left.
(*c*) If a linearly polarized radio wave is approaching you such that its electric field oscillates in a horizontal plane, to best detect this wave using an dipole antenna the antenna should be oriented vertically.

(*a*) False. A dipole antenna is oriented parallel to the electric field of an incoming wave so that the wave can induce an alternating current in the antenna.

(*b*) True. A loop antenna is oriented perpendicular to the magnetic field of an incoming wave so that the changing magnetic flux through the loop can induce a current in the loop. Orienting the loop antenna's plane so that it points either right or left satisfies this condition.

(*c*) False. The dipole antenna needs to be oriented parallel to the electric field of an incoming wave so that the wave can induce an alternating current in the antenna.

7 • A transmitter emits electromagnetic waves using an electric dipole antenna oriented vertically. (*a*) A receiver to detect these waves also uses an electric dipole antenna that is one mile from the transmitting antenna and at the same altitude. How should the receiver's electric dipole antenna be oriented for optimum signal reception? (*b*) A receiver to detect these waves uses a loop antenna that is one mile from the transmitting antenna and at the same altitude. How should the loop antenna be oriented for optimum signal reception?

Determine the Concept
(*a*) The electric dipole antenna should be oriented vertically.

(*b*) The loop antenna and the electric dipole transmitting antenna should be in the same vertical plane.

8 • Show that the expression $\vec{E}\times\vec{B}/\mu_0$ for the Poynting vector $\vec{S}$ (Equation 30-21) has units of watts per square meter (the SI units for electromagnetic wave intensity).

Determine the Concept We can that $\vec{E}\times\vec{B}/\mu_0$ has units of W/m^2 by substituting the SI units of $\vec{E}$, $\vec{B}$ and μ_0 and simplifying the resulting expression.

$$\frac{\frac{\text{N}}{\text{C}}\cdot\text{T}}{\frac{\text{T}\cdot\text{m}}{\text{A}}}=\frac{\frac{\text{N}}{\text{C}}}{\frac{\text{m}}{\text{A}}}=\frac{\text{N}}{\text{C}}\cdot\frac{\text{A}}{\text{m}}=\frac{\text{N}\cdot\text{m}}{\text{C}}\cdot\frac{\frac{\text{C}}{\text{s}}}{\text{m}^2}=\frac{\frac{\text{N}\cdot\text{m}}{\text{s}}}{\text{m}^2}=\frac{\frac{\text{J}}{\text{s}}}{\text{m}^2}=\boxed{\frac{\text{W}}{\text{m}^2}}$$

9 • [SSM] If a red light beam, a green light beam, and a violet light beam, all traveling in empty space, have the same intensity, which light beam carries more momentum? (*a*) the red light beam, (*b*) the green light beam, (*c*) the violet light beam, (*d*) They all have the same momentum. (*e*) You cannot determine which beam carries the most momentum from the data given.

Determine the Concept The momentum of an electromagnetic wave is directly proportional to its energy ($p=U/c$). Because the intensity of a wave is its energy per unit area and per unit time (the average value of its Poynting vector), waves with equal intensity have equal energy and equal momentum. $\boxed{(d)}$ is correct.

10 • If a red light plane wave, a green light plane wave, and a violet light plane wave, all traveling in empty space, have the same intensity, which wave has the largest peak electric field? (*a*) the red light wave, (*b*) the green light wave, (*c*) the violet light wave, (*d*) They all have the same peak electric field. (*e*) You cannot determine the largest peak electric field from the data given.

Determine the Concept The intensity of an electromagnetic wave is given by $I=\left|\vec{S}\right|_{\text{av}}=\dfrac{E_0B_0}{2\mu_0}$.

The intensity of an electromagnetic wave is given by:

$$I=\left|\vec{S}\right|_{\text{av}}=\frac{E_0B_0}{2\mu_0}$$

Because $E_0 = cB_0$:

$$\left|\vec{S}\right|_{\text{av}}=\frac{E_0^2}{2c\mu_0}$$

This result tells us that $\left|\vec{S}\right|_{\text{av}}\propto E_0^2$ independently of the wavelength of the electromagnetic radiation. Thus $\boxed{(d)}$ is correct.

11 • Two sinusoidal plane electromagnetic waves are identical except that wave A has a peak electric field that is three times the peak electric field of wave B. How do their intensities compare? (*a*) $I_{\text{A}}=\frac{1}{3}I_{\text{B}}$ (*b*) $I_{\text{A}}=\frac{1}{9}I_{\text{B}}$ (*c*) $I_{\text{A}}=3I_{\text{B}}$ (*d*) $I_{\text{A}}=9I_{\text{B}}$ (*e*) You cannot determine how their intensities compare from the data given.

Determine the Concept The intensity of an electromagnetic wave is given by $I=\left|\vec{S}\right|_{\text{av}}=\dfrac{E_0B_0}{2\mu_0}$.

Express the intensities of the two waves:

$$I_{\text{A}}=\frac{E_{0,\text{A}}B_{0,\text{A}}}{2\mu_0}\text{ and }I_{\text{B}}=\frac{E_{0,\text{B}}B_{0,\text{B}}}{2\mu_0}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{I_{\text{A}}}{I_{\text{B}}}=\frac{\dfrac{E_{0,\text{A}}B_{0,\text{A}}}{2\mu_0}}{\dfrac{E_{0,\text{B}}B_{0,\text{B}}}{2\mu_0}}=\frac{E_{0,\text{A}}B_{0,\text{A}}}{E_{0,\text{B}}B_{0,\text{B}}}$$

Because wave A has a peak electric field that is three times that of wave B, the peak magnetic field of A is also three times that of wave B. Hence:

$$\frac{I_{\text{A}}}{I_{\text{B}}}=\frac{(3E_{0,\text{B}})(3B_{0,\text{B}})}{E_{0,\text{B}}B_{0,\text{B}}}=9\Rightarrow I_{\text{A}}=9I_{\text{B}}$$

$\boxed{(d)}$ is correct.

Estimation and Approximation

12 •• In *laser cooling and trapping*, the forces associated with radiation pressure are used to slow down atoms from thermal speeds of hundreds of meters per second at room temperature to speeds of a few meters per second or slower. An isolated atom will absorb only radiation of specific frequencies. If the frequency of the laser-beam radiation is tuned so that the target atoms will absorb the radiation, then the radiation is absorbed during a process called *resonant absorption*. The cross-sectional area of the atom for resonant absorption is approximately equal to λ^2, where λ is the wavelength of the laser light. (*a*) Estimate the acceleration of a rubidium atom (molar mass 85 g/mol) in a laser beam whose wavelength is 780 nm and intensity is 10 W/m^2. (*b*) About how long would it take such a light beam to slow a rubidium atom in a gas at room temperature (300 K) to near-zero speed?

Picture the Problem We can use Newton's second law to express the acceleration of an atom in terms of the net force acting on the atom and the relationship between radiation pressure and the intensity of the beam to find the net force. Once we know the acceleration of an atom, we can use the definition of acceleration to find the stopping time for a rubidium atom at room temperature.

(*a*) Apply Newton's second law to the atom to obtain:

$$F_{\text{r}}=ma \qquad (1)$$

where F_{r} is the radiation force exerted by the laser beam.

The radiation pressure P_r and intensity of the beam I are related according to:

$$P_r = \frac{F_r}{A} = \frac{I}{c}$$

Solve for F_r to obtain:

$$F_r = \frac{IA}{c} = \frac{I\lambda^2}{c}$$

Substitute for F_r in equation (1) to obtain:

$$\frac{I\lambda^2}{c} = ma \Rightarrow a = \frac{I\lambda^2}{mc}$$

Substitute numerical values and evaluate a:

$$a = \frac{\left(10\,\text{W/m}^2\right)\left(780\,\text{nm}\right)^2}{\left(85\frac{\text{g}}{\text{mol}} \times \frac{1\,\text{mol}}{6.022\times10^{23}\,\text{particles}}\right)\left(2.998\times10^8\,\text{m/s}\right)} = 1.44\times10^5\,\text{m/s}^2$$
$$= \boxed{1.4\times10^5\,\text{m/s}^2}$$

(*b*) Using the definition of acceleration, express the stopping time Δt of the atom:

$$\Delta t = \frac{v_{\text{final}} - v_{\text{initial}}}{a}$$

Because $v_{\text{final}} \approx 0$:

$$\Delta t \approx \frac{-v_{\text{initial}}}{a}$$

Using the rms speed as the initial speed of an atom, relate v_{initial} to the temperature of the gas:

$$v_{\text{initial}} = v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

Substitute in the expression for the stopping time to obtain:

$$\Delta t = -\frac{1}{a}\sqrt{\frac{3kT}{m}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = -\frac{1}{-1.44\times10^5\,\text{m/s}^2}\sqrt{\frac{3\left(1.38\times10^{-23}\,\text{J/K}\right)\left(300\,\text{K}\right)}{\left(85\frac{\text{g}}{\text{mol}} \times \frac{1\,\text{mol}}{6.022\times10^{23}\,\text{particles}}\right)}} = \boxed{2.1\,\text{ms}}$$

13 •• **[SSM]** One of the first successful satellites launched by the United States in the 1950s was essentially a large spherical (aluminized) Mylar balloon from which radio signals were reflected. After several orbits around Earth,

scientists noticed that the orbit itself was changing with time. They eventually determined that radiation pressure from the sunlight was causing the orbit of this object to change—a phenomenon not taken into account in planning the mission. Estimate the ratio of the radiation-pressure force by the sunlight on the satellite to the gravitational force by Earth's gravity on the satellite.

Picture the Problem We can use the definition of pressure to express the radiation force on the balloon. We'll assume that the gravitational force on the balloon is approximately its weight at the surface of Earth, that the density of Mylar is approximately that of water and that the area receiving the radiation from the sunlight is the cross-sectional area of the balloon.

The radiation force acting on the balloon is given by:

$$F_r = P_r A$$

where A is the cross-sectional area of the balloon.

Because the radiation from the Sun is reflected, the radiation pressure is twice what it would be if it were absorbed:

$$P_r = \frac{2I}{c}$$

Substituting for P_r and A yields:

$$F_r = \frac{2I\left(\frac{1}{4}\pi d^2\right)}{c} = \frac{\pi d^2 I}{2c}$$

The gravitational force acting on the balloon when it is in a near-Earth orbit is approximately its weight at the surface of Earth:

$$F_g = w_{\text{balloon}} = m_{\text{balloon}}g = \rho_{\text{Mylar}}V_{\text{Mylar}}g$$
$$= \rho_{\text{Mylar}}A_{\text{surface, ballon}}t\,g$$

where t is the thickness of the Mylar skin of the balloon.

Because the surface area of the balloon is $4\pi r^2 = \pi d^2$:

$$F_g = \pi\rho_{\text{Mylar}}d^2 t\,g$$

Express the ratio of the radiation-pressure force to the gravitational force and simplify to obtain:

$$\frac{F_r}{F_g} = \frac{\frac{\pi d^2 I}{2c}}{\pi\rho_{\text{Mylar}}d^2 t\,g} = \frac{I}{2\rho_{\text{Mylar}}t\,gc}$$

Assuming the thickness of the Mylar skin of the balloon to be 1 mm, substitute numerical values and evaluate F_r/F_g:

$$\frac{F_r}{F_g}=\frac{1.35\,\frac{\text{kW}}{\text{m}^2}}{2\left(1.00\times10^3\,\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(1\text{ mm})\left(2.998\times10^8\,\frac{\text{m}}{\text{s}}\right)}\approx\boxed{2\times10^{-7}}$$

14 •• Some science fiction writers have described solar sails that could propel interstellar spaceships. Imagine a giant sail on a spacecraft subjected to radiation pressure from our Sun. (*a*) Explain why this arrangement works better if the sail is highly reflective rather than highly absorptive. (*b*) If the sail is assumed highly reflective, show that the force exerted by the sunlight on the spacecraft is given by $P_S A/(2\pi r^2 c)$ where P_S is the power output of the Sun (3.8×10^{26} W), A is the surface area of the sail, m is the total mass of the spacecraft, r is the distance from the Sun, and c is the speed of light. (Assume the area of the sail is much larger than the area of the spacecraft so that all the force is due to radiation pressure on the sail only.) (*c*) Using a reasonable value for A, compare the force on the spacecraft due to the radiation pressure and the force on the spacecraft due to the gravitational pull of the Sun. Does the result imply that such a system will work? Explain your answer.

Picture the Problem (*b*) We can use the definition of radiation pressure to show that the force exerted by the sunlight on the spacecraft is given by $P_S A/(2\pi r^2 c)$ where P_S is the power output of the Sun (3.8×10^{26} W), A is the surface area of the sail, m is the total mass of the spacecraft, r is the distance from the Sun, and c is the speed of light.

(*a*) If the sail is highly reflective rather than highly absorptive, the radiation force is doubled.

(*b*) Because the sail is highly reflective:

$$F_r=P_r A=\frac{2IA}{c}$$

where A is the area of the sail.

The intensity of the solar radiation on the sail is given by $I=\frac{P_s}{4\pi r^2}$.

Substituting for I yields:

$$F_r=\frac{2P_s A}{4\pi r^2 c}=\boxed{\frac{P_s A}{2\pi r^2 c}}$$

(*c*) Express the ratio of the force on the spacecraft due to the radiation pressure and the force on the spacecraft due the gravitational force of the Sun on the spacecraft:

$$\frac{F_r}{F_g}=\frac{\dfrac{P_S A}{2\pi r^2 c}}{\dfrac{GmM_S}{r^2}}=\frac{P_S A}{2\pi cGmM_S}$$

Assuming a 15-m diameter circular sail and a 500-kg spacecraft (values found using the internet), substitute numerical values and evaluate the ratio of the accelerations:

$$\frac{F_r}{F_g}=\frac{\left(3.8\times10^{26}\ \text{W}\right)\left(\dfrac{\pi}{4}(15\ \text{m})^2\right)}{2\pi\left(2.998\times10^{8}\ \dfrac{\text{m}}{\text{s}}\right)\left(6.673\times10^{-11}\ \dfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)(500\ \text{kg})\left(1.99\times10^{30}\ \text{kg}\right)}$$
$$=\boxed{5.4\times10^{-4}}$$

for A = 177 m^2 and m = 500 kg**.**

This scheme is not likely to work effectively. For any reasonable spacecraft mass, the surface mass density of the sail would to be extremely small (experimental sails have area densities of approximately 3 g/m^2) and the sail would have to be huge. Additionally, unless struts are built into the sail, it would collapse during use.

Maxwell's Displacement Current

15 • [SSM] A parallel-plate capacitor has circular plates and no dielectric between the plates. Each plate has a radius equal to 2.3 cm and the plates are separated by 1.1 mm. Charge is flowing onto the upper plate (and off of the lower plate) at a rate of 5.0 A. (*a*) Find the rate of change of the electric field strength in the region between the plates. (*b*) Compute the displacement current in the region between the plates and show that it equals 5.0 A.

Picture the Problem We can differentiate the expression for the electric field between the plates of a parallel-plate capacitor to find the rate of change of the electric field strength and the definitions of the conduction current and electric flux to compute I_d.

(*a*) Express the electric field strength between the plates of the parallel-plate capacitor:

$$E=\frac{Q}{\epsilon_0 A}$$

Differentiate this expression with respect to time to obtain an expression for the rate of change of the electric field strength:

$$\frac{dE}{dt}=\frac{d}{dt}\left[\frac{Q}{\epsilon_0 A}\right]=\frac{1}{\epsilon_0 A}\frac{dQ}{dt}=\frac{I}{\epsilon_0 A}$$

Substitute numerical values and evaluate dE/dt:

$$\frac{dE}{dt}=\frac{5.0\,\text{A}}{\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\pi\left(0.023\,\text{m}\right)^2}=3.40\times10^{14}\,\text{V/m}\cdot\text{s}$$
$$=\boxed{3.4\times10^{14}\,\text{V/m}\cdot\text{s}}$$

(*b*) Express the displacement current I_d:

$$I_d=\epsilon_0\frac{d\phi_e}{dt}$$

Substitute for the electric flux to obtain:

$$I_d=\epsilon_0\frac{d}{dt}[EA]=\epsilon_0 A\frac{dE}{dt}$$

Substitute numerical values and evaluate I_d:

$$I_d=\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\pi\left(0.023\,\text{m}\right)^2\left(3.40\times10^{14}\,\text{V/m}\cdot\text{s}\right)=\boxed{5.0\,\text{A}}$$

16 • In a region of space, the electric field varies with time as E = (0.050 N/C) sin (ωt), where ω = 2000 rad/s. Find the peak displacement current through a surface that is perpendicular to the electric field and has an area equal to 1.00 m^2.

Picture the Problem We can express the displacement current in terms of the electric flux and differentiate the resulting expression to obtain I_d in terms of dE/dt.

The displacement current I_d is given by:

$$I_d=\epsilon_0\frac{d\phi_e}{dt}$$

Substitute for the electric flux to obtain:

$$I_d=\epsilon_0\frac{d}{dt}[EA]=\epsilon_0 A\frac{dE}{dt}$$

Because $E=\left(0.050\,\text{N/C}\right)\sin 2000t$:

$$I_d=\epsilon_0 A\frac{d}{dt}\left[\left(0.050\,\text{N/C}\right)\sin 2000t\right]$$
$$=\left(2000\,\text{s}^{-1}\right)\epsilon_0 A\left(0.050\,\text{N/C}\right)\cos 2000t$$

I_d will have its maximum value when cos 2000t = 1. Hence:

$$I_{d,max} = \left(2000\,s^{-1}\right)\in_0 A\left(0.050\,N/C\right)$$

Substitute numerical values and evaluate $I_{d,max}$:

$$I_{d,\,max} = \left(2000\,s^{-1}\right)\left(8.854\times10^{-12}\,\frac{C^2}{N\cdot m^2}\right)\left(1.00\,m^2\right)\left(0.050\,\frac{N}{C}\right) = \boxed{0.89\,nA}$$

17 • For Problem 15, show that the magnetic field strength between the plates a distance r from the axis through the centers of both plates is given by $B = (1.9 \times 10^{-3}$ T/m$)r$.

Picture the Problem We can use Ampere's law to a circular path of radius r between the plates and parallel to their surfaces to obtain an expression relating B to the current enclosed by the amperian loop. Assuming that the displacement current is uniformly distributed between the plates, we can relate the displacement current enclosed by the circular loop to the conduction current I.

Apply Ampere's law to a circular path of radius r between the plates and parallel to their surfaces to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = 2\pi rB = \mu_0 I_{\text{enclosed}} = \mu_0 I$$

Assuming that the displacement current is uniformly distributed:

$$\frac{I}{\pi r^2} = \frac{I_d}{\pi R^2} \Rightarrow I = \frac{r^2}{R^2}I_d$$

where R is the radius of the circular plates.

Substituting for I yields:

$$2\pi rB = \frac{\mu_0 r^2}{R^2}I_d \Rightarrow B = \frac{\mu_0 r}{2\pi R^2}I_d$$

Substitute numerical values and evaluate B:

$$B(r) = \frac{\left(4\pi\times10^{-7}\,N/A^2\right)\left(5.0\,A\right)}{2\pi\left(0.023\,m\right)^2}r$$

$$= \boxed{\left(1.9\times10^{-3}\,\frac{T}{m}\right)r}$$

18 •• The capacitors referred to in this problem have only empty space between the plates. (a) Show that a parallel-plate capacitor has a displacement current in the region between its plates that is given by $I_d = C\,dV/dt$, where C is the capacitance and V is the potential difference between the plates. (b) A 5.00-nF

parallel-plate capacitor is connected to an ideal ac generator so the potential difference between the plates is given by $V = V_0 \cos \omega t$, where $V_0 = 3.00$ V and $\omega = 500\pi$ rad/s. Find the displacement current in the region between the plates as a function of time.

Picture the Problem We can use the definitions of the displacement current and electric flux, together with the expression for the capacitance of an air-core-parallel-plate capacitor to show that $I_d = C\,dV/dt$.

(*a*) Use its definition to express the displacement current I_d:

$$I_d = \epsilon_0 \frac{d\phi_e}{dt}$$

Substitute for the electric flux to obtain:

$$I_d = \epsilon_0 \frac{d}{dt}[EA] = \epsilon_0 A \frac{dE}{dt}$$

Because $E = V/d$:

$$I_d = \epsilon_0 A \frac{d}{dt}\left[\frac{V}{d}\right] = \frac{\epsilon_0 A}{d}\frac{dV}{dt}$$

The capacitance of an air-core-parallel-plate capacitor whose plates have area A and that are separated by a distance d is given by:

$$C = \frac{\epsilon_0 A}{d}$$

Substituting yields:

$$I_d = \boxed{C\frac{dV}{dt}}$$

(*b*) Substitute in the expression derived in (*a*) to obtain:

$$I_d = (5.00\,\text{nF})\frac{d}{dt}[(3.00\,\text{V})\cos 500\pi t] = -(5.00\,\text{nF})(3.00\,\text{V})(500\pi\,\text{s}^{-1})\sin 500\pi t$$
$$= \boxed{-(23.6\,\mu\text{A})\sin 500\pi t}$$

19 •• [SSM] There is a current of 10 A in a resistor that is connected in series with a parallel plate capacitor. The plates of the capacitor have an area of 0.50 m^2, and no dielectric exists between the plates. (*a*) What is the displacement current between the plates? (*b*) What is the rate of change of the electric field strength between the plates? (*c*) Find the value of the line integral $\oint_C \vec{B}\cdot d\vec{\ell}$, where the integration path C is a 10-cm-radius circle that lies in a plane that is parallel with the plates and is completely within the region between them.

Picture the Problem We can use the conservation of charge to find I_d, the definitions of the displacement current and electric flux to find dE/dt, and Ampere's law to evaluate $\vec{B}\cdot d\vec{\ell}$ around the given path.

(*a*) From conservation of charge we know that:

$$I_d = I = \boxed{10\,\text{A}}$$

(*b*) Express the displacement current I_d:

$$I_d = \epsilon_0 \frac{d\phi_e}{dt} = \epsilon_0 \frac{d}{dt}[EA] = \epsilon_0 A \frac{dE}{dt}$$

Substituting for dE/dt yields:

$$\frac{dE}{dt} = \frac{I_d}{\epsilon_0 A}$$

Substitute numerical values and evaluate dE/dt:

$$\frac{dE}{dt} = \frac{10\,\text{A}}{\left(8.85\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(0.50\,\text{m}^2\right)} = \boxed{2.3\times10^{12}\,\frac{\text{V}}{\text{m}\cdot\text{s}}}$$

(*c*) Apply Ampere's law to a circular path of radius *r* between the plates and parallel to their surfaces to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I_{\text{enclosed}}$$

Assuming that the displacement current is uniformly distributed and letting *A* represent the area of the circular plates yields:

$$\frac{I_{\text{enclosed}}}{\pi r^2} = \frac{I_d}{A} \Rightarrow I_{\text{enclosed}} = \frac{\pi r^2}{A} I_d$$

Substitute for I_{enclosed} to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = \frac{\mu_0 \pi r^2}{A} I_d$$

Substitute numerical values and evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$:

$$\oint_C \vec{B}\cdot d\vec{\ell} = \frac{\left(4\pi\times10^{-7}\,\text{N/A}^2\right)\pi\left(0.10\,\text{m}\right)^2\left(10\,\text{A}\right)}{0.50\,\text{m}^2} = \boxed{0.79\,\mu\text{T}\cdot\text{m}}$$

20 ••• Demonstrate the validity of the generalized form of Ampère's law (Equation 30-4) by showing that it gives the same result as the Biot–Savart law (Equation 27-3) in a specified situation. Figure 30-13 shows two momentarily

equal but opposite point charges ($+Q$ and $-Q$) on the x axis at $x = -a$ and $x = +a$, respectively. At the same instant there is a current I in the wire connecting them, as shown. Point P is on the y axis at $y = R$. (*a*) Use the Biot–Savart law to show that the magnitude of the magnetic field at point P is given by $B = \dfrac{\mu_0 Ia}{2\pi R}\dfrac{1}{\sqrt{R^2+a^2}}$. (*b*) Now consider a circular strip of radius r and width dr in the $x = 0$ plane that has its center at the origin. Show that the flux of the electric field through this strip is given by $E_x dA = \dfrac{Q\pi r}{\epsilon_0}\left(r^2+a^2\right)^{3/2} dr$. (*c*) Use the result from Part (*b*) to show that the total electric flux ϕ_e through a circular surface S of radius R. is given by $\phi_e = \dfrac{Q}{\epsilon_0}\left(1-\dfrac{a}{\sqrt{a^2+R^2}}\right)$. (*d*) Find the displacement current I_d through S, and show that $I + I_d = I\dfrac{a}{\sqrt{a^2+R^2}}$ (*e*) Finally, show that the generalized form of Ampere's law (Equation 30-4) gives the same result for the magnitude of the magnetic field as found in Part (*a*).

Picture the Problem We can follow the step-by-step instructions in the problem statement to show that Equation 30-4 gives the same result for B as that given in Part (*a*).

(*a*) Express the magnetic field strength at P using the expression for B due to a straight wire segment:

$$B_P = \frac{\mu_0}{4\pi}\frac{I}{R}\left(\sin\theta_1 + \sin\theta_2\right)$$

where

$$\sin\theta_1 = \sin\theta_2 = \frac{a}{\sqrt{R^2+a^2}}$$

Substitute for $\sin\theta_1$ and $\sin\theta_2$ to obtain:

$$B_P = \frac{\mu_0}{4\pi}\frac{I}{R}\frac{2a}{\sqrt{R^2+a^2}} = \boxed{\frac{\mu_0 Ia}{2\pi R}\frac{1}{\sqrt{R^2+a^2}}}$$

(*b*) Express the electric flux through the circular strip of radius r and width dr in the yz plane:

$$d\phi_e = E_x dA = E_x\left(2\pi r dr\right)$$

The electric field due to the dipole is:

$$E_x = \frac{2kQ}{r^2+a^2}\cos\theta_1 = \frac{2kQa}{\left(r^2+a^2\right)^{3/2}}$$

Substitute for E_x to obtain:

$$d\phi_e = E_x dA = \frac{2kQa}{\left(r^2+a^2\right)^{3/2}}\left(2\pi\, rdr\right) = \frac{2Qa}{4\pi \in_0 \left(r^2+a^2\right)^{3/2}}\left(2\pi\, rdr\right) = \boxed{\frac{Qa}{\in_0 \left(r^2+a^2\right)^{3/2}} rdr}$$

(*c*) Multiply both sides of the expression for $d\phi_e$ by $\in_0$:

$$\in_0 d\phi_e = \frac{Qa}{\left(r^2+a^2\right)^{3/2}} rdr$$

Integrate r from 0 to R to obtain:

$$\in_0 \phi_e = Qa\int_0^R \frac{rdr}{\left(r^2+a^2\right)^{3/2}} = Qa\left(\frac{-1}{\sqrt{R^2+a^2}}+\frac{1}{a}\right) = \boxed{Q\left(1-\frac{a}{\sqrt{R^2+a^2}}\right)}$$

(*d*) The displacement current is defined to be:

$$I_d = \in_0 \frac{d\phi_e}{dt} = \frac{d}{dt}\left[Q\left(1-\frac{a}{\sqrt{R^2+a^2}}\right)\right] = \left(1-\frac{a}{\sqrt{R^2+a^2}}\right)\frac{dQ}{dt} = -I\left(1-\frac{a}{\sqrt{R^2+a^2}}\right)$$

The total current is the sum of I and I_d:

$$I + I_d = I - I\left(1-\frac{a}{\sqrt{R^2+a^2}}\right) = \boxed{I\frac{a}{\sqrt{R^2+a^2}}}$$

(*e*) Apply Equation 30-4 (the generalized form of Ampere's law) to obtain:

$$\oint_C \vec{B}\cdot d\vec{\ell} = 2\pi RB = \mu_0\left(I+I_d\right)$$

Solving for B yields:

$$B = \frac{\mu_0}{2\pi R}\left(I+I_d\right)$$

Substitute for $I + I_d$ from (*d*) to obtain:

$$B = \frac{\mu_0}{2\pi R}\left(I\frac{a}{\sqrt{R^2+a^2}}\right) = \boxed{\frac{\mu_0 Ia}{2\pi R}\frac{1}{\sqrt{R^2+a^2}}}$$

Maxwell's Equations and the Electromagnetic Spectrum

21 • The color of the dominant light from the Sun is in the yellow-green region of the visible spectrum. Estimate the wavelength and frequency of the dominant light emitted by our Sun. *HINT: See Table 30-1.*

Picture the Problem We can find both the wavelength and frequency of the dominant light emitted by our Sun in Table 30-1.

Because the radiation from the Sun is yellow-green dominant, the dominant wavelength is approximately:

$$\lambda_{\text{yellow-green}} = \boxed{580\text{ nm}}$$

The corresponding frequency is:

$$f_{\text{yellow-green}} = \frac{c}{\lambda_{\text{yellow-green}}} = \frac{2.998\times10^8\text{ m/s}}{580\text{ nm}} = \boxed{5.17\times10^{14}\text{ Hz}}$$

22 • (*a*) What is the frequency of microwave radiation that has a 3.00-cm-long wavelength? (*b*) Using Table 30-1, estimate the ratio of the shortest wavelength of green light to the shortest wavelength of red light.

Picture the Problem We can use $c = f\lambda$ to find the frequency corresponding to the given wavelength.

(*a*) The frequency of an electromagnetic wave is the ratio of the speed of light in a vacuum to the wavelength of the wave:

$$f = \frac{c}{\lambda}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{2.998\times10^8\text{ m/s}}{3.00\times10^{-2}\text{ m}} = 9.993\times10^{10}\text{ Hz} = \boxed{9.99\text{GHz}}$$

(*b*) The ratio of the shortest wavelength green light to the shortest wavelength red light is:

$$\frac{\lambda_{\text{shortest green}}}{\lambda_{\text{shortest red}}} \approx \frac{520\text{ nm}}{620\text{ nm}} = \boxed{0.84}$$

23 • (*a*) What is the frequency of an X ray that has a 0.100-nm-long wavelength? (*b*) The human eye is sensitive to light that has a wavelength equal to 550 nm. What is the color and frequency of this light? Comment on how this answer compares to your answer for Problem 21.

Picture the Problem We can use $c = f\lambda$ to find the frequency corresponding to the given wavelengths and consult Table 30-1 to determine the color of light with a wavelength of 550 nm.

(*a*) The frequency of an X ray with a wavelength of 0.100 nm is:

$$f = \frac{c}{\lambda} = \frac{2.998\times10^{8}\text{ m/s}}{0.100\times10^{-9}\text{ m}}$$
$$= \boxed{3.00\times10^{18}\text{ Hz}}$$

(*b*) The frequency of light with a wavelength of 550 nm is:

$$f = \frac{2.998\times10^{8}\text{ m/s}}{550\text{nm}} = \boxed{5.45\times10^{14}\text{ Hz}}$$

Consulting Table 30-1, we see that the color of light that has a wavelength of 550 nm is yellow-green. This result is consistent with those of Problem 21 and is close to the wavelength of the peak output of the Sun. Because we see naturally by reflected sunlight, this result is not surprising.

Electric Dipole Radiation

24 •• Suppose a radiating electric dipole lies along the *z* axis. Let I_1 be the intensity of the radiation at a distance of 10 m and at angle of 90º. Find the intensity (in terms of I_1) at (*a*) a distance of 30 m and an angle of 90º, (*b*) a distance of 10 m and an angle of 45º, and (*c*) a distance of 20 m and an angle of 30º.

Picture the Problem We can use the intensity I_1 at a distance $r = 10$ m and at an angle $\theta = 90°$ to find the constant in the expression for the intensity of radiation from an electric dipole and then use the resulting equation to find the intensity at the given distances and angles.

Express the intensity of radiation as a function of *r* and θ:

$$I(\theta, r) = \frac{C}{r^2}\sin^2\theta \qquad (1)$$

where C is a constant.

Express $I(90°,10\text{ m})$:

$$I(90°,10\,\text{m}) = I_1 = \frac{C}{(10\,\text{m})^2}\sin^2 90° = \frac{C}{100\,\text{m}^2}$$

Solving for C yields:

$$C = (100\,\text{m}^2)I_1$$

Substitute in equation (1) to obtain:

$$I(\theta,r) = \frac{(100\,\text{m}^2)I_1}{r^2}\sin^2\theta \qquad (2)$$

(*a*) Evaluate equation (2) for $r = 30$ m and $\theta = 90°$:

$$I(90°,30\,\text{m}) = \frac{(100\,\text{m}^2)I_1}{(30\,\text{m})^2}\sin^2 90° = \boxed{\tfrac{1}{9}I_1}$$

(*b*) Evaluate equation (2) for $r = 10$ m and $\theta = 45°$:

$$I(45°,10\,\text{m}) = \frac{(100\,\text{m}^2)I_1}{(10\,\text{m})^2}\sin^2 45° = \boxed{\tfrac{1}{2}I_1}$$

(*c*) Evaluate equation (2) for $r = 20$ m and $\theta = 30°$:

$$I(30°,20\,\text{m}) = \frac{(100\,\text{m}^2)I_1}{(20\,\text{m})^2}\sin^2 30° = \boxed{\tfrac{1}{16}I_1}$$

25 •• (*a*) For the situation described in Problem 24, at what angle is the intensity at a distance of 5.0 m equal to I_1? (*b*) At what distance is the intensity equal to I_1 when $\theta = 45°$?

Picture the Problem We can use the intensity I_1 at a distance $r = 10$ m and at an angle $\theta = 90°$ to find the constant in the expression for the intensity of radiation from an electric dipole and then use the resulting equation to find the angle for a given intensity and distance and the distance corresponding to a given intensity and angle.

Express the intensity of radiation as a function of r and θ:

$$I(\theta,r) = \frac{C}{r^2}\sin^2\theta \qquad (1)$$

where C is a constant.

Express $I(90°,10\text{ m})$:

$$I(90°,10\,\text{m}) = I_1 = \frac{C}{(10\,\text{m})^2}\sin^2 90° = \frac{C}{100\,\text{m}^2}$$

Solving for C yields:

$$C = (100\,\text{m}^2)I_1$$

Substitute in equation (1) to obtain:

$$I(\theta,r) = \frac{(100\,\text{m}^2)I_1}{r^2}\sin^2\theta \qquad (2)$$

(*a*) For $r = 5$ m and $I(\theta,r) = I_1$:

$$I_1 = \frac{(100\,\text{m}^2)I_1}{(5.0\,\text{m})^2}\sin^2\theta \Rightarrow \sin^2\theta = \tfrac{1}{4}$$

Solve for θ to obtain:

$$\theta = \sin^{-1}\left(\tfrac{1}{2}\right) = \boxed{30°}$$

(*b*) For $\theta = 45°$ and $I(\theta,r) = I_1$:

$$I_1 = \frac{(100\,\text{m}^2)I_1}{r^2}\sin^2 45°$$

or

$$r^2 = \tfrac{1}{2}(100\,\text{m}^2)$$

Solve for r to obtain:

$$r = \sqrt{\tfrac{1}{2}(100\,\text{m}^2)} = \boxed{7.1\,\text{m}}$$

26 •• You and your engineering crew are in charge of setting up a wireless telephone network for a village in a mountainous region. The transmitting antenna of one station is an electric dipole antenna located atop a mountain 2.00 km above sea level. There is a nearby mountain that is 4.00 km from the antenna and is also 2.00 km above sea level. At that location, one member of the crew measures the intensity of the signal to be $4.00 \times 10^{-12}\ \text{W/m}^2$. What should be the intensity of the signal at the village that is located at sea level and 1.50 km from the transmitter?

Picture the Problem We can use the intensity I at a distance $r = 4.00$ km and at an angle $\theta = 90°$ to find the constant in the expression for the intensity of radiation from an electric dipole and then use the resulting equation to find the intensity at sea level and 1.50 km from the transmitter.

Express the intensity of radiation as a function of r and θ:

$$I(\theta,r) = \frac{C}{r^2}\sin^2\theta \qquad (1)$$

where C is a constant.

Use the given data to obtain:

$$4\times10^{-12}\ \text{W/m}^2 = \frac{C}{(4.00\,\text{km})^2}\sin^2 90° = \frac{C}{(4.00\,\text{km})^2}$$

Solving for C yields:

$$C = (4.00\,\text{km})^2\left(4.00\times10^{-12}\ \text{W/m}^2\right) = 6.40\times10^{-5}\ \text{W}$$

Substitute in equation (1) to obtain:

$$I(\theta,r) = \frac{6.40\times10^{-5}\ \text{W}}{r^2}\sin^2\theta \quad (2)$$

For a point at sea level and 1.50 km from the transmitter:

$$\theta = \tan^{-1}\left(\frac{2.00\,\text{km}}{1.50\,\text{km}}\right) = 53.1°$$

Evaluate I(53.1°,1.50 km):

$$I(53.1°,1.5\,\text{km}) = \frac{6.40\times10^{-5}\ \text{W}}{(1.50\,\text{km})^2}\sin^2 53.1° = \boxed{18.2\,\text{pW/m}^2}$$

27 ••• [SSM] A radio station that uses a vertical electric dipole antenna broadcasts at a frequency of 1.20 MHz and has a total power output of 500 kW. Calculate the intensity of the signal at a horizontal distance of 120 km from the station.

Picture the Problem The intensity of radiation from an electric dipole is given by $C(\sin^2\theta)/r^2$, where C is a constant whose units are those of power, r is the distance from the dipole to the point of interest, and θ is the angle between the antenna and the position vector $\vec{r}$. We can integrate the intensity to express the total power radiated by the antenna and use this result to evaluate C. Knowing C we can find the intensity at a horizontal distance of 120 km.

Express the intensity of the signal as a function of r and θ:

$$I(r,\theta) = C\frac{\sin^2\theta}{r^2}$$

At a horizontal distance of 120 km from the station:

$$I(120\,\text{km},90°) = C\frac{\sin^2 90°}{(120\,\text{km})^2} = \frac{C}{(120\,\text{km})^2} \quad (1)$$

From the definition of intensity we have:

$$dP = IdA$$

and

$$P_{\text{tot}} = \iint I(r,\theta)\,dA$$

where, in polar coordinates,

$$dA = r^2 \sin\theta\, d\theta\, d\phi$$

Substitute for dA to obtain:

$$P_{\text{tot}} = \int_0^{2\pi}\int_0^{\pi} I(r,\theta) r^2 \sin\theta\, d\theta\, d\phi$$

Substitute for $I(r,\theta)$:

$$P_{\text{tot}} = C\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi$$

From integral tables we find that:

$$\int_0^{\pi} \sin^3\theta d\theta = -\tfrac{1}{3}\cos\theta\left(\sin^2\theta + 2\right)\Big]_0^{\pi} = \frac{4}{3}$$

Substitute and integrate with respect to ϕ to obtain:

$$P_{\text{tot}} = \frac{4}{3}C\int_0^{2\pi} d\phi = \frac{4}{3}C[\phi]_0^{2\pi} = \frac{8\pi}{3}C$$

Solving for C yields:

$$C = \frac{3}{8\pi}P_{\text{tot}}$$

Substitute for P_{tot} and evaluate C to obtain:

$$C = \frac{3}{8\pi}(500\,\text{kW}) = 59.68\,\text{kW}$$

Substituting for C in equation (1) and evaluating I(120 km, 90°):

$$I(120\,\text{km}, 90°) = \frac{59.68\,\text{kW}}{(120\,\text{km})^2} = \boxed{4.14\,\mu\text{W/m}^2}$$

28 ••• Regulations require that licensed radio stations have limits on their broadcast power so as to avoid interference with signals from distant stations. You are in charge of checking compliance with the law. At a distance of 30.0 km from a radio station that broadcasts from a single vertical electric dipole antenna at a frequency of 800 kHz, the intensity of the electromagnetic wave is 2.00×10^{-13} W/m^2. What is the total power radiated by the station?

Picture the Problem The intensity of radiation from an electric dipole is given by $C(\sin^2\theta)/r^2$, where C is a constant whose units are those of power, r is the distance from the dipole to the point of interest, and θ is the angle between the electric dipole moment and the position vector $\vec{r}$. We can integrate the intensity to express

the total power radiated by the antenna and use this result to evaluate C. Knowing C we can find the total power radiated by the station.

From the definition of intensity we have:

$$dP = IdA$$

and

$$P_{\text{tot}} = \iint I(r,\theta)\,dA$$

where, in polar coordinates,

$$dA = r^2 \sin\theta\, d\theta\, d\phi$$

Substitute for dA to obtain:

$$P_{\text{tot}} = \int_0^{2\pi}\int_0^{\pi} I(r,\theta) r^2 \sin\theta\, d\theta\, d\phi \quad (1)$$

Express the intensity of the signal as a function of r and θ:

$$I(r,\theta) = C\frac{\sin^2\theta}{r^2} \quad (2)$$

Substitute for $I(r,\theta)$ in equation (1) to obtain:

$$P_{\text{tot}} = C\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi$$

From integral tables we find that:

$$\int_0^{\pi} \sin^3\theta d\theta = -\tfrac{1}{3}\cos\theta\left(\sin^2\theta + 2\right)\Big]_0^{\pi} = \frac{4}{3}$$

Substitute and integrate with respect to ϕ to obtain:

$$P_{\text{tot}} = \frac{4}{3}C\int_0^{2\pi} d\phi = \frac{4}{3}C[\phi]_0^{2\pi} = \frac{8\pi}{3}C$$

From equation (2) we have:

$$C = \frac{I(r,\theta)r^2}{\sin^2\theta}$$

Substitute for C in the expression for P_{tot} to obtain:

$$P_{\text{tot}} = \frac{8\pi}{3}\frac{I(r,\theta)r^2}{\sin^2\theta}$$

or, because $\theta = 90°$,

$$P_{\text{tot}} = \frac{8\pi}{3} I(r) r^2$$

Substitute numerical values and evaluate P_{tot}:

$$P_{\text{tot}} = \frac{8\pi}{3}\left(2.00\times10^{-13}\ \text{W/m}^2\right)\left(30.0\,\text{km}\right)^2 = \boxed{1.51\,\text{mW}}$$

29 ••• A small private plane approaching an airport is flying at an altitude of 2.50 km above sea level. As a flight controller at the airport, you know your

system uses a vertical electric dipole antenna to transmit 100 W at 24.0 MHz. What is the intensity of the signal at the plane's receiving antenna when the plane is 4.00 km from the airport? Assume the airport is at sea level.

Picture the Problem The intensity of radiation from the airport's vertical dipole antenna is given by $C(\sin^2\theta)/r^2$, where C is a constant whose units are those of power, r is the distance from the dipole to the point of interest, and θ is the angle between the electric dipole moment and the position vector $\vec{r}$. We can integrate the intensity to express the total power radiated by the antenna and use this result to evaluate C. Knowing C we can find the intensity of the signal at the plane's elevation and distance from the airport.

Express the intensity of the signal as a function of r and θ:

$$I(r,\theta) = C\frac{\sin^2\theta}{r^2} \qquad (1)$$

From the definition of intensity we have:

$$dP = IdA$$

and

$$P_{\text{tot}} = \iint I(r,\theta)dA$$

where, in polar coordinates,

$$dA = r^2 \sin\theta \, d\theta \, d\phi$$

Substitute for dA to obtain:

$$P_{\text{tot}} = \int_0^{2\pi}\int_0^{\pi} I(r,\theta) r^2 \sin\theta \, d\theta \, d\phi$$

Substituting for $I(r,\theta)$ yields:

$$P_{\text{tot}} = C\int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi$$

From integral tables we find that:

$$\int_0^{\pi} \sin^3\theta d\theta = -\tfrac{1}{3}\cos\theta\left(\sin^2\theta + 2\right)\Big]_0^{\pi} = \frac{4}{3}$$

Substitute and integrate with respect to ϕ to obtain:

$$P_{\text{tot}} = \frac{4}{3}C\int_0^{2\pi} d\phi = \frac{4}{3}C[\phi]_0^{2\pi} = \frac{8\pi}{3}C$$

Solving for C yields:

$$C = \frac{3}{8\pi}P_{\text{tot}}$$

Substitute for C in equation (1) to obtain:

$$I(r,\theta) = \frac{3P_{\text{tot}}}{8\pi}\frac{\sin^2\theta}{r^2}$$

At the elevation of the plane:

$$\theta = \tan^{-1}\left(\frac{4000\,\text{m}}{2500\,\text{m}}\right) = 58.0°$$

and

$$r = \sqrt{(2500\,\text{m})^2 + (4000\,\text{m})^2} = 4717\,\text{m}$$

Substitute numerical values and evaluate $I(4717\text{ m}, 58°)$:

$$I(4717\,\text{m}, 58.0°) = \frac{3(100\,\text{W})}{8\pi}\frac{\sin^2 58.0°}{(4717\,\text{m})^2}$$
$$= \boxed{386\ \text{nW/m}^2}$$

Energy and Momentum in an Electromagnetic Wave

30 • An electromagnetic wave has an intensity of 100 W/m^2. Find its (*a*) rms electric field strength, and (*b*) rms magnetic field strength.

Picture the Problem We can use $P_r = I/c$ to find the radiation pressure. The intensity of the electromagnetic wave is related to the rms values of its electric and magnetic field strengths according to $I = E_{rms}B_{rms}/\mu_0$, where $B_{rms} = E_{rms}/c$.

(*a*) Relate the intensity of the electromagnetic wave to E_{rms} and B_{rms}:

$$I = \frac{E_{rms}B_{rms}}{\mu_0}$$

or, because $B_{rms} = E_{rms}/c$,

$$I = \frac{E_{rms}\,E_{rms}/c}{\mu_0} = \frac{E_{rms}^2}{\mu_0 c}$$

Solving for E_{rms} yields:

$$E_{rms} = \sqrt{\mu_0 cI}$$

Substitute numerical values and evaluate E_{rms}:

$$E_{rms} = \sqrt{(4\pi\times10^{-7}\ \text{N/A}^2)(2.998\times10^8\ \text{m/s})(100\,\text{W/m}^2)} = \boxed{194\,\text{V/m}}$$

(*b*) Express B_{rms} in terms of E_{rms}:

$$B_{rms} = \frac{E_{rms}}{c}$$

Substitute numerical values and evaluate B_{rms}:

$$B_{rms} = \frac{194\,\text{V/m}}{2.998\times10^8\ \text{m/s}} = \boxed{647\,\text{nT}}$$

31 • **[SSM]** The amplitude of an electromagnetic wave's electric field is 400 V/m. Find the wave's (*a*) rms electric field strength, (*b*) rms magnetic field strength, (*c*) intensity and (*d*) radiation pressure (P_r).

Picture the Problem The rms values of the electric and magnetic fields are found from their amplitudes by dividing by the square root of two. The rms values of the electric and magnetic field strengths are related according to $B_{rms} = E_{rms}/c$. We can find the intensity of the radiation using $I = E_{rms}B_{rms}/\mu_0$ and the radiation pressure using $P_r = I/c$.

(*a*) Relate E_{rms} to E_0:

$$E_{rms} = \frac{E_0}{\sqrt{2}} = \frac{400\,\text{V/m}}{\sqrt{2}} = 282.8\,\text{V/m}$$
$$= \boxed{283\,\text{V/m}}$$

(*b*) Find B_{rms} from E_{rms}:

$$B_{rms} = \frac{E_{rms}}{c} = \frac{282.8\,\text{V/m}}{2.998\times10^8\ \text{m/s}}$$
$$= 0.9434\,\mu\text{T} = \boxed{943\ \text{nT}}$$

(*c*) The intensity of an electromagnetic wave is given by:

$$I = \frac{E_{rms}B_{rms}}{\mu_0}$$

Substitute numerical values and evaluate I:

$$I = \frac{(282.8\,\text{V/m})(0.9434\,\mu\text{T})}{4\pi\times10^{-7}\ \text{N/A}^2}$$
$$= 212.3\,\text{W/m}^2 = \boxed{212\,\text{W/m}^2}$$

(*d*) Express the radiation pressure in terms of the intensity of the wave:

$$P_r = \frac{I}{c}$$

Substitute numerical values and evaluate P_r:

$$P_r = \frac{212.3\,\text{W/m}^2}{2.998\times10^8\ \text{m/s}} = \boxed{708\ \text{nPa}}$$

32 • The rms value of an electromagnetic wave's electric field strength is 400 V/m. Find the wave's (*a*) rms magnetic field strength, (*b*) average energy density, and (*c*) intensity.

Picture the Problem Given E_{rms}, we can find B_{rms} using $B_{rms} = E_{rms}/c$. The average energy density of the wave is given by $u_{av} = E_{rms}B_{rms}/\mu_0 c$ and the intensity of the wave by $I = u_{av}c$.

(*a*) Express B_{rms} in terms of E_{rms}:

$$B_{rms} = \frac{E_{rms}}{c}$$

Substitute numerical values and evaluate B_{rms}:

$$B_{rms} = \frac{400\ \text{V/m}}{2.998\times10^{8}\ \text{m/s}} = 1.334\ \mu\text{T}$$
$$= \boxed{1.33\ \mu\text{T}}$$

(*b*) The average energy density u_{av} is given by:

$$u_{av} = \frac{E_{rms}B_{rms}}{\mu_0 c}$$

Substitute numerical values and evaluate u_{av}:

$$u_{av} = \frac{(400\ \text{V/m})(1.334\ \mu\text{T})}{(4\pi\times10^{-7}\ \text{N/A}^2)(2.998\times10^{8}\ \text{m/s})}$$
$$= 1.417\ \mu\text{J/m}^3 = \boxed{1.42\ \mu\text{J/m}^3}$$

(*c*) Express the intensity as the product of the average energy density and the speed of light in a vacuum:

$$I = u_{av}c$$

Substitute numerical values and evaluate I:

$$I = (1.417\ \mu\text{J/m}^3)(2.998\times10^{8}\ \text{m/s})$$
$$= \boxed{425\ \text{W/m}^2}$$

33 •• (*a*) An electromagnetic wave that has an intensity equal to 200 W/m^2 is normal to a black 20 cm by 30 cm rectangular card absorbs 100 percent of the wave. Find the force exerted on the card by the radiation. (*b*) Find the force exerted by the same wave if the card reflects 100 percent of the wave.

Picture the Problem We can find the force exerted on the card using the definition of pressure and the relationship between radiation pressure and the intensity of the electromagnetic wave. Note that, when the card reflects all the radiation incident on it, conservation of momentum requires that the force is doubled.

(*a*) Using the definition of pressure, express the force exerted on the card by the radiation:

$$F_r = P_r A$$

Relate the radiation pressure to the intensity of the wave:

$$P_r = \frac{I}{c}$$

Substitute for P_r to obtain:

$$F_r = \frac{IA}{c}$$

Substitute numerical values and evaluate F_r:

$$F_r = \frac{(200\,\text{W/m}^2)(0.20\,\text{m})(0.30\,\text{m})}{2.998\times10^8\ \text{m/s}} = \boxed{40\,\text{nN}}$$

(*b*) If the card reflects all of the radiation incident on it, the force exerted on the card is doubled:

$$F_r = \boxed{80\,\text{nN}}$$

34 •• Find the force exerted by the electromagnetic wave on the card in Part (*b*) of Problem 33 if both the incident and reflected rays are at angles of 30º to the normal.

Picture the Problem Only the normal component of the radiation pressure exerts a force on the card.

Using the definition of pressure, express the force exerted on the card by the radiation:

$$F_r = 2P_r A\cos\theta$$

where the factor of 2 is a consequence of the fact that the card reflects the radiation incident on it.

Relate the radiation pressure to the intensity of the wave:

$$P_r = \frac{I}{c}$$

Substitute for P_r to obtain:

$$F_r = \frac{2IA\cos\theta}{c}$$

Substitute numerical values and evaluate F_r:

$$F_r = \frac{2(200\,\text{W/m}^2)(0.20\,\text{m})(0.30\,\text{m})\cos 30^\circ}{2.998\times10^8\ \text{m/s}} = \boxed{69\,\text{nN}}$$

35 • **[SSM]** (*a*) For a given distance from a radiating electric dipole, at what angle (expressed as θ and measured from the dipole axis) is the intensity equal to 50 percent of the maximum intensity? (*b*) At what angle θ is the intensity equal to 1 percent of the maximum intensity?

Picture the Problem At a fixed distance from the electric dipole, the intensity of radiation is a function θ alone.

(*a*) The intensity of the radiation from the dipole is proportional to $\sin^2\theta$:

$$I(\theta) = I_0 \sin^2\theta \qquad (1)$$

where I_0 is the maximum intensity.

For $I = \frac{1}{2}I_0$: $\frac{1}{2}I_0 = I_0 \sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{2}$

Solving for θ yields: $\theta = \sin^{-1}\left(\sqrt{\frac{1}{2}}\right) = \boxed{45°}$

(*b*) For $I = 0.01I_0$: $0.01I_0 = I_0 \sin^2\theta \Rightarrow \sin^2\theta = 0.01$

Solving for θ yields: $\theta = \sin^{-1}\left(\sqrt{0.01}\right) = \boxed{5.7°}$

36 •• A laser pulse has an energy of 20.0 J and a beam radius of 2.00 mm. The pulse duration is 10.0 ns and the energy density is uniformly distributed within the pulse. (*a*) What is the spatial length of the pulse? (*b*) What is the energy density within the pulse? (*c*) Find the rms values of the electric and magnetic fields in the pulse.

Picture the Problem The spatial length *L* of the pulse is the product of its speed *c* and duration Δ*t*. We can find the energy density within the pulse using its definition (*u* = *U*/*V*). The electric amplitude of the pulse is related to the energy density in the beam according to $u = \epsilon_0 E^2$ and we can find *B* from *E* using *B* = *E*/*c*.

(*a*) The spatial length *L* of the pulse is the product of its speed *c* and duration Δ*t*:

$$L = c\Delta t$$

Substitute numerical values and evaluate *L*:

$$L = \left(2.998\times10^8\ \text{m/s}\right)\left(10.0\,\text{ns}\right) = 2.998\ \text{m} = \boxed{3.00\,\text{m}}$$

(*b*) The energy density within the pulse is the energy of the beam per unit volume:

$$u = \frac{U}{V} = \frac{U}{\pi r^2 L}$$

Substitute numerical values and evaluate *u*:

$$u = \frac{20.0\,\text{J}}{\pi\left(2.00\,\text{mm}\right)^2\left(2.998\,\text{m}\right)} = 530.9\,\text{kJ/m}^3 = \boxed{531\,\text{kJ/m}^3}$$

(*c*) *E* is related to *u* according to:

$$u = \epsilon_0 E_{\text{rms}}^2 \Rightarrow E_{\text{rms}} = \sqrt{\frac{u}{\epsilon_0}}$$

Substitute numerical values and evaluate E_{rms}:

$$E_{rms} = \sqrt{\frac{530.9\ \text{kJ/m}^3}{8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2}} = 244.9\,\text{MV/m} = \boxed{245\,\text{MV/m}}$$

Use $B_{rms} = E_{rms}/c$ to find B_{rms}:

$$B_{rms} = \frac{244.9\ \text{MV/m}}{2.998\times10^8\ \text{m/s}} = \boxed{0.817\,\text{T}}$$

37 •• **[SSM]** An electromagnetic plane wave has an electric field that is parallel to the y axis, and has a Poynting vector that is given by $\vec{S}(x,t) = (100\ \text{W/m}^2)\cos^2[kx-\omega t]\,\hat{i}$, where x is in meters, $k = 10.0$ rad/m, $\omega = 3.00 \times 10^9$ rad/s, and t is in seconds. (*a*) What is the direction of propagation of the wave? (*b*) Find the wavelength and frequency of the wave. (*c*) Find the electric and magnetic fields of the wave as functions of x and t.

Picture the Problem We can determine the direction of propagation of the wave, its wavelength, and its frequency by examining the argument of the cosine function. We can find E from $|\vec{S}| = E^2/\mu_0 c$ and B from $B = E/c$. Finally, we can use the definition of the Poynting vector and the given expression for $\vec{S}$ to find $\vec{E}$ and $\vec{B}$.

(*a*) Because the argument of the cosine function is of the form $kx-\omega t$, the wave propagates in the $+x$ direction.

(*b*) Examining the argument of the cosine function, we note that the wave number k of the wave is:

$$k = \frac{2\pi}{\lambda} = 10.0\,\text{m}^{-1} \Rightarrow \lambda = \boxed{0.628\,\text{m}}$$

Examining the argument of the cosine function, we note that the angular frequency ω of the wave is:

$$\omega = 2\pi f = 3.00\times10^9\ \text{s}^{-1}$$

Solving for f yields:

$$f = \frac{3.00\times10^9\ \text{s}^{-1}}{2\pi} = \boxed{477\,\text{MHz}}$$

(*c*) Express the magnitude of $\vec{S}$ in terms of E:

$$|\vec{S}| = \frac{E^2}{\mu_0 c} \Rightarrow E = \sqrt{\mu_0 c|\vec{S}|}$$

Substitute numerical values and evaluate E:

$$E = \sqrt{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(2.998\times10^{8}\ \text{m/s}\right)\left(100\ \text{W/m}^2\right)} = 194.1\ \text{V/m}$$

Because $\vec{S}(x,t) = \left(100\ \text{W/m}^2\right)\cos^2[kx-\omega t]\,\hat{\boldsymbol{i}}$ and $\vec{S} = \dfrac{1}{\mu_0}\vec{E}\times\vec{B}$:

$$\vec{E}(x,t) = \boxed{\left(194\ \text{V/m}\right)\cos[kx-\omega t]\,\hat{\boldsymbol{j}}}$$

where $k = 10.0$ rad/m and $\omega = 3.00 \times 10^9$ rad/s.

Use $B = E/c$ to evaluate B:

$$B = \frac{194.1\ \text{V/m}}{2.998\times10^{8}\ \text{m/s}} = 647.4\ \text{nT}$$

Because $\vec{S} = \dfrac{1}{\mu_0}\vec{E}\times\vec{B}$, the direction of $\vec{B}$ must be such that the cross product of $\vec{E}$ with $\vec{B}$ is in the $+x$ direction:

$$\vec{B}(x,t) = \boxed{\left(647\ \text{nT}\right)\cos[kx-\omega t]\,\hat{\boldsymbol{k}}}$$

where $k = 10.0$ rad/m and $\omega = 3.00 \times 10^9$ rad/s.

38 •• A parallel-plate capacitor is being charged. The capacitor consists of a pair of identical circular parallel plates that have radius b and a separation distance d. (*a*) Show that the displacement current in the capacitor gap has the same value as the conduction current in the capacitor leads. (*b*) What is the direction of the Poynting vector in the region between the capacitor plates? (*c*) Find an expression for the Poynting vector in this region and show that its flux into the region between the plates is equal to the rate of change of the energy stored in the capacitor.

Picture the Problem We can use the expression for the electric field strength between the plates of the parallel-plate capacitor and the definition of the displacement current to show that the displacement current in the capacitor is equal to the conduction current in the capacitor leads. In (*b*) we can use the definition of the Poynting vector and the directions of the electric and magnetic fields to determine the direction of the Poynting vector between the capacitor plates. In (*c*), we'll demonstrate that the flux of $\vec{S}$ into the region between the plates is equal to the rate of change of the energy stored in the capacitor by evaluating these quantities separately and showing that they are equal.

(*a*) The displacement current is proportional to the rate at which the flux is changing between the plates:

$$I_d = \epsilon_0\frac{d\phi_e}{dt} = \epsilon_0\frac{d}{dt}(AE) = \epsilon_0 A\frac{dE}{dt}$$

The electric field strength between the plates of the capacitor is given by:

$$E = \frac{Q}{\in_0 A}$$

where Q is the instantaneous charge on the capacitor plates.

Substituting for E yields:

$$I_d = \in_0 A \frac{d}{dt} \frac{Q}{\in_0 A} = \frac{dQ}{dt} = \boxed{I}$$

(b) Because $\vec{E}$ is perpendicular to the plates of the capacitor and $\vec{B}$ is tangent to circles that are concentric and whose center is through the middle of the capacitor plates, $\vec{S}$ points radially inward toward the center of the capacitor.

(c) The Poynting vector is:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \qquad (1)$$

Letting the direction of $\vec{E}$ be the $+x$ direction:

$$\vec{E} = E\hat{i}$$

where E is the electric field strength between the plates of the capacitor.

Apply Ampere's law to a closed circular path of radius $R \leq b$ to obtain:

$$B(2\pi R) = \mu_0 I_d$$

Substituting for I_d and simplifying yields:

$$B(2\pi R) = \mu_0 \in_0 \frac{d\phi_e}{dt} = \mu_0 \in_0 \frac{d}{dt} EA$$

$$= \mu_0 \in_0 \pi R^2 \frac{dE}{dt}$$

Solve for B to obtain:

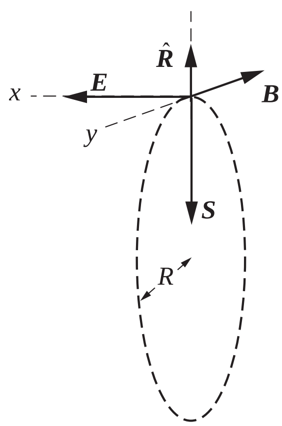
$$B = \frac{\mu_0 \in_0}{2} R \frac{dE}{dt}$$

and

$$\vec{B} = -\frac{\mu_0 \in_0}{2} R \frac{dE}{dt} \hat{j}$$

where $\hat{j}$ is a unit vector that is tangent to the concentric circles.

Substitute for $\vec{B}$ and $\vec{E}$ in equation (1) and simplify to obtain:



$$\vec{S} = \left(\frac{1}{\mu_0}E\right)\hat{i} \times \left(-\frac{\mu_0 \in_0}{2}R\frac{dE}{dt}\right)\hat{j}$$
$$= \frac{\in_0 E}{2}\frac{dE}{dt}R\left(\hat{i} \times -\hat{j}\right)$$
$$= \boxed{-\frac{\in_0 E}{2}\frac{dE}{dt}R\hat{R}}$$

where $R \leq b$, E is the electric field strength between the plates, R is the radial distance from the line joining the centers of the plates, $\hat{R}$ is a unit vector pointing radially outward from the line joining the centers of the plates, and b is the radius of the plates.

The rate at which energy is stored in the capacitor is:

$$\frac{dU}{dt} = \frac{d}{dt}(uV) = V\frac{d}{dt}\left(\in_0 E^2\right)$$
$$= Ad \in_0 E\frac{dE}{dt}$$

Because $E = \dfrac{Q}{\in_0 A}$:

$$\frac{dU}{dt} = Ad \in_0 \left(\frac{Q}{\in_0 A}\right)\frac{d}{dt}\left(\frac{Q}{\in_0 A}\right)$$
$$= \frac{Qd}{\in_0 A}\frac{dQ}{dt} = \frac{Qd}{\in_0 A}I$$

Consider a cylindrical surface of length d and radius b. Because $\vec{S}$ points inward, the energy flowing into the solenoid per unit time is:

$$\oint S_{\text{n}} dA = S(2\pi bd)$$
$$= \left(\tfrac{1}{2}\in_0 Eb\frac{dE}{dt}\right)(2\pi bd)$$
$$= \pi \in_0 Eb^2 d\frac{dE}{dt}$$

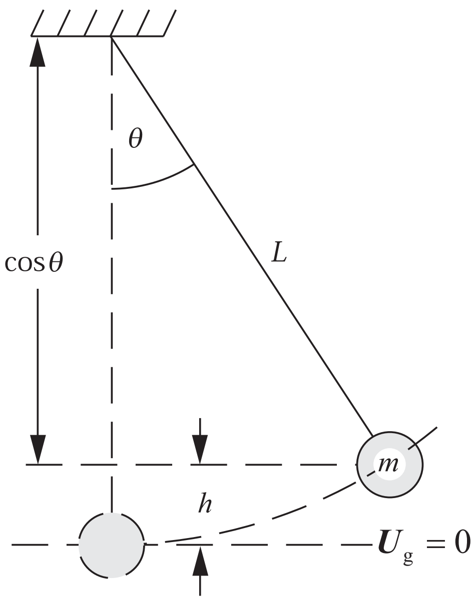
Substituting for E and simplifying yields:

$$\oint S_{\text{n}} dA = \pi \in_0 \left(\frac{Q}{\in_0 A}\right)b^2 d\frac{d}{dt}\left(\frac{Q}{\in_0 A}\right)$$
$$= \pi\left(\frac{Q}{\pi b^2}\right)\frac{b^2 d}{\in_0 A}\frac{dQ}{dt} = \frac{Qd}{\in_0 A}I$$

Because $\oint S_{\text{n}} dA = dU/dt$, we've proved that the flux of $\vec{S}$ into the region between the capacitor is equal to the rate of change of the energy stored in the capacitor.

39 •• **[SSM]** A pulsed laser fires a 1000-MW pulse that has a 200-ns duration at a small object that has a mass equal to 10.0 mg and is suspended by a fine fiber that is 4.00 cm long. If the radiation is completely absorbed by the object, what is the maximum angle of deflection of this pendulum? (Think of the system as a ballistic pendulum and assume the small object was hanging vertically before the radiation hit it.)

Picture the Problem The diagram shows the displacement of the pendulum bob, through an angle θ, as a consequence of the complete absorption of the radiation incident on it. We can use conservation of energy (mechanical energy is conserved *after* the collision) to relate the maximum angle of deflection of the pendulum to the initial momentum of the pendulum bob. Because the displacement of the bob during the absorption of the pulse is negligible, we can use conservation of momentum (conserved *during* the collision) to equate the momentum of the electromagnetic pulse to the initial momentum of the bob.



Apply conservation of energy to obtain:

$$K_f - K_i + U_f - U_i = 0$$

or, because $U_i = K_f = 0$ and $K_i = \dfrac{p_i^2}{2m}$,

$$-\frac{p_i^2}{2m} + U_f = 0$$

U_f is given by:

$$U_f = mgh = mgL(1-\cos\theta)$$

Substitute for U_f:

$$-\frac{p_i^2}{2m} + mgL(1-\cos\theta) = 0$$

Solve for θ to obtain:

$$\theta = \cos^{-1}\left(1 - \frac{p_i^2}{2m^2gL}\right)$$

Use conservation of momentum to relate the momentum of the electromagnetic pulse to the initial momentum p_i of the pendulum bob:

$$p_{\text{em wave}} = \frac{U}{c} = \frac{P\Delta t}{c} = p_i$$

where Δt is the duration of the pulse.

Substitute for p_i:

$$\theta = \cos^{-1}\left[1 - \frac{P^2(\Delta t)^2}{2m^2c^2gL}\right]$$

Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left[1 - \frac{(1000\,\text{MW})^2(200\,\text{ns})^2}{2(10.0\,\text{mg})^2(2.998\times10^8\ \text{m/s})^2(9.81\,\text{m/s}^2)(0.0400\,\text{m})}\right] = \boxed{6.10°}$$

Remarks: The solution presented here is valid only if the displacement of the bob during the absorption of the pulse is negligible. (Otherwise, the horizontal component of the momentum of the pulse-bob system is not conserved during the collision.) We can show that the displacement during the pulse-bob collision is small by solving for the speed of the bob after absorbing the pulse. Applying conservation of momentum ($mv = P(\Delta t)/c$) and solving for v gives $v = 6.67 \times 10^{-7}$ m/s. This speed is so slow compared to c, we can conclude that the duration of the collision is extremely close to 200 ns (the time for the pulse to travel its own length). Traveling at 6.67×10^{-7} m/s for 200 ns, the bob would travel 1.33×10^{-13} m—a distance 1000 times smaller that the diameter of a hydrogen atom. (Because 6.67×10^{-7} m/s is the maximum speed of the bob during the collision, the bob would actually travel less than 1.33×10^{-13} m during the collision.)

40 •• The mirrors used in a particular type of laser are 99.99% reflecting. (*a*) If the laser has an average output power of 15 W, what is the average power of the radiation incident on one of the mirrors? (*b*) What is the force due to radiation pressure on one of the mirrors?

Picture the Problem We can use the definitions of pressure and the relationship between radiation pressure and the intensity of the radiation to find the force due to radiation pressure on one of the mirrors.

(*a*) Because only about 0.01 percent of the energy inside the laser "leaks out", the average power of the radiation incident on one of the mirrors is:

$$P = \frac{15\,\text{W}}{1.0\times10^{-4}} = \boxed{1.5\times10^5\ \text{W}}$$

(*b*) Use the definition of radiation pressure to obtain:

$$P_r = \frac{F_r}{A}$$

where F_r is the force due to radiation pressure and A is the area of the mirror on which the radiation is incident.

The radiation pressure is also related to the intensity of the radiation:

$$P_r = \frac{2I}{c} = \frac{2P}{Ac}$$

where P is the power of the laser and the factor of 2 is due to the fact that the mirror is essentially totally reflecting.

Equate the two expression for the radiation pressure and solve for F_r:

$$\frac{F_r}{A} = \frac{2P}{Ac} \Rightarrow F_r = \frac{2P}{c}$$

Substitute numerical values and evaluate F_r:

$$F_r = \frac{2\left(1.5\times10^5\ \text{W}\right)}{2.998\times10^8\ \text{m/s}} = \boxed{1.0\,\text{mN}}$$

41 •• **[SSM]** (*a*) Estimate the force on Earth due to the pressure of the radiation on Earth by the Sun, and compare this force to the gravitational force of the Sun on Earth. (At Earth's orbit, the intensity of sunlight is 1.37 kW/m^2.) (*b*) Repeat Part (*a*) for Mars which is at an average distance of 2.28×10^8 km from the Sun and has a radius of 3.40×10^3 km. (*c*) Which planet has the larger ratio of radiation pressure to gravitational attraction.

Picture the Problem We can find the radiation pressure force from the definition of pressure and the relationship between the radiation pressure and the intensity of the radiation from the Sun. We can use Newton's law of gravitation to find the gravitational force the Sun exerts on Earth and Mars.

(*a*) The radiation pressure exerted on Earth is given by:

$$P_{\text{r, Earth}} = \frac{F_{\text{r, Earth}}}{A} \Rightarrow F_{\text{r, Earth}} = P_{\text{r, Earth}}A$$

where A is the cross-sectional area of Earth.

Express the radiation pressure in terms of the intensity of the radiation I from the Sun:

$$P_{\text{r, Earth}} = \frac{I}{c}$$

Substituting for $P_{\text{r, Earth}}$ and A yields:

$$F_{\text{r, Earth}} = \frac{I\pi R^2}{c}$$

Substitute numerical values and evaluate F_r:

$$F_{\text{r, Earth}}=\frac{\pi\left(1.37\ \text{kW/m}^2\right)\left(6.37\times10^6\ \text{m}\right)^2}{2.998\times10^8\ \text{m/s}}=5.825\times10^8\ \text{N}=\boxed{5.83\times10^8\ \text{N}}$$

The gravitational force exerted on Earth by the Sun is given by:

$$F_{\text{g, Earth}}=\frac{Gm_{\text{sun}}m_{\text{earth}}}{r^2}$$

where r is the radius of Earth's orbit.

Substitute numerical values and evaluate $F_{\text{g, Earth}}$:

$$F_{\text{g, Earth}}=\frac{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(1.99\times10^{30}\ \text{kg}\right)\left(5.98\times10^{24}\ \text{kg}\right)}{\left(1.50\times10^{11}\ \text{m}\right)^2}=3.529\times10^{22}\ \text{N}$$

Express the ratio of the force due to radiation pressure $F_{\text{r, Earth}}$ to the gravitational force $F_{\text{g, Earth}}$:

$$\frac{F_{\text{r, Earth}}}{F_{\text{g, Earth}}}=\frac{5.825\times10^8\ \text{N}}{3.529\times10^{22}\ \text{N}}=1.65\times10^{-14}$$

or

$$F_{\text{r, Earth}}=\boxed{\left(1.65\times10^{-14}\right)F_{\text{g, Earth}}}$$

(*b*) The radiation pressure exerted on Mars is given by:

$$P_{\text{r, Mars}}=\frac{F_{\text{r, Mars}}}{A}\Rightarrow F_{\text{r, Mars}}=P_{\text{r, Mars}}A$$

where A is the cross-sectional area of Mars.

Express the radiation pressure on Mars in terms of the intensity of the radiation I_{Mars} from the sun:

$$P_{\text{r, Mars}}=\frac{I_{\text{Mars}}}{c}$$

Substituting for $P_{\text{r, Mars}}$ and A yields:

$$F_{\text{r, Mars}}=\frac{I_{\text{Mars}}\pi R_{\text{Mars}}^2}{c}$$

Express the ratio of the solar constant at Earth to the solar constant at Mars:

$$\frac{I_{\text{Mars}}}{I_{\text{earth}}}=\left(\frac{r_{\text{earth}}}{r_{\text{Mars}}}\right)^2\Rightarrow I_{\text{Mars}}=I_{\text{earth}}\left(\frac{r_{\text{earth}}}{r_{\text{Mars}}}\right)^2$$

Substitute for I_{Mars} to obtain:

$$F_{\text{r, Mars}}=\frac{I_{\text{earth}}\pi R_{\text{Mars}}^2}{c}\left(\frac{r_{\text{earth}}}{r_{\text{Mars}}}\right)^2$$

Substitute numerical values and evaluate $F_{\text{r, Mars}}$:

$$F_{\text{r, Mars}} = \frac{\pi(1.37\,\text{kW/m}^2)(3.40\times10^3\ \text{km})^2}{2.998\times10^8\ \text{m/s}}\left(\frac{1.50\times10^{11}\ \text{m}}{2.28\times10^{11}\ \text{m}}\right)^2 = \boxed{7.18\times10^7\ \text{N}}$$

The gravitational force exerted on Mars by the Sun is given by:

$$F_{\text{g, Mars}} = \frac{Gm_{\text{sun}}m_{\text{Mars}}}{r^2} = \frac{Gm_{\text{sun}}(0.11m_{\text{Earth}})}{r^2}$$

where r is the radius of Mars' orbit.

Substitute numerical values and evaluate F_{g}

$$F_{\text{g, Mars}} = \frac{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})(0.11)(5.98\times10^{24}\ \text{kg})}{(2.28\times10^{11}\ \text{m})^2}$$

$$= 1.68\times10^{21}\ \text{N}$$

Express the ratio of the force due to radiation pressure $F_{\text{r, Mars}}$ to the gravitational force $F_{\text{g, Mars}}$:

$$\frac{F_{\text{r, Mars}}}{F_{\text{g, Mars}}} = \frac{7.18\times10^7\ \text{N}}{1.68\times10^{21}\ \text{N}} = 4.27\times10^{-14}$$

or

$$F_{\text{r, Mars}} = \boxed{(4.27\times10^{-14})F_{\text{g, Mars}}}$$

(c) Because the ratio of the radiation pressure force to the gravitational force is 1.65×10^{-14} for Earth and 4.27×10^{-14} for Mars, Mars has the larger ratio. The reason that the ratio is higher for Mars is that the dependence of the radiation pressure on the distance from the Sun is the same for both forces (r^{-2}), whereas the dependence on the radii of the planets is different. Radiation pressure varies as R^2, whereas the gravitational force varies as R^3 (assuming that the two planets have the same density, an assumption that is nearly true). Consequently, the ratio of the forces goes as $R^2/R^3 = R^{-1}$. Because Mars is smaller than Earth, the ratio is larger.

The Wave Equation for Electromagnetic Waves

42 • Show by direct substitution that Equation 30-8a is satisfied by the wave function $E_y = E_0 \sin(kx - \omega t) = E_0 \sin k(x - ct)$ where $c = \omega/k$.

Picture the Problem We can show that Equation 30-8a is satisfied by the wave function E_y by showing that the ratio of $\partial^2 E_y/\partial x^2$ to $\partial^2 E_y/\partial t^2$ is $1/c^2$ where $c = \omega/k$.

Differentiate $E_y = E_0 \sin(kx - \omega t)$ with respect to x:

$$\frac{\partial E_y}{\partial x} = \frac{\partial}{\partial x}\left[E_0 \sin(kx - \omega t)\right] = kE_0 \cos(kx - \omega t)$$

Evaluate the second partial derivative of E_y with respect to x:

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x}\left[kE_0 \cos(kx - \omega t)\right] = -k^2 E_0 \sin(kx - \omega t) \quad (1)$$

Differentiate $E_y = E_0 \sin(kx - \omega t)$ with respect to t:

$$\frac{\partial E_y}{\partial t} = \frac{\partial}{\partial t}\left[E_0 \sin(kx - \omega t)\right] = -\omega E_0 \cos(kx - \omega t)$$

Evaluate the second partial derivative of E_y with respect to t:

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial}{\partial t}\left[-\omega E_0 \cos(kx - \omega t)\right] = -\omega^2 E_0 \sin(kx - \omega t) \quad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{\dfrac{\partial^2 E_y}{\partial x^2}}{\dfrac{\partial^2 E_y}{\partial t^2}} = \frac{-k^2 E_0 \sin(kx - \omega t)}{-\omega^2 E_0 \sin(kx - \omega t)} = \frac{k^2}{\omega^2}$$

or

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

provided $c = \omega/k$.

43 • Use the values of μ_0 and $\in_0$ in SI units to compute $1/\sqrt{\in_0 \mu_0}$ and show that it is equal to 3.00×10^8 m/s.

Picture the Problem Substitute numerical values and evaluate c:

$$c = \frac{1}{\sqrt{\left(4\pi \times 10^{-7}\ \mathrm{N/A^2}\right)\left(8.854 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}\right)}} = \boxed{3.00 \times 10^8\ \mathrm{m/s}}$$

44 •• (*a*) Use Maxwell's equations to show for a plane wave, in which $\vec{E}$ and $\vec{B}$ are independent of y and z, that $\dfrac{\partial E_z}{\partial x} = \dfrac{\partial B_y}{\partial t}$ and $\dfrac{\partial B_y}{\partial x} = \mu_0 \in_0 \dfrac{\partial E_z}{\partial t}$.
(*b*) Show that E_z and B_y also satisfy the wave equation.

Picture the Problem We can use Figures 30-5 and 30-6 and a derivation similar to that in the text to obtain the given results.

In Figure 30-5, replace B_z by E_z. For Δx small:

$$E_z(x_2) = E_z(x_1) + \frac{\partial E_z}{\partial x}\Delta x$$

Evaluate the line integral of $\vec{E}$ around the rectangular area $\Delta x\Delta z$:

$$\oint \vec{E}\cdot d\vec{\ell} \approx -\frac{\partial E_z}{\partial x}\Delta x\Delta z \qquad (1)$$

Express the magnetic flux through the same area:

$$\int_S B_n\,dA = B_y\Delta x\Delta z$$

Apply Faraday's law to obtain:

$$\oint \vec{E}\cdot d\vec{\ell} \approx -\frac{\partial}{\partial t}\int_S B_n\,dA = -\frac{\partial}{\partial t}\left(B_y\Delta x\Delta z\right) = -\frac{\partial B_y}{\partial t}\Delta x\Delta z$$

Substitute in equation (1) to obtain:

$$-\frac{\partial E_z}{\partial x}\Delta x\Delta z = -\frac{\partial B_y}{\partial t}\Delta x\Delta z$$

or

$$\boxed{\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}}$$

In Figure 30-6, replace E_y by B_y and evaluate the line integral of $\vec{B}$ around the rectangular area $\Delta x\Delta z$:

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 \in_0 \int_S E_n\,dA$$

provided there are no conduction currents.

Evaluate these integrals to obtain:

$$\boxed{\frac{\partial B_y}{\partial x} = \mu_0 \in_0 \frac{\partial E_z}{\partial t}}$$

(*b*) Using the first result obtained in (*a*), find the second partial derivative of E_z with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial B_y}{\partial t}\right)$$

or

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial x}\right)$$

Use the second result obtained in (*a*) to obtain:

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}\right) = \mu_0 \in_0 \frac{\partial^2 E_z}{\partial t^2}$$

or, because $\mu_0 \in_0 = 1/c^2$,

$$\boxed{\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}}\,.$$

Using the second result obtained in (*a*), find the second partial derivative of B_y with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial B_y}{\partial x}\right) = \mu_0 \in_0 \frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial t}\right)$$

or

$$\frac{\partial^2 B_y}{\partial x^2} = \mu_0 \in_0 \frac{\partial}{\partial t}\left(\frac{\partial E_z}{\partial x}\right)$$

Use the first result obtained in (*a*) to obtain:

$$\frac{\partial^2 B_y}{\partial x^2} = \mu_0 \in_0 \frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial t}\right) = \mu_0 \in_0 \frac{\partial^2 B_y}{\partial t^2}$$

or, because $\mu_0 \in_0 = 1/c^2$,

$$\boxed{\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}}\,.$$

45 •• [SSM] Show that any function of the form $y(x, t) = f(x - vt)$ or $y(x, t) = g(x + vt)$ satisfies the wave Equation 30-7

Picture the Problem We can show that these functions satisfy the wave equations by differentiating them twice (using the chain rule) with respect to x and t and equating the expressions for the second partial of f with respect to u.

Let $u = x - vt$. Then:

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} = \frac{\partial f}{\partial u}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial f}{\partial u} = -v\frac{\partial f}{\partial u}$$

Express the second derivatives of f with respect to x and t to obtain:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} \text{ and } \frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial u^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{\dfrac{\partial^2 f}{\partial x^2}}{\dfrac{\partial^2 f}{\partial t^2}} = \frac{1}{v^2} \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

Let $u = x + vt$. Then:

$$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial f}{\partial u}=\frac{\partial f}{\partial u}$$

and

$$\frac{\partial f}{\partial t}=\frac{\partial u}{\partial t}\frac{\partial f}{\partial u}=v\frac{\partial f}{\partial u}$$

Express the second derivatives of f with respect to x and t to obtain:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\text{ and }\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 f}{\partial u^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{\dfrac{\partial^2 f}{\partial x^2}}{\dfrac{\partial^2 f}{\partial t^2}}=\frac{1}{v^2}\Rightarrow\boxed{\frac{\partial^2 f}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

General Problems

46 • An electromagnetic wave has a frequency of 100 MHz and is traveling in a vacuum. The magnetic field is given by $\vec{B}(z,t)=(1.00\times10^{-8}\text{ T})\cos(kz-\omega t)\hat{i}$. (*a*) Find the wavelength and the direction of propagation of this wave. (*b*) Find the electric field vector $\vec{E}(z,t)$. (*c*) Determine the Poynting vector, and use it to find the intensity of this wave.

Picture the Problem We can use $c = f\lambda$ to find the wavelength. Examination of the argument of the cosine function will reveal the direction of propagation of the wave. We can find the magnitude, wave number, and angular frequency of the electric vector from the given information and the result of (*a*) and use these results to obtain $\vec{E}$ (*z*, *t*). Finally, we can use its definition to find the Poynting vector.

(*a*) Relate the wavelength of the wave to its frequency and the speed of light:

$$\lambda=\frac{c}{f}$$

Substitute numerical values and evaluate λ:

$$\lambda=\frac{2.998\times10^8\text{ m/s}}{100\text{MHz}}=\boxed{3.00\text{m}}$$

From the sign of the argument of the cosine function and the spatial dependence on *z*, we can conclude that the wave propagates in the +z direction.

(*b*) Express the amplitude of $\vec{E}$:

$$E = cB = (2.998\times10^{8}\ \text{m/s})(10^{-8}\ \text{T}) = 3.00\,\text{V/m}$$

Find the angular frequency and wave number of the wave:

$$\omega = 2\pi f = 2\pi(100\,\text{MHz}) = 6.28\times10^{8}\ \text{s}^{-1}$$

and

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3.00\,\text{m}} = 2.09\,\text{m}^{-1}$$

Because $\vec{S}$ is in the positive *z* direction, $\vec{E}$ must be in the negative *y* direction in order to satisfy the Poynting vector expression:

$$\vec{E}(z,t) = \boxed{-(3.00\,\text{V/m})\cos\left[(2.09\,\text{m}^{-1})z - (6.28\times10^{8}\ \text{s}^{-1})t\right]\hat{j}}$$

(*c*) Use its definition to express and evaluate the Poynting vector:

$$\vec{S}(z,t) = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{-(3.00\,\text{V/m})(10^{-8}\ \text{T})}{4\pi\times10^{-7}\ \text{N/A}^2}\cos^2\left[(2.09\,\text{m}^{-1})z - (6.28\times10^{8}\ \text{s}^{-1})t\right](\hat{j}\times\hat{i})$$

or

$$\vec{S}(z,t) = \boxed{(23.9\,\text{mW/m}^2)\cos^2\left[(2.09\,\text{m}^{-1})z - (6.28\times10^{8}\ \text{s}^{-1})t\right]\hat{k}}$$

The intensity of the wave is the average magnitude of the Poynting vector. The average value of the square of the cosine function is 1/2:

$$I = \left|\vec{S}\right| = \tfrac{1}{2}(23.9\,\text{mW/m}^2) = \boxed{11.9\,\text{mW/m}^2}$$

47 •• [SSM] A circular loop of wire can be used to detect electromagnetic waves. Suppose the signal strength from a 100-MHz FM radio station 100 km distant is 4.0 μW/m^2, and suppose the signal is vertically polarized. What is the maximum rms voltage induced in your antenna, assuming your antenna is a 10.0-cm-radius loop?

Picture the Problem We can use Faraday's law to show that the maximum rms voltage induced in the loop is given by $\mathcal{E}_{\text{rms}} = A\omega B_0/\sqrt{2}$, where *A* is the area of the loop, B_0 is the amplitude of the magnetic field, and ω is the angular frequency of the wave. Relating the intensity of the radiation to B_0 will allow us to express $\mathcal{E}_{\text{rms}}$ as a function of the intensity.

The emf induced in the antenna is given by Faraday's law:

$$\mathcal{E} = -\frac{d\phi_m}{dt} = -\frac{d}{dt}\left(\vec{\boldsymbol{B}} \cdot A\hat{\boldsymbol{n}}\right) = -\frac{d}{dt}(BA)$$
$$= -A\frac{dB}{dt} = -\pi R^2 \frac{d}{dt}\left(B_0 \sin \omega t\right)$$
$$= -\pi R^2 \omega B_0 \cos \omega t = -\mathcal{E}_{peak} \cos \omega t$$

where $\mathcal{E}_{peak} = \pi R^2 \omega B_0$ and R is the radius of the loop antenna..

$\mathcal{E}_{rms}$ equals $\mathcal{E}_{peak}$ divided by the square root of 2:

$$\mathcal{E}_{rms} = \frac{\mathcal{E}_{peak}}{\sqrt{2}} = \frac{\pi R^2 \omega B_0}{\sqrt{2}} \qquad (1)$$

The intensity of the signal is given by:

$$I = \frac{E_0 B_0}{2\mu_0}$$

or, because $E_0 = cB_0$,

$$I = \frac{cB_0 B_0}{2\mu_0} = \frac{B_0^2 c}{2\mu_0}$$

Solving for B_0 yields:

$$B_0 = \sqrt{\frac{2\mu_0 I}{c}}$$

Substituting for B_0 and ω in equation (1) and simplifying yields:

$$\mathcal{E}_{rms} = \frac{\pi R^2 (2\pi f)\sqrt{\dfrac{2\mu_0 I}{c}}}{\sqrt{2}}$$
$$= 2\pi^2 R^2 f \sqrt{\frac{\mu_0 I}{c}}$$

Substitute numerical values and evaluate $\mathcal{E}_{rms}$:

$$\mathcal{E}_{rms} = 2\pi^2 (0.100\,\text{m})^2 (100\,\text{MHz}) \sqrt{\frac{\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(4.0\ \mu\text{W/m}^2\right)}{2.998\times10^8\ \text{m/s}}} = \boxed{2.6\,\text{mV}}$$

48 •• The electric field strength from a radio station some distance from the electric dipole transmitting antenna is given by $\left(1.00\times10^{-4}\ \text{N/C}\right)\cos\left(1.00\times10^{6}\ \text{rad/s}\right)t$, where t is in seconds. (*a*) What peak voltage is picked up on a 50.0-cm long wire oriented parallel with the electric field direction? (*b*) What is the maximum voltage that can be induced by this electromagnetic wave in a conducting loop of radius 20.0 cm? What orientation of the loop does this require?

Picture the Problem The voltage induced in the piece of wire is the product of the electric field and the length of the wire. The maximum rms voltage induced in the loop is given by $\mathcal{E} = A\omega B_0$, where A is the area of the loop, B_0 is the amplitude of the magnetic field, and ω is the angular frequency of the wave.

(*a*) Because E is independent of x:

$$V = E\ell$$

where ℓ is the length of the wire.

Substitute numerical values and evaluate V:

$$V = \left[\left(1.00\times10^{-4}\ \text{N/C}\right)\cos 10^6 t\right]\left(0.500\ \text{m}\right) = \left(50.0\ \mu\text{V}\right)\cos 10^6 t$$

and $V_{\text{peak}} = \boxed{50.0\ \mu\text{V}}$

(*b*) The maximum voltage induced in a loop is given by:

$$\mathcal{E} = \omega B_0 A$$

where A is the area of the loop and B_0 is the amplitude of the magnetic field.

Eliminate B_0 in favor of E_0 and substitute for A to obtain:

$$\mathcal{E} = \frac{\omega E_0 \pi R^2}{c}$$

Substitute numerical values and evaluate $\mathcal{E}$:

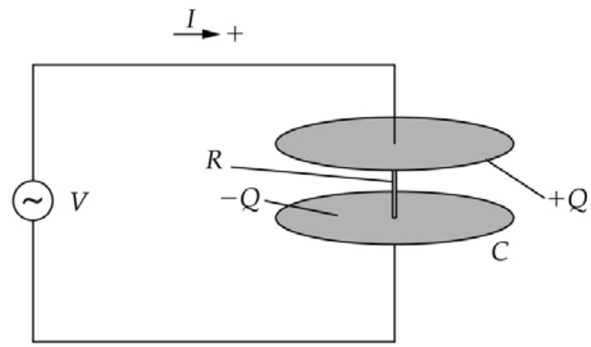
$$\mathcal{E} = \frac{\left(1.00\times10^6\ \text{s}^{-1}\right)\left(1.00\times10^{-4}\ \text{N/C}\right)\pi\left(0.200\ \text{m}\right)^2}{2.998\times10^8\ \text{m/s}} = \boxed{41.9\ \text{nV}}$$

The loop antenna should be oriented so the transmitting antenna lies in the plane of the loop.

49 ••• A parallel-plate capacitor has circular plates of radius a that are separated by a distance d. In the gap between the two plates is a thin straight wire of resistance R that connects the centers of the two plates. A time-varying voltage given by $V_0 \sin \omega t$ is applied across the plates. (*a*) What is the current drawn by this capacitor? (*b*) What is the magnetic field as a function of the radial distance r from the centerline within the capacitor plates? (*c*) What is the phase angle between the current drawn by the capacitor and the applied voltage?

Picture the Problem Some of the charge entering the capacitor passes through the resistive wire while the rest of it accumulates on the upper plate. The total current is the rate at which the charge passes through the resistive wire plus the rate at which it accumulates on the upper plate. The magnetic field between the capacitor plates is due to both the current in the resistive wire and the displacement current though a surface bounded by a circle a distance r from the

resistive wire. The phase difference between the current drawn by the capacitor and the applied voltage may be calculated using a phasor diagram.



(*a*) The current drawn by the capacitor is the sum of the conduction current through the resistance wire and *dQ/dt*, where *Q* is the charge on the upper plate of the capacitor:

$$I = I_c + \frac{dQ}{dt} \qquad (1)$$

Express the conduction current I_c in terms of the potential difference between the plates and the resistance of the wire:

$$I_c = \frac{V}{R} = \frac{V_0}{R}\sin\omega t$$

Because $Q = CV$:

$$\frac{dQ}{dt} = C\frac{dV}{dt} = \omega C V_0 \cos\omega t$$

Substitute in equation (1):

$$I = \frac{V_0}{R}\sin\omega t + \omega C V_0 \cos\omega t \qquad (2)$$

The capacitance of a parallel-plate capacitor with plate area *A* and plate separation *d* is given by:

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \pi a^2}{d}$$

Substituting for *C* in equation (2) gives:

$$I = \boxed{V_0\left(\frac{1}{R}\sin\omega t + \frac{\omega \epsilon_0 \pi a^2}{d}\cos\omega t\right)}$$

(*b*) Apply the generalized form of Ampere's law to a circular path of radius *r* centered within the plates of the capacitor, where I'_{d} is the displacement current through the flat surface *S* bounded by the path and I_{c} is the conduction current through the same surface:

$$\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\ell} = \mu_0 \left(I_{\mathrm{c}} + I'_{\mathrm{d}} \right)$$

By symmetry the line integral is *B* times the circumference of the circle of radius *r*:

$$B(2\pi r) = \mu_0 \left(I_{\mathrm{c}} + I'_{\mathrm{d}} \right) \qquad (3)$$

In the region between the capacitor plates there is a uniform electric field due to the surface charges $+Q$ and $-Q$. The associated displacement current through *S* is:

$$I'_{\mathrm{d}} = \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt} = \epsilon_0 \frac{d}{dt}(A'E)$$
$$= \epsilon_0 A' \frac{dE}{dt} = \epsilon_0 \pi r^2 \frac{dE}{dt}$$

provided $(r \le a)$

To evaluate the displacement current we first must evaluate *E* everywhere on *S*. Near the surface of a conductor, where σ is the surface charge density:

$E = \sigma / \epsilon_0$, where $\sigma = Q/A = Q/\left(\pi a^2\right)$

so

$$E = \frac{Q}{\epsilon_0 \pi a^2}$$

Substituting for *E* in the equation for I'_{d} gives:

$$I'_{\mathrm{d}} = \epsilon_0 \pi r^2 \frac{dE}{dt} = \epsilon_0 \pi r^2 \frac{d}{dt}\left(\frac{V}{d}\right)$$
$$= \frac{\epsilon_0 \pi r^2}{d} \frac{dV}{dt} = \frac{\epsilon_0 \pi r^2}{d} \frac{d}{dt}(V_0 \sin \omega t)$$
$$= \omega \frac{\epsilon_0 \pi r^2}{d} V_0 \cos \omega t$$

Solving for *B* in equation (3) and substituting for I_{c} and I'_{d} yields:

$$B(r) = \frac{\mu_0 \left(I_{\mathrm{c}} + I'_{\mathrm{d}} \right)}{2\pi r} = \frac{\mu_0}{2\pi r} \left(\frac{V_0}{R} \sin \omega t + \omega \frac{\epsilon_0 \pi r^2}{d} V_0 \cos \omega t \right)$$
$$= \boxed{\frac{\mu_0 V_0}{2\pi r} \left(\frac{1}{R} \sin \omega t + \omega \frac{\epsilon_0 \pi r^2}{d} \cos \omega t \right)}$$

(*c*) Both the charge Q and the conduction current I_c are in phase with V. However, dQ/dt, which is equal to the displacement current I_d through S for $r \geq a$, lags V by 90°. (Mathematically, cos ωt lags behind sin ωt by 90°.) The voltage V leads the current $I = I_c + I_d$ by phase angle δ. The current relation is expressed in terms of the current amplitudes:

$$I = I_c + I_d$$

or

$$I_{max}\sin(\omega t + \delta) = I_{c,max}\sin\omega t + I_{d,max}\cos\omega t$$

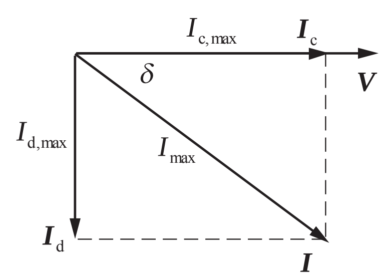
The values of the conduction and displacement current amplitudes are obtained by comparison with the answer to Part (*a*):

$$I_{c,max} = \frac{V_0}{R}$$

and

$$I_{d,max} = \frac{\omega \in_0 \pi a^2 V_0}{d}$$

A phasor diagram for adding the currents I_c and I_d is shown to the right. The conduction current I_c is in phase with the voltage V across the resistor and I_d lags behind it by 90°:



From the phasor diagram we have:

$$\tan\delta = \frac{I_{d,max}}{I_{c,max}} = \frac{V_0 \dfrac{\omega \in_0 \pi a^2}{d}}{V_0/R} = \frac{R\omega \in_0 \pi a^2}{d}$$

so

$$\delta = \boxed{\tan^{-1}\left(\frac{R\omega \in_0 \pi a^2}{d}\right)}$$

Remarks: The capacitor and the resistive wire are connected in parallel. The potential difference across each of them is the applied voltage V_0 sin ωt.

50 •• A 20-kW beam of electromagnetic radiation is normal to a surface that reflects 50 percent of the radiation. What is the force exerted by the radiation on this surface?

Picture the Problem The total force on the surface is the sum of the force due to the reflected radiation and the force due to the absorbed radiation. From the conservation of momentum, the force due to the 10 kW that are reflected is twice the force due to the 10 kW that are absorbed.

Express the total force on the surface:

$$F_{\text{tot}} = F_{\text{r}} + F_a$$

Substitute for F_{r} and F_{a} to obtain:

$$F_{\text{tot}} = \frac{2\left(\frac{1}{2}P\right)}{c} + \frac{\frac{1}{2}P}{c} = \frac{3P}{2c}$$

Substitute numerical values and evaluate F_{tot}:

$$F_{\text{tot}} = \frac{3(20\,\text{kW})}{2(2.998\times10^8\ \text{m/s})} = \boxed{0.10\,\text{mN}}$$

51 •• **[SSM]** The electric fields of two harmonic electromagnetic waves of angular frequency ω_1 and ω_2 are given by $\vec{E}_1 = E_{1,0}\cos(k_1x - \omega_1 t)\hat{j}$ and by $\vec{E}_2 = E_{2,0}\cos(k_2x - \omega_2 t + \delta)\hat{j}$. For the resultant of these two waves, find (*a*) the instantaneous Poynting vector and (*b*) the time-averaged Poynting vector. (*c*) Repeat Parts (*a*) and (*b*) if the direction of propagation of the second wave is reversed so that $\vec{E}_2 = E_{2,0}\cos(k_2x + \omega_2 t + \delta)\hat{j}$

Picture the Problem We can use the definition of the Poynting vector and the relationship between $\vec{B}$ and $\vec{E}$ to find the instantaneous Poynting vectors for each of the resultant wave motions and the fact that the time average of the cross product term is zero for $\omega_1 \neq \omega_2$, and ½ for the square of cosine function to find the time-averaged Poynting vectors.

(*a*) Because both waves propagate in the *x* direction:

$$\vec{E}\times\vec{B} = \mu_0 S\hat{i} \Rightarrow \vec{B} = B\hat{k}$$

Express B in terms of E_1 and E_2:

$$B = \frac{1}{c}(E_1 + E_2)$$

Substitute for E_1 and E_2 to obtain:

$$\vec{B}(x,t) = \frac{1}{c}\left[E_{1,0}\cos(k_1x - \omega_1 t) + E_{2,0}\cos(k_2x - \omega_2 t + \delta)\right]\hat{k}$$

The instantaneous Poynting vector for the resultant wave motion is given by:

$$\begin{aligned}\vec{S}(x,t) &= \frac{1}{\mu_0}\left(E_{1,0}\cos(k_1x-\omega_1t)+E_{2,0}\cos(k_2x-\omega_2t+\delta)\right)\hat{j}\\ &\qquad\times\frac{1}{c}\left(E_{1,0}\cos(k_1x-\omega_1t)+E_{2,0}\cos(k_2x-\omega_2t+\delta)\right)\hat{k}\\ &= \frac{1}{\mu_0c}\left(E_{1,0}\cos(k_1x-\omega_1t)+E_{2,0}\cos(k_2x-\omega_2t+\delta)\right)^2\left(\hat{j}\times\hat{k}\right)\\ &= \boxed{\begin{aligned}&\frac{1}{\mu_0c}\Big[E_{1,0}^2\cos^2(k_1x-\omega_1t)+2E_{1,0}E_{2,0}\cos(k_1x-\omega_1t)\\ &\qquad\times\cos(k_2x-\omega_2t+\delta)+E_{2,0}^2\cos^2(k_2x-\omega_2t+\delta)\Big]\hat{i}\end{aligned}}\end{aligned}$$

(*b*) The time average of the cross product term is zero for $\omega_1 \neq \omega_2$, and the time average of the square of the cosine terms is ½:

$$\vec{S}_{\text{av}} = \boxed{\frac{1}{2\mu_0c}\left[E_{1,0}^2+E_{2,0}^2\right]\hat{i}}$$

(*c*) In this case $\vec{B}_2 = -B\hat{k}$ because the wave with $k = k_2$ propagates in the $-\hat{i}$ direction. The magnetic field is then:

$$\vec{B}(x,t) = \frac{1}{c}\left[E_{1,0}\cos(k_1x-\omega_1t)-E_{2,0}\cos(k_2x+\omega_2t+\delta)\right]\hat{k}$$

The instantaneous Poynting vector for the resultant wave motion is given by:

$$\begin{aligned}\vec{S}(x,t) &= \frac{1}{\mu_0}\left(E_{1,0}\cos(k_1x-\omega_1t)+E_{2,0}\cos(k_2x-\omega_2t+\delta)\right)\hat{j}\\ &\qquad\times\frac{1}{c}\left(E_{1,0}\cos(k_1x-\omega_1t)-E_{2,0}\cos(k_2x+\omega_2t+\delta)\right)\hat{k}\\ &= \boxed{\frac{1}{\mu_0c}\left[E_{1,0}^2\cos^2(k_1x-\omega_1t)-E_{2,0}^2\cos^2(k_2x+\omega_2t+\delta)\right]\hat{i}}\end{aligned}$$

The time average of the square of the cosine terms is ½:

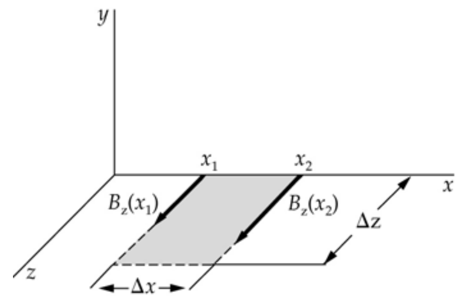
$$\vec{S}_{\text{av}} = \boxed{\frac{1}{2\mu_0c}\left[E_{1,0}^2-E_{2,0}^2\right]\hat{i}}$$

52 •• Show that $\frac{\partial B_z}{\partial x} = \mu_o \in_0 \frac{\partial E_n}{\partial t}$ (Equation 30-10) follows from $\oint_C \vec{B}\cdot d\vec{\ell} = -\mu_0 \in_0 \int_S \frac{\partial E_y}{\partial t}dA$ (Equation 30-6*d* with $I = 0$) by integrating along a

suitable curve C and over a suitable surface S in a manner that parallels the derivation of Equation 30-9.

Picture the Problem We'll choose the curve with sides Δx and Δz in the xy plane shown in the diagram and apply Equation 30-6*d* to show that

$$\frac{\partial B_z}{\partial x} = -\mu_0 \in_0 \frac{\partial E_y}{\partial t}.$$



Because Δx is very small, we can approximate the difference in B_z at the points x_1 and x_2 by:

$$B_z(x_2) - B_z(x_1) = \Delta B \approx \frac{\partial B_z}{\partial x}\Delta x$$

Then:

$$\oint_C \vec{B} \cdot d\vec{\ell} \approx \mu_0 \in_0 \frac{\partial E_y}{\partial t}\Delta x \Delta z$$

The flux of the electric field through this curve is approximately:

$$\int_S E_n dA = E_y \Delta x \Delta y$$

Apply Faraday's law to obtain:

$$\frac{\partial B_z}{\partial x}\Delta x \Delta z = -\mu_0 \in_0 \frac{\partial E_y}{\partial t}\Delta x \Delta z$$

or

$$\boxed{\frac{\partial B_z}{\partial x} = -\mu_0 \in_0 \frac{\partial E_y}{\partial t}}$$

53 •• For your backpacking excursions, you have purchased a radio capable of detecting a signal as weak as 1.00×10^{-14} W/m^2. This radio has a 2000-turn coil antenna that has a radius of 1.00 cm wound on an iron core that increases the magnetic field by a factor of 200. The broadcast frequency of the radio station is 1400 kHz. (*a*) What is the peak magnetic field strength of an electromagnetic wave of this minimum intensity? (*b*) What is the peak emf that it is capable of inducing in the antenna? (*c*) What would be the peak emf induced in a straight 2.00-m long metal wire oriented parallel to the direction of the electric field?

Picture the Problem We can use the relationship between the average value of the Poynting vector (the intensity), E_0, and B_0 to find B_0. The application of Faraday's law will allow us to find the emf induced in the antenna. The emf induced in a 2.00-m wire oriented in the direction of the electric field can be found using $\mathcal{E} = E\ell$ and the relationship between E and B.

(*a*) The intensity of the signal is related the amplitude of the magnetic field in the wave:

$$S_{av} = I = \frac{E_0B_0}{2\mu_0} = \frac{cB_0^2}{2\mu_0} \Rightarrow B_0 = \sqrt{\frac{2\mu_0 I}{c}}$$

Substitute numerical values and evaluate B_0:

$$B_0 = \sqrt{\frac{2\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(1.00\times10^{-14}\ \text{W/m}^2\right)}{2.998\times10^8\ \text{m/s}}} = \boxed{9.16\times10^{-15}\ \text{T}}$$

(*b*) Apply Faraday's law to the antenna coil to obtain:

$$\begin{aligned}|\mathcal{E}(t)| &= \frac{d}{dt}(BA) = A\frac{d}{dt}\left(NK_m B_0 \sin\omega t\right)\\ &= NK_m AB_0\omega\cos\omega t\\ &= \mathcal{E}_{peak}\cos\omega t\end{aligned}$$

where $\mathcal{E}_{peak} = NK_m AB_0\omega$

Substitute numerical values and evaluate $\mathcal{E}_{peak}$:

$$\mathcal{E}_{peak} = 2000(200)\pi(0.0100\,\text{m})^2\left(9.16\times10^{-15}\ \text{T}\right)\left[2\pi(1400\,\text{kHz})\right] = \boxed{10.1\,\mu\text{V}}$$

(*c*) The voltage induced in the wire is the product of its length ℓ and the amplitude of electric field E_0:

$$\mathcal{E} = E\ell$$

Relate E to B:

$$E = cB = cB_0\sin\omega t$$

Substitute for E to obtain:

$$\mathcal{E} = c\ell B_0\sin\omega t = \mathcal{E}_{peak}\sin\omega t$$

where $\mathcal{E}_{peak} = c\ell B_0$

Substitute numerical values and evaluate $\mathcal{E}_{peak}$:

$$\mathcal{E}_{peak} = \left(2.998\times10^8\ \text{m/s}\right)(2.00\,\text{m})\left(9.16\times10^{-15}\ \text{T}\right) = \boxed{5.49\,\mu\text{V}}$$

54 •• The intensity of the sunlight striking Earth's upper atmosphere is 1.37 kW/m^2. (*a*) Find the rms values of the magnetic and electric fields of this light. (*b*) Find the average power output of the Sun. (*c*) Find the intensity and the radiation pressure at the surface of the Sun.

Picture the Problem We can use $I = E_{rms}B_{rms}/\mu_0$ and $B_{rms} = E_{rms}/c$ to express E_{rms} in terms of I. We can then use $B_{rms} = E_{rms}/c$ to find B_{rms}. The average power output

of the Sun is given by $P_{av} = 4\pi R^2 I$ where R is the Earth-Sun distance. The intensity and the radiation pressure at the surface of the sun can be found from the definitions of these physical quantities.

(*a*) The intensity of the radiation is given by:

$$I = \frac{E_{rms}B_{rms}}{\mu_0} = \frac{E_{rms}^2}{c\mu_0} \Rightarrow E_{rms} = \sqrt{c\mu_0 I}$$

Substitute numerical values and evaluate E_{rms}:

$$E_{rms} = \sqrt{\left(2.998\times10^8\ \text{m/s}\right)\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(1.37\,\text{kW/m}^2\right)} = 718.4\ \text{V/m}$$
$$= \boxed{718\,\text{V/m}}$$

Use $B_{rms} = E_{rms}/c$ to evaluate B_{rms}:

$$B_{rms} = \frac{718.4\,\text{V/m}}{2.998\times10^8\ \text{m/s}} = \boxed{2.40\,\mu\text{T}}$$

(*b*) Express the average power output of the Sun in terms of the solar constant:

$$P_{av} = 4\pi R^2 I$$

where R is the Earth-sun distance.

Substitute numerical values and evaluate P_{av}:

$$P_{av} = 4\pi\left(1.50\times10^{11}\ \text{m}\right)^2\left(1.37\,\text{kW/m}^2\right)$$
$$= 3.874\times10^{26}\ \text{W}$$
$$= \boxed{3.87\times10^{26}\ \text{W}}$$

(*c*) Express the intensity at the surface of the Sun in terms of the sun's average power output and radius r:

$$I = \frac{P_{av}}{4\pi r^2}$$

Substitute numerical values (see Appendix B for the radius of the Sun) and evaluate I at the surface of the Sun:

$$I = \frac{3.874\times10^{26}\ \text{W}}{4\pi\left(6.96\times10^8\ \text{m}\right)^2}$$
$$= 6.363\times10^7\ \text{W/m}^2$$
$$= \boxed{6.36\times10^7\ \text{W/m}^2}$$

Express the radiation pressure in terms of the intensity:

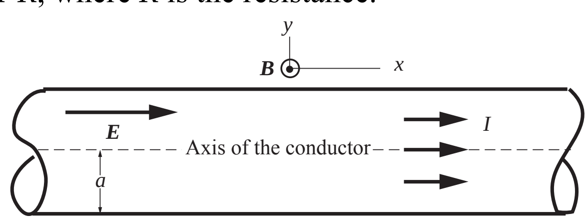
$$P_r = \frac{I}{c}$$

Substitute numerical values and evaluate P_r:

$$P_r = \frac{6.363\times10^7\ \text{W/m}^2}{2.998\times10^8\ \text{m/s}} = \boxed{0.212\,\text{Pa}}$$

55 ••• [SSM] A conductor in the shape of a long solid cylinder that has a length L, a radius a, and a resistivity ρ carries a steady current I that is uniformly distributed over its cross-section. (*a*) Use Ohm's law to relate the electric field $\vec{E}$ in the conductor to I, ρ, and a. (*b*) Find the magnetic field $\vec{B}$ just outside the conductor. (*c*) Use the results from Part (*a*) and Part (*b*) to compute the Poynting vector $\vec{S} = \left(\vec{E} \times \vec{B}\right)/\mu_0$ at $r = a$ (the edge of the conductor). In what direction is $\vec{S}$? (*d*) Find the flux $\oint S_n dA$ through the surface of the cylinder, and use this flux to show that the rate of energy flow into the conductor equals I^2R, where R is the resistance of the cylinder.

Picture the Problem A side view of the cylindrical conductor is shown in the diagram. Let the current be to the right (in the +x direction) and choose a coordinate system in which the +y direction is radially outward from the axis of the conductor. Then the +z direction is tangent to cylindrical surfaces that are concentric with the axis of the conductor (out of the plane of the diagram at the location indicated in the diagram). We can use Ohm's law to relate the electric field strength E in the conductor to I, ρ, and a and Ampere's law to find the magnetic field strength B just outside the conductor. Knowing $\vec{E}$ and $\vec{B}$ we can find $\vec{S}$ and, using its normal component, show that the rate of energy flow into the conductor equals I^2R, where R is the resistance.



(*a*) Apply Ohm's law to the cylindrical conductor to obtain:

$$V = IR = \frac{I\rho L}{A} = \frac{I\rho}{\pi a^2}L = EL$$

where $E = \dfrac{I\rho}{\pi a^2}$.

Because $\vec{E}$ is in the same direction as I:

$$\vec{E} = \boxed{\frac{I\rho}{\pi a^2}\hat{\boldsymbol{i}}}$$ where $\hat{\boldsymbol{i}}$ is a unit vector in the direction of the current.

(*b*) Applying Ampere's law to a circular path of radius a at the surface of the cylindrical conductor yields:

$$\oint_C \vec{B} \cdot d\vec{\ell} = B(2\pi a) = \mu_0 I_{\text{enclosed}} = \mu_0 I$$

Solve for the magnetic field strength B to obtain:

$$B = \frac{\mu_0 I}{2\pi a}$$

Apply a right-hand rule to determine the direction of $\vec{B}$ at the point of interest shown in the diagram:

$$\vec{B} = \boxed{\frac{\mu_0 I}{2\pi a}\hat{\theta}}$$ where $\hat{\boldsymbol{\theta}}$ is a unit vector perpendicular to $\hat{\boldsymbol{i}}$ and tangent to the surface of the conducting cylinder.

(*c*) The Poynting vector is given by:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Substitute for $\vec{E}$ and $\vec{B}$ and simplify to obtain:

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{I\rho}{\pi a^2}\right)\hat{\boldsymbol{i}} \times \left(\frac{\mu_0 I}{2\pi a}\right)\hat{\boldsymbol{k}} = -\frac{I^2\rho}{2\pi^2 a^3}\hat{\boldsymbol{j}}$$

Letting $\hat{\boldsymbol{r}}$ be a unit vector directed radially outward from the axis of the cylindrical conductor yields.

$$\vec{S} = \boxed{-\frac{I^2\rho}{2\pi^2 a^3}\hat{\boldsymbol{r}}}$$ where $\hat{\boldsymbol{r}}$ is a unit vector directed radially outward away from the axis of the conducting cylinder.

(*d*) The flux through the surface of the conductor into the conductor is:

$$\oint S_n dA = S(2\pi aL)$$

Substitute for S_n, the *inward* component of $\vec{S}$, and simplify to obtain:

$$\oint S_n dA = \frac{I^2\rho}{2\pi^2 a^3}(2\pi aL) = \frac{I^2\rho L}{\pi a^2}$$

Because $R = \frac{\rho L}{A} = \frac{\rho L}{\pi a^2}$:

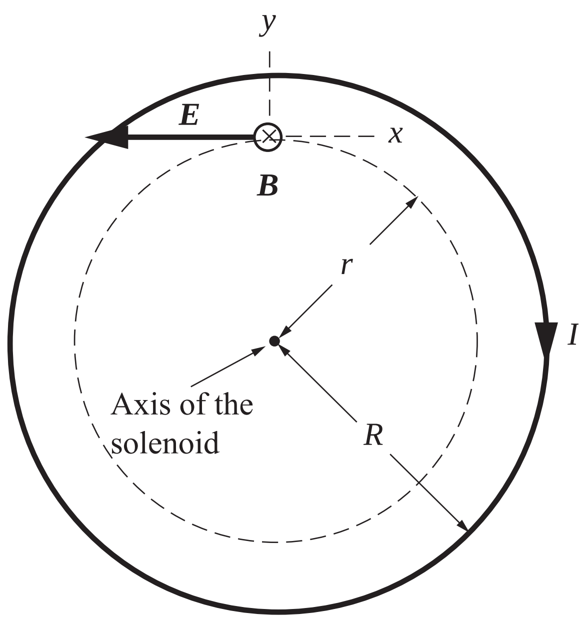
$$\oint S_n dA = \boxed{I^2 R}$$

Remarks: The equality of the two flow rates is a statement of the conservation of energy.

56 ••• A long solenoid that has n turns per unit length carries a current that increases linearly with time. The solenoid has radius R, length L, and the current I in the windings is given by $I = at$. (*a*) Find the induced electric field at a distance $r < R$ from the central axis of the solenoid. (*b*) Find the magnitude and direction of the Poynting vector $\vec{S}$ at $r = R$ (just inside the solenoid windings). (*c*) Calculate

the flux $\oint S_{\text{n}}dA$ into the region inside the solenoid, and show that this flux equals the rate of increase of the magnetic energy inside the solenoid.

Picture the Problem An end view of the solenoid is shown in the diagram. Let the current be clockwise and choose a coordinate system in which the +*x* direction is tangent to cylindrical surfaces concentric with the axis of the solenoid. Note that the +*z* direction is out of the plane of the diagram. We can use Faraday's law to express the induced electric field at a distance $r < R$ from the solenoid axis in terms of the rate of change of magnetic flux and $B = n\mu_0 at$ to express $\vec{B}$ in terms of the current in the windings of the solenoid. We can use the results of (*a*) to find the Poynting vector $\vec{S}$ at the cylindrical surface $r = R$ just inside the solenoid windings. In Part (*c*) we'll use the definition of flux and the expression for the magnetic energy in a given region to show that the flux of $\vec{S}$ into the solenoid equals the rate of increase of the magnetic energy inside the solenoid.



(*a*) Apply Faraday's law to a circular path of radius $r < R$ to relate the magnitude of the induced electric field to the magnitude of the rate of change of the magnetic flux:

$$\oint_C \vec{E}\cdot d\vec{\ell} = E(2\pi r) = -\frac{d\phi_{\text{m}}}{dt}$$

Solving for *E* yields:

$$E = -\frac{1}{2\pi r}\frac{d\phi_{\text{m}}}{dt} \qquad (1)$$

Express the magnetic field strength inside a long solenoid:

$$B = n\mu_0 I = n\mu_0 at$$

The magnetic flux through a circle of radius r is:

$$\phi_m = BA = n\mu_0 at\pi r^2$$

Substitute for ϕ_m in equation (1) and simplify to obtain:

$$E = -\frac{1}{2\pi r}\frac{d}{dt}\left[n\mu_0 at\pi r^2\right] = -\frac{n\mu_0 ar}{2}$$

The direction of $\vec{E}$ is such that it produces an emf that opposes the increase in the current, so if the current is clockwise, then $\vec{E}$ is in the opposite direction and:

$$\vec{E} = \boxed{-\tfrac{1}{2}n\mu_0 ar\hat{\theta}}$$

where $\hat{\theta}$ is a unit vector that is tangent to the circles that are concentric with the axis of the solenoid.

(*b*) Express $\vec{S}$ at $r = R$:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \qquad (2)$$

The magnitude of $\vec{B}$ is given by:

$$B = n\mu_0 I = n\mu_0 at$$

Applying a right-hand rule yields:

$$\vec{B} = -(n\mu_0 at)\hat{k}$$

Substitute for $\vec{E}$ and $\vec{B}$ in equation (2) and simplify to obtain:

$$\vec{S} = \frac{1}{\mu_0}\left(-\frac{n\mu_0 ar}{2}\right)\hat{i}\times(-n\mu_0 at)\hat{k} = -\frac{n^2\mu_0 a^2 Rt}{2}\hat{j}$$

Because $\vec{E}\times\vec{B}$ is a vector that points toward the axis of the solenoid, we can also write $\vec{S}$ as:

$$\vec{S} = \boxed{-\tfrac{1}{2}n^2\mu_0 a^2 Rt\hat{r}}$$

where $\hat{r}$ is a unit vector that points radially outward—away from the axis of the solenoid.

(*c*) Consider a cylindrical surface of length L and radius R. Because $\vec{S}$ points inward, the energy flowing into the solenoid per unit time is:

$$\oint S_n dA = 2\pi RLS = 2\pi RL\left(\frac{n^2\mu_0 a^2 Rt}{2}\right) = n^2\pi\mu_0 R^2 La^2 t$$

Express the magnetic energy in the solenoid:

$$U_B = u_m V = \frac{B^2}{2\mu_0}\left(\pi R^2 L\right) = \frac{\left(\mu_0 nat\right)^2}{2\mu_0}\left(\pi R^2 L\right) = \frac{n^2\pi\,\mu_0 R^2 La^2t^2}{2}$$

Evaluate dU_B/dt:

$$\frac{dU_B}{dt} = \frac{d}{dt}\left[\frac{n^2\pi\,\mu_0 R^2 La^2t^2}{2}\right] = \boxed{n^2\pi\,\mu_0 R^2 La^2 t} = \oint S_n\,dA$$

Remarks: The equality of the two flow rates is a statement of the conservation of energy.

57 ••• Small particles are be blown out of the solar system by the radiation pressure of sunlight. Assume that each particle is spherical, has a radius *r*, has a density of 1.00 g/cm^3, and absorbs all the radiation in a cross-sectional area of πr^2. Assume the particles are located at some distance *d* from the Sun, which has a total power output of 3.83×10^{26} W. (*a*) What is the critical value for the radius *r* of the particle for which the radiation force of repulsion just balances the gravitational force of attraction to the Sun? (*b*) Do particles that have radii larger than the critical value get ejected from the solar system, or is it only particles that have radii smaller than the critical value that get ejected? Explain your answer.

Picture the Problem We can use a condition for translational equilibrium to obtain an expression relating the forces due to gravity and radiation pressure that act on the particles. We can express the force due to radiation pressure in terms of the radiation pressure and the effective cross sectional area of the particles and the radiation pressure in terms of the intensity of the solar radiation. We can solve the resulting equation for *r*.

(*a*) Apply the condition for translational equilibrium to the particle:

$$F_r - F_g = 0$$

or, since $F_r = P_r A$ and $F_g = mg$,

$$P_r A - \frac{GM_s m}{R^2} = 0 \qquad (1)$$

The radiation pressure P_r depends on the intensity of the radiation *I*:

$$P_r = \frac{I}{c}$$

The intensity of the solar radiation at a distance R is:

$$I=\frac{P}{4\pi R^2}$$

Substitute for I to obtain:

$$P_\mathrm{r}=\frac{P}{4\pi R^2 c}$$

Substitute for P_r, A, and m in equation (1):

$$\frac{P}{4\pi R^2 c}\left(\pi r^2\right)-\frac{\frac{4}{3}\pi r^3\rho GM_\mathrm{s}}{R^2}=0$$

Solve for r to obtain:

$$r=\frac{3P}{16\pi\rho cGM_\mathrm{s}}$$

Substitute numerical values and evaluate r:

$$r=\frac{3\left(3.83\times10^{26}\ \mathrm{W}\right)}{16\pi\left(1.00\,\mathrm{g/cm^3}\right)\left(2.998\times10^{8}\ \mathrm{m/s}\right)\left(6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(1.99\times10^{30}\ \mathrm{kg}\right)}$$
$$=\boxed{574\ \mathrm{nm}}$$

(*b*) Because both the gravitational and radiation pressure forces decrease as the square of the distance from the Sun, it is then a comparison of grain mass to grain area. Since mass is proportional to volume and thus varies with the cube of the radius, the larger grains have more mass and thus experience a stronger gravitational than radiation-pressure force. The critical radius is an upper limit and so particles smaller than that radius will be blown out.

58 ••• When an electromagnetic wave at normal incidence on a perfectly conducting surface is reflected, the electric field of the reflected wave at the reflecting surface is equal and opposite to the electric field of the incident wave at the reflecting surface. (*a*) Explain why this assertion is valid. (*b*) Show that the superposition of incident and reflected waves results in a standing wave. (*c*) Are the magnetic fields of the incident waves and reflected waves at the reflecting surface equal and opposite as well? Explain your answer.

Picture the Problem

(*a*) At a perfectly conducting surface $\vec{E}=0$. Therefore, the sum of the electric fields of the incident and reflected wave must add to zero, and so $\vec{E}_\mathrm{i}=-\vec{E}_\mathrm{r}$.

(*b*) Let the incident and reflected waves be described by:

$$E_\mathrm{i}=E_{0y}\cos(\omega t-kx)$$
and
$$E_\mathrm{r}=-E_{0y}\cos(\omega t+kx)$$

Use the trigonometric identity cos($\alpha + \beta$) = cosαcosβ – sinαsinβ to obtain:

$$\begin{aligned} E_{\text{i}} + E_{\text{r}} &= E_{0y}\cos(\omega t - kx) - E_{0y}\cos(\omega t + kx) = E_{0y}[\cos(\omega t - kx) - \cos(\omega t + kx)] \\ &= E_{0y}[\cos\omega t\cos(-kx) - \sin\omega t\sin(-kx) - \cos\omega t\cos kx + \sin\omega t\sin(kx)] \\ &= E_{0y}[\cos\omega t\cos kx + \sin\omega t\sin kx - \cos\omega t\cos kx + \sin\omega t\sin kx] \\ &= \boxed{2E_{0y}\sin\omega t\sin kx} \end{aligned}$$

, the equation of a standing wave.

(c) Because $\vec{E} \times \vec{B} = \mu_0 \vec{S}$ and $\vec{S}$ is in the direction of propagation of the wave, we see that for the incident wave $B_{\text{i}} = B_z\cos(\omega t - kx)$. Since both $\vec{S}$ and E_y are reversed for the reflected wave $B_r = B_z\cos(\omega t + kx)$. So the magnetic field vectors are in the direction at the reflecting surface and add at that surface.
Hence $\vec{B} = 2\vec{B}_r$.

59 ••• [SSM] An intense point source of light radiates 1.00 MW isotropically (uniformly in all directions). The source is located 1.00 m above an infinite, perfectly reflecting plane. Determine the force that the radiation pressure exerts on the plane.

Picture the Problem Let the point source be a distance a above the plane. Consider a ring of radius r and thickness dr in the plane and centered at the point directly below the light source. Express the force on this elemental ring and integrate the resulting expression to obtain F.

The intensity anywhere along this infinitesimal ring is given by:

$$\frac{P}{4\pi(r^2 + a^2)}$$

The elemental force dF on the elemental ring of area $2\pi rdr$ is given by:

$$dF = \frac{P\,rdr}{c(r^2 + a^2)}\frac{a}{\sqrt{r^2 + a^2}} = \frac{Pardr}{c(r^2 + a^2)^{3/2}}$$

where we have taken into account that only the normal component of the incident radiation contributes to the force on the plane, and that the plane is a perfectly reflecting plane.

Integrate dF from $r = 0$ to $r = \infty$:

$$F = \frac{Pa}{c}\int_0^\infty \frac{rdr}{(r^2 + a^2)^{3/2}}$$

From integral tables:

$$\int_0^\infty \frac{r\,dr}{\left(r^2+a^2\right)^{3/2}} = \left.\frac{-1}{\sqrt{r^2+a^2}}\right]_0^\infty = \frac{1}{a}$$

Substitute to obtain:

$$F = \frac{Pa}{c}\left(\frac{1}{a}\right) = \frac{P}{c}$$

Substitute numerical values and evaluate F:

$$F = \frac{1.00\,\text{MW}}{2.998\times10^8\ \text{m/s}} = \boxed{3.34\,\text{mN}}$$

Chapter 31
Properties of Light

Conceptual Problems

1 • [SSM] A ray of light reflects from a plane mirror. The angle between the incoming ray and the reflected ray is 70°. What is the angle of reflection? (*a*) 70°, (*b*) 140°, (*c*) 35°, (*d*) Not enough information is given to determine the reflection angle.

Determine the Concept Because the angles of incidence and reflection are equal, their sum is 70° and the angle of reflection is 35°. $\boxed{(c)}$ is correct.

2 • A ray of light passes in air is incident on the surface of a piece of glass. The angle between the normal to the surface and the incident ray is 40°, and the angle between the normal and the refracted ray 28°. What is the angle between the incident ray and the refracted ray? (*a*) 12°, (*b*) 28°, (*c*) 40° ,(*d*) 68°

Determine the Concept The angle between the incident ray and the refracted ray is the difference between the angle of incidence and the angle of refraction. $\boxed{(a)}$ is correct.

3 • During a physics experiment, you are measuring refractive indices of different transparent materials using a red helium-neon laser beam. For a given angle of incidence, the beam has an angle of refraction equal to 28° in material A and an angle of refraction equal to 26° in material B. Which material has the larger index of refraction? (*a*) A, (*b*) B, (*c*) The indices of refraction are the same. (*d*) You cannot determine the relative magnitudes of the indices of refraction from the data given.

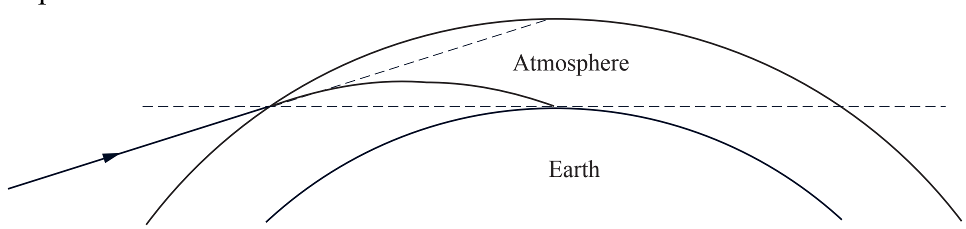
Determine the Concept The refractive index is a measure of the extent to which a material refracts light that passes through it. Because the angles of incidence are the same for materials A and B and the angle of refraction is smaller (more bending of the light) for material B, its index of refraction is larger than that of A. $\boxed{(b)}$ is correct.

4 • A ray of light passes from air into water, striking the surface of the water at an angle of incidence of 45º. Which, if any, of the following four quantities change as the light enters the water: (*a*) wavelength, (*b*) frequency, (*c*) speed of propagation, (*d*) direction of propagation, (*e*) none of the above?

Determine the Concept When light passes from air into water its wavelength changes ($\lambda_{\text{water}} = \lambda_{\text{air}}/n_{\text{water}}$), its speed changes ($v_{\text{water}} = c/n_{\text{water}}$), and the direction of its propagation changes in accordance with Snell's law. Its frequency does not change, so $\boxed{(a)}$, $\boxed{(c)}$ and $\boxed{(d)}$ change.

5 • Earth's atmosphere decreases in density as the altitude increases. As a consequence, the index of refraction of the atmosphere also decreases as altitude increases. Explain how one can see the Sun when it is below the horizon. (The horizon is the extension of a plane that is tangent to Earth's surface.) Why does the setting Sun appear flattened?

Determine the Concept The decrease in the index of refraction *n* of the atmosphere with altitude results in refraction of the light from the Sun, bending it toward the normal to the surface of Earth. Consequently, the Sun can be seen even after it is just below the horizon. The setting Sun appears flattened because refraction is greater for light from the lower part of the Sun than for light from the upper part.



6 • A physics student playing pocket billiards wants to strike her cue ball so that it hits a cushion and then hits the eight ball squarely. She chooses several points on the cushion and then measures the distances from each point to the cue ball and to the eight ball. She aims at the point for which the sum of these distances is least. (*a*) Will her cue ball hit the eight ball? (*b*) How is her method related to Fermat's principle? Neglect any effects due to ball rotation.

Determine the Concept (*a*) Yes. (*b*) Her procedure is based on Fermat's principle of least time. The ball presumably bounces off the cushion with an angle of reflection equal to the angle of incidence, just as a light ray would do if the cushion were a mirror. The least time would also be the shortest distance of travel for the light ray.

7 • **[SSM]** A swimmer at point *S* in Figure 31-53 develops a leg cramp while swimming near the shore of a calm lake and calls for help. A lifeguard at point *L* hears the call. The lifeguard can run 9.0 m/s and swim 3.0 m/s. She knows physics and chooses a path that will take the least time to reach the swimmer. Which of the paths shown in the figure does the lifeguard take?

Determine the Concept The path through point *D* is the path of least time. In analogy to the refraction of light, the ratio of the sine of the angle of incidence to the sine of the angle of refraction equals the ratio of the speeds of the lifeguard in each medium. Careful measurements from the figure show that path *LDS* is the path that best satisfies this criterion.

8 • Material A has a higher index of refraction than material B. Which material has the larger critical angle for total internal reflection when the material is in air? (*a*) A, (*b*) B, (*c*) The angles are the same. (*d*) You cannot compare the angles based on the data given.

Determine the Concept Because the product of the index of refraction on the incident side of the interface and the sine of the critical angle is equal to one (the index of refraction of air), the material with the smaller index of refraction will have the larger critical angle. $\boxed{(b)}$ is correct.

9 • **[SSM]** A human eye perceives color using a structure which is called a *cone* that is is located on the retina. Three types of molecules compose these cones and each type of molecule absorbs either red, green, or blue light by resonance absorption. Use this fact to explain why the color of an object that appears blue in air appears blue underwater, in spite of the fact that the wavelength of the light is shortened in accordance with Equation 31-6.

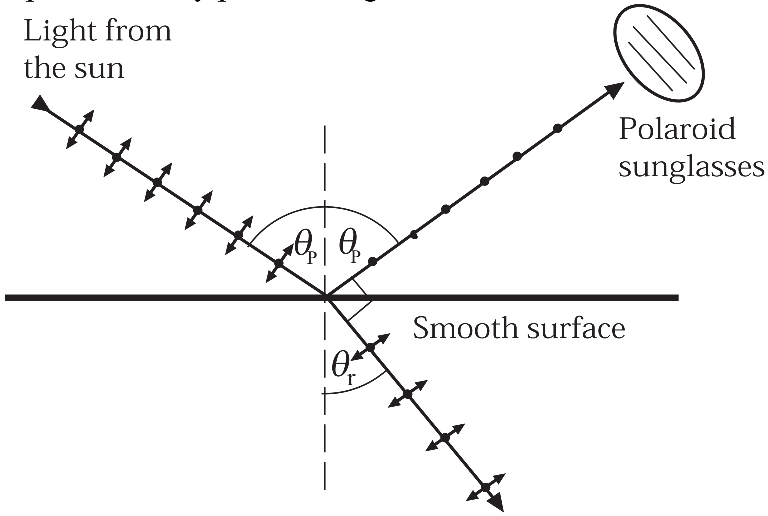
Determine the Concept In resonance absorption, the molecules respond to the frequency of the light through the Einstein photon relation $E = hf$. Neither the wavelength nor the frequency of the light within the eyeball depend on the index of refraction of the medium outside the eyeball. Thus, the color appears to be the same in spite of the fact that the wavelength has changed.

10 • Let θ be the angle between the transmission axes of two polarizing sheets. Unpolarized light of intensity I is incident upon the first sheet. What is the intensity of the light transmitted through both sheets? (*a*) $I\cos^2\theta$, (*b*) $(I\cos^2\theta)/2$, (*c*) $(I\cos^2\theta)/4$, (*d*) $I\cos\theta$, (*e*) $(I\cos\theta)/4$, (*f*) None of the above

Picture the Problem The intensity of the light transmitted by the second polarizer is given by $I_{\text{trans}} = I_0\cos^2\theta$, where $I_0 = \frac{1}{2}I$. Therefore, $I_{\text{trans}} = \frac{1}{2}I\cos^2\theta$ and $\boxed{(b)}$ is correct.

11 •• [SSM] Draw a diagram to explain how Polaroid sunglasses reduce glare from sunlight reflected from a smooth horizontal surface, such as the surface found on a pool of water. Your diagram should clearly indicate the direction of polarization of the light as it propagates from the Sun to the reflecting surface and then through the sunglasses into the eye.

Determine the Concept The following diagram shows unpolarized light from the sun incident on the smooth surface at the polarizing angle for that particular surface. The reflected light is polarized perpendicular to the plane of incidence, i.e., in the horizontal direction. The sunglasses are shown in the correct orientation to pass vertically polarized light and block the reflected sunlight.



12 • Why is it far less dangerous to stand in front of an intense beam of red light than in front of a very low-intensity beam of gamma rays?

Determine the Concept The red light photons contain considerably less energy than the gamma photons so, even though there are likely to be fewer photons in the low-intensity gamma beam, each one is potentially dangerous.

13 • Three energy states of an atom are A, B and C. State B is 2.0 eV above state A and state C is 3.00 eV above state B. Which atomic transition results in the emission of the shortest wavelength of light? (*a*) B → A, (*b*) C → B, (*c*) C → A, (*d*) A→ C

Determine the Concept Because the wavelength of the light emitted in an atomic transition is inversely proportional to the energy difference between the energy levels, the highest energy difference produces the shortest wavelength light. $\boxed{(c)}$ is correct.

14 • In Problem 13, if the atom is initially in state A, which transition results in the emission of the longest wavelength light? (*a*) A → B, (*b*) B → C, (*c*) A → C, (*d*) B→ A

Determine the Concept Because the energy required to induce an atomic transition varies inversely with the wavelength of the light that must be absorbed to induce the transition and the transition from A to B is the lowest energy transition, the transition B→ A results in the longest wavelength light. $\boxed{(d)}$ is correct.

15 • [SSM] What role does the helium play in a helium–neon laser?

Determine the Concept The population inversion between the state $E_{2,\mathrm{Ne}}$ and the state 1.96 eV below it (see Figure 31-51) is achieved by inelastic collisions between neon atoms and helium atoms excited to the state $E_{2,\mathrm{He}}$.

16 • When a beam of visible white light that passes through a gas of atomic hydrogen at room temperature is viewed with a spectroscope, dark lines are observed at the wavelengths of the hydrogen atom emission series. The atoms that participate in the resonance absorption then emit light of the same wavelength as they return to the ground state. Explain why the observed spectrum nevertheless exhibits pronounced dark lines.

Determine the Concept Although the excited atoms emit light of the same frequency on returning to the ground state, the light is emitted in a random direction, not exclusively in the direction of the incident beam. Consequently, the beam intensity is greatly diminished at this frequency.

17 • [SSM] Which of the following types of light would have the highest energy photons? (*a*) red (*b*) infrared (*c*) blue (*d*) ultraviolet

Determine the Concept The energy of a photon is directly proportional to the frequency of the light and inversely proportional to its wavelength. Of the portions of the electromagnetic spectrum include in the list of answers, ultraviolet light has the highest frequency. $\boxed{(d)}$ is correct.

Estimation and Approximation

18 • Estimate the time required for light to make the round trip during Galileo's experiment to measure the speed of light. Compare the time of the round trip to typical human response times. How accurate do you think this experiment is?

Picture the Problem We can use the distance, rate, and time relationship to estimate the time required to travel 6 km.

Express the distance d the light in Galileo's experiment traveled in terms of its speed c and the elapsed time Δt:

$$d = c\Delta t \Rightarrow \Delta t = \frac{d}{c}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{6\,\text{km}}{2.998\times10^{8}\ \text{m/s}} = \boxed{2\times10^{-5}\ \text{s}}$$

Because human reaction is approximately 0.3 s:

$$\frac{\Delta t_{\text{reaction}}}{\Delta t} = \frac{0.3\ \text{s}}{2\times10^{-5}\ \text{s}} \approx 2\times10^{4}$$

$$\text{or}\ \Delta t_{\text{reaction}} \approx \boxed{\left(2\times10^{4}\right)\Delta t}$$

Because human reaction time is so much longer than the travel time for the light, there was no way that Galileo's experiment could demonstrate that the speed of light was not infinite.

19 • Estimate the time delay in receiving a light on your retina when you are wearing eyeglasses compared to when you are not wearing your eyeglasses.

Determine the Concept We'll assume that the source of the photon is a distance L from your retina and express the difference in the photon's travel time when you are wearing your glasses. Let the thickness of your glasses by 2 mm and the index of refraction of the material from which they are constructed by 1.5.

When you are not wearing your glasses, the time required for a light photon, originating a distance L away, to reach your retina is given by:

$$\Delta t_0 = \frac{L}{c}$$

If glass of thickness d and index of refraction n is inserted in the path of the photon, its travel time becomes:

$$\Delta t = t_{\text{in air}} + t_{\text{in glass}} = \frac{L-d}{c} + \frac{d}{c/n}$$

$$= \frac{L+(n-1)d}{c} = \Delta t_0 + \frac{(n-1)d}{c}$$

The time delay is the difference between Δt and Δt_0:

$$t_{\text{delay}} = \Delta t - \Delta t_0 = \frac{(n-1)d}{c}$$

Substitute numerical values and evaluate t_{delay}:

$$t_{\text{delay}} = \frac{(1.5-1)(2\ \text{mm})}{2.998\times10^{8}\ \text{m/s}} \approx \boxed{3\ \text{ps}}$$

20 •• Estimate the number of photons that enter your eye if you look for a tenth of a second at the Sun. What energy is absorbed by your eye during that time, assuming that all the photons are absorbed? The total power output of the Sun is 4.2×10^{26} W.

Picture the Problem The rate at which photons enter your eye is the ratio of power incident on your pupil to the energy per photon. We'll assume that the electromagnetic radiation from the Sun is at 550 nm and that, therefore, its photons have energy (given by $E = hf = hc/\lambda$) of 2.25 eV.

The rate at which photons enter your eye is the ratio of the rate at which energy is incident on your pupil to the energy carried by each photon:

$$\frac{dN}{dt} = \frac{P_{\substack{\text{incident}\\ \text{on pupil}}}}{E_{\text{per photon}}} \quad (1)$$

The intensity of the radiation from the Sun is given by:

$$I_{\text{Sun}} = \frac{P_{\substack{\text{incident}\\ \text{on pupil}}}}{A_{\text{pupil}}} = \frac{P_{\text{Sun}}}{A_{\substack{\text{sphere at Earth's}\\ \text{distance from the Sun}}}}$$

Solving for $P_{\substack{\text{incident}\\ \text{on pupil}}}$ yields:

$$P_{\substack{\text{incident}\\ \text{on pupil}}} = \frac{A_{\text{pupil}}}{A_{\substack{\text{sphere at Earth's}\\ \text{distance from the Sun}}}} P_{\text{Sun}} \quad (2)$$

Substituting in equation (1) yields:

$$\frac{dN}{dt} = \frac{A_{\text{pupil}}}{A_{\substack{\text{sphere at Earth's}\\ \text{distance from the Sun}}}} \frac{P_{\text{Sun}}}{E_{\text{per photon}}}$$

Substitute for the two areas and simplify to obtain:

$$\frac{dN}{dt} = \frac{\frac{1}{4}\pi d_{\text{pupil}}^2}{4\pi R_{\text{Earth-Sun}}^2} \frac{P_{\text{Sun}}}{E_{\text{per photon}}} = \left(\frac{d_{\text{pupil}}}{4R_{\text{Earth-Sun}}}\right)^2 \frac{P_{\text{Sun}}}{E_{\text{per photon}}}$$

Substitute numerical values and evaluate dN/dt:

$$\frac{dN}{dt} = \left(\frac{1\text{ mm}}{4\left(1.50\times10^{11}\text{ m}\right)}\right)^2 \frac{4.2\times10^{26}\text{ W}}{2.25\text{ eV}\times\dfrac{1.602\times10^{-19}\text{ J}}{\text{eV}}} = 3.237\times10^{15}\text{ s}^{-1}$$

or, separating variables and integrating this expression,

$$N = \left(3.237\times10^{15}\text{ s}^{-1}\right)t$$

Evaluating N for $t = 0.1$ s yields:

$$N(0.1\,\text{s}) = (3.237\times10^{15}\ \text{s}^{-1})(0.1\,\text{s}) \approx \boxed{3\times10^{14}\ \text{photons}}$$

The energy deposited, assuming all the photons are absorbed, is the product of the rate at which energy is incident on the pupil and the time during which it is delivered:

$$E = P_{\substack{\text{incident}\\\text{on pupil}}}\,t$$

Substituting for $P_{\substack{\text{incident}\\\text{on pupil}}}$ from equation (2) yields:

$$E = \frac{A_{\text{pupil}}}{A_{\substack{\text{sphere at Earth's}\\\text{distance from the Sun}}}}P_{\text{Sun}}t$$

Substitute for the two areas and simplify to obtain:

$$E = \frac{\frac{1}{4}\pi d_{\text{pupil}}^2}{4\pi R_{\text{Earth-Sun}}^2}P_{\text{Sun}}t = \left(\frac{d_{\text{pupil}}}{4R_{\text{Earth-Sun}}}\right)^2 P_{\text{Sun}}t$$

Substitute numerical values and evaluate E for $t = 0.1$ s:

$$E(0.1\,\text{s}) = \left(\frac{1\,\text{mm}}{4(1.50\times10^{11}\ \text{m})}\right)^2 (4.2\times10^{26}\ \text{W})(0.1\,\text{s}) = 0.1167\ \text{mJ} \approx \boxed{0.1\,\text{mJ}}$$

21 •• Römer was observing the eclipses of Jupiter's moon Io with the hope that they would serve as a highly accurate clock that would be independent of longitude. (Prior to GPS, such a clock was needed for accurate navigation.) Io eclipses (enters the umbra of Jupiter's shadow) every 42.5 h. Assuming an eclipse of Io is observed on Earth on June 1 at midnight when Earth is at location A (as shown in Figure 31-54), predict the expected time of observation of an eclipse one-quarter of a year later when Earth is at location B, assuming (a) the speed of light is infinite and (b) the speed of light is 2.998×10^8 m/s.

Picture the Problem We can use the period of Io's motion and the position of Earth at B to find the number of eclipses of Io during Earth's movement and then use this information to find the number of days before a night-time eclipse. During the 42.5 h between eclipses of Jupiter's moon, Earth moves from A to B, increasing the distance from Jupiter by approximately the distance from Earth to the Sun, making the path for the light longer and introducing a delay in the onset of the eclipse.

(a) Find the time it takes Earth to travel from point A to point B:

$$t_{A\to B}=\frac{T_{\text{earth}}}{4}=\frac{365.24\,\text{d}}{4}\times\frac{24\,\text{h}}{\text{d}}$$
$$=2191.4\,\text{h}$$

Because there are 42.5 h between eclipses of Io, the number of eclipses N occurring in the time it takes for Earth to move from A to B is:

$$N=\frac{t_{A\to B}}{T_{\text{Io}}}=\frac{2191.4\,\text{h}}{42.5\,\text{h}}=51.56$$

Hence, in one-fourth of a year, there will be 51.56 eclipses. Because we want to find the next occurrence that happens in the evening hours, we'll use 52 as the number of eclipses. We'll also assume that Jupiter is visible so that the eclipse of Io can be observed at the time we determine.

Relate the time $t(N)$ at which the Nth eclipse occurs to N and the period T_{Io} of Io:

$$t(N)=NT_{\text{Io}}$$

Evaluate $t(52)$ to obtain:

$$t(52)=(52)\left(42.5\,\text{h}\times\frac{1\,\text{d}}{24\,\text{h}}\right)$$
$$=92.083\,\text{d}$$

Subtract the number of whole days to find the clock time t:

$$t=t(52)-92\,\text{d}=92.083\,\text{d}-92\,\text{d}$$
$$=0.083\,\text{d}\times\frac{24\,\text{h}}{\text{d}}=1.992\,\text{h}$$
$$\approx\boxed{2:00\,\text{a.m.}}$$

Because June, July, and August have 30, 31, and 31 d, respectively, the date is:

$$\boxed{\text{September 1}}$$

(b) Express the time delay Δt in the arrival of light from Io due to Earth's location at B:

$$\Delta t=\frac{r_{\text{earth-sun}}}{c}$$

Substitute numerical values and evaluate Δt:

$$\Delta t=\frac{1.5\times10^{11}\,\text{m}}{2.998\times10^{8}\,\text{m/s}}=500\,\text{s}\times\frac{1\,\text{min}}{60\,\text{s}}$$
$$=8.34\,\text{min}$$

Hence, the eclipse will actually occur at 2:08 a.m., September 1

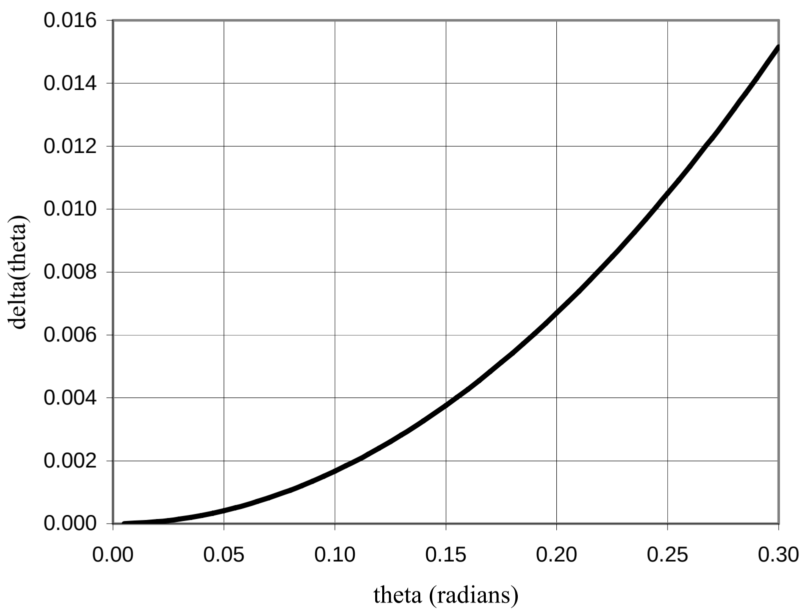
22 •• If the angle of incidence is small enough, the small angle approximation $\sin\theta \approx \theta$ may be used to simplify Snell's law of refraction. Determine the maximum value of the angle that would make the value for the angle differ by no more than one percent from the value for the sine of the angle. (This approximation will be used in connection with image formation by spherical surfaces in Chapter 32.)

Picture the Problem We can express the relative error in using the small angle approximation and then either use 1) trial-and-error methods, 2) a spreadsheet program, or 3) the Solver capability of a scientific calculator to solve the transcendental equation that results from setting the error function equal to 0.01.

Express the relative error δ in using the small angle approximation:

$$\delta(\theta) = \frac{\theta - \sin\theta}{\sin\theta} = \frac{\theta}{\sin\theta} - 1$$

A spreadsheet program was used to plot the following graph of $\delta(\theta)$.



From the graph, we can see that $\delta(\theta) < 1\%$ for $\theta \le 0.24$ radians. In degree measure, $\theta \le \boxed{14°}$

Remarks: Using the Solver program on a TI-85 gave $\theta = 0.244$ radians.

The Speed of Light

23 • Mission Control sends a brief wake-up call to astronauts in a spaceship that is far from Earth. 5.0 s after the call is sent, Mission Control can hear the groans of the astronauts. How far from Earth is the spaceship? (*a*) 7.5×10^8 m, (*b*) 15×10^8 m, (*c*) 30×10^8 m, (*d*) 45×10^8 m, (*e*) The spaceship is on the moon.

Picture the Problem We can use the distance, rate, and time relationship to find the distance to the spaceship.

Relate the distance d to the spaceship to the speed of electromagnetic radiation in a vacuum and to the time for the message to reach the astronauts:

$$d = c\Delta t$$

Noting that the time for the message to reach the astronauts is half the time for Mission Control to hear their response, substitute numerical values and evaluate d:

$$d = \left(2.998\times10^8\ \text{m/s}\right)\left(2.5\,\text{s}\right) = 7.5\times10^8\ \text{m}$$

and $\boxed{(a)}$ is correct.

24 • The distance from a point on the surface of Earth to a point on the surface of the moon is measured by aiming a laser light beam at a reflector on the surface of the moon and measuring the time required for the light to make a round trip. The uncertainty in the measured distance Δx is related to the uncertainty in the measured time Δt by $\Delta x = \frac{1}{2}c\Delta t$. If the time intervals can be measured to ±1.00 ns, (*a*) find the uncertainty of the distance. (*b*) Estimate the percentage uncertainty in the distance.

Picture the Problem We can use the given information that the uncertainty in the measured distance Δx is related to the uncertainty in the time Δt by $\Delta x = c\Delta t$ to evaluate Δx.

(*a*) The uncertainty in the distance is:

$$\Delta x = \tfrac{1}{2}c\Delta t$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \tfrac{1}{2}\left(2.998\times10^8\ \text{m/s}\right)\left(\pm1.00\,\text{ns}\right) = \boxed{\pm15.0\,\text{cm}}$$

(*b*)The percent uncertainty in the distance to the Moon is:

$$\frac{\Delta x_{\text{Earth to Moon}}}{x_{\text{Earth to Moon}}} = \frac{15.0\ \text{cm}}{3.84\times10^{8}\ \text{m}} \approx \boxed{10^{-8}\%}$$

25 •• [SSM] Ole Römer discovered the finiteness of the speed of light by observing Jupiter's moons. Approximately how sensitive would the timing apparatus need to be in order to detect a shift in the predicted time of the moon's eclipses that occur when the moon happens to be at perigee (3.63×10^{5} km) and those that occur when the moon is at apogee (4.06×10^{5} km)? Assume that an instrument should be able to measure to at least one-tenth the magnitude of the effect it is to measure.

Picture the Problem His timing apparatus would need to be sensitive enough to measure the difference in times for light to travel to Earth when the moon is at perigee and at apogee.

The sensitivity of the timing apparatus would need to be one-tenth of the difference in time for light to reach Earth from the two positions of the moon of Jupiter:

$$\text{Sensitivity} = \tfrac{1}{10}\Delta t$$

where Δt is the time required for light to travel between the two positions of the moon.

The time required for light to travel between the two positions of the moon is given by:

$$\Delta t = \frac{d_{\substack{\text{moon}\\ \text{at apogee}}} - d_{\substack{\text{moon}\\ \text{at perigee}}}}{c}$$

Substituting for Δt yields:

$$\text{Sensitivity} = \frac{d_{\substack{\text{moon}\\ \text{at apogee}}} - d_{\substack{\text{moon}\\ \text{at perigee}}}}{10c}$$

Substitute numerical values and evaluate the required sensitivity:

$$\Delta t = \frac{4.06\times10^{5}\ \text{km} - 3.63\times10^{5}\ \text{km}}{(10)(2.998\times10^{8}\ \text{m/s})} = \boxed{14\ \text{ms}}$$

Remarks: Instruments with this sensitivity did not exist in the 17th century.

Reflection and Refraction

26 • Calculate the fraction of light energy reflected from an air–water interface at normal incidence.

Picture the Problem Use the equation relating the intensity of reflected light at normal incidence to the intensity of the incident light and the indices of refraction of the media on either side of the interface.

Express the intensity I of the light reflected from an air-water interface at normal incidence in terms of the indices of refraction and the intensity I_0 of the incident light:

$$I=\left(\frac{n_{\text{air}}-n_{\text{water}}}{n_{\text{air}}+n_{\text{water}}}\right)^2 I_0$$

Solve for the ratio I/I_0:

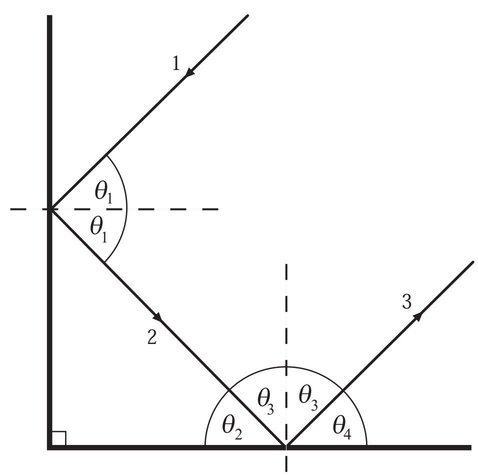
$$\frac{I}{I_0}=\left(\frac{n_{\text{air}}-n_{\text{water}}}{n_{\text{air}}+n_{\text{water}}}\right)^2$$

Substitute numerical values and evaluate I/I_0:

$$\frac{I}{I_0}=\left(\frac{1.00-1.33}{1.00+1.33}\right)^2=\boxed{2.0\%}$$

27 • A ray of light is incident on one of a pair of mirrors set at right angles to each A ray of light is incident on one of two mirrors that are set at right angles to each other. The plane of incidence is perpendicular to both mirrors. Show that after reflecting from each mirror, the ray will emerge traveling in the direction opposite to the incident direction, regardless of the angle of incidence.

Picture the Problem The diagram shows ray 1 incident on the vertical surface at an angle θ_1, reflected as ray 2, and incident on the horizontal surface at an angle of incidence θ_3. We'll prove that rays 1 and 3 are parallel by showing that $\theta_1 = \theta_4$, i.e., by showing that they make equal angles with the horizontal. Note that the law of reflection has been used in identifying equal angles of incidence and reflection.



We know that the angles of the right triangle formed by ray 2 and the two mirror surfaces add up to 180°:

$$\theta_2+90°+(90°-\theta_1)=180°$$

or

$$\theta_1=\theta_2$$

The sum of θ_2 and θ_3 is 90°:

$$\theta_3=90°-\theta_2$$

Because $\theta_1=\theta_2$:

$$\theta_3=90°-\theta_1$$

The sum of θ_4 and θ_3 is 90°:

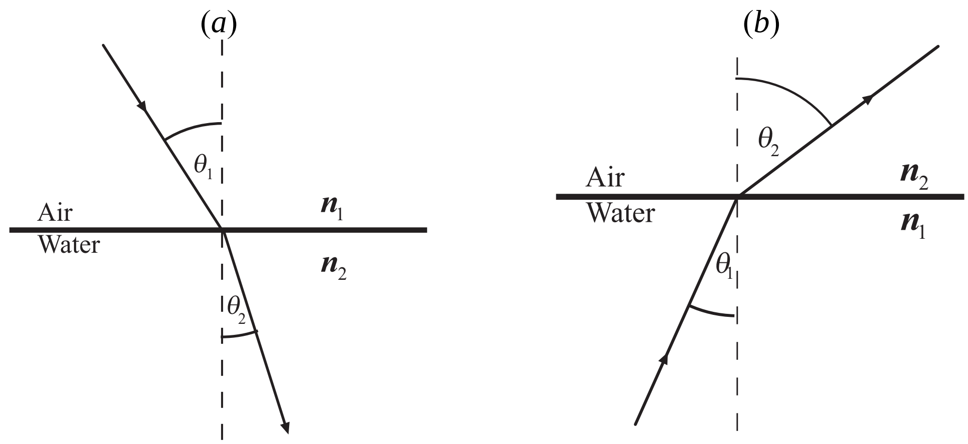
$$\theta_3+\theta_4=90°$$

Substitute for θ_3 to obtain:

$$(90° - \theta_1) + \theta_4 = 90° \Rightarrow \theta_1 = \boxed{\theta_4}$$

28 •• (*a*) A ray of light in air is incident on an air–water interface. Using a **spreadsheet** or graphing program, plot the angle of refraction as a function of the angle of incidence from 0º to 90º. (*b*) Repeat Part (*a*), but for a ray of light in water that is incident on a water–air interface. [For Part (*b*), there is no reflected ray for angles of incidence that are greater than the critical angle.]

Picture the Problem Diagrams showing the light rays for the two cases are shown below. In (*a*) the light travels from air into water and in (*b*) it travels from water into air.



(*a*) Apply Snell's law to the air-water interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where the angles of incidence and refraction are θ_1 and θ_2, respectively.

Solving for θ_2 yields:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

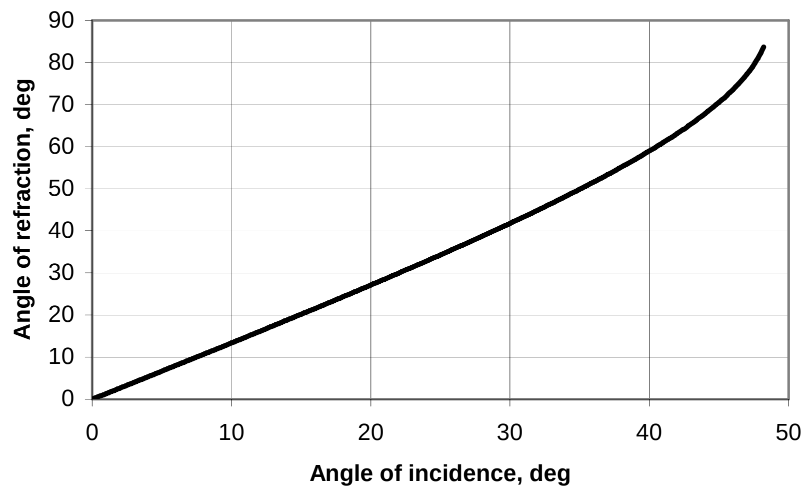
A spreadsheet program to graph θ_2 as a function of θ_1 is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
B1	1	n_1
B2	1.33333	n_2
A6	0	θ_1 (deg)
A7	A6 + 5	$\theta_1 + \Delta\theta$
B6	A6*PI()/180	$\theta_1 \times \dfrac{\pi}{180}$
C6	ASIN(($B$1/$B$2)*SIN(B6))	$\sin^{-1}\left(\dfrac{n_1}{n_2}\sin\theta_1\right)$
D6	C6*180/PI()	$\theta_2 \times \dfrac{180}{\pi}$

	A	B	C	D
1	n_1=	1		
2	n_2=	1.33333		
3				
4	θ_1	θ_1	θ_2	θ_2
5	(deg)	(rad)	(rad)	(deg)
6	0	0.00	0.000	0.00
7	1	0.02	0.013	0.75
8	2	0.03	0.026	1.50
9	3	0.05	0.039	2.25
21	87	1.52	0.847	48.50
22	88	1.54	0.847	48.55
23	89	1.55	0.848	48.58
24	90	1.57	0.848	48.59

A graph of θ_2 as a function of θ_1 follows:

(b) Change the contents of cell B1 to 1.33333 and the contents of cell B2 to 1 to obtain the following graph:



Note that as the angle of incidence approaches the critical angle for a water-air interface (48.6°), the angle of refraction approaches 90°.

29 •• The red light from a helium-neon laser has a wavelength of 632.8 nm in air. Find the (a) speed, (b) wavelength, and (c) frequency of helium-neon laser light in air, water, and glass. (The glass has an index of refraction equal to 1.50.)

Picture the Problem We can use the definition of the index of refraction to find the speed of light in the three media. The wavelength of the light in each medium is its wavelength in air divided by the index of refraction of the medium. The frequency of the helium-neon laser light is the same in all media and is equal to its value in air. The wavelength of helium-neon laser light in air is 632.8 nm.

The speed of light in a medium whose index of refraction is n is given by:

$$v = \frac{c}{n} \quad (1)$$

The wavelength of light in a medium whose index of refraction is n is given by:

$$\lambda_{n} = \frac{\lambda_{air}}{n} \quad (2)$$

The frequency of the light is equal to its frequency in air independently of the medium in which the light is propagating:

$$f = \frac{c}{\lambda} = \frac{2.998\times10^{8}\ \text{m/s}}{632.8\ \text{nm}} = \boxed{4.74\times10^{14}\ \text{Hz}}$$

Substitute numerical values in equation (1) and evaluate v_{water}:

$$v_{water} = \frac{2.998\times10^{8}\ \text{m/s}}{1.33} = \boxed{2.25\times10^{8}\ \text{m/s}}$$

Substitute numerical values in equation (2) and evaluate λ_{water}:

$$\lambda_{water} = \frac{632.8\ \text{nm}}{1.33} = \boxed{476\ \text{nm}}$$

The other speeds and wavelengths are found similarly and are summarized in the following table:

	(*a*) speed (m/s)	(*b*) wavelength (nm)	(*c*) frequency (Hz)
Air	$\boxed{3.00\times10^{8}}$	$\boxed{633}$	$\boxed{4.74\times10^{14}}$
Water	$\boxed{2.25\times10^{8}}$	$\boxed{476}$	$\boxed{4.74\times10^{14}}$
glass	$\boxed{2.00\times10^{8}}$	$\boxed{422}$	$\boxed{4.74\times10^{14}}$

30 •• The index of refraction for silicate flint glass is 1.66 for violet light that has a wavelength in air equal to 400 nm and 1.61 for red light that has a wavelength in air equal to 700 nm. A ray of 700-nm–wavelength red light and a ray of 400-nm-wavelength violet light both have angles of refraction equal to 30º upon entering the glass from air. (*a*) Which is greater, the angle of incidence of the ray of red light or the angle of incidence of the ray of violet light? Explain your answer. (*b*) What is the difference between the angles of incidence of the two rays?

Picture the Problem Let the subscript 1 refer to the air and the subscript 2 to the silicate glass and apply Snell's law to the air-glass interface.

(*a*) Because the index of refraction for violet light is larger than that of red light, for a given incident angle violet light would refract more than red light. Thus to exhibit the same refraction angle, violet light would require an angle of incidence larger than that of red light.

(*b*) Express the difference in their angles of incidence:

$$\Delta\theta = \theta_{1,\text{violet}} - \theta_{1,\text{red}} \qquad (1)$$

Apply Snell's law to the air-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solving for θ_1 yields:

$$\theta_1 = \sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_2\right)$$

Substitute for $\theta_{1,\text{violet}}$ and $\theta_{1,\text{red}}$ in equation (1) to obtain:

$$\Delta\theta = \sin^{-1}\left(\frac{n_{\text{violet}}}{n_{\text{air}}}\sin\theta_{2,\text{ violet}}\right) - \sin^{-1}\left(\frac{n_{\text{red}}}{n_{\text{air}}}\sin\theta_{2,\text{ red}}\right)$$

For $\theta_{2,\text{ violet}} = \theta_{2,\text{ red}} = 30°$:

$$\Delta\theta = \sin^{-1}\left(\frac{n_{\text{violet}}}{n_{\text{air}}}\sin 30°\right) - \sin^{-1}\left(\frac{n_{\text{red}}}{n_{\text{air}}}\sin 30°\right) = \sin^{-1}\left(\frac{n_{\text{violet}}}{2n_{\text{air}}}\right) - \sin^{-1}\left(\frac{n_{\text{red}}}{2n_{\text{air}}}\right)$$

Substitute numerical values and evaluate $\Delta\theta$:

$$\Delta\theta = \sin^{-1}\left(\frac{1.66}{2(1.00)}\right) - \sin^{-1}\left(\frac{1.61}{2(1.00)}\right) = 56.10° - 53.61° = \boxed{2.49°}$$

Remarks: Note that $\Delta\theta$ is positive. This means that the angle for violet light is greater than that for red light and confirms our answer in Part (*a*).

31 •• [SSM] A slab of glass that has an index of refraction of 1.50 is submerged in water that has an index of refraction of 1.33. Light in the water is incident on the glass. Find the angle of refraction if the angle of incidence is (*a*) 60º, (*b*) 45º, and (*c*) 30º.

Picture the Problem Let the subscript 1 refer to the water and the subscript 2 to the glass and apply Snell's law to the water-glass interface.

Apply Snell's law to the water-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solving for θ_2 yields:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

(*a*) Evaluate θ_2 for $\theta_1 = 60°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.50}\sin 60°\right) = \boxed{50°}$$

(*b*) Evaluate θ_2 for $\theta_1 = 45°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.50}\sin 45°\right) = \boxed{39°}$$

(*c*) Evaluate θ_2 for $\theta_1 = 30°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.50}\sin 30°\right) = \boxed{26°}$$

32 •• Repeat Problem 31 for a beam of light initially in the glass that is incident on the glass–water interface at the same angles.

Picture the Problem Let the subscript 1 refer to the glass and the subscript 2 to the water and apply Snell's law to the glass-water interface.

Apply Snell's law to the water-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solving for θ_2 yields:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

(*a*) Evaluate θ_2 for $\theta_1 = 60°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.50}{1.33}\sin 60°\right) = \boxed{78°}$$

(*b*) Evaluate θ_2 for $\theta_1 = 45°$:

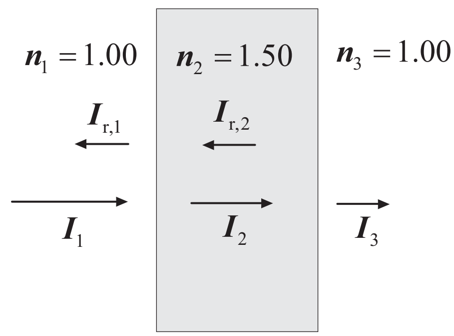
$$\theta_2 = \sin^{-1}\left(\frac{1.50}{1.33}\sin 45°\right) = \boxed{53°}$$

(*c*) Evaluate θ_2 for $\theta_1 = 30°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.50}{1.33}\sin 30°\right) = \boxed{34°}$$

33 •• A beam of light in air strikes a glass slab at normal incidence. The glass slab has an index of refraction of 1.50. (*a*) Approximately what percentage of the incident light intensity is transmitted through the slab (in one side and out the other)? (*b*) Repeat Part (*a*) if the glass slab is immersed in water.

Picture the Problem Let the subscript 1 refer to the medium to the left (air) of the first interface, the subscript 2 to glass, and the subscript 3 to the medium (air) to the right of the second interface. Apply the equation relating the intensity of reflected light at normal incidence to the intensity of the incident light and the indices of refraction of the media on either side of the interface to both interfaces. We'll neglect multiple reflections at glass-air interfaces.



(*a*) Express the intensity of the transmitted light in the second medium:

$$I_2 = I_1 - I_{r,1} = I_1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_1$$

$$= I_1\left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]$$

Express the intensity of the transmitted light in the third medium:

$$I_3 = I_2 - I_{r,2} = I_2 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2 I_2$$

$$= I_2\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

Substitute for I_2 to obtain:

$$I_3 = I_1\left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

Solve for the ratio I_3/I_1 to obtain:

$$\frac{I_3}{I_1} = \left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

Substitute numerical values and evaluate I_3/I_1:

$$\frac{I_3}{I_1} = \left[1 - \left(\frac{1.00 - 1.50}{1.00 + 1.50}\right)^2\right]\left[1 - \left(\frac{1.50 - 1.00}{1.50 + 1.00}\right)^2\right] = \boxed{92\%}$$

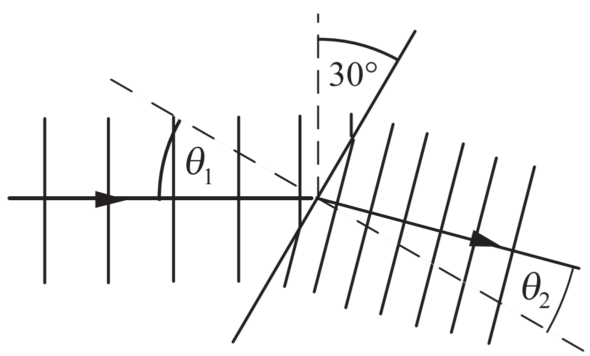
(*b*) With $n_1 = n_3 = 1.33$:

$$\frac{I_3}{I_1} = \left[1 - \left(\frac{1.33 - 1.50}{1.33 + 1.50}\right)^2\right]\left[1 - \left(\frac{1.50 - 1.33}{1.50 + 1.33}\right)^2\right] = \boxed{99\%}$$

34 •• This problem is a refraction analogy. A band is marching down a football field with a constant speed v_1. About midfield, the band comes to a section of muddy ground that has a sharp boundary making an angle of 30º with the 50-yd line, as shown in Figure 31-55. In the mud, each marcher moves at a speed equal to $\frac{1}{2}v_1$ in a direction perpendicular to the row of markers they are in. (*a*) Diagram how each line of marchers is bent as it encounters the muddy section of the field so that the band is eventually marching in a different direction. Indicate the original direction by a ray and the final direction by a second ray. (*b*) Find the angles between these rays and the line normal to the boundary. Is their direction of motion "bent" toward the normal or away from it? Explain your answer in terms of refraction.

Picture the Problem We can apply Snell's law to find the angle of refraction of the line of marchers as they enter the muddy section of the field

(*a*)



(*b*) Apply Snell's law at the interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solving for θ_2 yields:

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right]$$

The ratio of the indices of refraction is the reciprocal of the ratio of the speeds of the marchers in the two media:

$$\frac{n_1}{n_2} = \frac{\frac{v}{v_1}}{\frac{v}{v_2}} = \frac{v_2}{v_1} = \frac{\frac{1}{2}v_1}{v_1} = \frac{1}{2}$$

Because the left and right sides of the 30° angle and θ_1 are mutually perpendicular, θ_1 = 30°. Substitute numerical values and evaluate θ_2:

$$\theta_2 = \sin^{-1}\left[\tfrac{1}{2}\sin 30°\right] = \boxed{14°}$$

As the line enters the muddy field, its speed is reduced by half and the direction of the forward motion of the line is changed. In this case, the forward motion in the muddy field makes an angle θ_2 with respect to the normal to the boundary line. Note that the separation between successive lines in the muddy field is half that in the dry field.

35 •• [SSM] In Figure 31-56, light is initially in a medium that has an index of refraction n_1. It is incident at angle θ_1 on the surface of a liquid that has an index of refraction n_2. The light passes through the layer of liquid and enters glass that has an index of refraction n_3. If θ_3 is the angle of refraction in the glass, show that $n_1 \sin \theta_1 = n_3 \sin \theta_3$. That is, show that the second medium can be neglected when finding the angle of refraction in the third medium.

Picture the Problem We can apply Snell's law consecutively, first to the n_1-n_2 interface and then to the n_2-n_3 interface.

Apply Snell's law to the n_1-n_2 interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Apply Snell's law to the n_2-n_3 interface:

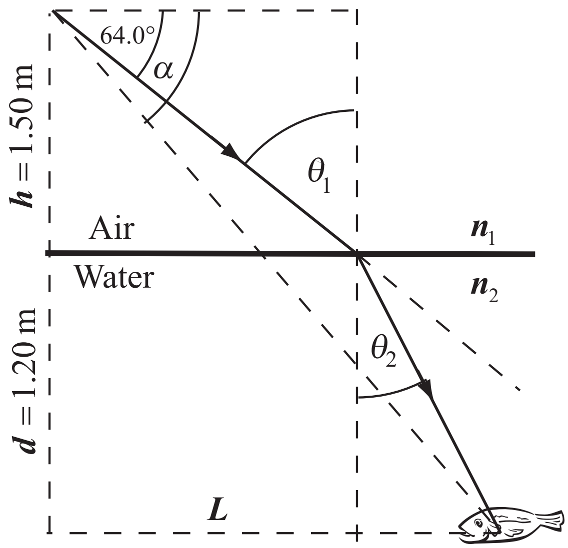
$$n_2 \sin\theta_2 = n_3 \sin\theta_3$$

Equate the two expressions for $n_2 \sin\theta_2$ to obtain:

$$\boxed{n_1 \sin\theta_1 = n_3 \sin\theta_3}$$

36 •• On a safari, you are spear fishing while wading in a river. You observe a fish gliding by you. If your line of sight to the fish is 64.0° degrees below the horizontal in air, and assuming the spear follows a straight-line path through the air and water after it is released, determine the angle below the horizontal that you should aim your spear gun in order to catch dinner. Assume the spear gun barrel is 1.50 m above the water surface, the fish is 1.20 m below the surface, and the spear travels in a straight line all the way to the fish.

Picture the Problem The following pictorial representation summarizes the information given in the problem statement. We can use the geometry of the diagram and apply Snell's law at the air-water interface to find the aiming angle α.



Use the pictorial representation to express the aiming angle α:

$$\alpha = \tan^{-1}\left[\frac{h+d}{L+\ell}\right]$$

Referring to the diagram, note that:

$L = h\tan\theta_1$ and $\ell = d\tan\theta_2$

Substituting for L and ℓ yields:

$$\alpha = \tan^{-1}\left[\frac{h+d}{h\tan\theta_1 + d\tan\theta_2}\right]$$

Apply Snell's law at the air-water interface to obtain:

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

Solving for θ_2 yields:

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right]$$

Substitute for θ_2 to obtain:

$$\alpha = \tan^{-1}\left[\frac{h+d}{h\tan\theta_1 + d\tan\left(\sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right]\right)}\right]$$

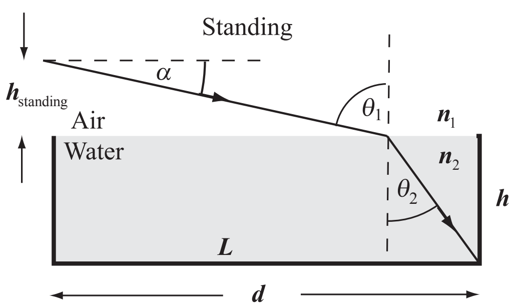
Noting that θ_1 is the complement of 64.0°, substitute numerical values and evaluate α:

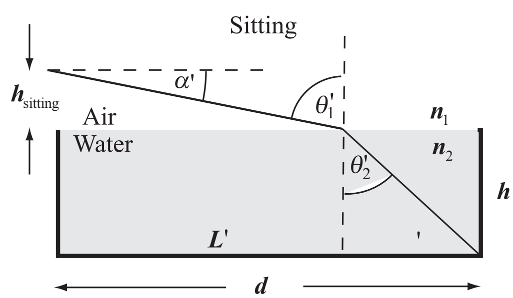
$$\alpha = \tan^{-1}\left[\frac{1.50\text{ m} + 1.20\text{ m}}{(1.50\text{ m})\tan 26.0° + (1.20\text{ m})\tan\left(\sin^{-1}\left[\frac{1.00}{1.33}\sin 26.0°\right]\right)}\right] = \boxed{66.9°}$$

That is, you should aim 66.9° below the horizontal.

37 ••• You are standing on the edge of a swimming pool and looking directly across at the opposite side. You notice that the bottom edge of the opposite side of the pool appears to be at an angle of 28° degrees below the horizontal. However, when you sit on the pool edge, the bottom edge of the opposite side of the pool appears to be at an angle of only 14° below the horizontal. Use these observations to determine the width and depth of the pool. *Hint: You will need to estimate the height of your eyes above the surface of the water when standing and sitting.*

Picture the Problem The following diagrams represent the situations when you are standing on the edge of the pool (the diagram to the left) and when you are sitting on the edge of the pool (the diagram to the right). We can use Snell's law and the geometry of the pool to determine the width and depth of the pool.





Use the fact that angles α and θ_1 are complementary, as are α' and θ_1' to determine θ_1 and θ_1':

$$\theta_1 = 90° - \alpha = 90° - 28° = 62°$$

and

$$\theta_1' = 90° - \alpha' = 90° - 14° = 76°$$

Express the distances L and L' in terms of θ_1 and θ_1':

$$L = h_{\text{standing}} \tan\theta_1$$

and

$$L' = h_{\text{sitting}} \tan\theta_1'$$

Assuming that your eyes are 1.7 m above the level of the water when you are standing and 0.7 m above the water when you are sitting, evaluate L and L':

$$L = (1.7\text{ m})\tan 62° = 3.197\text{ m}$$
and
$$L' = (0.7\text{ m})\tan 76° = 2.808\text{ m}$$

Referring to the pictorial representations, note that:

$$\tan\theta_2 = \frac{\ell}{h} = \frac{d-L}{h} \qquad (1)$$
and
$$\tan\theta_2' = \frac{\ell'}{h} = \frac{d-L'}{h}$$

Divide the first of these equations by the second to obtain:

$$\frac{\tan\theta_2}{\tan\theta_2'} = \frac{d-L}{d-L'}$$

Solving for d yields:

$$d = \frac{L'\tan\theta_2 - L\tan\theta_2'}{\tan\theta_2 - \tan\theta_2'} \qquad (2)$$

Apply Snell's law to the air-water interface when you are standing:

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right]$$

Substitute numerical values and evaluate θ_2:

$$\theta_2 = \sin^{-1}\left[\frac{1.00}{1.33}\sin 62°\right] = 41.60°$$

Apply Snell's law to the air-water interface when you are sitting:

$$\theta_2' = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1'\right]$$

Substitute numerical values and evaluate θ_2':

$$\theta_2' = \sin^{-1}\left[\frac{1.00}{1.33}\sin 76°\right] = 46.85°$$

Substitute numerical values in equation (2) and evaluate d:

$$d = \frac{(2.808\text{ m})\tan 41.60° - (3.197\text{ m})\tan 46.85°}{\tan 41.60° - \tan 46.85°} = 5.130\text{ m} = \boxed{5.1\text{ m wide}}$$

Solving equation (1) for h yields:

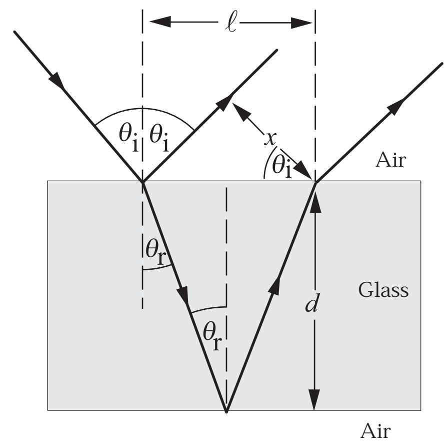
$$h = \frac{d-L}{\tan\theta_2}$$

Substitute numerical values and evaluate h:

$$h = \frac{5.130\text{ m} - 3.197\text{ m}}{\tan 41.60°} = \boxed{2.2\text{ m deep}}$$

38 ••• Figure 31-57 shows a beam of light incident on a glass plate of thickness *d* and index of refraction *n*. (*a*) Find the angle of incidence so that the separation *b* between the ray reflected from the top surface and the ray reflected from the bottom surface and exiting the top surface is a maximum. (*b*) What is this angle of incidence if the index of refraction of the glass is 1.60? (*c*) What is the separation of the two beams if the thickness of the glass plate is 4.0 cm?

Picture the Problem Let x be the perpendicular separation between the two rays and let ℓ be the separation between the points of emergence of the two rays on the glass surface. We can use the geometry of the refracted and reflected rays to express x as a function of ℓ, d, θ_r, and θ_i. Setting the derivative of the resulting equation equal to zero will yield the value of θ_i that maximizes x.



(*a*) Express ℓ in terms of d and the angle of refraction θ_r:

$$\ell = 2d\tan\theta_r$$

Express x as a function of ℓ, d, θ_r, and θ_i:

$$x = 2d\tan\theta_r\cos\theta_i$$

Differentiate x with respect to θ_i:

$$\frac{dx}{d\theta_i} = 2d\frac{d}{d\theta_i}\left(\tan\theta_r\cos\theta_i\right) = 2d\left(-\tan\theta_r\sin\theta_i + \sec^2\theta_r\cos\theta_i\frac{d\theta_r}{d\theta_i}\right) \quad (1)$$

Apply Snell's law to the air-glass interface:

$$n_1\sin\theta_i = n_2\sin\theta_r \quad (2)$$

or, because $n_1 = 1$ and $n_2 = n$,

$$\sin\theta_i = n\sin\theta_r$$

Differentiate implicitly with respect to θ_i to obtain:

$$\cos\theta_i d\theta_i = n\cos\theta_r d\theta_r$$

or

$$\frac{d\theta_r}{d\theta_i} = \frac{1}{n}\frac{\cos\theta_i}{\cos\theta_r}$$

Substitute in equation (1) to obtain:

$$\frac{dx}{d\theta_\mathrm{i}} = 2d\left(-\frac{\sin\theta_\mathrm{r}}{\cos\theta_\mathrm{r}}\sin\theta_\mathrm{i} + \frac{1}{n}\frac{\cos\theta_\mathrm{i}}{\cos^2\theta_\mathrm{r}}\frac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{r}}\right) = 2d\left(\frac{1}{n}\frac{\cos^2\theta_\mathrm{i}}{\cos^3\theta_\mathrm{r}} - \frac{\sin\theta_\mathrm{r}\sin\theta_\mathrm{i}}{\cos\theta_\mathrm{r}}\right)$$

Substitute $1-\sin^2\theta_\mathrm{i}$ for $\cos^2\theta_\mathrm{i}$ and $\frac{1}{n}\sin\theta_\mathrm{i}$ for $\sin\theta_\mathrm{r}$ to obtain:

$$\frac{dx}{d\theta_\mathrm{i}} = 2d\left(\frac{1-\sin^2\theta_\mathrm{i}}{n\cos^3\theta_\mathrm{r}} - \frac{\sin^2\theta_\mathrm{i}}{n\cos\theta_\mathrm{r}}\right)$$

Multiply the second term in parentheses by $\cos^2\theta_\mathrm{r}/\cos^2\theta_\mathrm{r}$ and simplify to obtain:

$$\frac{dx}{d\theta_\mathrm{i}} = 2d\left(\frac{1-\sin^2\theta_\mathrm{i}}{n\cos^3\theta_\mathrm{r}} - \frac{\sin^2\theta_\mathrm{i}\cos^2\theta_\mathrm{r}}{n\cos^3\theta_\mathrm{r}}\right) = \frac{2d}{n\cos^3\theta_\mathrm{r}}\left(1-\sin^2\theta_\mathrm{i} - \sin^2\theta_\mathrm{i}\cos^2\theta_\mathrm{r}\right)$$

Substitute $1-\sin^2\theta_\mathrm{r}$ for $\cos^2\theta_\mathrm{r}$:

$$\frac{dx}{d\theta_\mathrm{i}} = \frac{2d}{n\cos^3\theta_\mathrm{r}}\left[1-\sin^2\theta_\mathrm{i} - \sin^2\theta_\mathrm{i}\left(1-\sin^2\theta_\mathrm{r}\right)\right]$$

Substitute $\frac{1}{n}\sin\theta_\mathrm{i}$ for $\sin\theta_\mathrm{r}$ to obtain:

$$\frac{dx}{d\theta_\mathrm{i}} = \frac{2d}{n\cos^3\theta_\mathrm{r}}\left[1-\sin^2\theta_\mathrm{i} - \sin^2\theta_\mathrm{i}\left(1-\frac{1}{n^2}\sin^2\theta_\mathrm{i}\right)\right]$$

Factor out $1/n^2$, simplify, and set equal to zero to obtain:

$$\frac{dx}{d\theta_\mathrm{i}} = \frac{2d}{n^3\cos^3\theta_\mathrm{r}}\left[\sin^4\theta_\mathrm{i} - 2n^2\sin^2\theta_\mathrm{i} + n^2\right] = 0 \text{ for extrema}$$

If $dx/d\theta_\mathrm{i} = 0$, then it must be true that:

$$\sin^4\theta_\mathrm{i} - 2n^2\sin^2\theta_\mathrm{i} + n^2 = 0$$

Solve this quartic equation for θ_i to obtain:

$$\theta_\mathrm{i} = \boxed{\sin^{-1}\left(n\sqrt{1-\sqrt{1-\frac{1}{n^2}}}\right)}$$

(*b*) Evaluate θ_i for $n = 1.60$:

$$\theta_i = \sin^{-1}\left(1.6\sqrt{1-\sqrt{1-\frac{1}{(1.60)^2}}}\right) = \boxed{48.5°}$$

(*c*) In (*a*) we showed that:

$$x = 2d\tan\theta_r\cos\theta_i$$

Solve equation (2) for θ_r:

$$\theta_r = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Substitute numerical values and evaluate θ_r:

$$\theta_r = \sin^{-1}\left(\frac{1}{1.60}\sin 48.5°\right) = 27.9°$$

Substitute numerical values and evaluate *x*:

$$\boldsymbol{x} = 2(4.0\,\text{cm})\tan 27.9°\cos 48.5° = \boxed{2.8\,\text{cm}}$$

Total Internal Reflection

39 • **[SSM]** What is the critical angle for light traveling in water that is incident on a water–air interface?

Picture the Problem Let the subscript 1 refer to the water and the subscript 2 to the air and use Snell's law under total internal reflection conditions.

Use Snell's law to obtain:

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

When there is total internal reflection:

$$\theta_1 = \theta_c \text{ and } \theta_2 = 90°$$

Substitute to obtain:

$$n_1\sin\theta_c = n_2\sin 90° = n_2$$

Solving for θ_c yields:

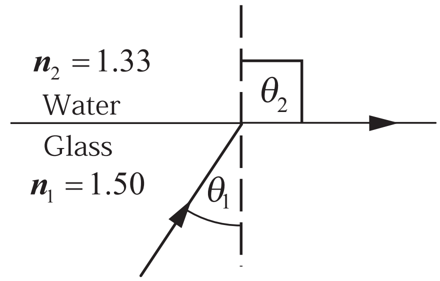
$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Substitute numerical values and evaluate θ_c:

$$\theta_c = \sin^{-1}\left(\frac{1.00}{1.33}\right) = \boxed{48.8°}$$

40 • A glass surface ($n = 1.50$) has a layer of water ($n = 1.33$) on it. Light in the glass is incident on the glass–water interface. Find the critical angle for total internal reflection.

Picture the Problem Let the index of refraction of glass be represented by n_1 and the index of refraction of water by n_2 and apply Snell's law to the glass-water interface under total internal reflection conditions.



Apply Snell's law to the glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

At the critical angle, $\theta_1 = \theta_c$ and $\theta_2 = 90°$:

$$n_1 \sin\theta_c = n_2 \sin 90°$$

Solve for θ_c:

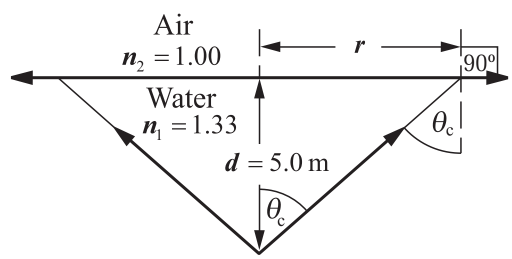
$$\theta_c = \sin^{-1}\left[\frac{n_2}{n_1}\sin 90°\right]$$

Substitute numerical values and evaluate θ_c:

$$\theta_c = \sin^{-1}\left[\frac{1.33}{1.50}\sin 90°\right] = \boxed{62.5°}$$

41 • A point source of light is located 5.0 m below the surface of a large pool of water. Find the area of the largest circle on the pool's surface through which light coming directly from the source can emerge.

Picture the Problem We can apply Snell's law to the water-air interface to express the critical angle θ_c in terms of the indices of refraction of water (n_1) and air (n_2) and then relate the radius of the circle to the depth d of the point source and θ_c.



Express the area of the circle whose radius is r:

$$A = \pi r^2$$

Relate the radius of the circle to the depth d of the point source and the critical angle θ_c:

$$r = d\tan\theta_c$$

Apply Snell's law to the water-air interface to obtain:

$$n_1 \sin\theta_c = n_2 \sin 90° = n_2$$

Solving for θ_c yields:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Substitute for r and θ_c to obtain:

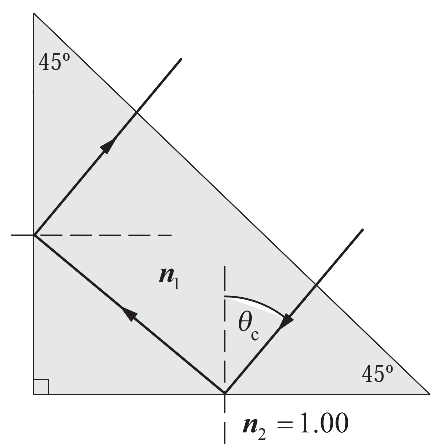
$$A = \pi[d\tan\theta_c]^2 = \pi\left[d\tan\left(\sin^{-1}\left\{\frac{n_2}{n_1}\right\}\right)\right]^2$$

Substitute numerical values and evaluate A:

$$A = \pi\left[(5.0\text{ m})\tan\left(\sin^{-1}\left\{\frac{1}{1.33}\right\}\right)\right]^2 = \boxed{1.0\times10^2\text{ m}^2}$$

42 •• Light traveling in air strikes the largest face of an isosceles-right-triangle prism at normal incidence. What is the speed of light in this prism if the prism is just barely able to produce total internal reflection?

Picture the Problem We can use the definition of the index of refraction to express the speed of light in the prism in terms of the index of refraction n_1 of the prism. The application of Snell's law at the prism-air interface will allow us to relate the index of refraction of the prism to the critical angle for total internal reflection. Finally, we can use the geometry of the isosceles-right-triangle prism to conclude that $\theta_c = 45°$.



Express the speed of light v in the prism in terms of its index of refraction n_1:

$$v = \frac{c}{n_1}$$

Apply Snell's law to the prism-air interface to obtain:

$$n_1\sin\theta_c = n_2\sin 90° = 1$$

Solving for n_1 yields:

$$n_1 = \frac{1}{\sin\theta_c}$$

Substitute for n_1 and simplify to obtain:

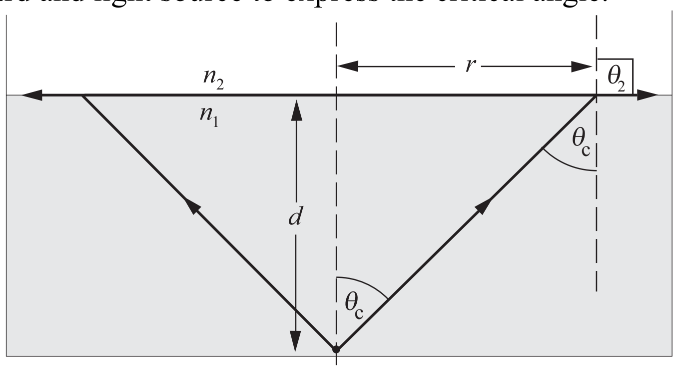
$$v = c\sin\theta_c$$

Substitute numerical values and evaluate v:

$$v = \left(2.998\times10^8\ \text{m/s}\right)\sin 45° = \boxed{2.1\times10^8\ \text{m/s}}$$

43 •• A point source of light is located at the bottom of a steel tank, and an opaque circular card of radius 6.00 cm is placed horizontally over it. A transparent fluid is gently added to the tank so that the card floats on the fluid surface with its center directly above the light source. No light is seen by an observer above the surface until the fluid is 5.00 cm deep. What is the index of refraction of the fluid?

Picture the Problem The observer above the surface of the fluid will not see any light until the angle of incidence of the light at the fluid-air interface is less than or equal to the critical angle for the two media. We can use Snell's law to express the index of refraction of the fluid in terms of the critical angle and use the geometry of card and light source to express the critical angle.



Apply Snell's law to the fluid-air interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Light is seen by the observer when $\theta_1 = \theta_c$ and $\theta_2 = 90°$:

$$n_1 \sin\theta_c = n_2 \sin 90° = n_2$$

Because the medium above the interface is air, $n_2 = 1$. Solve for n_1 to obtain:

$$n_1 = \frac{1}{\sin\theta_c}$$

From the geometry of the diagram:

$$\tan\theta_c = \frac{r}{d} \Rightarrow \theta_c = \tan^{-1}\left(\frac{r}{d}\right)$$

Substitute for θ_c to obtain:

$$n_1 = \frac{1}{\sin\left[\tan^{-1}\left(\frac{r}{d}\right)\right]}$$

Substitute numerical values and evaluate n_1:

$$n_1 = \frac{1}{\sin\left[\tan^{-1}\left(\frac{6.00\ \text{cm}}{5.00\ \text{cm}}\right)\right]} = \boxed{1.30}$$

44 •• An optical fiber allows rays of light to propagate long distances by using total internal reflection. Optical fibers are used extensively in medicine and in digital communications. As shown in Figure 31-58 the fiber consists of a core material that has an index of refraction n_2 and radius b surrounded by a cladding material that has an index of refraction $n_3 < n_2$. The *numerical aperture* of the fiber is defined as $\sin\theta_1$, where θ_1 is the angle of incidence of a ray of light that impinges on the center of the end of the fiber and then reflects off the core-cladding interface just at the critical angle. Using the figure as a guide, show that the numerical aperture is given by $\sin\theta_1 = \sqrt{n_2^2 - n_3^2}$ assuming the ray is initially in air. *Hint: Use of the Pythagorean theorem may be required.*

Picture the Problem We can use the geometry of the figure, the law of refraction at the air-n_1 interface, and the condition for total internal reflection at the n_1-n_2 interface to show that the numerical aperture is given by $\sin\theta_1 = \sqrt{n_2^2 - n_3^2}$.

Referring to the figure, note that:

$$\sin\theta_c = \frac{n_3}{n_2} = \frac{a}{c} \text{ and } \sin\theta_2 = \frac{b}{c}$$

Apply the Pythagorean theorem to the right triangle to obtain:

$$a^2 + b^2 = c^2 \text{ or } \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

Solving for $\frac{b}{c}$ yields:

$$\frac{b}{c} = \sqrt{1 - \frac{a^2}{c^2}}$$

Substitute for $\frac{a}{c}$ and $\frac{b}{c}$ to obtain:

$$\sin\theta_2 = \sqrt{1 - \frac{n_3^2}{n_2^2}}$$

Use the law of refraction to relate θ_1 and θ_2:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Substitute for $\sin\theta_2$, let $n_1 = 1$ (air), and simplify to obtain:

$$\sin\theta_1 = n_2\sqrt{1 - \frac{n_3^2}{n_2^2}} = \boxed{\sqrt{n_2^2 - n_3^2}}$$

45 •• **[SSM]** Find the maximum angle of incidence θ_1 of a ray that would propagate through an optical fiber that has a core index of refraction of 1.492, a core radius of 50.00 μm, and a cladding index of 1.489. See Problem 44.

Picture the Problem We can use the result of Problem 44 to find the maximum angle of incidence under the given conditions.

From Problem 44:

$$\sin\theta_1 = \sqrt{n_2^2 - n_3^2}$$

Solve for θ_1 to obtain:

$$\theta_1 = \sin^{-1}\left(\sqrt{n_2^2 - n_3^2}\right)$$

Substitute numerical values and evaluate θ_1:

$$\theta_1 = \sin^{-1}\left(\sqrt{(1.492)^2 - (1.489)^2}\right) = \boxed{5°}$$

46 •• Calculate the difference in time needed for two pulses of light to travel down 15.0 km of the fiber that is described in Problem 44. Assume that one pulse enters the fiber at normal incidence, and the second pulse enters the fiber at the maximum angle of incidence calculated in Problem 45. In fiber optics, this effect is known as *modal dispersion*.

Picture the Problem We can derive an expression for the difference in the travel times by expressing the travel time for a pulse that enters the fiber at the maximum angle of incidence and a pulse that enters the fiber at normal incidence. Examination of Figure 31-58 reveals that, if the length of the tube is *L*, the distance traveled by the pulse that enters at an angle θ_1 is the ratio of *c* to *a* multiplied by *L*.

The difference in the travel times Δt is given by:

$$\Delta t = t_{\theta_1} - t_{\substack{\text{normal}\\\text{incidence}}} \qquad (1)$$

The travel time for the pulse that enters the fiber at the maximum angle of incidence is:

$$t_{\theta_1} = \frac{\text{distance traveled}}{\text{speed in the medium}} = \frac{L\dfrac{c}{a}}{\dfrac{c}{n_2}}$$

The travel time for the pulse that enters the fiber normally is:

$$t_{\substack{\text{normal}\\\text{incidence}}} = \frac{L}{\dfrac{c}{n_2}}$$

Substitute for t_{θ_1} and $t_{\substack{\text{normal}\\\text{incidence}}}$ in equation (1) to obtain:

$$\Delta t = \frac{L\dfrac{c}{a}}{\dfrac{c}{n_2}} - \frac{L}{\dfrac{c}{n_2}}$$

Simplify to obtain:

$$\Delta t = \frac{n_2 L}{c}\left(\frac{c}{a} - 1\right) \qquad (2)$$

Referring to the figure, note that:

$$\sin\theta_c = \frac{a}{c}$$

From Snell's law, the sine of the critical angle is also given by:

$$\sin\theta_c = \frac{n_3}{n_2} \Rightarrow \frac{a}{c} = \frac{n_3}{n_2}$$

Substitute for c/a in equation (2) to obtain:

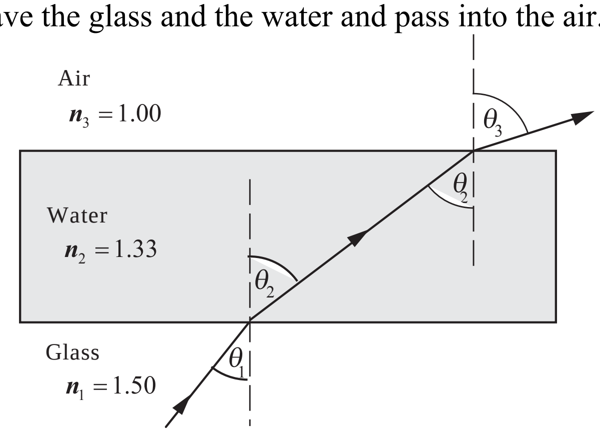
$$\Delta t = \frac{n_2 L}{c}\left(\frac{n_2}{n_3} - 1\right)$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{(1.492)(15\,\text{km})}{2.998\times10^8\ \text{m/s}}\left(\frac{1.492}{1.489} - 1\right) = \boxed{150\,\text{ns}}$$

47 ••• Investigate how a thin film of water on a glass surface affects the critical angle for total reflection. Use n = 1.50 for glass and n = 1.33 for water. (*a*) What is the critical angle for total internal reflection at the glass–water interface? (*b*) Does a range of incident angles exist such that the angles are greater than θ_c for glass-to-air refraction and for which the light rays will leave the glass, travel through the water and then pass into the air?

Picture the Problem Let the index of refraction of glass be represented by n_1, the index of refraction of water by n_2, and the index of refraction of air by n_3. We can apply Snell's law to the glass-water interface under total internal reflection conditions to find the critical angle for total internal reflection. The application of Snell's law to glass-air and glass-water interfaces will allow us to decide whether there are angles of incidence greater than θ_c for glass-to-air refraction for which light rays will leave the glass and the water and pass into the air.



(*a*) Apply Snell's law to the glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

At the critical angle, $\theta_1 = \theta_c$ and $\theta_2 = 90°$:

$$n_1 \sin\theta_c = n_2 \sin 90°$$

Solving for θ_c yields:

$$\theta_c = \sin^{-1}\left[\frac{n_2}{n_1}\sin 90°\right]$$

Substitute numerical values and evaluate θ_c:

$$\theta_c = \sin^{-1}\left[\frac{1.33}{1.50}\sin 90°\right] = \boxed{62.5°}$$

(*b*) Apply Snell's law to a water-air interface at the critical angle for a water-air interface:

$$n_2 \sin\theta_c = n_3 \sin 90°$$

Solving for θ_c yields:

$$\theta_c = \sin^{-1}\left(\frac{n_3}{n_2}\right)$$

Substitute numerical values and evaluate θ_c:

$$\theta_c = \sin^{-1}\left(\frac{1.00}{1.33}\right) = 48.8°$$

Apply Snell's law to a ray incident at the glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

and

$$\theta_1 = \sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_2\right)$$

For $\theta_2 = \theta_c$:

$$\theta_1 = \sin^{-1}\left[\left(\frac{n_2}{n_1}\right)\left(\frac{n_3}{n_2}\right)\right]\sin^{-1}\left(\frac{n_3}{n_1}\right)$$

Substitute numerical values and evaluate θ_1:

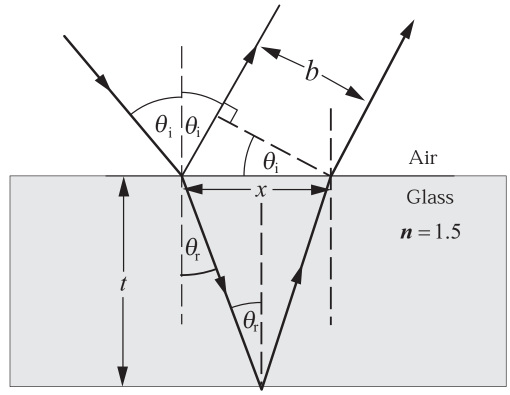
$$\theta_1 = \sin^{-1}\left(\frac{1.00}{1.50}\right) = 41.8°$$

Yes, if $\theta \geq 41.8°$, where θ is the angle of incidence for the rays in glass that are incident on the glass-water boundary, the rays will leave the glass through the water and pass into the air.

48 ••• A laser beam is incident on a plate of glass that is 3.0-cm thick (Figure 31-57). The glass has an index of refraction of 1.5 and the angle of incidence is 40°. The top and bottom surfaces of the glass are parallel. What is the distance *b*

between the beam formed by reflection off the top surface of the glass and the beam reflected off the bottom surface of the glass.

Picture the Problem The situation is shown in the adjacent figure. We can use the geometry of the diagram and trigonometric relationships to derive an expression for *d* in terms of the angles of incidence and refraction. Applying Snell's law will yield θ_r.



Express the distance *x* in terms of *t* and θ_r:

$$x = 2t\tan\theta_r$$

The separation of the reflected rays is:

$$b = x\cos\theta_i$$

Substitute for *x* to obtain:

$$b = 2t\tan\theta_r\cos\theta_i \qquad (1)$$

Apply Snell's law at the air-glass interface to obtain:

$$\sin\theta_i = n\sin\theta_r \Rightarrow \theta_r = \sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

Substitute for θ_r in equation (1) to obtain:

$$b = 2t\tan\left[\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)\right]\cos\theta_i$$

Substitute numerical values and evaluate *b*:

$$b = 2(3.0\,\text{cm})\tan\left[\sin^{-1}\left(\frac{\sin 40^\circ}{1.5}\right)\right]\cos 40^\circ$$

$$= \boxed{2.2\,\text{cm}}$$

Dispersion

49 • A beam of light strikes the plane surface of silicate flint glass at an angle of incidence of 45º. The index of refraction of the glass varies with wavelength (see Figure 31-59). How much smaller is the angle of refraction for violet light of wavelength 400 nm than the angle of refraction for red light of wavelength 700 nm?

Picture the Problem We can apply Snell's law of refraction to express the angles of refraction for red and violet light in silicate flint glass.

Express the difference between the angle of refraction for violet light and for red light:

$$\Delta\theta = \theta_{\mathrm{r,red}} - \theta_{\mathrm{r,violet}} \qquad (1)$$

Apply Snell's law of refraction to the interface to obtain:

$$\sin 45° = n \sin\theta_{\mathrm{r}} \Rightarrow \theta_{\mathrm{r}} = \sin^{-1}\left(\frac{1}{\sqrt{2}n}\right)$$

Substituting for θ_{r} in equation (1) yields:

$$\Delta\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}n_{\mathrm{red}}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}n_{\mathrm{violet}}}\right)$$

Substitute numerical values and evaluate $\Delta\theta$:

$$\Delta\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}(1.60)}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}(1.66)}\right) = 26.2° - 25.2° = \boxed{1.0°}$$

50 •• In many transparent materials, dispersion causes different colors (wavelengths) of light to travel at different speeds. This can cause problems in fiber-optic communications systems where pulses of light must travel very long distances in glass. Assuming a fiber is made of silicate crown glass (see Figure 31-59), calculate the difference in travel times that two short pulses of light take to travel 15.0 km in this fiber if the first pulse has a wavelength of 700 nm and the second pulse has a wavelength of 500 nm.

Picture the Problem The transit times will be different because the speed with which light of various wavelengths propagates in silicate crown glass is dependent on the index of refraction. We can use Figure 31-19 to estimate the indices of refraction for pulses of wavelengths 500 and 700 nm.

Express the difference in time needed for two short pulses of light to travel a distance L in the fiber:

$$\Delta t = \frac{L}{v_{500}} - \frac{L}{v_{700}}$$

Substitute for L, v_{500}, and v_{700} and simplify to obtain:

$$\Delta t = \frac{n_{500}L}{c} - \frac{n_{700}L}{c} = \frac{L}{c}\left(n_{500} - n_{700}\right)$$

Use Figure 31-59 to find the indices of refraction of silicate crown glass for the two wavelengths:

$n_{500} \approx 1.51$

and

$n_{700} \approx 1.50$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{15.0\,\text{km}}{2.998\times10^{8}\ \text{m/s}}(1.51-1.50)$$
$$\approx \boxed{0.5\,\mu\text{s}}$$

Polarization

51 • **[SSM]** What is the polarizing angle for light in air that is incident on (*a*) water (n = 1.33), and (*b*) glass (n = 1.50)?

Picture the Problem The polarizing angle is given by Brewster's law: $\tan\theta_\text{p} = n_2/n_1$ where n_1 and n_2 are the indices of refraction on the near and far sides of the interface, respectively.

Use Brewster's law to obtain:

$$\theta_\text{p} = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

(*a*) For n_1 = 1 and n_2 = 1.33:

$$\theta_\text{p} = \tan^{-1}\left(\frac{1.33}{1.00}\right) = \boxed{53.1°}$$

(*b*) For n_1 = 1 and n_2 = 1.50:

$$\theta_\text{p} = \tan^{-1}\left(\frac{1.50}{1.00}\right) = \boxed{56.3°}$$

52 • Light that is horizontally polarized is incident on a polarizing sheet. It is observed that only 15 percent of the intensity of the incident light is transmitted through the sheet. What angle does the transmission axis of the sheet make with the horizontal?

Picture the Problem The intensity of the transmitted light I is related to the intensity of the incident light I_0 and the angle the transmission axis makes with the horizontal θ according to $I = I_0\cos^2\theta$.

Express the intensity of the transmitted light in terms of the intensity of the incident light and the angle the transmission axis makes with the horizontal:

$$I = I_0\cos^2\theta \Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{I}{I_0}}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left(\sqrt{0.15}\right) = \boxed{67°}$$

53 • Two polarizing sheets have their transmission axes crossed so that no light gets through. A third sheet is inserted between the first two so that its transmission axis makes an angle θ with the transmission axis of the first sheet. Unpolarized light of intensity I_0 is incident on the first sheet. Find the intensity of the light transmitted through all three sheets if (*a*) $\theta = 45^\circ$ and (*b*) $\theta = 30^\circ$.

Picture the Problem Let I_n be the intensity after the *n*th polarizing sheet and use $I = I_0\cos^2\theta$ to find the intensity of the light transmitted through all three sheets for $\theta = 45^\circ$ and $\theta = 30^\circ$.

(*a*) The intensity of the light between the first and second sheets is:

$$I_1 = \tfrac{1}{2}I_0$$

The intensity of the light between the second and third sheets is:

$$I_2 = I_1\cos^2\theta_{1,2} = \tfrac{1}{2}I_0\cos^2 45^\circ = \tfrac{1}{4}I_0$$

The intensity of the light that has passed through the third sheet is:

$$I_3 = I_2\cos^2\theta_{2,3} = \tfrac{1}{4}I_0\cos^2 45^\circ = \boxed{\tfrac{1}{8}I_0}$$

(*b*) The intensity of the light between the first and second sheets is:

$$I_1 = \tfrac{1}{2}I_0$$

The intensity of the light between the second and third sheets is:

$$I_2 = I_1\cos^2\theta_{1,2} = \tfrac{1}{2}I_0\cos^2 30^\circ = \tfrac{3}{8}I_0$$

The intensity of the light that has passed through the third sheet is:

$$I_3 = I_2\cos^2\theta_{2,3} = \tfrac{3}{8}I_0\cos^2 60^\circ = \boxed{\tfrac{3}{32}I_0}$$

54 • A horizontal 5.0 mW laser beam that is vertically polarized is incident on a sheet that is oriented with its transmission axis vertical. Behind the first sheet is a second sheet that is oriented so that its transmission axis makes an angle of 27° with respect to the vertical. What is the power of the beam transmitted through the second sheet?

Picture the Problem Because the light is polarized in the vertical direction and the first polarizer is also vertically polarized, no loss of intensity results from the first transmission. We can use Malus's law to find the intensity of the light after it has passed through the second polarizer.

The intensity of the beam is the ratio of its power to cross-sectional area:

$$I = \frac{P}{A}$$

Express the intensity of the light between the first and second polarizers:

$$I_1 = I_0 \text{ and } P_1 = P_0$$

Express the law of Malus in terms of the power of the beam:

$$\frac{P}{A} = \frac{P_0}{A}\cos^2\theta \Rightarrow P = P_0\cos^2\theta$$

Express the power of the beam after the second transmission:

$$P_2 = P_1\cos^2\theta_{1,2} = P_0\cos^2\theta_{12}$$

Substitute numerical values and evaluate P_2:

$$P_2 = (5.0\,\text{mW})\cos^2 27° = \boxed{4.0\,\text{mW}}$$

55 •• **[SSM]** The polarizing angle for light in air that is incident on a certain substance is 60º. (*a*) What is the angle of refraction of light incident at this angle? (*b*) What is the index of refraction of this substance?

Picture the Problem Assume that light is incident in air (n_1 = 1.00). We can use the relationship between the polarizing angle and the angle of refraction to determine the latter and Brewster's law to find the index of refraction of the substance.

(*a*) At the polarizing angle, the sum of the angles of polarization and refraction is 90°:

$$\theta_p + \theta_r = 90° \Rightarrow \theta_r = 90° - \theta_p$$

Substitute for θ_p to obtain:

$$\theta_r = 90° - 60° = \boxed{30°}$$

(*b*) From Brewster's law we have:

$$\tan\theta_p = \frac{n_2}{n_1}$$

or, because n_1 = 1.00,

$$n_2 = \tan\theta_p$$

Substitute for θ_p and evaluate n_2:

$$n_2 = \tan 60° = \boxed{1.7}$$

56 •• Two polarizing sheets have their transmission axes crossed so that no light is transmitted. A third sheet is inserted so that its transmission axis makes an angle θ with the transmission axis of the first sheet. (*a*) Derive an expression for the intensity of the transmitted light as a function of θ. (*b*) Show that the intensity transmitted through all three sheets is maximum when θ = 45º.

Picture the Problem Let I_n be the intensity after the *n*th polarizing sheet and use $I = I_0 \cos^2\theta$ to find the intensity of the light transmitted through the three sheets.

(*a*) The intensity of the light between the first and second sheets is:

$$I_1 = \tfrac{1}{2} I_0$$

The intensity of the light between the second and third sheets is:

$$I_2 = I_1 \cos^2\theta_{1,2} = \tfrac{1}{2} I_0 \cos^2\theta$$

Express the intensity of the light that has passed through the third sheet and simplify to obtain:

$$\begin{aligned} I_3 &= I_2 \cos^2\theta_{2,3} \\ &= \tfrac{1}{2} I_0 \cos^2\theta \cos^2(90° - \theta) \\ &= \tfrac{1}{2} I_0 \cos^2\theta \sin^2\theta \\ &= \tfrac{1}{8} I_0 (2\cos\theta\sin\theta)^2 \\ &= \boxed{\tfrac{1}{8} I_0 \sin^2 2\theta} \end{aligned}$$

(*b*) Because the sine function is a maximum when its argument is 90°, the maximum value of I_3 occurs when $\theta = 45°$.

57 •• If the middle polarizing sheet in Problem 56 is rotating at an angular speed ω about an axis parallel with the light beam, find an expression for the intensity transmitted through all three sheets as a function of time. Assume that $\theta = 0$ at time $t = 0$.

Picture the Problem Let I_n be the intensity after the *n*th polarizing sheet, use $I = I_0 \cos^2\theta$ to find the intensity of the light transmitted through each sheet, and replace θ with ωt.

The intensity of the light between the first and second sheets is:

$$I_1 = \tfrac{1}{2} I_0$$

The intensity of the light between the second and third sheets is:

$$I_2 = I_1 \cos^2\theta_{1,2} = \tfrac{1}{2} I_0 \cos^2\omega t$$

Express the intensity of the light that has passed through the third sheet and simplify to obtain:

$$\begin{aligned} I_3 &= I_2 \cos^2\theta_{2,3} \\ &= \tfrac{1}{2} I_0 \cos^2\omega t \cos^2(90° - \omega t) \\ &= \tfrac{1}{2} I_0 \cos^2\omega t \sin^2\omega t \\ &= \tfrac{1}{8} I_0 (2\cos\omega t\sin\omega t)^2 \\ &= \boxed{\tfrac{1}{8} I_0 \sin^2 2\omega t} \end{aligned}$$

58 •• A stack of $N + 1$ ideal polarizing sheets is arranged so that each sheet is rotated by an angle of $\pi/(2N)$ rad with respect to the preceding sheet. A linearly polarized light wave of intensity I_0 is incident normally on the stack. The incident light is polarized along the transmission axis of the first sheet and is therefore perpendicular to the transmission axis of the last sheet in the stack.
(*a*) Show that the intensity of the light transmitted through the entire stack is given by $I_0 \cos^{2N}\left[\pi/(2N)\right]$. (*b*) Using a **spreadsheet** or graphing program, plot the transmitted intensity as a function of N for values of N from 2 to 100. (*c*) What is the direction of polarization of the transmitted beam in each case?

Picture the Problem Let I_n be the intensity after the nth polarizing sheet and use $I = I_0 \cos^2\theta$ to find the ratio of I_{n+1} to I_n.

(*a*) Find the ratio of I_{n+1} to I_n:

$$\frac{I_{n+1}}{I_n} = \cos^2\frac{\pi}{2N}$$

Because there are N such reductions of intensity:

$$\frac{I_{N+1}}{I_1} = \frac{I_{N+1}}{I_0} = \cos^{2N}\left(\frac{\pi}{2N}\right)$$

and

$$I_{N+1} = \boxed{I_0 \cos^{2N}\left(\frac{\pi}{2N}\right)}$$

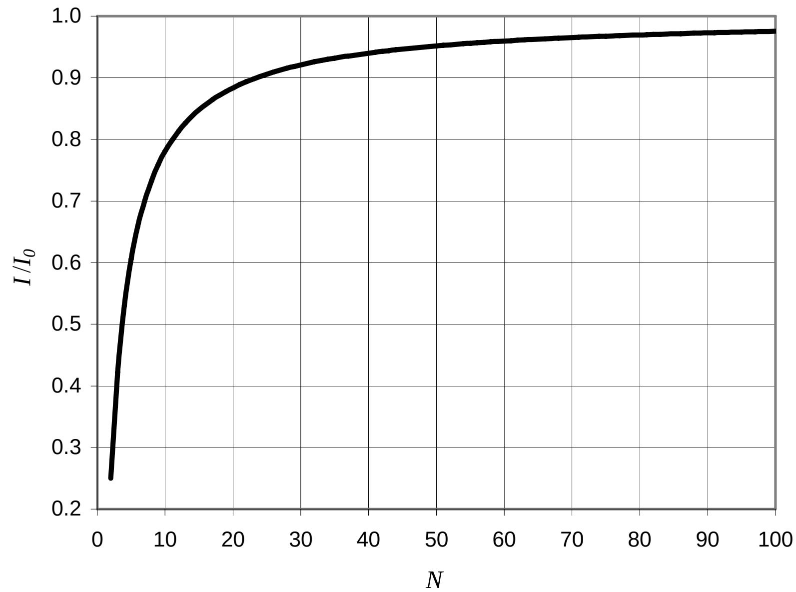
(*b*) A spreadsheet program to graph I_{N+1}/I_0 as a function of N is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	2	N
A3	A2 + 1	$N + 1$
B2	(cos(PI()/(2*A2))^(2*A2)	$\cos^{2N}\left(\frac{\pi}{2N}\right)$

	A	B
1	N	I/I_0
2	2	0.250
3	3	0.422
4	4	0.531
5	5	0.605
95	95	0.974
96	96	0.975
97	97	0.975
98	98	0.975
99	99	0.975

100	100	0.976

A graph of I/I_0 as a function of N follows.



(c) The transmitted light, if any, is polarized parallel to the transmission axis of the last sheet. (For $N = 2$ there is no transmitted light.)

Remarks: The correct values for the intensity of the transmitted light would be significantly smaller than predicted above because of reflection losses.

59 •• [SSM] The device described in Problem 58 could serve as a *polarization rotator*, which changes the linear plane of polarization from one direction to another. The efficiency of such a device is measured by taking the ratio of the output intensity at the desired polarization to the input intensity. The result of Problem 58 suggests that the highest efficiency is achieved by using a large value for the number N. A small amount of intensity is lost regardless of the input polarization when using a real polarizer. For each polarizer, assume the transmitted intensity is 98 percent of the amount predicted by the law of Malus and use a **spreadsheet** or graphing program to determine the optimum number of sheets you should use to rotate the polarization 90º.

Picture the Problem Let I_n be the intensity after the nth polarizing sheet and use $I = I_0\cos^2\theta$ to find the ratio of I_{n+1} to I_n. Because each sheet introduces a 2% loss of intensity, the net transmission after N sheets $(0.98)^N$.

Find the ratio of I_{n+1} to I_n:

$$\frac{I_{n+1}}{I_n} = (0.98)\cos^2\frac{\pi}{2N}$$

Because there are N such reductions of intensity:

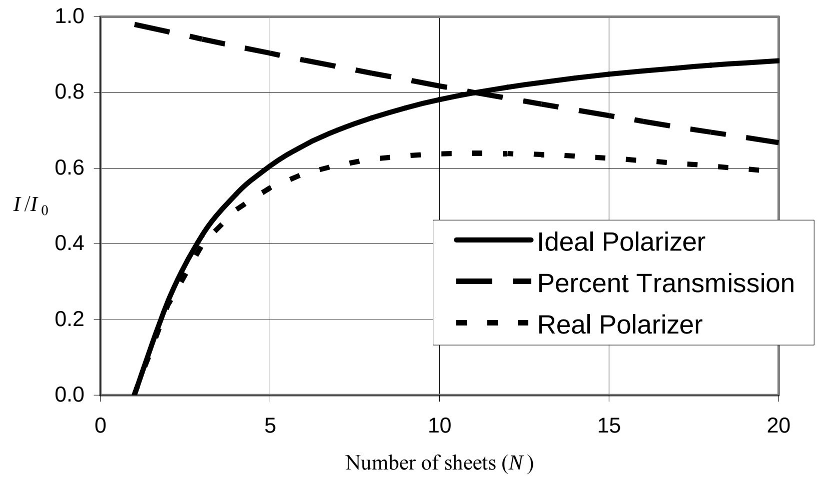
$$\frac{I_{N+1}}{I_0}=(0.98)^N\cos^{2N}\left(\frac{\pi}{2N}\right)$$

A spreadsheet program to graph I_{N+1}/I_0 for an ideal polarizer as a function of N, the percent transmission, and I_{N+1}/I_0 for a real polarizer as a function of N is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A3	1	N
B2	(cos(PI()/(2*A2))^(2*A2)	$\cos^{2N}\left(\frac{\pi}{2N}\right)$
C3	(0.98)^A3	$(0.98)^N$
D4	B3*C3	$(0.98)^N\cos^{2N}\left(\frac{\pi}{2N}\right)$

	A	B	C	D
1		Ideal	Percent	Real
2	N	Polarizer	Transmission	Polarizer
3	1	0.000	0.980	0.000
4	2	0.250	0.960	0.240
5	3	0.422	0.941	0.397
6	4	0.531	0.922	0.490
7	5	0.605	0.904	0.547
8	6	0.660	0.886	0.584
9	7	0.701	0.868	0.608
10	8	0.733	0.851	0.624
11	9	0.759	0.834	0.633
12	10	0.781	0.817	0.638
13	11	0.798	0.801	0.639
14	12	0.814	0.785	0.638
15	13	0.827	0.769	0.636
16	14	0.838	0.754	0.632
17	15	0.848	0.739	0.626
18	16	0.857	0.724	0.620
19	17	0.865	0.709	0.613
20	18	0.872	0.695	0.606
21	19	0.878	0.681	0.598
22	20	0.884	0.668	0.590

A graph of I/I_0 as a function of N for the quantities described above follows:



Inspection of the table, as well as of the graph, tells us that the optimum number of sheets is $\boxed{11.}$

Remarks: The correct values for the intensity of the transmitted light would be significantly smaller than predicted above because of reflection losses.

60 •• Show mathematically that a linearly polarized wave can be thought of as a superposition of a right and a left circularly polarized wave.

Picture the Problem A circularly polarized wave is said to be *right circularly polarized* if the electric and magnetic fields rotate clockwise when viewed along the direction of propagation and *left circularly polarized* if the fields rotate counterclockwise.

For a circularly polarized wave, the x and y components of the electric field are given by:

$$E_x = E_0 \cos\omega t$$

and

$$E_y = E_0 \sin\omega t \text{ or } E_y = -E_0 \sin\omega t$$

for left and right circular polarization, respectively.

For a wave polarized along the x axis:

$$\vec{E}_{\text{right}} + \vec{E}_{\text{left}} = E_0 \cos\omega t\,\hat{i} + E_0 \cos\omega t\,\hat{i}$$

$$= \boxed{2E_0 \cos\omega t\,\hat{i}}$$

61 •• Suppose that the middle sheet in Problem 53 is replaced by two polarizing sheets. If the angle between the transmission axes in the second, third, and fourth sheets in the stack make angles of 30º, 60º and 90º, respectively, with the transmission axis of the first sheet, (a) what is the intensity of the transmitted

light? (*b*) How does this intensity compare with the intensity obtained in Part (*a*) of Problem 53?

Picture the Problem Let I_n be the intensity after the *n*th polarizing sheet and use $I = I_0\cos^2\theta$ to find the intensity of the light transmitted by the four sheets.

(*a*) The intensity of the light between the first and second sheets is:

$$I_1 = \tfrac{1}{2}I_0$$

The intensity of the light between the second and third sheets is:

$$I_2 = I_1\cos^2\theta_{1,2} = \tfrac{1}{2}I_0\cos^2 30° = \tfrac{3}{8}I_0$$

The intensity of the light between the third and fourth sheets is:

$$I_3 = I_2\cos^2\theta_{2,3} = \tfrac{3}{8}I_0\cos^2 30° = \tfrac{9}{32}I_0$$

The intensity of the light to the right of the fourth sheet is:

$$I_4 = I_3\cos^2\theta_{3,4} = \tfrac{9}{32}I_0\cos^2 30° = \tfrac{27}{128}I_0$$
$$= \boxed{0.211I_0}$$

(*b*) The intensity with four sheets at angles of 0°, 30°, 30° and 90° is greater the intensity of three sheets at angles of 0°, 45° and 90° by a factor of 1.69.

Remarks: We could also apply the result obtained in Problem 58(*a*) to solve this problem.

62 •• Show that the electric field of a circularly polarized wave propagating parallel with the *x* axis can be expressed by

$$\vec{E} = E_0\sin(kx+\omega t)\hat{j} + E_0\cos(kx+\omega t)\hat{k}.$$

Picture the Problem A circularly polarized wave may be resolved into two linearly polarized waves, of equal amplitude, 90 degrees apart and with their planes of polarization at right angles to each other. The components of the electric field $\vec{E}$ of a circularly polarized wave propagating parallel to the *x* axis are given by $E_x = E_0\sin\omega t$ and $E_y = E_0\cos\omega t$.

The electric field $\vec{E}$ is the vector sum of its components:

$$\vec{E} = E_x\hat{j} + E_y\hat{k}$$

For an electric field propagating parallel with the *x* axis and in the −*x* direction:

$$E_x = E_0\sin(kx+\omega t)$$
and
$$E_y = E_0\cos(kx+\omega t)$$

Substituting for E_x and E_y yields:

$$\vec{E} = \boxed{E_0 \sin(kx + \omega t)\hat{j} + E_0 \cos(kx + \omega t)\hat{k}}$$

63 •• [SSM] A circularly polarized wave is said to be *right circularly polarized* if the electric and magnetic fields rotate clockwise when viewed along the direction of propagation and *left circularly polarized* if the fields rotate counterclockwise. (*a*) What is the sense of the circular polarization for the wave described by the expression in Problem 62? (*b*) What would be the expression for the electric field of a circularly polarized wave traveling in the same direction as the wave in Problem 60, but with the fields rotating in the opposite direction?

Picture the Problem We can apply the given definitions of right and left circular polarization to the electric field and magnetic fields of the wave.

(*a*) The electric field of the wave in Problem 62 is:

$$\vec{E} = E_0 \sin(kx + \omega t)\hat{j} + E_0 \cos(kx + \omega t)\hat{k}$$

The corresponding magnetic field is:

$$\vec{B} = B_0 \sin(kx + \omega t)\hat{k} - B_0 \cos(kx + \omega t)\hat{j}$$

Because these fields rotate clockwise when viewed along the direction of propagation, the wave is right circularly polarized.

(*b*) For a left circularly polarized wave traveling in the opposite direction:

$$\vec{E} = \boxed{E_0 \sin(kx + \omega t)\hat{j} - E_0 \cos(kx + \omega t)\hat{k}}$$

Sources of Light

64 • A helium–neon laser emits light that has a wavelength equal to 632.8 nm and has a power output of 4.00 mW. How many photons are emitted per second by this laser?

Picture the Problem We can express the number of photons emitted per second as the ratio of the power output of the laser and energy of a single photon.

Relate the number of photons per second n to the power output of the pulse and the energy of a single photon E_{photon}:

$$n = \frac{P}{E_{\text{photon}}}$$

The energy of a photon is given by:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substitute for E_{photon} to obtain:

$$n = \frac{\lambda P}{hc}$$

Substitute numerical values and evaluate n:

$$n = \frac{(632.8\,\text{nm})(4.00\,\text{mW})}{1240\,\text{eV}\cdot\text{nm}}\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}} = \boxed{1.27\times10^{16}\ \text{photons/s}}$$

65 • The first excited state of an atom of a gas is 2.85 eV above the ground state. (*a*) What is the maximum wavelength of radiation for resonance absorption by atoms of the gas that are in the ground state? (*b*) If the gas is irradiated with monochromatic light that has a wavelength of 320 nm, what is the wavelength of the Raman scattered light?

Picture the Problem We can use the Einstein equation for photon energy to find the wavelength of the radiation for resonance absorption. We can use the same relationship, with $E_{\text{Raman}} = E_{\text{inc}} - \Delta E$ where ΔE is the energy for resonance absorption, to find the wavelength of the Raman scattered light.

(*a*) Use the Einstein equation for photon energy to relate the wavelength of the radiation to energy of the first excited state:

$$\lambda = \frac{hc}{E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{2.85\,\text{eV}} = \boxed{435\,\text{nm}}$$

(*b*) The wavelength of the Raman scattered light is given by:

$$\lambda_{\text{Raman}} = \frac{1240\,\text{eV}\cdot\text{nm}}{E_{\text{Raman}}}$$

Relate the energy of the Raman scattered light E_{Raman} to the energy of the incident light E_{inc}:

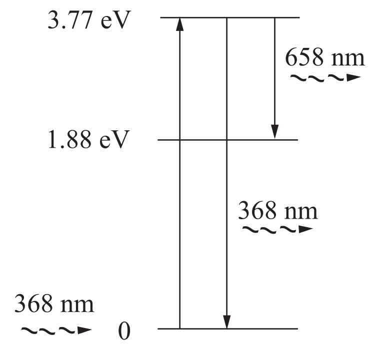
$$\begin{aligned} E_{\text{Raman}} &= E_{\text{inc}} - \Delta E \\ &= \frac{1240\,\text{eV}\cdot\text{nm}}{320\,\text{nm}} - 2.85\,\text{eV} \\ &= 1.025\,\text{eV} \end{aligned}$$

Substitute numerical values and evaluate λ_{Raman}:

$$\lambda_{\text{Raman}} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.025\,\text{eV}} = \boxed{1210\,\text{nm}}$$

66 •• A gas is irradiated with monochromatic ultraviolet light that has a wavelength of 368 nm. Scattered light that has a wavelength equal to 368 nm is observed, and scattered light that has a wavelength of 658-nm is also observed. Assuming that the gas atoms were in their ground state prior to irradiation, find the energy difference between the ground state and the excited state obtained by the irradiation.

Picture the Problem The incident radiation will excite atoms of the gas to higher energy states. The scattered light that is observed is a consequence of these atoms returning to their ground state. The energy difference between the ground state and the atomic state excited by the irradiation is given by $\Delta E = hf = \dfrac{hc}{\lambda}$.



The energy difference between the ground state and the atomic state excited by the irradiation is given by:

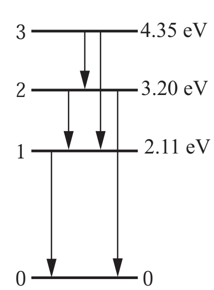
$$\Delta E = hf = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{\lambda}$$

Substitute 368 nm for λ and evaluate ΔE:

$$\Delta E = \frac{1240\,\text{eV}\cdot\text{nm}}{368\,\text{nm}} = \boxed{3.37\,\text{eV}}$$

67 •• **[SSM]** Sodium has excited states 2.11 eV, 3.20 eV, and 4.35 eV above the ground state. Assume that the atoms of the gas are all in the ground state prior to irradiation. (*a*) What is the maximum wavelength of radiation that will result in resonance fluorescence? What is the wavelength of the fluorescent radiation? (*b*) What wavelength will result in excitation of the state 4.35 eV above the ground state? If that state is excited, what are the possible wavelengths of resonance fluorescence that might be observed?

Picture the Problem The ground state and the three excited energy levels are shown in the diagram to the right. Because the wavelength is related to the energy of a photon by $\lambda = hc/\Delta E$, longer wavelengths correspond to smaller energy differences.



(*a*) The maximum wavelength of radiation that will result in resonance fluorescence corresponds to an excitation to the 3.20 eV level followed by decays to the 2.11 eV level and the ground state:

$$\lambda_{\max} = \frac{1240\,\text{eV}\cdot\text{nm}}{3.20\,\text{eV}} = \boxed{388\,\text{nm}}$$

The fluorescence wavelengths are:

$$\lambda_{2\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{3.20\,\text{eV} - 2.11\,\text{eV}} = \boxed{1140\,\text{nm}}$$

and

$$\lambda_{1\to0} = \frac{1240\,\text{eV}\cdot\text{nm}}{2.11\,\text{eV} - 0} = \boxed{588\,\text{nm}}$$

(*b*) For excitation:

$$\lambda_{0\to3} = \frac{1240\,\text{eV}\cdot\text{nm}}{4.35\,\text{eV}} = 285\,\text{nm}$$

(**not in visible the spectrum**)

The fluorescence wavelengths corresponding to the possible transitions are:

$$\lambda_{3\to2} = \frac{1240\,\text{eV}\cdot\text{nm}}{4.35\,\text{eV} - 3.20\,\text{eV}} = 1080\,\text{nm}$$

(**not in visible spectrum**)

$$\lambda_{2\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{3.20\,\text{eV} - 2.11\,\text{eV}} = 1140\,\text{nm}$$

(**not in visible spectrum**)

$$\lambda_{1\to0} = \frac{1240\,\text{eV}\cdot\text{nm}}{2.11\,\text{eV} - 0} = \boxed{588\,\text{nm}}$$

$$\lambda_{3\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{4.35\,\text{eV} - 2.11\,\text{eV}} = \boxed{554\,\text{nm}}$$

and

$$\lambda_{2\to0} = \frac{1240\,\text{eV}\cdot\text{nm}}{3.20\,\text{eV} - 0} = 388\,\text{nm}$$

(**not in visible spectrum**)

68 •• Singly ionized helium is a hydrogen-like atom that has a nuclear charge of +2*e*. Its energy levels are given by $E_n = -4E_0/n^2$, where $n = 1, 2, \ldots$ and $E_0 = 13.6$ eV. If a beam of visible white light is sent through a gas of singly ionized helium, at what wavelengths will dark lines be found in the spectrum of the transmitted radiation? (Assume that the ions of the gas are all in the same state with energy E_1 prior to irradiation.)

Determine the Concept The energy difference between the ground state and the first excited state is $3E_0$ = 40.8 eV, corresponding to a wavelength of 30.4 nm. This is in the far ultraviolet, well outside the visible range of wavelengths. There will be no dark lines in the transmitted radiation.

69 • **[SSM]** A pulse from a ruby laser has an average power of 10 MW and lasts 1.5 ns. (*a*) What is the total energy of the pulse? (*b*) How many photons are emitted in this pulse?

Picture the Problem We can use the definition of power to find the total energy of the pulse. The ratio of the total energy to the energy per photon will yield the number of photons emitted in the pulse.

(*a*) Use the definition of power to obtain:

$$E = P\Delta t$$

Substitute numerical values and evaluate *E*:

$$E = (10\,\text{MW})(1.5\,\text{ns}) = \boxed{15\,\text{mJ}}$$

(*b*) Relate the number of photons *N* to the total energy in the pulse and the energy of a single photon E_{photon}:

$$N = \frac{E}{E_{\text{photon}}}$$

The energy of a photon is given by:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substitute for E_{photon} to obtain:

$$N = \frac{\lambda E}{hc}$$

Substitute numerical values (the wavelength of light emitted by a ruby laser is 694.3 nm) and evaluate *N*:

$$N = \frac{(694.3\,\text{nm})(15\,\text{mJ})}{1240\,\text{eV}\cdot\text{nm}}\,\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}} = \boxed{5.2\times10^{16}}$$

General Problems

70 • A beam of red light that has a wavelength of 700 nm in air travels in water. (*a*) What is the wavelength in water? (*b*) Does a swimmer underwater observe the same color or a different color for this light?

Picture the Problem We can use $v = f\lambda$ and the definition of the index of refraction to relate the wavelength of light in a medium whose index of refraction is n to the wavelength of light in air.

(a) The wavelength λ_n of light in a medium whose index of refraction is n is given by:

$$\lambda_n = \frac{v}{f} = \frac{c}{nf} = \frac{\lambda_0}{n}$$

Substitute numerical values and evaluate λ_{water}:

$$\lambda_{\text{water}} = \frac{700\,\text{nm}}{n_{\text{water}}} = \frac{700\,\text{nm}}{1.33} = \boxed{526\,\text{nm}}$$

(b) Because the color observed depends on the frequency of the light, a swimmer observes the same color in air and in water.

71 •• **[SSM]** The critical angle for total internal reflection for a substance is 48°. What is the polarizing angle for this substance?

Picture the Problem We can use Snell's law, under critical angle and polarization conditions, to relate the polarizing angle of the substance to the critical angle for internal reflection.

Apply Snell's law, under critical angle conditions, to the interface:

$$n_1 \sin\theta_c = n_2 \quad (1)$$

Apply Snell's law, under polarization conditions, to the interface:

$$n_1 \sin\theta_p = n_2 \sin(90° - \theta_p) = n_2 \cos\theta_p$$

or

$$\tan\theta_p = \frac{n_2}{n_1} \Rightarrow \theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right) \quad (2)$$

Solve equation (1) for the ratio of n_2 to n_1:

$$\frac{n_2}{n_1} = \sin\theta_c$$

Substitute for n_2/n_1 in equation (2) to obtain:

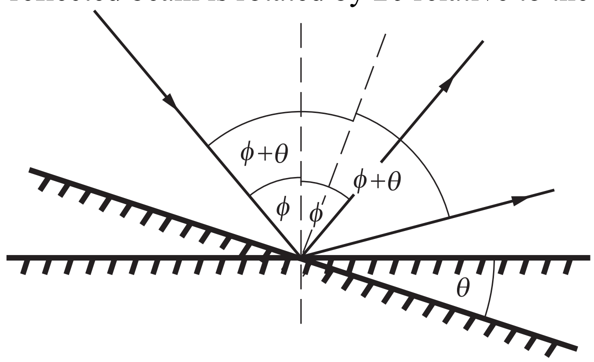
$$\theta_p = \tan^{-1}(\sin\theta_c)$$

Substitute numerical values and evaluate θ_p:

$$\theta_p = \tan^{-1}(\sin 48°) = \boxed{37°}$$

72 •• Show that when a flat mirror is rotated through an angle θ about an axis in the plane of the mirror, a reflected beam of light (from a fixed incident beam) that is perpendicular to the rotation axis is rotated through 2θ.

Picture the Problem Let ϕ be the initial angle of incidence. Since the angle of reflection with the normal to the mirror is also ϕ, the angle between incident and reflected rays is 2ϕ. If the mirror is now rotated by a further angle θ, the angle of incidence is increased by θ to $\phi + \theta$, and so is the angle of reflection. Consequently, the reflected beam is rotated by 2θ relative to the incident beam.



73 •• **[SSM]** Use Figure 31-59 to calculate the critical angles for light initially in silicate flint glass that is incident on a glass–air interface if the light is (*a*) violet light of wavelength 400 nm and (*b*) red light of wavelength 700 nm.

Picture the Problem We can apply Snell's law at the glass-air interface to express θ_c in terms of the index of refraction of the glass and use Figure 31-59 to find the index of refraction of the glass for the given wavelengths of light.

Apply Snell's law at the glass-air interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

If $\theta_1 = \theta_c$ and $n_2 = 1$:

$$n_1 \sin\theta_c = \sin 90° = 1$$

and

$$\theta_c = \sin^{-1}\left(\frac{1}{n_1}\right)$$

(*a*) For violet light of wavelength 400 nm, $n_1 = 1.67$:

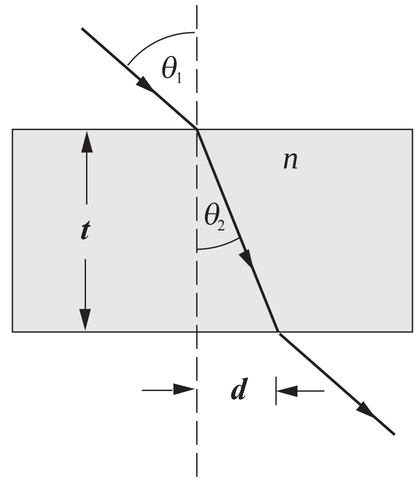
$$\theta_c = \sin^{-1}\left(\frac{1}{1.67}\right) = \boxed{36.8°}$$

(*b*) For red light of wavelength 700 nm, $n_1 = 1.60$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.60}\right) = \boxed{38.7°}$$

74 •• Light is incident on a slab of transparent material at an angle θ_1, as shown in Figure 31-60. The slab has a thickness t and an index of refraction n. Show that $n = \sin(\theta_1)/\left[\tan^{-1}(d/t)\right]$, where d is the distance shown in the figure.

Picture the Problem We can apply Snell's law at the air-slab interface to express the index of refraction n in terms of θ_1 and θ_2 and then use the geometry of the figure to relate θ_2 to t and d.



Applying Snell's law to the first interface yields:

$$\sin\theta_1 = n\sin\theta_2 \Rightarrow n = \frac{\sin\theta_1}{\sin\theta_2}$$

From the diagram:

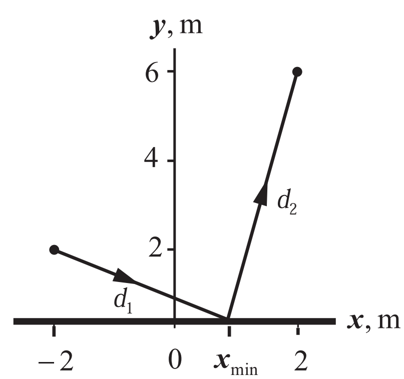
$$d = t\tan\theta_2 \Rightarrow \theta_2 = \tan^{-1}\left(\frac{d}{t}\right)$$

Substitute for θ_2 to obtain:

$$n = \boxed{\frac{\sin\theta_1}{\sin\left[\tan^{-1}\left(\frac{d}{t}\right)\right]}}$$

75 •• A ray of light begins at the point (–2.00 m, 2.00 m, 0.00 m), strikes a mirror in the $y = 0$ plane at some point $(x, 0, 0)$, and reflects through the point (2.00 m, 6.00 m, 0.00 m). (*a*) Find the value of x that makes the total distance traveled by the ray a minimum. (*b*) What is the angle of incidence on the reflecting plane? (*c*) What is the angle of reflection?

Picture the Problem We can write an expression for the total distance traveled by the light as a function of x and set the derivative of this expression equal to zero to find the value of x that minimizes the distance traveled by the light. The adjacent figure shows the two points and the reflecting surface. The x and y coordinates are in meters.



(*a*) Express the total distance D traveled by the light:

$$D = d_1 + d_2 = \sqrt{(x+2)^2 + 4} + \sqrt{(2-x)^2 + 36}$$

Differentiate D with respect to x:

$$\frac{dD}{dx}=\frac{d}{dx}\left[\sqrt{(x+2)^2+4}+\sqrt{(2-x)^2+36}\right]$$
$$=\tfrac{1}{2}\left[(x+2)^2+4\right]^{-\frac{1}{2}}2(x+2)+\tfrac{1}{2}\left[(2-x)^2+36\right]^{-\frac{1}{2}}2(2-x)(-1)$$

Setting this expression equal to zero for extreme values yields:

$$\frac{x_{\text{min}}+2}{\sqrt{(x_{\text{min}}+2)^2+4}}-\frac{2-x_{\text{min}}}{\sqrt{(2-x_{\text{min}})^2+36}}=0$$

Solve for x_{min} to obtain: $\quad x_{\text{min}}=\boxed{-1.00\,\text{m}}$

(*b*) Letting the coordinates of the point at which the ray originates be (x_1, y_1) and the coordinates of the terminal point be (x_2, y_2), the tangent of the angle of incidence is given by:

$$\tan\theta_{\text{i}}=\left|\frac{x-x_1}{y-y_1}\right|\Rightarrow\theta_{\text{i}}=\tan^{-1}\left(\left|\frac{x-x_1}{y-y_1}\right|\right)$$

where $(x, y) = (-1.00\text{ m}, 0)$.

Substitute numerical values and evaluate θ_{i}:

$$\theta_{\text{i}}=\tan^{-1}\left[\left|\frac{-1.00\,\text{m}-(-2.00\,\text{m})}{0-2.00\,\text{m}}\right|\right]=\boxed{26.6^\circ}$$

(*c*) Letting the coordinates of the point at which the ray originates be (x, y) and the coordinates of the terminal point be (x_2, y_2), the tangent of the angle of reflection is given by:

$$\tan\theta_{\text{r}}=\frac{x_2-x}{y_2-y}\Rightarrow\theta_{\text{r}}=\tan^{-1}\left(\frac{x_2-x}{y_2-y}\right)$$

where $(x, y) = (-1.00\text{ m}, 0)$.

Substitute numerical values and evaluate θ_{r}:

$$\theta_{\text{r}}=\tan^{-1}\left[\frac{2.00\,\text{m}-(-1.00\,\text{m})}{6.00\,\text{m}-0}\right]=\boxed{26.6^\circ}$$

76 •• To produce a polarized laser beam a plate of transparent material, (Figure 31-61) is placed in the laser cavity and oriented so the light strikes it at the polarizing angle. Such a plate is called a Brewster window. Show that if θ_{P1} is the

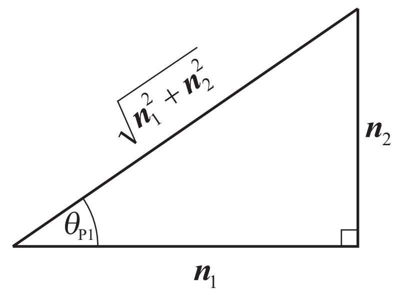
polarizing angle for the n_1 to n_2 interface, then θ_{P2} is the polarizing angle for the n_2 to n_1 interface.

Picture the Problem Let the angle of refraction at the first interface by θ_1 and the angle of refraction at the second interface be θ_2. We can apply Snell's law at each interface and eliminate θ_1 and n_2 to show that $\theta_2 = \theta_{P2}$.

Apply Brewster's law at the n_1-n_2 interface:

$$\tan\theta_{P1} = \frac{n_2}{n_1}$$

Draw a reference triangle consistent with Brewster's law:



Apply Snell's law at the n_1-n_2 interface:

$$n_1 \sin\theta_{P1} = n_2 \sin\theta_1$$

Solve for θ_1 to obtain:

$$\theta_1 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_{P1}\right)$$

Referring to the reference triangle we note that:

$$\theta_1 = \sin^{-1}\left(\frac{n_1}{n_2}\frac{n_2}{\sqrt{n_1^2 + n_2^2}}\right)$$

$$= \sin^{-1}\left(\frac{n_1}{\sqrt{n_1^2 + n_2^2}}\right)$$

that is, θ_1 is the complement of θ_{p1}.

Apply Snell's law at the n_2-n_1 interface:

$$n_2 \sin\theta_1 = n_1 \sin\theta_2$$

Solve for θ_2 to obtain:

$$\theta_2 = \sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_1\right)$$

Refer to the reference triangle again to obtain:

$$\theta_2 = \sin^{-1}\left(\frac{n_2}{n_1}\frac{n_1}{\sqrt{n_1^2+n_2^2}}\right)$$

$$= \sin^{-1}\left(\frac{n_2}{\sqrt{n_1^2+n_2^2}}\right) = \boxed{\theta_{P2}}$$

Equate these expressions for $n_2 \sin\theta_1$ to obtain:

$$n_1 \sin\theta_P = n_1 \sin\theta_2 \Rightarrow \theta_2 = \boxed{\theta_P}$$

77 •• [SSM] From the data provided in Figure 31-59, calculate the polarization angle for an air–glass interface, using light of wavelength 550 nm in each of the four types of glass shown.

Picture the Problem We can use Brewster's law in conjunction with index of refraction data from Figure 31-59 to calculate the polarization angles for the air-glass interface.

From Brewster's law we have:

$$\theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

or, for $n_1 = 1$,

$$\theta_p = \tan^{-1}(n_2)$$

For silicate flint glass, $n_2 \approx 1.62$ and:

$$\theta_{\text{p,silicate flint}} = \tan^{-1}(1.62) = \boxed{58.3°}$$

For borate flint glass, $n_2 \approx 1.57$ and:

$$\theta_{\text{p,borate flint}} = \tan^{-1}(1.57) = \boxed{57.5°}$$

For quartz glass, $n_2 \approx 1.54$ and:

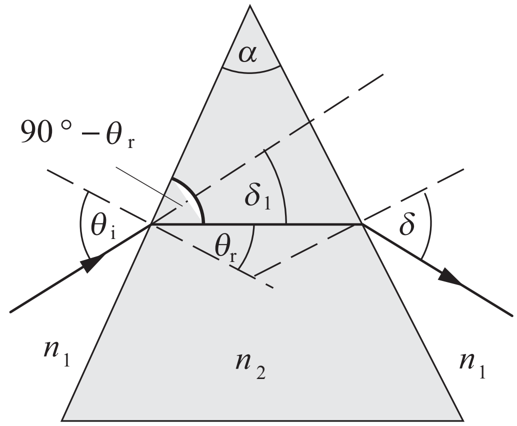
$$\theta_{\text{p, quartz}} = \tan^{-1}(1.54) = \boxed{57.0°}$$

For silicate crown glass, $n_2 \approx 1.51$ and:

$$\theta_{\text{p,silicate crown}} = \tan^{-1}(1.51) = \boxed{56.5°}$$

78 •• A light ray passes through a prism with an apex angle of α, as shown in Figure 31-63. The ray and the bisector of the apex angle bisect at right angles. Show that the *angle of deviation* δ is related to the apex angle and the index of refraction of the prism material by $\sin\left[\frac{1}{2}(\alpha+\delta)\right] = n\sin\left(\frac{1}{2}\alpha\right)$.

Picture the Problem The diagram to the right shows the angles of incidence, refraction, and deviation at the first interface. We can use the geometry of this symmetric passage of the light to express θ_r in terms of α and δ_1 in terms of θ_r and α. We can then use a symmetry argument to express the deviation at the second interface and the total deviation δ. Finally, we can apply Snell's law at the first interface to complete the derivation of the given expression.



With respect to the normal to the left face of the prism, let the angle of incidence be θ_i and the angle of refraction be θ_r. From the geometry of the figure, it is evident that:

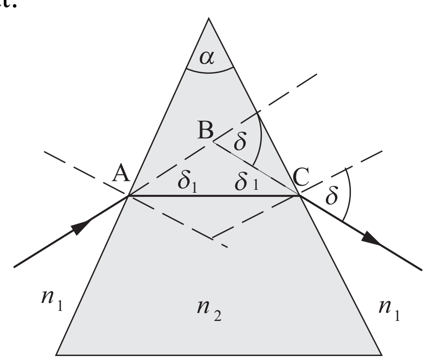
$$\theta_r = \tfrac{1}{2}\alpha$$

Express the angle of deviation at the refracting surface:

$$\delta_1 = \theta_i - \theta_r = \theta_i - \tfrac{1}{2}\alpha$$

Referring to triangle ABC, we see that:

$$\delta = 2\delta_1 = 2\left(\theta_1 - \tfrac{1}{2}\alpha\right) = 2\theta_i - \alpha$$



Solving for θ_i yields:

$$\theta_i = \tfrac{1}{2}(\alpha + \delta)$$

Apply Snell's law, with $n_1 = 1$ and $n_2 = n$, to the first interface:

$$\sin\theta_i = n\sin\tfrac{1}{2}\alpha$$

Substitute for θ_1 to obtain:

$$\boxed{\sin\frac{\alpha+\delta}{2}=n\sin\frac{\alpha}{2}} \quad (1)$$

79 •• **[SSM]** (*a*) For light rays inside a transparent medium that is surrounded by a vacuum, show that the polarizing angle and the critical angle for total internal reflection satisfy $\tan\theta_p = \sin\theta_c$. (*b*) Which angle is larger, the polarizing angle or the critical angle for total internal reflection?

Picture the Problem We can apply Snell's law at the critical angle and the polarizing angle to show that $\tan\theta_p = \sin\theta_c$.

(*a*) Apply Snell's law at the medium-vacuum interface:

$$n_1\sin\theta_1 = n_2\sin\theta_r$$

For $\theta_1 = \theta_c$, $n_1 = n$, and $n_2 = 1$:

$$n\sin\theta_c = \sin 90° = 1$$

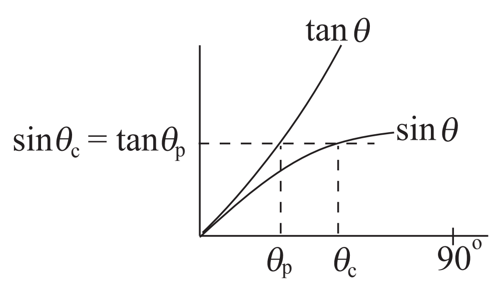
For $\theta_1 = \theta_p$, $n_1 = n$, and $n_2 = 1$:

$$\tan\theta_p = \frac{n_2}{n_1} = \frac{1}{n} \Rightarrow n\tan\theta_p = 1$$

Because both expressions equal one:

$$\boxed{\tan\theta_p = \sin\theta_c}$$

(*b*) For any value of θ:



$$\tan\theta > \sin\theta \Rightarrow \boxed{\theta_c > \theta_p}$$

80 •• Light in air is incident on the surface of a transparent substance at an angle of 58° with the normal. The reflected and refracted rays are observed to be mutually perpendicular. (*a*) What is the index of refraction of the transparent substance? (*b*) What is the critical angle for total internal reflection in this substance?

Picture the Problem Let the numeral 1 refer to the side of the interface from which the light is incident and the numeral 2 to the refraction side of the interface. We can apply Snell's law, under the conditions described in the problem statement, at the interface to derive an expression for n as a function of the angle of incidence (also the polarizing angle).

(*a*) Apply Snell's law at the air-medium interface:

$$\sin\theta_1 = n\sin\theta_2$$

Because the reflected and refracted rays are mutually perpendicular:

$$\theta_1 + \theta_2 = 90° \Rightarrow \theta_2 = 90° - \theta_1$$

Substitute for θ_2 to obtain:

$$\sin\theta_1 = n\sin(90° - \theta_1) = n\cos\theta_1$$

or

$$n = \tan\theta_1 = \tan\theta_p$$

Substitute for θ_p and evaluate n:

$$n = \tan 58° = \boxed{1.6}$$

(*b*) Apply Snell's law at the interface under conditions of total internal reflection:

$$n_2\sin\theta_c = n_1\sin 90° = n_1$$

Because $n_1 = 1$:

$$\theta_c = \sin^{-1}\left(\frac{1}{n_2}\right) = \sin^{-1}\left(\frac{1}{n}\right)$$

Substitute for n and evaluate θ_c:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.6}\right) = \boxed{39°}$$

81 •• A light ray in dense flint glass that has an index of refraction of 1.655 is incident on the glass surface. An unknown liquid condenses on the surface of the glass. Total internal reflection on the glass–liquid interface occurs for a minimum angle of incidence on the glass–liquid interface of 53.7º. (*a*) What is the refractive index of the unknown liquid? (*b*) If the liquid is removed, what is the minimum angle of incidence for total internal reflection? (*c*) For the angle of incidence found in Part (*b*), what is the angle of refraction of the ray into the liquid film? Does a ray emerge from the liquid film into the air above? Assume the glass and liquid have parallel planar surfaces.

Picture the Problem We can apply Snell's law at the glass–liquid and liquid–air interfaces to find the refractive index of the unknown liquid, the minimum angle of incidence (glass-air interface) for total internal reflection, and the angle of refraction of a ray into the liquid film.

(*a*) Apply Snell's law, under critical-angle conditions, at the glass–liquid interface:

$$\sin\theta_c = \frac{n_{liquid}}{n_{glass}} \Rightarrow n_{liquid} = n_{glass}\sin\theta_c$$

Substitute numerical values and evaluate n_{liquid}:

$$n_{\text{liquid}} = (1.655)\sin 53.7° = \boxed{1.33}$$

(*b*) With the liquid removed:

$$\theta_{\text{c}} = \sin^{-1}\left(\frac{1}{n_{\text{glass}}}\right)$$

Substitute numerical values and evaluate θ_{c}:

$$\theta_{\text{c}} = \sin^{-1}\left(\frac{1}{1.655}\right) = \boxed{37.2°}$$

(*c*) Apply Snell's law at the glass–liquid interface:

$$n_{\text{glass}} \sin\theta_1 = n_{\text{liquid}} \sin\theta_2$$

Solve for θ_2:

$$\theta_2 = \sin^{-1}\left[\frac{n_{\text{glass}}}{n_{\text{liquid}}}\sin\theta_1\right]$$

Letting θ_1 be the angle of incidence found in Part (b) yields:

$$\theta_2 = \sin^{-1}\left[\left(\frac{n_{\text{glass}}}{n_{\text{liquid}}}\right)\left(\frac{1}{n_{\text{glass}}}\right)\right] = \sin^{-1}\left(\frac{1}{n_{\text{liquid}}}\right)$$

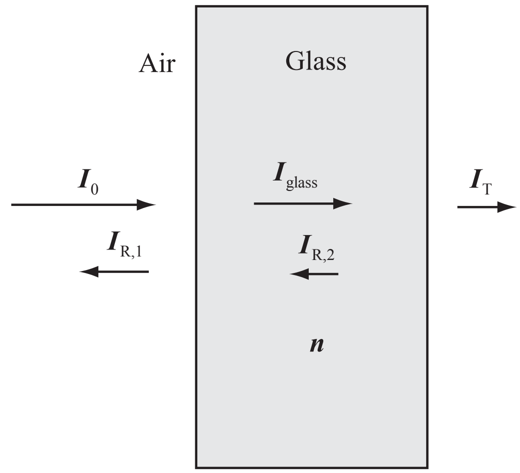
Substitute numerical values and evaluate θ_2:

$$\theta_2 = \sin^{-1}\left(\frac{1}{1.333}\right) = \boxed{48.6°}$$

Because 48.6° is also the angle of incidence at the liquid-air interface and because it is equal to the critical angle for total internal reflection at this interface, no light will emerge into the air.

82 ••• (*a*) Show that for normally incident light, the intensity transmitted through a glass slab that has an index of refraction of *n* and is surrounded by air is approximately given by $I_{\text{T}} = I_0\left[4n/(n+1)^2\right]^2$. (*b*) Use the Part (*a*) result to find the ratio of the transmitted intensity to the incident intensity through *N* parallel slabs of glass for light of normal incidence. (*c*) How many slabs of a glass that has an index of refraction of 1.5 are required to reduce the intensity to 10 percent of the incident intensity?

Picture the Problem We'll neglect multiple reflections at the glass-air interfaces. We can use the expression (Equation 31-7) for the reflected intensity at an interface to express the intensity of the light in the glass slab as the difference between the intensity of the incident beam and the reflected beam. Repeating this analysis at the glass-air interface will lead to the desired result.



(*a*) Express the intensity of the light transmitted into the glass:

$$I_{\text{glass}} = I_0 - I_{\text{R},1}$$

where $I_{\text{R},1}$ is the intensity of the light reflected at the air-glass interface.

The intensity of the light reflected at the air-glass interface is given by Equation 31-7:

$$I_{\text{R},1} = \left(\frac{1-n}{1+n}\right)^2 I_0$$

Substitute and simplify to obtain:

$$I_{\text{glass}} = I_0 - \left(\frac{1-n}{1+n}\right)^2 I_0 = I_0\left[1-\left(\frac{1-n}{1+n}\right)^2\right] = I_0\left[\frac{4n}{(1+n)^2}\right]$$

Express the intensity of the light transmitted at the glass-air interface:

$$I_{\text{T}} = I_{\text{glass}} - I_{\text{R},2}$$

where $I_{\text{R},2}$ is the intensity of the light reflected at the glass-air interface.

The intensity of the light reflected at the glass-air interface is:

$$I_{\text{R},2} = \left(\frac{1-n}{1+n}\right)^2 I_{\text{glass}} = \left(\frac{1-n}{1+n}\right)^2\left[\frac{4n}{(1+n)^2}\right] I_0$$

Substitute and simplify to obtain:

$$\begin{aligned} I_{\mathrm{T}} &= I_0\left[\frac{4n}{(1+n)^2}\right]-\left(\frac{1-n}{1+n}\right)^2\left[\frac{4n}{(1+n)^2}\right]I_0 \\ &= I_0\left[1-\left(\frac{1-n}{1+n}\right)^2\right]\left[\frac{4n}{(1+n)^2}\right] \\ &= I_0\left[\frac{4n}{(1+n)^2}\right]\left[\frac{4n}{(1+n)^2}\right] \\ &= \boxed{I_0\left[\frac{4n}{(1+n)^2}\right]^2} \end{aligned}$$

(b) From Part (a), each slab reduces the intensity by the factor:

$$\left[\frac{4n}{(n+1)^2}\right]^2$$

For N slabs:

$$I_{\mathrm{t}} = I_0\left[\frac{4n}{(n+1)^2}\right]^{2N}$$

and

$$\frac{I_{\mathrm{t}}}{I_0} = \boxed{\left[\frac{4n}{(n+1)^2}\right]^{2N}} \qquad (1)$$

(c) Begin the solution of equation (1) for N by taking the logarithm (arbitrarily to base 10) of both sides of the equation:

$$\begin{aligned} \log\left(\frac{I_{\mathrm{t}}}{I_0}\right) &= \log\left[\frac{4n}{(n+1)^2}\right]^{2N} \\ &= 2N\log\left[\frac{4n}{(n+1)^2}\right] \end{aligned}$$

Solve for N:

$$N = \frac{\log\left(\frac{I_{\mathrm{t}}}{I_0}\right)}{2\log\left[\frac{4n}{(n+1)^2}\right]}$$

Substitute numerical values and evaluate N:

$$N = \frac{\log(0.1)}{2\log\left[\frac{4(1.5)}{(1.5+1)^2}\right]} = 28.2 \approx \boxed{28}$$

83 ••• Equation 31-14 gives the relation between the angle of deviation ϕ_d of a light ray incident on a spherical drop of water in terms of the incident angle θ_1 and the index of refraction of water. (*a*) Assume that $n_{air} = 1$, and derive an expression for $d\phi_d/d\theta_1$. *Hint:* If $y = \sin^{-1} x$, then $dy/dx = (1 - x^2)^{-1/2}$. (*b*) Use this result to show that the angle of incidence for minimum deviation θ_{1m} is given by $\cos\theta_{1m} = \sqrt{\frac{1}{3}(n^2 - 1)}$. (*c*) The index of refraction for a certain red light in water is 1.3318 and that the index of refraction for a certain blue light in water is 1.3435. Use the result of Part (*a*) to find the angular separation of these colors in the primary rainbow.

Picture the Problem (*a*) We can follow the directions given in the problem statement and use the hint to establish the given result. (*b*) Treating the result of Part (*a*) as an extreme-value problem will lead to the given result. (*c*) We can use Equation 31-14 and the result of Part (*b*) to find the angular separation of these colors in the primary rainbow.

(*a*) Equation 31-14 is:

$$\phi_d = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{n_{air}\sin\theta_1}{n_{water}}\right)$$

For $n_{air} = 1$ and $n_{water} = n$:

$$\phi_d = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

Use the hint to differentiate ϕ_d with respect to θ_1:

$$\frac{d\phi_d}{d\theta_1} = \frac{d}{d\theta_1}\left[\pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]$$

$$= \boxed{2 - \frac{4\cos\theta_1}{\sqrt{n^2 - \sin^2\theta_1}}}$$

(*b*) Set $d\phi_d/d\theta_1 = 0$:

$$2 - \frac{4\cos\theta_1}{\sqrt{n^2 - \sin^2\theta_1}} = 0 \text{ for extrema}$$

Simplify to obtain:

$$16\cos^2\theta_1 = 4\left(n^2 - \sin^2\theta_1\right)$$

Replace $\sin^2\theta_1$ with $1 - \cos^2\theta_1$ and simplify to obtain:

$$12\cos^2\theta_1 = 4n^2 - 4$$

Solve for $\cos\theta_1 = \cos\theta_{1m}$:

$$\cos\theta_{1m} = \boxed{\sqrt{\frac{n^2 - 1}{3}}}$$

(*c*) Express the angular separation $\Delta\phi$ of blue and red:

$$\Delta\phi = \phi_{\text{d,blue}} - \phi_{\text{d,red}} \tag{1}$$

From Equation 31-18, with $n_{\text{air}} = 1$ and $n_{\text{water}} = n$:

$$\phi_{\text{d}} = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

From Part (*b*):

$$\theta_{1\text{m}} = \cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right]$$

Substitute to obtain:

$$\phi_{\text{d}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right]\right\}}{n}\right)$$

Evaluate ϕ_{d} for blue light in water:

$$\phi_{\text{d,blue}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{(1.3435)^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{(1.3435)^2-1}{3}}\right]\right\}}{1.3435}\right)$$

$$= 139.42^\circ$$

Evaluate ϕ_{d} for red light in water:

$$\phi_{\text{d,red}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{(1.3318)^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{(1.3318)^2-1}{3}}\right]\right\}}{1.3318}\right)$$

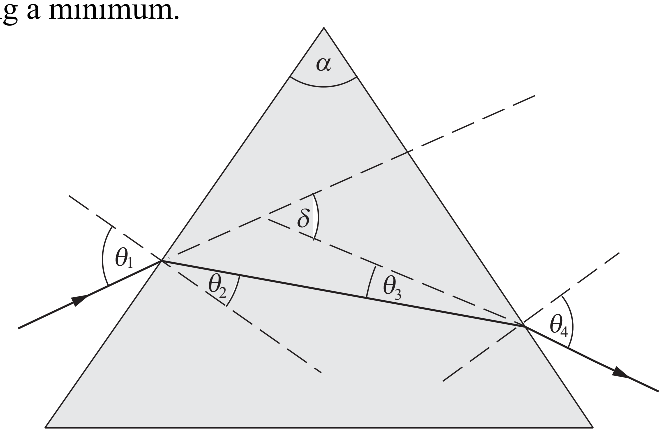
$$= 137.75^\circ$$

Substitute numerical values in equation (1) and evaluate $\Delta\phi$:

$$\Delta\phi = 139.42 - 137.75° = \boxed{1.67°}$$

84 ••• Show that the angle of deviation δ is a minimum if the angle of incidence is such that the ray and the bisector of the apex angle α (Figure 31-62) intersect at right angles.

Picture the Problem The figure below shows the prism and the path of the ray through it. The dashed lines are the normals to the prism faces. The triangle formed by the interior ray and the prism faces has interior angles of α, 90° − θ_2, and 90° − θ_3. Consequently, $\theta_2 + \theta_3 = \alpha$. We can apply Snell's law at both interfaces to express the angle of deviation δ as a function of θ_3 and then set the derivative of this function equal to zero to find the conditions on θ_3 and θ_2 that result in δ being a minimum.



Express the angle of deviation:

$$\delta = \theta_1 + \theta_2 - \alpha \qquad (1)$$

Apply Snell's law to relate θ_1 to θ_2 and θ_3 to θ_4:

$$\sin\theta_1 = n\sin\theta_2 \qquad (2)$$

and

$$n\sin\theta_3 = \sin\theta_4 \qquad (3)$$

Solve equation (2) for θ_1 and equation (3) for θ_4:

$$\theta_1 = \sin^{-1}(n\sin\theta_2)$$

and

$$\theta_4 = \sin^{-1}(n\sin\theta_3)$$

Substitute in equation (1) to obtain:

$$\delta = \sin^{-1}(n\sin\theta_2) + \sin^{-1}(n\sin\theta_3) - \alpha = \sin^{-1}[n\sin(\theta_3 - \alpha)] + \sin^{-1}(n\sin\theta_3) - \alpha$$

Note that the only variable in this expression is θ_3. To determine the condition that minimizes δ, take the derivative of δ with respect to θ_3 and set it equal to zero.

$$\frac{d\delta}{d\theta_3}=\frac{d}{d\theta_3}\left\{\sin^{-1}\left[n\sin(\theta_3-\alpha)\right]+\sin^{-1}(n\sin\theta_3)-\alpha\right\}$$

$$=-\frac{n\cos(\alpha-\theta_3)}{\sqrt{1-\left[n\sin(\alpha-\theta_3)\right]^2}}+\frac{n\cos\theta_3}{\sqrt{1-(n\sin\theta_3)^2}}=0 \text{ for extrema}$$

This equation is satisfied provided: $\alpha-\theta_3=\theta_3 \Rightarrow \theta_3=\frac{1}{2}\alpha$

Because $\theta_2=\alpha-\theta_3$: $\theta_2=\alpha-\frac{1}{2}\alpha=\frac{1}{2}\alpha$

Because $\boldsymbol{\theta}_2=\boldsymbol{\theta}_3$, we can conclude that the deviation angle is a minimum if the ray p through the prism symmetrically.

Remarks: Setting $d\delta/d\theta_3$ = 0 establishes the condition on θ_3 that δ is either a maximum or a minimum. To establish that δ is indeed a minimum when $\theta_3=\theta_2=\frac{1}{2}\alpha$, we can either show that $d^2\delta/d\theta_3^2$, evaluated at $\theta_3=\frac{1}{2}\alpha$, is positive or, alternatively, plot a graph of $\delta(\theta_3)$ to show that it is concave upward at $\theta_3=\frac{1}{2}\alpha$.

Chapter 32
Optical Images

Conceptual Problems

1 • Can a virtual image be photographed? If so, give an example. If not, explain why.

Determine the Concept Yes. Note that a virtual image is "seen" because the eye focuses the diverging rays to form a real image on the retina. For example, you can photograph the virtual image of yourself in a flat mirror and get a perfectly good picture.

2 • Suppose the *x*, *y*, and *z* axes of a 3-D coordinate system are painted a different color. One photograph is taken of the coordinate system and another is taken of its image in a plane mirror. Is it possible to tell that one of the photographs is of a mirror image? Or could both photographs be of the real coordinate system from different angles?

Determine the Concept Yes; the coordinate system and its mirror image will have opposite handedness. That is, if the coordinate system is a right-handed coordinate system, then the mirror image will be a left-handed coordinate system, and vice versa.

3 • **[SSM]** True or False

(*a*) The virtual image formed by a concave mirror is always smaller than the object.
(*b*) A concave mirror always forms a virtual image.
(*c*) A convex mirror never forms a real image of a real object.
(*d*) A concave mirror never forms an enlarged real image of an object.

(*a*) False. The size of the virtual image formed by a concave mirror when the object is between the focal point and the vertex of the mirror depends on the distance of the object from the vertex and is always larger than the object.

(*b*) False. When the object is outside the focal point, the image is real.

(*c*) True.

(*d*) False. When the object is between the center of curvature and the focal point, the image is enlarged and real.

4 • An ant is crawling along the axis of a convex mirror that has a radius of curvature *R*. At what object distances, if any, will the mirror produce (*a*) an upright image, (*b*) a virtual image, (*c*) an image smaller than the object, and (*d*) an image larger than the object?

Determine the Concept Let *s* be the object distance and *f* the focal length of the mirror.

(*a*) If $s < \frac{1}{2}R$, the image is virtual, upright, and larger than the object.

(*b*) If $s < \frac{1}{2}R$, the image is virtual, upright, and larger than the object.

(*c*) If $s > R$, the image is real, inverted, and smaller than the object.

(*d*) If $\frac{1}{2}R < s < R$, the image is real, inverted, and larger than the object.

5 • [SSM] An ant is crawling along the axis of a concave mirror that has a radius of curvature *R*. At what object distances, if any, will the mirror produce (*a*) an upright image, (*b*) a virtual image, (*c*) an image smaller than the object, and (*d*) an image larger than the object?

Determine the Concept
(*a*) The mirror will produce an upright image for all object distances.

(*b*) The mirror will produce a virtual image for all object distances.

(*c*) The mirror will produce an image that is that is smaller than the object for all object distances.

(*d*) The mirror will never produce an enlarged image.

6 •• Convex mirrors are often used for rearview mirrors on cars and trucks to give a wide-angle view. "Warning, objects are closer than they appear." is written below the mirrors. Yet, according to a ray diagram, the image distance for distant objects is much shorter than the object distance. Why then do they appear more distant?

Determine the Concept They appear more distant because in a convex mirror the angular sizes of the images are smaller than they would be in a flat mirror. The angular size of the image is $\dfrac{y'}{|s'|+L}$, where y' is the height of the image, s' is the distance between the image and the mirror, and L is the distance between the observer and the mirror. For a flat mirror, $y' = y$ and $s' = -s$.

We wish to prove that, for a convex mirror:

$$\frac{y'}{|s'|+L} < \frac{y}{s+L} \qquad (1)$$

If equation (1) is valid, then:

$$\frac{|s'|+L}{y'} > \frac{s+L}{y}$$

or

$$\frac{|s'|}{y'} + \frac{L}{y'} > \frac{s}{y} + \frac{L}{y} \qquad (2)$$

Because the lateral magnification is $m = \frac{y'}{y} = -\frac{s'}{s}$:

$$\frac{y'}{|s'|} = \frac{y}{s} \qquad (3)$$

Combining this with equation (2) yields:

$$\frac{L}{y'} > \frac{L}{y} \Rightarrow y' < y \qquad (4)$$

It follows that, if $y' < y$, then:

$$|s'| < s$$

To show that $|s'| < s$ we begin with the mirror equation:

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$$

Solving for s' yields:

$$s' = s\frac{f}{s-f}$$

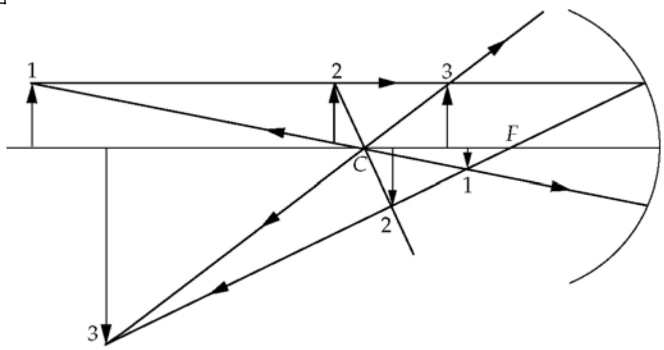
Because f is negative and s is positive:

$$|s'| = s\frac{|f|}{s+|f|} \Rightarrow |s'| < s$$

Thus, we have proven that, if equation (1) is valid, then equation (4) is valid. To prove that equation 1 is valid, we merely follow this proof in reverse order.

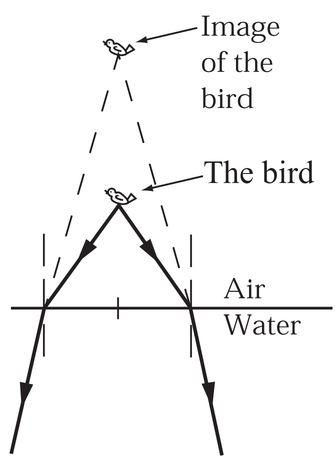
7 • As an ant on the axis of a concave mirror crawls from a great distance to the focal point of a concave mirror, the image of the ant moves from (*a*) a great distance toward the focal point and is always real, (*b*) the focal point to a great distance from the mirror and is always real, (*c*) the focal point to the center of curvature of the mirror and is always real, (*d*) the focal point to a great distance from the mirror and changes from a real image to a virtual image.

Picture the Problem The ray diagram shows three object positions 1, 2, and 3 as the moves from a great distance to the focal point F of a concave mirror. The real images corresponding to each of these object positions are labeled with the same numeral. $\boxed{(b)}$ is correct.



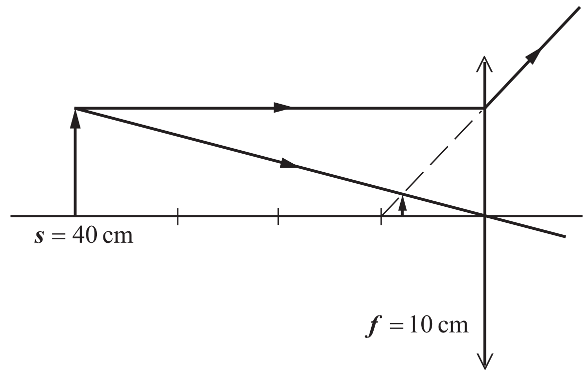
8 • A kingfisher bird that is perched on a branch a few feet above the water is viewed by a scuba diver submerged beneath the surface of the water directly below the bird. Does the bird appear to the diver to be closer to or farther from the surface than the actual bird? Explain your answer using a ray diagram.

Picture the Problem The diagram shows two rays (from the bundle of rays) of light refracted at the air-water interface. Because the index of refraction of water is greater than that of air, the rays are bent toward the normal. The diver will, therefore, think that the rays are diverging from a point above the bird and so the bird appears to be farther from the surface than it actually is.



9 • An object is placed on the axis of a diverging lens whose focal length has a magnitude of 10 cm. The distance from the object to the lens is 40 cm. The image is (a) real, inverted, and diminished, (b) real, inverted, and enlarged, (c) virtual, inverted, and diminished, (d) virtual, upright, and diminished, (e) virtual, upright, and enlarged.

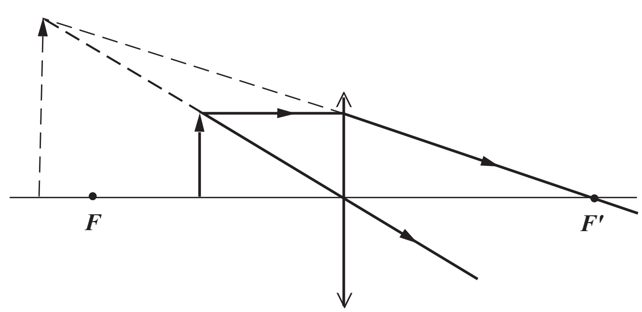
Picture the Problem We can use a ray diagram to determine the general features of the image. In the diagram shown, the parallel ray and central ray have been used to locate the image.



From the diagram, we see that the image is virtual (only one of the rays from the head of the object actually pass through the head of the image), upright, and diminished. $\boxed{(d)}$ is correct.

10 •• If an object is placed between the focal point of a converging lens and the optical center of the lens, the image is (*a*) real, inverted, and enlarged, (*b*) virtual, upright, and diminished, (*c*) virtual, upright, and enlarged, (*d*) real, inverted, and diminished.

Picture the Problem We can use a ray diagram to determine the general features of the image. In the diagram shown, the ray parallel to the principle axis and the central ray have been used to locate the image. Note that the object has been placed farther to the right than suggested in the problem statement. This was done so that the image would not be so large and so far to the left as to make the diagram excessively large.



From the diagram, we see that the image is virtual (neither ray from the head of the object passes through the head of the image), upright, and enlarged. $\boxed{(c)}$ is correct.

11 • A converging lens is made of glass that has an index of refraction of 1.6. When the lens is in air, its focal length is 30 cm. When the lens is immersed in water, its focal length (*a*) is greater than 30 cm, (*b*) is between zero and 30 cm, (*c*) is equal to 30 cm, (*d*) has a negative value.

Picture the Problem We can apply the lens maker's equation to the air-glass lens and to the water-glass lens to find the ratio of their focal lengths.

Apply the lens maker's equation to the air-glass interface:

$$\frac{1}{f_{\text{air}}}=(1.6-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Apply the lens maker's equation to the water-glass interface:

$$\frac{1}{f_{\text{water}}}=(1.6-1.33)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Divide the first of these equations by the second to obtain:

$$\frac{f_{\text{water}}}{f_{\text{air}}}=\frac{(1.6-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{(1.6-1.33)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}=2.2$$

or $f_{\text{water}}=2.2\,f_{\text{air}}$ and $\boxed{(a)}$ is correct.

12 • True or false:

(*a*) A virtual image cannot be displayed on a screen.
(*b*) A negative image distance implies that the image is virtual.
(*c*) All rays parallel to the axis of a spherical mirror are reflected through a single point.
(*d*) A diverging lens cannot form a real image from a real object.
(*e*) The image distance for a converging lens is always positive.

(*a*) True.

(*b*) True.

(*c*) False. Where the rays intersect the axis of a spherical mirror depends on how far from the axis they are reflected from the mirror.

(*d*) True.

(*e*) False. The image distance for a virtual image is negative.

13 • Both the human eye and the digital camera work by forming real images on light-sensitive surfaces. The eye forms a real image on the retina and the camera forms a real image on a CCD array. Explain the difference between the ways in which these two systems *accommodate*. That is, the difference between how an eye adjusts and how a camera adjusts (or can be adjusted) to form a focused image for objects at both large and short distances from the camera.

Determine the Concept The muscles in the eye change the thickness of the lens and thereby change the focal length of the lens to accommodate objects at different distances. A camera lens, on the other hand, has a fixed focal length so that focusing is accomplished by varying the distance between the lens and the light-sensitive surface.

14 • If an object is 25 cm in front of the naked eye of a farsighted person, an image (*a*) would be formed behind the retina if it were not for the fact that that the light is blocked (by the back of the eyeball) and the corrective contact lens should be convex, (*b*) would be formed behind the retina if it were not for the fact that that the light is blocked (by the back of the eyeball) and the corrective contact lens should be concave, (*c*) is formed in front of the retina and the corrective contact lens should be convex, (*d*) is formed in front of the retina and the corrective contact lens should be concave.

Determine the Concept The eye muscles of a farsighted person lack the ability to shorten the focal length of the lens in the eye sufficiently to form an image on the retina of the eye. A convex lens (a lens that is thicker in the middle than at the circumference) will bring the image forward onto the retina. $\boxed{(a)}$ is correct.

15 •• Explain the following statement: A microscope is an object magnifier, but a telescope is an angle magnifier. *Hint: Take a look at the ray diagram for each magnifier and use it to explain the difference in adjectives.*

Determine the Concept The objective lens of a microscope ordinarily produces an image that is larger than the object being viewed, and that image is angularly magnified by the eyepiece. The objective lens of a telescope, on the other hand, ordinarily produces an image that is smaller than the object being viewed (see Figure 32-52), and that image is angularly magnified by the eyepiece. The telescope never produces a real image that is larger than the object.

Estimation and Approximation

16 • Estimate the location and size of the image of your face when you hold a shiny new tablespoon a foot in front of your face and with the convex side toward you.

Picture the Problem We can model the spoon as a convex spherical mirror and use the mirror equation and the lateral magnification equation to estimate the location and size of the image of your face.

The mirror equation is: $$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

Because $f = r/2$, where r is the radius of curvature of the mirror:

$$\frac{1}{s}+\frac{1}{s'}=\frac{2}{r}\Rightarrow s'=\frac{rs}{r-2s}$$

Let the distance from your face to the surface of the spoon be 15 cm. If the radius of curvature of the spoon is 2 cm, then:

$$s'=\frac{(2.0\text{ cm})(15\text{ cm})}{2.0\text{ cm}-2(15\text{ cm})}\approx -1.1\text{ cm}$$

or $\boxed{\text{about } 1.1\text{ cm behind the spoon}}$

The lateral magnification equation is:

$$m=\frac{y'}{y}=-\frac{s'}{s}\Rightarrow y'=-\left(\frac{s'}{s}\right)y$$

Assume your face is 25 cm long. Substitute numerical values and evaluate y':

$$y'=-\left(\frac{-1.1\text{ cm}}{15\text{ cm}}\right)(25\text{ cm})$$

$$\approx \boxed{2\text{ cm}}$$

17 • Estimate the focal length of the ″mirror″ produced by the surface of the water in the reflection pools in front of the Lincoln Memorial on a still night.

Determine the Concept The surface of the ″mirror″ produced by the surface of the water in the Lincoln Memorial reflecting pool is approximately spherical with a radius of curvature approximately equal to the radius of Earth. The focal length of a spherical mirror is half its radius of curvature. Hence $f = \boxed{\tfrac{1}{2}R_{\text{Earth}}}$

18 •• Estimate the maximum value that could be obtained for the magnifying power of a simple magnifier, using Equation 32-20. *Hint: Think about the smallest focal length lens that could be made from glass and still be used as a magnifier*.

Picture the Problem Because the focal length of a spherical lens depends on its radii of curvature and the magnification depends on the focal length, there is a practical upper limit to the magnification.

Use Equation 32-20 to relate the magnification M of a simple magnifier to its focal length f:

$$M=\frac{x_{\text{np}}}{f}$$

Use the lens-maker's equation to relate the focal length of a lens to its radii of curvature and the index of refraction of the material from which it is constructed:

$$\frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

For a plano-convex lens, $r_2 = \infty$. Hence:

$$\frac{1}{f} = \frac{n-1}{r_1} \Rightarrow f = \frac{r_1}{n-1}$$

Substitute in the expression for M and simplify to obtain:

$$M = \frac{(n-1)x_{\text{np}}}{r_1}$$

Note that the smallest reasonable value for r_1 will maximize M.

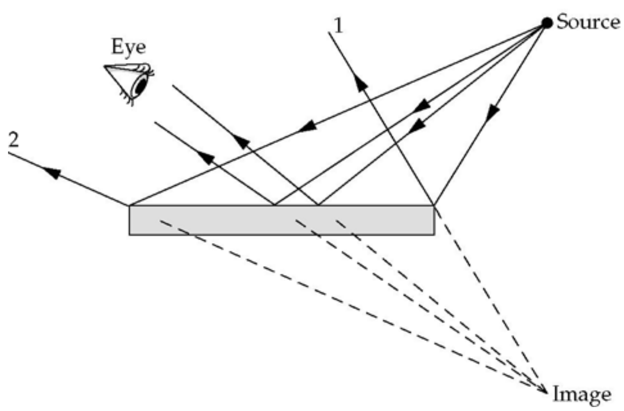
A reasonable smallest value for the radius of a magnifier is 1.0 cm. Use this value and $n = 1.5$ to estimate M_{max}:

$$M_{\text{max}} = \frac{(1.5-1)(25\,\text{cm})}{1.0\,\text{cm}} = \boxed{13}$$

Plane Mirrors

19 • **[SSM]** The image of the object point P in Figure 32-57 is viewed by an eye, as shown. Draw rays from the object point that reflect from the mirror and enter the eye. If the object point and the mirror are fixed in their locations, indicate the range of locations where the eye can be positioned and still see the image of the object point.

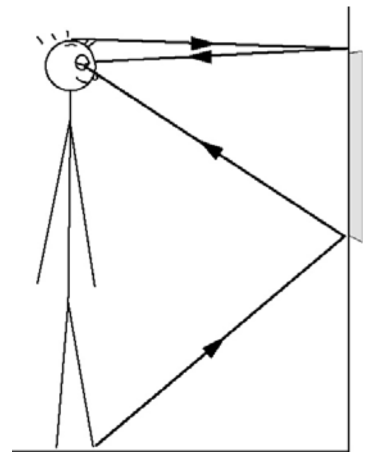
Determine the Concept Rays from the source that are reflected by the mirror are shown in the following diagram. The reflected rays appear to diverge from the image. The eye can see the image if it is in the region between rays 1 and 2.



20 • You are 1.62 m tall and want to be able to see your full image in a vertical plane mirror. (a) What is the minimum height of the mirror that will meet your needs? (b) How far above the floor should the bottom of mirror in (a) be placed, assuming that the top of your head is 14 cm above your eye level? Use a ray diagram to explain your answer.

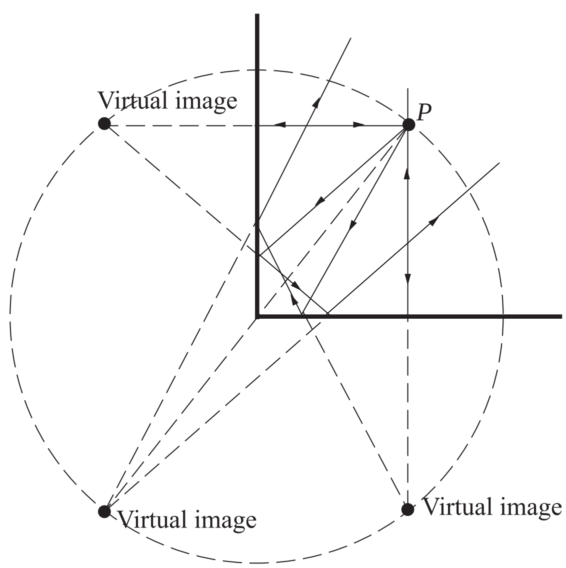
Determine the Concept A ray diagram showing rays from your feet and the top of your head reaching your eyes is shown to the right. (*a*) The mirror must be half your height, i.e., $\boxed{81\,\text{cm}}$

(*b*) The top of the minimum-height mirror must be 7 cm below the top of your head, or 155 cm above the floor. The bottom of the mirror should be placed $155\,\text{cm} - 81\,\text{cm} = \boxed{74\,\text{cm}}$ above the floor.



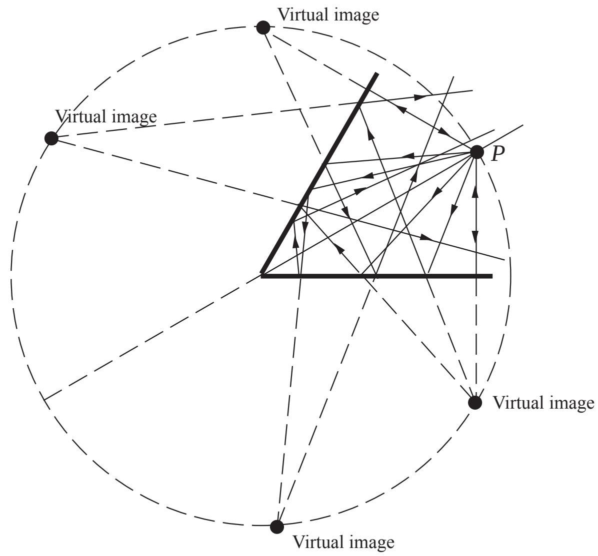
21 •• [SSM] (*a*) Two plane mirrors make an angle of 90º. The light from a point object that is arbitrarily positioned in front of the mirrors produces images at three locations. For each image location, draw two rays from the object that, after one or two reflections, appear to come from the image location. (*b*) Two plane mirrors make an angle of 60º with each other. Draw a sketch to show the location of all the images formed of an object on the bisector of the angle between the mirrors. (*c*) Repeat Part (*b*) for an angle of 120º.

Determine the Concept
(*a*) Draw rays of light from the object (*P*) that satisfy the law of reflection at the two mirror surfaces. Three virtual images are formed, as shown in the following figure. The eye should be to the right and above the mirrors in order to see these images.

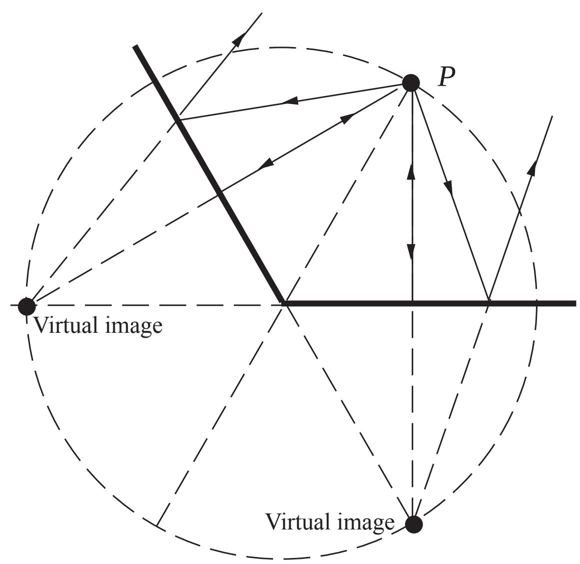


(*b*) The following diagram shows selected rays emanating from a point object (*P*) located on the bisector of the 60° angle between the mirrors. These rays form the two virtual images below the horizontal mirror. The construction details for the

two virtual images behind the mirror that is at an angle of 60° with the horizontal mirror have been omitted due to the confusing detail their inclusion would add to the diagram.



(c) The following diagram shows selected rays emanating from a point object (P) on the bisector of the 120° angle between the mirrors. These rays form the two virtual images at the intersection of the dashed lines (extensions of the reflected rays):



22 •• Show that the mirror equation (Equation 32-4 where $f = r/2$) yields the correct image distance and magnification for a plane mirror.

Picture the Problem We can use Equation 32-4 and the definition of the lateral magnification of an image to show that the mirror equation yields the correct image distance and magnification for a plane mirror.

The mirror equation is: $$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

Because $f=r/2$: $$\frac{1}{s}+\frac{1}{s'}=\frac{2}{r}$$

For a plane mirror, $r=\infty$. Hence: $$\frac{1}{s}+\frac{1}{s'}=0 \Rightarrow s'=\boxed{-s}$$

The lateral magnification of the image is given by: $$M=-\frac{s'}{s}=-\frac{-s}{s}=\boxed{+1}$$

23 •• When two plane mirrors are parallel, such as on opposite walls in a barber shop, multiple images arise because each image in one mirror serves as an object for the other mirror. An object is placed between parallel mirrors separated by 30 cm. The object is 10 cm in front of the left mirror and 20 cm in front of the right mirror. (*a*) Find the distance from the left mirror to the first four images in that mirror. (*b*) Find the distance from the right mirror to the first four images in that mirror. (*c*) Explain why each more distant image becomes fainter and fainter.

Determine the Concept (*a*) The first image in the mirror on the left is 10 cm behind the mirror. The mirror on the right forms an image 20 cm behind that mirror or 50 cm from the left mirror. This image will result in a second image 50 cm behind the left mirror. The first image in the left mirror is 40 cm from the right mirror and forms an image 40 cm behind the right mirror or 70 cm from the left mirror. That image gives an image 70 cm behind the left mirror. The fourth image behind the left mirror is 110 cm behind that mirror.

(*a*) For the mirror on the left, the images are located 10 cm, 50 cm, 70 cm, and 110 cm behind the mirror on the left

(*b*) For the mirror on the right, the images are located 20 cm, 40 cm, 80 cm, and 100 cm behind the mirror on the right.

(*c*) The successive images are dimmer because the light travels farther to form them. The intensity falls of inversely with the square of the distance the light travels. In addition, at each reflection a small percentage of the light intensity is lost. Real mirrors are not 100% reflecting.

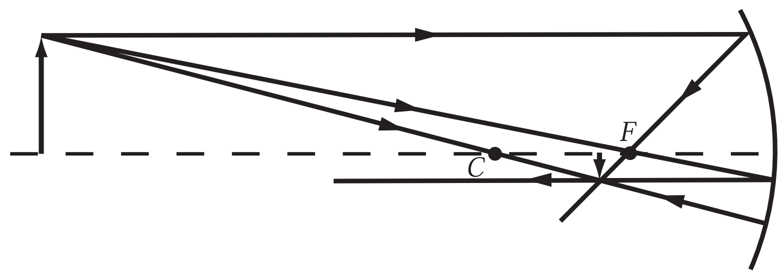
Spherical Mirrors

24 • A concave mirror has a radius of curvature equal to 24 cm. Use ray diagrams to locate the image, if it exists, for an object near the axis at distances of

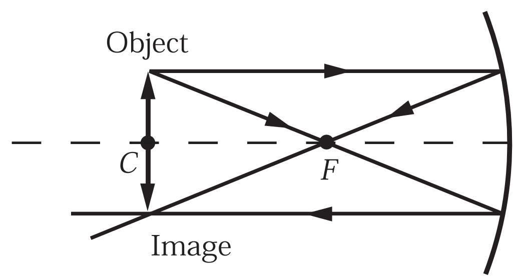
(*a*) 55 cm, (*b*) 24 cm, (*c*) 12 cm, and (*d*) 8.0 cm from the mirror. For each case, state whether the image is real or virtual; upright or inverted; and enlarged, reduced, or the same size as the object.

Picture the Problem The easiest rays to use in locating the image are (1) the ray parallel to the principal axis and passes through the focal point of the mirror, (2) the ray that passes through the center of curvature of the spherical mirror and is reflected back on itself, and (3) the ray that passes through the focal point of the spherical mirror and is reflected parallel to the principal axis. We can use any two of these rays emanating from the top of the object to locate the image of the object.

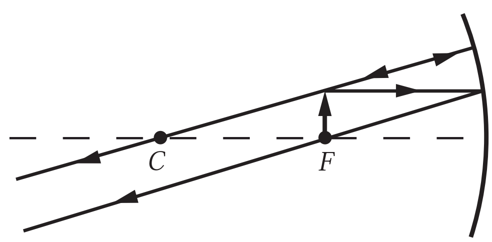
(*a*) The ray diagram follows. The image is real, inverted, and reduced.



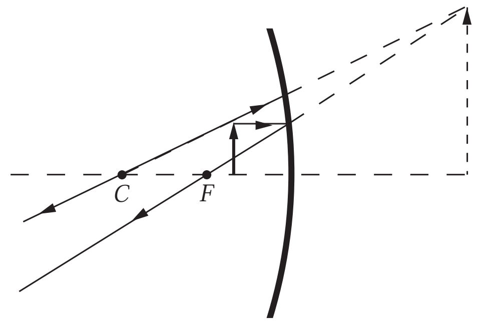
(*b*) The ray diagram is shown to the right. The image is real, inverted, and the same size as the object.



(*c*) The ray diagram is shown to the right. The object is at the focal plane of the mirror. The emerging rays are parallel and do not form an image.



(*d*) The ray diagram is shown to the right. The image is virtual, erect, and enlarged.



25 • **[SSM]** (*a*) Use the mirror equation (Equation 32-4 where $f = r/2$) to calculate the image distances for the object distances and mirror of Problem 24. (*b*) Calculate the magnification for each given object distance.

Picture the Problem In describing the images, we must indicate where they are located, how large they are in relationship to the object, whether they are real or virtual, and whether they are upright or inverted. The object distance *s*, the image distance *s*′, and the focal length of a mirror are related according to $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$, where $f = \frac{1}{2}r$ and *r* is the radius of curvature of the mirror. In this problem, $f = 12$ cm because *r* is positive for a concave mirror.

(*a*) Solve the mirror equation for *s*′:

$$s' = \frac{fs}{s-f} \text{ where } \boldsymbol{f} = \frac{\boldsymbol{r}}{2}$$

When $s = 55$ cm:

$$s' = \frac{(12\,\text{cm})(55\,\text{cm})}{55\,\text{cm} - 12\,\text{cm}} = 15.35\,\text{cm}$$

$$= \boxed{15\,\text{cm}}$$

When $s = 24$ cm:

$$s' = \frac{(12\,\text{cm})(24\,\text{cm})}{24\,\text{cm} - 12\,\text{cm}} = \boxed{24\,\text{cm}}$$

When $s = 12$ cm:

$$s' = \frac{(12\,\text{cm})(12\,\text{cm})}{12\,\text{cm} - 12\,\text{cm}} \boxed{\text{is undefined}}$$

When $s = 8.0$ cm:

$$s' = \frac{(12\,\text{cm})(8.0\,\text{cm})}{8.0\,\text{cm} - 12\,\text{cm}} = -24.0\,\text{cm}$$

$$= \boxed{-0.2\,\text{m}}$$

(*b*) The lateral magnification of the image is:

$$m=-\frac{s'}{s}$$

When $s = 55$ cm, the lateral magnification of the image is:

$$m=-\frac{s'}{s}=-\frac{15.35\,\text{cm}}{55\,\text{cm}}=\boxed{-0.28}$$

When $s = 24$ cm, the lateral magnification of the image is:

$$m=-\frac{s'}{s}=-\frac{24\,\text{cm}}{24\,\text{cm}}=\boxed{-1.0}$$

When $s = 12$ cm:

$$\boxed{m \text{ is undefined.}}$$

When $s = 8.0$ cm, the lateral magnification of the image is:

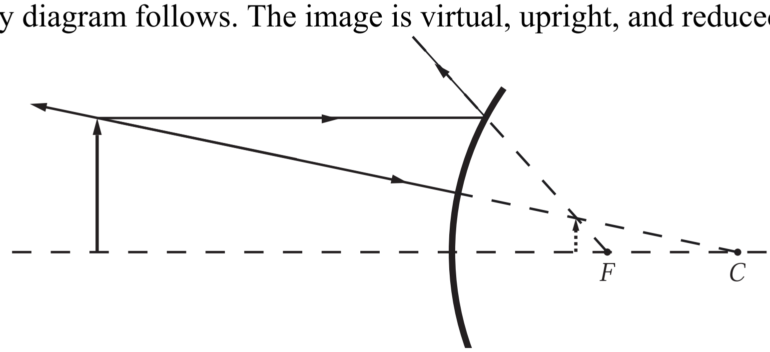
$$m=-\frac{s'}{s}=-\frac{-24\,\text{cm}}{8.0\,\text{cm}}=\boxed{3.0}$$

Remarks: These results are in excellent agreement with those obtained graphically in Problem 24.

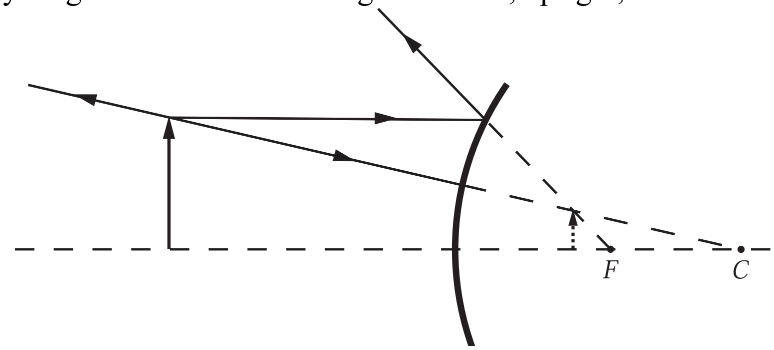
26 • A convex mirror has a radius of curvature that has a magnitude equal to 24 cm. Use ray diagrams to locate the image, if it exists, for an object near the axis at distances of (*a*) 55 cm, (*b*) 24 cm, (*c*) 12 cm, (*d*) 8.0 cm, (*e*) 1.0 cm from the mirror. For each case, state whether the image is real or virtual; upright or inverted; and enlarged, reduced, or the same size as the object.

Picture the Problem The easiest rays to use in locating the image are (1) the ray parallel to the principal axis and passes through the focal point of the mirror, (2) the ray that passes through the center of curvature of the spherical mirror and is reflected back on itself, and (3) the ray that passes through the focal point of the spherical mirror and is reflected parallel to the principal axis. We can use any two of these rays emanating from the top of the object to locate the image of the object. Note that, due to space limitations, the following ray diagrams are not to the same scale as those in Problem 24.

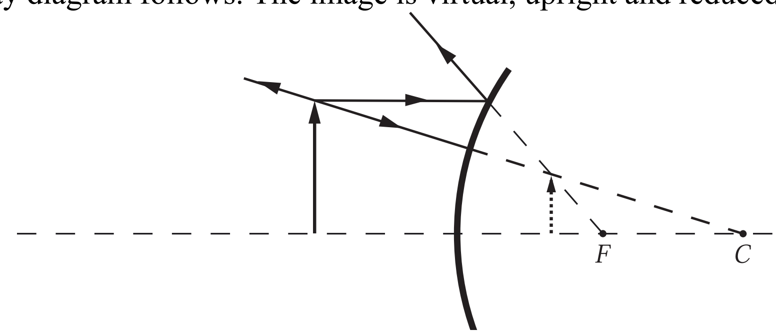
(*a*) The ray diagram follows. The image is virtual, upright, and reduced.



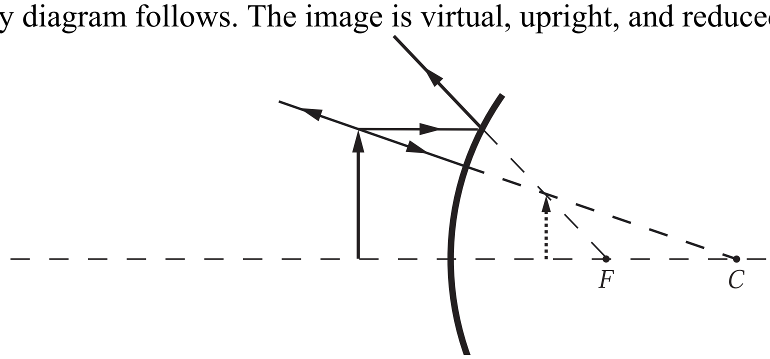
(*b*) The ray diagram follows. The image is virtual, upright, and reduced.



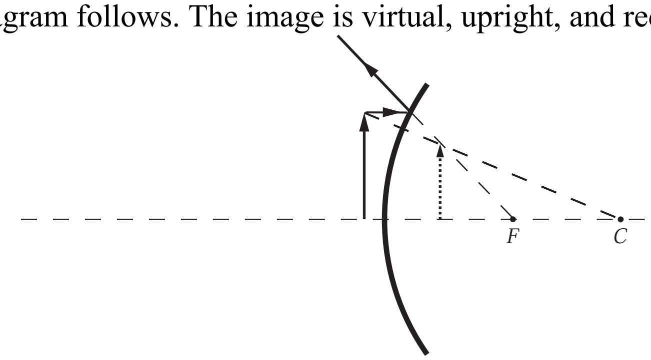
(*c*) The ray diagram follows. The image is virtual, upright and reduced.



(*d*) The ray diagram follows. The image is virtual, upright, and reduced.



(*e*) The ray diagram follows. The image is virtual, upright, and reduced.



27 • (*a*) Use the mirror equation (Equation 32-4 where $f = r/2$) to calculate the image distances for the object distances and mirror of Problem 26.
(*b*) Calculate the magnification for each given object distance.

Picture the Problem In describing the images, we must indicate where they are located, how large they are in relationship to the object, whether they are real or virtual, and whether they are upright or inverted. The object distance *s*, the image distance *s*′, and the focal length of a mirror are related according to $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$, where $f = \frac{1}{2}r$ and *r* is the radius of curvature of the mirror. In this problem, $f = -12$ cm because *r* is negative for a convex mirror.

(*a*) Solve the mirror equation for *s*′:

$$s' = \frac{fs}{s-f}$$

When *s* = 55 cm:

$$s' = \frac{(-12\,\text{cm})(55\,\text{cm})}{55\,\text{cm}-(-12\,\text{cm})} = -9.85\,\text{cm} = \boxed{-9.9\,\text{cm}}$$

When *s* = 24 cm:

$$s' = \frac{(-12\,\text{cm})(24\,\text{cm})}{24\,\text{cm}-(-12\,\text{cm})} = \boxed{-8.0\,\text{cm}}$$

When *s* = 12 cm:

$$s' = \frac{(-12\,\text{cm})(12\,\text{cm})}{12\,\text{cm}-(-12\,\text{cm})} = \boxed{-6.0\,\text{cm}}$$

When *s* = 8.0 cm:

$$s' = \frac{(-12\,\text{cm})(8.0\,\text{cm})}{8.0\,\text{cm}-(-12\,\text{cm})} = \boxed{-4.8\,\text{cm}}$$

(*b*) The lateral magnification is given by:

$$m = -\frac{s'}{s}$$

When *s* = 55 cm, the lateral magnification of the image is:

$$m = -\frac{-9.85\,\text{cm}}{55\,\text{cm}} = \boxed{0.18}$$

When *s* = 24 cm, the lateral magnification of the image is:

$$m = -\frac{-8.0\,\text{cm}}{24\,\text{cm}} = \boxed{0.33}$$

When *s* = 12 cm, the lateral magnification of the image is:

$$m = -\frac{-6.0\,\text{cm}}{12\,\text{cm}} = \boxed{0.50}$$

When s = 8.0 cm, the lateral magnification of the image is:

$$m = -\frac{-4.80\,\text{cm}}{8.0\,\text{cm}} = \boxed{0.60}$$

Remarks: These results are in excellent agreement with those obtained graphically in Problem 26.

28 •• Use the mirror equation (Equation 32-4 where $f = r/2$) to prove that a convex mirror cannot form a real image of a real object, no matter where the object is placed.

Picture the Problem We can solve the mirror equation for $1/s'$ and then examine the implications of $f < 0$ and $s > 0$.

Solve the mirror equation for $1/s'$:

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} = \frac{s-f}{sf}$$

For a convex mirror:

$$f < 0$$

For $s > 0$, the numerator is positive and the denominator negative. Consequently:

$$\frac{1}{s'} < 0 \Rightarrow \boxed{s' < 0}$$

29 • **[SSM]** A dentist wants a small mirror that will produce an upright image that has a magnification of 5.5 when the mirror is located 2.1 cm from a tooth. (*a*) Should the mirror be concave or convex? (*b*) What should the radius of curvature of the mirror be?

Picture the Problem We can use the mirror equation and the definition of the lateral magnification to find the radius of curvature of the dentist's mirror.

(*a*) The mirror must be concave. A convex mirror always produces a diminished virtual image.

(*b*) Express the mirror equation:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} = \frac{2}{r} \Rightarrow r = \frac{2ss'}{s'+s} \qquad (1)$$

The lateral magnification of the mirror is given by:

$$m = -\frac{s'}{s} \Rightarrow s' = -ms$$

Substitute for s' in equation (1) to obtain:

$$r = \frac{-2ms}{1-m}$$

Substitute numerical values and evaluate r:

$$r = \frac{-2(5.5)(2.1\,\text{cm})}{1-5.5} = \boxed{5.1\,\text{cm}}$$

30 •• Convex mirrors are used in many stores to provide a wide angle of surveillance for a reasonable mirror size. Your summer job is at a local convenience store that uses the mirror shown in Figure 32-58. This setup allows you (or the clerk) to survey the entire store when you are 5.0 m from the mirror. The mirror has a radius of curvature equal to 1.2 m. Assume all rays are paraxial. (*a*) If a customer is 10 m from the mirror, how far from the mirror is his image? (*b*) Is the image in front of or behind the mirror? (*c*) If the customer is 2.0 m tall, how tall is his image?

Picture the Problem We can use the mirror equation and the relationship between the focal length of a mirror and its radius of curvature to find the location of the image. We can then use the definition of the lateral magnification of the mirror to find the height of the image formed in the mirror.

(*a*) Solve the mirror equation for s':

$$s' = \frac{fs}{s-f} \text{ where } f = \tfrac{1}{2}r$$

Substitute for f and simplify to obtain:

$$s' = \frac{\frac{1}{2}rs}{s-\frac{1}{2}r} = \frac{rs}{2s-r}$$

Substitute numerical values and evaluate s':

$$s' = \frac{(-1.2\,\text{m})(10\,\text{m})}{2(10\,\text{m})-(-1.2\,\text{m})} = -56.6\,\text{cm}$$

and

$$\boxed{\text{the image is 57 cm from the mirror.}}$$

(*b*) Because the image distance is negative:

$$\boxed{\text{the image is behind the mirror.}}$$

(*c*) The lateral magnification of the mirror is given by Equation 32-5:

$$m = \frac{y'}{y} = -\frac{s'}{s} \Rightarrow y' = -\frac{s'}{s}y$$

Substitute numerical values and evaluate y':

$$y' = -\frac{-0.566\,\text{m}}{10\,\text{m}}(2.0\,\text{m}) = \boxed{11\,\text{cm}}$$

31 •• A certain telescope uses a concave spherical mirror that has a radius equal to 8.0 m. Find the location and diameter of the image of the moon formed by this mirror. The moon has a diameter of 3.5×10^6 m and is 3.8×10^8 m from Earth.

Picture the Problem We can use the mirror equation to locate the image formed in this mirror and the expression for the lateral magnification of the mirror to find the diameter of the image.

Solve the mirror equation for the location of the image of the moon:

$$s' = \frac{fs}{s - f}$$

Because $f = \frac{1}{2}r$:

$$s' = \frac{\frac{1}{2}rs}{s - \frac{1}{2}r} = \frac{rs}{2s - r}$$

Substitute numerical values and evaluate s':

$$s' = \frac{(8.0\,\text{m})(3.8\times10^{8}\,\text{m})}{2(3.8\times10^{8}\,\text{m}) - 8.0\,\text{m}} = \boxed{4.0\,\text{m}}$$

Express the lateral magnification of the mirror:

$$m = \frac{y'}{y} = -\frac{s'}{s} \Rightarrow y' = -\frac{s'}{s}y$$

Substitute numerical values and evaluate y':

$$y' = -\frac{4.0\,\text{m}}{3.8\times10^{8}\,\text{m}}(3.5\times10^{6}\,\text{m})$$
$$= -3.7\,\text{cm}$$

The 3.7-cm-diameter image is 4.0 m in front of the mirror.

32 •• A piece of a thin spherical shell that has a radius of curvature of 100 cm is silvered on both sides. The concave side of the piece forms a real image 75 cm from the piece. The piece is then turned around so that its convex side faces the object. The piece is moved so that the image is now 35 cm from the piece on the concave side. (*a*) How far was the piece moved? (*b*) Was it moved toward the object or away from the object?

Picture the Problem We can use the mirror equation to find the focal length of the piece of the thin spherical shell and then apply it a second time to find the object position after the piece has been moved.

(*a*) Solve the mirror equation for *f* to obtain:

$$f = \frac{ss'}{s' + s}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{(100\,\text{cm})(75\,\text{cm})}{75\,\text{cm} + 100\,\text{cm}} = 42.86\,\text{cm}$$

Solving the mirror equation for s yields:

$$s=\frac{fs'}{s'-f}$$

With the piece turned around, $f=-42.86$ cm and $s'=-35$ cm. Substitute numerical values and evaluate s:

$$s=\frac{(-42.86\,\text{cm})(-35\,\text{cm})}{-35\,\text{cm}-(-42.86\,\text{cm})}\approx 191\,\text{cm}$$

The distance d the mirror moved is:

$$d=191\,\text{cm}-100\,\text{cm}=\boxed{91\,\text{cm}}$$

(b) The piece was moved *away from* the object.

33 •• Two light rays parallel to the optic axis of a concave mirror strike that mirror as shown in Figure 32-59. This mirror has a radius of curvature equal to 5.0 m. They then strike a small spherical mirror that is 2.0 m from the large mirror. The light rays finally meet at the vertex of the large mirror. Note: The small mirror is shown as planar, so as not to give away the answer, but it is *not* actually planar. (a) What is the radius of curvature of the small mirror? (b) Is that mirror convex or concave? Explain your answer.

Picture the Problem Denote the larger mirror on the right using the numeral 1, the smaller mirror using the numeral 2, and the distance between the mirrors by d. Applying the mirror equation to the two mirrors will lead us to an expression for the radius of curvature of the small mirror.

(a) Apply the mirror equation to mirror 1:

$$\frac{1}{s_1}+\frac{1}{s_1'}=\frac{2}{r_1}$$

Because $s_1=\infty$:

$$\frac{1}{s_1'}=\frac{2}{r_1}\Rightarrow s_1'=\tfrac{1}{2}r_1$$

The image formed by mirror #1 serves as a virtual object for mirror 2. Apply the mirror equation to mirror 2 to obtain:

$$\frac{1}{s_2}+\frac{1}{s_2'}=\frac{2}{r_2}$$

or, because $s_2=d-s'_1=d-\tfrac{1}{2}r_1$,

$$\frac{1}{d-\tfrac{1}{2}r_1}+\frac{1}{s_2'}=\frac{2}{r_2}$$

Because the image distance for mirror 2 is d:

$$\frac{2}{2d-r_1}+\frac{1}{d}=\frac{2}{r_2}$$

Substituting numerical values yields:

$$\frac{2}{2(2.0\text{ m})-5.0\text{ m}}+\frac{1}{2.0\text{ m}}=\frac{2}{r_2}$$

Finally, solve for r_2 to obtain:

$$r_2=\boxed{-1.3\text{ m}}$$

(*b*) Because $f_2=\frac{1}{2}r_2<0$, the small mirror is $\boxed{\text{convex}}$.

Images Formed by Refraction

34 •• A very long 1.75-cm-diameter glass rod has one end ground and polished to a convex spherical surface that has a 7.20-cm radius. The glass material has an index of refraction of 1.68. (*a*) A point object in air is on the axis of the rod and 30.0 cm from the spherical surface. Find the location of the image and state whether the image is real or virtual. (*b*) Repeat Part (*a*) for a point object in air, on the axis, and 5.00 cm from the spherical surface. Draw a ray diagram for each case.

Picture the Problem We can use the equation for refraction at a single surface to find the images corresponding to these three object positions. The signs of the image distances will tell us whether the images are real or virtual and the ray diagrams will confirm the correctness of our analytical solutions.

Use the equation for refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{r} \quad (1)$$

Solving for s' yields:

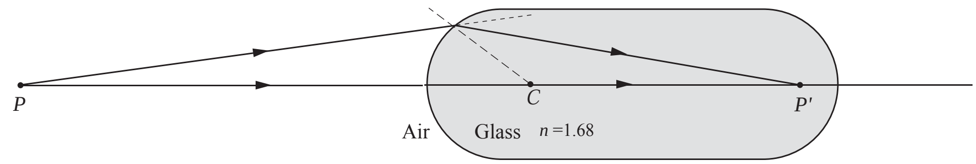
$$s'=\frac{n_2}{\frac{n_2-n_1}{r}-\frac{n_1}{s}}$$

(*a*) Substitute numerical values (s = 30.0 cm, n_1 = 1.00, n_2 = 1.68 and r = 7.20 cm) and evaluate s':

$$s'=\frac{1.68}{\frac{1.68-1.00}{7.20}-\frac{1.00}{30.0}}=\boxed{27\text{ cm}}$$

where the positive distance tells us that the image is 27 cm in back of the surface and is $\boxed{\text{real.}}$

In the following ray diagram, the object is at P and the image at P'.

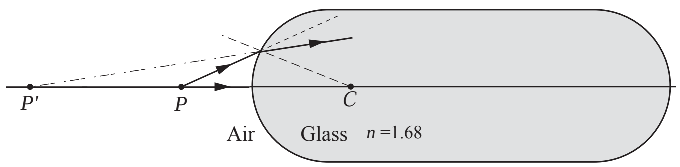


(*b*) Substitute numerical values (s = 5.00 cm, n_1 = 1.00, n_2 = 1.68 and r = 7.20 cm) and evaluate s':

$$s' = \frac{1.68}{\frac{1.68-1.00}{7.20} - \frac{1.00}{5.00}} = \boxed{-16\text{ cm}}$$

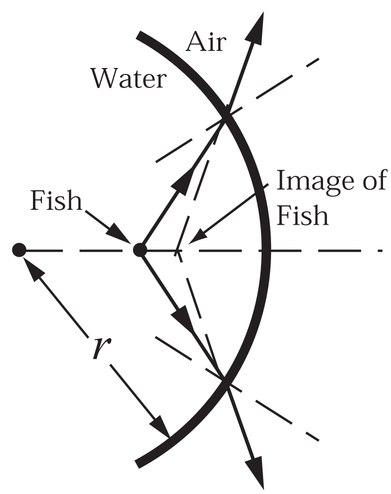
where the minus sign tells us that the image is 16 cm in front of the surface and is $\boxed{\text{virtual.}}$

In the following ray diagram, the object is at P and the image at P'.



35 • **[SSM]** A fish is 10 cm from the front surface of a spherical fish bowl of radius 20 cm. (*a*) How far behind the surface of the bowl does the fish appear to someone viewing the fish from in front of the bowl? (*b*) By what distance does the fish's apparent location change (relative to the front surface of the bowl) when it swims away to 30 cm from the front surface?

Picture the Problem The diagram shows two rays (from the bundle of rays) of light refracted at the water-air interface. Because the index of refraction of air is less than that of water, the rays are bent away from the normal. The fish will, therefore, appear to be closer than it actually is. We can use the equation for refraction at a single surface to find the distance s'. We'll assume that the glass bowl is thin enough that we can ignore the refraction of the light passing through it.



(*a*) Use the equation for refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{r}$$

Solving for s' yields:

$$s'=\frac{n_2}{\dfrac{n_2-n_1}{r}-\dfrac{n_1}{s}}$$

Substitute numerical values ($s = -10$ cm, $n_1 = 1.33$, $n_2 = 1.00$ and $r = 20$ cm) and evaluate s':

$$s'=\frac{1.00}{\dfrac{1.00-1.33}{20\text{ cm}}-\dfrac{1.33}{-10\text{ cm}}}=\boxed{8.6\text{ cm}}$$

and the image is 8.6 cm from the front surface of the bowl.

(*b*) For $s = 30$ cm:

$$s'=\frac{1.00}{\dfrac{1.00-1.33}{20\text{ cm}}-\dfrac{1.33}{-30\text{ cm}}}=35.9\text{ cm}$$

The change in the fish's apparent location is:

$$35.9\text{ cm}-8.6\text{ cm}\approx\boxed{27\text{ cm}}$$

36 •• A very long 1.75-cm-diameter glass rod has one end ground and polished to a concave spherical surface that has a 7.20-cm radius. The glass material has an index of refraction of 1.68. A point object in air is on the axis of the rod and 15.0 cm from the spherical surface. Find the location of the image and state whether the image is real or virtual. Draw a ray diagram.

Picture the Problem We can use the equation for refraction at a single surface to find the images corresponding to these three object positions. The signs of the image distances will tell us whether the images are real or virtual and the ray diagrams will confirm the correctness of our analytical solutions.

Use the equation for refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{r} \qquad (1)$$

Solving for s' yields:

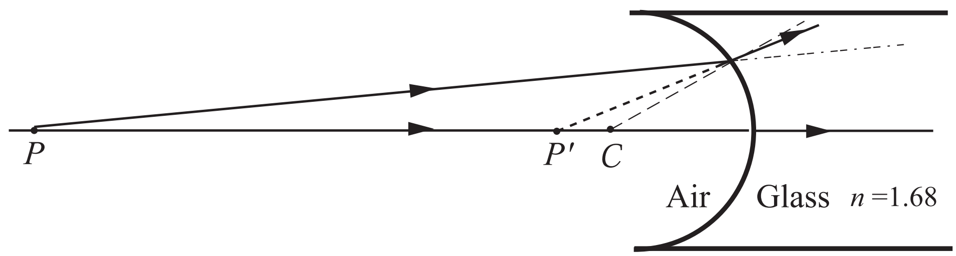
$$s'=\frac{n_2}{\dfrac{n_2-n_1}{r}-\dfrac{n_1}{s}}$$

Substitute numerical values (s = 15.0 cm, n_1 = 1.00, n_2 = 1.68, and r = −7.20 cm) and evaluate s':

$$s' = \frac{1.68}{\dfrac{1.68-1.00}{-7.20\text{ cm}} - \dfrac{1.00}{15.0\text{ cm}}} = \boxed{-10}$$

where the minus sign tells us that the image is 10 cm in front of the surface of the rod and is $\boxed{\text{virtual.}}$

In the following ray diagram, the object is at P and the virtual image is at P'.



37 •• [SSM] Repeat Problem 34 for when the glass rod and the object are immersed in water and (*a*) the object is 6.00 cm from the spherical surface, and (*b*) the object is 12.0 cm from the spherical surface.

Picture the Problem We can use the equation for refraction at a single surface to find the images corresponding to these three object positions. The signs of the image distances will tell us whether the images are real or virtual and the ray diagrams will confirm the correctness of our analytical solutions.

Use the equation for refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r} \qquad (1)$$

Solving for s' yields:

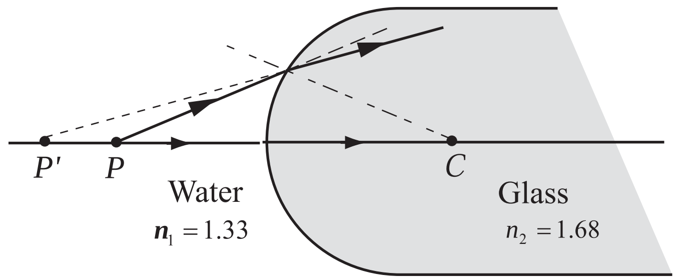
$$s' = \frac{n_2}{\dfrac{n_2 - n_1}{r} - \dfrac{n_1}{s}}$$

(*a*) Substitute numerical values (s = 6.00 cm, n_1 = 1.33, n_2 = 1.68, and r = 7.20 cm) and evaluate s':

$$s' = \frac{1.68}{\dfrac{1.68-1.33}{7.20\text{ cm}} - \dfrac{1.33}{6.00\text{ cm}}} = \boxed{-9.7\text{ cm}}$$

where the negative distance tells us that the image is 9.7 cm in front of the surface and is virtual.

In the following ray diagram, the object is at P and the virtual image is at P'.

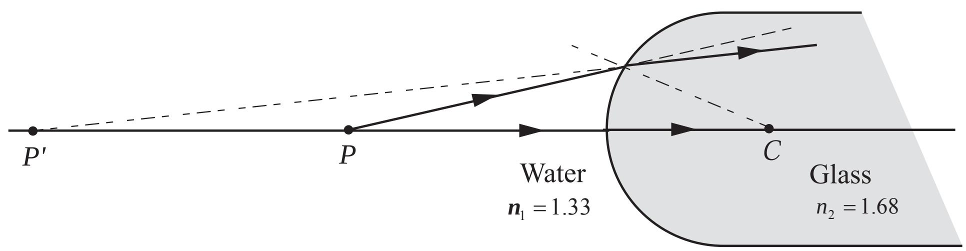


(*b*) Substitute numerical values (s = 12.0 cm, n_1 = 1.33, n_2 = 1.68, and r = 7.20 cm) and evaluate s':

$$s' = \frac{1.68}{\frac{1.68-1.33}{7.20\,\text{cm}} - \frac{1.33}{12.0\,\text{cm}}} = \boxed{-27\,\text{cm}}$$

where the negative distance tells us that the image is 27 cm in front of the surface and is virtual.

In the following ray diagram, the object is at P and the virtual image is at P'.



38 •• Repeat Problem 36 for when the glass rod and the object are immersed in water and the object is 20 cm from the spherical surface.

Picture the Problem We can use the equation for refraction at a single surface to find the images corresponding to these three object positions. The signs of the image distances will tell us whether the images are real or virtual and the ray diagrams will confirm the correctness of our analytical solutions.

Use the equation for refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r} \qquad (1)$$

Solving for s' yields:

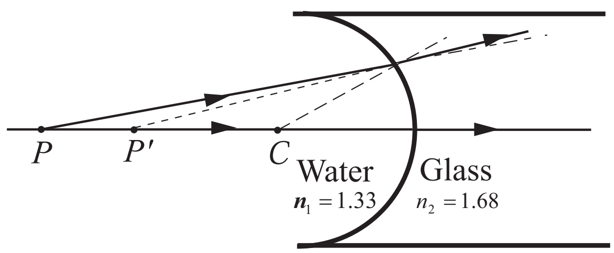
$$s' = \frac{n_2}{\frac{n_2 - n_1}{r} - \frac{n_1}{s}}$$

Substitute numerical values (s = 20 cm, n_1 = 1.33, n_2 = 1.68, and r = −7.20 cm) and evaluate s':

$$s' = \frac{1.68}{\frac{1.68 - 1.33}{-7.20\text{ cm}} - \frac{1.33}{20\text{ cm}}} = \boxed{-15}$$

where the minus sign tells us that the image is 15 cm in front of the surface of the rod and is virtual

In the following ray diagram the object is at P and the virtual image is at P'.



39 •• A rod that is 96.0-cm long is made of glass that has an index of refraction equal to 1.60. The rod has its ends ground to convex spherical surfaces that have radii equal to 8.00 cm and 16.0 cm. An object is in air on the long axis of the rod 20.0 cm from the end that has the 8.00-cm radius. (*a*) Find the image distance due to refraction at the 8.00-cm-radius surface. (*b*) Find the position of the final image due to refraction at both surfaces. (*c*) Is the final image real or virtual?

Picture the Problem (*a*) We can use the equation for refraction at a single surface to find the images due to refraction at the ends of the glass rod. (*b*) The image formed by the refraction at the first surface will serve as the object for the second surface. (*c*) The sign of the final image distance will tell us whether the image is real or virtual.

(*a*) Use the equation for refraction at a single surface to relate the image and object distances at the surface whose radius is 8.00 cm:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$$

Solving for s' yields:

$$s' = \frac{n_2}{\frac{n_2 - n_1}{r} - \frac{n_1}{s}}$$

Substitute numerical values (s = 20.0 cm, n_1 = 1.00, n_2 = 1.60, and r = 8.00 cm) and evaluate s':

$$s' = \frac{1.60}{\frac{1.60-1.00}{8.00\text{ cm}} - \frac{1.00}{20.0\text{ cm}}} = \boxed{64\text{ cm}}$$

(b) The object for the second surface is 96 cm − 64 cm = 32 cm from the surface whose radius is −16.0 cm. Substitute numerical values (note that now n_1 = 1.60 and n_2 = 1.00) and evaluate s':

$$s' = \frac{1.00}{\frac{1.00-1.60}{-16.0\text{ cm}} - \frac{1.60}{32\text{ cm}}} = \boxed{-80\text{ cm}}$$

(c) The final image is inside the rod and 96 cm − 80 cm = 16 cm from the surface whose radius of curvature is 8.00 cm and is virtual.

40 •• Repeat Problem 39 for an object in air on the axis of the glass rod 20.0 cm from the end that has the 16.0-cm radius.

Picture the Problem (a) We can use the equation for refraction at a single surface to find the images due to refraction at the ends of the glass rod. (b) The image formed by the refraction at the first surface will serve as the object for the second surface. (c) The sign of the final image distance will tell us whether the image is real or virtual.

(a) Use the equation for refraction at a single surface to relate the image and object distances at the surface that has a 16.0 cm radius:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$$

Solving for s' yields:

$$s' = \frac{n_2}{\frac{n_2-n_1}{r} - \frac{n_1}{s}}$$

Substitute numerical values (s = 20.0 cm, n_1 = 1.00, n_2 = 1.60 and r = 16.0 cm) evaluate s':

$$s' = \frac{1.60}{\frac{1.60-1.00}{16.0\text{ cm}} - \frac{1.00}{20.0\text{ cm}}} = -128\text{ cm} = \boxed{-1.3\times10^{2}\text{ cm}}$$

where the minus sign tells us that the image is 1.3×10^2 cm to the left of the surface whose radius of curvature is 16.0 cm.

(*b*) The object for the second surface is 96 cm + 128 cm = 224 cm from the surface whose radius of curvature is −8.00 cm. Substitute numerical values (note that now n_1 = 1.60 and n_2 = 1.00) and evaluate s':

$$s' = \frac{1.00}{\frac{1.00-1.60}{-8.00\text{ cm}} - \frac{1.60}{224\text{ cm}}} = 14.7\text{ cm}$$

$$= \boxed{15\text{ cm}}$$

(*c*) The final image is outside the rod, 15 cm from the end whose radius of curvature is 8.00 cm, and is real.

Thin Lenses

41 • [SSM] A double concave lens that has an index of refraction equal to 1.45 has radii whose magnitudes are equal to 30.0 cm and 25.0 cm. An object is located 80.0 cm to the left of the lens. Find (*a*) the focal length of the lens, (*b*) the location of the image, and (*c*) the magnification of the image. (*d*) Is the image real or virtual? Is the image upright or inverted?

Picture the Problem We can use the lens-maker's equation to find the focal length of the lens and the thin-lens equation to locate the image. We can use $m = -\frac{s'}{s}$ to find the lateral magnification of the image.

(*a*) The lens-maker's equation is:

$$\frac{1}{f} = \left(\frac{n}{n_{\text{air}}} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where the numerals 1 and 2 denote the first and second surfaces, respectively.

Substitute numerical values to obtain:

$$\frac{1}{f} = \left(\frac{1.45}{1.00} - 1\right)\left(\frac{1}{-30.0\text{ cm}} - \frac{1}{25.0\text{ cm}}\right)$$

Solving for f yields:

$$f = -30.3\text{ cm} = \boxed{-30\text{ cm}}$$

(*b*) Use the thin-lens equation to relate the image and object distances:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow s' = \frac{fs}{s-f}$$

Substitute numerical values and evaluate s':

$$s' = \frac{(-30.3\,\text{cm})(80.0\,\text{cm})}{80.0\,\text{cm} - (-30.3\,\text{cm})} = -21.98\,\text{cm}$$
$$= -22\,\text{cm}$$

The image is 22 cm from the lens and on the same side of the lens as the object.

(*c*) The lateral magnification of the image is given by:

$$m = -\frac{s'}{s}$$

Substitute numerical values and evaluate m:

$$m = -\frac{-21.98\,\text{cm}}{80.0\,\text{cm}} = \boxed{0.27}$$

(*d*) Because $s' < 0$ and $m > 0$, the image is $\boxed{\text{virtual and upright.}}$

42 • The following thin lenses are made of glass that has an index of refraction equal to 1.60. Make a sketch of each lens and find each focal length in air: (*a*) r_1 = 20.0 cm and r_2 = 10.0 cm, (*b*) r_1 = 10.0 cm and r_2 = 20.0 cm, and (*c*) r_1 = –10.0 cm and r_2 = –20.0 cm.

Picture the Problem We can use the lens-maker's equation to find the focal length of each of the lenses described in the problem statement.

The lens-maker's equation is:

$$\frac{1}{f} = \left(\frac{n}{n_{\text{air}}} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where the numerals 1 and 2 denote the first and second surfaces, respectively.

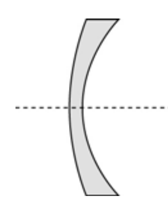
(*a*) For r_1 = 20.0 cm and r_2 = 10.0 cm:

$$\frac{1}{f} = \left(\frac{1.60}{1.00} - 1\right)\left(\frac{1}{20.0\,\text{cm}} - \frac{1}{10.0\,\text{cm}}\right)$$

and

$$f = \boxed{-33\,\text{cm}}$$

A sketch of the lens is shown to the right:



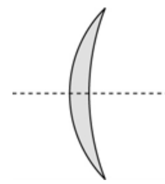
(*b*) For r_1 = 10.0 cm and r_2 = 20.0 cm:

$$\frac{1}{f}=\left(\frac{1.60}{1.00}-1\right)\left(\frac{1}{10.0\,\text{cm}}-\frac{1}{20.0\,\text{cm}}\right)$$

and

$$f=\boxed{33\,\text{cm}}$$

A sketch of the lens is shown to the right:

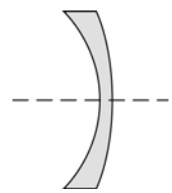


(*c*) For r_1 = –10.0 cm and r_2 = –20.0 cm:

$$\frac{1}{f}=\left(\frac{1.60}{1.00}-1\right)\left(\frac{1}{-10.0\,\text{cm}}-\frac{1}{-20.0\,\text{cm}}\right)$$

and $f=\boxed{-33\,\text{cm}}$

A sketch of the lens is shown to the right:



Remarks: Note that the lenses that are thicker on their axis than on their circumferences are positive (converging) lenses and those that are thinner on their axis are negative (diverging) lenses.

43 • The following four thin lenses are made of glass that has an index of refraction of 1.5. The radii given are *magnitudes*. Make a sketch of each lens and find each focal length in air: (*a*) double-convex that has radii of curvature equal to 15 cm and 26 cm, (*b*) plano-convex that has a radius of curvature equal to 15 cm, (*c*) double concave that has radii of curvature equal to 15 cm, and (*d*) plano-concave that has a radius of curvature equal to 26 cm.

Picture the Problem We can use the lens-maker's equation to find the focal length of each of the lenses.

The lens-maker's equation is:

$$\frac{1}{f}=\left(\frac{n}{n_{\text{air}}}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

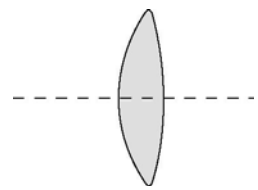
where the numerals 1 and 2 denote the first and second surfaces, respectively.

(*a*) Because the lens is double convex, the radius of curvature of the second surface is negative. Hence $r_1 = 15$ cm and $r_2 = -26$ cm:

$$\frac{1}{f} = \left(\frac{1.50}{1.00} - 1\right)\left(\frac{1}{15\,\text{cm}} - \frac{1}{-26\,\text{cm}}\right)$$

and $f = \boxed{19\,\text{cm}}$

A double convex lens is shown to the right:

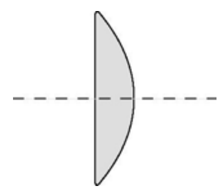


(*b*) Because the lens is plano-convex, the radius of curvature of the second surface is negative. Hence $r_1 = \infty$ and $r_2 = -15$ cm:

$$\frac{1}{f} = \left(\frac{1.50}{1.00} - 1\right)\left(\frac{1}{\infty} - \frac{1}{-15\,\text{cm}}\right)$$

and $f = \boxed{30\,\text{cm}}$

A plano-convex lens is shown to the right:

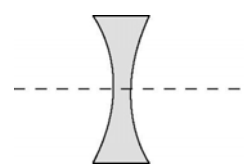


(*c*) Because the lens is double concave, the radius of curvature of its first surface is negative. Hence $r_1 = -15$ cm and $r_2 = +15$ cm:

$$\frac{1}{f} = \left(\frac{1.50}{1.00} - 1\right)\left(\frac{1}{-15\,\text{cm}} - \frac{1}{15\,\text{cm}}\right)$$

and $f = \boxed{-15\,\text{cm}}$

A double concave lens is shown to the right:

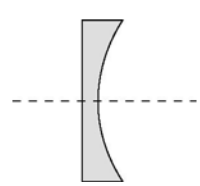


(*d*) Because the lens is plano-concave, the radius of curvature of its second surface is positive. Hence $r_1 = \infty$ and $r_2 = +26$ cm:

$$\frac{1}{f} = \left(\frac{1.50}{1.00} - 1\right)\left(\frac{1}{\infty} - \frac{1}{26\,\text{cm}}\right)$$

and $f = \boxed{-52\,\text{cm}}$

A plano-concave lens is shown to the right:



44 • Find the focal length of a glass lens that has an index of refraction equal to 1.62, a concave surface that has a radius of curvature of magnitude 100 cm, and a convex surface that has a radius of curvature of magnitude 40.0 cm.

Picture the Problem We can use the lens-maker's equation to find the focal length of the lens.

The lens-maker's equation is:

$$\frac{1}{f}=\left(\frac{n}{n_{\text{air}}}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

where the numerals 1 and 2 denote the first and second surfaces, respectively.

Substitute numerical values to obtain:

$$\frac{1}{f}=\left(\frac{1.62}{1.00}-1\right)\left(\frac{1}{-100\,\text{cm}}-\frac{1}{-40.0\,\text{cm}}\right)$$

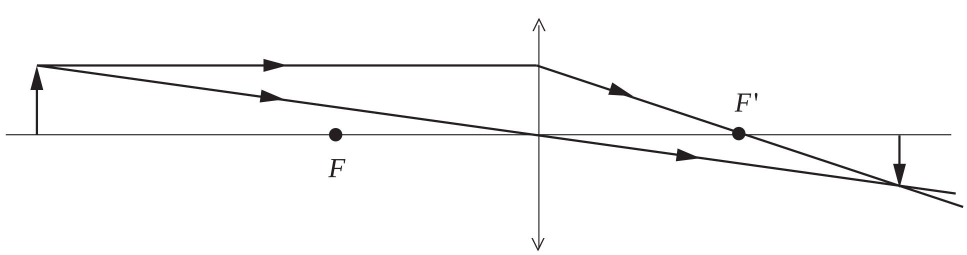
Solving for f yields:

$$f=\boxed{1.1\ \text{m}}$$

45 •• **[SSM]** (*a*) An object that is 3.00 cm high is placed 25.0 cm in front of a thin lens that has a power equal to 10.0 D. Draw a ray diagram to find the position and the size of the image and check your results using the thin-lens equation. (*b*) Repeat Part (*a*) if the object is placed 20.0 cm in front of the lens. (*c*) Repeat Part (*a*) for an object placed 20.0 cm in front of a thin lens that has a power equal to –10.0 D.

Picture the Problem We can find the focal length of each lens from the definition of the power *P*, in diopters, of a lens ($P = 1/f$). The thin-lens equation can be applied to find the image distance for each lens and the size of the image can be found from the magnification equation $m=\frac{y'}{y}=-\frac{s'}{s}$.

(*a*) The parallel and central rays were used to locate the image in the diagram shown below.



The image is real, inverted, and diminished.

Solving the thin-lens equation for s' yields:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}\Rightarrow s'=\frac{fs}{s-f} \qquad (1)$$

Use the definition of the power of the lens to find its focal length:

$$f=\frac{1}{P}=\frac{1}{10.0\,\text{m}^{-1}}=0.100\,\text{m}=10.0\,\text{cm}$$

Substitute numerical values and evaluate s':

$$s'=\frac{(10.0\,\text{cm})(25.0\,\text{cm})}{25.0\,\text{cm}-10.0\,\text{cm}}=16.7\,\text{cm}$$

Use the lateral magnification equation to relate the height of the image y' to the height y of the object and the image and object distances:

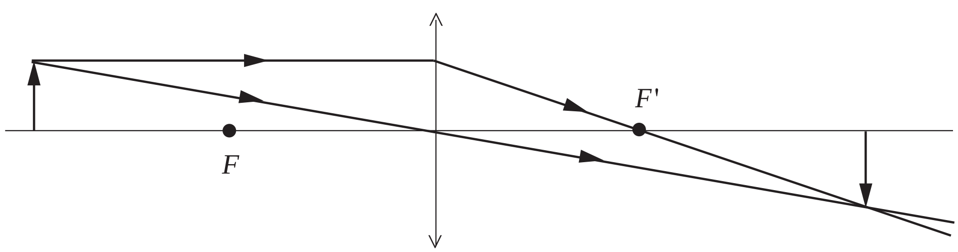
$$m=\frac{y'}{y}=-\frac{s'}{s}\Rightarrow y'=-\frac{s'}{s}y \qquad (2)$$

Substitute numerical values and evaluate y':

$$y'=-\frac{16.7\,\text{cm}}{25.0\,\text{cm}}(3.00\,\text{cm})=-2.00\,\text{cm}$$

$s'=16.7\,\text{cm}$, $y'=-2.00\,\text{cm}$. Because $s'>0$, the image is real, and because $y'/y=-0.67$ cm, the image is inverted and diminished. These results confirm those obtained graphically.

(*b*) The parallel and central rays were used to locate the image in the following diagram.



The image is real and inverted and appears to be the same size as the object.

Use the definition of the power of the lens to find its focal length:

$$f=\frac{1}{10.0\,\text{m}^{-1}}=0.100\,\text{m}=10.0\,\text{cm}$$

Substitute numerical values in equation (1) and evaluate s':

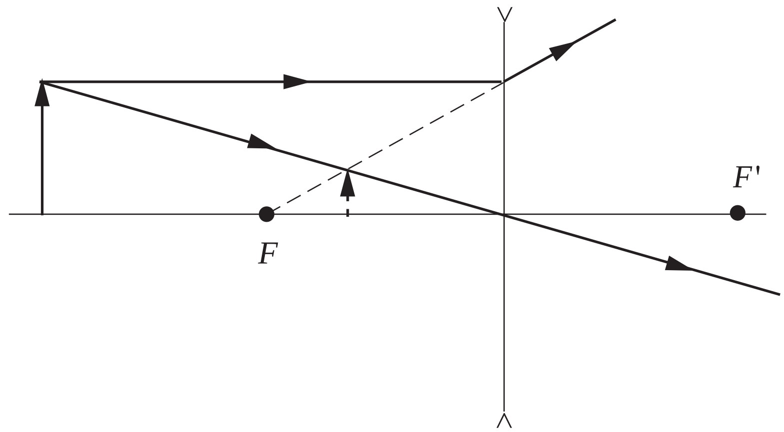
$$s'=\frac{(10.0\,\text{cm})(20.0\,\text{cm})}{20.0\,\text{cm}-10.0\,\text{cm}}=\boxed{20.0\,\text{cm}}$$

Substitute numerical values in equation (2) and evaluate y':

$$y'=-\frac{20.0\,\text{cm}}{20.0\,\text{cm}}(3.00\,\text{cm})=\boxed{-3.00\,\text{cm}}$$

Because $s' > 0$ and $y' = -3.00$ cm, the image is real, inverted, and the same size as the object. These results confirm those obtained from the ray diagram.

(*c*) The parallel and central rays were used to locate the image in the following diagram.



The image is virtual, upright, and diminished.

Use the definition of the power of the lens to find its focal length:

$$f = \frac{1}{-10.0\,\text{m}^{-1}} = -0.100\,\text{m} = -10.0\,\text{cm}$$

Substitute numerical values in equation (1) and evaluate s':

$$s' = \frac{(-10.0\,\text{cm})(20.0\,\text{cm})}{20.0\,\text{cm} - (-10.0\,\text{cm})} = \boxed{-6.67\,\text{cm}}$$

Substitute numerical values in equation (2) and evaluate y':

$$y' = -\frac{-6.67\,\text{cm}}{20.0\,\text{cm}}(3.00\,\text{cm}) = \boxed{1.00\,\text{cm}}$$

Because $s' < 0$ and $y' = 1.00$ cm, the image is virtual, erect, and about one-third the size of the object. These results are consistent with those obtained graphically.

46 •• The lens-maker's equation has three design parameters. They consist of the index of refraction of the lens and the radii of curvature for its two surfaces. Thus, there are many ways to design a lens that has a particular focal length in air. Use the lens-maker's equation to design three different thin converging lenses, each having a focal length of 27.0 cm and each made from glass that has an index of refraction of 1.60. Sketch each of your designs.

Picture the Problem We can use the lens-maker's equation to obtain a relationship between the two radii of curvature of the lenses we are to design.

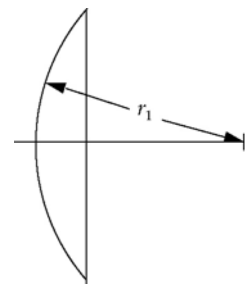
For a thin lens of focal length 27.0 cm and index of refraction of 1.60:

$$\frac{1}{27.0\,\text{cm}} = (1.60-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

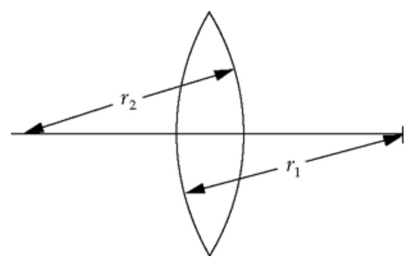
or

$$\frac{1}{r_1}-\frac{1}{r_2} = \frac{1}{16.2\,\text{cm}}$$

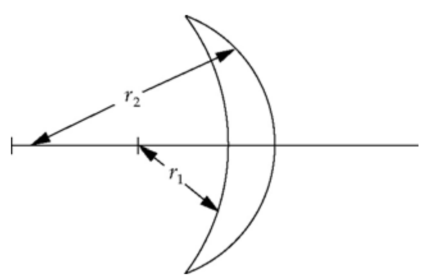
One solution is a plano-convex lens (one with a flat surface and a convex surface). Let $r_2 = \infty$. Then $r_1 = \boxed{16.2\,\text{cm}}$ and $r_2 = \boxed{\infty}$.



Another design is a double convex lens (one with both surfaces convex and radii of curvature that are equal in magnitude) obtained by letting $r_2 = -r_1$. Then $r_1 = \boxed{32.4\,\text{cm}}$ and $r_2 = \boxed{-32.4\,\text{cm}}$.



A third possibility is a double convex lens with unequal curvature, e.g., let $r_2 = -12.0$ cm. Then $r_1 = \boxed{-6.89\,\text{cm}}$ and $r_2 = \boxed{-12.0\,\text{cm}}$.



47 •• Repeat Problem 46, but for a diverging lens that has a focal length in air of the same magnitude.

Picture the Problem We can use the lens-maker's equation to obtain a relationship between the two radii of curvature of the lenses we are to design.

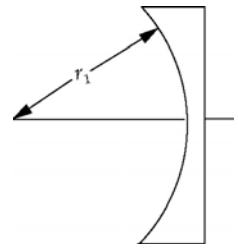
For a thin lens of focal length –27.0 cm and index of refraction of 1.60:

$$\frac{1}{-27.0\,\text{cm}} = (1.60-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

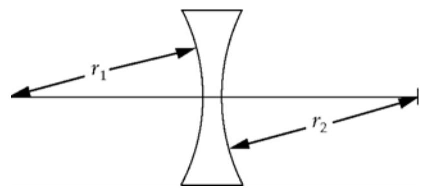
or

$$\frac{1}{r_1}-\frac{1}{r_2} = \frac{1}{-16.2\,\text{cm}}$$

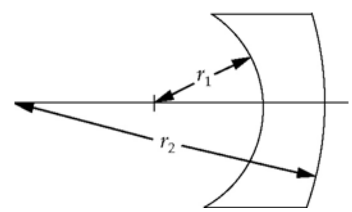
One solution is a plano-concave lens (one with a flat surface and a concave surface), Let $r_2 = \infty$. Then $r_1 = \boxed{-16.2\,\text{cm}}$ and $r_2 = \boxed{\infty}$.



Another design is a biconcave lens (one with both surfaces concave) by letting $r_2 = -r_1$. Then $r_1 = \boxed{-32.4\,\text{cm}}$ and $r_2 = \boxed{32.4\,\text{cm}}$.



A third possibility is a lens with $r_2 = -8.10$ cm.
Then $r_1 = \boxed{-5.40\,\text{cm}}$ and $r_2 = \boxed{-8.10\,\text{cm}}$.



48 •• (*a*) What is meant by a negative object distance? Describe a specific situation in which a negative object distance can occur. (*b*) Find the image distance and the magnification for a thin lens in air when the object distance is –20 cm and the lens is a converging lens that has a focal length of 20 cm. Describe the image—is it virtual or real, upright or inverted? (*c*) Repeat Part (*b*) if the object distance is, instead, –10 cm, and the lens is diverging and has a focal length (magnitude) of 30 cm.

Picture the Problem The parallel and central rays were used to locate the image in the diagram shown below. The power *P* of the lens, in diopters, can be found from $P = 1/f$ and the lateral magnification of the image from $m = -\frac{s'}{s}$.

(*a*) A negative object distance means that the object is virtual; i.e., that extensions of light rays, and not the light rays themselves, converge on the object rather than diverge from it. A virtual object can occur in a two-lens system when converging rays from the first lens are incident on the second lens before they converge to form an image.

(*b*) The thin-lens equation is:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}\Rightarrow s'=\frac{fs}{s-f}$$

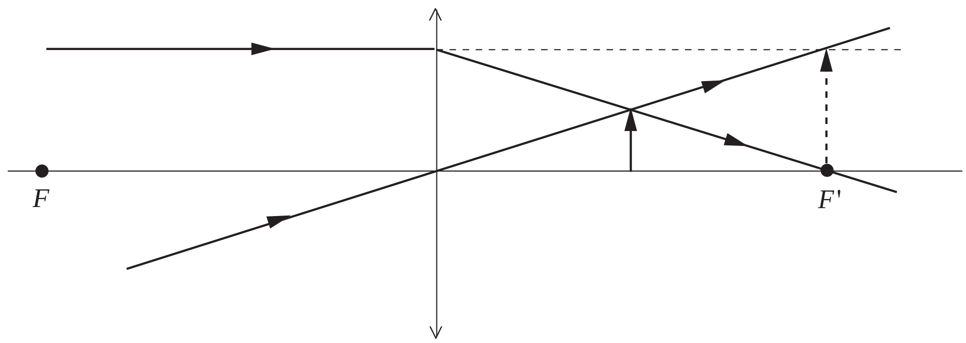
Substitute numerical values and evaluate s':

$$s' = \frac{(20\,\text{cm})(-20\,\text{cm})}{-20\,\text{cm}-(20\,\text{cm})} = 10\,\text{cm}$$

The lateral magnification is given by:

$$m = -\frac{s'}{s} = -\frac{10\text{ cm}}{-20\text{ cm}} = 0.50$$

The parallel and central rays were used to locate the image in the following ray diagram:



$s' = 10$ cm, $m = 0.50$. Because $s' > 0$, the image is real, and because $m > 0$, the image is erect. In addition, it is one-half the size of the virtual object.

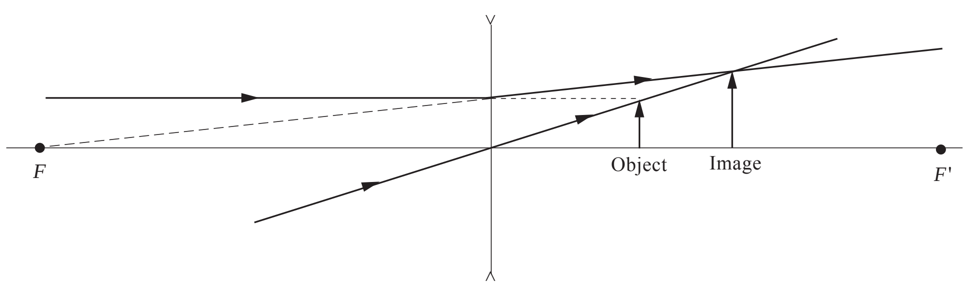
(*c*) Proceed as in Part (*b*) with $s = -10$ cm and $f = -30$ cm:

$$s' = \frac{(-30\,\text{cm})(-10\,\text{cm})}{-10\,\text{cm}-(-30\,\text{cm})} = 15\,\text{cm}$$

and

$$m = -\frac{s'}{s} = -\frac{15\,\text{cm}}{-10\,\text{cm}} = 1.5$$

The parallel and central rays were used to locate the image in the following ray diagram:

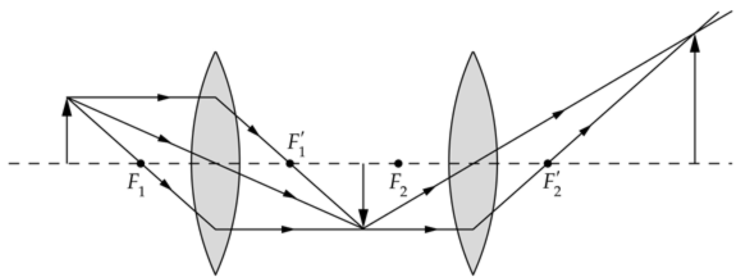


$s' > 0$ and $m = 1.5$. Because $s' > 0$, the image is real, and because $m > 0$, the image is erect. In addition, it is one and one-half times the size of the virtual object.

49 •• **[SSM]** Two converging lenses, each having a focal length equal to 10 cm, are separated by 35 cm. An object is 20 cm to the left of the first lens. (*a*) Find the position of the final image using both a ray diagram and the thin-lens equation. (*b*) Is the final image real or virtual? Is the final image upright or inverted? (*c*) What is the overall lateral magnification of the final image?

Picture the Problem We can apply the thin-lens equation to find the image formed in the first lens and then use this image as the object for the second lens.

(*a*) The parallel, central, and focal rays were used to locate the image formed by the first lens and the parallel and central rays to locate the image formed by the second lens.



Apply the thin-lens equation to express the location of the image formed by the first lens:

$$s_1' = \frac{f_1 s_1}{s_1 - f_1} \quad (1)$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(10\,\text{cm})(20\,\text{cm})}{20\,\text{cm} - 10\,\text{cm}} = 20\,\text{cm}$$

Find the lateral magnification of the first image:

$$m_1 = -\frac{s_1'}{s} = -\frac{20\,\text{cm}}{20\,\text{cm}} = -1.0$$

Because the lenses are separated by 35 cm, the object distance for the second lens is 35 cm – 20 cm = 15 cm. Equation (1) applied to the second lens is:

$$s_2' = \frac{f_2 s_2}{s_2 - f_2}$$

Substitute numerical values and evaluate s_2':

$$s_2' = \frac{(10\,\text{cm})(15\,\text{cm})}{15\,\text{cm} - 10\,\text{cm}} = 30\,\text{cm}$$

and the final image is $\boxed{85\,\text{cm}}$ to the right of the object.

Find the lateral magnification of the second image:

$$m_2 = -\frac{s_2'}{s} = -\frac{30\,\text{cm}}{15\,\text{cm}} = -2.0$$

(*b*) Because $s'_2 > 0$, the image is real and because $m = m_1 m_2 = 2.0$, the image is erect and twice the size of the object.

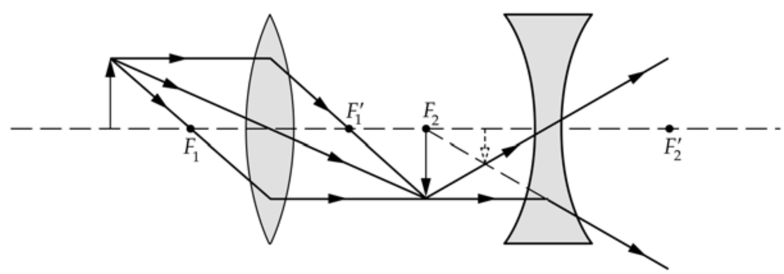
(*c*) The overall lateral magnification of the image is the product of the magnifications of each image:

$$m = m_1 m_2 = (-1.0)(-2.0) = \boxed{2.0}$$

50 •• Repeat Problem 49 but with the second lens replaced by a diverging lens that has a focal length equal to −15 cm.

Picture the Problem We can apply the thin-lens equation to find the image formed in the first lens and then use this image as the object for the second lens.

(*a*) The parallel, central, and focal rays were used to locate the image formed by the first lens and the parallel and central rays to locate the image formed by the second lens.



Apply the thin-lens equation to express the location of the image formed by the first lens:

$$s_1' = \frac{f_1 s_1}{s_1 - f_1} \quad (1)$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(10\,\text{cm})(20\,\text{cm})}{20\,\text{cm} - 10\,\text{cm}} = 20\,\text{cm}$$

Find the lateral magnification of the first image:

$$m_1 = -\frac{s'_1}{s} = -\frac{20\,\text{cm}}{20\,\text{cm}} = -1.0$$

Because the lenses are separated by 35 cm, the object distance for the second lens is 35 cm – 20 cm = 15 cm. Equation (1) applied to the second lens is:

$$s_2' = \frac{f_2 s_2}{s_2 - f_2}$$

Substitute numerical values and evaluate s_2':

$$s_2' = \frac{(-15\,\text{cm})(15\,\text{cm})}{15\,\text{cm} - (-15\,\text{cm})} = -7.5\,\text{cm}$$

and the final image is $\boxed{48\,\text{cm}}$ to the right of the object.

Find the lateral magnification of the second image:

$$m_2 = -\frac{s_2'}{s} = -\frac{-7.5\,\text{cm}}{15\,\text{cm}} = 0.50$$

(*b*) Because $s'_2 < 0$ and $m = m_1m_2 = -0.50$, the image is virtual, inverted, and half the height of the object.

(*c*) The overall lateral magnification of the image is the product of the magnifications of each image:

$$m = m_1 m_2 = (-1.0)(0.50) = \boxed{-0.50}$$

51 •• (*a*) Show that to obtain a magnification of magnitude $|m|$ using a converging thin lens of focal length *f*, the object distance must be equal to $\left(1 + |m|^{-1}\right)f$. (*b*) You want to use a digital camera which has a lens whose focal length is 50.0 mm to take a picture of a person 1.75 m tall. How far from the camera lens should you have that person stand so that the image size on the light receiving sensors of your camera is 24.0 mm?

Picture the Problem We can use the thin-lens equation and the definition of the lateral magnification to derive the result.

(*a*) Start with the thin-lens equation:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Express the magnitude of the lateral magnification of the image and solve for s': (Note that this ensures that s' is positive for this situation, as it must be because the image is real.)

$$|m| = \left|-\frac{s'}{s}\right| \Rightarrow s' = |m|s$$

Substitute for s' to obtain:

$$\frac{1}{s} + \frac{1}{|m|s} = \frac{1}{f} \Rightarrow s = \boxed{\left(1 + |m|^{-1}\right)f}$$

(*b*) The magnitude of the magnification *m* is:

$$m = \left|-\frac{y'}{y}\right| = \frac{24.0\,\text{mm}}{1.75\,\text{m}} = 0.0137$$

Substitute numerical values and evaluate s:

$$s=\frac{(0.0137+1)(50.0\,\text{mm})}{0.0137}=\boxed{3.70\,\text{m}}$$

52 •• A converging lens has a focal length of 12.0 cm. (*a*) Using a **spreadsheet** program or graphing calculator, plot the image distance as a function of the object distance, for object distances ranging from $1.10f$ to $10.0f$ where f is the focal length. (*b*) On the same graph used in Part (*a*), but using a different y axis, plot the magnification of the lens as a function of the object distance. (*c*) What type of image is produced for this range of object distances—real or virtual, upright or inverted? (*d*) Discuss the significance of any asymptotic limits your graphs have.

Picture the Problem We can plot the first graph by solving the thin-lens equation for the image distance s' and the second graph by using the definition of the magnification of the image.

(*a*) and (*b*) Solve the thin-lens equation for s' to obtain:

$$s'=\frac{fs}{s-f}$$

The magnification of the image is given by:

$$m=-\frac{s'}{s}$$

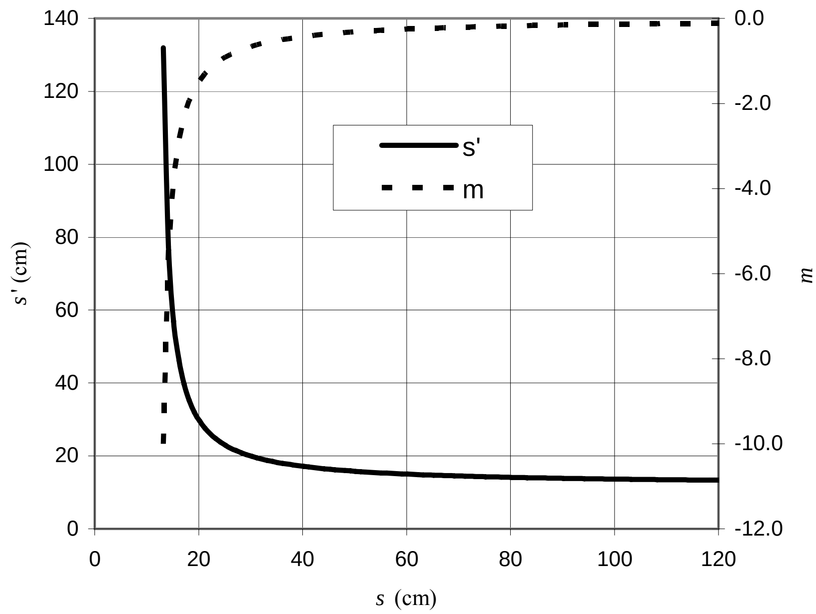
A spreadsheet program to calculate s' as a function of s is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
B1	12	f
A4	13.2	s
A5	A4 + 1	$s+\Delta s$
B4	$B$1*A4/(A4 – $B$1)	$\frac{fs}{s-f}$
C5	–B4/A4	$-\frac{s'}{s}$

	A	B	C
1	f=	12	cm
2			
3	s	s'	m
4	13.2	132.00	–10.00
5	14.2	77.45	–5.45
6	15.2	57.00	–3.75
7	16.2	46.29	–2.86
8	17.2	39.69	–2.31
9	18.2	35.23	–1.94

108	117.2	13.37	−0.11
109	118.2	13.36	−0.11
110	119.2	13.34	−0.11
111	120.2	13.33	−0.11

A graph of s' and m as functions of s follows.



(c) The images are real and inverted for this range of object distances.

(d) The equations for the horizontal and vertical asymptotes of the graph of s' versus s are $s' = f$ and $s = f$, respectively. These indicate that as the object moves away from the lens image distance the image moves toward the first focal point, and that as the object moves toward the second focal point the image distance becomes large without limit. The equations for the horizontal and vertical asymptotes of the graph of m versus s are $m = 0$ and $s = f$, respectively. These indicate that as the object moves away from the lens, the image size of the image approaches zero, and that as the object moves toward the second focal point the image becomes large without limit.

53 •• A converging lens has a focal length of 12.0 cm. (a) Using a **spreadsheet** program or graphing calculator, plot the image distance as a function of the object distance, for object distances ranging from $0.010f$ to $s = 0.90f$, where f is the focal length. (b) On the same graph used in Part (a), but using a different y axis, plot the magnification of the lens as a function of the object distance.
(c) What type of image is produced for this range of object distances—real or virtual, upright or inverted? (d) Discuss the significance of any asymptotic limits your graphs have.

Picture the Problem We can plot the first graph by solving the thin-lens equation for the image distance s' and the second graph by using the definition of the magnification of the image.

(*a*) and (*b*) Solve the thin-lens equation for s' to obtain:

$$s' = \frac{fs}{s-f}$$

The magnification of the image is given by:

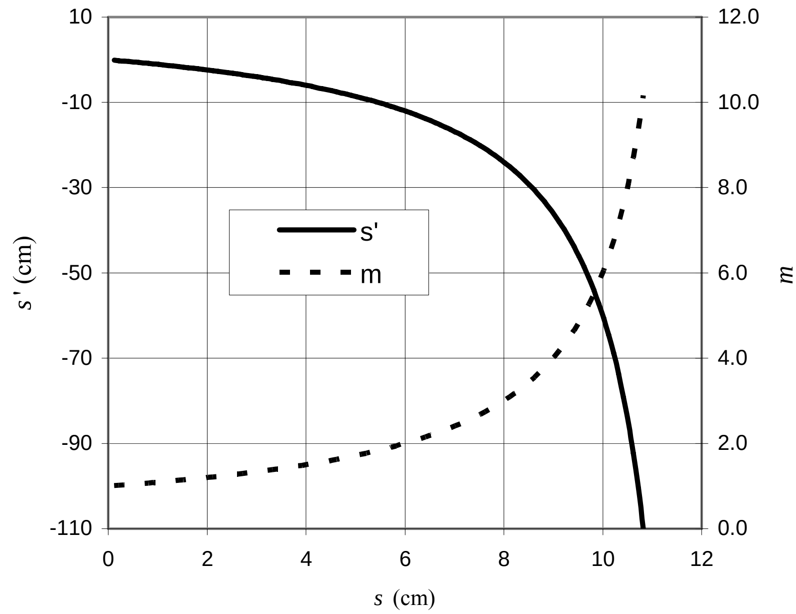
$$m = -\frac{s'}{s}$$

A spreadsheet program to calculate s' as a function of s is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
B1	12	f
A4	0.12	s
A5	A4 + 0.1	$s + \Delta s$
B4	$B$1*A4/(A4 – $B$1)	$\frac{fs}{s-f}$
C5	–B4/A4	$-\frac{s'}{s}$

	A	B	C
1	*f*=	12	cm
2			
3	*s*	*s'*	*m*
4	0.12	–0.12	1.01
5	0.22	–0.22	1.02
6	0.32	–0.33	1.03
7	0.42	–0.44	1.04
8	0.52	–0.54	1.05
9	0.62	–0.65	1.05
108	10.52	–85.30	8.11
109	10.62	–92.35	8.70
110	10.72	–100.50	9.37
111	10.82	–110.03	10.17

A graph of s' and m as functions of s follows.



(c) The images are virtual and erect for this range of object distances.

(d) The equation for the vertical asymptote of the graph of s' versus s is $f = s$. This indicates that, as the object moves toward the second focal point, the magnitude of the image distance becomes large without limit. The equation for the vertical asymptote of the graph of m versus s is $s = f$. This indicates that, as the object moves toward the second focal point, the image becomes large without limit. In addition, as s approaches zero, s' approaches negative infinity and m approaches 1, so as the object moves toward the lens the image becomes the same size as the object and the magnitude of the image distance increases without limit

54 •• An object is 15.0 cm in front of a converging lens that has a focal length equal to 15.0 cm. A second converging lens that also has a focal length equal to15.0 cm is located 20.0 cm in back of the first. (a) Find the location of the final image and describe its properties (for example, real and inverted) and (b) draw a ray diagram to corroborate your answers to Part (a).

Picture the Problem We can apply the thin-lens equation to find the image formed in the first lens and then use this image as the object for the second lens.

(a) Apply the thin-lens equation to express the location of the image formed by the first lens:

$$s_1' = \frac{f_1 s_1}{s_1 - f_1} \qquad (1)$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(15.0\,\text{cm})(15.0\,\text{cm})}{15.0\,\text{cm} - 15.0\,\text{cm}} = \infty$$

i.e., no image is formed by the first lens.

With $s_1' = \infty$, the thin-lens equation applied to the second lens becomes:

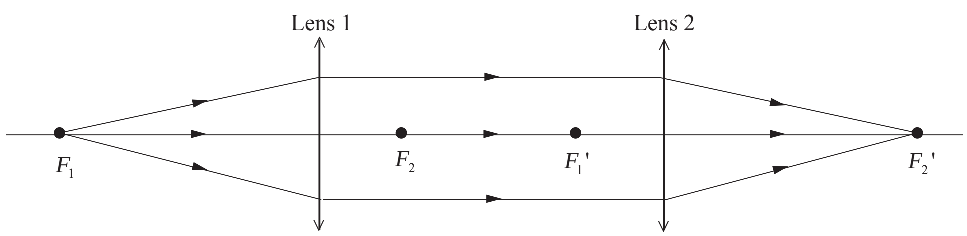
$$\frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow s_2' = f_2 = \boxed{15.0\,\text{cm}}$$

The overall magnification of the two-lens system is the magnification of the second lens:

$$m = m_2 = -\frac{s_2'}{s_1} = -\frac{15.0\text{ cm}}{15.0\text{ cm}} = \boxed{-1.00}$$

The final image is 50 cm from the object, real, inverted, and the same size as the object.

(*b*) A corroborating ray diagram is shown below:



55 •• **[SSM]** An object is 15.0 cm in front of a converging lens that has a focal length equal to 15.0 cm. A diverging lens that has a focal length whose magnitude is equal to 15.0 cm is located 20.0 cm in back of the first. (*a*) Find the location of the final image and describe its properties (for example, real and inverted) and (*b*) draw a ray diagram to corroborate your answers to Part (*a*).

Picture the Problem We can apply the thin-lens equation to find the image formed in the first lens and then use this image as the object for the second lens.

Apply the thin-lens equation to express the location of the image formed by the first lens:

$$s_1' = \frac{f_1 s_1}{s_1 - f_1} \quad (1)$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(15.0\,\text{cm})(15.0\,\text{cm})}{15.0\,\text{cm} - 15.0\,\text{cm}} = \infty$$

With $s'_1 = \infty$, the thin-lens equation applied to the second lens becomes:

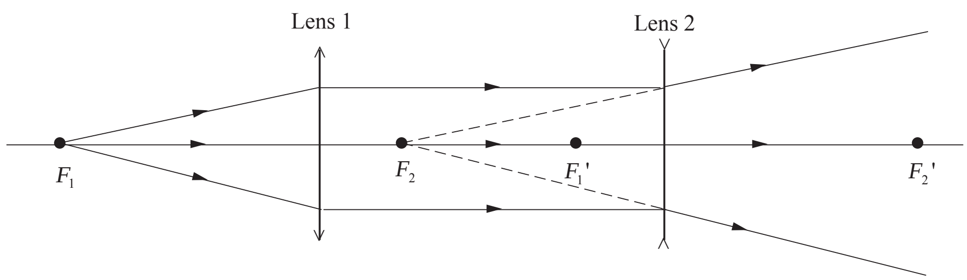
$$\frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow s_2' = f_2 = \boxed{-15.0\,\text{cm}}$$

The overall magnification of the two-lens system is the magnification of the second lens:

$$m = m_2 = -\frac{s_2'}{s_1} = -\frac{-15.0\,\text{cm}}{15.0\,\text{cm}} = \boxed{1.00}$$

The final image is 20 cm from the object, virtual, erect, and the same size as the object.

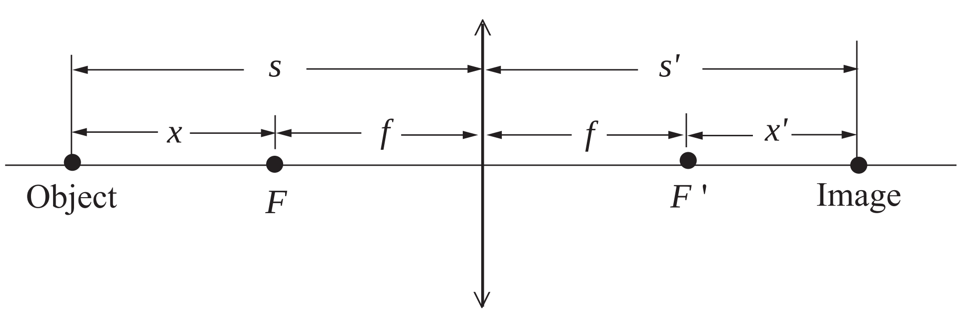
A corroborating ray diagram follows:



56 ••• In a convenient form of the thin-lens equation used by Newton, the object and image distances x and x' are measured from the focal points F and F', and not from the center of the lens. (a) Indicate x and x' on a sketch of a lens and show that if $x = s - f$ and $x' = s' - f$, the thin-lens equation (Equation 32-12) can be rewritten as $xx' = f^2$. (b) Show that the lateral magnification is given by $m = -x'/f = -f/x$.

Picture the Problem We can substitute $x = s - f$ and $x' = s' - f$ in the thin-lens equation and the equation for the lateral magnification of an image to obtain Newton's equations.

(a) The variables x, f, s ,and s' are shown in the following sketch:



Express the thin-lens equation: $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$

If $x = s - f$ and $x' = s' - f$, then: $\frac{1}{x+f}+\frac{1}{x'+f}=\frac{1}{f}$

Expand this expression to obtain:

$$f\left(x'+x+2f\right)=\left(x+f\right)\left(x'+f\right)$$
$$=xx'+xf+x'f+f^2$$

or, simplifying, $\boxed{xx' = f^2}$ (1)

(*b*) The lateral magnification is:

$$m=-\frac{s'}{s}$$

or, because $x = s - f$ and $x' = s' - f$,

$$m=-\frac{x'+f}{x+f}$$

Solve equation (1) for x: $x=\frac{f^2}{x'}$

Substitute for x and simplify to obtain:

$$m=-\frac{x'+f}{\frac{f^2}{x'}+f}=-\frac{x'+f}{\frac{f\left(f+x'\right)}{x'}}=\boxed{-\frac{x'}{f}}$$

The lateral magnification is also given by:

$$m=-\frac{x'+f}{x+f}$$

From equation (1) we have: $x'=\frac{f^2}{x}$

Substitute to obtain:

$$m=-\frac{\frac{f^2}{x}+f}{x+f}=-\frac{f\left(\frac{f}{x}+1\right)}{x\left(1+\frac{f}{x}\right)}=\boxed{-\frac{f}{x}}$$

57 ••• [SSM] In *Bessel's method* for finding the focal length f of a lens, an object and a screen are separated by distance L, where $L > 4f$. It is then possible to place the lens at either of two locations, both between the object and the screen, so that there is an image of the object on the screen, in one case magnified and in the other case reduced. Show that if the distance between these two lens locations is D, then the focal length is given by $f = \frac{1}{4}\left(L^2 - D^2\right)/L$. *Hint: Refer to Figure 32-60.*

Picture the Problem The ray diagram shows the two lens positions and the corresponding image and object distances (denoted by the numerals 1 and 2). We can use the thin-lens equation to relate the two sets of image and object distances to the focal length of the lens and then use the hint to express the relationships between these distances and the distances D and L to eliminate s_1, s_1', s_2, and s_2' and obtain an expression relating f, D, and L.

Relate the image and object distances for the two lens positions to the focal length of the lens:

$$\frac{1}{s_1}+\frac{1}{s_1'}=\frac{1}{f} \text{ and } \frac{1}{s_2}+\frac{1}{s_2'}=\frac{1}{f}$$

Solve for f to obtain:

$$f=\frac{s_1 s_1'}{s_1+s_1'}=\frac{s_2 s_2'}{s_2+s_2'} \quad (1)$$

The distances D and L can be expressed in terms of the image and object distances:

$$L = s_1 + s_1' = s_2 + s_2'$$
and
$$D = s_2 - s_1 = s_1' - s_2'$$

Substitute for the sums of the image and object distances in equation (1) to obtain:

$$f=\frac{s_1 s_1'}{L}=\frac{s_2 s_2'}{L}$$

From the hint:

$$s_1 = s_2' \text{ and } s_1' = s_2$$

Hence $L = s_1 + s_2$ and:

$$L - D = 2s_1 \text{ and } L + D = 2s_2$$

Take the product of $L - D$ and $L + D$ to obtain:

$$\begin{aligned}(L-D)(L+D) &= L^2 - D^2\\ &= 4s_1 s_2 = 4s_1 s_1'\end{aligned}$$

From the thin-lens equation:

$$4s_1 s_2 = rs_1 s_1' = 4fL$$

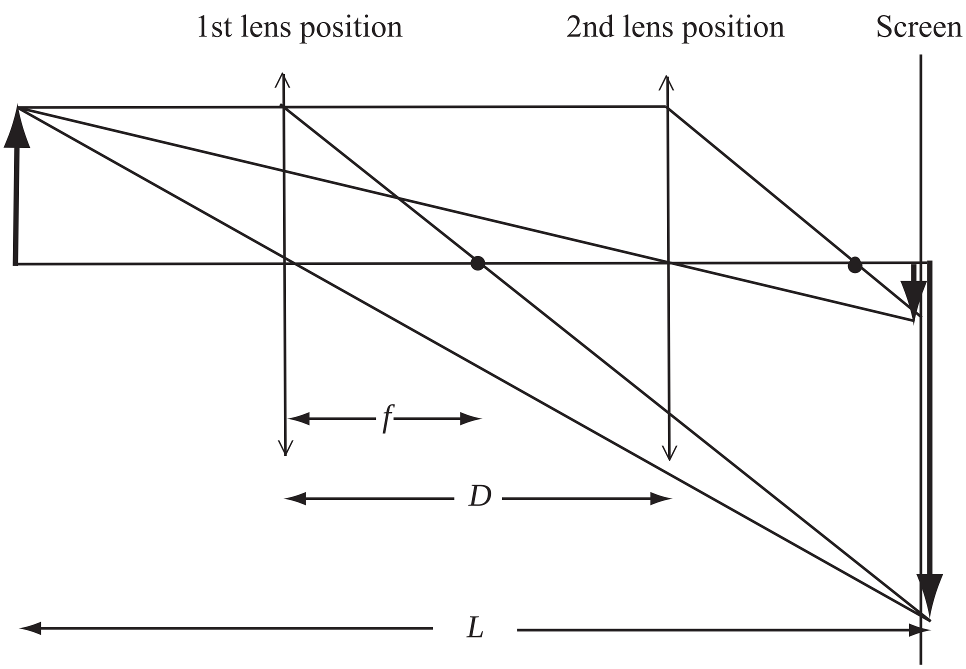
Substitute to obtain:

$$4fL = L^2 - D^2 \Rightarrow f = \boxed{\frac{L^2 - D^2}{4L}}$$

58 •• You are working for an optician during the summer. The optician needs to measure an unknown focal length and you suggest using *Bessel's method* (see Problem 56) which you used during a physics lab. You set the object-to-image distance at 1.70 m. The lens position is adjusted to get a sharp image on the screen. A second sharp image is found when the lens is moved a distance of 72 cm from its first location. (*a*) Sketch the ray diagram for the two locations. (*b*) Find the focal length of the lens using Bessel's method. (*c*) What are the two locations of the lens with respect to the object? (*d*) What are the magnifications of the images when the lens is in the two locations?

Picture the Problem We can use results obtained in Problem 57 to find the focal length of the lens and the two locations of the lens with respect to the object. Let 1 refer to the first lens position and 2 to the second lens position.

(*a*) The distances defined in Problem 57 have been identified in the following ray diagram. Note that, for clarity, the two images have been slightly offset from the plane of the screen.



(*b*) From Problem 57 we have:

$$f = \frac{L^2 - D^2}{4L}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{(170\,\text{cm})^2 - (72\,\text{cm})^2}{4(170\,\text{cm})} = \boxed{35\,\text{cm}}$$

(*c*) Solve the thin-lens equation for the image distance to obtain:

$$s' = \frac{fs}{f - s} \qquad (1)$$

In Problem 57 it was established that:

$$L - D = 2s_1 \text{ and } L + D = 2s_2$$

Solve for s_1 and s_2:

$$s_1 = \frac{L-D}{2} \text{ and } s_2 = \frac{L+D}{2}$$

Substitute numerical values and evaluate s_1 and s_2:

$$s_1 = \frac{170\,\text{cm} - 72\,\text{cm}}{2} = \boxed{49\,\text{cm}}$$

and

$$s_2 = \frac{170\,\text{cm} + 72\,\text{cm}}{2} = \boxed{121\,\text{cm}}$$

(*d*) The magnification when the lens was in its first position is given by:

$$m_1 = -\frac{s'_1}{s_1} = -\frac{D - s_1}{s_1}$$

Substituting numerical values yields:

$$m_1 = -\frac{170\text{ cm} - 49\text{ cm}}{49\text{ cm}} = \boxed{-2.5}$$

The magnification when the lens was in its second position is given by:

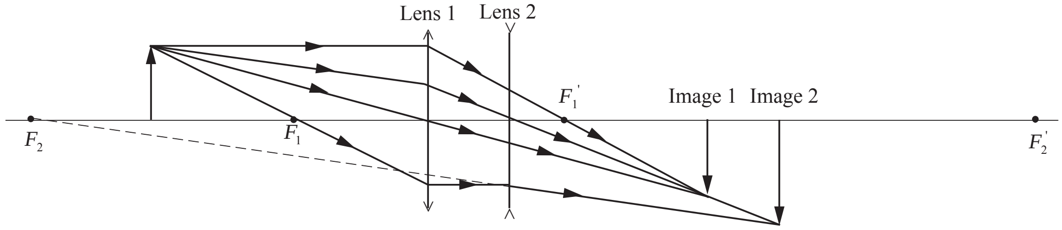
$$m_2 = -\frac{s'_2}{s_2} = -\frac{L - s_2}{s_2}$$

Substitute numerical values and evaluate m_2:

$$m_2 = -\frac{170\text{ cm} - 121\text{ cm}}{121\text{ cm}} = \boxed{-0.41}$$

59 ••• An object is 17.5 cm to the left of a lens that has a focal length of +8.50 cm. A second lens, which has a focal length of −30.0 cm, is 5.00 cm to the right of the first lens. (*a*) Find the distance between the object and the final image formed by the second lens. (*b*) What is the overall magnification? (*c*) Is the final image real or virtual? Is the final image upright or inverted?

Picture the Problem The ray diagram shows four rays from the head of the object that locate images I_1 and I_2. We can use the thin-lens equation to find the location of the image formed in the positive lens and then, knowing the separation of the two lenses, determine the object distance for the second lens and apply the thin lens a second time to find the location of the final image.



(*a*) Express the object-to-image distance *d*:

$$d = s_1 + 5\,\text{cm} + s_2' \qquad (1)$$

Apply the thin-lens equation to the positive lens:

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \Rightarrow s_1' = \frac{f_1 s_1}{s_1 - f_1}$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(8.50\,\text{cm})(17.5\,\text{cm})}{17.5\,\text{cm} - 8.50\,\text{cm}} = 16.53\,\text{cm}$$

Find the object distance for the negative lens:

$$s_2 = 5.00\,\text{cm} - s_1' = 5.00\,\text{cm} - 16.53\,\text{cm}$$
$$= -11.53\,\text{cm}$$

The image distance s_2' is given by:

$$s_2' = \frac{f_2 s_2}{s_2 - f_2}$$

Substitute numerical values and evaluate s_2':

$$s_2' = \frac{(-30.0\,\text{cm})(-11.53\,\text{cm})}{-11.53\,\text{cm} - (-30.0\,\text{cm})} = 18.73\,\text{cm}$$

Substitute numerical values in equation (1) and evaluate *d*:

$$d = 17.5\,\text{cm} + 5.00\,\text{cm} + 18.73\,\text{cm}$$
$$= \boxed{41\,\text{cm}}$$

and the final image is 18.7 cm to the right of the second lens.

(*b*) The overall lateral magnification is given by:

$$m = m_1 m_2$$

Express m_1 and m_2:

$$m_1 = -\frac{s_1'}{s_1} \text{ and } m_2 = -\frac{s_2'}{s_2}$$

Substitute for m_1 and m_2 to obtain:

$$m = \left(-\frac{s_1'}{s_1}\right)\left(-\frac{s_2'}{s_2}\right) = \frac{s_1' s_2'}{s_1 s_2}$$

Substitute numerical values and evaluate *m*:

$$m = \frac{(16.53\,\text{cm})(18.73\,\text{cm})}{(17.5\,\text{cm})(-11.53\,\text{cm})} = \boxed{-1.53}$$

(*c*) Because $m < 0$, the image is inverted. Because $s_2' > 0$, the image is real.

Aberrations

60 • Chromatic aberration is a common defect of (*a*) concave and convex lenses, (*b*) concave lenses only, (*c*) concave and convex mirrors, (*d*) all lenses and mirrors.

Determine the Concept Chromatic aberrations are a consequence of the differential refraction of light of differing wavelengths by lenses. $\boxed{(a)}$ is correct.

61 • Discuss some of the reasons why most telescopes that are used by astronomers are reflecting rather than refracting telescopes.

Determine the Concept Reasons for the preference for reflectors include: 1) no chromatic aberrations, 2) cheaper to shape one side of a piece of glass than to shape both sides, 3) reflectors can be more easily supported from rear instead of edges, preventing sagging and focal length changes, and 4) support from rear makes larger sizes easier to handle.

62 • A symmetric double-convex lens has radii of curvature equal to 10.0 cm. It is made from glass that has an index of refraction equal to 1.530 for blue light and to 1.470 for red light. Find the focal length of this lens for (*a*) red light and (*b*) blue light.

Picture the Problem We can use the lens-maker's equation to find the focal length the this lens for the two colors of light.

The lens-maker's equation relates the radii of curvature and the index of refraction to the focal length of the lens:

$$\frac{1}{f}=\left(\frac{n}{n_{\text{air}}}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

(*a*) For red light:

$$\frac{1}{f_{\text{red}}}=\left(\frac{1.470}{1.00}-1\right)\left(\frac{1}{10.0\,\text{cm}}-\frac{1}{-10.0\,\text{cm}}\right)$$

and $f_{\text{red}}=\boxed{10.6\,\text{cm}}$

(*b*) For blue light:

$$\frac{1}{f_{\text{blue}}}=\left(\frac{1.530}{1.00}-1\right)\left(\frac{1}{10.0\,\text{cm}}-\frac{1}{-10.0\,\text{cm}}\right)$$

and $f_{\text{blue}}=\boxed{9.43\,\text{cm}}$

The Eye

63 • Find the change in the focal length of the eye when an object originally at 3.0 m is brought to 30 cm from the eye.

Picture the Problem We can apply the thin-lens equation for the two values of s to find Δf.

Express the change Δf in the focal length:

$$\Delta f = f_{s=0.30\,\text{m}} - f_{s=3.0\,\text{m}}$$

Solve the thin-lens equation for s:

$$f = \frac{ss'}{s'+s}$$

Substitute for f to obtain:

$$\Delta f = \frac{s_{0.30\,\text{m}}s'_{0.30\,\text{m}}}{s'_{0.30\,\text{m}} + s_{0.30\,\text{m}}} - \frac{s_{3.0\,\text{m}}s'_{3.0\,\text{m}}}{s'_{3.0\,\text{m}} + s_{3.0\,\text{m}}}$$

Because $s'_{3.0\,\text{m}} = s'_{0.30\,\text{m}}$:

$$\Delta f = s'_{3.0\,\text{m}}\left[\frac{s_{0.30\,\text{m}}}{s'_{0.30\,\text{m}} + s_{0.30\,\text{m}}} - \frac{s_{3.0\,\text{m}}}{s'_{3.0\,\text{m}} + s_{3.0\,\text{m}}}\right]$$

Substitute numerical values and evaluate Δf:

$$\Delta f = (2.5\,\text{cm})\left[\frac{30\,\text{cm}}{2.5\,\text{cm}+30\,\text{cm}} - \frac{300\,\text{cm}}{2.5\,\text{cm}+300\,\text{cm}}\right] = -0.172\,\text{cm} = \boxed{-1.7\,\text{mm}}$$

64 • A farsighted person requires lenses that have powers equal to 1.75 D to read comfortably from a book that is 25.0 cm from his eyes. What is that person's near point without the lenses?

Picture the Problem We can use the thin-lens equation and the definition of the power of a lens to express the near point distance as a function of P and s.

From the thin-lens equation we have:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}=P \Rightarrow s'=\frac{s}{Ps-1}$$

Substitute numerical values and evaluate s':

$$s' = \frac{25.0\,\text{cm}}{(1.75\,\text{m}^{-1})(0.250\,\text{m})-1} = -44.4\,\text{cm}$$

The person's near point without lenses is $\boxed{44.4\text{ cm}}$.

65 • [SSM] If two point objects close together are to be seen as two distinct objects, the images must fall on the retina on two different cones that are not adjacent. That is, there must be an unactivated cone between them. The separation of the cones is about 1.00 μm. Model the eye as a uniform 2.50-cm-diameter sphere that has a refractive index of 1.34. (*a*) What is the smallest angle the two points can subtend? (See Figure 32-61.) (*b*) How close together can two points be if they are 20.0 m from the eye?

Picture the Problem We can use the relationship between a distance measured along the arc of a circle and the angle subtended at its center to approximate the smallest angle the two points can subtend and the separation of the two points 20.0 m from the eye.

(*a*) Relate θ_{min} to the diameter of the eye and the distance between the activated cones:

$$d_{\text{eye}}\theta_{\text{min}} \approx 2.00\,\mu\text{m} \Rightarrow \theta_{\text{min}} = \frac{2.00\,\mu\text{m}}{d_{\text{eye}}}$$

Substitute numerical values and evaluate θ_{min}:

$$\theta_{\text{min}} = \frac{2.00\,\mu\text{m}}{2.50\,\text{cm}} = \boxed{80.0\,\mu\text{rad}}$$

(*b*) Let D represent the separation of the points R = 20.0 m from the eye to obtain:

$$D = R\theta_{\text{min}} = (20.0\,\text{m})(80.0\,\mu\text{rad}) = \boxed{1.60\,\text{mm}}$$

66 • Suppose the eye were designed like a camera that has a lens of fixed focal length equal to 2.50 cm that could move toward or away from the retina. Approximately how far would the lens have to move to focus the image of an object 25.0 cm from the eye onto the retina? *Hint: Find the distance from the retina to the image behind it for an object at 25.0 cm.*

Picture the Problem We can use the thin-lens equation to find the distance from the lens to the image and then take their difference to find the distance the lens would have to be moved.

Express the distance d that the lens would have to move:

$$d = s' - f$$

Solve the thin-lens equation for s':

$$s' = \frac{fs}{s-f}$$

Substitute for s' to obtain:

$$d=\frac{fs}{s-f}-f$$

Substitute numerical values and evaluate d:

$$d=\frac{(2.50\,\text{cm})(25.0\,\text{cm})}{25.0\,\text{cm}-2.50\,\text{cm}}-2.50\,\text{cm}$$
$$=0.28\,\text{cm}$$

That is, the lens would have to move $\boxed{0.28\,\text{cm}}$ toward the object.

67 •• **[SSM]** The Model Eye I: A simple model for the eye is a lens that has a variable power P located a fixed distance d in front of a screen, with the space between the lens and the screen filled by air. This "eye" can focus for all values of object distance s such that $x_{np} \le s \le x_{fp}$ where the subscripts on the variables refer to "near point" and "far point" respectively. This "eye" is said to be normal if it can focus on very distant objects. (*a*) Show that for a normal "eye," of this type, the required minimum value of P is given by $P_{min}=1/d$. (*b*) Show that the maximum value of P is given by $P_{max}=1/x_{np}+1/d$. (*c*) The difference between the maximum and minimum powers, symbolized by A, is defined as $A=P_{max}-P_{min}$ and is called the *accommodation*. Find the minimum power and accommodation for this model eye that has a screen distance of 2.50 cm, a far point distance of infinity and a near point distance of 25.0 cm.

Picture the Problem The thin-lens equation relates the image and object distances to the power of a lens.

(*a*) Use the thin-lens equation to relate the image and object distances to the power of the lens:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}=P \qquad (1)$$

Because $s'=d$ and, for a distance object, $s=\infty$:

$$P_{min}=\frac{1}{s'}=\boxed{\frac{1}{d}}$$

(*b*) If x_{np} is the closest distance an object could be and still remain in clear focus on the screen, equation (1) becomes:

$$P_{max}=\boxed{\frac{1}{x_{np}}+\frac{1}{d}}$$

(*c*) Use our result in (*a*) to obtain:

$$P_{min}=\frac{1}{2.50\,\text{cm}}=\boxed{40.0\,\text{D}}$$

Use the results of (a) and (b) to express the accommodation of the model eye:

$$A = P_{\max} - P_{\min} = \frac{1}{x_{\text{np}}} + \frac{1}{d} - \frac{1}{d} = \frac{1}{x_{\text{np}}}$$

Substitute numerical values and evaluate A:

$$A = \frac{1}{25.0\,\text{cm}} = \boxed{4.00\,\text{D}}$$

68 •• The Model Eye II: In an eye that exhibits nearsightedness, the eye cannot focus on distant objects. (a) Show that for a nearsighted model eye capable of focusing out to a maximum distance x_{fp}, the minimum value of the power P is greater than that of a normal eye (that has a far point at infinity) and is given by $P_{\min} = 1/x_{\text{fp}} + 1/d$. (b) To correct for nearsightedness, a contact lens may be placed directly in front of the model-eye's lens. What power contact lens would be needed to correct the vision of a nearsighted model eye that has a far point distance of 50.0 cm?

Picture the Problem The thin-lens equation relates the image and object distances to the power of a lens.

(a) Use the thin-lens equation to relate the image and object distances to the power of the lens:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} = P$$

Because $s' = d$ and $s = x_{\text{fp}}$:

$$P_{\min} = \boxed{\frac{1}{x_{\text{fp}}} + \frac{1}{d}}$$

(b) To correct for the nearsightedness of this eye, we need a lens that will form an image 25.0 cm in front of the eye of an object at the eye's far point:

$$P_{\min} = \frac{1}{50.0\,\text{cm}} + \frac{1}{-25.0\,\text{cm}} = \boxed{-2.00\,\text{D}}$$

69 •• The Model Eye III: In an eye that exhibits farsightedness, the eye may be able to focus on distant objects but cannot focus on close objects. (a) Show that for a farsighted model eye capable of focusing only as close as a distance x'_{np}, the maximum value of the power P is given by $P_{\max} = 1/x'_{\text{np}} + 1/d$. (b) Show that, compared to a model eye capable of focusing as close as a distance x_{np} (where $x_{\text{np}} < x'_{\text{np}}$), the maximum power of the farsighted lens is too small by $1/x_{\text{np}} - 1/x'_{\text{np}}$. (c) What power contact lens would be needed to correct the vision of a farsighted model eye, with x'_{np} = 150 cm, so that the eye may focus on objects as close as 15 cm?

Picture the Problem The thin-lens equation relates the image and object distances to the power of a lens.

(*a*) Use the thin-lens equation to relate the image and object distances to the power of the lens:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}=P$$

Because $s' = d$ and $s = x'_{np}$:

$$P'_{max}=\boxed{\frac{1}{x'_{np}}+\frac{1}{d}} \qquad (1)$$

(*b*) For a normal eye:

$$P_{max}=\frac{1}{x_{np}}+\frac{1}{d} \qquad (2)$$

The amount by which the power of the lens is too small is the difference between equations (2) and (1):

$$P_{max}-P'_{max}=\frac{1}{x_{np}}+\frac{1}{d}-\left(\frac{1}{x'_{np}}+\frac{1}{d}\right)$$
$$=\boxed{\frac{1}{x_{np}}-\frac{1}{x'_{np}}}$$

(*c*) For x_{np} = 15 cm and x'_{np} = 150 cm:

$$P_{max}-P'_{max}=\frac{1}{15\,\text{cm}}-\frac{1}{150\,\text{cm}}$$
$$=\boxed{6.0\,\text{D}}$$

70 •• A person who has a near point of 80 cm needs to read from a computer screen that is only 45 cm from her eye. (*a*) Find the focal length of the lenses in reading glasses that will produce an image of the screen at a distance of 80 cm from her eye. (*b*) What is the power of the lenses?

Picture the Problem We can use the thin-lens equation to find *f* and the definition of the power of a lens to find *P*.

(*a*) Solve the thin-lens equation for *f*:

$$f=\frac{ss'}{s'+s}$$

Noting that $s' < 0$, substitute numerical values and evaluate *f*:

$$f=\frac{(45\,\text{cm})(-80\,\text{cm})}{-80\,\text{cm}+45\,\text{cm}}=103\,\text{cm}$$
$$=\boxed{1.0\,\text{m}}$$

(*b*) Use the definition of the power of a lens to obtain:

$$P=\frac{1}{f}=\frac{1}{1.03\,\text{m}}=\boxed{0.97\,\text{D}}$$

71 •• A nearsighted person cannot focus clearly on objects that are more distant than 2.25 m from her eye. What power lenses are required for her to see distant objects clearly?

Picture the Problem We can use the thin-lens equation to find *f* and the definition of the power of a lens to find *P*.

Express the required power of the lens:

$$P = \frac{1}{f}$$

The thin-lens equation is:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

For $s = \infty$:

$$\frac{1}{s'} = \frac{1}{f} \Rightarrow f = s'$$

Substitute for *f* to obtain:

$$P = \frac{1}{s'}$$

Substitute for s' and evaluate *P*:

$$P = \frac{1}{2.25\,\text{m}} = \boxed{0.444\,\text{D}}$$

72 •• Because the index of refraction of the lens of the eye is not very different from that of the surrounding material, most of the refraction takes place at the cornea, where the index changes abruptly from 1.00 (air) to approximately 1.38. (*a*) Modeling the cornea, aqueous humor, lens and vitreous humor is a transparent homogeneous solid sphere that has an index of refraction of 1.38, calculate the sphere's radius if it focuses parallel light on the retina a distance 2.50 cm away. (*b*) Do you expect your result to be larger or smaller than the actual radius of the cornea? Explain your answer.

Picture the Problem We can use the equation for refraction at a single surface (Equation 32-6) with $s = \infty$ to find the radius of the cornea modeled as a homogeneous sphere with an index of refraction of 1.38.

(*a*) Use Equation 32-6 to relate the radius of the cornea to its index of refraction and that of air:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$$

Because $n_2 = n$, $n_1 = 1$, and $s = \infty$:

$$\frac{n}{s'} = \frac{n-1}{r} \Rightarrow r = \frac{s'(n-1)}{n} = \left(1 - \frac{1}{n}\right)s'$$

Substitute numerical values and evaluate r:

$$r = \left(1 - \frac{1}{1.38}\right)(2.50\,\text{cm}) = \boxed{0.688\,\text{cm}}$$

(*b*) The calculated value for the distance to the retina is too large. The eye is not a homogeneous sphere, and in a real eye additional refraction occurs at the lens. By modeling the eye as a homogeneous solid sphere we are ignoring the refraction that takes place at the lens.

73 •• The near point of a certain person's eyes is 80 cm. Reading glasses are prescribed so that he can read a book at 25 cm from his eye. The glasses are 2.0 cm from the eye. What diopter lenses should be used in the glasses?

Picture the Problem We can use the definition of the power of a lens and the thin-lens equation to find the power of the lens that should be used in the glasses.

Express the power of the lens that should be used in the glasses:

$$P = P_{\text{eye}} + P_{\text{lens}} = \frac{1}{f_{\text{eye}}} + \frac{1}{f_{\text{glasses}}} \quad (1)$$

Because the glasses are 2.0 cm from the eye:

$$s' = -80\,\text{cm} + 2.0\,\text{cm} = -78\,\text{cm}$$

and

$$s = 25\,\text{cm} - 2.0\,\text{cm} = 23\,\text{cm}$$

Apply the thin-lens equation to the eye with $s' = \infty$:

$$\frac{1}{s} = \frac{1}{f_{\text{eye}}} \Rightarrow f_{\text{eye}} = s$$

Apply the thin-lens equation to the glasses with $s = \infty$:

$$\frac{1}{s'} = \frac{1}{f_{\text{glasses}}} \Rightarrow f_{\text{glasses}} = s'$$

Substitute for f_{eye} and f_{glasses} in equation (1) to obtain:

$$P = \frac{1}{s} + \frac{1}{s'}$$

Substitute numerical values and evaluate P:

$$P = \frac{1}{0.23\,\text{m}} + \frac{1}{-0.78\,\text{m}} = \boxed{3.1\,\text{D}}$$

74 ••• At age 45, a person is fitted for reading glasses that have a power equal to 2.10 D in order to read at 25 cm. By the time she reaches the age of 55, she discovers herself holding her newspaper at a distance of 40 cm in order to see it clearly with her glasses on. (*a*) Where was her near point at age 45? (*b*) Where is her near point at age 55? (*c*) What power is now required for the lenses of her reading glasses so that she can again read at 25 cm? Assume the glasses are placed 2.2 cm from her eyes.

Picture the Problem We can use the thin-lens equation and the distance from her eyes to her glasses to derive an expression for the location of her near point.

(a) Express her near point, x_{np}, at age 45 in terms of the location of her glasses:

$$x_{np} = |s'| + 2.2\,\text{cm} \qquad (1)$$

Because the glasses are 2.2 cm from her eye:

$$s = 25\ \text{cm} - 2.2\ \text{cm} = 22.8\ \text{cm}$$

Apply the thin-lens equation to the glasses:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{\text{glasses}}} = P$$

Solving for s' yields:

$$s' = \frac{s}{Ps - 1} = \frac{1}{P - \frac{1}{s}}$$

Substitute in equation (1) to obtain:

$$x_{np} = \left|\frac{1}{P - \frac{1}{s}}\right| + 2.2\,\text{cm} \qquad (2)$$

Substitute numerical values and evaluate x_{np}:

$$x_{np} = \left|\frac{1}{2.10\,\text{m}^{-1} - \frac{1}{0.228\,\text{m}}}\right| + 2.2\,\text{cm} = \boxed{46\,\text{cm}}$$

(b) At age 55:

$$s = 40\ \text{cm} - 2.2\ \text{cm} = 37.8\ \text{cm}$$

Substitute numerical values in equation (2) and evaluate s':

$$x_{np} = \left|\frac{1}{2.10\,\text{m}^{-1} - \frac{1}{0.378\,\text{m}}}\right| + 2.2\,\text{cm} = \boxed{1.9\ \text{m}}$$

(c) Solve the thin-lens equation for f:

$$f = \frac{ss'}{s' + s}$$

From the definition of P:

$$P = \frac{1}{f} = \frac{s' + s}{s's}$$

For s = 22.8 cm and s' = 183.3 cm:

$$P = \frac{183.3\,\text{cm} + 22.8\,\text{cm}}{(183.3\,\text{cm})(22.8\,\text{cm})} = \boxed{4.9\,\text{D}}$$

The Simple Magnifier

75 • **[SSM]** What is the magnifying power of a lens that has a focal length equal to 7.0 cm when the image is viewed at infinity by a person whose near point is at 35 cm?

Picture the Problem We can use the definition of the magnifying power of a lens to find the magnifying power of this lens.

The magnifying power of the lens is given by:

$$M = \frac{x_{\text{np}}}{f}$$

Substitute numerical values and evaluate M:

$$M = \frac{35\,\text{cm}}{7.0\,\text{cm}} = \boxed{5.0}$$

76 •• A lens that has a focal length equal to 6.0 cm is used as a simple magnifier by one person whose near point is 25 cm and by another person whose near point is 40 cm. (*a*) What is the effective magnifying power of the lens for each person? (*b*) Compare the sizes of the images on the retinas when each person looks at the same object with the magnifier.

Picture the Problem Let the numerals 1 and 2 denote the 1st and 2nd persons, respectively. We can use the definition of magnifying power to find the effective magnifying power of the lens for each person. The relative sizes of the images on the retinas of the two persons is given by the ratio of the effective magnifying powers.

(*a*) The magnifying power of the lens is given by:

$$M = \frac{x_{\text{np}}}{f}$$

Substitute numerical values and evaluate M_1 and M_2:

$$M_1 = \frac{25\,\text{cm}}{6.0\,\text{cm}} = 4.17 = \boxed{4.2}$$

and

$$M_2 = \frac{40\,\text{cm}}{6.0\,\text{cm}} = 6.67 = \boxed{6.7}$$

If $x_{np} = 25$ cm, then $M = 4.2$, and if $x_{np} = 40$ cm, then $M = 6.7$.

(*b*) From the definition of magnifying power we have:

$$\frac{M_1}{M_2} = \frac{\dfrac{y_1}{f}}{\dfrac{y_2}{f}} = \frac{y_1}{y_2}$$

Substitute for M_1 and M_2 and evaluate the ratio of y_1 to y_2:

$$\frac{y_1}{y_2} = \frac{4.17}{6.67} = \boxed{0.63}$$

For a person with $x_{np} = 40$ cm, the image on the retina is 1.6 times larger than it is for a person with $x_{np} = 25$ cm.

77 •• [SSM] In your botany class, you examine a leaf using a convex 12-D lens as a simple magnifier. What is the angular magnification of the leaf if image formed by the lens (*a*) is at infinity and (*b*) is at 25 cm?

Picture the Problem We can use the definition of angular magnification to find the expected angular magnification if the final image is at infinity and the thin-lens equation and the expression for the magnification of a thin lens to find the angular magnification when the final image is at 25 cm.

(*a*) Express the angular magnification when the final image is at infinity:

$$M = \frac{x_{np}}{f} = x_{np}P$$

where P is the power of the lens.

Substitute numerical values and evaluate M:

$$M = (25\,\text{cm})(12\,\text{m}^{-1}) = \boxed{3.0}$$

(*b*) Express the magnification of the lens when the final image is at 25 cm:

$$m = -\frac{s'}{s}$$

Solve the thin-lens equation for s:

$$s = \frac{fs'}{s' - f}$$

Substitute for s and simplify to obtain:

$$m = -\frac{s'}{\dfrac{fs'}{s'-f}} = -\frac{s'-f}{f} = -\frac{s'}{f} + 1$$
$$= 1 - s'P$$

Substitute numerical values and evaluate m:

$$m = 1 - (-0.25\,\text{m})(12\,\text{m}^{-1}) = \boxed{4.0}$$

The Microscope

78 •• Your laboratory microscope objective has a focal length of 17.0 mm. It forms an image of a tiny specimen at 16.0 cm from its second focal point. (*a*) How far from the objective is the specimen located? (*b*) What is the magnifying power for you if your near point distance is 25.0 cm and the focal length of the eyepiece is 51.0 mm?

Picture the Problem We can use the thin-lens equation to find the location of the object and the expression for the magnifying power of a microscope to find the magnifying power of the given microscope for a person whose near point is at 25.0 cm.

(*a*) Using the thin-lens equation, relate the object distance s to the focal length of the objective lens f_0:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_0} \Rightarrow s = \frac{f_0 s'}{s' - f_0}$$

The image distance for the image formed by the objective lens is:

$$s' = f_0 + L = 1.70\,\text{cm} + 16.0\,\text{cm} = 17.7\,\text{cm}$$

Substitute numerical values and evaluate s:

$$s = \frac{(1.70\,\text{cm})(17.7\,\text{cm})}{17.7\,\text{cm} - 1.70\,\text{cm}} = \boxed{18.8\ \text{mm}}$$

(*b*) Express the magnifying power of a microscope:

$$M = -\frac{L}{f_0}\frac{x_{\text{np}}}{f_{\text{e}}}$$

Substitute numerical values and evaluate M:

$$M = -\left(\frac{16.0\,\text{cm}}{1.70\,\text{cm}}\right)\left(\frac{25.0\,\text{cm}}{5.10\,\text{cm}}\right) = \boxed{-46.1}$$

79 •• **[SSM]** Your laboratory microscope objective has a focal length of 17.0 mm. It forms an image of a tiny specimen at 16.0 cm from its second focal point. (*a*) How far from the objective is the specimen located? (*b*) What is the magnifying power for you if your near point distance is 25.0 cm and the focal length of the eyepiece is 51.0 mm?

Picture the Problem The lateral magnification of the objective is $m_{\text{o}} = -L/f_{\text{o}}$ and the magnifying power of the microscope is $M = m_{\text{o}} M_{\text{e}}$.

(*a*) The lateral magnification of the objective is given by:

$$m_o = -\frac{L}{f_o}$$

Substitute numerical values and evaluate m_o:

$$m_o = -\frac{16\,\text{cm}}{8.5\,\text{mm}} = -18.8 = \boxed{-19}$$

(*b*) The magnifying power of the microscope is given by:

$$M = m_o M_e$$

where M_e is the angular magnification of the lens.

Substitute numerical values and evaluate M:

$$M = (-18.8)(10) = \boxed{-1.9\times10^2}$$

80 •• A crude, symmetric handheld microscope consists of two 20-D lenses fastened at the ends of a tube 30 cm long. (*a*) What is the tube length of this microscope? (*b*) What is the lateral magnification of the objective? (*c*) What is the magnifying power of the microscope? (*d*) How far from the objective should the object be placed?

Picture the Problem We can find the tube length from the length of the tube to which the lenses are fastened and the focal lengths of the objective and eyepiece. We can use their definitions to find the lateral magnification of the objective and the magnifying power of the microscope. The distance of the object from the objective can be found using the thin-lens equation.

(*a*) The tube length L is given by:

$$L = D - f_o - f_e$$
$$= 0.30\,\text{m} - \frac{2}{20\,\text{m}^{-1}} = \boxed{20\,\text{cm}}$$

(*b*) The lateral magnification of the objective m_o is given by:

$$m_o = -\frac{L}{f_o} = -\frac{20\,\text{cm}}{5.0\,\text{cm}} = \boxed{-4.0}$$

(*c*) The magnifying power of the microscope is given by:

$$M = m_o M_e = m_o \frac{x_{np}}{f_e}$$

Substitute numerical values and evaluate M:

$$M = (-4)\frac{25\,\text{cm}}{5.0\,\text{cm}} = \boxed{-20}$$

(*d*) From the thin-lens equation we have:

$$\frac{1}{s_o} + \frac{1}{s_o'} = \frac{1}{f_o}$$

where $s_o' = f_o + L$

Substitute for s_o' to obtain:

$$\frac{1}{s_o}+\frac{1}{f_o+L}=\frac{1}{f_o}\Rightarrow s_o=\frac{f_o(f_o+L)}{L}$$

Substitute numerical values and evaluate s_o:

$$s_o=\frac{(5.0\,\text{cm})(5.0\,\text{cm}+20\,\text{cm})}{20\,\text{cm}}=\boxed{6.3\,\text{cm}}$$

81 •• A compound microscope has an objective lens that has a power of 45 D and an eyepiece that has a power of 80 D. The lenses are separated by 28 cm. Assuming that the final image is formed by the microscopy is 25 cm from the eye, what is the magnifying power?

Picture the Problem The magnifying power of a compound microscope is the product of the magnifying powers of the objective and the eyepiece.

Express the magnifying power of the microscope in terms of the magnifying powers of the objective and eyepiece:

$$M=m_o m_e \qquad (1)$$

The magnification of the eyepiece is given by:

$$m_e=\frac{x_{np}}{f_e}+1=P_e x_{np}+1$$

The magnification of the objective is given by:

$$m_o=-\frac{L}{f_o}$$

where $L=D-f_o-f_e$

Substitute for L to obtain:

$$m_o=-\frac{D-f_o-f_e}{f_o}$$

Substitute for m_e and m_o in equation (1) to obtain:

$$M=(P_e x_{np}+1)\left(-\frac{D-f_o-f_e}{f_o}\right)$$

Substitute numerical values and evaluate M:

$$M=[(80\,\text{D})(0.25\,\text{m})+1]\left(-\frac{28\,\text{cm}-2.22\,\text{cm}-1.25\,\text{cm}}{2.22\,\text{cm}}\right)=\boxed{-230}$$

82 ••• A microscope has a magnifying power of 600. The eyepiece has an angular magnification of 15.0. The objective lens is 22.0 cm from the eyepiece. Calculate (*a*) the focal length of the eyepiece, (*b*) the location of the object so that it is in focus for a normal relaxed eye, and (*c*) the focal length of the objective lens.

Picture the Problem We can find the focal length of the eyepiece from its angular magnification and the near point of a normal eye. The location of the object such that it is in focus for a normal relaxed eye can be found from the lateral magnification of the eyepiece and the magnifying power of the microscope. Finally, we can use the thin-lens equation to find the focal length of the objective lens.

(*a*) The focal length of the eyepiece is related to its angular magnifying power:

$$M_e = \frac{x_{np}}{f_e} \Rightarrow f_e = \frac{x_{np}}{M_e}$$

Substitute numerical values and evaluate f_e:

$$f_e = \frac{25.0\,\text{cm}}{15.0} = \boxed{1.67\,\text{cm}}$$

(*b*) Relate s to s' through the lateral magnification of the objective:

$$m_o = -\frac{s'}{s} \Rightarrow s = -\frac{s'}{m_o}$$

Relate the magnifying power of the microscope M to the lateral magnification of its objective m_0 and the angular magnification of its eyepiece M_e:

$$M = m_o M_e \Rightarrow m_o = \frac{M}{M_e}$$

Substitute for m_o to obtain:

$$s = -\frac{s' M_e}{M}$$

Evaluate s':

$$s' = 22\,\text{cm} - f_e$$
$$= 22\,\text{cm} - 1.67\,\text{cm} = 20.33\,\text{cm}$$

Substitute numerical values and evaluate s:

$$s = -\frac{(20.33\,\text{cm})(15.0)}{-600} = \boxed{0.508\,\text{cm}}$$

(*c*) Solving the thin-lens equation for f_o yields:

$$f_o = \frac{ss'}{s' + s}$$

Substitute numerical values and evaluate f_o:

$$f_o = \frac{(0.508\,\text{cm})(20.33\,\text{cm})}{20.33\,\text{cm} + 0.508\,\text{cm}} = \boxed{0.496\,\text{cm}}$$

The Telescope

83 • **[SSM]** You have a simple telescope that has an objective which has a focal length of 100 cm and an eyepiece which has a focal length of 5.00 cm. You are using it to look at the moon, which subtends an angle of about 9.00 mrad. (*a*) What is the diameter of the image formed by the objective? (*b*) What angle is subtended by the image formed at infinity by the eyepiece? (*c*) What is the magnifying power of your telescope?

Picture the Problem Because of the great distance to the moon, its image formed by the objective lens is at the focal point of the objective lens and we can use $D = f_o\theta$ to find the diameter D of the image of the moon. Because angle subtended by the final image at infinity is given by $\theta_e = M\theta_o = M\theta$, we can solve (*b*) and (*c*) together by first using $M = -f_o/f_e$ to find the magnifying power of the telescope.

(*a*) Relate the diameter D of the image of the moon to the image distance and the angle subtended by the moon:

$$D = s_o'\theta$$

Because the image of the moon is at the focal point of the objective lens:

$$s_o' = f_o \text{ and } D = f_o\theta$$

Substitute numerical values and evaluate D:

$$D = (100\,\text{cm})(9.00\,\text{mrad}) = \boxed{9.00\,\text{mm}}$$

(*b*) and (*c*) Relate the angle subtended by the final image at infinity to the magnification of the telescope and the angle subtended at the objective:

$$\theta_e = M\theta_o = M\theta$$

Express the magnifying power of the telescope:

$$M = -\frac{f_o}{f_e}$$

Substitute numerical values and evaluate M and θ_e:

$$M = -\frac{100\,\text{cm}}{5.00\,\text{cm}} = \boxed{-20.0}$$

and

$$\theta_e = |-20.0|(9.00\,\text{mrad}) = \boxed{180\ \text{mrad}}$$

84 • The objective lens of the refracting telescope at the Yerkes Observatory has a focal length of 19.5 m. The moon subtends an angle of about 9.00 mrad. When the telescope is used to look at the moon, what is the diameter of the image of the moon formed by the objective?

Picture the Problem Because of the great distance to the moon, its image formed by the objective lens is at the focal point of the objective lens and we can use $D = f_o\theta$ to find the diameter D of the image of the moon.

Relate the diameter D of the image of the moon to the image distance and the angle subtended by the moon:

$$D = s_o'\theta$$

Because the image of the moon is at the focal point of the objective lens:

$$s_o' = f_o \text{ and } D = f_o\theta$$

Substitute numerical values and evaluate D:

$$D = (19.5\,\text{m})(9.00\,\text{mrad}) = \boxed{17.6\,\text{cm}}$$

85 •• The 200-in (5.08-m) diameter mirror of the reflecting telescope at Mt. Palomar has a focal length of 16.8 m. (*a*) By what factor is the light-gathering power increased over the 40.0-in (1.02-m) diameter refracting lens of the Yerkes Observatory telescope? (*b*) If the focal length of the eyepiece is 1.25 cm, what is the magnifying power of the 200-in telescope?

Picture the Problem Because the light-gathering power of a mirror is proportional to its area, we can compare the light-gathering powers of these mirrors by finding the ratio of their areas. We can use the ratio of the focal lengths of the objective and eyepiece lenses to find the magnifying power of the Palomar telescope.

(*a*) Express the ratio of the light-gathering powers of the Palomar and Yerkes mirrors:

$$\frac{P_{\text{Palomar}}}{P_{\text{Yerkes}}}=\frac{A_{\substack{\text{Palomar}\\\text{mirror}}}}{A_{\substack{\text{Yerkes}\\\text{mirror}}}}=\frac{\frac{\pi}{4}d^2_{\substack{\text{Palomar}\\\text{mirror}}}}{\frac{\pi}{4}d^2_{\substack{\text{Yerkes}\\\text{mirror}}}}=\frac{d^2_{\substack{\text{Palomar}\\\text{mirror}}}}{d^2_{\substack{\text{Yerkes}\\\text{mirror}}}}$$

Substitute numerical values and evaluate $P_{\text{Palomar}}/P_{\text{Yerkes}}$:

$$\frac{P_{\text{Palomar}}}{P_{\text{Yerkes}}}=\frac{(200\,\text{in})^2}{(40.0\,\text{in})^2}=\boxed{25.0}$$

(*b*) Express the magnifying power of the Palomar telescope:

$$M=-\frac{f_o}{f_e}$$

Substitute numerical values and evaluate M:

$$M=-\frac{16.8\,\text{m}}{1.25\,\text{cm}}=\boxed{-1340}$$

86 •• An astronomical telescope has a magnifying power of 7.0. The two lenses are 32 cm apart. Find the focal length of each lens.

Picture the Problem We can use the expression for the magnifying power of a telescope and the fact that the length of a telescope is the sum of focal lengths of its objective and eyepiece lenses to obtain simultaneous equations in f_o and f_e.

The magnifying power of the telescope is given by:

$$M=-\frac{f_o}{f_e}=-7$$

The length of the telescope is the sum of the focal lengths of the objective and eyepiece lenses:

$$L=f_o+f_e=32\,\text{cm}$$

Solve these equations simultaneously to obtain:

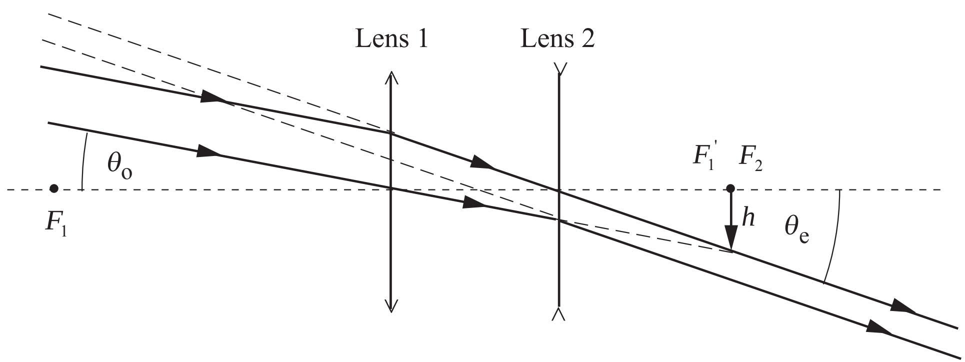
$$f_o=\boxed{28\,\text{cm}}\text{ and } f_e=\boxed{4\,\text{cm}}$$

87 •• **[SSM]** A disadvantage of the astronomical telescope for terrestrial use (for example, at a football game) is that the image is inverted. A Galilean telescope uses a converging lens as its objective, but a diverging lens as its eyepiece. The image formed by the objective is at the second focal point of the eyepiece (the focal point on the refracted side of the eyepiece), so that the final image is virtual, upright, and at infinity. (*a*) Show that the magnifying power is given by $M = -f_o/f_e$, where f_o is the focal length of the objective and f_e is that of the

eyepiece (which is negative). (*b*) Draw a ray diagram to show that the final image is indeed virtual, upright, and at infinity.

Picture the Problem The magnification of a telescope is the ratio of the angle subtended at the eyepiece lens to the angle subtended at the objective lens. We can use the geometry of the ray diagram to express both θ_e and θ_o.

(*b*) The ray diagram is shown below:



(*a*) Express the magnifying power M of the telescope:

$$M = \frac{\theta_e}{\theta_o}$$

Because the image formed by the objective lens is at the focal point, F'_1:

$$\theta_o = \frac{h}{f_o}$$

where we have assumed that $\theta_o << 1$ so that $\tan \theta_o \approx \theta_o$.

Express the angle subtended by the eyepiece:

$$\theta_e = \frac{h}{f_e}$$ where f_e is negative.

Substitute to obtain:

$$M = \frac{\frac{h}{f_e}}{\frac{h}{f_o}} = \frac{f_o}{f_e}$$

and $M = \boxed{-\frac{f_o}{f_e}}$ is positive.

Remarks: Because the object for the eyepiece is at its focal point, the image is at infinity. As is also evident from the ray diagram, the image is virtual and upright.

88 •• A Galilean telescope (see Problem 87) is designed so that the final image is at the near point, which is 25 cm (rather than at infinity). The focal length of the objective is 100 cm and the focal length of the eyepiece is –5.0 cm. (*a*) If the object distance is 30.0 m, where is the image of the objective? (*b*) What is the object distance for the eyepiece so that the final image is at the near point? (*c*) How far apart are the lenses? (*d*) If the object height is 1.5 m, what is the height of the final image? What is the angular magnification?

Picture the Problem We can use the thin-lens equation to find the image distance for the objective lens and the object distance for the eyepiece lens. The separation of the lenses is the sum of these distances. We can use the definition of the angular magnification and the angles subtended at the objective and eyepiece lenses to find the height of the final image.

(*a*) Solve the thin-lens equation for s_o':

$$s_o' = \frac{f_o s_o}{s_o - f_o}$$

Substitute numerical values and evaluate s_o':

$$s_o' = \frac{(1.00\,\text{m})(30.0\,\text{m})}{30.0\,\text{m} - 1.00\,\text{m}} = \boxed{103\text{ cm}}$$

(*b*) Solve the thin-lens equation for s_e:

$$s_e = \frac{f_e s_e'}{s_e' - f_e}$$

Noting that $s_e' = -25$ cm, substitute numerical values and evaluate s_e:

$$s_e = \frac{(-5.0\,\text{cm})(-25\,\text{cm})}{-25\,\text{cm} - (-5.0\,\text{cm})_e} = -6.25\,\text{cm}$$

$$\boxed{-6.3\,\text{cm}}$$

where the minus sign tells us that the object of the eyepiece is virtual.

(*c*) Express the separation D of the lenses:

$$D = s_o' + s_e$$

Substitute numerical values and evaluate D:

$$D = 103\,\text{cm} - 6\,\text{cm} \approx \boxed{97\text{ cm}}$$

(*d*) Express the height h' of the final image in terms of the magnification M of the telescope:

$$h' = Mh$$

The magnification of the telescope is the product of the magnifications of the objective and eyepiece lenses:

$$M = m_o m_e = \frac{s_o'}{s_o}\frac{s_e'}{s_e}$$

Substitute for M to obtain:

$$h' = \frac{s_o'}{s_o}\frac{s_e'}{s_e}h$$

Substitute numerical values and evaluate h':

$$h' = \left(\frac{103.45\,\text{cm}}{3000\,\text{cm}}\right)\left(\frac{-25\,\text{cm}}{-6.25\,\text{cm}}\right)(1.5\,\text{m}) = \boxed{21\,\text{cm}}$$

Express the angular magnification of the telescope:

$$M = \frac{\theta_e}{\theta_o}$$

The angle subtended by the object is:

$$\theta_o = \frac{h}{s_o}$$

The angle subtended by the image is:

$$\theta_e = \tan^{-1}\left(\frac{h'}{s_e}\right)$$

Substitute for θ_e and θ_o and simplify to obtain:

$$M = \frac{\tan^{-1}\left(\frac{h'}{s_e}\right)}{\frac{h}{s_o}} = \frac{s_o}{h}\tan^{-1}\left(\frac{h'}{s_e}\right)$$

Substitute numerical values and evaluate M:

$$M = \frac{30\,\text{m}}{1.5\,\text{m}}\tan^{-1}\left(\frac{20.7\,\text{cm}}{6.25\,\text{cm}}\right) = \boxed{26}$$

89 ••• If you look into the wrong end of a telescope, that is, into the objective, you will see distant objects reduced in angular size. For a refracting telescope that has an objective with a focal length equal to 2.25 m and an eyepiece with a focal length equal to 1.50 cm, by what factor is the angular size of the object changed?

Picture the Problem The roles of the objective and eyepiece lenses are reversed.

Express the magnifying power of the ″wrong end″ telescope:

$$M = -\frac{f_e}{f_o}$$

Substitute numerical values and evaluate M:

$$M = -\frac{1.50\,\text{cm}}{2.25\,\text{m}} = \boxed{-6.67\times10^{-3}}$$

General Problems

90 • To focus a camera, the distance between the lens and the image-sensing surface is varied. A wide-angle lens of a camera has a focal length of 28 mm. By how much must the lens move to change from focusing on an object at infinity to an object at a distance of 5.00 m in front of the camera?

Picture the Problem We can express the distance Δs that the lens must move as the difference between the image distances when the object is at 30 m and when it is at infinity and then express these image distances using the thin-lens equation.

Express the distance Δs that the lens must move to change from focusing on an object at infinity to one at a distance of 5 m:

$$\Delta s = s'_5 - s'_\infty$$

Solve the thin-lens equation for s' to obtain:

$$s' = \frac{fs}{s-f}$$

Substitute s'_5 and s'_∞ and simplify to obtain:

$$\Delta s = \frac{fs_5}{s_5 - f} - \frac{fs_\infty}{s_\infty - f} = \frac{fs_5}{s_5 - f} - \frac{f}{1 - f/s_\infty} = f\left[\frac{s_5}{s_5 - f} - 1\right]$$

Substitute numerical values and evaluate Δs:

$$\Delta s = (28\,\text{mm})\left[\frac{5.0\,\text{m}}{5.00\,\text{m} - 0.028\,\text{m}} - 1\right] = \boxed{0.16\,\text{mm}}$$

91 • A converging lens that has a focal length equal to 10 cm is used to obtain an image that is twice the height of the object. Find the object and image distances if (a) the image is to be upright and (b) the image is to be inverted. Draw a ray diagram for each case.

Picture the Problem We can use the thin-lens and magnification equations to obtain simultaneous equations that we can solve to find the image and object distances for the two situations described in the problem statement.

(*a*) Use the thin-lens equation to relate the image and object distances to the focal length of the lens:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

Because the image is twice as large as the object:

$$m=-\frac{s'}{s}\Rightarrow s'=-2s$$

Substitute for s' to obtain:

$$\frac{1}{s}+\frac{1}{-2s}=\frac{1}{f}\Rightarrow s=\tfrac{1}{2}f$$

Substitute numerical values and evaluate s and s':

$$s=\tfrac{1}{2}(10\,\text{cm})=\boxed{5.0\,\text{cm}}$$

and

$$s'=-2(5.0\,\text{cm})=\boxed{-10\,\text{cm}}$$

(*b*) If the image is inverted, then:

$$s'=2s\ \text{ and }\ \frac{1}{s}+\frac{1}{2s}=\frac{1}{f}\Rightarrow s=\tfrac{3}{2}f$$

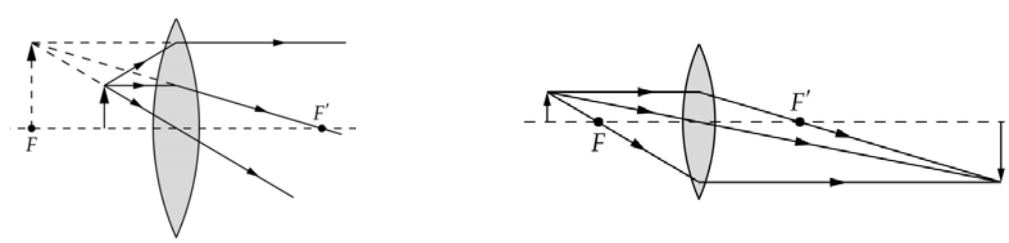
Substitute numerical values and evaluate s and s':

$$s=\tfrac{1}{2}\left[3(10\,\text{cm})\right]=\boxed{15\,\text{cm}}$$

and

$$s'=2(15\,\text{cm})=\boxed{30\,\text{cm}}$$

The ray diagrams for (*a*) (left) and (*b*) (right) follow:

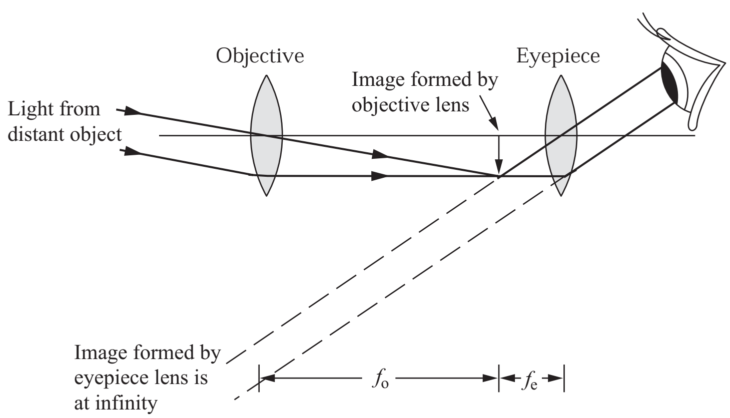


92 •• You are given two converging lenses that have focal lengths of 75 mm and 25 mm. (*a*) Show how the lenses should be arranged to form a telescope. State which lens to use as the objective, which lens to use as the eyepiece, how far apart to place the lenses and, what angular magnification you expect. (*b*) Draw a ray diagram to show how rays from a distant object are refracted by the two lenses.

Picture the Problem In an astronomical telescope the eyepiece (short focal length) and objective (long focal length) lenses are separated by the sum of their focal lengths. Given these two lenses, we'll use the 25 mm lens as the eyepiece lens and the 75 mm lens as the objective lens and mount them 100 mm apart. The angular magnification is then $M = -\frac{f_0}{f_e} = -\frac{75\,\text{mm}}{25\,\text{mm}} = -3$.

(*a*) The lens with a focal length of 75 mm should be the objective. The two lenses should be separated by 100 mm. The angular magnification is –3.

(*b*) A ray diagram showing how rays from a distant object are refracted by the two lenses follows. A real and inverted image of the distant object is formed by the objective lens near its second focal point. The eyepiece lens forms an enlarged and inverted image of the image formed by the objective lens.



93 •• [SSM] (*a*) Show how the same two lenses in Problem 92 should be arranged to form a compound microscope that has a tube length of 160 mm. State which lens to use as the objective, which lens to use as the eyepiece, how far apart to place the lenses, and what overall magnification you expect to get, assuming the user has a near point of 25 cm. (*b*) Draw a ray diagram to show how rays from a close object are refracted by the lenses.

Picture the Problem

Because the focal lengths appear in the magnification formula as a product, it would appear that it does not matter in which order we use them. The usual arrangement would be to use the shorter focal length lens as the objective but we get the same magnification in the reverse order. What difference does it make then? None in this problem. However, it is generally true that the smaller the focal length of a lens, the smaller its diameter. This condition makes it harder to use the shorter focal length lens, with its smaller diameter, as the eyepiece lens.

In a compound microscope, the lenses are separated by:

$$\delta = L + f_e + f_0$$

Substitute numerical values and evaluate δ:

$$\delta = 16\,\text{cm} + 7.5\,\text{cm} + 2.5\,\text{cm} = 26\,\text{cm}$$

The overall magnification of a compound microscope is given by:

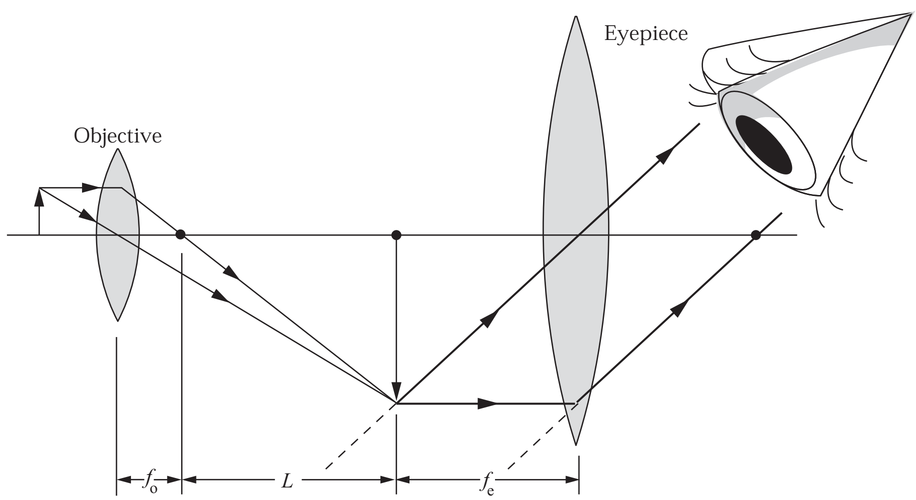
$$M = m_0 M_e = -\frac{L}{f_0}\frac{x_{np}}{f_e}$$

Substitute numerical values and evaluate M:

$$M = -\left(\frac{16\,\text{cm}}{7.5\,\text{cm}}\right)\left(\frac{25\,\text{cm}}{2.5\,\text{cm}}\right) = -21$$

The lens with a focal length of 25 mm should be the objective. The two lenses should be separated by 210 mm. The angular magnification is –21.

(*b*) A ray diagram showing how rays from a near-by object are refracted by the lenses follows. A real and inverted image of the near-by object is formed by the objective lens at the first focal point of the eyepiece lens. The eyepiece lens forms an inverted and virtual image of this image at infinity.



94 •• On a vacation, you are scuba-diving and using a diving mask that has a face plate and that bulges outward with a radius of curvature of 0.50 m. As a result, a convex spherical surface exists between the water and the air in the mask. A fish swims by you 2.5 m in front of your mask. (*a*) How far in front of the mask does the fish appear to be? (*b*) What is the lateral magnification of the image of the fish?

Picture the Problem We can use the equation for refraction at a single surface to locate the image of the fish and the expression for the magnification due to refraction at a spherical surface to find the magnification of the image.

(*a*) Use the equation describing refraction at a single surface to relate the image and object distances:

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{r}$$

Solving for s' yields:

$$s'=\frac{n_2 rs}{(n_2-n_1)s-n_1 r}$$

Substitute numerical values and evaluate s':

$$s'=\frac{(1)(0.50\,\text{m})(2.5\,\text{m})}{(1-1.33)(2.5\,\text{m})-(1.33)(0.50\,\text{m})}$$
$$=-0.839\,\text{m}=\boxed{-84\,\text{cm}}$$

Note that the fish appears to be only 84 cm in front of your mask.

(*b*) Express the magnification due to refraction at a spherical surface:

$$m=-\frac{n_1 s'}{n_2 s}$$

Substitute numerical values and evaluate m:

$$m=-\frac{(1.33)(-0.839\,\text{m})}{(1)(2.5\,\text{m})}=\boxed{0.45}$$

Note that the fish appears to be smaller than it actually is.

95 •• [SSM] A 35-mm digital camera has a rectangular array of CCDs (light sensors) that is 24 mm by 36 mm. It is used to take a picture of a person 175-cm tall so that the image just fills the height (24 mm) of the CCD array. How far should the person stand from the camera if the focal length of the lens is 50 mm?

Picture the Problem We can use the thin-lens equation and the definition of the magnification of an image to determine where the person should stand.

Use the thin-lens equation to relate s and s':

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

The magnification of the image is given by:

$$m=-\frac{s'}{s}=-\frac{2.4\,\text{cm}}{175\,\text{cm}}=-1.37\times10^{-2}$$

and

$$s'=-ms$$

Substitute for s' to obtain:

$$\frac{1}{s}-\frac{1}{ms}=\frac{1}{f}\Rightarrow s=\left(1-\frac{1}{m}\right)f$$

Substitute numerical values and evaluate s:

$$s=\left(1-\frac{1}{-1.37\times10^{-2}}\right)(50\,\text{mm})=\boxed{3.7\,\text{m}}$$

96 •• A 35-mm camera that has interchangeable lenses is used to take a picture of a hawk that has a wing span of 2.0 m. The hawk is 30 m away. What would be the ideal focal length of the lens used so that the image of the wings just fills the width of the light sensitive area of the camera, which is 36 mm?

Picture the Problem We can use the thin-lens equation and the definition of the magnification of an image to determine the ideal focal length of the lens.

Use the thin-lens equation to relate s and s':

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

The magnification of the image is given by:

$$m=-\frac{s'}{s}=-\frac{3.6\,\text{cm}}{200\,\text{cm}}=-1.80\times10^{-2}$$

and

$$s'=-ms$$

Substitute for s' to obtain:

$$\frac{1}{s}-\frac{1}{ms}=\frac{1}{f}\Rightarrow f=\frac{s}{1-\frac{1}{m}}$$

Substitute numerical values and evaluate f:

$$f=\frac{30\,\text{m}}{1-\frac{1}{-1.80\times10^{-2}}}=\boxed{53\,\text{cm}}$$

97 •• An object is placed 12.0 cm in front of a lens that has a focal length equal to 10.0 cm. A second lens that has a focal length equal to 12.5 cm is placed 20.0 cm in back of the first lens. (*a*) Find the position of the final image. (*b*) What is the magnification of the image? (*c*) Sketch a ray diagram showing the final image.

Picture the Problem Let the numeral 1 refer to the first lens and the numeral 2 to the second lens. We apply the thin-lens equation twice; once to locate the image formed by the first lens and a second time to find the image formed by the second lens. The magnification of the image is the product of the magnifications produced by the two lenses.

(*a*) Solve the thin-lens equation for the location of the image formed by the first lens:

$$s_1' = \frac{f_1 s_1}{s_1 - f_1}$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(10\,\text{cm})(12\,\text{cm})}{12\,\text{cm} - 10\,\text{cm}} = 60\,\text{cm}$$

Because the second lens is 20 cm to the right of the first lens:

$$s_2 = 20\,\text{cm} - 60\,\text{cm} = -40\,\text{cm}$$

Solve the thin-lens equation for the location of the image formed by the second lens:

$$s_2' = \frac{f_2 s_2}{s_2 - f_2}$$

Substitute numerical values and evaluate s_2':

$$s_2' = \frac{(12.5\,\text{cm})(-40\,\text{cm})}{-40\,\text{cm} - 12.5\,\text{cm}} = \boxed{9.5\,\text{cm}}$$

i.e., the final image is 9.5 cm in back of the second lens.

(*b*) Express the magnification of the final image:

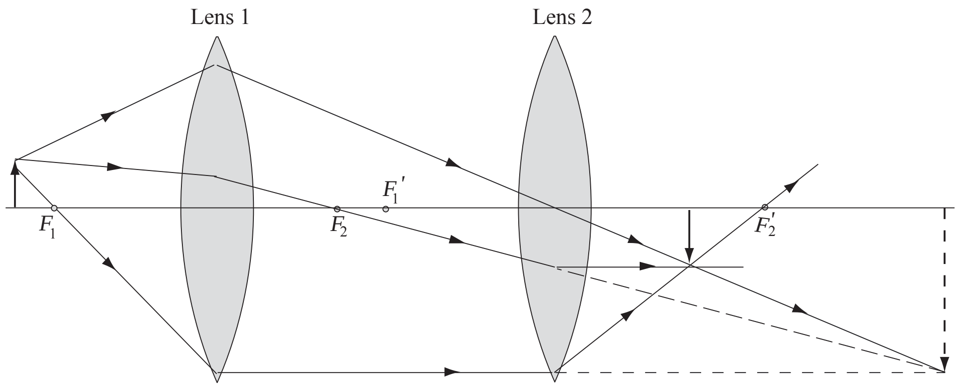
$$m = m_1 m_2 = \left(-\frac{s_1'}{s_1}\right)\left(-\frac{s_2'}{s_2}\right) = \frac{s_1' s_2'}{s_1 s_2}$$

Substitute numerical values and evaluate *m*:

$$m = \frac{(60\,\text{cm})(9.52\,\text{cm})}{(12\,\text{cm})(-40\,\text{cm})} = \boxed{-1.2}$$

i.e., the final image is about 20% larger than the object and is inverted.

(*c*) The enlarged, inverted image formed by the first lens serves as a virtual object for the second lens. The image formed from this virtual object is the real, inverted image shown in the diagram.



98 •• (*a*) Show that if f_a is the focal length of a thin lens in air, its focal length in water is given by $f_w = (n_w/n_a)(n-n_a)/(n-n_w)\,f_a$ where n_w is the index of refraction of water, n is that of the lens material and n_a is that of air.
(*b*) Calculate the focal length in air and in water of a double concave lens that has an index of refraction of 1.50 and radii of magnitude 30 cm and 35 cm.

Picture the Problem We can apply the equation for refraction at a surface to both surfaces of the lens and add the resulting equations to obtain an equation relating the image and object distances to the indices of refraction. We can then use the lens maker's equation to complete the derivation of the given relationship between f' and f.

(*a*) Relate s and s' at the water-lens interface:

$$\frac{n_w}{s}+\frac{n}{s_1'}=\frac{n-n_w}{r_1}$$

Relate s and s' at the lens-water interface:

$$\frac{n}{-s_1'}+\frac{n}{s'}=\frac{n_w-n}{r_2}$$

Add these equations to obtain:

$$n_w\left(\frac{1}{s}+\frac{1}{s'}\right)=(n-n_w)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Let $\dfrac{1}{f_w}=\dfrac{1}{s}+\dfrac{1}{s'}$ to obtain:

$$\frac{n_w}{f_w}=(n-n_w)\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \qquad (1)$$

The lens-maker's equation is:

$$\frac{1}{f_a}=\left(\frac{n}{n_a}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

and

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{1}{\left(\frac{n}{n_a}-1\right)f_a}=\frac{n_a}{(n-n_a)f_a}$$

Substitute for $\frac{1}{r_1}-\frac{1}{r_2}$ in equation (1) to obtain:

$$\frac{n_w}{f_w}=(n-n_w)\left(\frac{n_a}{(n-n_a)f_a}\right)$$

Solving for f' yields:

$$f_w=\boxed{\frac{n_w(n-n_a)}{n_a(n-n_w)}f_a}$$

(*b*) Use the lens-maker's equation to find the focal length of the lens in air:

$$\frac{1}{f_a}=\left(\frac{1.50}{1.00}-1\right)\left(\frac{1}{-30\,\text{cm}}-\frac{1}{35\,\text{cm}}\right)$$

and

$$f=-32.3\,\text{cm}=\boxed{-32\,\text{cm}}$$

Use the result derived in Part (*a*) to find f_w:

$$f_w=\frac{(1.33)(1.50-1)}{1.50-1.33}(-32.3\,\text{cm})$$

$$=\boxed{-1.3\,\text{m}}$$

99 •• While parked in your car, you see a jogger in your rear view mirror, which is convex and has a radius of curvature whose magnitude is equal to 2.00 m. The jogger is 5.00 m from the mirror and is approaching at 3.50 m/s. How fast is the image of the jogger moving relative to the mirror?

Picture the Problem The speed of the jogger as seen in the mirror is $v'=ds'/dt$. We can use the mirror equation to derive an expression for v' in terms of f and ds/dt.

Solve the mirror equation for s':

$$s'=\left(\frac{1}{f}-\frac{1}{s}\right)^{-1} \qquad (1)$$

Differentiate s' with respect to time to obtain:

$$v' = \frac{ds'}{dt} = \frac{d}{dt}\left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = -\left(\frac{1}{f} - \frac{1}{s}\right)^{-2}\left(\frac{1}{s^2}\right)\frac{ds}{dt}$$

Simplify this result to obtain:

$$v' = -\left(\frac{s'}{s}\right)^2 v \qquad (2)$$

Rewrite equation (1) in terms of r:

$$s' = \left(\frac{2}{r} - \frac{1}{s}\right)^{-1}$$

Find s' when $s = 5.00$ m:

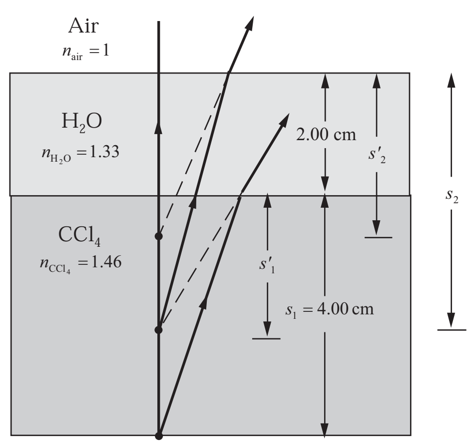
$$s' = \left(\frac{2}{-2.00\,\text{m}} - \frac{1}{5.00\,\text{m}}\right)^{-1} = -0.8333\,\text{m}$$

Use equation (2) to find $|v'|$ when $|v| = 3.50$ m/s:

$$|v'| = \left|-\left(\frac{-0.8333\,\text{m}}{5.00\,\text{m}}\right)^2(3.50\,\text{m/s})\right| = \boxed{9.72\,\text{cm/s}}$$

100 •• A 2.00-cm-thick layer of water ($n = 1.33$) floats on top of a 4.00-cm-thick layer of carbon tetrachloride ($n = 1.46$) in a tank. How far below the top surface of the water does the bottom of the tank appear, according to an observer looking down from above at normal incidence?

Picture the Problem Let the numeral 1 denote the CCl_4-H_2O interface and the numeral 2 the H_2O-air interface. We can locate the final image by applying the equation for refraction at a single surface to both interfaces. The ray diagram shown below shows a spot at the bottom of the tank and the rays of light emanating from it that form the intermediate and final images.



Use the equation for refraction at a single surface to relate s and s' at the CCl_4-H_2O interface:

$$\frac{n_{CCl_4}}{s_1}+\frac{n_{H_2O}}{s_1'}=\frac{n_{H_2O}-n_{CCl_4}}{r}$$

or, because $r = \infty$,

$$\frac{n_{CCl_4}}{s_1}+\frac{n_{H_2O}}{s_1'}=0 \Rightarrow s_1' = -\frac{n_{H_2O}s_1}{n_{CCl_4}} \quad (1)$$

Substitute numerical values and evaluate s_1':

$$s_1' = -\frac{(1.33)(4.00\text{ cm})}{1.46} = -3.64\text{ cm}$$

The depth of this image, as viewed from the H_2O-air interface is:

$$s_2 = 2.00\text{ cm} - s_1' = 2.00\text{ cm} - (-3.64\text{ cm}) = 5.64\text{ cm}$$

At the H_2O-air interface equation (1) becomes:

$$s_2' = -\frac{n_{air}s_2}{n_{H_2O}}$$

Substitute numerical values and evaluate s_2':

$$s_2' = -\frac{(1)(5.64\text{ cm})}{1.33} = -4.24\text{ cm}$$

$\boxed{\text{The apparent depth is } 4.24\text{ cm.}}$

101 ••• [SSM] An object is 15.0 cm in front of a thin converging lens that has a focal length equal to 10.0 cm. A concave mirror that has a radius equal to 10.0 cm is 25.0 cm in back of the lens. (*a*) Find the position of the final image formed by the mirror-lens combination. (*b*) Is the image real or virtual? Is the image upright or inverted? (*c*) On a diagram, show where your eye must be to see this image.

Picture the Problem We can use the thin-lens equation to locate the first image formed by the converging lens, the mirror equation to locate the image formed in the mirror, and the thin-lens equation a second time to locate the final image formed by the converging lens as the rays pass back through it.

(*a*) Solve the thin-lens equation for s_1':

$$s_1' = \frac{fs_1}{s_1 - f}$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(10.0\text{ cm})(15.0\text{ cm})}{15.0\text{ cm} - 10.0\text{ cm}} = 30\text{ cm}$$

Because the image formed by the converging lens is behind the mirror:

$$s_2 = 25\text{ cm} - 30\text{ cm} = -5\text{ cm}$$

Solve the mirror equation for s_2':

$$s_2' = \frac{fs_2}{s_2 - f}$$

Substitute numerical values and evaluate s_2':

$$s_2' = \frac{(5\,\text{cm})(-5\,\text{cm})}{-5\,\text{cm} - 5\,\text{cm}} = 2.50\,\text{cm} \text{ and the}$$

image is 22.5 cm from the lens; i.e., $s_3 = 22.5$ cm.

Solve the thin-lens equation for s_3':

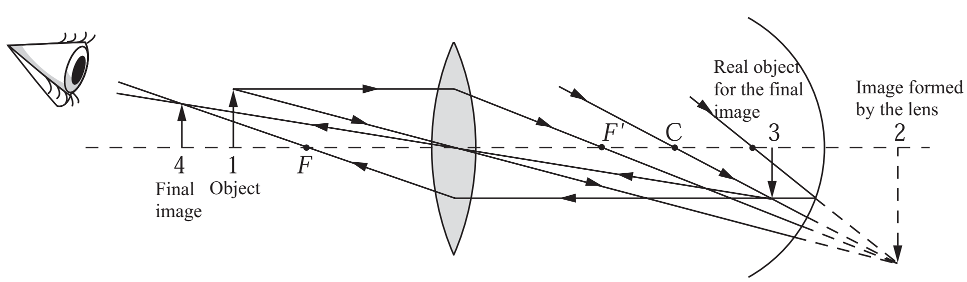
$$s_3' = \frac{fs_3}{s_3 - f}$$

Substitute numerical values and evaluate s_3':

$$s_3' = \frac{(10.0\,\text{cm})(22.5\,\text{cm})}{22.5\,\text{cm} - 10.0\,\text{cm}} = 18\,\text{cm}$$

The final position of the image is 18 cm from the lens, on the same side as the original object.

(*b*) The ray diagram is shown below. The numeral 1 represents the object. The parallel and central rays from 1 are shown; one passes through the center of the converging lens, the other is paraxial and then passes through the focal point F'. The two rays intersect behind the mirror, and the image formed there, identified by the numeral 2, serves as a virtual object for the mirror. Two rays are shown emanating from this virtual image, one through the center of the mirror, the other passing through its focal point (halfway between C and the mirror surface) and then continuing as a paraxial ray. These two rays intersect in front of the mirror, forming a real image, identified by the numeral 3. Finally, the image 3 serves as a real object for the lens; again we show two rays, a paraxial ray that then passes through the focal point F and a ray through the center of the lens. These two rays intersect to form the final $\boxed{\text{real, upright,}}$ and diminished image, identified as 4.



(*c*) To see this image the eye must be to the left of the final image.

102 ••• When a bright light source is placed 30 cm in front of a lens, there is an upright image 7.5 cm from the lens. There is also a faint inverted image 6.0 cm from the lens on the incident-light side due to reflection from the front surface of the lens. When the lens is turned around, this weaker, inverted image is 10 cm in front of the lens. Find the index of refraction of the lens.

Picture the Problem The mirror surfaces must be concave to create inverted images on reflection. Therefore, the lens is a diverging lens. Let the numeral 1 denote the lens in its initial orientation and the numeral 2 the lens in its second orientation. We can use the mirror equation to find the magnitudes of the radii of the lens' surfaces, the thin-lens equation to find its focal length, and the lens maker's equation to find its index of refraction.

Solve the mirror equation for $|r_1|$:

$$|r_1| = \frac{2s_1s_1'}{s_1' + s_1}$$

Substitute numerical values and evaluate $|r_1|$:

$$|r_1| = \frac{2(30\,\text{cm})(6.0\,\text{cm})}{6.0\,\text{cm} + 30\,\text{cm}} = 10.0\,\text{cm}$$

Solve the mirror equation for $|r_2|$:

$$|r_2| = \frac{2s_2s_2'}{s_2' + s_2}$$

Substitute numerical values and evaluate $|r_2|$:

$$|r_2| = \frac{2(30\,\text{cm})(10\,\text{cm})}{10\,\text{cm} + 30\,\text{cm}} = 15.0\,\text{cm}$$

Solve the thin-lens equation for *f*:

$$f = \frac{ss'}{s' + s}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{(30\,\text{cm})(-7.5\,\text{cm})}{-7.5\,\text{cm} + 30\,\text{cm}} = -10.0\,\text{cm}$$

Solve the lens-maker's equation for n to obtain:

$$n = \frac{n_\text{a}}{f\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} + n_\text{air}$$

Because the lens is a diverging lens, $r_1 = -10$ cm and $r_2 = 15$ cm. Substitute numerical values and evaluate n:

$$n = \frac{1.00}{(-10\,\text{cm})\left(\frac{1}{-10\,\text{cm}} - \frac{1}{15\,\text{cm}}\right)} + 1.00$$

$$= \boxed{1.6}$$

103 ••• A concave mirror that has a radius of curvature equal to 50.0 cm is oriented with its axis vertical. The mirror is filled with water that has an index of refraction equal to 1.33 and a maximum depth of 1.00 cm. At what height above the vertex of the mirror must an object be placed so that its image is at the same position as the object?

Picture the Problem Assume that the object is very small compared to r so that all incident and reflected rays traverse 1 cm of water. The problem involves two refractions at the air–water interface and one reflection at the mirror. Let the numeral 1 refer to the first refraction at the air–water interface, the numeral 2 to the reflection in the mirror surface, and the numeral 3 to the second refraction at the water–air interface.

Use the equation for refraction at a single surface to relate s_1 and s_1':

$$\frac{n_1}{s_1}+\frac{n_2}{s_1'}=\frac{n_2-n_1}{r}$$

or, because $r=\infty$,

$$\frac{n_1}{s_1}+\frac{n_2}{s_1'}=0\Rightarrow s_1'=-\frac{n_2 s_1}{n_1}$$

Let $n_2 = n$. Because $n_1 = 1$:

$$s_1'=-ns_1$$

Find the object distance for the mirror:

$$s_2=1-s_1'=1+ns_1$$

where 1 has units of cm.

Solve the mirror equation for s_2':

$$s_2'=\left(\frac{2}{r}-\frac{1}{s_2}\right)^{-1}$$

Substitute for s_2:

$$s_2'=\left(\frac{2}{r}-\frac{1}{1+ns_1}\right)^{-1}$$

Find the object distance s_3 for the water–air interface:

$$s_3=1-s_2'=1-\left(\frac{2}{r}-\frac{1}{1+ns_1}\right)^{-1}$$

Use the equation for refraction at a single surface to relate s_3 and s_3':

$$\frac{n_1}{s_3}+\frac{n_2}{s_3'}=\frac{n_2-n_1}{r}$$

or, because $r=\infty$,

$$\frac{n_1}{s_3}+\frac{n_2}{s_3'}=0\Rightarrow s_3'=-\frac{n_2 s_3}{n_1}$$

Because $n_2 = 1$ and $n_1 = n$:

$$s_3{}' = -\frac{s_3}{n} = -\frac{1-\left(\frac{2}{r}-\frac{1}{1+ns_1}\right)^{-1}}{n}$$

Equate $s_3{}'$ and s_1:

$$s_1 = -\frac{1-\left(\frac{2}{r}-\frac{1}{1+ns_1}\right)^{-1}}{n}$$

Simplifying yields:

$$s_1^2 + \frac{2-r}{n}s_1 + \frac{1-r}{n^2} = 0$$

Substitute numerical values and simplify:

$$s_1^2 + \frac{2.00\,\text{cm} - 50.0\,\text{cm}}{1.33}s_1 + \frac{1.00\,\text{cm} - 50.0\,\text{cm}}{(1.33)^2} = 0$$

or

$$s_1^2 - 36.09s_1 - 27.70 = 0$$

where s_1 is in cm.

Solve for the positive value of s_1 to obtain:

$$s_1 = \boxed{37\,\text{cm}}$$

104 ••• A concave side of a lens has a radius of curvature of magnitude equal to 17.0 cm, and the convex side of the lens that has a radius of curvature of magnitude equal to 8.00 cm. The focal length of the lens in air is 27.5 cm. When the lens is placed in a liquid that has an unknown index of refraction, the focal length increases to 109 cm. What is the index of refraction of the liquid?

Picture the Problem We can use the lens maker's equation, in conjunction with the result given in Problem 108, to find the index of refraction of the liquid.

Solve the lens-maker's equation for n:

$$n = \frac{n_{\text{air}}}{f\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} + n_{\text{air}}$$

Substitute numerical values and evaluate n:

$$n = \frac{1.00}{(27.5\,\text{cm})\left(\frac{1}{-17.0\,\text{cm}} - \frac{1}{-8.00\,\text{cm}}\right)} + 1.00 = 1.55$$

From Problem 98, the focal length of the lens in the liquid, f_L, is related to the focal length of the lens in air, f, according to:

$$f_L = \frac{n_L(n - n_{air})}{n_{air}(n - n_L)} f$$

Solving for n_L yields:

$$n_L = \frac{n_{air} n f_L}{(n - n_{air}) f + n_{air} f_L}$$

Substitute numerical values and evaluate n_L:

$$n_L = \frac{(1.55)(1.00)(109\,\text{cm})}{(1.55 - 1.00)(27.5\,\text{cm}) + (1.00)(109\,\text{cm})} = \boxed{1.4}$$

105 ••• A solid glass ball of radius 10.0 cm has an index of refraction equal to 1.500. The right half of the ball is silvered so that it acts as a concave mirror (Figure 32-63). Find the position of the final image formed for an object located at (*a*) 40.0 cm, and (*b*) 30.0 cm to the left of the center of the ball.

Picture the Problem The problem involves two refractions and one reflection. We can use the refraction at spherical surface equation and the mirror equation to find the images formed in the two refractions and one reflection. Let the numeral 1 refer to the first refraction at the air-glass interface, the numeral 2 to the reflection from the silvered surface, and the numeral 3 refer to the refraction at the glass-air interface.

(*a*) The image and object distances for the refraction at the surface of the ball are related according to:

$$\frac{n_1}{s_1} + \frac{n_2}{s_1'} = \frac{n_2 - n_1}{r}$$

Solve for s_1' to obtain:

$$s_1' = \frac{n_2 r s_1}{(n_2 - n_1)s_1 - n_1 r}$$

Substitute numerical values and evaluate s_1':

$$s_1' = \frac{(1.500)(10.0\,\text{cm})(30.0\,\text{cm})}{(1.500 - 1)(30.0\,\text{cm}) - (1)(10.0\,\text{cm})} = 90.0\,\text{cm}$$

The object for the mirror surface is behind the mirror and its distance from the surface of the mirror is:

$$s_2 = 20.0\,\text{cm} - 90.0\,\text{cm} = -70.0\,\text{cm}$$

Use the mirror equation to relate s_2 and s_2':

$$\frac{1}{s_2}+\frac{1}{s_2'}=\frac{2}{r}\Rightarrow s_2'=\frac{rs_2}{2s_2-r}$$

Substitute numerical values and evaluate s_2':

$$s_2'=\frac{(10.0\,\text{cm})(-70.0\,\text{cm})}{2(-70.0\,\text{cm})-10.0\,\text{cm}}=4.67\,\text{cm}$$

The object for the second refraction at the glass-air interface is in front of the mirrored surface and its distance from the glass-air interface is:

$$s_3=20.0\,\text{cm}-4.67\,\text{cm}=15.3\,\text{cm}$$

The image and object distances for the second refraction are related according to:

$$\frac{n_1}{s_3}+\frac{n_2}{s_3'}=\frac{n_2-n_1}{r}$$

Solve for s_3' to obtain:

$$s_3'=\frac{n_2rs_3}{(n_2-n_1)s_3-n_1r}$$

Noting that $r=-10.0$ cm, substitute numerical values and evaluate s_3':

$$s_3'=\frac{(1)(-10.0\,\text{cm})(15.3\,\text{cm})}{(1.500-1)(15.3\,\text{cm})-(1.500)(-10.0\,\text{cm})}=-6.76\,\text{cm}$$

The final image is 3.2 cm to the left of the center of the ball.

(*b*) With $s_1 = 20.0$ cm:

$$s_1'=\frac{(1.500)(10.0\,\text{cm})(20.0\,\text{cm})}{(1.500-1)(20.0\,\text{cm})-(1)(10.0\,\text{cm})}$$
$$=\text{undefined}$$

Because s_1' is undefined, no image is formed when the object is 20.0 cm to the left of the glass ball. (Alternatively, an image is formed an infinite distance to the left of the ball.)

106 ••• (*a*) Show that a small change dn in the index of refraction of a lens material produces a small change in the focal length df given approximately by $df/f=-dn/(n-n_{\text{air}})$. (*b*) Use this result to estimate the focal length of a thin lens for blue light, for which $n = 1.530$, if the focal length for red light, for which $n = 1.470$, is 20.0 cm.

Picture the Problem We can solve the lens maker's equation for f and then differentiate with respect to n and simplify to obtain $df/f=-dn/(n-n_{\text{air}})$.

(*a*) The lens maker's equation is:

$$\frac{1}{f}=\left(\frac{n}{n_{\text{air}}}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \qquad (1)$$

Differentiating both sides gives:

$$-\frac{df}{f^2}=\frac{dn}{n_{\text{air}}}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \qquad (2)$$

Diving equation (2) by equation (1) and simplifying yields:

$$\frac{-\dfrac{df}{f^2}}{\dfrac{1}{f}}=\frac{\dfrac{dn}{n_{\text{air}}}\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)}{\left(\dfrac{n}{n_{\text{air}}}-1\right)\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)}=\frac{\dfrac{dn}{n_{\text{air}}}}{\dfrac{n}{n_{\text{air}}}-1}=\frac{\dfrac{dn}{n_{\text{air}}}}{\dfrac{n-n_{\text{air}}}{n_{\text{air}}}}$$

Simplifying both sides yields:

$$\frac{df}{f}=\boxed{-\frac{dn}{n-n_{\text{air}}}}$$

(*b*) Express the focal length for blue light in terms of the focal length for red light:

$$f_{\text{blue}}=f_{\text{red}}+\Delta f \qquad (1)$$

Approximate *df*/*f* by $\Delta f/f$ and *dn* by Δn to obtain:

$$\frac{\Delta f}{f}\approx -\frac{\Delta n}{n-n_{\text{air}}}$$

Solving for Δf yields :

$$\Delta f=-\frac{f\,\Delta n}{n-n_{\text{air}}}$$

Substitute for Δf in equation (1) to obtain:

$$f_{\text{blue}}=f_{\text{red}}-\frac{f_{\text{red}}\Delta n}{n-n_{\text{air}}}$$

$$=f_{\text{red}}\left(1-\frac{\Delta n}{n_{\text{red}}-n_{\text{air}}}\right)$$

Substitute numerical values and evaluate f_{blue}:

$$f_{\text{blue}}=(20.0\,\text{cm})\left(1-\frac{1.530-1.470}{1.470-1.00}\right)=\boxed{17\,\text{cm}}$$

107 ••• **[SSM]** The lateral magnification of a spherical mirror or a thin lens is given by $m = -s'/s$. Show that for objects of small horizontal extent, the longitudinal magnification is approximately $-m^2$. *Hint:* Show that $ds'/ds = -s'^2/s^2$.

Picture the Problem Examine the amount by which the image distance s' changes due to a change in s.

Solve the thin-lens equation for s':

$$s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1}$$

Differentiate s' with respect to s:

$$\frac{ds'}{ds} = \frac{d}{ds}\left[\left(\frac{1}{f} - \frac{1}{s}\right)^{-1}\right] = -\frac{1}{\left(\frac{1}{f} - \frac{1}{s}\right)^2}\frac{1}{s^2} = -\frac{s'^2}{s^2} = -m^2$$

$$\boxed{\text{The image of an object of length } \Delta s \text{ will have a length } -m^2\Delta s.}$$

Chapter 33
Interference and Diffraction

Conceptual Problems

1 • A phase difference due to path-length difference is observed for monochromatic visible light. Which phase difference requires the least (minimum) path length difference? (*a*) 90° (*b*) 180° (*c*) 270° (*d*) the answer depends on the wavelength of the light.

Determine the Concept The phase difference δ due to a path difference Δr are related according to $\delta/2\pi = \Delta r/\lambda$. Therefore, the least path length difference corresponds to the smallest phase difference. $\boxed{(a)}$ is correct.

2 • Which of the following pairs of light sources are coherent: (*a*) two candles, (*b*) one point source and its image in a plane mirror, (*c*) two pinholes uniformly illuminated by the same point source, (*d*) two headlights of a car, (*e*) two images of a point source due to reflection from the front and back surfaces of a soap film.

Determine the Concept Coherent sources have a constant phase difference. The pairs of light sources that satisfy this criterion are (*b*), (*c*), and (*e*).

3 • **[SSM]** The spacing between Newton's rings decreases rapidly as the diameter of the rings increases. Explain qualitatively why this occurs.

Determine the Concept The thickness of the air space between the flat glass and the lens is approximately proportional to the square of *d*, the diameter of the ring. Consequently, the separation between adjacent rings is proportional to 1/*d*.

4 • If the angle of a wedge-shaped air film, such as that in Example 32-2 is too large, fringes are not observed. Why?

Determine the Concept There are two possible reasons that fringes might not be observed. (1) The distance between adjacent fringes is so small that the fringes are not resolved by the eye. (2) Twice the thickness of the air space is greater than the coherence length of the light. If this is the case, fringes would be observed in the region close to the point where the thickness of the air space approaches zero.

5 • Why must a film that is used to observe interference colors be thin?

Determine the Concept Colors are observed when the light reflected off the front and back surfaces of the film interfere destructively for some wavelengths and

constructively for other wavelengths. For this interference to occur, the phase difference between the light reflected off the front and back surfaces of the film must be constant. This means that twice the thickness of the film must be less than the coherence length of the light. The film is called a thin film if twice its thickness is less than the coherence length of the light.

6 • A loop of wire is dipped in soapy water and held up so that the soap film is vertical. (*a*) Viewed by reflection with white light, the top of the film appears black. Explain why. (*b*) Below the black region are colored bands. Is the first band red or violet?

(*a*) The phase change due to reflection from the front surface of the film is 180°; the phase change due to reflection from the back surface of the film is 0°. As the film thins toward the top, the phase change due to the path length difference between the two reflected waves (the phase difference associated with the film's thickness) becomes negligible and the two reflected waves interfere destructively.

(*b*) The first constructive interference will arise when twice the thickness of the film is equal to half the wavelength of the color with the shortest wavelength. Therefore, the first band will be violet (shortest visible wavelength).

7 • **[SSM]** A two-slit interference pattern is formed using monochromatic laser light with a wavelength of 640 nm. At the second maximum from the central maximum, what is the path-length difference between the light coming from each of the slits? (*a*) 640 nm (*b*) 320 nm (*c*) 960 nm (*d*) 1280 nm.

Determine the Concept For constructive interference, the path difference is an integer multiple of λ; that is, $\Delta r = m\lambda$. For $m = 2$, $\Delta r = 2(640\text{ nm})$. $\boxed{(d)}$ is correct.

8 • A two-slit interference pattern is formed using monochromatic laser light with a wavelength of 640 nm. At the first minimum from the central maximum, what is the path-length difference between the light coming from each of the slits? (*a*) 640 nm (*b*) 320 nm (*c*) 960 nm (*d*) 1280 nm.

Determine the Concept For destructive interference, the path difference is an odd-integer multiple of $\frac{1}{2}\lambda$; that is $\Delta r = m\left(\frac{1}{2}\lambda\right), m = 1, 3, 5, \ldots$. For the first minimum, $m = 1$ and $\Delta r = \frac{1}{2}(640\text{ nm}) = 320\text{ nm}$. $\boxed{(b)}$ is correct.

9 • A two-slit interference pattern is formed using monochromatic laser light with a wavelength of 450 nm. What happens to the distance between the first maximum and the central maximum as the two slits are moved closer together? (*a*) The distance increases. (*b*) The distance decreases. (*c*) The distance remains the same.

Determine the Concept The relationship between the slit separation d and the angular position θ_m of each maximum is given by $d\sin\theta_m = m\lambda, m = 0,1,2,...$ (Equation 33-2). Because d and $\sin\theta_m$ are inversely proportional for a given wavelength and interference maximum (value of m), decreasing d increases $\sin\theta_m$ and θ_m. $\boxed{(a)}$ is correct.

10 • A two-slit interference pattern is formed using two different monochromatic lasers, one green and one red. Which color light has its first maximum closer to the central maximum? (*a*) Green, (*b*) red, (*c*) both maxima are in the same location.

Determine the Concept The relationship between the slit separation d, the angular position θ_m of each maximum, and the wavelength of the light illuminating the slits is given by $d\sin\theta_m = m\lambda,\ m = 0,1,2,...$ (Equation 33-2). Because λ and $\sin\theta_m$ are directly proportional for a given interference maximum (value of m) and the wavelength of green light is shorter than the wavelength of red light, $\boxed{(a)}$ is correct.

11 • A single slit diffraction pattern is formed using monochromatic laser light with a wavelength of 450 nm. What happens to the distance between the first maximum and the central maximum as the slit is made narrower? (*a*) The distance increases. (*b*) The distance decreases. (*c*) The distance remains the same.

Determine the Concept The relationship between the slit width a, the angular position θ_m of each maximum, and the wavelength of the light illuminating the slit is given by $a\sin\theta_m = m\lambda,\ m = 1,2,3,...$ (Equation 33-11). Because a and $\sin\theta_m$ are inversely proportional for a given diffraction maximum (value of m), narrowing the slit increases $\sin\theta_m$ and θ_m. $\boxed{(a)}$ is correct.

12 • Equation 33-2 which is $d \sin \theta_\mathrm{m} = m\lambda$, and Equation 33-11, which is $a \sin \theta_\mathrm{m} = m\lambda$, are sometimes confused. For each equation, define the symbols and explain the equation's application.

Determine the Concept Equation 33-2 expresses the condition for an intensity maximum in two-slit interference. Here d is the slit separation, λ the wavelength of the light, m an integer, and θ_m the angle at which the interference maximum appears. Equation 33-11 expresses the condition for an intensity minimum in single-slit diffraction. Here a is the width of the slit, λ the wavelength of the light, and θ_m the angle at which the minimum appears, and m is a nonzero integer.

13 • When a diffraction grating is illuminated by white light, the first-order maximum of green light (*a*) is closer to the central maximum than the first-order maximum of red light. (*b*) is closer to the central maximum than the first-order maximum of blue light. (*c*) overlaps the second-order maximum of red light. (*d*) overlaps the second-order maximum of blue light.

Picture the Problem We can solve $d\sin\theta = m\lambda$ for θ with $m = 1$ to express the location of the first-order maximum as a function of the wavelength of the light.

The interference maxima in a diffraction pattern are at angles θ given by:

$$d\sin\theta = m\lambda$$

where d is the separation of the slits and $m = 0, 1, 2, \ldots$

Solve for the angular location θ_1 of the first-order maximum :

$$\theta_1 = \sin^{-1}\left(\frac{\lambda}{d}\right)$$

Because $\lambda_{\text{green light}} < \lambda_{\text{red light}}$:

$\theta_{\text{green light}} < \theta_{\text{red light}}$ and $\boxed{(a)}$ is correct.

14 • A double-slit interference experiment is set up in a chamber that can be evacuated. Using light from a helium-neon laser, an interference pattern is observed when the chamber is open to air. As the chamber is evacuated, one will note that (*a*) the interference fringes remain fixed. (*b*) the interference fringes move closer together. (*c*) the interference fringes move farther apart. (*d*) the interference fringes disappear completely.

Determine the Concept

The distance on the screen to *m*th bright fringe is given by:

$$y_m = m\frac{\lambda_n L}{d}$$

where L is the distance from the slits to the screen, λ_n is the wavelength of the light in a medium whose index of refraction is n, and d is the separation of the slits.

The separation of the interference fringes is given by:

$$y_{m+1} - y_m = (m+1)\frac{\lambda_n L}{d} - m\frac{\lambda_n L}{d} = \frac{\lambda_n L}{d}$$

Because the index of refraction of a vacuum is slightly less than the index of refraction of air, the removal of air increases λ_n and, hence, $y_m - y_{m-1}$. $\boxed{(c)}$ is correct.

15 • **[SSM]** True or false:

(*a*) When waves interfere destructively, the energy is converted into heat energy.
(*b*) Interference patterns are observed only if the relative phases of the waves that superimpose remain constant.
(*c*) In the Fraunhofer diffraction pattern for a single slit, the narrower the slit, the wider the central maximum of the diffraction pattern.
(*d*) A circular aperture can produce both a Fraunhofer diffraction pattern and a Fresnel diffraction pattern.
(*e*) The ability to resolve two point sources depends on the wavelength of the light.

(*a*) False. When destructive interference of light waves occurs, the energy is no longer distributed evenly. For example, light from a two-slit device forms a pattern with very bright and very dark parts. There is practically no energy at the dark fringes and a great deal of energy at the bright fringe. The total energy over the entire pattern equals the energy from one slit plus the energy from the second slit. Interference re-distributes the energy.

(*b*) True.

(*c*) True. The width of the central maximum in the diffraction pattern is given by $\theta_m = \sin^{-1}\frac{m\lambda}{a}$ where *a* is the width of the slit. Hence, the narrower the slit, the wider the central maximum of the diffraction pattern.

(*d*) True.

(*e*) True. The critical angle for the resolution of two sources is directly proportional to the wavelength of the light emitted by the sources ($\alpha_c = 1.22\frac{\lambda}{D}$).

16 • You observe two very closely-spaced sources of white light through a circular opening using various filters. Which color filter is most likely to prevent your resolving the images on your retinas as coming from two distinct sources? (*a*) red (*b*) yellow (*c*) green (*d*) blue (*e*) The filter choice is irrelevant.

Determine the Concept The condition for the resolution of the two sources is given by Rayleigh's criterion: $\alpha_c = 1.22\lambda/D$ (Equation 33-25), where α_c is the critical angular separation and D is the diameter of the aperture. The larger the critical angle required for resolution, the less likely it is that you can resolve the sources as being two distinct sources. Because α_c and λ are directly proportional, the filter that passes the shorter wavelength light would be most likely to resolve the sources and the longer wavelength (red) would be most likely to prevent resolving the sources. $\boxed{(a)}$ is correct.

17 •• Explain why the ability to distinguish the two headlights of an oncoming car, at a given distance, is easier for the human eye at night than during daylight hours. Assume the headlights of the oncoming car are on during both daytime and nighttime hours.

Determine the Concept The condition for the resolution of the two sources is given by Rayleigh's criterion: $\alpha_c = 1.22\lambda/D$ (Equation 33-25), where α_c is the critical angular separation, D is the diameter of the aperture, and λ is the wavelength of the light coming from the objects, in this case headlights, to be resolved. Because the diameter of the pupils of your eyes are larger at night, the critical angle is smaller at night, which means that at night you can resolve the light as coming from two distinct sources when they are at a greater distance.

Estimation and Approximation

18 • It is claimed that the Great Wall of China is the only man made object that can be seen from space with the naked eye. Check to see if this claim is true, based on the resolving power of the human eye. Assume the observers are in low-Earth orbit that has an altitude of about 250 km.

Picture the Problem We'll assume that the diameter of the pupil of the eye is 5.0 mm and use the best-case scenario (the minimum resolvable width varies directly with the wavelength of the light reflecting from the object) that the wavelength of light is 400 nm (the lower limit for the human eye). Then we can use the expression for the minimum angular separation of two objects than can be resolved by the eye and the relationship between this angle and the width of an object and the distance from which it is viewed to support the claim.

Relate the width w of an object that can be seen at an altitude h to the critical angular separation α_c:

$$\tan\alpha_c = \frac{w}{h} \Rightarrow w = h\tan\alpha_c$$

The minimum angular separation α_c of two point objects that can just be resolved by an eye depends on the diameter D of the eye and the wavelength λ of light:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

Substitute for α_c in the expression for w to obtain:

$$w = h\tan\left(1.22\frac{\lambda}{D}\right)$$

Substitute numerical values and evaluate w_{min} for an altitude of 250 km:

$$w_{min} = (250\,\text{km})\tan\left(1.22\left(\frac{400\,\text{nm}}{5.0\,\text{mm}}\right)\right) \approx 24\,\text{m}$$

This claim is probably false. Because the minimum width that is resolvable from low-Earth orbit (250 km) is 24 m and the width of the Great Wall is 5 to 8 m high and 5 m wide, so this claim is likely false. However, it is easily seen using binoculars, and pictures can be taken of it using a camera. This is because both binoculars and cameras have apertures that are larger than the pupil of the human eye. (The Chinese astronaut Yang Liwei reported that he was not able to see the wall with the naked eye during the first Chinese manned space flight in 2003.)

19 •• [SSM] (*a*) Estimate how close an approaching car at night on a flat, straight stretch of highway must be before its headlights can be distinguished from the single headlight of a motorcycle. (*b*) Estimate how far ahead of you a car is if its two red taillights merge to look as if they were one.

Picture the Problem Assume a separation of 1.5 m between typical automobile headlights and tail lights, a nighttime pupil diameter of 5.0 mm, 550 nm for the wavelength of the light (as an average) emitted by the headlights, 640 nm for red taillights, and apply the Rayleigh criterion.

(*a*) The Rayleigh criterion is given by Equation 33-25:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

where D is the separation of the headlights (or tail lights).

The critical angular separation is also given by:

$$\alpha = \frac{d}{L}$$

where d is the separation of head lights (or tail lights) and L is the distance to approaching or receding automobile.

Equate these expressions for α_c to obtain:

$$\frac{d}{L} = 1.22\frac{\lambda}{D} \Rightarrow L = \frac{Dd}{1.22\lambda}$$

Substitute numerical values and evaluate L:

$$L = \frac{(5.0\,\text{mm})(1.5\,\text{m})}{1.22(550\,\text{nm})} \approx \boxed{11\,\text{km}}$$

(b) For red light:

$$L = \frac{(5.0\,\text{mm})(1.5\,\text{m})}{1.22(640\,\text{nm})} \approx \boxed{9.6\,\text{km}}$$

20 •• A small loudspeaker is located at a large distance to the east from you. The loudspeaker is driven by a sinusoidal current whose frequency can be varied. Estimate the lowest frequency for which your ears would receive the sound waves exactly out of phase when you are facing north.

Picture the Problem If your ears receive the sound exactly out of phase, the waves arriving at your ear that is farthest from the speaker must be traveling one-half wavelength farther than the waves arriving at your ear that is nearest the speaker. The lowest frequency corresponds to the longest wavelength. Assume that the speaker and your ears are on the same line and let the distance between your ears be about 20 cm. Take the speed of sound in air to be 343 m/s.

The frequency received by your ears is given by:

$$f = \frac{v}{\lambda}$$

Let Δr be the distance between your ears (the path difference) to obtain:

$$\Delta r = \tfrac{1}{2}\lambda \Rightarrow \lambda = 2\Delta r$$

Substituting for λ yields:

$$f = \frac{v}{2\Delta r}$$

Substitute numerical values and evaluate f:

$$f = \frac{343\,\text{m/s}}{2(20\,\text{cm})} = 0.86\,\text{kHz}$$

or between 0.80 and 0.90 kHz.

21 •• **[SSM]** Estimate the maximum distance a binary star system could be resolvable by the human eye. Assume the two stars are about fifty times further apart than Earth and Sun are. Neglect atmospheric effects. (A test similar to this ″eye test″ was used in ancient Rome to test for eyesight acuity before entering the army. A normal eye could just barely resolve two well-known close-together stars in the sky. Anyone who could not tell there were two stars was rejected. In this case, the stars were not a binary system, but the principle is the same.)

Picture the Problem Assume that the diameter of a pupil at night is 5.0 mm and that the wavelength of light is in the middle of the visible spectrum at about 550 nm. We can use the Rayleigh criterion for the separation of two sources and the geometry of the Earth-binary star system to derive an expression for the distance to the binary stars.

If the distance between the binary stars is represented by *d* and the Earth-star distance by *L*, then their angular separation is given by:

$$\alpha = \frac{d}{L}$$

The critical angular separation of the two sources is given by the Rayleigh criterion:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

For $\alpha = \alpha_c$:

$$\frac{d}{L} = 1.22\frac{\lambda}{D} \Rightarrow L = \frac{Dd}{1.22\lambda}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{(5.0\text{ mm})(50)(1.5\times10^{11}\text{ m})}{1.22(550\text{ nm})}$$

$$\approx 5.59\times10^{13}\text{ km}\times\frac{1\,c\cdot\text{y}}{9.461\times10^{15}\text{ m}}$$

$$\boxed{5.9\,c\cdot\text{y}}$$

Phase Difference and Coherence

22 • Light of wavelength 500 nm is incident normally on a film of water 1.00 μm thick. (*a*) What is the wavelength of the light in the water? (*b*) How many wavelengths are contained in the distance 2*t*, where *t* is the thickness of the film? (*c*) What is the phase difference between the wave reflected from the top of the air–water interface and the wave reflected from the bottom of the water–air interface in the region where the two reflected waves superpose?

Picture the Problem The wavelength of light in a medium whose index of refraction is *n* is the ratio of the wavelength of the light in air divided by *n*. The number of wavelengths of light contained in a given distance is the ratio of the distance to the wavelength of light in the given medium. The difference in phase between the two waves is the sum of a π phase shift in the reflected wave and a phase shift due to the additional distance traveled by the wave reflected from the bottom of the water–air interface.

(*a*) Express the wavelength of light in water in terms of the wavelength of light in air:

$$\lambda_{\text{water}} = \frac{\lambda_{\text{air}}}{n_{\text{water}}} = \frac{500\,\text{nm}}{1.33} = \boxed{376\,\text{nm}}$$

(*b*) Relate the number of wavelengths *N* to the thickness *t* of the film and the wavelength of light in water:

$$N=\frac{2t}{\lambda_{\text{water}}}=\frac{2(1.00\ \mu\text{m})}{376\,\text{nm}}=\boxed{5.32}$$

(*c*) Express the phase difference as the sum of the phase shift due to reflection and the phase shift due to the additional distance traveled by the wave reflected from the bottom of the water–air interface:

$$\delta=\delta_{\text{reflection}}+\delta_{\text{additional distance traveled}}$$
$$=\pi+\frac{2t}{\lambda_{\text{water}}}2\pi=\pi+2\pi N$$

Substitute for *N* and evaluate δ:

$$\delta=\pi\,\text{rad}+2\pi(5.32\,\text{rad})=11.64\pi\,\text{rad}$$
$$=\boxed{11.6\pi\,\text{rad}}$$

or, subtracting 11.64π rad from 12π rad,

$$\delta=0.4\pi\,\text{rad}=\boxed{1.1\ \text{rad}}$$

23 •• [SSM] Two coherent microwave sources both produce waves of wavelength 1.50 cm. The sources are located in the $z = 0$ plane, one at $x = 0$, $y = 15.0$ cm and the other at $x = 3.00$ cm, $y = 14.0$ cm. If the sources are in phase, find the difference in phase between these two waves for a receiver located at the origin.

Picture the Problem The difference in phase depends on the path difference according to $\delta=\frac{\Delta r}{\lambda}2\pi$. The path difference is the difference in the distances of (0, 15.0 cm) and (3.00 cm, 14.0 cm) from the origin.

Relate a path difference Δr to a phase shift δ:

$$\delta=\frac{\Delta r}{\lambda}2\pi$$

The path difference Δr is:

$$\Delta r=15.0\,\text{cm}-\sqrt{(3.00\,\text{cm})^2+(14.0\,\text{cm})^2}$$
$$=0.682\,\text{cm}$$

Substitute numerical values and evaluate δ:

$$\delta=\left(\frac{0.682\,\text{cm}}{1.50\,\text{cm}}\right)2\pi\approx\boxed{2.9\,\text{rad}}$$

Interference in Thin Films

24 • A wedge-shaped film of air is made by placing a small slip of paper between the edges of two flat plates of glass. Light of wavelength 700 nm is

incident normally on the glass plates, and interference fringes are observed by reflection. (*a*) Is the first fringe near the point of contact of the plates dark or bright? Why? (*b*) If there are five dark fringes per centimeter, what is the angle of the wedge?

Picture the Problem Because the *m*th fringe occurs when the path difference 2*t* equals *m* wavelengths, we can express the additional distance traveled by the light in air as an $m\lambda$. The thickness of the wedge, in turn, is related to the angle of the wedge and the distance from its vertex to the *m*th fringe.

(*a*) The first fringe is dark because the phase difference due to reflection by the bottom surface of the top plate and the top surface of the bottom plate is 180°

(*b*) The *m*th fringe occurs when the path difference 2*t* equals *m* wavelengths:

$$2t = m\lambda$$

Relate the thickness of the air wedge to the angle of the wedge:

$$\theta = \frac{t}{x} \Rightarrow t = x\theta$$

where we've used a small-angle approximation to replace an arc length by the length of a chord.

Substitute for *t* to obtain:

$$2x\theta = m\lambda \Rightarrow \theta = \frac{m\lambda}{2x} = \frac{1}{2}\frac{m}{x}\lambda$$

Substitute numerical values and evaluate θ:

$$\theta = \frac{1}{2}\left(\frac{5}{\text{cm}}\right)(700\,\text{nm}) = \boxed{1.75\times10^{-4}\text{ rad}}$$

25 •• [SSM] The diameters of fine fibers can be accurately measured using interference patterns. Two optically flat pieces of glass of length *L* are arranged with the wire between them, as shown in Figure 33-40. The setup is illuminated by monochromatic light, and the resulting interference fringes are observed. Suppose that *L* is 20.0 cm and that yellow sodium light (wavelength of 590 nm) is used for illumination. If 19 bright fringes are seen along this 20.0-cm distance, what are the limits on the diameter of the wire? *Hint: The nineteenth fringe might not be right at the end, but you do not see a twentieth fringe at all.*

Picture the Problem The condition that one sees *m* fringes requires that the path difference between light reflected from the bottom surface of the top slide and the top surface of the bottom slide is an integer multiple of a wavelength of the light.

The *m*th fringe occurs when the path difference 2*d* equals *m* wavelengths:

$$2d = m\lambda \Rightarrow d = \frac{m\lambda}{2}$$

Because the nineteenth (but not the twentieth) bright fringe can be seen, the limits on *d* must be:

$$\left(m - \tfrac{1}{2}\right)\frac{\lambda}{2} < d < \left(m + \tfrac{1}{2}\right)\frac{\lambda}{2}$$

where $m = 19$

Substitute numerical values to obtain:

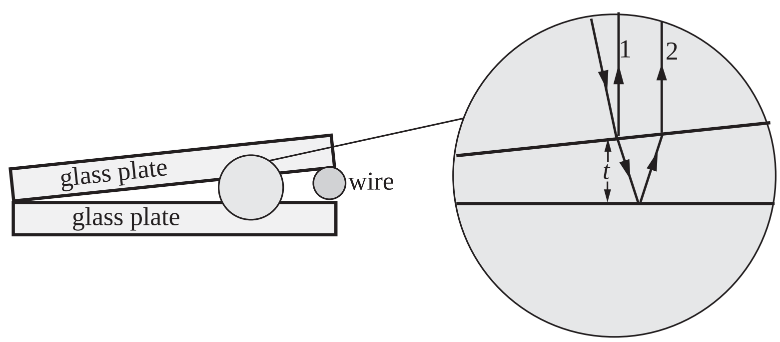
$$\left(19 - \tfrac{1}{2}\right)\frac{590\,\text{nm}}{2} < d < \left(19 + \tfrac{1}{2}\right)\frac{590\,\text{nm}}{2}$$

or

$$\boxed{5.5\,\mu\text{m} < d < 5.8\,\mu\text{m}}$$

26 •• Light that has a wavelength equal to 600 nm is used to illuminate two glass plates at normal incidence. The plates are 22 cm in length, touch at one end, and are separated at the other end by a wire that has a radius of 0.025 mm. How many bright fringes appear along the total length of the plates?

Picture the Problem The light reflected from the top surface of the bottom plate (wave 2 in the diagram) is phase shifted relative to the light reflected from the bottom surface of the top plate (wave 1 in the diagram). This phase difference is the sum of a phase shift of π (equivalent to a $\lambda/2$ path difference) resulting from reflection plus a phase shift due to the additional distance traveled.



Relate the extra distance traveled by wave 2 to the distance equivalent to the phase change due to reflection and to the condition for constructive interference:

$$2t + \tfrac{1}{2}\lambda = \lambda, 2\lambda, 3\lambda, \ldots$$

or

$$2t = \tfrac{1}{2}\lambda, \tfrac{3}{2}\lambda, \tfrac{5}{2}\lambda, \ldots$$

and

$$2t = \left(m + \tfrac{1}{2}\right)\lambda \text{ where } m = 0, 1, 2, \ldots,$$

$0 \le t \le 2r$ and λ is the wavelength of light in air.

Solving for m gives:

$$m=\frac{2t}{\lambda}-\frac{1}{2}$$

where m = 0, 1, 2, …and $0\le t\le 2r$

Solve for the highest value of m:

$$m_{\max}=\operatorname{int}\left(\frac{2(2r)}{\lambda}-\frac{1}{2}\right)=\operatorname{int}\left(\frac{4r}{\lambda}-\frac{1}{2}\right)$$

where r is the radius of the wire.

Substitute numerical values and evaluate m:

$$m_{\max}=\operatorname{int}\left(\frac{4(0.025\,\text{mm})}{600\,\text{nm}}-\frac{1}{2}\right)$$
$$=\operatorname{int}(166.2)=166$$

Because we start counting from m = 0, the number of bright fringes is $m_{\max}$ + 1.

$$N=m_{\max}+1=\boxed{167}$$

27 •• A thin film having an index of refraction of 1.50 is surrounded by air. It is illuminated normally by white light. Analysis of the reflected light shows that the wavelengths 360, 450, and 602 nm are the only missing wavelengths in or near the visible portion of the spectrum. That is, for these wavelengths, there is destructive interference. (*a*) What is the thickness of the film? (*b*) What visible wavelengths are brightest in the reflected interference pattern? (*c*) If this film were resting on glass with an index of refraction of 1.60, what wavelengths in the visible spectrum would be missing from the reflected light?

Picture the Problem (*a*) We can use the condition for destructive interference in a thin film to find the thickness of the film. (*b*) and (*c*) Once we've found the thickness of the film, we can use the condition for constructive interference to find the wavelengths in the visible portion of the spectrum that will be brightest in the reflected interference pattern and the condition for destructive interference to find the wavelengths of light missing from the reflected light when the film is placed on glass with an index of refraction greater than that of the film.

(*a*) Express the condition for destructive interference in the thin film:

$$2t + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\lambda', \tfrac{3}{2}\lambda', \tfrac{5}{2}\lambda', ...$$

or

$$2t = \lambda', 2\lambda', 3\lambda', ...$$

or

$$2t = m\lambda' = m\frac{\lambda}{n} \qquad (1)$$

where m = 1, 2, 3, … and λ' is the wavelength of the light in the film.

Solving for λ yields:

$$\lambda = \frac{2nt}{m}$$

Substitute for the missing wavelengths to obtain:

$$450\,\text{nm} = \frac{2nt}{m} \text{ and } 360\,\text{nm} = \frac{2nt}{m+1}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{450\,\text{nm}}{360\,\text{nm}} = \frac{\frac{2nt}{m}}{\frac{2nt}{m+1}} = \frac{m+1}{m}$$

Solving for m gives:

$$m = 4 \text{ for } \lambda = 450\,\text{nm}$$

Solve equation (1) for t to obtain:

$$t = \frac{m\lambda}{2n}$$

Substitute numerical values and evaluate t:

$$t = \frac{4(450\,\text{nm})}{2(1.50)} = \boxed{600\,\text{nm}}$$

(*b*) The condition for constructive interference in the thin film is:

$$2t + \tfrac{1}{2}\lambda' = \lambda', 2\lambda', 3\lambda', ...$$

or

$$2t = \tfrac{1}{2}\lambda', \tfrac{3}{2}\lambda', \tfrac{5}{2}\lambda', ... = \left(m + \tfrac{1}{2}\right)\lambda'$$

where λ' is the wavelength of light in the oil and m = 0, 1, 2, …

Substitute for λ' to obtain:

$$2t = \left(m + \tfrac{1}{2}\right)\frac{\lambda}{n} \Rightarrow \lambda = \frac{2nt}{m + \frac{1}{2}}$$

where n is the index of refraction of the film.

Substitute numerical values and simplify to obtain:

$$\lambda = \frac{2(1.50)(600\,\text{nm})}{m + \frac{1}{2}} = \frac{1800\,\text{nm}}{m + \frac{1}{2}}$$

Substitute numerical values for m and evaluate λ to obtain the following table:

m	0	1	2	3	4	5
λ (nm)	3600	1200	720	514	400	327

From the table, we see that the only wavelengths in the visible spectrum are 720 nm, 514 nm, and 400 nm.

(*c*) Because the index of refraction of the glass is greater than that of the film, the light reflected from the film-glass interface will be shifted by $\frac{1}{2}\lambda$ (as is the wave reflected from the top surface) and the condition for destructive interference becomes:

$$2t = \tfrac{1}{2}\lambda', \tfrac{3}{2}\lambda', \tfrac{5}{2}\lambda', ...$$

or

$$2t = \left(m + \tfrac{1}{2}\right)\frac{\lambda}{n}$$

where n is the index of refraction of the film and m = 0, 1, 2, …

Solving for λ yields:

$$\lambda = \frac{2nt}{m + \frac{1}{2}}$$

Substitute numerical values and simplify to obtain:

$$\lambda = \frac{2(1.5)(600\,\text{nm})}{m + \frac{1}{2}} = \frac{1800\,\text{nm}}{m + \frac{1}{2}}$$

Substitute numerical values for m and evaluate λ to obtain the following table:

m	0	1	2	3	4	5
λ (nm)	3600	1200	720	514	400	327

From the table we see that the missing wavelengths in the visible spectrum are 720 nm, 514 nm, and 400 nm.

28 •• A drop of oil (refractive index of 1.22) floats on water (refractive index of 1.33). When reflected light is observed from above, as shown in Figure 33-41 what is the thickness of the drop at the point where the second red fringe, counting from the edge of the drop, is observed? Assume red light has a wavelength of 650 nm.

Picture the Problem Because there is a $\frac{1}{2}\lambda$ phase change due to reflection at both the air-oil and oil-water interfaces, the condition for constructive interference is that twice the thickness of the oil film equal an integer multiple of the wavelength of light in the film.

The condition for constructive interference is:

$$2t = \lambda', 2\lambda', 3\lambda', ...$$

or

$$2t = m\lambda' \qquad (1)$$

where λ' is the wavelength of light in the oil and m = 1, 2, 3, …

Substitute for λ' to obtain:

$$2t = m\frac{\lambda}{n} \Rightarrow t = \frac{m\lambda}{2n}$$

Substitute numerical values and evaluate t:

$$t = \frac{(2)(650\,\text{nm})}{2(1.22)} = \boxed{533\,\text{nm}}$$

29 •• [SSM] A film of oil that has an index of refraction of 1.45 rests on an optically flat piece of glass with an index of refraction of 1.60. When illuminated by white light at normal incidence, light of wavelengths 690 nm and 460 nm is predominant in the reflected light. Determine the thickness of the oil film.

Picture the Problem Because there is a $\frac{1}{2}\lambda$ phase change due to reflection at both the air-oil and oil-glass interfaces, the condition for constructive interference is that twice the thickness of the oil film equal an integer multiple of the wavelength of light in the film.

Express the condition for constructive interference:

$$2t = \lambda', 2\lambda', 3\lambda', ... = m\lambda' \qquad (1)$$

where λ' is the wavelength of light in the oil and m = 0, 1, 2, …

Substitute for λ' to obtain:

$$2t = m\frac{\lambda}{n} \Rightarrow \lambda = \frac{2nt}{m}$$

where n is the index of refraction of the oil.

Substitute for the predominant wavelengths to obtain:

$$690\,\text{nm} = \frac{2nt}{m} \text{ and } 460\,\text{nm} = \frac{2nt}{m+1}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{690\,\text{nm}}{460\,\text{nm}} = \frac{\frac{2nt}{m}}{\frac{2nt}{m+1}} = \frac{m+1}{m} \Rightarrow m = 2$$

Solve equation (1) for t:

$$t = \frac{m\lambda}{2n}$$

Substitute numerical values and evaluate t:

$$t=\frac{(2)(690\,\text{nm})}{2(1.45)}=\boxed{476\,\text{nm}}$$

30 •• A film of oil that has an index of refraction 1.45 floats on water. When illuminated with white light at normal incidence, light of wavelengths 700 nm and 500 nm is predominant in the reflected light. Determine the thickness of the oil film.

Picture the Problem Because the index of refraction of air is less than that of the oil, there is a phase shift of π rad ($\frac{1}{2}\lambda$) in the light reflected at the air-oil interface. Because the index of refraction of the oil is greater than that of the glass, there is no phase shift in the light reflected from the oil-glass interface. We can use the condition for constructive interference to determine m for λ = 700 nm and then use this value in our equation describing constructive interference to find the thickness t of the oil film.

Express the condition for constructive interference between the waves reflected from the air-oil interface and the oil-glass interface:

$$2t+\tfrac{1}{2}\lambda'=\lambda',2\lambda',3\lambda',...$$

or

$$2t=\tfrac{1}{2}\lambda',\tfrac{3}{2}\lambda',\tfrac{5}{2}\lambda',...=\left(m+\tfrac{1}{2}\right)\lambda' \qquad (1)$$

where λ' is the wavelength of light in the oil and m = 0, 1, 2, …

Substitute for λ' and solve for λ to obtain:

$$\lambda=\frac{2nt}{m+\frac{1}{2}}$$

Substitute the predominant wavelengths to obtain:

$$700\,\text{nm}=\frac{2nt}{m+\frac{1}{2}} \text{ and } 500\,\text{nm}=\frac{2nt}{m+\frac{3}{2}}$$

Divide the first of these equations by the second to obtain:

$$\frac{700\,\text{nm}}{500\,\text{nm}}=\frac{\dfrac{2nt}{m+\frac{1}{2}}}{\dfrac{2nt}{m+\frac{3}{2}}}=\frac{m+\frac{3}{2}}{m+\frac{1}{2}}\Rightarrow m=2$$

Solve equation (1) for t:

$$t=\left(m+\tfrac{1}{2}\right)\frac{\lambda}{2n}$$

Substitute numerical values and evaluate t:

$$t=\left(2+\tfrac{1}{2}\right)\frac{700\,\text{nm}}{2(1.45)}=\boxed{603\,\text{nm}}$$

Newton's Rings

31 •• [SSM] A Newton's ring apparatus consists of a plano-convex glass lens with radius of curvature R that rests on a flat glass plate, as shown in Figure 33-42. The thin film is air of variable thickness. The apparatus is illuminated from above by light from a sodium lamp that has a wavelength of 590 nm. The pattern is viewed by reflected light. (*a*) Show that for a thickness t the condition for a bright (constructive) interference ring is $2t = \left(m + \frac{1}{2}\right)\lambda$ where m = 0, 1, 2, . . .
(*b*) Show that for $t << R$, the radius r of a fringe is related to t by $r = \sqrt{2tR}$.
(*c*) For a radius of curvature of 10.0 m and a lens diameter of 4.00 cm. How many bright fringes would you see in the reflected light? (*d*) What would be the diameter of the sixth bright fringe? (*e*) If the glass used in the apparatus has an index of refraction n = 1.50 and water replaces the air between the two pieces of glass, explain qualitatively the changes that will take place in the bright-fringe pattern.

Picture the Problem This arrangement is essentially identical to a ″thin film″ configuration, except that the ″film″ is air. A phase change of 180° ($\frac{1}{2}\lambda$) occurs at the top of the flat glass plate. We can use the condition for constructive interference to derive the result given in (*a*) and use the geometry of the lens on the plate to obtain the result given in (*b*). We can then use these results in the remaining parts of the problem.

(*a*) The condition for constructive interference is:

$$2t + \tfrac{1}{2}\lambda = \lambda, 2\lambda, 3\lambda, ...$$

or

$$2t = \tfrac{1}{2}\lambda, \tfrac{3}{2}\lambda, \tfrac{5}{2}\lambda, ... = \left(m + \tfrac{1}{2}\right)\lambda$$

where λ is the wavelength of light in air and m = 0, 1, 2, …

Solving for t yields:

$$t = \boxed{\left(m + \tfrac{1}{2}\right)\frac{\lambda}{2}, m = 0, 1, 2, ...} \qquad (1)$$

(*b*) From Figure 33-42 we have:

$$r^2 + (R - t)^2 = R^2$$

or

$$R^2 = r^2 + R^2 - 2Rt + t^2$$

For $t << R$ we can neglect the last term to obtain:

$$R^2 \approx r^2 + R^2 - 2Rt \Rightarrow r = \boxed{\sqrt{2Rt}} \qquad (2)$$

(*c*) Square equation (2) and substitute for t from equation (1) to obtain:

$$r^2 = \left(m + \tfrac{1}{2}\right)R\lambda \Rightarrow m = \frac{r^2}{R\lambda} - \frac{1}{2}$$

Substitute numerical values and evaluate m:

$$m = \frac{(2.00\,\text{cm})^2}{(10.0\,\text{m})(590\,\text{nm})} - \frac{1}{2} = 67$$

and so there will be $\boxed{68}$ bright fringes.

(d) The diameter of the m^{th} fringe is:

$$D = 2r = 2\sqrt{\left(m + \tfrac{1}{2}\right)R\lambda}$$

Noting that m = 5 for the sixth fringe, substitute numerical values and evaluate D:

$$D = 2\sqrt{\left(5 + \tfrac{1}{2}\right)(10.0\,\text{m})(590\,\text{nm})} = \boxed{1.14\,\text{cm}}$$

(e) The wavelength of the light in the film becomes λ_{air}/n = 444 nm. The separation between fringes is reduced (the fringes would become more closely spaced.) and the number of fringes that will be seen is increased by a factor of 1.33.

32 •• A plano-convex glass lens of radius of curvature 2.00 m rests on an optically flat glass plate. The arrangement is illuminated from above with monochromatic light of 520-nm wavelength. The indexes of refraction of the lens and plate are 1.60. Determine the radii of the first and second bright fringe from the center in the reflected light.

Picture the Problem This arrangement is essentially identical to a ″thin film″ configuration, except that the ″film″ is air. A phase change of 180° ($\frac{1}{2}\lambda$) occurs at the top of the flat glass plate. We can use the condition for constructive interference and the results from Problem 31(b) to determine the radii of the first and second bright fringes in the reflected light.

The condition for constructive interference is:

$$2t + \tfrac{1}{2}\lambda = \lambda, 2\lambda, 3\lambda, \ldots$$

or

$$2t = \tfrac{1}{2}\lambda, \tfrac{3}{2}\lambda, \tfrac{5}{2}\lambda, \ldots = \left(m + \tfrac{1}{2}\right)\lambda$$

where λ is the wavelength of light in air and m = 0, 1, 2, …

Solving for t gives:

$$t = \left(m + \tfrac{1}{2}\right)\frac{\lambda}{2},\ m = 0, 1, 2, \ldots$$

In Problem 31(b) it was shown that:

$$r = \sqrt{2tR}$$

Substitute for t to obtain:

$$r = \sqrt{\left(m + \tfrac{1}{2}\right)\lambda R}$$

The first fringe corresponds to $m = 0$:

$$r = \sqrt{\tfrac{1}{2}(520\,\text{nm})(2.00\,\text{m})} = \boxed{0.721\,\text{mm}}$$

The second fringe corresponds to $m = 1$:

$$r = \sqrt{\tfrac{3}{2}(520\,\text{nm})(2.00\,\text{m})} = \boxed{1.25\,\text{mm}}$$

33 ••• Suppose that before the lens of Problem 32 is placed on the plate, a film of oil of refractive index 1.82 is deposited on the plate. What will then be the radii of the first and second bright fringes?

Picture the Problem This arrangement is essentially identical to a ″thin film″ configuration, except that the ″film″ is oil. A phase change of 180° ($\frac{1}{2}\lambda$) occurs at lens-oil interface. We can use the condition for constructive interference and the results from Problem 31(*b*) to determine the radii of the first and second bright fringes in the reflected light.

The condition for constructive interference is:

$$2t + \tfrac{1}{2}\lambda' = \lambda', 2\lambda', 3\lambda', ...$$

or

$$2t = \tfrac{1}{2}\lambda', \tfrac{3}{2}\lambda', \tfrac{5}{2}\lambda', ... = \left(m + \tfrac{1}{2}\right)\lambda'$$

where λ' is the wavelength of light in the oil and $m = 0, 1, 2, \ldots$

Substitute for λ' and solve for t:

$$t = \left(m + \tfrac{1}{2}\right)\frac{\lambda}{2n}, m = 0, 1, 2, ...$$

where λ is the wavelength of light in air.

In Problem 31(*b*) it was shown that:

$$r = \sqrt{2tR}$$

Substitute for t to obtain:

$$r = \sqrt{\left(m + \tfrac{1}{2}\right)\frac{\lambda R}{n}}$$

The first fringe corresponds to $m = 0$:

$$r = \sqrt{\frac{1}{2}\frac{(520\,\text{nm})(2.00\,\text{m})}{1.82}} = \boxed{0.535\,\text{mm}}$$

The second fringe corresponds to $m = 1$:

$$r = \sqrt{\frac{3}{2}\frac{(520\,\text{nm})(2.00\,\text{m})}{1.82}} = \boxed{0.926\,\text{mm}}$$

Two-Slit Interference Patterns

34 • Two narrow slits separated by 1.00 mm are illuminated by light of wavelength 600 nm, and the interference pattern is viewed on a screen 2.00 m

away. Calculate the number of bright fringes per centimeter on the screen in the region near the center fringe.

Picture the Problem The number of bright fringes per unit distance is the reciprocal of the separation of the fringes. We can use the expression for the distance on the screen to the *m*th fringe to find the separation of the fringes.

Express the number N of bright fringes per centimeter in terms of the separation of the fringes:

$$N=\frac{1}{\Delta y} \qquad (1)$$

Express the distance on the screen to the *m*th and (m + 1)st bright fringe:

$$y_m=m\frac{\lambda L}{d} \text{ and } y_{m+1}=(m+1)\frac{\lambda L}{d}$$

Subtract the first of these equations from the second to obtain:

$$\Delta y=\frac{\lambda L}{d}$$

Substitute in equation (1) to obtain:

$$N=\frac{d}{\lambda L}$$

Substitute numerical values and evaluate N:

$$N=\frac{1.00\,\text{mm}}{(600\,\text{nm})(2.00\,\text{m})}=\boxed{8.33\,\text{cm}^{-1}}$$

35 • **[SSM]** Using a conventional two-slit apparatus with light of wavelength 589 nm, 28 bright fringes per centimeter are observed near the center of a screen 3.00 m away. What is the slit separation?

Picture the Problem We can use the expression for the distance on the screen to the *m*th and (m + 1)st bright fringes to obtain an expression for the separation Δy of the fringes as a function of the separation of the slits d. Because the number of bright fringes per unit length N is the reciprocal of Δy, we can find d from N, λ, and L.

Express the distance on the screen to the *m*th and (m + 1)st bright fringe:

$$y_m=m\frac{\lambda L}{d} \text{ and } y_{m+1}=(m+1)\frac{\lambda L}{d}$$

Subtract the first of these equations from the second to obtain:

$$\Delta y=\frac{\lambda L}{d}\Rightarrow d=\frac{\lambda L}{\Delta y}$$

Because the number of fringes per unit length N is the reciprocal of Δy:

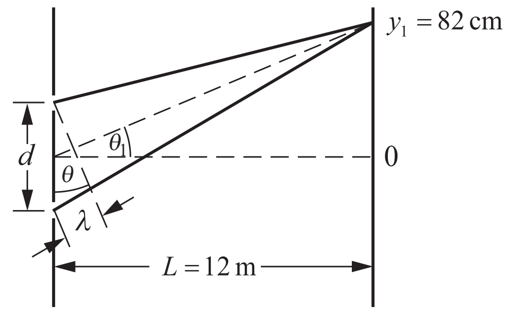
$$d=N\lambda L$$

Substitute numerical values and evaluate d:

$$d = (28\,\text{cm}^{-1})(589\,\text{nm})(3.00\,\text{m}) = \boxed{4.95\,\text{mm}}$$

36 • Light of wavelength 633 nm from a helium–neon laser is shone normally on a plane containing two slits. The first interference maximum is 82 cm from the central maximum on a screen 12 m away. (*a*) Find the separation of the slits. (*b*) How many interference maxima is it, in principle, possible to observe?

Picture the Problem We can use the geometry of the setup, represented to the right, to find the separation of the slits. To find the number of interference maxima that, in principle, can be observed, we can apply the equation describing two-slit interference maxima and require that $\sin\theta \le 1$.



Because $d << L$, we can approximate $\sin\theta_1$ as:

$$\sin\theta_1 \approx \frac{\lambda}{d} \Rightarrow d \approx \frac{\lambda}{\sin\theta_1} \qquad (1)$$

From the right triangle whose sides are L and y_1 we have:

$$\sin\theta_1 = \frac{82\,\text{cm}}{\sqrt{(12\,\text{m})^2 + (82\,\text{cm})^2}} = 0.06817$$

Substitute numerical values in equation (1) and evaluate d:

$$d \approx \frac{633\,\text{nm}}{0.06817} = 9.29\,\mu\text{m} = \boxed{9.3\,\mu\text{m}}$$

(*b*) The equation describing two-slit interference maxima is:

$$d\sin\theta = m\lambda, m = 0, 1, 2, \ldots$$

Because $\sin\theta \le 1$ determines the maximum number of interference fringes that can be seen:

$$d = m_{\text{max}}\lambda \Rightarrow m_{\text{max}} = \frac{d}{\lambda}$$

Substitute numerical values and evaluate m_{max}:

$$m_{\text{max}} = \frac{9.29\,\mu\text{m}}{633\,\text{nm}} = 14$$ because m must be an integer.

Because there are 14 fringes on either side of the central maximum:

$$N = 2m_{\text{max}} + 1 = 2(14) + 1 = \boxed{29}$$

37 •• Two narrow slits are separated by a distance *d*. Their interference pattern is to be observed on a screen a large distance *L* away. (*a*) Calculate the spacing between successive maxima near the center fringe for light of wavelength 500 nm, when *L* is 1.00 m and *d* is 1.00 cm. (*b*) Would you expect to be able to observe the interference of light on the screen for this situation? (*c*) How close together should the slits be placed for the maxima to be separated by 1.00 mm for this wavelength and screen distance?

Picture the Problem We can use the equation for the distance on a screen to the *m*th bright fringe to derive an expression for the spacing of the maxima on the screen. In (c) we can use this same relationship to express the slit separation *d*.

(*a*) Express the distance on the screen to the *m*th and (*m* + 1)st bright fringe:

$$y_m = m\frac{\lambda L}{d} \text{ and } y_{m+1} = (m+1)\frac{\lambda L}{d}$$

Subtract the first of these equations from the second to obtain:

$$\Delta y = \frac{\lambda L}{d} \qquad (1)$$

Substitute numerical values and evaluate Δy:

$$\Delta y = \frac{(500\,\text{nm})(1.00\,\text{m})}{1.00\,\text{cm}} = \boxed{50.0\,\mu\text{m}}$$

(*b*) According to the Raleigh criterion you could resolve them, but not by much.

(*c*) Solve equation (1) for *d* to obtain:

$$d = \frac{\lambda L}{\Delta y}$$

Substitute numerical values and evaluate *d*:

$$d = \frac{(500\,\text{nm})(1.00\,\text{m})}{1.00\,\text{mm}} = \boxed{0.500\,\text{mm}}$$

38 •• Light is incident at an angle ϕ with the normal to a vertical plane containing two slits of separation *d* (Figure 33-43). Show that the interference maxima are located at angles θ_m given by $\sin\theta_m + \sin\phi = m\lambda/d$.

Picture the Problem Let the separation of the slits be *d*. We can find the total path difference when the light is incident at an angle ϕ and set this result equal to an integer multiple of the wavelength of the light to obtain the given equation.

Express the total path difference:

$$\Delta\ell = d\sin\phi + d\sin\theta_m$$

The condition for constructive interference is:

$$\Delta\ell = m\lambda$$

where *m* is an integer.

Substitute to obtain: $d\sin\phi + d\sin\theta_{\mathrm{m}} = m\lambda$

Divide both sides of the equation by d to obtain:

$$\boxed{\sin\phi + \sin\theta_{\mathrm{m}} = \frac{m\lambda}{d}}$$

39 •• [SSM] White light falls at an angle of 30º to the normal of a plane containing a pair of slits separated by 2.50 μm. What visible wavelengths give a bright interference maximum in the transmitted light in the direction normal to the plane? (See Problem 38.)

Picture the Problem Let the separation of the slits be d. We can find the total path difference when the light is incident at an angle ϕ and set this result equal to an integer multiple of the wavelength of the light to relate the angle of incidence on the slits to the direction of the transmitted light and its wavelength.

Express the total path difference: $\Delta\ell = d\sin\phi + d\sin\theta$

The condition for constructive interference is: $\Delta\ell = m\lambda$ where m is an integer.

Substitute to obtain: $d\sin\phi + d\sin\theta = m\lambda$

Divide both sides of the equation by d to obtain:

$$\sin\phi + \sin\theta = \frac{m\lambda}{d}$$

Set $\theta = 0$ and solve for λ:

$$\lambda = \frac{d\sin\phi}{m}$$

Substitute numerical values and simplify to obtain:

$$\lambda = \frac{(2.50\,\mu\text{m})\sin 30^\circ}{m} = \frac{1.25\,\mu\text{m}}{m}$$

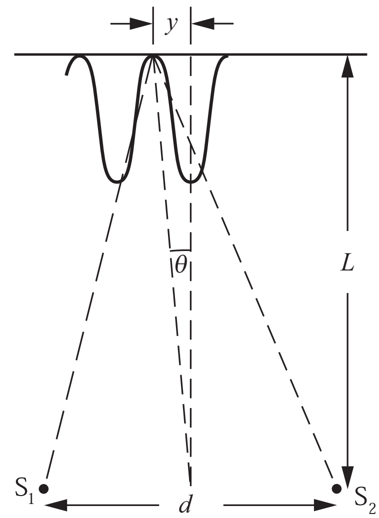
Evaluate λ for positive integral values of m:

m	λ (nm)
1	1250
2	625
3	417
4	313

From the table we can see that 625 nm and 417 nm are in the visible portion of the electromagnetic spectrum.

40 •• Two small loudspeakers are separated by 5.0 cm, as shown in Figure 33-44. The speakers are driven in phase with a sine wave signal of frequency 10 kHz. A small microphone is placed a distance 1.00 m away from the speakers on the axis running through the middle of the two speakers, and the microphone is then moved perpendicular to the axis. Where does the microphone record the first minimum and the first maximum of the interference pattern from the speakers? The speed of sound in air is 343 m/s.

Picture the Problem The diagram shows the two speakers, S_1 and S_2, the central-bright image and the first-order image to the left of the central-bright image. The distance *y* is measured from the center of the central-bright image. We can apply the conditions for constructive and destructive interference from two sources and use the geometry of the speakers and microphone to find the distance to the first interference minimum and the distance to the first interference maximum.



Relate the distance Δy to the first minimum from the center of the central maximum to θ and the distance L from the speakers to the plane of the microphone:

$$\tan\theta = \frac{y}{L} \Rightarrow y = L\tan\theta \qquad (1)$$

Interference minima occur where:

$$d\sin\theta = \left(m + \tfrac{1}{2}\right)\lambda$$
where m = 0, 1, 2, 3, …

Solve for θ to obtain:

$$\theta = \sin^{-1}\left[\frac{\left(m+\frac{1}{2}\right)\lambda}{d}\right]$$

Relate the wavelength λ of the sound waves to the speed of sound v and the frequency f of the sound:

$$\lambda = \frac{v}{f}$$

Substitute for λ in the expression for θ to obtain:

$$\theta = \sin^{-1}\left[\frac{\left(m+\frac{1}{2}\right)v}{df}\right]$$

Substituting for θ in equation (1) yields:

$$y = L\tan\left\{\sin^{-1}\left[\frac{\left(m+\frac{1}{2}\right)v}{df}\right]\right\} \qquad (2)$$

Noting that the first minimum corresponds to $m = 0$, substitute numerical values and evaluate Δy:

$$y_{\text{1st min}} = (1.00\,\text{m})\tan\left\{\sin^{-1}\left[\frac{\left(\frac{1}{2}\right)(343\,\text{m/s})}{(5.0\,\text{cm})(10\,\text{kHz})}\right]\right\} = \boxed{37\,\text{cm}}$$

The maxima occur where:

$$d\sin\theta = m\lambda$$
where $m = 1, 2, 3, \ldots$

For diffraction maxima, equation (2) becomes:

$$y = L\tan\left\{\sin^{-1}\left[\frac{mv}{af}\right]\right\}$$

Noting that the first maximum corresponds to $m = 1$, substitute numerical values and evaluate y:

$$y_{\text{1st max}} = (1.00\,\text{m})\tan\left\{\sin^{-1}\left[\frac{(1)(343\,\text{m/s})}{(5.0\,\text{cm})(10\,\text{kHz})}\right]\right\} = \boxed{94\,\text{cm}}$$

Diffraction Pattern of a Single Slit

41 • Light that has a 600-nm wavelength is incident on a long narrow slit. Find the angle of the first diffraction minimum if the width of the slit is (*a*) 1.0 mm, (*b*) 0.10 mm, and (*c*) 0.010 mm.

Picture the Problem We can use the expression locating the first zeroes in the intensity to find the angles at which these zeroes occur as a function of the slit width *a*.

The first zeroes in the intensity occur at angles given by:

$$\sin\theta = \frac{\lambda}{a} \Rightarrow \theta = \sin^{-1}\left(\frac{\lambda}{a}\right)$$

(*a*) For $a = 1.0$ mm:

$$\theta = \sin^{-1}\left(\frac{600\,\text{nm}}{1.0\,\text{mm}}\right) = \boxed{0.60\,\text{mrad}}$$

(*b*) For $a = 0.10$ mm:

$$\theta = \sin^{-1}\left(\frac{600\,\text{nm}}{0.10\,\text{mm}}\right) = \boxed{6.0\,\text{mrad}}$$

(*c*) For $a = 0.010$ mm:

$$\theta = \sin^{-1}\left(\frac{600\,\text{nm}}{0.010\,\text{mm}}\right) = \boxed{60\,\text{mrad}}$$

42 • Plane microwaves are incident on the thin metal sheet that has a long, narrow slit of width 5.0 cm in it. The microwave radiation strikes the sheet at normal incidence. The first diffraction minimum is observed at $\theta = 37°$. What is the wavelength of the microwaves?

Picture the Problem We can use the expression locating the first zeroes in the intensity to find the wavelength of the radiation as a function of the angle at which the first diffraction minimum is observed and the width of the plate.

The first zeroes in the intensity occur at angles given by:

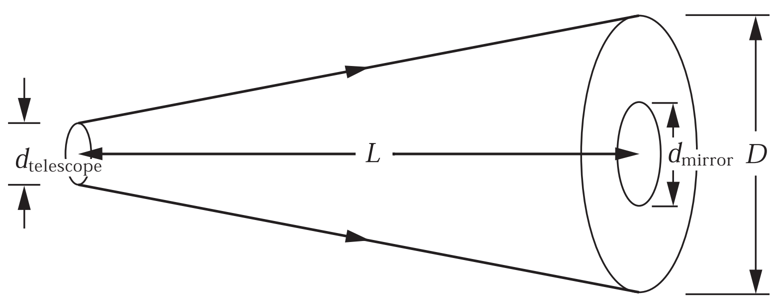
$$\sin\theta = \frac{\lambda}{a} \Rightarrow \lambda = a\sin\theta$$

Substitute numerical values and evaluate λ:

$$\lambda = (5.0\,\text{cm})\sin 37° = \boxed{3.0\,\text{cm}}$$

43 ••• **[SSM]** Measuring the distance to the moon (lunar ranging) is routinely done by firing short-pulse lasers and measuring the time it takes for the pulses to reflect back from the moon. A pulse is fired from Earth. To send the pulse out, the pulse is expanded so that it fills the aperture of a 6.00-in-diameter telescope. Assuming the only thing spreading the beam out is diffraction and that the light wavelength is 500 nm, how large will the beam be when it reaches the Moon, 3.82×10^5 km away?

Picture the Problem The diagram shows the beam expanding as it travels to the moon and that portion of it that is reflected from the mirror on the moon expanding as it returns to Earth. We can express the diameter of the beam at the moon as the product of the beam divergence angle and the distance to the moon and use the equation describing diffraction at a circular aperture to find the beam divergence angle.



Relate the diameter D of the beam when it reaches the moon to the distance to the moon L and the beam divergence angle θ:

$$D \approx \theta L \qquad (1)$$

The angle θ subtended by the first diffraction minimum is related to the wavelength λ of the light and the diameter of the telescope opening $d_{\text{telescope}}$ by:

$$\sin\theta = 1.22\frac{\lambda}{d_{\text{telescope}}}$$

Because $\theta << 1$, $\sin\theta \approx \theta$ and:

$$\theta \approx 1.22\frac{\lambda}{d_{\text{telescope}}}$$

Substitute for θ in equation (1) to obtain:

$$D = \frac{1.22L\lambda}{d_{\text{telescope}}}$$

Substitute numerical values and evaluate D:

$$D = \left(3.82\times10^{8}\text{ m}\right)\left[\frac{1.22(500\text{ nm})}{6.00\text{ in}\times\frac{2.54\text{ cm}}{\text{in}}\times\frac{1\text{ m}}{10^{2}\text{ cm}}}\right] = \boxed{1.53\text{ km}}$$

Interference-Diffraction Pattern of Two Slits

44 • How many interference maxima will be contained in the central diffraction maximum in the interference–diffraction pattern of two slits if the separation of the slits is exactly 5 times their width? How many will there be if the slit separation is an integral multiple of the slit width (that is $d = na$) for any value of n?

Picture the Problem We need to find the value of m for which the mth interference maximum coincides with the first diffraction minimum. Then there will be $N = 2m - 1$ fringes in the central maximum.

The number of fringes N in the central maximum is:

$$N = 2m - 1 \qquad (1)$$

Relate the angle θ_1 of the first diffraction minimum to the width a of the slits of the diffraction grating:

$$\sin\theta_1 = \frac{\lambda}{a}$$

Express the angle θ_m corresponding to the mth interference maxima in terms of the separation d of the slits:

$$\sin\theta_m = \frac{m\lambda}{d}$$

Because we require that $\theta_1 = \theta_m$, we can equate these expressions to obtain:

$$\frac{m\lambda}{d} = \frac{\lambda}{a} \Rightarrow m = \frac{d}{a}$$

Substituting for d and simplifying yields:

$$m = \frac{5a}{a} = 5$$

Substitute for m in equation (1) to obtain:

$$N = 2(5) - 1 = \boxed{9}$$

If $d = na$:

$$m = \frac{d}{a} = \frac{na}{a} = n \text{ and } N = \boxed{2n - 1}$$

45 •• [SSM] A two-slit Fraunhofer interference–diffraction pattern is observed using light that has a wavelength equal to 500 nm. The slits have a separation of 0.100 mm and an unknown width. (*a*) Find the width if the fifth interference maximum is at the same angle as the first diffraction minimum. (*b*) For that case, how many bright interference fringes will be seen in the central diffraction maximum?

Picture the Problem We can equate the sine of the angle at which the first diffraction minimum occurs to the sine of the angle at which the fifth interference maximum occurs to find a. We can then find the number of bright interference fringes seen in the central diffraction maximum using $N = 2m - 1$.

(*a*) Relate the angle θ_1 of the first diffraction minimum to the width a of the slits of the diffraction grating:

$$\sin\theta_1 = \frac{\lambda}{a}$$

Express the angle θ_5 corresponding to the m^{th} fifth interference maxima maximum in terms of the separation d of the slits:

$$\sin\theta_5 = \frac{5\lambda}{d}$$

Because we require that $\theta_1 = \theta_{m5}$, we can equate these expressions to obtain:

$$\frac{5\lambda}{d} = \frac{\lambda}{a} \Rightarrow a = \frac{d}{5}$$

Substituting the numerical value of d yields:

$$a = \frac{0.100\,\text{mm}}{5} = \boxed{20.0\,\mu\text{m}}$$

(*b*) Because $m = 5$:

$$N = 2m - 1 = 2(5) - 1 = \boxed{9}$$

46 •• A two-slit Fraunhofer interference–diffraction pattern is observed using light that has a wavelength equal to 700 nm. The slits have widths of 0.010 mm and are separated by 0.20 mm. How many bright fringes will be seen in the central diffraction maximum?

Picture the Problem We can equate the sine of the angle at which the first diffraction minimum occurs to the sine of the angle at which the *m*th interference maximum occurs to find *m*. We can then find the number of bright interference fringes seen in the central diffraction maximum using $N = 2m - 1$.

The number of fringes *N* in the central maximum is:

$$N = 2m - 1 \quad (1)$$

Relate the angle θ_1 of the first diffraction minimum to the width *a* of the slits of the diffraction grating:

$$\sin\theta_1 = \frac{\lambda}{a}$$

Express the angle θ_m corresponding to the *m*th interference maxima in terms of the separation *d* of the slits:

$$\sin\theta_m = \frac{m\lambda}{d}$$

Because we require that $\theta_1 = \theta_m$, we can equate these expressions to obtain:

$$\frac{m\lambda}{d} = \frac{\lambda}{a} \Rightarrow m = \frac{d}{a}$$

Substitute for *m* in equation (1) to obtain:

$$N = \frac{2d}{a} - 1$$

Substitute numerical values and evaluate *N*:

$$N = \frac{2(0.20\,\text{mm})}{0.010\,\text{mm}} - 1 = \boxed{39}$$

47 •• Suppose that the *central* diffraction maximum for two slits has 17 interference fringes for some wavelength of light. How many interference fringes would you expect in the diffraction maximum adjacent to one side of the central diffraction maximum?

Picture the Problem There are 8 interference fringes on each side of the central maximum. The secondary diffraction maximum is half as wide as the central one. It follows that it will contain 8 interference maxima.

48 •• Light that has a wavelength equal to 550 nm illuminates two slits that both have widths equal to 0.030 mm and separations equal to 0.15 mm. (*a*) How many interference maxima fall within the full width of the central diffraction maximum? (*b*) What is the ratio of the intensity of the third interference maximum to one side of the center interference maximum to the intensity of the center interference maximum?

Picture the Problem We can equate the sine of the angle at which the first diffraction minimum occurs to the sine of the angle at which the *m*th interference maximum occurs to find *m*. We can then find the number of bright interference fringes seen in the central diffraction maximum using $N = 2m - 1$. In (*b*) we can use the expression relating the intensity in a single-slit diffraction pattern to phase constant $\phi = \frac{2\pi}{\lambda} a \sin\theta$ to find the ratio of the intensity of the third interference maximum to one side of the center interference maximum.

(*a*) The number of fringes *N* in the central maximum is:

$$N = 2m - 1 \qquad (1)$$

Relate the angle θ_1 of the first diffraction minimum to the width *a* of the slits of the diffraction grating:

$$\sin\theta_1 = \frac{\lambda}{a}$$

Express the angle θ_m corresponding to the *m*th interference maxima in terms of the separation *d* of the slits:

$$\sin\theta_m = \frac{m\lambda}{d}$$

Because we require that $\theta_1 = \theta_m$, we can equate these expressions to obtain:

$$\frac{m\lambda}{d} = \frac{\lambda}{a} \Rightarrow m = \frac{d}{a}$$

Substitute in equation (1) to obtain:

$$N = \frac{2d}{a} - 1$$

Substitute numerical values and evaluate *N*:

$$N = \frac{2(0.15\,\text{mm})}{0.030\,\text{mm}} - 1 = \boxed{9}$$

(*b*) Express the intensity for a single-slit diffraction pattern as a function of the phase difference ϕ:

$$I = I_0\left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)^2 \qquad (2)$$

where $\phi = \frac{2\pi}{\lambda} a \sin\theta$

For $m = 3$:

$$\sin\theta_3 = \frac{3\lambda}{d}$$

and

$$\phi = \frac{2\pi}{\lambda}a\sin\theta_3 = \frac{2\pi}{\lambda}a\left(\frac{3\lambda}{d}\right) = 6\pi\left(\frac{a}{d}\right)$$

Substitute numerical values and evaluate ϕ:

$$\phi = 6\pi\left(\frac{0.030\,\text{mm}}{0.15\,\text{mm}}\right) = \frac{6\pi}{5}$$

Solve equation (2) for the ratio of I_3 to I_0:

$$\frac{I_3}{I_0} = \left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)^2$$

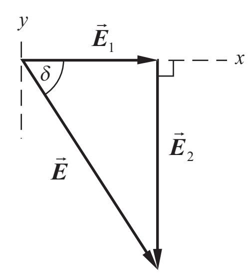
Substitute numerical values and evaluate I_3/I_0:

$$\frac{I_3}{I_0} = \left[\frac{\sin\frac{1}{2}\left(\frac{6\pi}{5}\right)}{\frac{1}{2}\left(\frac{6\pi}{5}\right)}\right]^2 = \boxed{0.25}$$

Using Phasors to Add Harmonic Waves

49 • [SSM] Find the resultant of the two waves whose electric fields at a given location vary with time as follows: $\vec{E}_1 = 2A_0\sin\omega t\,\hat{i}$ and $\vec{E}_2 = 3A_0\sin\left(\omega t + \frac{3}{2}\pi\right)\hat{i}$.

Picture the Problem Chose the coordinate system shown in the phasor diagram. We can use the standard methods of vector addition to find the resultant of the two waves.



The resultant of the two waves is of the form:

$$\vec{E} = \left|\vec{E}\right|\sin(\omega t + \delta)\boldsymbol{i} \qquad (1)$$

The magnitude of $\left|\vec{E}\right|$ is:

$$\left|\vec{E}\right| = \sqrt{(2A_0)^2 + (3A_0)^2} = 3.6A_0$$

The phase angle δ is:

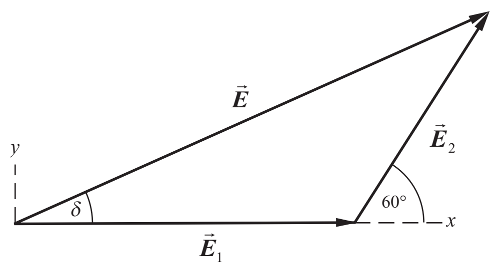
$$\delta = \tan^{-1}\left(\frac{-3A_0}{2A_0}\right) = -0.98\,\text{rad}$$

Substitute for $\left|\vec{E}\right|$ and δ in equation (1) to obtain:

$$\vec{E} = \boxed{3.6A_0 \sin(\omega t - 0.98\,\text{rad})\boldsymbol{i}}$$

50 • Find the resultant of the two waves whose electric fields at a given location vary with time as follows: $\vec{E}_1 = 4A_0 \sin \omega t\,\hat{\boldsymbol{i}}$ and $\vec{E}_2 = 3A_0 \sin\left(\omega t + \frac{1}{6}\pi\right)\hat{\boldsymbol{i}}$.

Picture the Problem Chose the coordinate system shown in the phasor diagram. We can use the standard methods of vector addition to find the resultant of the two waves.



The resultant of the two waves is of the form:

$$\vec{E} = \left|\vec{E}\right| \sin(\omega t + \delta)\boldsymbol{i} \qquad (1)$$

Apply the law of cosines to the magnitudes of the scalars to obtain:

$$\left|\vec{E}\right|^2 = \left|\vec{E}_1\right|^2 + \left|\vec{E}_2\right|^2 - 2\left|\vec{E}_1\right|\left|\vec{E}_2\right|\cos 120°$$

or

$$\left|\vec{E}\right| = \sqrt{\left|\vec{E}_1\right|^2 + \left|\vec{E}_2\right|^2 - 2\left|\vec{E}_1\right|\left|\vec{E}_2\right|\cos 120°}$$

Substitute for $\left|\vec{E}_1\right|$ and $\left|\vec{E}_2\right|$ and evaluate $\left|\vec{E}\right|$ to obtain:

$$\left|\vec{E}\right| = \sqrt{(4A_0)^2 + (3A_0)^2 - 2(4A_0)(3A_0)\cos 120°} = 6.08A_0$$

Applying the law of sines yields:

$$\frac{\sin\delta}{\left|\vec{E}_2\right|} = \frac{\sin 120°}{\left|\vec{E}\right|}$$

Solve for δ to obtain:

$$\delta = \sin^{-1}\left[\frac{\left|\vec{E}_2\right|\sin 120°}{\left|\vec{E}\right|}\right]$$

Substitute numerical values and evaluate δ:

$$\delta = \sin^{-1}\left[\frac{3A_0 \sin 120°}{6.08A_0}\right] = 0.43\,\text{rad}$$

Substitute for $|\vec{E}|$ and δ in equation (1) to obtain:

$$\vec{E} = \boxed{6.1A_0\sin(\omega t + 0.43\,\text{rad})\boldsymbol{i}}$$

Remarks: We could have used the law of cosines to find R and the law of sines to find δ.

51 •• Monochromatic light is incident on a sheet with a long narrow slit (Figure 33-45). Let I_0 be the intensity at the central maximum of the diffraction pattern on a distant screen, and let I be the intensity at the second intensity maximum from the central intensity maximum. The distance from this second intensity maximum to the far edge of the slit is longer than the distance from the second intensity maximum to the near edge of the slit by approximately 2.5 wavelengths. What is the ratio if I to I_0?

Picture the Problem We can evaluate the expression for the intensity for a single-slit diffraction pattern at the second secondary maximum to express I_2 in terms of I_0.

The intensity at the second secondary maximum is given by:

$$I = I_0\left[\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right]^2 \Rightarrow \frac{I}{I_0} = \left[\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right]^2$$

where

$$\phi = \frac{2\pi}{\lambda}a\sin\theta$$

At this second secondary maximum:

$$a\sin\theta = \frac{5}{2}\lambda \text{ and } \phi = \frac{2\pi}{\lambda}\left(\frac{5\lambda}{2}\right) = 5\pi$$

Substitute for ϕ and evaluate $\frac{I}{I_0}$:

$$\frac{I}{I_0} = \left[\frac{\sin\left(\frac{5\pi}{2}\right)}{\frac{5\pi}{2}}\right]^2 = \boxed{0.0162}$$

52 •• Monochromatic light is incident on a sheet that has three long narrow parallel equally spaced slits a distance d apart. (*a*) Show that the positions of the interference minima on a screen a large distance L away from the sheet that has the three equally spaced slits (spacing d, with $d >> \lambda$) are given approximately by $y_m = m\lambda L/3d$ where m = 1, 2, 4, 5, 7, 8, 10, . . . that is, m is *not* a multiple of 3. (*b*) For a screen distance of 1.00 m, a light wavelength of 500 nm, and a source spacing of 0.100 mm, calculate the width of the principal interference maxima (the distance between successive minima) for three sources.

Picture the Problem We can use phasor concepts to find the phase angle δ in terms of the number of phasors N (three in this problem) forming a closed polygon of N sides at the minima and then use this information to express the path difference Δr for each of these locations. Applying a small angle approximation, we can obtain an expression for y that we can evaluate for enough of the path differences to establish the pattern given in the problem statement.

(*a*) Express the phase angle δ in terms of the number of phasors N forming a closed polygon of N sides:

$$\delta = m\left(\frac{2\pi}{N}\right)$$

where m = 1, 2, 3, 4, 5, 6, ,7, …

For three equally spaced sources, the phase angle is:

$$\delta = m\left(\frac{2\pi}{3}\right)$$

Express the path difference corresponding to this phase angle to obtain:

$$\Delta r = \left(\frac{\lambda}{2\pi}\right)\delta = m\frac{\lambda}{3} \qquad (1)$$

Interference maxima occur for: m = 3, 6, 9, 12, …

Interference minima occur for: m = 1, 2, 4, 5, 7, 8, …
(Note that m is not a multiple of 3.)

Express the path difference Δr in terms of $\sin\theta$ and the separation d of the slits:

$$\Delta r = d\sin\theta$$

or, provided the small angle approximation is valid,

$$\Delta r = \frac{yd}{L} \Rightarrow y = \frac{L}{d}\Delta r$$

Substituting for Δr from equation (1) yields:

$$\boxed{y_{\text{min}} = \frac{m\lambda L}{3d},\ m = 1,2,4,5,7,8,...}$$

(*b*) For L = 1.00 m, λ = 500 nm, and d = 0.100 mm:

$$2y_{\text{min}} = \frac{2(500\,\text{nm})(1.00\,\text{m})}{3(0.100\,\text{mm})} = \boxed{3.33\,\text{mm}}$$

53 •• [SSM] Monochromatic light is incident on a sheet that has four long narrow parallel equally spaced slits a distance d apart. (*a*) Show that the positions of the interference minima on a screen a large distance L away from four equally spaced sources (spacing d, with $d >> \lambda$) are given by $y_m = m\lambda L/4d$ where m = 1, 2, 3, 5, 6, 7, 9, 10, . . . that is, m is *not* a multiple of 4. (*b*) For a screen distance of 2.00 m, light wavelength of 600 nm, and a source

spacing of 0.100 mm, calculate the width of the principal interference maxima (the distance between successive minima) for four sources. Compare this width with that for two sources with the same spacing.

Picture the Problem We can use phasor concepts to find the phase angle δ in terms of the number of phasors N (four in this problem) forming a closed polygon of N sides at the minima and then use this information to express the path difference Δr for each of these locations. Applying a small angle approximation, we can obtain an expression for y that we can evaluate for enough of the path differences to establish the pattern given in the problem statement.

(*a*) Express the phase angle δ in terms of the number of phasors N forming a closed polygon of N sides:

$$\delta = m\left(\frac{2\pi}{N}\right)$$

where m = 1, 2, 3, 4, 5, 6, ,7, …

For four equally spaced sources, the phase angle is:

$$\delta = m\left(\frac{\pi}{2}\right)$$

Express the path difference corresponding to this phase angle to obtain:

$$\Delta r = \left(\frac{\lambda}{2\pi}\right)\delta = m\frac{\lambda}{4} \qquad (1)$$

Interference maximum occur for: m = 3, 6, 9, 12, …

Interference minima occur for: m = 1, 2, 4, 5, 7, 8, …
(Note that m is not a multiple of 3.)

Express the path difference Δr in terms of $\sin\theta$ and the separation d of the slits:

$$\Delta r = d\sin\theta$$

or, provided the small angle approximation is valid,

$$\Delta r = \frac{yd}{L} \Rightarrow y = \frac{L}{d}\Delta r$$

Substituting for Δr from equation (1) yields:

$$\boxed{y_{\min} = \frac{m\lambda L}{4d},\ m = 1,2,3,5,6,7,9,...}$$

(*b*) For L = 2.00 m, λ = 600 nm, d = 0.100 mm, and n = 1:

$$2y_{\min} = \frac{2(600\,\text{nm})(2.00\,\text{m})}{4(0.100\,\text{mm})} = \boxed{6.00\,\text{mm}}$$

For two slits:

$$2y_{\min} = \frac{2\left(m+\frac{1}{2}\right)\lambda L}{d}$$

For L = 2.00 m, λ = 600 nm, d = 0.100 mm, and m = 0:

$$2y_{\text{min}} = \frac{(600\,\text{nm})(2.00\,\text{m})}{0.100\,\text{mm}} = 12.0\,\text{mm}$$

The width for four sources is half the width for two sources.

54 •• Light of wavelength 480 nm falls normally on four slits. Each slit is 2.00 μm wide and the center-to-center separation between it and the next slit is 6.00 μm. (*a*) Find the angular width of the of the central intensity maximum of the single-slit diffraction pattern on a distant screen. (*b*) Find the angular position of all interference intensity maxima that lie inside the central diffraction maximum. (*c*) Find the angular width of the central interference. That is, find the angle between the first intensity minima on either side of the central intensity maximum. (*d*) Sketch the relative intensity as a function of the sine of the angle.

Picture the Problem We can use $\sin\theta = \lambda/a$ to find the first zeros in the intensity pattern. The four-slit interference maxima occur at angles given by $d\sin\theta = m\lambda$, where m = 0,1,2, … . In (*c*) we can use the result of Problem 53 to find the angular spread between the central interference maximum and the first interference minimum on either side of it. In (*d*) we'll use a phasor diagram for a four-slit grating to find the resultant amplitude at a given point in the intensity pattern as a function of the phase constant δ, that, in turn, is a function of the angle θ that determines the location of a point in the interference pattern.

(*a*) The first zeros in the intensity occur at angles given by:

$$\sin\theta = \frac{\lambda}{a} \Rightarrow \theta = \sin^{-1}\left(\frac{\lambda}{a}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left(\frac{480\,\text{nm}}{2.00\,\mu\text{m}}\right) = \boxed{242\,\text{mrad}}$$

(*b*) The four-slit interference maxima occur at angles given by:

$$d\sin\theta = m\lambda \Rightarrow \theta_m = \sin^{-1}\left[\frac{m\lambda}{d}\right]$$

where m = 0, 1, 2, 3, …

Substitute numerical values to obtain:

$$\theta_m = \sin^{-1}\left[\frac{m(480\,\text{nm})}{6.00\,\mu\text{m}}\right] = \sin^{-1}(0.0800m)$$

Evaluate θ_m for m = 0, 1, 2, and 3:

$$\theta_0 = \sin^{-1}[0(0.0800)] = \boxed{0}$$
$$\theta_1 = \sin^{-1}[1(0.0800)] = \boxed{\pm 80.1\,\text{mrad}}$$
$$\theta_2 = \sin^{-1}[2(0.0800)] = \boxed{\pm 161\,\text{mrad}}$$
$$\theta_3 = \sin^{-1}[3(0.0800)] = \pm 0.242\,\text{rad}$$

where θ_3 will not be seen as it coincides with the first minimum in the diffraction pattern.

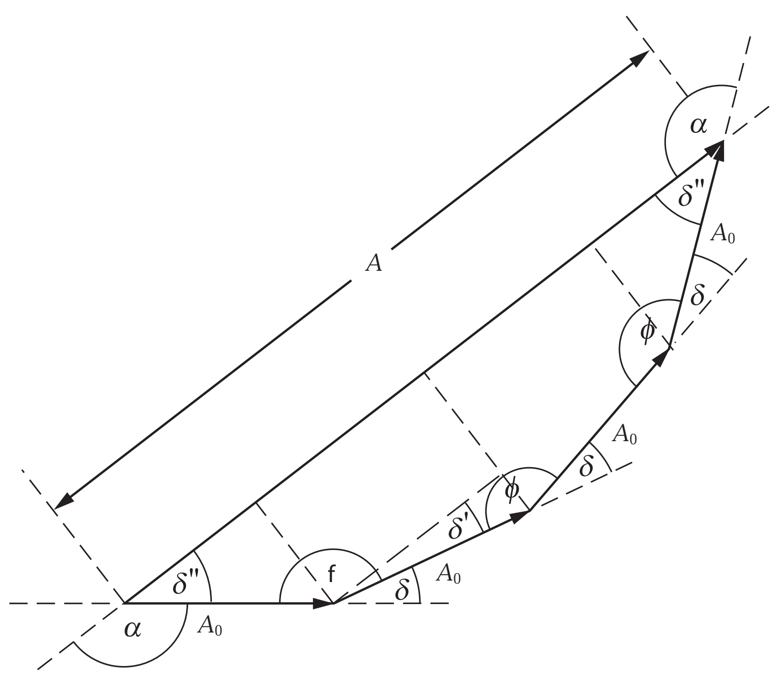
(c) From Problem 53:

$$\theta_{\text{min}} = \frac{n\lambda}{4d}$$

For n = 1:

$$\theta_{\text{min}} = \frac{480\,\text{nm}}{4(6.00\,\mu\text{m})} = \boxed{20\,\text{mrad}}$$

(d) Use the phasor method to show the superposition of four waves of the same amplitude A_0 and constant phase difference $\delta = \dfrac{2\pi}{\lambda} d \sin\theta$.



Express A in terms of δ' and δ'':

$$A = 2(A_0 \cos\delta'' + A_0 \cos\delta') \qquad (1)$$

Because the sum of the external angles of a polygon equals 2π:

$$2\alpha + 3\delta = 2\pi$$

Examining the phasor diagram we see that: $\alpha + \delta'' = \pi$

Eliminate α and solve for δ'' to obtain: $\delta'' = \frac{3}{2}\delta$

Because the sum of the internal angles of a polygon of *n* sides is $(n-2)\pi$: $3\phi + 2\delta'' = 3\pi$

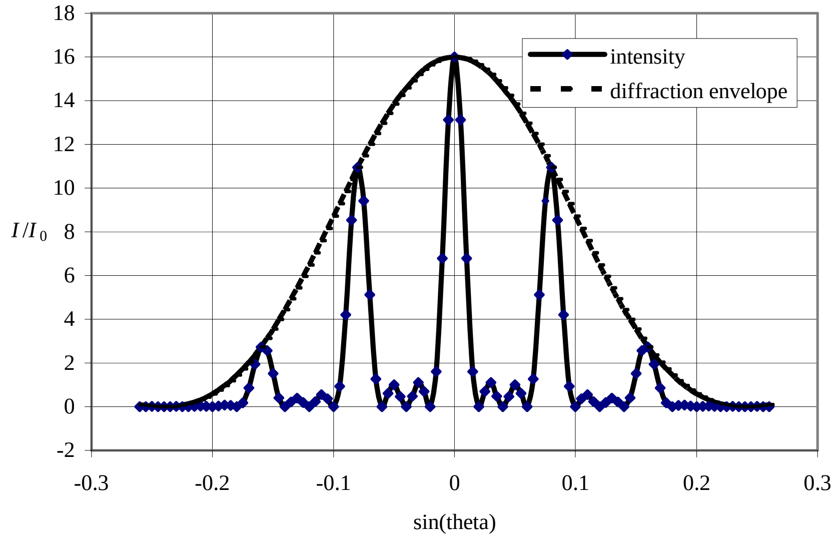
From the definition of a straight angle we have: $\phi - \delta' + \delta = \pi$

Eliminate ϕ between these equations to obtain: $\delta' = \frac{1}{2}\delta$

Substitute for δ'' and δ' in equation (1) to obtain: $A = 2A_0\left(\cos\frac{3}{2}\delta + \cos\frac{1}{2}\delta\right)$

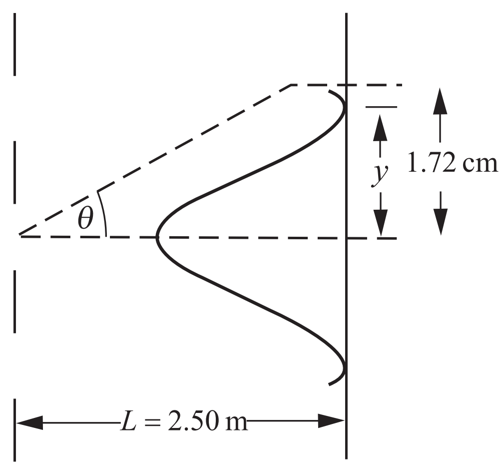
Because the intensity is proportional to the square of the amplitude of the resultant wave: $I = 4I_0\left(\cos\frac{3}{2}\delta + \cos\frac{1}{2}\delta\right)^2$

The following graph of I/I_0 as a function of $\sin\theta$ was plotted using a spreadsheet program. The diffraction envelope was plotted using $\frac{I}{I_0} = 4^2\left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)^2$, where $\phi = \frac{2\pi}{\lambda}a\sin\theta$. Note the excellent agreement with the results calculated in (*a*), (*b*) and (*c*).



55 ••• [SSM] Three slits, each separated from its neighbor by 60.0 μm, are illuminated at the central intensity maximum by a coherent light source of wavelength 550 nm. The slits are extremely narrow. A screen is located 2.50 m from the slits. The intensity is 50.0 mW/m^2. Consider a location 1.72 cm from the central maximum. (*a*) Draw a phasor diagram suitable for the addition of the three harmonic waves at that location. (*b*) From the phasor diagram, calculate the intensity of light at that location.

Picture the Problem We can find the phase constant δ from the geometry of the diagram to the right. Using the value of δ found in this fashion we can express the intensity at the point 1.72 cm from the centerline in terms of the intensity on the centerline. On the centerline, the amplitude of the resultant wave is 3 times that of each individual wave and the intensity is 9 times that of each source acting separately.



(*a*) Express δ for the adjacent slits:

$$\delta = \frac{2\pi}{\lambda} d \sin\theta$$

For $\theta << 1$, $\sin\theta \approx \tan\theta \approx \theta$:

$$\sin\theta \approx \tan\theta = \frac{y}{L}$$

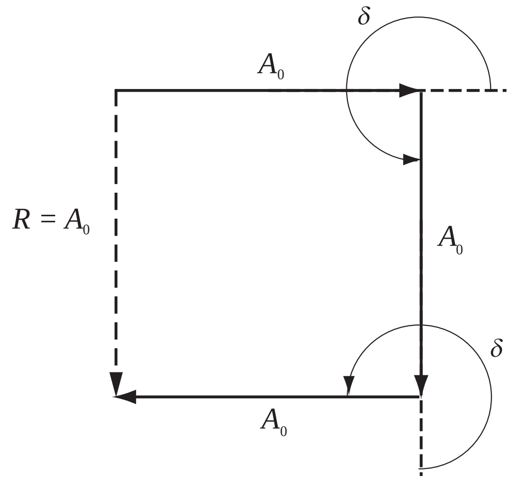
Substitute to obtain:

$$\delta = \frac{2\pi dy}{\lambda L}$$

Substitute numerical values and evaluate δ:

$$\delta = \frac{2\pi(60.0\,\mu\text{m})(1.72\,\text{cm})}{(550\,\text{nm})(2.50\,\text{m})} = \frac{3\pi}{2}\,\text{rad} = 270°$$

The three phasors, 270° apart, are shown in the diagram to the right. Note that they form three sides of a square. Consequently, their sum, shown as the resultant R, equals the magnitude of one of the phasors.



(b) Express the intensity at the point 1.72 cm from the centerline:

$$I \propto R^2$$

Because $I_0 \propto 9R^2$:

$$\frac{I}{I_0} = \frac{R^2}{9R^2} \Rightarrow I = \frac{I_0}{9}$$

Substitute for I_0 and evaluate I:

$$I = \frac{50.0\,\text{mW/m}^2}{9} = \boxed{5.56\,\text{mW/m}^2}$$

56 ••• In single-slit Fraunhofer diffraction, the intensity pattern (Figure 33-11) consists of a broad central maximum with a sequence of secondary maxima to either side of the central maximum. This intensity is given by $I = I_0\left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)^2$, where ϕ is the phase difference between the wavelets arriving from the opposite edges of the slits. Calculate the values of ϕ for the first three secondary maxima to one side of the central maximum by finding the values of ϕ for which $dI/d\phi$ is equal to zero. Check your results by comparing your answers with approximate values for ϕ of 3π, 5π and 7π. (That these values for ϕ are approximately correct at the secondary intensity maxima is discussed in the discussion surrounding Figure 33-27.)

Picture the Problem We can use the phasor diagram shown in Figure 33-26 to determine the first three values of ϕ that produce secondary intensity maxima. Setting the derivative of Equation 33-19 equal to zero will yield a transcendental equation whose roots are the values of ϕ corresponding to the intensity maxima in the diffraction pattern.

Referring to Figure 33-26 we see that the first subsidiary maximum occurs where:

$$\phi = 3\pi$$

An intensity minimum occurs where:

$$\phi = 4\pi$$

Another intensity maximum occurs where:

$$\phi = 5\pi$$

Thus, secondary intensity maxima occur where:

$$\phi = (2n+1)\pi,\ n = 1, 2, 3, \ldots$$

and the first three secondary intensity maxima are at $\phi = 3\pi, 5\pi,$ and 7π

The intensity in the single-slit diffraction pattern is given by:

$$I = I_0\left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)^2$$

Set the derivative of this expression equal to zero for extreme values (relative minima and maxima):

$$\frac{dI}{d\phi} = 2I_0\left(\frac{\sin\frac{1}{2}\phi}{\frac{1}{2}\phi}\right)\left[\frac{\frac{1}{4}\phi\cos\frac{1}{2}\phi - \frac{1}{2}\sin\frac{1}{2}\phi}{\left(\frac{1}{2}\phi\right)^2}\right] = 0 \text{ for extrema}$$

Simplify to obtain the transcendental equation:

$$\tan\tfrac{1}{2}\phi = \tfrac{1}{2}\phi$$

Solve this equation numerically (use the ″Solver″ function of your calculator or trial-and-error methods) to obtain:

$$\phi = 2.86\pi,\ 4.92\pi,\ \text{and}\ 6.94\pi$$

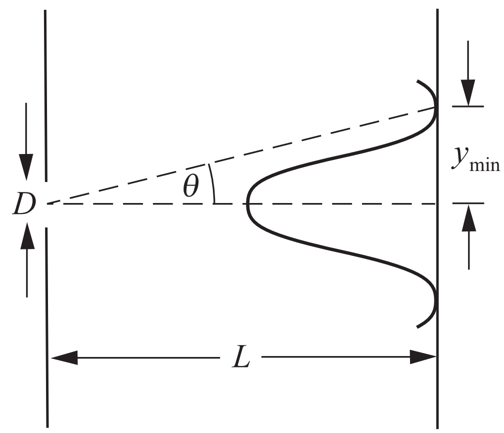
At the three intensity minima $\phi = 2\pi$, 4π, and 6π , and at the three intensity maxima $\phi = 2.86\pi$, 4.92π, and 6.94π. At the intensity maxima $\phi \approx 3\pi$, 5π, and 7π.

Remarks: Note that our results in (*b*) are smaller than the approximate values found in (*a*) by 4.9%, 1.6%, and 0.86% and that the agreement improves as *n* increases.

Diffraction and Resolution

57 • [SSM] Light that has a wavelength equal to 700 nm is incident on a pinhole of diameter 0.100 mm. (*a*) What is the angle between the central maximum and the first diffraction minimum for a Fraunhofer diffraction pattern? (*b*) What is the distance between the central maximum and the first diffraction minimum on a screen 8.00 m away?

Picture the Problem We can use $\theta = 1.22\dfrac{\lambda}{D}$ to find the angle between the central maximum and the first diffraction minimum for a Fraunhofer diffraction pattern and the diagram to the right to find the distance between the central maximum and the first diffraction minimum on a screen 8 m away from the pinhole.



(*a*) The angle between the central maximum and the first diffraction minimum for a Fraunhofer diffraction pattern is given by:

$$\theta = 1.22\frac{\lambda}{D}$$

Substitute numerical values and evaluate θ:

$$\theta = 1.22\left(\frac{700\,\text{nm}}{0.100\,\text{mm}}\right) = \boxed{8.54\,\text{mrad}}$$

(*b*) Referring to the diagram, we see that:

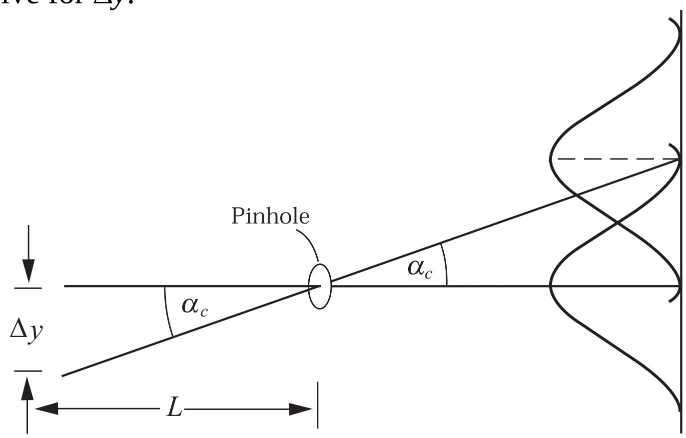
$$y_{\text{min}} = L\tan\theta$$

Substitute numerical values and evaluate y_{min}:

$$y_{\text{min}} = (8.00\,\text{m})\tan(8.54\,\text{mrad}) = \boxed{6.83\,\text{cm}}$$

58 • Two sources of light that both have wavelengths equal to 700 nm are 10.0 m away from the pinhole of Problem 57. How far apart must the sources be for their diffraction patterns to be resolved by Rayleigh's criterion?

Picture the Problem We can apply Rayleigh's criterion to the overlapping diffraction patterns and to the diameter D of the pinhole to obtain an expression that we can solve for Δy.



Rayleigh's criterion is satisfied provided:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

Relate α_c to the separation Δy of the light sources:

$$\alpha_c \approx \frac{\Delta y}{L} \text{ provided } \alpha_c << 1.$$

Equate these expressions to obtain:

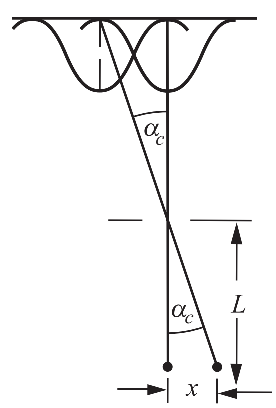
$$\frac{\Delta y}{L} = 1.22\frac{\lambda}{D} \Rightarrow \Delta y = 1.22\frac{\lambda L}{D}$$

Substitute numerical values and evaluate Δy:

$$\Delta y = 1.22\frac{(700\,\text{nm})(10.0\,\text{m})}{0.100\,\text{mm}} = \boxed{8.54\,\text{cm}}$$

59 • Two sources of light that both have wavelengths of 700 nm are separated by a horizontal distance x. They are 5.00 m from a vertical slit of width 0.500 mm. What is the smallest value of x for which the diffraction pattern of the sources can be resolved by Rayleigh's criterion?

Picture the Problem We can use Rayleigh's criterion for slits and the geometry of the diagram to the right showing the overlapping diffraction patterns to express x in terms of λ, L, and the width a of the slit.



Referring to the diagram, relate α_c, L, and x:

$$\alpha_c \approx \frac{x}{L}$$

For slits, Rayleigh's criterion is:

$$\alpha_c = \frac{\lambda}{a}$$

Equate these two expressions to obtain:

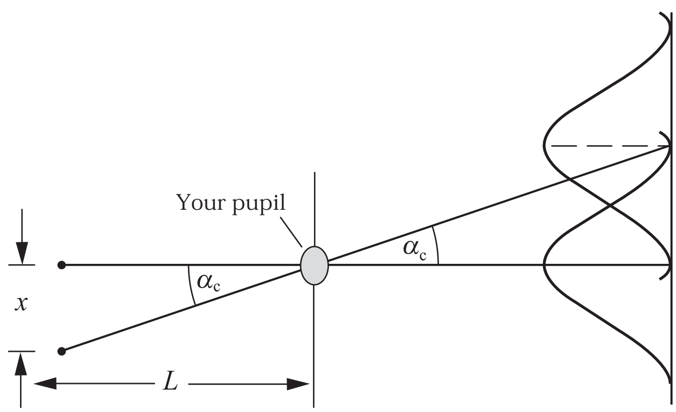
$$\frac{x}{L} = \frac{\lambda}{a} \Rightarrow x = \frac{\lambda L}{a}$$

Substitute numerical values and evaluate x:

$$x = \frac{(700\,\text{nm})(5.00\,\text{m})}{0.500\,\text{mm}} = \boxed{7.00\,\text{mm}}$$

60 •• The ceiling of your lecture hall is probably covered with acoustic tile, which has small holes separated by about 6.0 mm. (*a*) Using light that has a wavelength of 500 nm, how far could you be from this tile and still resolve these holes? Assume the diameter of the pupil of your eye is about 5.0 mm. (*b*) Could you resolve these holes better using red light or using violet light? Explain your answer.

Picture the Problem We can use Rayleigh's criterion for circular apertures and the geometry of the diagram to the right showing the overlapping diffraction patterns to express L in terms of λ, x, and the diameter D of your pupil.



(*a*) Referring to the diagram, relate α_c, L, and x:

$$\alpha_c \approx \frac{x}{L} \text{ provided } \alpha \ll 1$$

For circular apertures, Rayleigh's criterion is:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

Equate these two expressions to obtain:

$$\frac{x}{L} = 1.22\frac{\lambda}{D} \Rightarrow L = \frac{xD}{1.22\lambda}$$

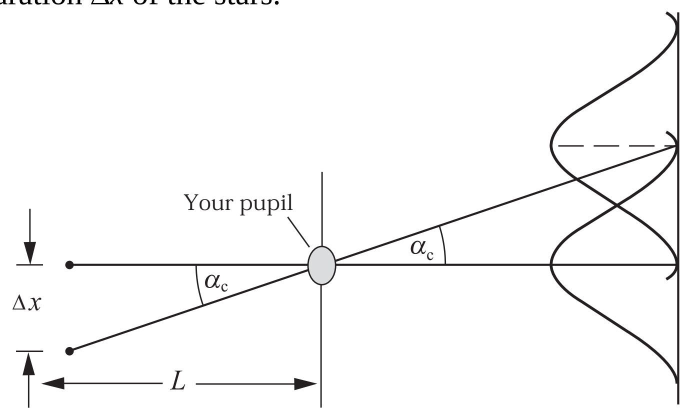
Substitute numerical values and evaluate L:

$$L = \frac{(6.0\,\text{mm})(5.0\,\text{mm})}{1.22(500\,\text{nm})} = \boxed{49\,\text{m}}$$

(b) Because L is inversely proportional to λ, the holes can be resolved better with violet light which has a shorter wavelength. The critical angle for resolution is proportional to the wavelength. Thus, the shorter the wavelength the farther away you can be and still resolve the two images.

61 •• [SSM] The telescope on Mount Palomar has a diameter of 200 in. Suppose a double star were 4.00 light-years away. Under ideal conditions, what must be the minimum separation of the two stars for their images to be resolved using light that has a wavelength equal to 550 nm?

Picture the Problem We can use Rayleigh's criterion for circular apertures and the geometry of the diagram to obtain an expression we can solve for the minimum separation Δx of the stars.



Rayleigh's criterion is satisfied provided:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

Relate α_c to the separation Δx of the light sources:

$$\alpha_c \approx \frac{\Delta x}{L} \text{ because } \alpha_c << 1$$

Equate these expressions to obtain:

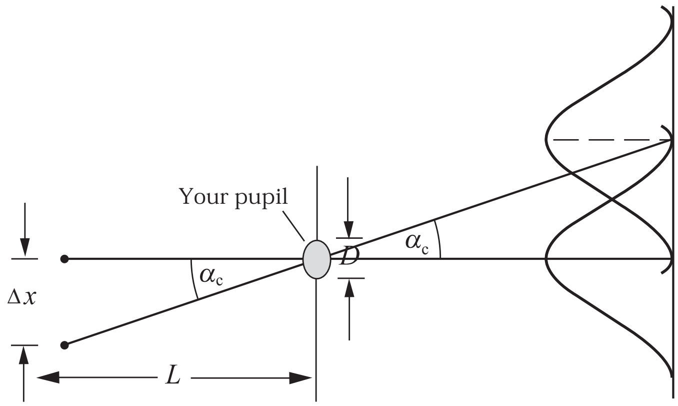
$$\frac{\Delta x}{L} = 1.22\frac{\lambda}{D} \Rightarrow \Delta x = 1.22\frac{\lambda L}{D}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = 1.22\left(\frac{(550\,\text{nm})\left(4c\cdot\text{y}\times\dfrac{9.461\times10^{15}\,\text{m}}{1c\cdot\text{y}}\right)}{200\,\text{in}\times\dfrac{2.54\,\text{cm}}{1\,\text{in}}}\right) = \boxed{5.00\times10^{9}\,\text{m}}$$

62 •• The star Mizar in Ursa Major is a binary system of stars of nearly equal magnitudes. The angular separation between the two stars is 14 seconds of arc. What is the minimum diameter of the pupil that allows resolution of the two stars using light that has a wavelength equal to 550 nm?

Picture the Problem We can use Rayleigh's criterion for circular apertures and the geometry of the diagram to obtain an expression we can solve for the minimum diameter *D* of the pupil that allows resolution of the binary stars.



Rayleigh's criterion is satisfied provided:

$$\alpha_c = 1.22\frac{\lambda}{D} \Rightarrow D = 1.22\frac{\lambda}{\alpha_c}$$

Substitute numerical values and evaluate *D*:

$$D = 1.22\left(\frac{550\,\text{nm}}{14'' \times \frac{1^\circ}{3600''} \times \frac{\pi\,\text{rad}}{180^\circ}}\right)$$
$$= \boxed{9.9\,\text{mm}} \approx 1\,\text{cm}$$

Diffraction Gratings

63 • **[SSM]** A diffraction grating that has 2000 slits per centimeter is used to measure the wavelengths emitted by hydrogen gas. (*a*) At what angles in the first-order spectrum would you expect to find the two violet lines that have wavelengths of 434 nm and 410 nm? (*b*) What are the angles if the grating has 15 000 slits per centimeter?

Picture the Problem We can solve $d\sin\theta = m\lambda$ for θ with m = 1 to express the location of the first-order maximum as a function of the wavelength of the light.

(*a*) The interference maxima in a diffraction pattern are at angles θ given by:

$$d\sin\theta = m\lambda$$

where *d* is the separation of the slits and *m* = 0, 1, 2, …

Solve for the angular location θ_m of the maxima :

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right)$$

Relate the number of slits N per centimeter to the separation d of the slits:

$$N = \frac{1}{d}$$

Substitute for d to obtain:

$$\theta_m = \sin^{-1}(mN\lambda) \qquad (1)$$

Evaluate θ for λ = 434 nm and m =1:

$$\theta_1 = \sin^{-1}\left[\left(2000\,\text{cm}^{-1}\right)(434\,\text{nm})\right] = \boxed{86.9\,\text{mrad}}$$

Evaluate θ for λ = 410 nm and m = 1:

$$\theta_1 = \sin^{-1}\left[\left(2000\,\text{cm}^{-1}\right)(410\,\text{nm})\right] = \boxed{82.1\,\text{mrad}}$$

(b) Use equation (1) to evaluate θ for λ = 434 nm and m = 1:

$$\theta_1 = \sin^{-1}\left[\left(15000\,\text{cm}^{-1}\right)(434\,\text{nm})\right] = \boxed{709\,\text{mrad}}$$

Evaluate θ for λ = 410 nm and m = 1:

$$\theta_1 = \sin^{-1}\left[\left(15000\,\text{cm}^{-1}\right)(410\,\text{nm})\right] = \boxed{662\,\text{mrad}}$$

64 • Using a diffraction grating that has 2000 lines per centimeter, two other lines in the first-order hydrogen spectrum are found at angles of 9.72×10^{-2} rad and 1.32×10^{-1} rad. What are the wavelengths of these lines?

Picture the Problem We can solve $d\sin\theta = m\lambda$ for λ with m = 1 to express the location of the first-order maximum as a function of the angles at which the first-order images are found.

The interference maxima in a diffraction pattern are at angles θ given by:

$$d\sin\theta = m\lambda \Rightarrow \lambda = \frac{d\sin\theta}{m}$$

where d is the separation of the slits and m = 0, 1, 2, …

Relate the number of slits N per centimeter to the separation d of the slits:

$$N = \frac{1}{d}$$

Let $m = 1$ and substitute for d to obtain:

$$\lambda = \frac{\sin\theta}{N}$$

Substitute numerical values and evaluate λ_1 for $\theta_1 = 9.72\times10^{-2}$ rad:

$$\lambda_1 = \frac{\sin\left(9.72\times10^{-2}\text{ rad}\right)}{2000\text{ cm}^{-1}} = \boxed{485\text{ nm}}$$

Substitute numerical values and evaluate λ_1 for $\theta_2 = 1.32\times10^{-1}$ rad:

$$\lambda_1 = \frac{\sin\left(1.32\times10^{-1}\text{ rad}\right)}{2000\text{ cm}^{-1}} = \boxed{658\text{ nm}}$$

65 • The colors of many butterfly wings and beetle carapaces are due to effects of diffraction. The *Morpho* butterfly has structural elements on its wings that effectively act as a diffraction grating with spacing 880 nm. At what angle will the first diffraction maximum occur for normally incident light diffracted by the butterfly's wings? Assume the light is blue with a wavelength of 440 nm.

Picture the Problem We can use the grating equation to find the angle at which normally incident blue light will be diffracted by the butterfly's wings.

The grating equation is:

$d\sin\theta = m\lambda$, where m = 1, 2, 3, …

Solve for θ to obtain:

$$\theta = \sin^{-1}\left[\frac{m\lambda}{d}\right]$$

Substitute numerical values and evaluate θ_1:

$$\theta = \sin^{-1}\left[\frac{(1)(440\text{ nm})}{880\text{ nm}}\right] = \boxed{30.0^\circ}$$

66 •• A diffraction grating that has 2000 slits per centimeter is used to analyze the spectrum of mercury. (*a*) Find the angular separation in the first-order spectrum of the two lines of wavelength 579 nm and 577 nm. (*b*) How wide must the beam on the grating be for these lines to be resolved?

Picture the Problem We can use the grating equation to find the angular separation of the first-order spectrum of the two lines. In Part (*b*) we can apply the definition of the resolving power of the grating to find the width of the grating that must be illuminated for the lines to be resolved.

(*a*) Express the angular separation in the first-order spectrum of the two lines:

$$\Delta\theta = \theta_{579} - \theta_{577}$$

Solve the grating equation for θ:

$$\theta = \sin^{-1}\left(\frac{m\lambda}{d}\right)$$

Substitute for θ_{579} and θ_{577} to obtain:

$$\Delta\theta = \sin^{-1}\left[\frac{m(579\,\text{nm})}{\frac{1}{2000\,\text{cm}^{-1}}}\right] - \sin^{-1}\left[\frac{m(577\,\text{nm})}{\frac{1}{2000\,\text{cm}^{-1}}}\right] = \sin^{-1}[0.1158m] - \sin^{-1}[0.1154m]$$

For $m = 1$:

$$\Delta\theta = \sin^{-1}[0.1158m] - \sin^{-1}[0.1154m] = 6.65° - 6.63° = \boxed{0.02°}$$

(*b*) The width of the beam necessary for these lines to be resolved is given by:

$$w = Nd \qquad (1)$$

Relate the resolving power of the diffraction grating to the number of slits *N* that must be illuminated in order to resolve these wavelengths in the *m*th order:

$$\frac{\lambda}{\Delta\lambda} = mN$$

For $m = 1$:

$$N = \frac{\lambda}{\Delta\lambda}$$

Substitute for *N* in equation (1) to obtain:

$$w = \frac{\lambda d}{\Delta\lambda}$$

Letting λ be the average of the two wavelengths, substitute numerical values and evaluate *w*:

$$w = \frac{(578\,\text{nm})\left(\frac{1}{2000\,\text{cm}^{-1}}\right)}{2\,\text{nm}} = \boxed{1\text{mm}}$$

67 •• [SSM] A diffraction grating that has 4800 lines per centimeter is illuminated at normal incidence with white light (wavelength range of 400 nm to 700 nm). How many orders of spectra can one observe in the transmitted light? Do any of these orders overlap? If so, describe the overlapping regions.

Picture the Problem We can use the grating equation $d\sin\theta = m\lambda, m = 1, 2, 3, \ldots$ to express the order number in terms of the slit separation *d*, the wavelength of the light λ, and the angle θ.

The interference maxima in the diffraction pattern are at angles θ given by:

$$d\sin\theta = m\lambda \Rightarrow m = \frac{d\sin\theta}{\lambda}$$

where m = 1, 2, 3, …

If one is to see the complete spectrum, it must be true that:

$$\sin\theta \le 1 \Rightarrow m \le \frac{d}{\lambda}$$

Evaluate m_{max}:

$$m_{max} = \frac{\dfrac{1}{4800\,\text{cm}^{-1}}}{\lambda_{max}} = \frac{\dfrac{1}{4800\,\text{cm}^{-1}}}{700\,\text{nm}} = 2.98$$

Because m_{max} = 2.98, one can see the complete spectrum only for m = 1 and 2.

Express the condition for overlap:

$$m_1\lambda_1 \ge m_2\lambda_2$$

One can see the complete spectrum for only the first and second order spectra. That is, only for m = 1 and 2. Because 700 nm < 2 × 400 nm, there is no overlap of the second-order spectrum into the first-order spectrum; however, there is overlap of long wavelengths in the second order with short wavelengths in the third-order spectrum.

68 •• A square diffraction grating that has an area of 25.0 cm^2 has a resolution of 22 000 in the fourth order. At what angle should you look to see a wavelength of 510 nm in the fourth order?

Picture the Problem We can use the grating equation and the resolving power of the grating to derive an expression for the angle at which you should look to see a wavelength of 510 nm in the fourth order.

The interference maxima in the diffraction pattern are at angles θ given by:

$$d\sin\theta = m\lambda, \quad \text{where } m = 1, 2, 3, \ldots \qquad (1)$$

The resolving power R is given by:

$$R = mN$$

where N is the number of slits and m is the order number.

Relate d to the width w of the grating:

$$d = \frac{w}{N}$$

Substitute for N and simplify to obtain:

$$d = \frac{mw}{R}$$

Substitute for d in equation (1) to obtain:

$$\frac{mw}{R}\sin\theta = m\lambda \Rightarrow \theta = \sin^{-1}\left(\frac{R\lambda}{w}\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \sin^{-1}\left[\frac{(22{,}000)(510\,\text{nm})}{5.00\,\text{cm}}\right] = \boxed{13.0°}$$

69 •• Sodium light that has a wavelength equal to 589 nm falls normally on a 2.00-cm-square diffraction grating ruled with 4000 lines per centimeter. The Fraunhofer diffraction pattern is projected onto a screen a distance of 1.50 m from the grating by a 1.50-m-focal-length lens that is placed immediately in front of the grating. Find (*a*) the distance of the first and second order intensity maxima from the central intensity maximum, (*b*) the width of the central maximum, and (*c*) the resolution in the first order. (Assume the entire grating is illuminated.)

Picture the Problem The distance on the screen to the *m*th bright fringe can be found using $d\sin\theta = m\lambda$ (where d is the slit separation and m = 0, 1, 2, …) and the geometry of the grating and projection screen. We can use $\theta_{\text{min}} = \lambda/Nd = \Delta y/2L$ to find the width of the central maximum and $R = mN$, where N is the number of slits in the grating, to find the resolving power in the first order.

(*a*) The angle θ at which maxima occur is related to the slit separation d, the wavelength of the incident light λ, and the order number m according to:

$$d\sin\theta = m\lambda \qquad (1)$$

where m = 0, 1, 2, … .

θ is also related to the distance to the screen L and the positions of the intensity maxima y_m:

$$\sin\theta = \frac{y_m}{\sqrt{L^2 + y_m^2}}$$

Substituting for $\sin\theta$ in equation (1) yields:

$$\frac{dy_m}{\sqrt{L^2 + y_m^2}} = m\lambda \Rightarrow y_m = \frac{m\lambda L}{\sqrt{d^2 - m^2\lambda^2}}$$

Substituting numerical values yields:

$$y_m = \frac{m(589\,\text{nm})(1.50\,\text{m})}{\sqrt{\left(\frac{1.00\,\text{cm}}{4000}\right)^2 - m^2(589\,\text{nm})^2}}$$

Evaluate this expression for $m = 1$ and $m = 2$ to obtain:

$$y_1 = \boxed{36.4\,\text{cm}} \text{ and } y_2 = \boxed{80.1\,\text{cm}}$$

(*b*) The angle $\theta_{\min}$ that locates the first minima in the diffraction pattern is given by:

$$\theta_{\min} = \frac{\lambda}{Nd} = \frac{\Delta y}{2L} \Rightarrow \Delta y = \frac{2L\lambda}{Nd}$$

where Δy is the width of the central maximum.

Substitute numerical values and evaluate Δy:

$$\Delta y = \frac{2(1.50\,\text{m})(589\,\text{nm})}{(8000\,\text{lines})\left(\dfrac{1}{4000\,\text{cm}^{-1}}\right)} = \boxed{88.4\,\mu\text{m}}$$

(*c*) The resolving power R in the mth order is given by:

$$R = mN$$

Substitute numerical values and evaluate R:

$$R = (1)(8000) = \boxed{8000}$$

70 •• The spectrum of neon is exceptionally rich in the visible region. Among the many lines are two lines at wavelengths of 519.313 nm and 519.322 nm. If light from a neon discharge tube is normally incident on a transmission grating with 8400 lines per centimeter and the spectrum is observed in second order, what must be the width of the grating that is illuminated, so that these two lines can be resolved?

Picture the Problem The width of the grating w is the product of its number of lines N and the separation of its slits d. Because the resolution of the grating is a function of the average wavelength, the difference in the wavelengths, and the order number, we can express w in terms of these quantities.

Express the width w of the grating as a function of the number of lines N and the slit separation d:

$$w = Nd$$

The resolving power R of the grating is given by:

$$R = \frac{\lambda}{\Delta\lambda} = mN \Rightarrow N = \frac{\lambda}{m\Delta\lambda}$$

Substitute for N in the expression for w to obtain:

$$w = \frac{\lambda d}{m\Delta\lambda}$$

Letting λ be the average of the given wavelengths, substitute numerical values and evaluate w:

$$w=\frac{\frac{1}{2}(519.313\,\text{nm}+519.322\,\text{nm})\left(\dfrac{1}{8400\,\text{cm}^{-1}}\right)}{2(519.322\,\text{nm}-519.313\,\text{nm})}=\boxed{3\,\text{cm}}$$

71 •• [SSM] Mercury has several stable isotopes, among them ^{198}Hg and ^{202}Hg. The strong spectral line of mercury, at about 546.07 nm, is a composite of spectral lines from the various mercury isotopes. The wavelengths of this line for ^{198}Hg and ^{202}Hg are 546.07532 nm and 546.07355 nm, respectively. What must be the resolving power of a grating capable of resolving these two isotopic lines in the third-order spectrum? If the grating is illuminated over a 2.00-cm-wide region, what must be the number of lines per centimeter of the grating?

Picture the Problem We can use the expression for the resolving power of a grating to find the resolving power of the grating capable of resolving these two isotopic lines in the third-order spectrum. Because the total number of the slits of the grating N is related to width w of the illuminated region and the number of lines per centimeter of the grating and the resolving power R of the grating, we can use this relationship to find the number of lines per centimeter of the grating.

The resolving power of a diffraction grating is given by:

$$R=\frac{\lambda}{|\Delta\lambda|}=mN \qquad (1)$$

Substitute numerical values and evaluate R:

$$R=\frac{546.07532}{|546.07532-546.07355|}=3.0852\times10^{5}=\boxed{3.09\times10^{5}}$$

Express n, be the number of lines per centimeter of the grating, in terms of the total number of slits N of the grating and the width w of the grating:

$$n=\frac{N}{w}$$

From equation (1) we have:

$$N=\frac{R}{m}$$

Substitute for N to obtain:

$$n=\frac{R}{mw}$$

Substitute numerical values and evaluate n:

$$n = \frac{3.0852\times10^{5}}{(3)(2.00\,\text{cm})} = \boxed{5.14\times10^{4}\ \text{cm}^{-1}}$$

72 ••• A diffraction grating has n lines per unit length. Show that the angular separation ($\Delta\theta$) of two lines of wavelengths λ and $\lambda + \Delta\lambda$ is approximately $\Delta\theta = \Delta\lambda \Big/ \sqrt{\frac{1}{n^2m^2} - \lambda^2}$ where m is the order number.

Picture the Problem We can differentiate the grating equation implicitly and approximate $d\theta/d\lambda$ by $\Delta\theta/\Delta\lambda$ to obtain an expression $\Delta\theta$ as a function of m, n, $\Delta\lambda$, and $\cos\theta$. We can use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ and the grating equation to write $\cos\theta$ in terms of n, m, and λ. Making these substitutions will yield the given equation.

The grating equation is:

$$d\sin\theta = m\lambda,\ m = 0, 1, 2, \ldots \qquad (1)$$

Differentiate both sides of this equation with respect to λ:

$$\frac{d}{d\lambda}(d\sin\theta) = \frac{d}{d\lambda}(m\lambda)$$

or

$$d\cos\theta\frac{d\theta}{d\lambda} = m$$

Because $n = 1/d$:

$$\cos\theta\frac{d\theta}{d\lambda} = nm \Rightarrow n = \frac{1}{m}\cos\theta\frac{d\theta}{d\lambda}$$

Approximate $d\theta/d\lambda$ by $\Delta\theta/\Delta\lambda$:

$$n = \frac{1}{m}\frac{\Delta\theta}{\Delta\lambda}\cos\theta$$

Solving for $\Delta\theta$ yields:

$$\Delta\theta = \frac{nm\Delta\lambda}{\cos\theta}$$

Substitute for $\cos\theta$ to obtain:

$$\Delta\theta = \frac{nm\Delta\lambda}{\sqrt{1-\sin^2\theta}}$$

From equation (1):

$$\sin\theta = \frac{m\lambda}{d} = nm\lambda$$

Substituting for $\sin\theta$ yields:

$$\Delta\theta = \frac{nm\Delta\lambda}{\sqrt{1-n^2m^2\lambda^2}}$$

Simplify by dividing the numerator and denominator by *nm* to obtain:

$$\Delta\theta = \frac{\Delta\lambda}{\frac{1}{nm}\sqrt{1-n^2m^2\lambda^2}} = \frac{\Delta\lambda}{\sqrt{\frac{1-n^2m^2\lambda^2}{n^2m^2}}} = \boxed{\frac{\Delta\lambda}{\sqrt{\frac{1}{n^2m^2}-\lambda^2}}}$$

73 ••• [SSM] For a diffraction grating in which all the surfaces are normal to the incident radiation, most of the energy goes into the zeroth order, which is useless from a spectroscopic point of view, since in zeroth order all the wavelengths are at 0º. Therefore, modern reflection gratings have shaped, or *blazed*, grooves, as shown in Figure 33-45. This shifts the specular reflection, which contains most of the energy, from the zeroth order to some higher order. (*a*) Calculate the *blaze angle* ϕ_m in terms of the groove separation *d*, the wavelength λ, and the order number *m* in which specular reflection is to occur for $m = 1, 2, \ldots$. (*b*) Calculate the proper blaze angle for the specular reflection to occur in the second order for light of wavelength 450 nm incident on a grating with 10 000 lines per centimeter.

Picture the Problem We can use the grating equation and the geometry of the grating to derive an expression for ϕ_m in terms of the order number *m*, the wavelength of the light λ, and the groove separation *d*.

(*a*) Because $\theta_i = \theta_r$, application of the grating equation yields:

$$d\sin(2\theta_i) = m\lambda, \quad \text{where } m = 0, 1, 2, \ldots \qquad (1)$$

Because ϕ and θ_i have their left and right sides mutually perpendicular:

$$\theta_i = \phi_m$$

Substitute for θ_i to obtain:

$$d\sin(2\phi_m) = m\lambda$$

Solving for ϕ_m yields:

$$\phi_m = \boxed{\tfrac{1}{2}\sin^{-1}\left(m\frac{\lambda}{d}\right)}$$

(*b*) For $m = 2$:

$$\phi_2 = \tfrac{1}{2}\sin^{-1}\left(2\frac{450\,\text{nm}}{\frac{1}{10{,}000\,\text{cm}^{-1}}}\right) = \boxed{32.1^\circ}$$

74 ••• In this problem, you will derive the relation $R = \lambda/|\Delta\lambda| = mN$ (Equation 33-27) for the resolving power of a diffraction grating containing *N* slits separated by a distance *d*. To do this, you will calculate the angular

separation between the intensity maximum and intensity minimum for some wavelength λ and set it equal to the angular separation of the mth-order maximum for two nearby wavelengths. (a) First show that the phase difference ϕ between the waves from two adjacent slits is given by $\phi = \dfrac{2\pi d}{\lambda}\sin\theta$. ($b$) Next differentiate this expression to show that a small change in angle $d\theta$ results in a change in phase of $d\phi$ given by $d\phi = \dfrac{2\pi d}{\lambda}\cos\theta\, d\theta$. ($c$) Then for N slits, the angular separation between an interference maximum and an interference minimum corresponds to a phase change of $d\phi = 2\pi/N$. Use this to show that the angular separation $d\theta$ between the intensity maximum and intensity minimum for some wavelength λ is given by $d\theta = \dfrac{\lambda}{Nd\cos\theta}$. ($d$) Next use the fact that the angle of the mth-order interference maximum for wavelength λ is specified by $d\sin\theta = m\lambda$ (Equation 33-26). Compute the differential of each side of this equation to show that angular separation of the mth-order maximum for two nearly equal wavelengths differing by $d\lambda$ is given by $d\theta = \dfrac{m\,d\lambda}{d\cos\theta}$. ($e$) According to Rayleigh's criterion, two wavelengths will be resolved in the mth order if the angular separation of the wavelengths, given by the Part (d) result, equals the angular separation of the interference maximum and the interference minimum given by the Part (c) result. Use this to arrive at $R = \lambda/|\Delta\lambda| = mN$ (Equation 33-27) for the resolving power of a grating.

Picture the Problem We can follow the procedure outlined in the problem statement to obtain $R = \lambda/|\Delta\lambda| = mN$.

(a) Express the relationship between the phase difference ϕ and the path difference Δr:

$$\frac{\phi}{2\pi} = \frac{\Delta r}{\lambda} \Rightarrow \phi = \frac{2\pi\Delta r}{\lambda}$$

Because $\Delta r = d\sin\theta$:

$$\phi = \boxed{\frac{2\pi d}{\lambda}\sin\theta}$$

(b) Differentiate this expression with respect to θ to obtain:

$$\frac{d\phi}{d\theta} = \frac{d}{d\theta}\left[\frac{2\pi d}{\lambda}\sin\theta\right] = \frac{2\pi d}{\lambda}\cos\theta$$

Solve for $d\phi$:

$$d\phi = \boxed{\frac{2\pi d}{\lambda}\cos\theta\, d\theta}$$

(c) From Part (b):

$$d\theta = \frac{\lambda d\phi}{2\pi d\cos\theta}$$

Substitute $2\pi/N$ for $d\phi$ to obtain:

$$d\theta = \boxed{\frac{\lambda}{Nd\cos\theta}}$$

(*d*) Equation 33-26 is:

$$d\sin\theta = m\lambda,\ m = 0, 1, 2, \ldots$$

Differentiate this expression implicitly with respect to λ to obtain:

$$\frac{d}{d\lambda}[d\sin\theta] = \frac{d}{d\lambda}[m\lambda]$$

or

$$d\cos\theta\frac{d\theta}{d\lambda} = m$$

Solve for $d\theta$ to obtain:

$$d\theta = \boxed{\frac{md\lambda}{d\cos\theta}}$$

(*e*) Equate the two expressions for $d\theta$ obtained in (*c*) and (*d*):

$$\frac{\lambda}{Nd\cos\theta} = \frac{md\lambda}{d\cos\theta}$$

Approximating $d\lambda$ by $\Delta\lambda$ and allowing for the possibility that $\Delta\lambda < 0$ yields:

$$\frac{\lambda}{Nd\cos\theta} = \frac{m|\Delta\lambda|}{d\cos\theta}$$

Solving for $R = \lambda/\Delta\lambda$ yields:

$$R = \boxed{\frac{\lambda}{|\Delta\lambda|} = mN}$$

General Problems

75 • **[SSM]** Naturally occurring coronas (brightly colored rings) are sometimes seen around the Moon or the Sun when viewed through a thin cloud. (Warning: When viewing a sun corona, be sure that the entire sun is blocked by the edge of a building, a tree, or a traffic pole to safeguard your eyes.) These coronas are due to diffraction of light by small water droplets in the cloud. A typical angular diameter for a coronal ring is about 10º. From this, estimate the size of the water droplets in the cloud. Assume that the water droplets can be modeled as opaque disks with the same radius as the droplet, and that the Fraunhofer diffraction pattern from an opaque disk is the same as the pattern from an aperture of the same diameter. (This last statement is known as *Babinet's principle*.)

Picture the Problem We can use $\sin\theta = 1.22\lambda/D$ to relate the diameter D of the opaque-disk water droplets to the angular diameter θ of a coronal ring and to the wavelength of light. We'll assume a wavelength of 500 nm.

The angle θ subtended by the first diffraction minimum is related to the wavelength λ of light and the diameter D of the opaque-disk water droplet:

$$\sin\theta = 1.22\frac{\lambda}{D}$$

Because of the great distance to the cloud of water droplets, $\theta << 1$ and:

$$\theta \approx 1.22\frac{\lambda}{D} \Rightarrow D = \frac{1.22\lambda}{\theta}$$

Substitute numerical values and evaluate D:

$$D = \frac{1.22(500\,\text{nm})}{10°\times\frac{\pi\,\text{rad}}{180°}} = \boxed{3.5\,\mu\text{m}}$$

76 • An artificial corona (see Problem 75) can be made by placing a suspension of polystyrene microspheres in water. Polystyrene microspheres are small, uniform spheres made of plastic with an index of refraction equal to 1.59. Assuming that the water has an index of refractive equal to 1.33, what is the angular diameter of such an artificial corona if 5.00-μm-diameter particles are illuminated by light from a helium–neon laser with wavelength in air of 632.8 nm?

Picture the Problem We can use $\sin\theta = 1.22\lambda_n/D$ to relate the diameter D of a microsphere to the angular diameter θ of a coronal ring and to the wavelength of light in water.

The angle θ subtended by the first diffraction minimum is related to the wavelength λ_n of light in water and the diameter D of the microspheres:

$$\sin\theta = 1.22\frac{\lambda_n}{D} = 1.22\frac{\lambda}{nD}$$

Because $\theta << 1$:

$$\theta \approx 1.22\frac{\lambda}{nD}$$

Substitute numerical values and evaluate θ:

$$\theta \approx \frac{(1.22)(632.8\,\text{nm})}{(1.33)(5.00\,\mu\text{m})} = 0.116\,\text{rad}$$

$$= \boxed{6.65°}$$

77 • Coronas (see Problem 74) can be caused by pollen grains, typically of birch or pine. Such grains are irregular in shape, but they can be treated as if they had an average diameter of about 25 μm. What is the angular diameter (in degrees) of the corona for blue light? What is the diameter (in degrees) of the corona for red light?

Picture the Problem We can use $\sin\theta = 1.22\lambda/D$ to relate the diameter D of a pollen grain to the angular diameter θ of a coronal ring and to the wavelength of light. We'll assume a wavelength of 450 nm for blue light and 650 nm for red light.

The angle α subtended by the first diffraction intensity minima is related to the wavelength λ of light and to the diameter D of the microspheres:

$$\sin\alpha = 1.22\frac{\lambda}{D}$$

or, because $\alpha = \frac{1}{2}\theta$ where θ is the angular diameter of the coronal ring,

$$\sin\tfrac{1}{2}\theta = 1.22\frac{\lambda}{D}$$

Because $\theta << 1$, $\sin\theta \approx \theta$ and:

$$\tfrac{1}{2}\theta \approx 1.22\frac{\lambda}{D} \Rightarrow \theta \approx 2.44\frac{\lambda}{D}$$

Substitute numerical values and evaluate θ for red light:

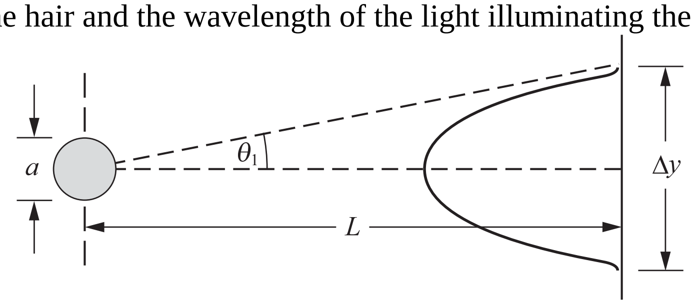
$$\theta_{\text{red}} \approx \frac{2.44(650\,\text{nm})}{25\,\mu\text{m}} = 6.344\times10^{-2}\text{ rad} = \boxed{3.6°}$$

Substitute numerical values and evaluate θ for blue light:

$$\theta_{\text{blue}} \approx \frac{2.44(450\,\text{nm})}{25\,\mu\text{m}} = 4.392\times10^{-2}\text{ rad} = \boxed{2.5°}$$

78 • Light from a He-Ne laser (632.8 nm wavelength) is directed upon a human hair, in an attempt to measure its diameter by examining the diffraction pattern. The hair is mounted in a frame 7.5 m from a wall, and the central diffraction maximum is measured to be 14.6 cm wide. What is the diameter of the hair? (The diffraction pattern of a hair with diameter d is the same as the diffraction pattern of a single slit with width $a = d$. See Babinet's principle, Problem 75.)

Picture the Problem The diagram shows the hair whose diameter $d = a$, the screen a distance L from the hair, and the separation Δy of the first diffraction peak from the center. We can use the geometry of the experiment to relate Δy to L and a and the condition for diffraction maxima to express θ_1 in terms of the diameter of the hair and the wavelength of the light illuminating the hair.



Relate θ to Δy:

$$\tan\theta = \frac{\frac{1}{2}\Delta y}{L} \Rightarrow \theta = \tan^{-1}\left(\frac{\Delta y}{2L}\right)$$

Diffraction maxima occur where:

$$a\sin\theta = \left(m+\tfrac{1}{2}\right)\lambda \Rightarrow a = \frac{\left(m+\frac{1}{2}\right)\lambda}{\sin\theta}$$

where m = 1, 2, 3, …

Substituting for θ yields:

$$a = \frac{\left(m+\frac{1}{2}\right)\lambda}{\sin\left[\tan^{-1}\left(\frac{\Delta y}{2L}\right)\right]}$$

Substitute numerical values and evaluate a for m = 1:

$$a = \frac{\left(1+\frac{1}{2}\right)(632.8\,\text{nm})}{\sin\left[\tan^{-1}\left(\frac{14.6\,\text{cm}}{2(7.5\,\text{m})}\right)\right]} = \boxed{98\,\mu\text{m}}$$

79 • **[SSM]** A long, narrow horizontal slit lies 1.00 μm above a plane mirror, which is in the horizontal plane. The interference pattern produced by the slit and its image is viewed on a screen 1.00 m from the slit. The wavelength of the light is 600 nm. (*a*) Find the distance from the mirror to the first maximum. (*b*) How many dark bands per centimeter are seen on the screen?

Picture the Problem We can apply the condition for constructive interference to find the angular position of the first maximum on the screen. Note that, due to reflection, the wave from the image is 180° out of phase with that from the source.

(*a*) Because $y_0 << L$, the distance from the mirror to the first maximum is given by:

$$y_0 = L\theta_0 \qquad (1)$$

Express the condition for constructive interference:

$$d\sin\theta = \left(m+\tfrac{1}{2}\right)\lambda$$

where m = 0, 1, 2, …

Solving for θ yields:

$$\theta = \sin^{-1}\left[\left(m+\tfrac{1}{2}\right)\frac{\lambda}{d}\right]$$

For the first maximum, m = 0 and:

$$\theta_0 = \sin^{-1}\left[\frac{\lambda}{2d}\right]$$

Substitute in equation (1) to obtain:

$$y_0 = L\sin^{-1}\left[\frac{\lambda}{2d}\right]$$

Because the image of the slit is as far behind the mirror's surface as the slit is in front of it, $d = 2.00\ \mu\text{m}$. Substitute numerical values and evaluate y_0:

$$y_0 = (1.00\,\text{m})\sin^{-1}\left[\frac{600\,\text{nm}}{2(2.00\,\mu\text{m})}\right]$$
$$= \boxed{15.1\,\text{cm}}$$

(*b*) The separation of the fringes on the screen is given by:

$$\Delta y = \frac{\lambda L}{d}$$

The number of dark bands per unit length is the reciprocal of the fringe separation:

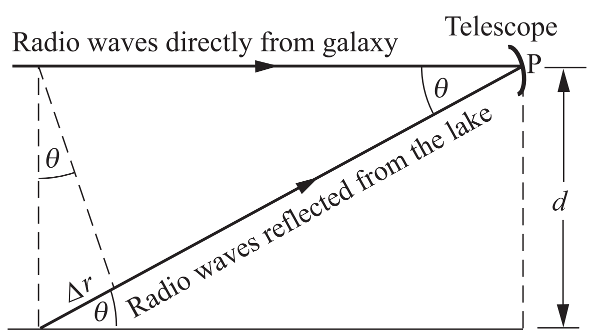
$$n = \frac{1}{\Delta y} = \frac{d}{\lambda L}$$

Substitute numerical values and evaluate n:

$$n = \frac{2.00\,\mu\text{m}}{(600\,\text{nm})(1.00\,\text{m})} = \boxed{3.33\ \text{m}^{-1}}$$

80 • A radio telescope is situated at the edge of a lake. The telescope is looking at light from a radio galaxy that is just rising over the horizon. If the height of the antenna is 20 m above the surface of the lake, at what angle above the horizon will the radio galaxy be when the telescope is centered in the first intensity interference maximum of the radio waves? Assume the wavelength of the radio waves is 20 cm. *Hint: The interference is caused by the light reflecting off the lake and remember that this reflection will result in a 180º phase shift.*

Picture the Problem The radio waves from the galaxy reach the telescope by two paths; one coming directly from the galaxy and the other reflected from the surface of the lake. The radio waves reflected from the surface of the lake are phase shifted 180°, relative to the radio waves reaching the telescope directly, by reflection from the surface of the lake. We can use the condition for constructive interference of two waves to find the angle above the horizon at which the radio waves from the galaxy will interfere constructively.



Because the reflected radio waves are phase shifted by 180°, the condition for constructive interference at point P is:

$$\Delta r = \left(m + \tfrac{1}{2}\right)\lambda$$
where $m = 0, 1, 2, \ldots$

Referring to the figure, note that:

$$\sin\theta \approx \frac{\Delta r}{d} \Rightarrow \theta = \sin^{-1}\left[\frac{\Delta r}{d}\right]$$

Substitute for Δr to obtain:

$$\theta = \sin^{-1}\left[\frac{\left(m + \frac{1}{2}\right)\lambda}{d}\right]$$

Noting that $m = 0$ for the first interference maximum, substitute numerical values and evaluate θ_0:

$$\theta_0 = \sin^{-1}\left[\frac{\frac{1}{2}(20\,\text{cm})}{20\,\text{m}}\right] = 5.00\times10^{-3}\ \text{rad}$$
$$= \boxed{0.29°}$$

81 • The diameter of the radio telescope at Arecibo, Puerto Rico, is 300 m. What is the smallest angular separation of two objects that this telescope can detect when it is tuned to detect microwaves of 3.2-cm wavelength?

Picture the Problem The resolving power of a telescope is the ability of the instrument to resolve two objects that are close together. Hence we can use Rayleigh's criterion to find the resolving power of the Arecibo telescope.

Rayleigh's criterion for resolution is:

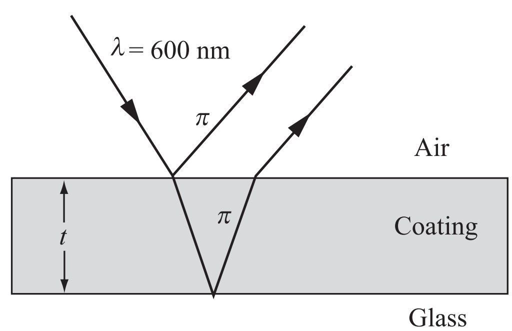
$$\alpha_c = 1.22\frac{\lambda}{D}$$

Substitute numerical values and evaluate α_c:

$$\alpha_c = 1.22\left(\frac{3.2\,\text{cm}}{300\,\text{m}}\right) = \boxed{0.13\,\text{mrad}}$$

82 •• A thin layer of a transparent material that has an index of refraction of 1.30 is used as a nonreflective coating on the surface of glass that has an index of refraction of 1.50. What should the minimum thickness of the material be for the material to be nonreflecting for light that has a wavelength 600 nm?

Picture the Problem Note that reflection at both surfaces involves a phase shift of π rad. We can apply the condition for destructive interference to find the thickness t of the nonreflective coating.



The condition for destructive interference is:

$$2t = \left(m+\tfrac{1}{2}\right)\lambda_{\text{coating}} = \left(m+\tfrac{1}{2}\right)\frac{\lambda_{\text{air}}}{n_{\text{coating}}}$$

Solve for t to obtain:

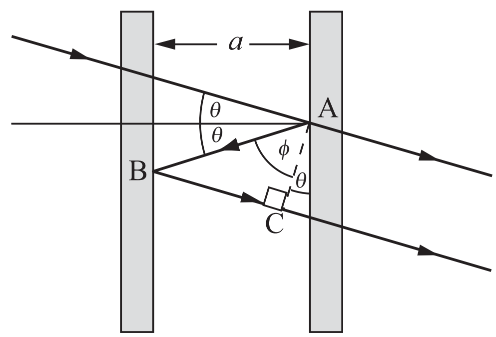
$$t = \left(m+\tfrac{1}{2}\right)\frac{\lambda_{\text{air}}}{2n_{\text{coating}}}$$

Evaluate t for $m = 0$:

$$t = \left(\tfrac{1}{2}\right)\frac{600\,\text{nm}}{2(1.30)} = \boxed{115\,\text{nm}}$$

83 •• [SSM] A *Fabry–Perot interferometer* (Figure 33-47) consists of two parallel, half-silvered mirrors that face each other and are separated by a small distance a. A half-silvered mirror is one that transmits 50% of the incident intensity and reflects 50% of the incident intensity. Show that when light is incident on the interferometer at an angle of incidence θ, the transmitted light will have maximum intensity when $2a = m\lambda/\cos\,\theta$.

Picture the Problem The *Fabry-Perot interferometer* is shown in the figure. For constructive interference in the transmitted light the path difference must be an integral multiple of the wavelength of the light. This path difference can be found using the geometry of the interferometer.



For constructive interference we require that:

$$\Delta r = m\lambda, m = 0, 1, 2, \ldots \qquad (1)$$

The path difference between the two rays that emerge from the interferometer is given by:

$$\Delta r = \text{AB} + \text{BC} = \frac{a}{\cos\theta} + \text{AB}\sin\phi$$

From the geometry of the interferometer:

$$2\theta + \phi = 90° \Rightarrow \phi = 90° - 2\theta$$

Substituting for ϕ and AB and simplifying gives:

$$\Delta r = \frac{a}{\cos\theta} + \frac{a}{\cos\theta}\sin(90° - 2\theta)$$
$$= \frac{a}{\cos\theta} + \frac{a}{\cos\theta}\cos 2\theta$$
$$= \frac{a}{\cos\theta}(1 + \cos 2\theta)$$

Using the trigonometric identity $\cos 2\theta = 2\cos^2\theta - 1$ and simplifying yields:

$$\Delta r = \frac{a}{\cos\theta}(1 + 2\cos^2\theta - 1) = 2a\cos\theta$$

Substitute for Δr in equation (1) to obtain:

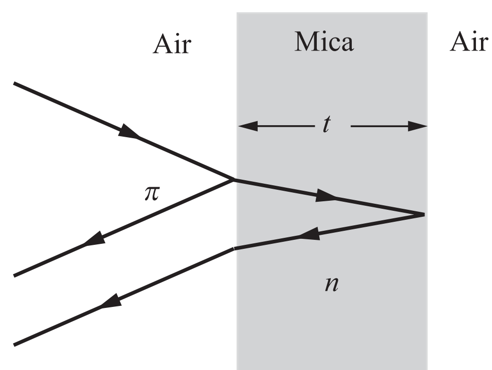
$$2a\cos\theta = m\lambda, m = 0, 1, 2, ...$$

Solving for $2a$ gives:

$$2a = \boxed{\frac{m\lambda}{\cos\theta}} \text{ where } m = 0, 1, 2, \ldots$$

84 •• A mica sheet 1.20 μm thick is suspended in air. In reflected light, there are gaps in the visible spectrum at 421, 474, 542, and 633 nm. Find the index of refraction of the mica sheet.

Picture the Problem The gaps in the spectrum of the visible light are the result of destructive interference between the incident light and the reflected light. Noting that there is a π rad phase shift at the first air-mica interface, we can use the condition for destructive interference to find the index of refraction n of the mica sheet.



Because there is a π rad phase shift at the first air-mica interface, the condition for destructive interference is:

$$2t = m\lambda_{\text{mica}} = m\frac{\lambda_{\text{air}}}{n}, m = 1, 2, 3, ...$$

Solving for n yields:

$$n = m\frac{\lambda_{\text{air}}}{2t} \qquad (1)$$

For λ = 474 nm:

$$2t = (474\,\text{nm})m$$

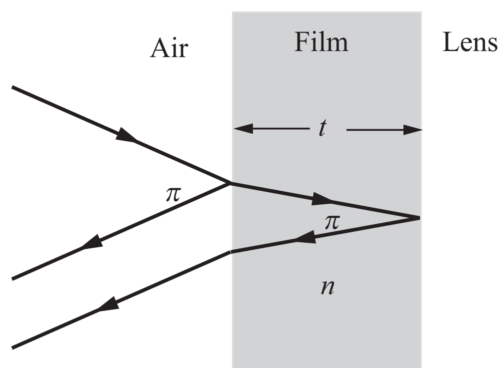
For λ = 421 nm: $$2t = (421\,\text{nm})(m+1)$$

Equate these two expressions and solve for m to obtain: $m = 8$ for λ = 474 nm

Substitute numerical values in equation (1) and evaluate n: $$n = 8\left(\frac{474\,\text{nm}}{2(1.20\,\mu\text{m})}\right) = \boxed{1.58}$$

85 •• [SSM] A camera lens is made of glass that has an index of refraction of 1.60. This lens is coated with a magnesium fluoride film (index of refraction 1.38) to enhance its light transmission. The purpose of this film is to produce zero reflection for light of wavelength 540 nm. Treat the lens surface as a flat plane and the film as a uniformly thick flat film. (*a*) What minimum thickness of this film will accomplish its objective? (*b*) Would there be destructive interference for any other visible wavelengths? (*c*) By what factor would the reflection for light of 400 nm wavelength be reduced by the presence of this film? Neglect the variation in the reflected light amplitudes from the two surfaces.

Picture the Problem Note that the light reflected at both the air-film and film-lens interfaces undergoes a π rad phase shift. We can use the condition for destructive interference between the light reflected from the air-film interface and the film-lens interface to find the thickness of the film. In (*c*) we can find the factor by which light of the given wavelengths is reduced by this film from $I \propto \cos^2 \frac{1}{2}\delta$.



(*a*) Express the condition for destructive interference between the light reflected from the air-film interface and the film-lens interface: $$2t = \left(m+\tfrac{1}{2}\right)\lambda_{\text{film}} = \left(m+\tfrac{1}{2}\right)\frac{\lambda_{\text{air}}}{n} \quad (1)$$ where $m = 0, 1, 2, \ldots$

Solving for t gives: $$t = \left(m+\tfrac{1}{2}\right)\frac{\lambda_{\text{air}}}{2n}$$

Evaluate t for $m = 0$: $$t = \left(\frac{1}{2}\right)\frac{540\,\text{nm}}{2(1.38)} = 97.83\,\text{nm} = \boxed{97.8\,\text{nm}}$$

(*b*) Solve equation (1) for λ_{air} to obtain:

$$\lambda_{\text{air}} = \frac{2tn}{m+\frac{1}{2}}$$

Evaluate λ_{air} for $m = 1$:

$$\lambda_{\text{air}} = \frac{2(97.8\,\text{nm})(1.38)}{1+\frac{1}{2}} = 180\,\text{nm}$$

No; because 180 nm is not in the visible portion of the spectrum.

(*c*) Express the reduction factor f as a function of the phase difference δ between the two reflected waves:

$$f = \cos^2 \tfrac{1}{2}\delta \qquad (2)$$

Relate the phase difference to the path difference Δr:

$$\frac{\delta}{2\pi} = \frac{\Delta r}{\lambda_{\text{film}}} \Rightarrow \delta = 2\pi\left(\frac{\Delta r}{\lambda_{\text{film}}}\right)$$

Because $\Delta r = 2t$:

$$\delta = 2\pi\left(\frac{2t}{\lambda_{\text{film}}}\right)$$

Substitute in equation (2) to obtain:

$$f = \cos^2\left[\tfrac{1}{2}2\pi\left(\frac{2t}{\lambda_{\text{film}}}\right)\right] = \cos^2\left[\frac{2\pi\, t}{\lambda_{\text{film}}}\right]$$

$$= \cos^2\left[\frac{2\pi\, nt}{\lambda_{\text{air}}}\right]$$

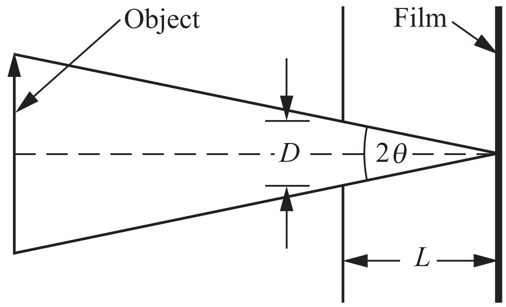
Evaluate f for $\lambda = 400$ nm:

$$f_{400} = \cos^2\left[\frac{2\pi(1.38)(97.83\,\text{nm})}{400\,\text{nm}}\right]$$

$$= \boxed{0.273}$$

86 •• In a pinhole camera, the image is fuzzy because of geometry (rays arrive at the film after passing through different parts of the pinhole) and because of diffraction. As the pinhole is made smaller, the fuzziness due to geometry is reduced, but the fuzziness due to diffraction is increased. The optimum size of the pinhole for the sharpest possible image occurs when the spread due to diffraction equals the spread due to the geometric effects of the pinhole. Estimate the optimum size of the pinhole if the distance from the pinhole to the film is 10.0 cm and the wavelength of the light is 550 nm.

Picture the Problem As indicated in the problem statement, we can find the optimal size of the pinhole by equating the angular width of the object at the film and the angular width of the diffraction pattern.



Express the angular width of the a distant object at the film in terms of the diameter D of the pinhole and the distance L from the pinhole to the object:

$$2\theta = \frac{D}{L} \Rightarrow \theta = \frac{D}{2L}$$

Using Rayleigh's criterion, express the angular width of the diffraction pattern:

$$\theta_{\text{diffraction}} = 1.22\frac{\lambda}{D}$$

Equate these two expressions to obtain:

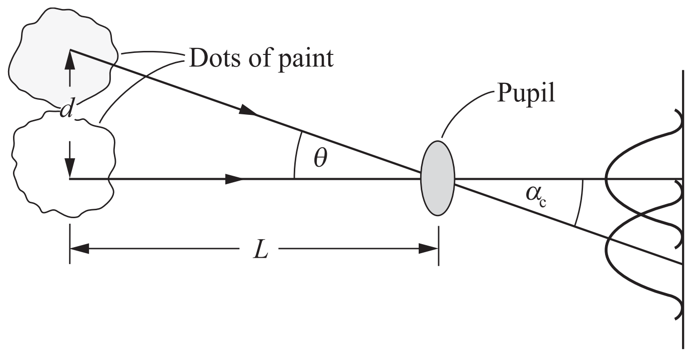
$$\frac{D}{2L} = 1.22\frac{\lambda}{D} \Rightarrow D = \sqrt{2.44\lambda L}$$

Substitute numerical values and evaluate D:

$$D = \sqrt{2.44(550\,\text{nm})(10.0\,\text{cm})} = \boxed{0.366\,\text{mm}}$$

87 •• **[SSM]** The Impressionist painter Georges Seurat used a technique called *pointillism*, in which his paintings are composed of small, closely spaced dots of pure color, each about 2.0 mm in diameter. The illusion of the colors blending together smoothly is produced in the eye of the viewer by diffraction effects. Calculate the minimum viewing distance for this effect to work properly. Use the wavelength of visible light that requires the *greatest* distance between dots, so that you are sure the effect will work for *all* visible wavelengths. Assume the pupil of the eye has a diameter of 3.0 mm.

Picture the Problem We can use the geometry of the dots and the pupil of the eye and Rayleigh's criterion to find the greatest viewing distance that ensures that the effect will work for all visible wavelengths.



Referring to the diagram, express the angle subtended by the adjacent dots:

$$\theta \approx \frac{d}{L}$$

Letting the diameter of the pupil of the eye be D, apply Rayleigh's criterion to obtain:

$$\alpha_c = 1.22\frac{\lambda}{D}$$

Set $\theta = \alpha_c$ to obtain:

$$\frac{d}{L} = 1.22\frac{\lambda}{D} \Rightarrow L = \frac{Dd}{1.22\lambda}$$

Evaluate L for the *shortest* wavelength light in the visible portion of the spectrum:

$$L = \frac{(3.0\,\text{mm})(2.0\,\text{mm})}{(1.22)(400\,\text{nm})} = \boxed{12\,\text{m}}$$

Chapter 34
Wave-Particle Duality and Quantum Physics

Conceptual Problems

1 • [SSM] The quantized character of electromagnetic radiation is observed by (*a*) the Young double-slit experiment, (*b*) diffraction of light by a small aperture, (*c*) the photoelectric effect, (*d*) the J. J. Thomson cathode-ray experiment.

Determine the Concept The Young double-slit experiment and the diffraction of light by a small aperture demonstrated the wave nature of electromagnetic radiation. J. J. Thomson's experiment showed that the rays of a cathode-ray tube were deflected by electric and magnetic fields and therefore must consist of electrically charged particles. Only the photoelectric effect requires an explanation based on the quantization of electromagnetic radiation. $\boxed{(c)}$ is correct.

2 •• Two monochromatic light sources, A and B, emit the same number of photons per second. The wavelength of A is $\lambda_A = 400$ nm, and the wavelength of B is $\lambda_B = 600$ nm. The power radiated by source B (*a*) is equal to the power of source A, (*b*) is less than the power of source A, (*c*) is greater than the power of source A, (*d*) cannot be compared to the power from source A using the available data.

Determine the Concept Since the power radiated by a source is the energy radiated per unit area and per unit time, it is directly proportional to the energy. The energy radiated varies inversely with the wavelength ($E = hc/\lambda$); i.e., the longer the wavelength, the less energy is associated with the electromagnetic radiation. $\boxed{(b)}$ is correct.

3 • [SSM] The work function of a surface is ϕ. The threshold wavelength for emission of photoelectrons from the surface is equal to (*a*) hc/ϕ, (*b*) ϕ/hf, (*c*) hf/ϕ, (*d*) none of above.

Determine the Concept The work function is equal to the minimum energy required to remove an electron from the material. A photon that has that energy also has the threshold wavelength required for photoemission. Thus, $hf = \phi$. In addition, $c = f\lambda$. It follows that $hc/\lambda_t = \phi$, so $\lambda_t = hc/\phi$ and $\boxed{(a)}$ is correct.

4 •• When light of wavelength λ_1 is incident on a certain photoelectric cathode, no electrons are emitted, no matter how intense the incident light is. Yet, when light of wavelength $\lambda_2 < \lambda_1$ is incident, electrons are emitted, even when the incident light has low intensity. Explain this observation.

Determine the Concept We can use Einstein's photoelectric equation to explain this observation.

Einstein's photoelectric equation is:

$$K_{\max} = hf - \phi$$

where ϕ, called the work function, is a characteristic of the particular metal.

Because $f = c/\lambda$:

$$K_{\max} = \frac{hc}{\lambda} - \phi$$

We're given that when light of wavelength λ_1 is incident on a certain photoelectric cathode; no electrons are emitted independently of the intensity of the incident light. Hence, we can conclude that:

$$\frac{hc}{\lambda_1} - \phi < 0 \Rightarrow \frac{hc}{\lambda_1} < \phi$$

When light of wavelength $\lambda_2 < \lambda_1$ is incident, electrons are emitted, electrons are emitted independently of the intensity of the incident light. Hence, we can conclude that:

$$\frac{hc}{\lambda_2} - \phi > 0 \Rightarrow \frac{hc}{\lambda_2} > \phi$$

Thus, photons with wavelengths greater than the threshold wavelength $\lambda_t = c/f_t$, where f_t is the threshold frequency, do not have enough energy to eject an electron from the metal.

5 • True or false:

(*a*) The wavelength of an electron's matter wave varies inversely with the momentum of the electron.
(*b*) Electrons can undergo diffraction.
(*c*) Neutrons can undergo diffraction.

(*a*) True. The de Broglie wavelength of an electron is given by $\lambda = h/p$.

(*b*) True

(*c*) True

6 • If the wavelength of an electron is equal to the wavelength of a proton, then (*a*) the speed of the proton is greater than the speed of the electron, (*b*) the speeds of the proton and the electron are equal, (*c*) the speed of the proton is less than the speed of the electron, (*d*) the energy of the proton is greater than the energy of the electron, (*e*) Both (*a*) and (*d*) are correct.

Determine the Concept If the de Broglie wavelengths of an electron and a proton are equal, their momenta must be equal. Because $m_p > m_e$, $v_p < v_e$. $\boxed{(c)}$ is correct.

7 • A proton and an electron have equal kinetic energies. It follows that the wavelength of the proton is (*a*) greater than the wavelength of the electron, (*b*) equal to the wavelength of the electron, (*c*) less than the wavelength of the electron.

Determine the Concept The kinetic energy of a particle can be expressed, in terms of its momentum, as $K = p^2/2m$. We can use the equality of the kinetic energies and the fact that $m_e < m_p$ to determine the relative sizes of their de Broglie wavelengths.

Express the equality of the kinetic energies of the proton and electron in terms of their momenta and masses:

$$\frac{p_p^2}{2m_p} = \frac{p_e^2}{2m_e}$$

Use the de Broglie relation for the wavelength of matter waves to obtain:

$$\frac{h^2}{2m_p\lambda_p^2} = \frac{h^2}{2m_e\lambda_e^2} \Rightarrow m_p\lambda_p^2 = m_e\lambda_e^2$$

Because $m_e < m_p$:

$\lambda_p^2 < \lambda_e^2$ and $\lambda_e > \lambda_p$

and $\boxed{(c)}$ is correct.

8 • The parameter x is the position of a particle, $<x>$ is the expectation value of x, and $P(x)$ is the probability density function. Can the expectation value of x ever equal a value of x for which $P(x)$ is zero? If yes, give a specific example. If no, explain why not.

Determine the Concept Yes. Consider a particle in a one-dimensional box of length L that is on the x axis on the interval $0 <x\ L$. The wave function for a particle in the n = 2 state (the lowest state above the ground state) is given by $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(2\pi\frac{x}{L}\right)$ (see Figure 31-13). The expectation value of x is $L/2$ and $P(L/2) = 0$ (see Figure 31-14).

9 •• It was once believed that if two identical experiments are done on identical systems under the same conditions, the results must be identical. Explain how this statement can be modified so that it is consistent with quantum physics.

Determine the Concept According to quantum theory, the average value of many measurements of the same quantity will yield the expectation value of that quantity. However, any single measurement may differ from the expectation value.

10 •• A six-sided die has the numeral 1 painted on three sides and the numeral 2 painted on the other three sides. (*a*) What is the probability of a 1 coming up when the die is thrown? (*b*) What is the expectation value of the numeral that comes up when the die is thrown? (*c*) What is the expectation value of the cube of the numeral that comes up when the die is thrown?

Determine the Concept The probability of a particular event occurring is the number of ways that event can occur divided by the number of possible outcomes. The expectation value, on the other hand, is the average value of the experimental results.

(*a*) Find the probability of a 1 coming up when the die is thrown:

$$P(1) = \frac{3}{6} = \boxed{\frac{1}{2}}$$

(*b*) Find the average value of a large number of throws of the die:

$$\langle n \rangle = \frac{3\times 1 + 3\times 2}{6} = \boxed{1.5}$$

(*c*) The expectation value of the cube of the numeral that comes up when the die is thrown is:

$$\langle n \rangle = \frac{3\times 1^3 + 3\times 2^3}{6} = \boxed{4.5}$$

Estimation and Approximation

11 •• **[SSM]** During an advanced physics lab, students use X rays to measure the Compton wavelength, λ_C. The students obtain the following wavelength shifts $\lambda_2 - \lambda_1$ as a function of scattering angle θ:

θ	45°	75°	90°	135°	180°
$\lambda_2 - \lambda_1$	0.647 pm	1.67 pm	2.45 pm	3.98 pm	4.95 pm

Use their data to estimate the value for the Compton wavelength. Compare this number with the accepted value.

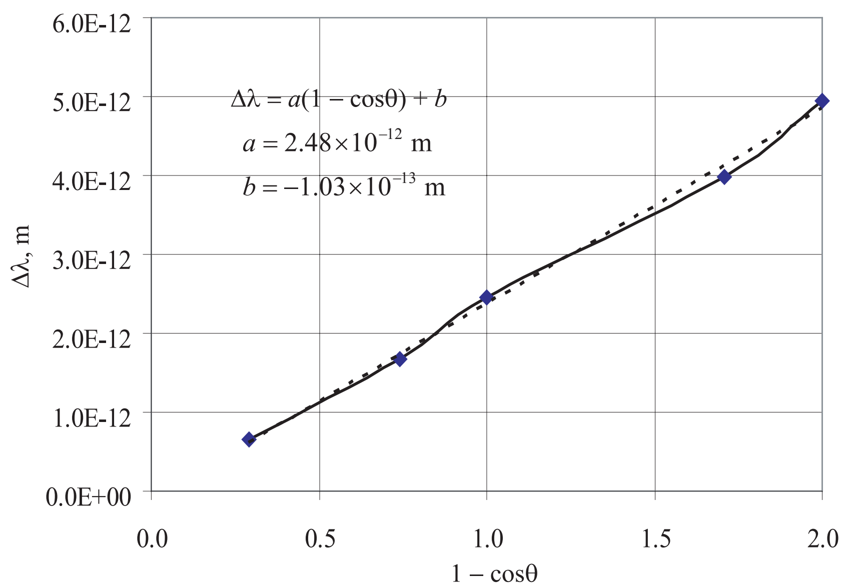
Picture the Problem From the Compton-scattering equation we have $\lambda_2 - \lambda_1 = \lambda_C(1-\cos\theta)$, where $\lambda_C = h/m_e c$ is the Compton wavelength. Note that this equation is of the form $y = mx + b$ provided we let $y = \lambda_2 - \lambda_1$ and $x = 1 - \cos\theta$. Thus, we can linearize the Compton equation by plotting $\Delta\lambda = \lambda_2 - \lambda_1$ as a function of $1-\cos\theta$. The slope of the resulting graph will yield an experimental value for the Compton wavelength.

(*a*) The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A3	45	θ (deg)
B3	1 – cos(A3*PI()/180)	$1-\cos\theta$
C3	6.47E^–13	$\Delta\lambda = \lambda_2 - \lambda_1$

θ	$1-\cos\theta$	$\lambda_2-\lambda_1$
(deg)		
45	0.293	6.47E–13
75	0.741	1.67E–12
90	1.000	2.45E–12
135	1.707	3.98E–12
180	2.000	4.95E–12

The following graph was plotted from the data shown in the above table. Excel's ″Add Trendline″ was used to fit a linear function to the data and to determine the regression constants.



From the trend line we note that the experimental value for the Compton wavelength $\lambda_{C,exp}$ is:

$$\lambda_{C,exp} = \boxed{2.48\,\text{pm}}$$

The Compton wavelength is given by:

$$\lambda_C = \frac{h}{m_e c} = \frac{hc}{m_e c^2}$$

Substitute numerical values and evaluate λ_C:

$$\lambda_C = \frac{1240\,\text{eV}\cdot\text{nm}}{5.11\times10^5\,\text{eV}} = 2.43\,\text{pm}$$

Express the percent difference between λ_C and $\lambda_{C,exp}$:

$$\%\,\text{diff} = \frac{\lambda_{C,exp} - \lambda_{exp}}{\lambda_{exp}} = \frac{\lambda_{C,exp}}{\lambda_{exp}} - 1$$

$$= \frac{2.48\,\text{pm}}{2.43\,\text{pm}} - 1 \approx \boxed{2\%}$$

12 •• Students in a physics lab are trying to determine the value of Planck's constant *h*, using a photoelectric apparatus similar to the one shown in Figure 34-2. The students are using a helium–neon laser that has a tunable wavelength as the light source. The data that the students obtain for the maximum electron kinetic energies are

λ	544 nm	594 nm	604 nm	612 nm	633 nm
K_{max}	0.360 eV	0.199 eV	0.156 eV	0.117 eV	0.062 eV

(*a*) Using a **spreadsheet** program or graphing calculator, plot K_{max} versus light frequency. (*b*) Use the graph to estimate the value of Planck's constant. (*Note:* You may wish to use a feature of your **spreadsheet** program or graphing calculator to obtain the best straight-line fit to the data.) (*c*) Compare your result with the accepted value for Planck's constant.

Picture the Problem From Einstein's photoelectric equation we have $K_{max} = hf - \phi$, which is of the form $y = mx + b$, where the slope is *h* and the K_{max}-intercept is the work function. Hence we should plot a graph of K_{max} versus *f* in order to obtain a straight line whose slope will be an experimental value for Planck's constant.

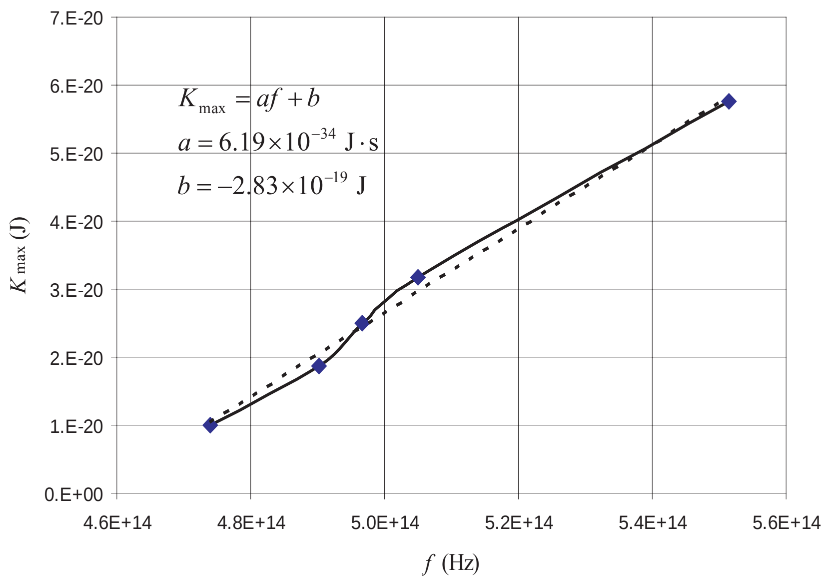
(*a*) The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A3	544	λ (nm)
B3	0.36	K_{max} (eV)
C3	A3*10^–19	λ (m)

D3	3*10^8/C3	c/λ
E3	B3*1.6*10^−19	K_{max} (J)

λ (nm)	K_{max} (eV)	λ (m)	$f = c/\lambda$ (Hz)	K_{max} (J)
544	0.36	5.44E−07	5.51E+14	5.76E−20
594	0.199	5.94E−07	5.05E+14	3.18E−20
604	0.156	6.04E−07	4.97E+14	2.50E−20
612	0.117	6.12E−07	4.90E+14	1.87E−20
633	0.062	6.33E−07	4.74E+14	9.92E−21

The following graph was plotted from the data shown in the above table. Excel's ″Add Trendline″ was used to fit a linear function to the data and to determine the regression constants.



(b) From the trend line we note that the experimental value for Planck's constant is:

$$h_{exp} = \boxed{6.19\times10^{-34}\ \mathrm{J\cdot s}}$$

(c) Express the percent difference between h_{exp} and h:

$$\%\,\mathrm{diff} = \frac{h - h_{exp}}{h} = 1 - \frac{h_{exp}}{h}$$

$$= 1 - \frac{6.19\times10^{-34}\ \mathrm{J\cdot s}}{6.63\times10^{-34}\ \mathrm{J\cdot s}} \approx \boxed{6.6\%}$$

13 •• The cathode that was used by the students in the experiment described in Problem 12 is constructed from one of the following metals:

Metals	Tungsten	Silver	Potassium	Cesium
Work function	4.58 eV	2.4 eV	2.1 eV	1.9 eV

Determine which metal composes the cathode by using the same data given in Problem 12. (*a*) Using a **spreadsheet** program or graphing calculator, plot K_{max} versus frequency. (*b*) Use the graph to estimate the value of the work function based on the students' data. (*Note:* You may wish to use a feature of your **spreadsheet** program or graphing calculator to obtain the best straight-line fit to the data.) (*c*) Which metal was most likely used for the cathode in their experiment?

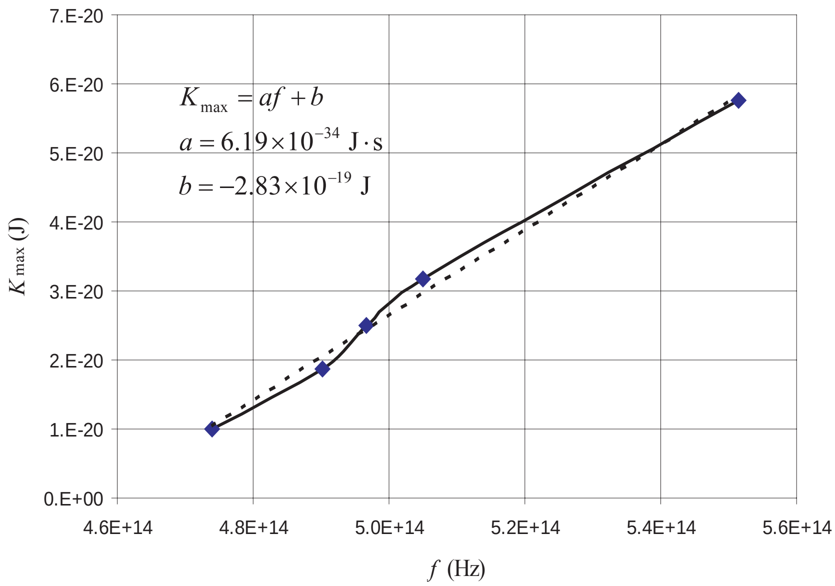
Picture the Problem From Einstein's photoelectric equation we have $K_{max} = hf - \phi$, which is of the form $y = mx + b$, where the slope is h and the K_{max}-intercept is the work function. Hence we should plot a graph of K_{max} versus f in order to obtain a straight line whose intercept will be an experimental value for the work function.

(*a*) The spreadsheet solution is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Formula/Content	Algebraic Form
A3	544	λ (nm)
B3	0.36	K_{max} (eV)
C3	A3*10^−19	λ (m)
D3	3*10^8/C3	c/λ
E3	B3*1.6*10^−19	K_{max} (J)

λ (nm)	K_{max} (eV)	λ (m)	$f = c/\lambda$ (Hz)	K_{max} (J)
544	0.36	5.44E−07	5.51E+14	5.76E−20
594	0.199	5.94E−07	5.05E+14	3.18E−20
604	0.156	6.04E−07	4.97E+14	2.50E−20
612	0.117	6.12E−07	4.90E+14	1.87E−20
633	0.062	6.33E−07	4.74E+14	9.92E−21

The following graph was plotted from the data shown in the above table. Excel's ″Add Trendline″ was used to fit a linear function to the data and to determine the regression constants.



(*b*) From the trend line we note that the experimental value for the work function ϕ is:

$$\phi_{exp} = 2.83\times10^{-19}\ \text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = \boxed{1.77\,\text{eV}}$$

(*c*) The value of ϕ_{exp} = 1.77 eV is closest to the work function for $\boxed{\text{cesium}}$.

The Particle Nature of Light: Photons

14 • Find the photon energy in electron volts for light of wavelength (*a*) 450 nm, (*b*) 550 nm, and (*c*) 650 nm.

Picture the Problem We can use $E = hc/\lambda$ to find the photon energy when we are given the wavelength of the radiation.

(*a*) Express the photon energy as a function of wavelength and evaluate E for λ = 450 nm:

$$E = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{450\,\text{nm}} = \boxed{2.76\,\text{eV}}$$

(*b*) For λ = 550 nm:

$$E = \frac{1240\,\text{eV}\cdot\text{nm}}{550\,\text{nm}} = \boxed{2.25\,\text{eV}}$$

(*c*) For λ = 650 nm:

$$E=\frac{1240\,\text{eV}\cdot\text{nm}}{650\,\text{nm}}=\boxed{1.91\,\text{eV}}$$

15 • Find the photon energy in electron volts for an electromagnetic wave of frequency (*a*) 100 MHz in the FM radio band and (*b*) 900 kHz in the AM radio band.

Picture the Problem We can find the photon energy for an electromagnetic wave of a given frequency f from $E = hf$ where h is Planck's constant.

(*a*) For f = 100 MHz:

$$E=hf=\left(6.626\times10^{-34}\text{ J}\cdot\text{s}\right)\left(100\times10^{6}\text{ s}^{-1}\right)$$
$$=6.626\times10^{-26}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J}}$$
$$=\boxed{4.14\times10^{-7}\text{ eV}}$$

(*b*) For f = 900 kHz:

$$E=hf=\left(6.626\times10^{-34}\text{ J}\cdot\text{s}\right)\left(900\times10^{3}\text{ s}^{-1}\right)$$
$$=5.963\times10^{-28}\text{ J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\text{ J}}$$
$$=\boxed{3.72\times10^{-9}\text{ eV}}$$

16 • What are the frequencies of photons that have the following energies (*a*) 1.00 eV, (*b*) 1.00 keV, and (*c*) 1.00 MeV?

Picture the Problem The energy of a photon, in terms of its frequency, is given by $E=hf$.

(*a*) Express the frequency of a photon in terms of its energy and evaluate f for E = 1.00 eV:

$$f=\frac{E}{h}=\frac{1.00\,\text{eV}}{4.136\times10^{-15}\text{ eV}\cdot\text{s}}$$
$$=\boxed{2.42\times10^{14}\text{ Hz}}$$

(*b*) For E = 1.00 keV:

$$f=\frac{1.00\,\text{keV}}{4.136\times10^{-15}\text{ eV}\cdot\text{s}}$$
$$=\boxed{2.42\times10^{17}\text{ Hz}}$$

(*c*) For E = 1.00 MeV:

$$f=\frac{1.00\,\text{MeV}}{4.136\times10^{-15}\text{ eV}\cdot\text{s}}$$
$$=\boxed{2.42\times10^{20}\text{ Hz}}$$

17 • Find the photon energy in electron volts if the wavelength is (*a*) 0.100 nm (about 1 atomic diameter) and (*b*) 1.00 fm (1 fm = 10^{-15} m, about 1 nuclear diameter).

Picture the Problem We can use $E = hc/\lambda$ to find the photon energy when we are given the wavelength of the radiation.

The energy of a photon as a function of its wavelength is given by:

$$E = \frac{hc}{\lambda}$$

(*a*) Substitute numerical values and evaluate E for λ = 0.100 nm:

$$E = \frac{1240\,\text{eV}\cdot\text{nm}}{0.100\,\text{nm}} = \boxed{12.4\,\text{keV}}$$

(*b*) Substitute numerical values and evaluate E for λ =1.00 fm = 1.00 × 10^{-6} nm:

$$E = \frac{1240\,\text{eV}\cdot\text{nm}}{1.00\times10^{-6}\,\text{nm}} = \boxed{1.24\,\text{GeV}}$$

18 •• The wavelength of red light emitted by a 3.00-mW helium–neon laser is 633 nm. If the diameter of the laser beam is 1.00 mm, what is the density of photons in the beam? Assume that the intensity is uniformly distributed across the beam.

Picture the Problem We can express the density of photons in the beam as the number of photons per unit volume. The number of photons per unit volume is, in turn, the ratio of the power of the laser to the energy of the photons and the volume occupied by the photons emitted in one second is the product of the cross-sectional area of the beam and the speed at which the photons travel, i.e., the speed of light.

Express the density of photons in the beam as a function of the number of photons emitted per second and the volume occupied by those photons:

$$\rho = \frac{N}{V}$$

Relate the number of photons emitted per second to the power of the laser and the energy of the photons:

$$N = \frac{P}{E} = \frac{P\lambda}{hc}$$

Express the volume containing the photons emitted in one second as a function of the cross sectional area of the beam:

$$V = Ac$$

Substitute for N and V and simplify to obtain:

$$\rho = \frac{P\lambda}{hc^2 A} = \frac{P\lambda}{hc^2\left(\frac{\pi}{4}d^2\right)} = \frac{4P\lambda}{\pi hc^2 d^2}$$

where d is the diameter of the laser beam.

Substitute numerical values and evaluate ρ:

$$\rho = \frac{4(3.00\,\text{mW})(633\,\text{nm})}{\pi(6.626\times10^{-34}\,\text{J}\cdot\text{s})(2.998\times10^{8}\,\text{m/s})^2(1.00\,\text{mm})^2} = \boxed{4.06\times10^{13}\,\text{m}^{-3}}$$

19 • **[SSM]** Lasers used in a telecommunications network typically produce light that has a wavelength near 1.55 μm. How many photons per second are being transmitted if such a laser has an output power of 2.50 mW?

Picture the Problem The number of photons per second is the ratio of the power of the laser to the energy of the photons.

Relate the number of photons emitted per second to the power of the laser and the energy of the photons:

$$N = \frac{P}{E} = \frac{P}{\frac{hc}{\lambda}} = \frac{P\lambda}{hc}$$

Substitute numerical values and evaluate N:

$$N = \frac{(2.50\,\text{mW})(1.55\,\mu\text{m})}{(6.626\times10^{-34}\,\text{J}\cdot\text{s})(2.998\times10^{8}\,\text{m/s})} = \boxed{1.95\times10^{16}\,\text{s}^{-1}}$$

The Photoelectric Effect

20 • The work function for tungsten is 4.58 eV. (*a*) Find the threshold frequency and wavelength for the photoelectric effect to occur when monochromatic electromagnetic radiation is incident on the surface of a sample of tungsten. Find the maximum kinetic energy of the electrons if the wavelength of the incident light is (*b*) 200 nm and (*c*) 250 nm.

Picture the Problem The threshold wavelength and frequency for emission of photoelectrons is related to the work function of a metal through $\phi = hf_t = hc/\lambda_t$. We can use Einstein's photoelectric equation $K_{max} = \dfrac{hc}{\lambda} - \phi$ to find the maximum kinetic energy of the electrons for the given wavelengths of the incident light.

(*a*) Express the threshold frequency in terms of the work function for tungsten and evaluate f_t:

$$f_t = \frac{\phi}{h} = \frac{4.58\,\text{eV}}{4.136\times10^{-15}\,\text{eV}\cdot\text{s}} = 1.107\times10^{15}\,\text{Hz} = \boxed{1.11\times10^{15}\,\text{Hz}}$$

Using $c = f\lambda$, express the threshold wavelength in terms of the threshold frequency and evaluate λ_t:

$$\lambda_t = \frac{c}{f_t} = \frac{2.998\times10^{8}\,\text{m/s}}{1.107\times10^{15}\,\text{Hz}} = \boxed{271\,\text{nm}}$$

(*b*) Using Einstein's photoelectric equation, relate the maximum kinetic energy of the electrons to their wavelengths and evaluate K_{max}:

$$K_{max} = E - \phi = hf - \phi = \frac{hc}{\lambda} - \phi = \frac{1240\,\text{eV}\cdot\text{nm}}{200\,\text{nm}} - 4.58\,\text{eV} = \boxed{1.62\,\text{eV}}$$

(*c*) Evaluate K_{max} for λ = 250 nm:

$$K_{max} = \frac{1240\,\text{eV}\cdot\text{nm}}{250\,\text{nm}} - 4.58\,\text{eV} = \boxed{0.380\,\text{eV}}$$

21 • When monochromatic ultraviolet light that has a wavelength equal to 300 nm is incident on a sample of potassium, the emitted electrons have maximum kinetic energy of 2.03 eV. (*a*) What is the energy of an incident photon? (*b*) What is the work function for potassium? (*c*) What would be the maximum kinetic energy of the electrons if the incident electromagnetic radiation had a wavelength of 430 nm? (*d*) What is the maximum wavelength of incident electromagnetic radiation that will result in the photoelectric emission of electrons by a sample of potassium?

Picture the Problem (*a*) We can use the Einstein equation for photon energy to find the energy of an incident photon. (*b*) and (*c*) We can use the photoelectric equation to relate the work function for potassium to the maximum energy of the photoelectrons. (*d*) The maximum wavelength of incident electromagnetic radiation that will result in the photoelectric emission of electrons can be found from $\lambda_t = hc/\phi$.

(*a*) Use the Einstein equation for photon energy to relate the energy of the incident photon to its wavelength:

$$E = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{300\,\text{nm}} = \boxed{4.13\,\text{eV}}$$

(*b*) Using Einstein's photoelectric equation, relate the work function for potassium to the maximum kinetic energy of the photoelectrons:

$$K_{max} = E - \phi$$

Solve for and evaluate ϕ:

$$\phi = E - K_{max} = 4.13\,\text{eV} - 2.03\,\text{eV} = \boxed{2.10\,\text{eV}}$$

(*c*) Proceeding as in (*b*) with $E = hc/\lambda$ gives:

$$K_{max} = \frac{hc}{\lambda} - \phi = \frac{1240\,\text{eV}\cdot\text{nm}}{430\,\text{nm}} - 2.10\,\text{eV} = \boxed{0.78\,\text{eV}}$$

(*d*) Express the maximum wavelength as a function of potassium's work function and evaluate λ_t:

$$\lambda_t = \frac{hc}{\phi} = \frac{1240\,\text{eV}\cdot\text{nm}}{2.10\,\text{eV}} = \boxed{590\,\text{nm}}$$

22 • The maximum wavelength of electromagnetic radiation that will result in the photoelectric emission of electrons from a sample of silver is 262 nm. (*a*) Find the work function for silver. (*b*) Find the maximum kinetic energy of the electrons if the incident radiation has a wavelength of 175 nm.

Picture the Problem (*a*) We can find the work function for silver using $\phi = hc/\lambda_t$ and (*b*) the maximum kinetic energy of the electrons using Einstein's photoelectric equation.

(*a*) Express the work function for silver as a function of the maximum wavelength of the electromagnetic radiation that will result in photoelectric emission of electrons:

$$\phi = \frac{hc}{\lambda_t}$$

Substitute numerical values and evaluate ϕ:

$$\phi = \frac{1240\,\text{eV}\cdot\text{nm}}{262\,\text{nm}} = \boxed{4.73\,\text{eV}}$$

(*b*) Using Einstein's photoelectric equation, relate the work function for silver to the maximum kinetic energy of the photoelectrons:

$$K_{\max} = E - \phi = \frac{hc}{\lambda} - \phi$$

Substitute numerical values and evaluate $K_{\max}$:

$$K_{\max} = \frac{1240\,\text{eV}\cdot\text{nm}}{175\,\text{nm}} - 4.73\,\text{eV} = \boxed{2.36\,\text{eV}}$$

23 • The work function for cesium is 1.90 eV. (*a*) Find the minimum frequency and maximum wavelength of electromagnetic radiation that will result in the photoelectric emission of electrons from a sample of cesium. Find the maximum kinetic energy of the electrons if the wavelength of the incident radiation is (*b*) 250 nm and (*c*) 350 nm.

Picture the Problem We can find the minimum frequency and maximum wavelength for cesium using $\phi = hf_{\text{t}} = hc/\lambda_{\text{t}}$ and the maximum kinetic energy of the electrons using Einstein's photoelectric equation.

(*a*) Use the Einstein equation for photon energy to express the maximum wavelength for cesium:

$$\lambda_{\text{t}} = \frac{hc}{\phi}$$

Substitute numerical values and evaluate λ_{t}:

$$\lambda_{\text{t}} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.90\,\text{eV}} = \boxed{653\,\text{nm}}$$

Use $c = f\lambda$ to find the maximum frequency:

$$f_{\text{t}} = \frac{c}{\lambda_{\text{t}}} = \frac{2.998\times10^{8}\ \text{m/s}}{653\,\text{nm}} = \boxed{4.59\times10^{14}\ \text{Hz}}$$

(*b*) Using Einstein's photoelectric equation, relate the maximum kinetic energy of the photoelectrons to the wavelength of the incident light:

$$K_{\max} = \frac{hc}{\lambda} - \phi$$

Evaluating $K_{\max}$ for $\lambda = 250$ nm gives:

$$K_{\max} = \frac{1240\,\text{eV}\cdot\text{nm}}{250\,\text{nm}} - 1.90\,\text{eV} = \boxed{3.06\,\text{eV}}$$

(c) Proceed as in Part (b) with $\lambda = 350$ nm to obtain:

$$K_{max} = \frac{1240\,\text{eV}\cdot\text{nm}}{350\,\text{nm}} - 1.90\,\text{eV} = \boxed{1.64\,\text{eV}}$$

24 •• When a surface is illuminated with electromagnetic radiation of wavelength 780 nm, the maximum kinetic energy of the emitted electrons is 0.37 eV. What is the maximum kinetic energy if the surface is illuminated using radiation of wavelength 410 nm?

Picture the Problem We can use Einstein's photoelectric equation to find the work function of this surface and then apply it a second time to find the maximum kinetic energy of the photoelectrons when the surface is illuminated with light of wavelength 410 nm.

Use Einstein's photoelectric equation to relate the maximum kinetic energy of the emitted electrons to their total energy and the work function of the surface:

$$K_{max} = \frac{hc}{\lambda} - \phi \qquad (1)$$

The work function of the surface is given by:

$$\phi = E - K_{max} = \frac{hc}{\lambda} - K_{max}$$

Substitute numerical values and evaluate ϕ:

$$\phi = \frac{1240\,\text{eV}\cdot\text{nm}}{780\,\text{nm}} - 0.37\,\text{eV} = 1.22\,\text{eV}$$

Substitute for ϕ and λ in equation (1) and evaluate K_{max}:

$$K_{max} = \frac{1240\,\text{eV}\cdot\text{nm}}{410\,\text{nm}} - 1.22\,\text{eV} = \boxed{1.80\,\text{eV}}$$

Compton Scattering

25 • Find the shift in wavelength of photons scattered by free stationary electrons at $\theta = 60°$. (Assume that the electrons are initially moving with negligible speed and are virtually free of (unattached to) any atoms or molecules.)

Picture the Problem We can calculate the shift in wavelength using the Compton relationship $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$.

The shift in wavelength is given by:

$$\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$$

Substitute numerical values and evaluate $\Delta\lambda$:

$$\Delta\lambda = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})}(1-\cos60°) = \boxed{1.2\,\text{pm}}$$

26 • When photons are scattered by electrons in a carbon sample, the shift in wavelength is 0.33 pm. Find the scattering angle. (Assume that the electrons are initially moving with negligible speed and are virtually free of (unattached to) any atoms or molecules.)

Picture the Problem We can calculate the scattering angle using the Compton relationship $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$.

Using the Compton scattering equation, relate the shift in wavelength to the scattering angle:

$$\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$$

Solving for θ yields:

$$\theta = \cos^{-1}\left(1-\frac{m_e c}{h}\Delta\lambda\right)$$

Substitute numerical values and evaluate θ:

$$\theta = \cos^{-1}\left(1-\frac{(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})}{6.626\times10^{-34}\ \text{J}\cdot\text{s}}(0.33\,\text{pm})\right) = \boxed{30°}$$

27 • The photons in a monochromatic beam are scattered by electrons. The wavelength of the photons that are scattered at an angle of 135° with the direction of the incident photon beam is 2.3 percent less than the wavelength of the incident photons. What is the wavelength of the incident photons?

Picture the Problem We can calculate the shift in wavelength using the Compton relationship $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$.

Express the wavelength of the incident photons in terms of the fractional change in wavelength:

$$\frac{\Delta\lambda}{\lambda} = 0.023 \Rightarrow \lambda = \frac{\Delta\lambda}{0.023}$$

Using the Compton scattering equation, relate the shift in wavelength to the scattering angle:

$$\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$$

Substituting for $\Delta\lambda$ yields:

$$\lambda = \frac{h}{0.023 m_e c}(1-\cos\theta)$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{(0.023)(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})}(1-\cos135°) = \boxed{0.18\,\text{nm}}$$

28 • Compton used photons of wavelength 0.0711 nm. (*a*) What is the energy of one of these photons? (*b*) What is the wavelength of the photons scattered in the direction opposite to the direction of the incident photons? (*c*) What is the energy of the photon scattered in this direction?

Picture the Problem We can use the Einstein equation for photon energy to find the energy of both the incident and scattered photon and the Compton scattering equation to find the wavelength of the scattered photon.

(*a*) Use the Einstein equation for photon energy to obtain:

$$E = \frac{hc}{\lambda_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.0711\,\text{nm}} = \boxed{17.4\,\text{keV}}$$

(*b*) Express the wavelength of the scattered photon in terms of its pre-scattering wavelength and the shift in its wavelength during scattering:

$$\lambda_2 = \lambda_1 + \Delta\lambda = \lambda_1 + \frac{h}{m_e c}(1-\cos\theta)$$

Substitute numerical values and evaluate λ_2:

$$\lambda_2 = 0.0711\,\text{nm} + \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})}(1-\cos180°) = \boxed{0.0760\,\text{nm}}$$

(*c*) Use the Einstein equation for photon energy to obtain:

$$E = \frac{hc}{\lambda_2} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.0760\,\text{nm}} = \boxed{16.3\,\text{keV}}$$

29 • For the photons used by Compton (see Problem 28), find the momentum of the incident photon and the momentum of the photon scattered in the direction opposite to the direction of the incident photons. Use the

conservation of momentum to find the momentum of the recoil electron in this case.

Picture the Problem Compton used X rays of wavelength 71.1 pm. Let the direction the incident photon (and the recoiling electron) is moving be the positive direction. We can use $p = h/\lambda$ to find the momentum of the incident photon and the conservation of momentum to find its momentum after colliding with the electron.

Use the expression for the momentum of a photon to find the momentum of Compton's photons:

$$p_1 = \frac{h}{\lambda_1} = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{71.1\,\text{pm}} = \boxed{9.32\times10^{-24}\ \text{kg}\cdot\text{m/s}}$$

Using the Compton scattering equation, relate the shift in wavelength to the scattering angle:

$$\lambda_2 = \lambda_1 + \lambda_C(1-\cos\theta)$$

Substitute numerical values and evaluate λ_2:

$$\lambda_2 = 71.1\,\text{pm} + \left(2.43\times10^{-12}\ \text{m}\right)(1-\cos180°) = 76.0\,\text{pm}$$

Apply conservation of momentum to obtain:

$$p_1 = p_e - p_2 \Rightarrow p_e = p_1 - p_2$$

Substitute for p_1 and p_2 and evaluate p_e:

$$p_e = 9.32\times10^{-24}\ \text{kg}\cdot\text{m/s} - \left(-\frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{76.0\,\text{pm}}\right) = \boxed{1.80\times10^{-23}\ \text{kg}\cdot\text{m/s}}$$

30 •• A beam of photons have a wavelength equal to 6.00 pm is scattered by electrons initially at rest. A photon in the beam is scattered in a direction perpendicular to the direction of the incident beam. (*a*) What is the change in wavelength of the photon? (*b*) What is the kinetic energy of the electron?

Picture the Problem We can calculate the shift in wavelength using the Compton relationship $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta) = \lambda_C(1-\cos\theta)$ and use conservation of energy to find the kinetic energy of the scattered electron.

(*a*) Use the Compton scattering equation to find the change in wavelength of the photon:

$$\Delta\lambda = \lambda_C(1-\cos\theta) = (2.43\times10^{-12}\ \mathrm{m})(1-\cos 90°) = \boxed{2.43\,\mathrm{pm}}$$

(*b*) Use conservation of energy to relate the change in the kinetic energy of the electron to the energies of the incident and scattered photon:

$$\Delta E_e = \frac{hc}{\lambda_1} - \frac{hc}{\lambda_2}$$

Find the wavelength of the scattered photon:

$$\lambda_2 = \lambda_1 + \Delta\lambda = 6.00\,\mathrm{pm} + 2.43\,\mathrm{pm} = 8.43\,\mathrm{pm}$$

Substitute numerical values and evaluate the kinetic energy of the electron (equal to the change in its energy since it was stationary prior to the collision with the photon):

$$\Delta E_e = 1240\,\mathrm{eV\cdot nm}\left(\frac{1}{6.00\,\mathrm{pm}} - \frac{1}{8.43\,\mathrm{pm}}\right) = \boxed{60\,\mathrm{keV}}$$

Electrons and Matter Waves

31 • An electron is moving at 2.5 × 10^5 m/s. Find the electron's de Broglie wavelength.

Picture the Problem We can use its definition to find the de Broglie wavelength of this electron.

Use its definition to express the de Broglie wavelength of the electron in terms of its momentum:

$$\lambda = \frac{h}{p} = \frac{h}{m_e v}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{6.626\times10^{-34}\ \mathrm{J\cdot s}}{(9.109\times10^{-31}\ \mathrm{kg})(2.5\times10^{5}\ \mathrm{m/s})} = \boxed{2.9\,\mathrm{nm}}$$

32 • An electron has a wavelength of 200 nm. Find (*a*) the magnitude of its momentum and (*b*) its kinetic energy.

Picture the Problem We can find the momentum of the electron from the de Broglie equation and its kinetic energy from $\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm}$, where K is in eV.

(*a*) Use the de Broglie relation to express the momentum of the electron:

$$p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{200\,\text{nm}}$$
$$= \boxed{3.31\times10^{-27}\ \text{kg}\cdot\text{m/s}}$$

(*b*) Use the electron wavelength equation to relate the electron's wavelength to its kinetic energy:

$$\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm}$$

Solving for K gives:

$$K = \left(\frac{1.226\,\text{eV}^{1/2}\text{nm}}{200\,\text{nm}}\right)^2$$
$$= \boxed{3.76\times10^{-5}\ \text{eV}}$$

33 •• An electron, a proton, and an alpha particle each have a kinetic energy of 150 keV. Find (*a*) the magnitudes of their momenta and (*b*) their de Broglie wavelengths.

Picture the Problem The momenta of these particles can be found from their kinetic energies. The values for the electron, however, are only approximately correct because at the given energy the speed of the electron is a significant fraction of the speed of light. The solution presented is valid only in the non-relativistic limit $v << c$. The de Broglie wavelengths of the particles are given by $\lambda = h/p$.

(*a*) The momentum of a particle p, in terms of its kinetic energy K, is given by:

$$p = \sqrt{2mK}$$

Substitute numerical values and evaluate p_e:

$$p_e = \sqrt{2m_eK} = \sqrt{2\left(9.109\times10^{-31}\ \text{kg}\right)\left(150\,\text{keV}\times\frac{1.602\times10^{-19}\ \text{C}}{\text{eV}}\right)}$$
$$= 2.092\times10^{-22}\ \text{N}\cdot\text{s} = \boxed{2.09\times10^{-22}\ \text{N}\cdot\text{s}}$$

Substitute numerical values and evaluate p_p:

$$p_p = \sqrt{2m_pK} = \sqrt{2(1.673\times10^{-27}\text{ kg})\left(150\text{ keV}\times\frac{1.602\times10^{-19}\text{ C}}{\text{eV}}\right)}$$
$$= 8.967\times10^{-21}\text{ N}\cdot\text{s} = \boxed{8.97\times10^{-21}\text{ N}\cdot\text{s}}$$

Substitute numerical values and evaluate p_α:

$$p_\alpha = \sqrt{2m_\alpha K} = \sqrt{2\left(4\text{ u}\times\frac{1.661\times10^{-27}\text{ kg}}{\text{u}}\right)\left(150\text{ keV}\times\frac{1.602\times10^{-19}\text{ C}}{\text{eV}}\right)}$$
$$= 1.787\times10^{-20}\text{ N}\cdot\text{s} = \boxed{1.79\times10^{-20}\text{ N}\cdot\text{s}}$$

(*b*) The de Broglie wavelengths of the particles are given by:

$$\lambda = \frac{h}{p} \qquad (1)$$

Substitute numerical values in equation (1) and evaluate λ_e:

$$\lambda_e = \frac{h}{p_e} = \frac{6.626\times10^{-34}\text{ J}\cdot\text{s}}{2.092\times10^{-21}\text{ N}\cdot\text{s}} = \boxed{3.17\text{ pm}}$$

Substitute numerical values in equation (1) and evaluate λ_p:

$$\lambda_p = \frac{h}{p_p} = \frac{6.626\times10^{-34}\text{ J}\cdot\text{s}}{8.967\times10^{-21}\text{ N}\cdot\text{s}} = \boxed{73.9\text{ fm}}$$

Substitute numerical values in equation (1) and evaluate λ_α:

$$\lambda_\alpha = \frac{h}{p_\alpha} = \frac{6.63\times10^{-34}\text{ J}\cdot\text{s}}{1.79\times10^{-20}\text{ N}\cdot\text{s}} = \boxed{37.0\text{ fm}}$$

34 • A neutron in a reactor has kinetic energy of approximately 0.020 eV. Calculate the wavelength of this neutron.

Picture the Problem The wavelength associated with a particle of mass *m* and kinetic energy *K* is given by $\lambda = \dfrac{1240\text{ eV}\cdot\text{nm}}{\sqrt{2mc^2K}}$.

Substituting numerical values yields:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\sqrt{2(940\,\text{MeV})(0.020\,\text{eV})}} = \boxed{0.20\,\text{nm}}$$

35 • Find the wavelength of a proton that has a kinetic energy of 2.00 MeV.

Picture the Problem The wavelength associated with a particle of mass *m* and kinetic energy *K* is given by $\lambda = \dfrac{1240\,\text{eV}\cdot\text{nm}}{\sqrt{2mc^2K}}$.

Substituting numerical values yields:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\sqrt{2(938\,\text{MeV})(2.00\,\text{MeV})}} = \boxed{20.2\,\text{fm}}$$

36 • What is the kinetic energy of a proton whose wavelength is (*a*) 1.00 nm and (*b*) 1.00 fm?

Picture the Problem We can solve Equation 34-15 ($\lambda = \dfrac{1240\,\text{eV}\cdot\text{nm}}{\sqrt{2mc^2K}}$) for the kinetic energy of the proton and use the rest energy of a proton mc^2 = 938 MeV to simplify our computation.

Solving Equation 34-15 for the kinetic energy of the proton gives:

$$K = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{2mc^2\lambda^2}$$

(*a*) Substitute numerical values and evaluate *K* for λ = 1.00 nm:

$$K = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{2(938\,\text{MeV})(1.00\,\text{nm})^2} = \boxed{0.820\,\text{meV}}$$

(*b*) Evaluate *K* for λ = 1.00 fm:

$$K = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{2(938\,\text{MeV})(1.00\,\text{fm})^2} = \boxed{820\,\text{MeV}}$$

37 • The kinetic energy of the electrons in the electron beam in a run of Davisson and Germer's experiment was 54 eV. Calculate the wavelength of the electrons in the beam.

Picture the Problem If *K* is in electron volts, the wavelength of a particle is given by $\lambda = \dfrac{1.226}{\sqrt{K}}\,\text{nm}$ provided *K* is in eV.

Evaluate λ for $K = 54$ eV:

$$\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm} = \frac{1.226}{\sqrt{54}}\,\text{nm} = \boxed{0.17\,\text{nm}}$$

38 • The distance between Li^+ and Cl^- ions in a LiCl crystal is 0.257 nm. Find the energy of electrons that have a wavelength equal to this spacing.

Picture the Problem We can use $\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm}$, where K is in eV, to find the energy of electrons whose wavelength is λ.

Relate the wavelength of the electrons to their kinetic energy:

$$\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm}$$

Solving for K yields:

$$K = \left(\frac{1.226\,\text{nm}\cdot\text{eV}^{1/2}}{\lambda}\right)^2$$

Substituting numerical values and evaluating K gives:

$$K = \left(\frac{1.226\,\text{nm}\cdot\text{eV}^{1/2}}{0.257\,\text{nm}}\right)^2 = \boxed{22.8\,\text{eV}}$$

39 • **[SSM]** An electron microscope uses electrons that have energies equal to 70 keV. Find the wavelength of these electrons.

Picture the Problem We can approximate the wavelength of 70-keV electrons using $\lambda = \frac{1.226}{\sqrt{K}}\,\text{nm}$, where K is in eV. This solution is, however, only approximately correct because at the given energy the speed of the electron is a significant fraction of the speed of light. The solution presented is valid only in the non-relativistic limit $v << c$.

Relate the wavelength of the electrons to their kinetic energy:

$$\lambda = \frac{1.23}{\sqrt{K}}\,\text{nm}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1.226}{\sqrt{70\times10^3\,\text{eV}}}\,\text{nm} = \boxed{4.6\,\text{pm}}$$

40 • What is the de Broglie wavelength of a neutron that has a speed of 1.00×10^6 m/s?

Picture the Problem We can use its definition to calculate the de Broglie wavelength of a neutron with speed 10^6 m/s.

Use its definition to express the de Broglie wavelength of the neutron:

$$\lambda_n = \frac{h}{p_n} = \frac{h}{m_n v_n}$$

Substitute numerical values and evaluate λ_n:

$$\lambda_n = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{\left(1.675\times10^{-27}\ \text{kg}\right)\left(1.00\times10^{6}\ \text{m/s}\right)} = \boxed{0.396\ \text{pm}}$$

A Particle in a Box

41 •• (*a*) Find the energy of the ground state ($n = 1$) and the first two excited states of a neutron in a one-dimensional box of length $L = 1.00\times10^{-15}$ m = 1.00 fm (about the diameter of an atomic nucleus). Make an energy-level diagram for this system. Calculate the wavelength of electromagnetic radiation emitted when the neutron makes a transition from (*b*) $n = 2$ to $n = 1$, (*c*) $n = 3$ to $n = 2$, and (*d*) $n = 3$ to $n = 1$.

Picture the Problem We can find the ground-state energy using $E_1 = \frac{h^2}{8mL^2}$ and the energies of the excited states using $E_n = n^2E_1$. The wavelength of the electromagnetic radiation emitted when the neutron transitions from one state to another is given by the Einstein equation for photon energy ($E = \frac{hc}{\lambda}$).

(*a*) Express the ground-state energy:

$$E_1 = \frac{h^2}{8m_n L^2}$$

Substitute numerical values and evaluate E_1:

$$E_1 = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(1.6749\times10^{-27}\ \text{kg}\right)\left(1.00\times10^{-15}\ \text{m}\right)^2}\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = 204.53\,\text{MeV}$$
$$= \boxed{205\,\text{MeV}}$$

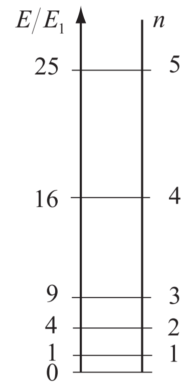
Find the energies of the first two excited states:

$$E_2 = 2^2E_1 = 4(204.53\,\text{MeV}) = \boxed{818\,\text{MeV}}$$

and

$$E_3 = 3^2E_1 = 9(204.53\,\text{MeV}) = \boxed{1.84\,\text{GeV}}$$

The energy-level diagram for this system is shown to the right:



(*b*) Relate the wavelength of the electromagnetic radiation emitted during a neutron transition to the energy released in the transition:

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E}$$

For the $n = 2$ to $n = 1$ transition:

$$\Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1$$

Substitute numerical values and evaluate $\lambda_{2\to1}$:

$$\lambda_{2\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{3E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{3(204.53\,\text{MeV})} = \boxed{2.02\,\text{fm}}$$

(*c*) For the $n = 3$ to $n = 2$ transition:

$$\Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1$$

Substitute numerical values and evaluate $\lambda_{3\to2}$:

$$\lambda_{3\to2} = \frac{1240\,\text{eV}\cdot\text{nm}}{5E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{5(204.53\,\text{MeV})} = \boxed{1.21\,\text{fm}}$$

(*d*) For the $n = 3$ to $n = 1$ transition:

$$\Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1$$

Substitute numerical values and evaluate $\lambda_{3\to1}$:

$$\lambda_{3\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{8E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{8(204.53\,\text{MeV})} = \boxed{0.758\,\text{fm}}$$

42 •• (*a*) Find the energy of the ground state ($n = 1$) and the first two excited states of a neutron in a one-dimensional box of length 0.200 nm (about the diameter of a H_2 molecule). Calculate the wavelength of electromagnetic radiation emitted when the neutron makes a transition from (*b*) $n = 2$ to $n = 1$, (*c*) $n = 3$ to $n = 2$, and (*d*) $n = 3$ to $n = 1$.

Picture the Problem We can find the ground-state energy using $E_1 = \dfrac{h^2}{8mL^2}$ and the energies of the excited states using $E_n = n^2 E_1$. The wavelength of the electromagnetic radiation emitted when the neutron transitions from one state to another is given by the Einstein equation for photon energy ($E = \dfrac{hc}{\lambda}$).

(*a*) Express the ground-state energy:

$$E_1 = \frac{h^2}{8m_\mathrm{n}L^2}$$

Substitute numerical values and evaluate E_1:

$$E_1 = \frac{\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)^2}{8\left(1.6749\times10^{-27}\ \mathrm{kg}\right)\left(0.200\,\mathrm{nm}\right)^2}\frac{1\,\mathrm{eV}}{1.602\times10^{-19}\ \mathrm{J}} = 5.1133\,\mathrm{meV} = \boxed{5.11\,\mathrm{meV}}$$

Find the energies of the first two excited states:

$$E_2 = 2^2 E_1 = 4(5.1133\,\mathrm{meV}) = \boxed{20.5\,\mathrm{meV}}$$

and

$$E_3 = 3^2 E_1 = 9(5.1133\,\mathrm{meV}) = \boxed{46.0\,\mathrm{meV}}$$

(*b*) Relate the wavelength of the electromagnetic radiation emitted during a neutron transition to the energy released in the transition:

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\mathrm{eV\cdot nm}}{\Delta E}$$

For the $n = 2$ to $n = 1$ transition:

$$\Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1$$

Substitute numerical values and evaluate $\lambda_{2\to1}$:

$$\lambda_{2\to1} = \frac{1240\,\mathrm{eV\cdot nm}}{3E_1} = \frac{1240\,\mathrm{eV\cdot nm}}{3(5.1133\,\mathrm{meV})} = \boxed{80.8\,\mu\mathrm{m}}$$

(*c*) For the $n = 3$ to $n = 2$ transition:

$$\Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1$$

Substitute numerical values and evaluate $\lambda_{3\to2}$:

$$\lambda_{3\to2} = \frac{1240\,\mathrm{eV\cdot nm}}{5E_1} = \frac{1240\,\mathrm{eV\cdot nm}}{5(5.1133\,\mathrm{meV})} = \boxed{48.5\,\mu\mathrm{m}}$$

(*d*) For the $n = 3$ to $n = 1$ transition: $\Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1$

Substitute numerical values and evaluate $\lambda_{3\to1}$:

$$\lambda_{3\to1} = \frac{1240\,\text{eV}\cdot\text{nm}}{8E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{8(5.1133\,\text{meV})} = \boxed{30.3\,\mu\text{m}}$$

Calculating Probabilities and Expectation Values

43 •• A particle is in the ground state of a one-dimensional box that has length L. (The box has one end at the origin and the other end on the positive x axis.) Determine the probability of finding the particle in the interval of length $\Delta x = 0.002L$ and centered at (*a*) $\frac{1}{4}x = L$, (*b*) $x = \frac{1}{2}L$, and (*c*) $x = \frac{3}{4}L$. (Because Δx is very small you need not do any integration.)

Picture the Problem The probability of finding the particle in some range Δx is $\psi^2 dx$. The interval $\Delta x = 0.002L$ is so small that we can neglect the variation in $\psi(x)$ and just compute $\psi^2\Delta x$.

Express the probability of finding the particle in the interval Δx:

$$P = P(x)\Delta x = \psi^2(x)\Delta x$$

Express the wave function for a particle in the ground state:

$$\psi_1(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

Substitute for $\psi_1(x)$ to obtain:

$$P = \frac{2}{L}\sin^2\frac{\pi x}{L}\Delta x = \frac{2}{L}\left(\sin^2\frac{\pi x}{L}\right)(0.002L) = 0.004\sin^2\left(\frac{\pi x}{L}\right)$$

(*a*) Evaluate P at $x = 4L$:

$$P(4L) = 0.004\sin^2\frac{\pi(4L)}{2L} = 0.004\sin^2(2\pi) = \boxed{0}$$

(*b*) Evaluate P at $x = L/2$:

$$P(L/2) = 0.004\sin^2\frac{\pi L}{2L} = 0.004\sin^2\left(\frac{\pi}{2}\right) = \boxed{1}$$

(c) Evaluate P at $x = 3L/4$:

$$P(3L/4) = 0.004\sin^2\left(\frac{3\pi L}{4L}\right) = 0.004\sin^2\left(\frac{3\pi}{4}\right) = \boxed{0.002}$$

44 •• A particle is in the second excited state (n = 3) of a one-dimensional box that has length L. (The box has one end at the origin and the other end on the positive x axis.) Determine the probability of finding the particle in the interval of length $\Delta x = 0.002L$ and centered at (a) $x = \frac{1}{3}L$, (b) $x = \frac{1}{2}L$, and (c) $x = \frac{2}{3}L$. (Because Δx is very small you need not do any integration.)

Picture the Problem The probability of finding the particle in some range dx is $\psi^2 dx$. The interval $\Delta x = 0.002L$ is so small that we can neglect the variation in $\psi(x)$ and just compute $\psi^2\Delta x$.

Express the probability of finding the particle in the interval Δx:

$$P = P(x)\Delta x = \psi^2(x)\Delta x$$

Express the wave function for a particle in its second excited state:

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

Substitute for $\psi_2(x)$ to obtain:

$$P = \frac{2}{L}\sin^2\frac{2\pi x}{L}\Delta x = \frac{2}{L}\left(\sin^2\frac{2\pi x}{L}\right)(0.002L) = 0.004\sin^2\left(\frac{2\pi x}{L}\right)$$

(a) Evaluate P at $x = L/3$:

$$P(L/3) = 0.004\sin^2\left(\frac{2\pi L}{3L}\right) = 0.004\sin^2\left(\frac{2\pi}{3}\right) = \boxed{0.003}$$

(b) Evaluate P at $x = L/2$:

$$P(L/2) = 0.004\sin^2\left(\frac{2\pi L}{2L}\right) = 0.004\sin^2(\pi) = \boxed{0}$$

(c) Evaluate P at $x = 2L/3$:

$$P(2L/3) = 0.004\sin^2\left(\frac{4\pi L}{3L}\right) = 0.004\sin^2\left(\frac{4\pi}{3}\right) = \boxed{0.003}$$

45 •• A particle is in the first excited ($n = 2$) state of a one-dimensional box that has length L. (The box has one end at the origin and the other end on the positive x axis.) Find (a) $\langle x\rangle$ and (b) $\langle x^2\rangle$.

Picture the Problem We'll use $\langle f(x)\rangle = \int f(x)\psi^2(x)\,dx$ with $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$.

(a) Express $\psi(x)$ for the $n = 2$ state:

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

Express $\langle x\rangle$ using the $n = 2$ wave function:

$$\langle x\rangle = \int_0^L \frac{2x}{L}\sin^2\frac{2\pi x}{L}\,dx$$

Change variables by letting $\theta = \frac{2\pi x}{L}$. Then:

$$x = \frac{L}{2\pi}\theta,\ d\theta = \frac{2\pi}{L}dx,\ dx = \frac{L}{2\pi}d\theta$$

and the limits on θ are 0 and 2π.

Substitute to obtain:

$$\langle x\rangle = \frac{2}{L}\int_0^L\left(\frac{L}{2\pi}\theta\right)\sin^2\theta\left(\frac{L}{2\pi}d\theta\right) = \frac{L}{2\pi^2}\int_0^{2\pi}\theta\sin^2\theta\,d\theta$$

Use a table of integrals to evaluate the integral:

$$\langle x\rangle = \frac{L}{2\pi^2}\left[\frac{\theta^2}{4} - \frac{\theta\sin 2\theta}{4} - \frac{\cos 2\theta}{8}\right]_0^{2\pi} = \frac{L}{2\pi^2}\left[\pi^2 - \frac{1}{8} + \frac{1}{8}\right] = \boxed{\frac{L}{2}}$$

(b) Express $\langle x^2\rangle$ using the $n = 2$ wave function:

$$\langle x^2\rangle = \int_0^L \frac{2x^2}{L}\sin^2\frac{2\pi x}{L}\,dx$$

Change variables as in (*a*) and substitute to obtain:

$$\langle x^2\rangle = \frac{2}{L}\int_0^L \left(\frac{L}{2\pi}\theta\right)^2 \sin^2\theta\left(\frac{L}{2\pi}d\theta\right) = \frac{L^2}{4\pi^3}\int_0^{2\pi}\theta^2\sin^2\theta\, d\theta$$

Use a table of integrals to evaluate the integral:

$$\langle x^2\rangle = \frac{L^2}{4\pi^3}\left[\frac{\theta^3}{6} - \left(\frac{\theta^2}{4}-\frac{1}{8}\right)\sin 2\theta - \frac{\theta\cos 2\theta}{4}\right]_0^{2\pi} = \frac{L^2}{4\pi^3}\left[\frac{4\pi^3}{3}-\frac{\pi}{2}\right]$$

$$= L^2\left(\frac{1}{3}-\frac{1}{8\pi^2}\right) = \boxed{0.321L^2}$$

46 •• A particle in a one-dimensional box that has length *L* is in the first excited state ($n = 2$). (The box has one end at the origin and the other end on the positive *x* axis.) (*a*) Sketch $\psi^2(x)$ versus *x* for this state. (*b*) What is the expectation value <*x*> for this state? (*c*) What is the probability of finding the particle in some small region *dx* centered at $x = L/2$? (*d*) Are your answers for Part (*b*) and Part (*c*) contradictory? If not, explain why your answers are not contradictory.

Picture the Problem We'll use $\langle f(x)\rangle = \int f(x)\psi^2(x)dx$ with $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$. In Part (*c*) we'll use $P(x) = \psi_2^2(x)$ to determine the probability of finding the particle in some small region *dx* centered at $x = \frac{1}{2}L$.

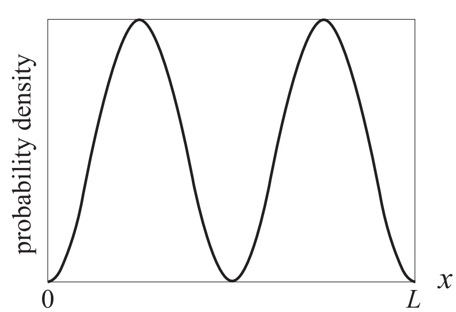
(*a*) Express the wave function for a particle in its first excited state:

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

Square both sides of the equation to obtain:

$$\psi_2^2(x) = \frac{2}{L}\sin^2\frac{2\pi x}{L}$$

The graph of $\psi^2(x)$ as a function of *x* is shown to the right:



(*b*) Express $\langle x\rangle$ using the $n = 2$ wave function:

$$\langle x\rangle = \int_0^L \frac{2x}{L}\sin^2\frac{2\pi x}{L}\,dx$$

Change variables by letting $\theta = \dfrac{2\pi x}{L}$. Then:

$$x = \frac{L}{2\pi}\theta,$$

$$d\theta = \frac{2\pi}{L}dx, \text{ and}$$

$$dx = \frac{L}{2\pi}d\theta$$

and the limits on θ are 0 and 2π.

Substitute in the expression for $\langle x\rangle$ to obtain:

$$\langle x\rangle = \frac{2}{L}\int_0^L\left(\frac{L}{2\pi}\theta\right)\sin^2\theta\left(\frac{L}{2\pi}d\theta\right)$$

$$= \frac{L}{2\pi^2}\int_0^{2\pi}\theta\sin^2\theta\,d\theta$$

Using a table of integrals, evaluate the integral:

$$\langle x\rangle = \frac{L}{2\pi^2}\left[\frac{\theta^2}{4} - \frac{\theta\sin 2\theta}{4} - \frac{\cos 2\theta}{8}\right]_0^{2\pi}$$

$$= \frac{L}{2\pi^2}\left[\pi^2 - \frac{1}{8} + \frac{1}{8}\right] = \boxed{\frac{L}{2}}$$

(*c*) Express $P(x)$:

$$P(x) = \psi_2^2(x) = \frac{2}{L}\sin^2\frac{2\pi x}{L}$$

Evaluate $P(L/2)$:

$$P\left(\frac{L}{2}\right) = \frac{2}{L}\sin^2\frac{2\pi}{L}\cdot\frac{L}{2}$$

$$= \frac{2}{L}\sin^2\pi = 0$$

Because $P(L/2) = 0$:

$$P\left(\frac{L}{2}\right)dx = \boxed{0}$$

(*d*) The answers to Parts (*b*) and (*c*) are not contradictory. Part (*b*) states that the average value of measurements of the position of the particle will yield $L/2$ even though the probability that any one measurement of position will yield this value is zero.

47 •• A particle of mass *m* has a wave function given by $\psi(x) = Ae^{-|x|/a}$, where *A* and *a* are positive constants. (*a*) Find the normalization constant *A*. (*b*) Calculate the probability of finding the particle in the region $-a \le x \le a$.

Picture the Problem We can find the constant *A* by applying the normalization condition $\int_{-\infty}^{\infty}\psi^2(x)dx = 1$ and finding the value for *A* that satisfies this condition. As soon as we have found the normalization constant, we can calculate the probability of the finding the particle in the region $-a \le x \le a$ using $P = \int_{-a}^{a}\psi^2(x)dx$.

(*a*) The normalization condition is:

$$\int_{-\infty}^{\infty}\psi^2(x)dx = 1$$

Substitute $\psi(x) = Ae^{-|x|/a}$:

$$\int_{-\infty}^{\infty}\left(Ae^{-|x|/a}\right)^2 dx = 2A^2\int_0^{\infty}e^{-2x/a}dx$$

From integral tables:

$$\int_0^{\infty}e^{-\alpha x}dx = \frac{1}{\alpha}$$

Therefore:

$$2A^2\int_0^{\infty}e^{-2x/a}dx = 2A^2\left(\frac{a}{2}\right) = aA^2 = 1$$

Solving for *A* yields:

$$A = \boxed{\frac{1}{\sqrt{a}}}$$

(*b*) Express the normalized wave function:

$$\psi(x) = \frac{1}{\sqrt{a}}e^{-|x|/a}$$

The probability of finding the particle in the region $-a \le x \le a$ is:

$$P = \int_{-a}^{a}\psi^2(x)dx = 2\int_0^{a}\frac{1}{a}e^{-2x/a}dx$$

$$= \frac{2}{a}\int_0^{a}e^{-2x/a}dx = 1 - e^{-2} = \boxed{0.865}$$

48 •• A one-dimensional box is on the *x*-axis in the region of $0 \le x \le L$. A particle in this box is in its ground state. Calculate the probability that the particle will be found in the region (*a*) $0 < x < \frac{1}{2}L$, (*b*) $0 < x < \frac{1}{3}L$, and (*c*) $0 < x < \frac{3}{4}L$.

Picture the Problem The probability density for the particle in its ground state is given by $P(x)=\frac{2}{L}\sin^2\frac{\pi}{L}x$. We'll evaluate the integral of $P(x)$ between the limits specified in (*a*), (*b*), and (*c*).

Express $P(x)$ for $0 < x < d$:

$$P(x)=\frac{2}{L}\int_0^d \sin^2\frac{\pi}{L}x\,dx$$

Change variables by letting $\theta=\frac{\pi}{L}x$. Then:

$$x=\frac{L}{\pi}\theta,\ d\theta=\frac{\pi}{L}dx,\text{ and}$$

$$dx=\frac{L}{\pi}d\theta$$

Substitute to obtain:

$$P(x)=\frac{2}{L}\int_0^{\theta'}\sin^2\theta\left(\frac{L}{\pi}d\theta\right)$$

$$=\frac{2}{\pi}\int_0^{\theta'}\sin^2\theta d\theta$$

Use a table of integrals to evaluate $\int_0^{\theta'}\sin^2\theta\,d\theta$:

$$P(x)=\frac{2}{\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\theta'} \qquad (1)$$

(*a*) Noting that the limits on θ are 0 and $\pi/2$, evaluate equation (1) over the interval $0 < x < \frac{1}{2}L$:

$$P(x)=\frac{2}{\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi/2}=\frac{2}{\pi}\left[\frac{\pi}{4}\right]$$

$$=\boxed{0.500}$$

(*b*) Noting that the limits on θ are 0 and $\pi/3$, evaluate equation (1) over the interval $0 < x < L/3$:

$$P(x)=\frac{2}{\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi/3}$$

$$=\frac{2}{\pi}\left[\frac{\pi}{6}-\frac{\sin 2\pi/3}{4}\right]$$

$$=\frac{1}{3}-\frac{\sqrt{3}}{4\pi}=\boxed{0.196}$$

(*c*) Noting that the limits on θ are 0 and $3\pi/4$, Evaluate equation (1) over the interval $0 < x < 3L/4$:

$$P(x) = \frac{2}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{3\pi/4}$$
$$= \frac{2}{\pi}\left[\frac{3\pi}{8} - \frac{\sin 6\pi/4}{4}\right]$$
$$= \frac{3}{4} + \frac{1}{2\pi} = \boxed{0.909}$$

49 •• A one-dimensional box is on the x-axis in the region of $0 \le x \le L$. A particle in this box is in its first excited state. Calculate the probability that the particle will be found in the region (*a*) $0 < x < \frac{1}{2}L$, (*b*) $0 < x < \frac{1}{3}L$, and (*c*) $0 < x < \frac{3}{4}L$.

Picture the Problem The probability density for the particle in its first excited state is given by $P(x) = \frac{2}{L}\sin^2\frac{2\pi}{L}x$. We'll evaluate the integral of $P(x)$ between the limits specified in (*a*), (*b*), and (*c*).

Express $P(x)$ for $0 < x < d$:

$$P(x) = \frac{2}{L}\int_0^d \sin^2\frac{2\pi}{L}x\,dx$$

Change variables by letting $\theta = \frac{2\pi}{L}x$. Then:

$$x = \frac{L}{2\pi}\theta,\ d\theta = \frac{2\pi}{L}dx, \text{ and}$$
$$dx = \frac{L}{2\pi}d\theta$$

Substitute to obtain:

$$P(x) = \frac{2}{L}\int_0^{\theta'} \sin^2\theta\left(\frac{L}{2\pi}d\theta\right)$$
$$= \frac{1}{\pi}\int_0^{\theta'} \sin^2\theta\,d\theta$$

Use a table of integrals to evaluate $\int_0^{\theta'} \sin^2\theta\,d\theta$:

$$P(x) = \frac{1}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\theta'} \qquad (1)$$

(*a*) Noting that the limits on θ are 0 and π, evaluate equation (1) over the interval $0 < x < ½L$:

$$P(x) = \frac{1}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}\right]$$
$$= \boxed{0.500}$$

(*b*) Noting that the limits on θ are 0 and $2\pi/3$, evaluate equation (1) over the interval $0 < x < L/3$:

$$P(x) = \frac{1}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} = \frac{1}{\pi}\left[\frac{2\pi}{6} - \frac{\sin 4\pi/3}{4}\right] = \frac{1}{3} + \frac{\sqrt{3}/2}{4\pi} = \boxed{0.402}$$

(*c*) Noting that the limits on θ are 0 and $3\pi/2$, evaluate equation (1) over the interval $0 < x < 3L/4$:

$$P(x) = \frac{1}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{3\pi/2} = \frac{1}{\pi}\left[\frac{3\pi}{4}\right] = \boxed{0.750}$$

50 •• The classical probability distribution function for a particle in a one-dimensional box on the *x* axis in the region $0 < x < L$ is given by $P(x) = 1/L$. Use this expression to show that $\langle x \rangle = \frac{1}{2}L$ and $\langle x^2 \rangle = \frac{1}{3}L^2$ for a classical particle in the box.

Picture the Problem Classically, $\langle x \rangle = \int xP(x)dx$ and $\langle x^2 \rangle = \int x^2 P(x)dx$.

Evaluate $\langle x \rangle$ with $P(x) = 1/L$:

$$\langle x \rangle = \int_0^L \frac{x}{L}dx = \left[\frac{x^2}{2L}\right]_0^L = \boxed{\tfrac{1}{2}L}$$

Evaluate $\langle x^2 \rangle$ with $P(x) = 1/L$:

$$\langle x^2 \rangle = \int_0^L \frac{x^2}{L}dx = \left[\frac{x^3}{3L}\right]_0^L = \boxed{\tfrac{1}{3}L^2}$$

51 •• A one-dimensional box is on the *x*-axis in the region of $0 \le x \le L$. (*a*) The wave functions for a particle in the box are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$$

For a particle in the *n*th state, show that $\langle x \rangle = \frac{1}{2}L$ and $\langle x^2 \rangle = L^2/3 - L^2/(2n^2\pi^2)$. (*b*) Compare these expressions for $\langle x \rangle$ and $\langle x^2 \rangle$, for $n >> 1$, with the expressions for $\langle x \rangle$ and $\langle x^2 \rangle$ for the classical distribution of Problem 50.

Picture the Problem We'll use $\langle f(x) \rangle = \int f(x)\psi^2(x)dx$ with $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ to show that $\langle x \rangle = \frac{L}{2}$ and $\langle x^2 \rangle = \frac{L^2}{3} = \frac{L^2}{2n^2\pi^2}$.

(*a*) Express $\langle x\rangle$ for a particle in the *n*th state:

$$\langle x\rangle=\int_0^L \frac{2x}{L}\sin^2\frac{n\pi x}{L}\,dx$$

Change variables by letting $\theta=\frac{n\pi x}{L}$. Then:

$$x=\frac{L}{n\pi}\theta,\ d\theta=\frac{n\pi}{L}dx,\ dx=\frac{L}{n\pi}d\theta$$

and the limits on θ are 0 and $n\pi$.

Substitute to obtain:

$$\langle x\rangle=\frac{2}{L}\int_0^{n\pi}\left(\frac{L}{n\pi}\theta\right)\sin^2\theta\left(\frac{L}{n\pi}d\theta\right)=\frac{2L}{n^2\pi^2}\int_0^{n\pi}\theta\sin^2\theta\,d\theta$$

Use a table of integrals to evaluate the integral:

$$\begin{aligned}\langle x\rangle&=\frac{2L}{n^2\pi^2}\left[\frac{\theta^2}{4}-\frac{\theta\sin 2\theta}{4}-\frac{\cos 2\theta}{8}\right]_0^{n\pi}\\&=\frac{2L}{n^2\pi^2}\left[\frac{n^2\pi^2}{4}-\frac{n\pi\sin 2n\pi}{4}-\frac{\cos 2n\pi}{8}+\frac{1}{8}\right]=\frac{2L}{n^2\pi^2}\left[\frac{n^2\pi^2}{4}-\frac{1}{8}+\frac{1}{8}\right]\\&=\boxed{\frac{L}{2}}\end{aligned}$$

Express $\langle x^2\rangle$ for a particle in the *n*th state:

$$\langle x^2\rangle=\int_0^L \frac{2x^2}{L}\sin^2\frac{n\pi x}{L}\,dx$$

Change variables by letting $\theta=\frac{n\pi x}{L}$. Then:

$$x=\frac{L}{n\pi}\theta,\ d\theta=\frac{n\pi}{L}dx,\ dx=\frac{L}{n\pi}d\theta$$

and the limits on θ are 0 and $n\pi$.

Substitute to obtain:

$$\langle x^2\rangle=\frac{2}{L}\int_0^{n\pi}\left(\frac{L}{n\pi}\theta\right)^2\sin^2\theta\left(\frac{L}{n\pi}d\theta\right)=\frac{2L^2}{n^3\pi^3}\int_0^{n\pi}\theta^2\sin^2\theta\,d\theta$$

Use a table of integrals to evaluate the integral:

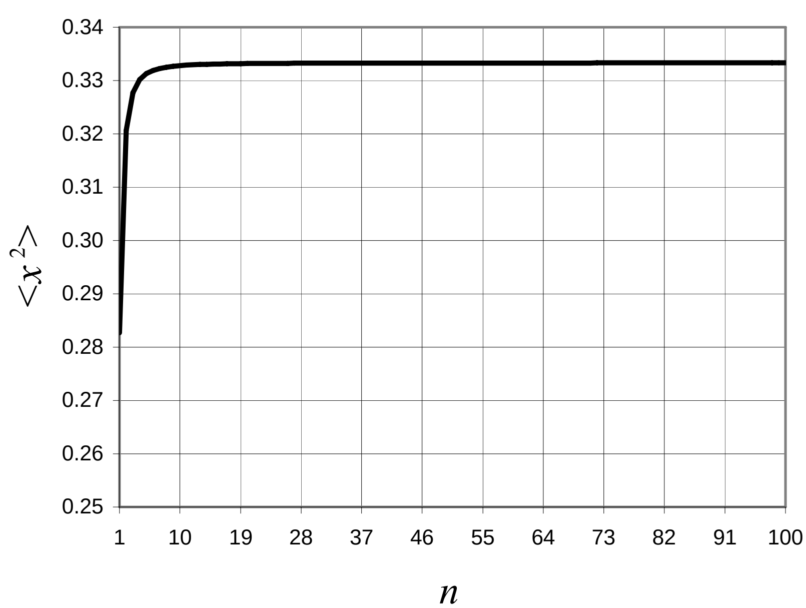
$$\langle x^2 \rangle = \frac{2L^2}{n^3\pi^3}\left[\frac{\theta^3}{6} - \left(\frac{\theta^2}{4} - \frac{1}{8}\right)\sin 2\theta - \frac{\theta\cos 2\theta}{4}\right]_0^{n\pi} = \frac{2L^2}{n^3\pi^3}\left[\frac{n^3\pi^3}{6} - \frac{n\pi}{4}\right]$$

$$= \boxed{\frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}}$$

(*b*) For large values of *n* the result agrees with the classical value of $L^2/3$ given in Problem 50.

52 •• (*a*) Use a **spreadsheet** program or graphing calculator to plot $\langle x^2 \rangle$ as a function of the quantum number *n* for the particle in the box described in Problem 48 and for values of *n* from 1 to 100. Assume *L* = 1.00 m for your graph. Refer to Problem 51. (*b*) Comment on the significance of any asymptotic limits that your graph shows.

Picture the Problem (*a*) From Problem 51 we have $\langle x \rangle = \frac{L}{2}$ and $\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$. A spreadsheet program was used to plot the following graph of $<x^2>$ as a function of *n*.



(*b*) As $n \to \infty, \langle x^2 \rangle \to \dfrac{L^2}{3}$

53 •• The wave functions for a particle of mass *m* in a one-dimensional box of length *L centered at the origin* (so that the ends are at $x = \pm\frac{1}{2}L$) are given by

$$\psi(x) = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \qquad n = 1, 3, 5, 7, \ldots$$

and

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad n = 2, 4, 6, 8, \ldots$$

Calculate $\langle x \rangle$ and $\langle x^2 \rangle$ for the ground state ($n = 1$).

Picture the Problem For the ground state, $n = 1$ and so we'll evaluate $\langle f(x) \rangle = \int f(x)\psi^2(x)dx$ using $\psi_1(x) = \sqrt{\dfrac{2}{L}}\cos\dfrac{\pi x}{L}$.

Because $\psi_1^2(x)$ is an even function of x, $x\psi_1^2(x)$ is an odd function of x. It follows that the integral of $x\psi_1^2(x)$ between $-L/2$ and $L/2$ is zero. Thus:

$$\langle x \rangle = \boxed{0} \text{ for all values of } n.$$

Express $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\cos^2\frac{\pi}{L}x\,dx$$

Change variables by letting $\theta = \dfrac{\pi x}{L}$. Then:

$$x = \frac{L}{\pi}\theta,\ d\theta = \frac{\pi}{L}dx,\ dx = \frac{L}{\pi}d\theta$$

and the limits on θ are $-\pi/2$ and $\pi/2$.

Substitute for x and dx to obtain:

$$\langle x^2 \rangle = \frac{2}{L}\int_{-\pi/2}^{\pi/2}\left(\frac{L}{\pi}\theta\right)^2\cos^2\theta\left(\frac{L}{\pi}d\theta\right)$$

$$= \frac{2L^2}{\pi^3}\int_{-\pi/2}^{\pi/2}\theta^2\cos^2\theta\,d\theta$$

Use a trigonometric identity to rewrite the integrand:

$$\langle x^2 \rangle = \frac{2L^2}{\pi^3}\int_{-\pi/2}^{\pi/2}\theta^2\left(1-\sin^2\theta\right)d\theta$$

$$= \frac{2L^2}{\pi^3}\left[\int_{-\pi/2}^{\pi/2}\theta^2 d\theta - \int_{-\pi/2}^{\pi/2}\theta^2\sin^2\theta\,d\theta\right]$$

Evaluate the second integral by looking it up in the tables:

$$\langle x^2\rangle = \frac{2L^2}{\pi^3}\left[\frac{\theta^3}{3} - \left\{\frac{\theta^3}{6} - \left(\frac{\theta^2}{4} - \frac{1}{8}\right)\sin 2\theta - \frac{\theta\cos 2\theta}{4}\right\}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2L^2}{\pi^3}\left[\frac{\theta^3}{6} + \left(\frac{\theta^2}{4} - \frac{1}{8}\right)\sin 2\theta + \frac{\theta\cos 2\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2L^2}{\pi^3}\left[\frac{\pi^3}{48} + \left(\frac{\pi^2}{16} - \frac{1}{8}\right)\sin\pi + \frac{\pi\cos\pi}{8}\right.$$

$$\left.+\frac{\pi^3}{48} + \left(\frac{\pi^2}{16} - \frac{1}{8}\right)\sin\pi + \frac{\pi\cos\pi}{8}\right]$$

$$= \frac{2L^2}{\pi^3}\left[\frac{\pi^3}{24} - \frac{\pi}{4}\right] = \boxed{L^2\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]}$$

Remarks: The result differs from that of Example 34-8. Since we have shifted the origin by $\Delta x = L/2$, we could have arrived at the above result, without performing the integration, by subtracting $(\Delta x)^2 = L^2/4$ from $\langle x^2\rangle$.

54 •• Calculate $<x>$ and $<x^2>$ for the first excited state ($n = 2$) of the box described in Problem 53.

Picture the Problem For the first excited state, $n = 2$, and so we'll evaluate $\langle f(x)\rangle = \int f(x)\psi^2(x)dx$ using $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$.

Because $\psi_2^2(x)$ is an even function of x, $x\psi_2^2(x)$ is an odd function of x. It follows that the integral of $x\psi_2^2(x)$ between $-L/2$ and $L/2$ is zero. Thus:

$$\langle x\rangle = \boxed{0}$$

Express $\langle x^2\rangle$:

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\frac{2\pi}{L}x\,dx$$

Change variables by letting $\theta = \frac{2\pi x}{L}$. Then:

$x = \frac{L}{2\pi}\theta, d\theta = \frac{2\pi}{L}dx, dx = \frac{L}{2\pi}d\theta$ and the limits on θ are $-\pi$ and π.

Substitute to obtain:

$$\langle x^2 \rangle = \frac{2}{L}\int_{-\pi}^{\pi}\left(\frac{L}{2\pi}\theta\right)^2 \sin^2\theta\left(\frac{L}{2\pi}d\theta\right)$$
$$= \frac{L^2}{4\pi^3}\int_{-\pi}^{\pi}\theta^2\sin^2\theta\,d\theta$$

Evaluate the integral by looking it up in the tables:

$$\langle x^2 \rangle = \frac{L^2}{4\pi^3}\left[\frac{\theta^3}{6} - \left(\frac{\theta^2}{4} - \frac{1}{8}\right)\sin 2\theta - \frac{\theta\cos 2\theta}{4}\right]_{-\pi}^{\pi}$$
$$= \frac{L^2}{4\pi^3}\left[\frac{\pi^3}{6} - \left(\frac{\pi^2}{4} - \frac{1}{8}\right)\sin 2\pi - \frac{\pi\cos 2\pi}{4}\right.$$
$$\left. + \frac{\pi^3}{6} - \left(\frac{\pi^2}{4} - \frac{1}{8}\right)\sin 2\pi - \frac{\pi\cos 2\pi}{4}\right]$$
$$= \frac{L^2}{4\pi^3}\left[\frac{\pi^3}{6} - \frac{\pi}{4} + \frac{\pi^3}{6} - \frac{\pi}{4}\right]$$
$$= \frac{L^2}{4\pi^3}\left[\frac{\pi^3}{3} - \frac{\pi}{2}\right] = \boxed{L^2\left[\frac{1}{12} - \frac{1}{8\pi^2}\right]}$$

Remarks: The result differs from that of Example 34-8. Since we have shifted the origin by $\Delta x = L/2$, we could have arrived at the above result, without performing the integration, by subtracting $(\Delta x)^2 = L^2/4$ from $\langle x^2 \rangle$.

General Problems

55 • [SSM] Photons in a uniform 4.00-cm-diameter light beam have wavelengths equal to 400 nm and the beam has an intensity of 100 W/m^2.
(*a*) What is the energy of each photon in the beam? (*b*) How much energy strikes an area of 1.00 cm^2 perpendicular to the beam in 1 s? (*c*) How many photons strike this area in 1.00 s?

Picture the Problem We can use the Einstein equation for photon energy to find the energy of each photon in the beam. The intensity of the energy incident on the surface is the ratio of the power delivered by the beam to its delivery time. Hence, we can express the energy incident on the surface in terms of the intensity of the beam.

(a) Use the Einstein equation for photon energy to express the energy of each photon in the beam:

$$E_{\text{photon}} = hf = \frac{hc}{\lambda}$$

Substitute numerical values and evaluate E_{photon}:

$$E_{\text{photon}} = \frac{1240\,\text{eV}\cdot\text{nm}}{400\,\text{nm}} = 3.100\,\text{eV}$$
$$= \boxed{3.10\,\text{eV}}$$

(b) Relate the energy incident on a surface of area A to the intensity of the beam:

$$E = IA\Delta t$$

Substitute numerical values and evaluate E:

$$E = \left(100\,\text{W/m}^2\right)\left(1.00\times10^{-4}\,\text{m}^2\right)\left(1.00\,\text{s}\right)$$
$$= 0.0100\,\text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}$$
$$= 6.242\times10^{16}\,\text{eV} = \boxed{6.24\times10^{16}\,\text{eV}}$$

(c) Express the number of photons striking this area in 1.00 s as the ratio of the total energy incident on the surface to the energy delivered by each photon:

$$N = \frac{E}{E_{\text{photon}}} = \frac{6.242\times10^{16}\,\text{eV}}{3.100\,\text{eV}}$$
$$= \boxed{2.08\times10^{16}}$$

56 • A 1-μg particle is moving with a speed of approximately 1 mm/s in a one-dimensional box that has a length equal to 1 cm. Calculate the approximate value of the quantum number n of the state occupied by the particle.

Picture the Problem The particle's nth-state energy is $E_n = n^2\dfrac{h^2}{8mL^2}$. We can find n by solving this equation for n and substituting the particle's kinetic energy for E_n.

Express the energy of the particle when it is in its nth state:

$$E_n = n^2\frac{h^2}{8mL^2} \Rightarrow n = \frac{L}{h}\sqrt{8mE_n}$$

Express the energy (kinetic) of the particle:

$$E_n = \tfrac{1}{2}mv^2$$

Substitute for E_n and simplify to obtain:

$$n = \frac{2mvL}{h}$$

Substitute numerical values and evaluate n:

$$n = \frac{2(10^{-9}\text{ kg})(1\times10^{-3}\text{ m/s})(10^{-2}\text{ m})}{6.626\times10^{-34}\text{ J}\cdot\text{s}} \approx \boxed{3\times10^{19}}$$

57 • (*a*) For the particle and box of Problem 56, find Δx and Δp_x, assuming that these uncertainties are given by $\Delta x/L = 0.01$ percent and $\Delta p_x/p_x = 0.01$ percent. (*b*) What is $(\Delta x \Delta p_x)/h$?

Picture the Problem We can use the fact that the uncertainties are given by $\Delta x/L = 0.01$ percent and $\Delta p_x/p_x = 0.01$ percent to find Δx and Δp.

(*a*) Assuming that $\Delta x/L = 0.01$ percent, find Δx:

$$\Delta x = 10^{-4}(L) = 10^{-4}(10^{-2}\text{ m}) = \boxed{1\,\mu\text{m}}$$

Assuming that $\Delta p_x/p_x = 0.01$ percent, find Δp:

$$\Delta p_x = 10^{-4}mv = 10^{-4}(10^{-9}\text{ kg})(10^{-3}\text{ m/s}) = \boxed{10^{-16}\text{ kg}\cdot\text{m/s}}$$

(*b*) Evaluate $(\Delta x \Delta p_x)/h$:

$$\frac{\Delta x \Delta p_x}{h} = \frac{(1\,\mu\text{m})(10^{-16}\text{ kg}\cdot\text{m/s})}{6.626\times10^{-34}\text{ J}\cdot\text{s}} = \boxed{2\times10^{11}}$$

58 • In 1987, a laser at Los Alamos National Laboratory produced a flash that lasted 1×10^{-12} s and had a power of 5×10^{15} W. Estimate the number of emitted photons, assuming they all had wavelengths equal to 400 nm.

Picture the Problem We can estimate the number of emitted photons from the ratio of the total energy in the flash to the energy of a single photon.

Letting N be the number of emitted photons, express the ratio of the total energy in the flash to the energy of a single photon:

$$N = \frac{E}{E_{\text{photon}}}$$

Relate the energy in the flash to the power produced:

$$E = P\Delta t$$

Express the energy of a single photon as a function of its wavelength:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substituting for E and E_{photon} and simplifying gives:

$$N = \frac{P\Delta t\lambda}{hc}$$

Substitute numerical values and evaluate N:

$$N = \frac{(5\times10^{15}\ \text{W})(1\times10^{-12}\ \text{s})(400\times10^{-9}\ \text{m})}{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})} \approx \boxed{1\times10^{22}}$$

59 • You cannot "see" anything smaller than the wavelength of the wave used to make the observation. What is the minimum energy of an electron needed in an electron microscope to "see" an atom that has a diameter of about 0.1 nm?

Picture the Problem We can use the electron wavelength equation $\lambda = \frac{1.23}{\sqrt{K}}\text{nm}$, where K is in eV to find the minimum energy required to see an atom.

Relate the energy of the electron to the size of an atom (the wavelength of the electron):

$$\lambda = \frac{1.23}{\sqrt{K}}\text{nm} \Rightarrow K = \frac{(1.23\,\text{eV}^{1/2}\cdot\text{nm})^2}{\lambda^2}$$

provided K is in eV.

Substitute numerical values and evaluate K:

$$K = \frac{(1.23\,\text{eV}^{1/2}\cdot\text{nm})^2}{(0.1\,\text{nm})^2} \approx \boxed{0.2\,\text{keV}}$$

60 • A common flea that has a mass of 0.008 g can jump vertically as high as 20 cm. Estimate the de Broglie wavelength for the flea immediately after takeoff.

Picture the Problem The flea's de Broglie wavelength is $\lambda = h/p$, where p is the flea's momentum immediately after takeoff. We can use a constant acceleration equation to find the flea's speed and, hence, momentum immediately after takeoff.

Express the de Broglie wavelength of the flea immediately after takeoff:

$$\lambda = \frac{h}{p} = \frac{h}{mv_0}$$

Using a constant acceleration equation, express the height the flea can jump as a function of its takeoff speed:

$$v^2 = v_0^2 + 2a\Delta y$$

or, since $v = 0$ and $a = -g$,

$$v_0 = \sqrt{2g\Delta y}$$

Substitute to obtain:

$$\lambda = \frac{h}{m\sqrt{2g\Delta y}}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{\left(8\times10^{-6}\ \text{kg}\right)\sqrt{2\left(9.81\,\text{m/s}^2\right)\left(0.2\,\text{m}\right)}} \approx \boxed{4\times10^{-29}\ \text{m}}$$

61 •• [SSM] A 100-W source radiates light of wavelength 600 nm uniformly in all directions. An eye that has been adapted to the dark has a 7-mm-diameter pupil and can detect the light if at least 20 photons per second enter the pupil. How far from the source can the light be detected under these rather extreme conditions?

Picture the Problem We can relate the fraction of the photons entering the eye to ratio of the area of the pupil to the area of a sphere of radius R. We can find the number of photons emitted by the source from the rate at which it emits and the energy of each photon which we can find using the Einstein equation.

Letting r be the radius of the pupil, $N_{\text{entering eye}}$ the number of photons per second entering the eye, and N_{emitted} the number of photons emitted by the source per second, express the fraction of the light energy entering the eye at a distance R from the source:

$$\frac{N_{\text{entering eye}}}{N_{\text{emitted}}} = \frac{A_{\text{eye}}}{4\pi R^2} = \frac{\pi r^2}{4\pi R^2} = \frac{r^2}{4R^2}$$

Solving for R yields:

$$R = \frac{r}{2}\sqrt{\frac{N_{\text{emitted}}}{N_{\text{entering eye}}}} \qquad (1)$$

Find the number of photons emitted by the source per second:

$$N_{\text{emitted}} = \frac{P}{E_{\text{photon}}}$$

Using the Einstein equation, express the energy of the photons:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substitute numerical values and evaluate E_{photon}:

$$E_{\text{photon}} = \frac{1240\,\text{eV}\cdot\text{nm}}{600\,\text{nm}} = 2.07\,\text{eV}$$

Substitute and evaluate N_{emitted}:

$$N_{\text{emitted}} = \frac{100\,\text{W}}{(2.07\,\text{eV})\left(1.602\times10^{-19}\,\frac{\text{J}}{\text{eV}}\right)} = 3.02\times10^{20}\,\text{s}^{-1}$$

Substitute for N_{emitted} in equation (1) and evaluate R:

$$R = \frac{3.5\,\text{mm}}{2}\sqrt{\frac{3.02\times10^{20}\,\text{s}^{-1}}{20\,\text{s}^{-1}}} \approx \boxed{7\times10^{3}\,\text{km}}$$

62 •• The diameter of the pupil of an eye under room-light conditions is approximately 5 mm. Find the intensity of light that has a wavelength equal to 600 nm so that 1 photon per second passes through the pupil.

Picture the Problem The intensity of the light such that one photon per second passes through the pupil is the ratio of the energy of one photon to the product of the area of the pupil and time interval during which the photon passes through the pupil. We'll use the Einstein equation to express the energy of the photon.

Use its definition to relate the intensity of the light to the energy of a 600-nm photon:

$$I_{1\,\text{photon}} = \frac{P}{A} = \frac{E_{1\,\text{photon}}}{A\Delta t}$$

Using the Einstein equation, express the energy of a 600-nm photon:

$$E_{1\,\text{photon}} = \frac{hc}{\lambda}$$

Substitute for $E_{1\,\text{photon}}$ to obtain:

$$I_{1\,\text{photon}} = \frac{hc}{\lambda A\Delta t}$$

Substitute numerical values and evaluate $I_{1\,\text{photon}}$:

$$I_{1\,\text{photon}} = \frac{(1240\,\text{eV}\cdot\text{nm})(1.602\times10^{-19}\,\text{J/eV})}{(600\,\text{nm})\left[\frac{\pi}{4}(5\times10^{-3}\,\text{m})^2\right](1\,\text{s})} \approx \boxed{2\times10^{-14}\,\text{W/m}^2}$$

63 •• A 100-W incandescent light bulb radiates 2.6 W of visible light uniformly in all directions. (*a*) Find the intensity of the light from the bulb at a distance of 1.5 m. (*b*) If the average wavelength of the visible light is 650 nm, and counting only those photons in the visible spectrum, find the number of photons per second that strike a surface that has an area equal to 1.0 cm^2, is oriented so that the line to the bulb is perpendicular to the surface, and is a distance of 1.5 m from the bulb.

Picture the Problem We can find the intensity at a distance of 1.5 m directly from its definition. The number of photons striking the surface each second can be found from the ratio of the energy incident on the surface to the energy of a 650-nm photon.

(*a*) Use its definition to express the intensity of the light as a function of distance from the light bulb:

$$I = \frac{P}{A} = \frac{P}{4\pi R^2}$$

Substitute numerical data to obtain:

$$I = \frac{2.6\,\text{W}}{4\pi(1.5\,\text{m})^2} = 91.96\,\text{mW/m}^2$$
$$= \boxed{92\,\text{mW/m}^2}$$

(*b*) Express the number of photons per second that strike the surface as the ratio of the energy incident on the surface to the energy of a 650-nm photon:

$$N = \frac{IA}{E_{\text{photon}}}$$

where A is the area of the surface.

Use the Einstein equation to express the energy of the 650-nm photons:

$$E = \frac{hc}{\lambda}$$

Substituting for E yields:

$$N = \frac{IA\lambda}{hc}$$

Substitute numerical values and evaluate N:

$$N = \frac{(91.96\,\text{mW/m}^2)(1.0\times10^{-4}\,\text{m}^2)(650\,\text{nm})}{(1240\,\text{eV}\cdot\text{nm})\left(1.602\times10^{-19}\,\frac{\text{J}}{\text{eV}}\right)} = \boxed{3.0\times10^4}$$

64 •• When light of wavelength λ_1 is incident on the cathode of a photoelectric tube, the maximum kinetic energy of the emitted electrons is 1.8 eV. If the wavelength is reduced to $\frac{1}{2}\lambda_1$, the maximum kinetic energy of the emitted electrons is 5.5 eV. Find the work function ϕ of the cathode material.

Picture the Problem The maximum kinetic energy of the photoelectrons is related to the frequency of the incident photons and the work function of the cathode material through the Einstein equation. We can apply this equation to the two sets of data and solve the resulting equations simultaneously for the work function.

Using the Einstein equation, relate the maximum kinetic energy of the emitted electrons to the frequency of the incident photons and the work function of the cathode material:

$$K_{\text{max}} = hf - \phi = \frac{hc}{\lambda} - \phi$$

Substitute numerical values for the light of wavelength λ_1:

$$1.8\,\text{eV} = \frac{hc}{\lambda_1} - \phi \qquad (1)$$

Substitute numerical values for the light of wavelength $\frac{1}{2}\lambda_1$:

$$5.5\,\text{eV} = \frac{hc}{\frac{1}{2}\lambda_1} - \phi = \frac{2hc}{\lambda_1} - \phi \qquad (2)$$

Solve equations (1) and (2) simultaneously for ϕ to obtain:

$$\phi = \boxed{1.9\,\text{eV}}$$

65 •• An incident photon of energy E_i undergoes Compton scattering at an angle of θ. Show that the energy E_s of the scattered photon is given by

$$E_s = \frac{E_i}{1 + \left(E_i/m_e c^2\right)(1 - \cos\theta)}$$

Picture the Problem We can use the Einstein equation to express the energy of the scattered photon in terms of its wavelength and the Compton scattering equation to relate this wavelength to the scattering angle and the pre-scattering wavelength.

Express the energy of the scattered photon E_s as a function of their wavelength λ_s:

$$E_s = \frac{hc}{\lambda_s}$$

Express the wavelength of the scattered photon as a function of the scattering angle θ:

$$\lambda_s = \frac{h}{m_e c}(1 - \cos\theta) + \lambda$$

where λ is the wavelength of the incident photon.

Substitute for λ and simplify to obtain:

$$E_s = \frac{hc}{\frac{h}{m_e c}(1-\cos\theta)+\lambda}$$

$$= \frac{\frac{hc}{\lambda}}{1+\frac{hc}{m_e c^2 \lambda}(1-\cos\theta)}$$

$$= \boxed{\frac{E_i}{1+\left(\frac{E_i}{m_e c^2}\right)(1-\cos\theta)}}$$

66 •• A particle is confined to a one-dimensional box. While the particle makes a transition from the state n to the state $n-1$, radiation of 114.8 nm is emitted. While the particle makes the transition from the state $n-1$ to the state $n-2$, radiation of wavelength 147 nm is emitted. The ground-state energy of the particle is 1.2 eV. Determine n.

Picture the Problem While we can work with either of the transitions described in the problem statement, we'll use the first transition in which radiation of wavelength 114.8 nm is emitted. We can express the energy released in the transition in terms of the difference between the energies in the two states and solve the resulting equation for n.

Express the energy of the emitted radiation as the particle goes from the nth to $n-1$ state:

$$\Delta E = E_n - E_{n-1}$$

Express the energy of the particle in nth state:

$$E_n = n^2 E_1$$

Express the energy of the particle in the $n-1$ state:

$$E_{n-1} = (n-1)^2 E_1$$

Substitute for E_n and E_{n-1} and simplify to obtain:

$$\Delta E = n^2 E_1 - (n-1)^2 E_1$$

$$= (2n-1)E_1 = \frac{hc}{\lambda}$$

Solving for n yields:

$$n = \frac{hc}{2\lambda E_1} + \frac{1}{2}$$

Substitute numerical values and evaluate n:

$$n = \frac{1240\,\text{eV}\cdot\text{nm}}{2(114.8\,\text{nm})(1.2\,\text{eV})} + \frac{1}{2} = \boxed{5}$$

67 •• [SSM] The Pauli exclusion principle states that no more than one electron may occupy a particular quantum state at a time. Electrons intrinsically occupy two spin states. Therefore, if we wish to model an atom as a collection of electrons trapped in a one-dimensional box, no more than two electrons in the box can have the same value of the quantum number n. Calculate the energy that the most energetic electron(s) would have for the uranium atom that has an atomic number 92. Assume the box has a length of 0.050 nm and the electrons are in the lowest possible energy states. How does this energy compare to the rest energy of the electron?

Picture the Problem We can use the expression for the energy of a particle in a well to find the energy of the most energetic electron in the uranium atom.

Relate the energy of an electron in the uranium atom to its quantum number n:

$$E_n = n^2\left(\frac{h^2}{8m_e L^2}\right)$$

Substitute numerical values and evaluate E_{92}:

$$E_{92} = (92)^2\left[\frac{(6.626\times10^{-34}\,\text{J}\cdot\text{s})^2}{8(9.109\times10^{-31}\,\text{kg})(0.050\,\text{nm})^2} \times \frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right] = 1.273\,\text{MeV}$$
$$= \boxed{1.3\,\text{MeV}}$$

The rest energy of an electron is:

$$m_e c^2 = (9.109\times10^{-31}\,\text{kg})(2.998\times10^8\,\text{m/s})^2\left(\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right) = 0.512\,\text{MeV}$$

Express the ratio of E_{92} to $m_e c^2$:

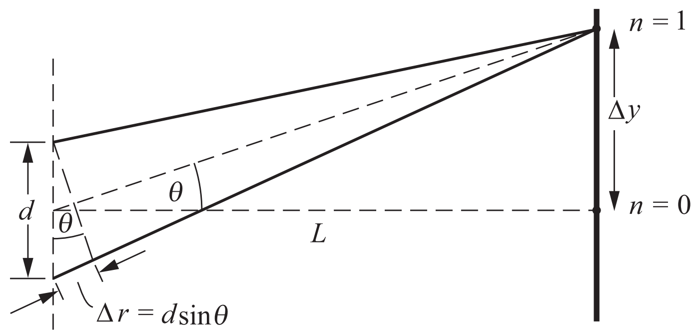
$$\frac{E_{92}}{m_e c^2} = \frac{1.273\,\text{MeV}}{0.512\,\text{MeV}} \approx 2.5$$

The energy of the most energetic electron is approximately 2.5 times the rest-energy of an electron.

68 •• A beam of electrons that each have the same kinetic energy illuminates a pair of slits separated by a distance 54 nm. The beam forms bright and dark fringes on a screen located a distance 1.5 m beyond the two slits. The arrangement is otherwise identical to that used in the optical two-slit interference

experiment described in Chapter 33 and in Figure 33-7 and the fringes have the appearance shown in Figure 34-18*d*. The bright fringes are found to be separated by a distance of 0.68 mm. What is the kinetic energy of the electrons in the beam?

Picture the Problem We can express the kinetic energy of an electron in the beam in terms of its momentum. We can use the de Broglie relationship to relate the electron's momentum to its wavelength and use the condition for constructive interference to find λ.



Express the kinetic energy of an electron in terms of its momentum:

$$K = \frac{p^2}{2m} \quad (1)$$

Using the de Broglie relationship, relate the momentum of an electron to its wavelength:

$$p = \frac{h}{\lambda}$$

Substitute for p in equation (1) to obtain:

$$K = \frac{h^2}{2m\lambda^2} \quad (2)$$

The condition for constructive interference is:

$$d\sin\theta = n\lambda \Rightarrow \lambda = \frac{d\sin\theta}{n}$$

where d is the slit separation and $n = 0, 1, 2, \ldots$

For $\theta << 1$, $\sin\theta$ is also given by:

$$\sin\theta \approx \frac{\Delta y}{L}$$

Substitute for $\sin\theta$ to obtain:

$$\lambda = \frac{d\Delta y}{nL}$$

Substitute for λ in equation (2) to obtain:

$$K = \frac{n^2L^2h^2}{2md^2(\Delta y)^2}$$

Substitute numerical values (n = 1) and evaluate K:

$$K = \frac{(1)^2(1.5\,\text{m})^2\left(6.626\times10^{-34}\,\text{J}\cdot\text{s}\right)^2}{2\left(9.109\times10^{-31}\,\text{kg}\right)(54\,\text{nm})^2(0.68\,\text{mm})^2}\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}} = \boxed{2.5\,\text{keV}}$$

69 •• When a surface is illuminated by light of wavelength λ, the maximum kinetic energy of the emitted electrons is 1.20 eV. If the wavelength $\lambda' = 0.800\lambda$ is used, the maximum kinetic energy increases to 1.76 eV. For wavelength $\lambda' = 0.600\lambda$, the maximum kinetic energy of the emitted electrons is 2.676 eV. Determine the work function of the surface and the wavelength λ.

Picture the Problem The maximum kinetic energy of the photoelectrons is related to the frequency of the incident photons and the work function of the illuminated surface through the Einstein equation. We can apply this equation to either set of data and solve the resulting equations simultaneously for the work function of the surface and the wavelength of the incident photons.

Using the Einstein equation, relate the maximum kinetic energy of the emitted electrons to the frequency of the incident photons and the work function of the cathode material:

$$K_{\text{max}} = hf - \phi = \frac{hc}{\lambda} - \phi$$

Substitute numerical values for the light of wavelength λ:

$$1.20\,\text{eV} = \frac{hc}{\lambda} - \phi$$

Substitute numerical values for the light of wavelength λ':

$$1.76\,\text{eV} = \frac{hc}{\lambda'} - \phi = \frac{hc}{0.800\lambda} - \phi$$

Solve these equations simultaneously for ϕ to obtain:

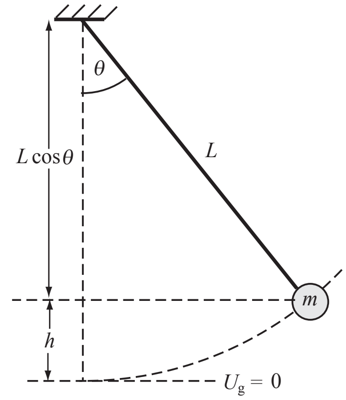
$$\phi = \boxed{1.04\,\text{eV}}$$

Substitute in either of the equations and solve for λ:

$$\lambda = \boxed{554\,\text{nm}}$$

70 •• A simple pendulum has a length equal to 1.0 m and has a bob that has a mass equal to 0.30 kg. The energy of this oscillator is quantized, and the allowed values of the energy are given by $E_n = \left(n + \frac{1}{2}\right)hf_0$, where n is an integer and f_0 is the frequency of the pendulum. (a) Find n if the angular amplitude is 1.0°. (b) Find n such that E_{n+1} exceeds E_n by 0.010 percent.

Picture the Problem The diagram shows the pendulum when its angular displacement is θ. The energy of the oscillator is equal to its initial potential energy $mgh = mgL(1 - \cos\theta)$. We can find n by equating this initial energy to $E_n = \left(n + \frac{1}{2}\right)hf_0$ and solving for n.



(a) Express the nth-state energy as a function of the frequency of the pendulum:

$$E_n = \left(n + \tfrac{1}{2}\right)hf_0$$

Because the small-amplitude frequency of the pendulum is given by $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{L}}$:

$$E_n = \left(n + \tfrac{1}{2}\right)\frac{h}{2\pi}\sqrt{\frac{g}{L}}$$

The energy of the pendulum is also equal to its initial gravitational potential energy:

$$E_n = mgL(1 - \cos\theta)$$

Equating these expressions gives:

$$mgL(1 - \cos\theta) = \left(n + \tfrac{1}{2}\right)\frac{h}{2\pi}\sqrt{\frac{g}{L}}$$

Solving for n gives:

$$n = \frac{2\pi m\sqrt{g}L^{3/2}(1 - \cos\theta)}{h} - \frac{1}{2}$$

Substitute numerical values and evaluate n:

$$n = \frac{2\pi(0.30\,\text{kg})\sqrt{9.81\,\text{m/s}^2}\,(1.0\,\text{m})^{3/2}(1 - \cos 1.0°)}{6.626\times10^{-34}\,\text{J}\cdot\text{s}} - \frac{1}{2}$$
$$= 1.357\times10^{30} = \boxed{1.4\times10^{30}}$$

(b) Because E_{n+1} exceeds E_n by 0.010 percent:

$$\frac{E_{n+1} - E_n}{E_n} = 1.0\times10^{-4}$$

Substituting for E_{n+1} and E_n gives:

$$\frac{\left(n+1+\frac{1}{2}\right)hf_0-\left(n+\frac{1}{2}\right)hf_0}{\left(n+\frac{1}{2}\right)hf_0}=1.0\times10^{-4}$$

or

$$\left(n+1+\tfrac{1}{2}\right)-\left(n+\tfrac{1}{2}\right)=1.0\times10^{-4}\left(n+\tfrac{1}{2}\right)$$

Solving for n yields:

$$n=1.0\times10^{4}-\tfrac{1}{2}=\boxed{1.0\times10^{4}}$$

71 •• **[SSM]** (*a*) Show that for large *n*, the fractional difference in energy between state *n* and state *n* + 1 for a particle in a one-dimensional box is given approximately by $\left(E_{n+1}-E_n\right)/E_n\approx 2/n$ (*b*) What is the approximate percentage energy difference between the states n_1 = 1000 and n_2 = 1001? (*c*) Comment on how this result is related to Bohr's correspondence principle.

Picture the Problem We can use the fact that the energy of the *n*th state is related to the energy of the ground state according to $E_n=n^2E_1$ to express the fractional change in energy in terms of *n* and then examine this ratio as *n* grows without bound.

(*a*) Express the ratio $(E_{n+1}-E_n)/E_n$:

$$\frac{E_{n+1}-E_n}{E_n}=\frac{(n+1)^2-n^2}{n^2}=\frac{2n+1}{n^2}$$

$$=\frac{2}{n}+\frac{1}{n^2}\approx\boxed{\frac{2}{n}}$$

for $n >> 1$.

(*b*) Evaluate $\dfrac{E_{1001}-E_{1000}}{E_{1000}}$:

$$\frac{E_{1001}-E_{1000}}{E_{1000}}\approx\frac{2}{1000}=\boxed{0.2\%}$$

(*c*) Classically, the energy is continuous. For very large values of *n,* the energy difference between adjacent levels is infinitesimal.

72 •• A mode-locked, titanium–sapphire laser has a wavelength of 850 nm and produces 100 million pulses of light each second. Each pulse has a duration of 125 femtoseconds (1 fs = 10^{-15} s) and consists of 5×10^9 photons. What is the average power produced by the laser?

Picture the Problem We can apply the definition of power in conjunction with the de Broglie equation for the energy of a photon to derive an expression for the average power produced by the laser.

The average power produced by the laser is:

$$P = \frac{\Delta E}{\Delta t}$$

Use the de Broglie equation to express the energy of the emitted photons:

$$\Delta E = Nhf = \frac{Nhc}{\lambda}$$

where N is number of photons in each pulse.

Substitute for ΔE to obtain:

$$P = \frac{Nhc}{\lambda \Delta t}$$

Substitute numerical values and evaluate P:

$$P = \frac{(5\times10^{9})(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{(850\,\text{nm})(10^{-8}\ \text{s})} \approx \boxed{0.1\,\text{W}}$$

Remarks: Note that the pulse length has no bearing on the solution.

73 •• This problem estimates the time lag in the photoelectric effect that is expected classically but not observed. Let the intensity of the incident radiation falling on an atom be 0.010 W/m^2. (*a*) If the area presented by the atom is 0.010 nm^2, find the energy per second falling on an atom. (*b*) If the work function is 2.0 eV, how long would it take for this much energy to fall on the atom if the radiation energy was distributed uniformly rather than in compact packets (photons)?

Picture the Problem We can find the rate at which energy is delivered to the atom using the definitions of power and intensity. We can also use the definition of power to determine how much time is required for an amount of energy equal to the work function to fall on one atom.

(*a*) Relate the energy per second (power) falling on an atom to the intensity of the incident radiation:

$$P = \frac{\Delta E}{\Delta t} = IA$$

Substitute numerical values and evaluate P:

$$\begin{aligned} P &= (0.010\,\text{W/m}^2)(0.010\times10^{-18}\ \text{m}^2) \\ &= 10^{-22}\ \frac{\text{J}}{\text{s}} \times \frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} \\ &= 6.242\times10^{-4}\ \text{eV/s} \\ &= \boxed{6.2\times10^{-4}\ \text{eV/s}} \end{aligned}$$

(*b*) Classically:

$$\Delta t = \frac{\Delta E}{P} = \frac{\phi}{P}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2.0\,\text{eV}}{6.242\times10^{-4}\ \text{eV/s}} = 3204\,\text{s}$$

$$\approx \boxed{53\,\text{min}}$$

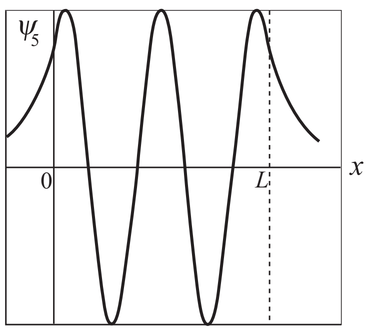
Chapter 35
Applications of the Schrödinger Equation

Conceptual Problems

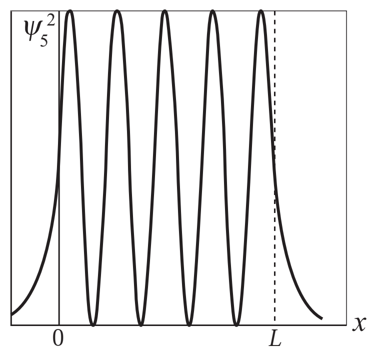
1 • Sketch (*a*) the wave function and (*b*) the probability density function for the $n = 5$ state of the finite square-well potential.

Determine the Concept Looking at the graphs in the text for the $n = 1$, 2, and 3 states, we note that the $n = 5$ state graph of the wave function must have five extrema in the region $0 < x < L$ and decay in toward zero in the regions $x < 0$ and $x > L$.

(*a*)



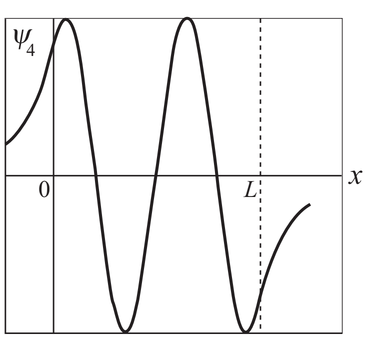
(*b*)



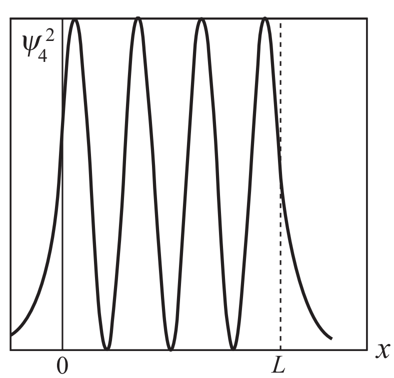
2 • Sketch (*a*) the wave function and (*b*) the probability density function for the $n = 4$ state of the finite square-well potential.

Determine the Concept Looking at the graphs in the text for the $n = 1$, 2, and 3 states, we note that the $n = 4$ state graph of the wave function must have four extrema in the region $0 < x < L$ and decay in toward zero in the regions $x < 0$ and $x > L$.

(*a*)



(*b*)



The Schrödinger Equation

3 •• Show that if $\psi_1(x)$ and $\psi_2(x)$ are each solutions to the time-independent Schrödinger equation (Equation 35-4), then $\psi_3(x) = \psi_1(x) + \psi_2(x)$ is also a solution. This result, known as the superposition principle, applies to the solutions of all linear equations.

Picture the Problem We can show that $\psi_3(x)$ is a solution to the time-independent Schrödinger equation by differentiating it twice and substituting in Equation 35-4.

Equation 35-4 is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x)$$

Because $\psi_1(x)$ and $\psi_2(x)$ are solutions of Equation 35-4:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2}+U(x)\psi_1(x)=E\psi_1(x)$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2(x)}{dx^2}+U(x)\psi_2(x)=E\psi_2(x)$$

Add these equations to obtain:

$$-\frac{\hbar^2}{2m}\left[\frac{d^2\psi_1(x)}{dx^2}+\frac{d^2\psi_2(x)}{dx^2}\right]+U(x)[\psi_1(x)+\psi_2(x)]=E[\psi_1(x)+\psi_2(x)] \quad (1)$$

Differentiate $\psi_3(x)=\psi_1(x)+\psi_2(x)$ twice with respect to x to obtain:

$$\frac{d\psi_3(x)}{dx}=\frac{d\psi_1(x)}{dx}+\frac{d\psi_2(x)}{dx}$$

and

$$\frac{d^2\psi_3(x)}{dx^2}=\frac{d^2\psi_1(x)}{dx^2}+\frac{d^2\psi_2(x)}{dx^2}$$

Substitute in equation (1) to obtain:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_3(x)}{dx^2}+U(x)\psi_3(x)=E\psi_3(x)$$

which shows that $\psi_3(x)=\psi_1(x)+\psi_2(x)$ satisfies Equation 35-4.

The Harmonic Oscillator

4 •• The harmonic oscillator problem may be used to describe the vibrations of molecules. For example, the hydrogen molecule H_2 is found to have equally spaced vibrational energy levels separated by 8.7×10^{-20} J. What value of the force constant of the spring would be needed to get this energy spacing,

assuming that half the molecule can be modeled as a hydrogen atom attached to one end of a spring that has its other end fixed? Hint: The spacing for the energy levels of this half-molecule would be half of the spacing for the energy levels of the complete molecule. In addition, the force constant of a spring is inversely proportional to its relaxed length, so if half of the spring has force constant *k*, the entire spring has a force constant that is equal to $\frac{1}{2}k$.

Picture the Problem We can relate the spring constant to the mass of the hydrogen atom and its angular frequency and then use the relationship between the allowed energy levels and the angular frequency ω to derive an expression for the force constant *k*.

The force constant *k* is related to the mass *m* of the hydrogen molecule and its angular frequency ω:

$$k = m\omega^2 \qquad (1)$$

Relate the energy spacing ΔE to the angular frequency ω:

$$\Delta E = hf = \frac{h\omega}{2\pi} = \hbar\omega \Rightarrow \omega = \frac{\Delta E}{\hbar}$$

Substitute for ω in equation (1) to obtain:

$$k = m\left(\frac{\Delta E}{\hbar}\right)^2$$

Substitute numerical values and evaluate *k*:

$$k = \left(1\,\text{u} \times \frac{1.661\times10^{-27}\,\text{kg}}{\text{u}}\right)\left(\frac{8.7\times10^{-20}\,\text{J}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}}\right)^2 = \boxed{1.1\,\text{kN/m}}$$

Remarks: Our result is very similar to the stiffness constant of typical macroscopic springs. Note that strictly speaking one should use the reduced mass of a hydrogen molecule rather than the simpler model of a single atom attached to a fixed point.

5 •• [SSM] Use the procedure of Example 35-1 to verify that the energy of the first excited state of the harmonic oscillator is $E_1 = \frac{3}{2}\hbar\omega_0$. (Note: *Rather than solve for a again, use the step-8 result* $a = \frac{1}{2}m\omega_0/\hbar$ *obtained in Example 35-1.)*

Picture the Problem We can differentiate $\psi(x)$ twice and substitute in the Schrödinger equation for the harmonic oscillator. Substitution of the given value for *a* will lead us to an expression for E_1.

The wave function for the first excited state of the harmonic oscillator is:

$$\psi_1(x) = A_1 x e^{-ax^2}$$

Compute $d\psi_1(x)/dx$:

$$\frac{d\psi_1(x)}{dx} = \frac{d}{dx}\left[A_1 x e^{-ax^2}\right] = A_1 e^{-ax^2}$$

Compute $d^2\psi_1(x)/dx^2$:

$$\frac{d^2\psi_1(x)}{dx^2} = \frac{d}{dx}\left[A_1 e^{-ax^2}\right] = -2axA_1e^{-ax^2} - 4axA_1e^{-ax^2} + 4a^2x^3A_1e^{-ax^2}$$
$$= \left(4a^2x^3 - 6ax\right)A_1e^{-ax^2}$$

Substitute in the Schrödinger equation to obtain:

$$-\frac{\hbar^2}{2m}\left[\left(4a^2x^3 - 6ax\right)A_1e^{-ax^2}\right] + \tfrac{1}{2}m\omega_0^2x^2A_1xe^{-ax^2} = E_1A_1xe^{-ax^2}$$

Dividing out $A_1e^{-ax^2}$ (one can do this because the exponential function is never zero) yields:

$$-\frac{\hbar^2}{2m}\left[\left(4a^2x^3 - 6ax\right)\right] + \tfrac{1}{2}m\omega_0^2x^3 = E_1x$$

or

$$-\frac{\hbar^2}{2m}\left(4a^2x^3\right) + \frac{\hbar^2}{2m}(6ax) + \tfrac{1}{2}m\omega_0^2x^3 = E_1x$$

Substitute for a to obtain:

$$-\frac{\hbar^2}{2m}4\left(\frac{m\omega_0}{2\hbar}\right)^2x^3 + \frac{\hbar^2}{2m}6\left(\frac{m\omega_0}{2\hbar}\right)x + \tfrac{1}{2}m\omega_0^2x^3 = E_1x$$

Solve for E_1 to obtain:

$$E_1 = \boxed{\tfrac{3}{2}\hbar\omega_0} = 3E_0$$

6 •• Show that the expectation value $< x >= \int x|\psi|^2 dx$ is zero for both the ground state and the first excited state of the harmonic oscillator.

Determine the Concept The integral $\langle x\rangle = \int x|\psi|^2 dx = 0$ because the integrand is an odd function of x for the ground state as well as any excited state of the harmonic oscillator.

7 •• Verify that the normalization constant A_0 in the ground-state harmonic-oscillator wave function $\psi_0(x) = A_0 e^{-ax^2}$ (Equation 35-23) is given by $A_0 = (2m\omega_0/h)^{1/4}$.

Picture the Problem We must show that if $\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = \int_{-\infty}^{\infty} \left|A_0 e^{-ax^2}\right|^2 dx = 1$, then $A_0 = (2m\omega_0/h)^{1/4}$.

We need to show that:

$$\int_{-\infty}^{\infty} \left|A_0 e^{-ax^2}\right|^2 dx = A_0^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = 1$$

Because $\int_{-\infty}^{\infty} e^{-2ax^2} dx = 2\int_{0}^{\infty} e^{-2ax^2} dx$:

$$A_0^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = 2A_0^2 \int_{0}^{\infty} e^{-2ax^2} dx$$

Letting $b = 2a$ yields:

$$2A_0^2 \int_{0}^{\infty} e^{-bx^2} dx = 1$$

From integral tables (see Table D-5):

$$\int_{0}^{\infty} e^{-bx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{b}} = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

Substituting for $\int_{0}^{\infty} e^{-bx^2} dx$ gives:

$$2A_0^2 \left(\frac{1}{2}\sqrt{\frac{\pi}{2a}}\right) = 1$$

or

$$A_0^2 \sqrt{\frac{\pi}{2a}} = 1$$

Solving for A_0 yields:

$$A_0 = \left(\frac{2a}{\pi}\right)^{1/4}$$

From Example 35-1:

$$a = \frac{m\omega_0}{2\hbar}$$

Substitute for a to obtain:

$$A_0 = \left(\frac{2m\omega_0}{2\hbar\pi}\right)^{1/4} = \boxed{\left(\frac{2m\omega_0}{h}\right)^{1/4}}$$

8 •• Using the result of Problem 7, show that for the ground state of the harmonic oscillator $\langle x^2 \rangle = \int x^2 |\psi|^2 dx = h/(2m\omega_0) = 1/(4a)$. Use this result to show that the average potential energy equals half the total energy.

Picture the Problem We are required to evaluate $\langle x^2 \rangle = \int x^2 |\psi_0(x)|^2$ with $\psi_0(x) = A_0 e^{-ax^2}$, where $a = \dfrac{m\omega_0}{2\hbar}$. We can then use $U_{\text{av}} = \frac{1}{2} m\omega_0^2 \langle x^2 \rangle$ to find the average potential energy of the harmonic oscillator.

We need to evaluate:

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi|^2 dx$$

For the ground state of the harmonic oscillator:

$$|\psi|^2 = A_0^2 e^{-2ax^2}$$

Substitute for $|\psi|^2$ to obtain:

$$\langle x^2 \rangle = A_0^2 \int_{-\infty}^{+\infty} x^2 e^{-2ax^2} dx$$
$$= 2A_0^2 \int_0^{+\infty} x^2 e^{-2ax^2} dx$$

Use the appropriate integral from the inside of the back cover of the text to obtain:

$$\langle x^2 \rangle = 2A_0^2 \frac{1}{4}\sqrt{\frac{\pi}{(2a)^3}} = \frac{A_0^2}{4a}\sqrt{\frac{\pi}{2a}} \qquad (1)$$

The normalization condition is:

$$1 = \int_{-\infty}^{+\infty} |\psi|^2 dx = A_0^2 \int_{-\infty}^{+\infty} e^{-2ax^2} dx$$
$$= 2A_0^2 \int_0^{+\infty} e^{-2ax^2} dx$$

Again, use the appropriate integral from the inside of the back cover of the text to obtain:

$$1 = 2A_0^2 \frac{1}{2}\sqrt{\frac{\pi}{2a}} = A_0^2 \sqrt{\frac{\pi}{2a}}$$

Solving for A_0^2 yields:

$$A_0^2 = \sqrt{\frac{2a}{\pi}}$$

Substitute in equation (1) to obtain:

$$\langle x^2 \rangle = \frac{1}{4a}\sqrt{\frac{2a}{\pi}}\sqrt{\frac{\pi}{2a}} = \boxed{\frac{1}{4a}} \qquad (2)$$

From Example 36-1: $a = \frac{m\omega_0}{2\hbar}$

Substitute for a in equation (2) to obtain: $\langle x^2 \rangle = \frac{2\hbar}{4m\omega_0} = \boxed{\frac{\hbar}{2m\omega_0}}$

The average potential energy of the oscillator is: $U_{\text{av}} = \frac{1}{2} m\omega_0^2 \langle x^2 \rangle$

Substitute for $\langle x^2 \rangle$ and simplify:

$$U_{\text{av}} = \frac{1}{2} m\omega_0^2 \left(\frac{\hbar}{2m\omega_0} \right) = \boxed{\tfrac{1}{4}\hbar\omega_0}$$
$$= \tfrac{1}{2} E_0$$

9 •• The quantity $\sqrt{<x^2> - <x>^2}$ is a measure of the average spread in the location of a particle. (*a*) Consider an electron trapped in a harmonic oscillator potential. Its lowest energy level is found to be 2.1×10^{-4} eV. Calculate $\sqrt{<x^2> - <x>^2}$ for this electron. (See Problems 6 and 8.) (*b*) Now consider an electron trapped in an infinite square-well potential. If the width of the well is equal to $\sqrt{<x^2> - <x>^2}$, what would be the lowest energy level for this electron?

Picture the Problem We can combine the result for $\sqrt{\langle x^2 \rangle}$ from Problem 8 and the result for $\langle x \rangle$ from Problem 6 to obtain an expression for $\sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. The lowest energy of the electron in an infinite potential well is given by $E_1 = \frac{h^2}{8mL^2}$.

(*a*) From Problem 8 we have: $\langle x^2 \rangle = \frac{\hbar}{2m\omega_0}$ (1)

The ground-state energy is given by: $E_0 = \frac{1}{2}\hbar\omega_0 \Rightarrow \omega_0 = \frac{2E_0}{\hbar}$

Substitute in equation (1) and simplify to obtain: $\langle x^2 \rangle = \frac{\hbar^2}{4mE_0}$

From Problem 6 we have: $\langle x \rangle = 0$ (2)

Substitute equations (1) and (2) in the expression $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$ to obtain:

$$\sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar^2}{4mE_0} - 0} = \frac{\hbar}{2}\sqrt{\frac{1}{mE_0}}$$

Substitute numerical values and evaluate $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$:

$$\sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{2}\sqrt{\frac{1}{(9.109\times10^{-31}\ \text{kg})\left(2.1\times10^{-4}\ \text{eV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)}}$$

$$= 9.53\,\text{nm} = \boxed{9.5\,\text{nm}}$$

(*b*) The lowest energy of an electron trapped in an infinite potential well is:

$$E_1 = \frac{h^2}{8mL^2}$$

Letting $L = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ yields:

$$E_1 = \frac{h^2}{8m\left(\sqrt{\langle x^2\rangle - \langle x\rangle^2}\right)^2}$$

Substitute numerical values and evaluate E_1:

$$E_1 = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.109\times10^{-31}\ \text{kg})(9.53\times10^{-9}\ \text{m})^2}\frac{1\,\text{eV}}{1.602\times10^{-19}\ \text{J}} = \boxed{4.1\,\text{meV}}$$

10 ••• Classically, the average kinetic energy of the harmonic oscillator equals the average potential energy. Assume that this result is also true for the quantum mechanical harmonic oscillator, and use this result, along with the result of Problem 8, to determine the expectation value of p_x^2 (where $p_x = mv_x$) for the ground state of the harmonic oscillator.

Picture the Problem We can begin by equating the average kinetic energy of the harmonic oscillator and its average potential energy and solving for $\langle p_x^2\rangle$ and then evaluating and substituting for $\langle x^2\rangle$.

According to the problem statement:

$$\frac{\langle p_x^2\rangle}{2m} = \frac{1}{2}k\langle x^2\rangle \Rightarrow \langle p_x^2\rangle = mk\langle x^2\rangle$$

Because $\omega_0^2 = \dfrac{k}{m}$:

$$\left\langle p_x^2 \right\rangle = m^2\omega_0^2 \left\langle x^2 \right\rangle \qquad (1)$$

We need to evaluate:

$$\left\langle x^2 \right\rangle = \int x^2 |\psi|^2 = A_0^2 \int_{-\infty}^{\infty} x^2 e^{-2ax^2}\,dx \qquad (2)$$

where $a = \dfrac{m\omega_0}{2\hbar}$

Let $y^2 = 2ax^2$. Then $x^2 = \dfrac{y^2}{2a}$ and:

$$dx = \frac{ydy}{2ax}$$

With appropriate substitutions, the integral becomes:

$$\int_{-\infty}^{\infty} \frac{y^2}{2a} e^{-y^2} \frac{ydy}{2ax} = \int_{-\infty}^{\infty} \frac{y^3}{2a} e^{-y^2} \frac{\sqrt{2a}dy}{2ay} = \frac{1}{(2a)^{3/2}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy$$

From integral tables (see Table D-5):

$$\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{1}{2}\sqrt{\pi}$$

In Problem 7 it was given that:

$$A_0 = \left(\frac{2m\omega_0}{h}\right)^{1/4}$$

Substituting in equation (2) yields:

$$\left\langle x^2 \right\rangle = \frac{1}{(2a)^{3/2}} \left(\frac{2m\omega_0}{h}\right)^{1/2} \left(\frac{1}{2}\sqrt{\pi}\right)$$

Substitute for a and simplify:

$$\left\langle x^2 \right\rangle = \frac{1}{\left(\frac{m\omega_0}{\hbar}\right)^{3/2}} \left(\frac{2m\omega_0}{h}\right)^{1/2} \left(\frac{1}{2}\sqrt{\pi}\right)$$

$$= \frac{\hbar}{2m\omega_0}$$

Substitute for $\left\langle x^2 \right\rangle$ in equation (1) and simplify to obtain:

$$\left\langle p_x^2 \right\rangle = m^2\omega_0^2 \left(\frac{\hbar}{2m\omega_0}\right) = \boxed{\tfrac{1}{2}\hbar m\omega_0}$$

11 ••• We know that for the classical harmonic oscillator, $p_{x,\,\mathrm{av}} = 0$. It can be shown that for the quantum mechanical harmonic oscillator, $\left\langle p_x \right\rangle = 0$. Use the results of Problem 6, Problem 8, and Problem 10 to determine the uncertainty

product $\Delta x\ \Delta p_x$ for the ground state of the harmonic oscillator. The uncertainties are defined by $(\Delta x)^2 = \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle$ and $(\Delta p_x)^2 = \left\langle \left(p_x - \langle p_x \rangle\right)^2 \right\rangle$.

Picture the Problem We can use the definition of the standard deviation as the uncertainty of Δx and Δp and the results of Problems 6, 8, and 10 to determine the uncertainty product $\Delta x\ \Delta p$ for the ground state of the harmonic oscillator.

The square of the uncertainty of Δp is given by:

$$\begin{aligned}(\Delta p_x)^2 &= \left\langle \left(p_x - \langle p_x \rangle\right)^2 \right\rangle \\ &= \left\langle p_x^2 - 2p_x\langle p_x \rangle + \langle p_x \rangle^2 \right\rangle \\ &= \left\langle p_x^2 \right\rangle - 2\langle p_x \rangle\langle p_x \rangle + \langle p_x \rangle^2 \\ &= \left\langle p_x^2 \right\rangle - \langle p_x \rangle^2\end{aligned}$$

or, because $\langle p_x \rangle = 0$,

$$(\Delta p_x)^2 = \left\langle p_x^2 \right\rangle$$

Similarly, the square of the uncertainty of Δx is given by:

$$(\Delta x)^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

or, because $\langle x \rangle = 0$ (Problem 6),

$$(\Delta x)^2 = \left\langle x^2 \right\rangle$$

We have (from Problems 8 and 10) for the ground state of the harmonic oscillator:

$$\left\langle x^2 \right\rangle = (\Delta x)^2 = \frac{\hbar}{2m\omega_0}$$

and

$$\left\langle p^2 \right\rangle = (\Delta p)^2 = \frac{\hbar m\omega_0}{2}$$

Express the product of $(\Delta x)^2$ and $(\Delta p)^2$:

$$(\Delta x)^2 (\Delta p)^2 = \left(\frac{\hbar}{2m\omega_0}\right)\left(\frac{\hbar m\omega_0}{2}\right) = \frac{\hbar^2}{4}$$

Take the square root of both sides of the equation to obtain:

$$\boxed{\Delta x \Delta p = \frac{\hbar}{2}}$$

Reflection and Transmission of Electron Waves: Barrier Penetration

12 •• A particle of energy E approaches a step barrier of height U_0. What should be the ratio E/U_0 so that the reflection coefficient is 1/2?

Picture the Problem We can use the energies in the regions $U = 0$ and $U = U_0$ to express the ratio of the potential energy to the total energy in terms of the ratio of the wave numbers. We can also express this ratio in terms of the reflection coefficient R to obtain an expression for the ratio of E to U in terms of R.

In the region $U = 0$:

$$E = \frac{\hbar^2 k_1^2}{2m} \Rightarrow k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

In the region $U = U_0$:

$$E - U_0 = \frac{\hbar^2 k_2^2}{2m} \Rightarrow k_2 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

Let r equal the ratio of k_2 to k_1:

$$r = \frac{k_2}{k_1} = \frac{\sqrt{\frac{2m(E-U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}}} = \sqrt{1 - \frac{U_0}{E}}$$

Letting $U_0 = U$, solve for U/E:

$$\frac{U}{E} = 1 - r^2 \qquad (1)$$

Write the reflection coefficient R as a function of $r = \frac{k_2}{k_1}$:

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{\left(1 - \frac{k_2}{k_1}\right)^2}{\left(1 + \frac{k_2}{k_1}\right)^2} = \frac{(1-r)^2}{(1+r)^2}$$

Solving for r gives:

$$r = \frac{1 - \sqrt{R}}{1 + \sqrt{R}}$$

Substitute for r in equation (1):

$$\frac{U}{E} = 1 - \left(\frac{1 - \sqrt{R}}{1 + \sqrt{R}}\right)^2$$

and

$$\frac{E}{U} = \left[1 - \left(\frac{1 - \sqrt{R}}{1 + \sqrt{R}}\right)^2\right]^{-1}$$

Substitute a numerical value for R and evaluate E/U:

$$\frac{E}{U} = \left[1 - \left(\frac{1 - \sqrt{0.5}}{1 + \sqrt{0.5}}\right)^2\right]^{-1} = \boxed{1.03}$$

13 •• [SSM] A particle that has mass m is traveling in the direction of increasing x. The potential energy of the particle is equal to zero everywhere in the region $x < 0$ and is equal to U_0 everywhere in the region $x > 0$, where $U_0 > 0$. (*a*) Show that if the total energy is $E = \alpha U_0$, where $\alpha \geq 1$, then the wave number k_2 in the region $x > 0$ is given by $k_2 = k_1\sqrt{(\alpha - 1)/\alpha}$, where k_1 is the wave number in the region $x < 0$. (*b*) Using a **spreadsheet** program or graphing calculator, graph the reflection coefficient R and the transmission coefficient T as functions of α, for $1 \leq \alpha \leq 5$.

Picture the Problem We can use the total energy of the particle in the region $x > 0$ to express k_2 in terms of α and k_1. Knowing k_2 in terms of k_1, we can use $R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ to find R and $T = 1 - R$ to determine the transmission coefficient T.

(*a*) Using conservation of energy, express the energy of the particle in the region $x > 0$:

$$\frac{\hbar^2 k_2^2}{2m} + U_0 = \alpha U_0$$

Solving for k_2 gives:

$$k_2 = \frac{\sqrt{2mU_0(\alpha - 1)}}{\hbar}$$

From the equation for the total energy of the particle:

$$k_1 = \frac{\sqrt{2m\alpha U_0}}{\hbar}$$

Express the ratio of k_2 to k_1:

$$\frac{k_2}{k_1} = \frac{\frac{\sqrt{2mU_0(\alpha - 1)}}{\hbar}}{\frac{\sqrt{2m\alpha U_0}}{\hbar}} = \sqrt{\frac{\alpha - 1}{\alpha}}$$

and $k_2 = \boxed{\sqrt{\frac{\alpha - 1}{\alpha}}k_1}$

(*b*) The reflection coefficient R is given by:

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

Factor k_1 from the numerator and denominator to obtain:

$$R = \frac{\left(1 - \frac{k_2}{k_1}\right)^2}{\left(1 + \frac{k_2}{k_1}\right)^2}$$

Substitute the result from Part (a) for k_2/k_1:

$$R = \frac{\left(1-\sqrt{\frac{\alpha-1}{\alpha}}\right)^2}{\left(1+\sqrt{\frac{\alpha-1}{\alpha}}\right)^2} = \left(\frac{1-\sqrt{\frac{\alpha-1}{\alpha}}}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)^2$$

The transmission coefficient is given by:

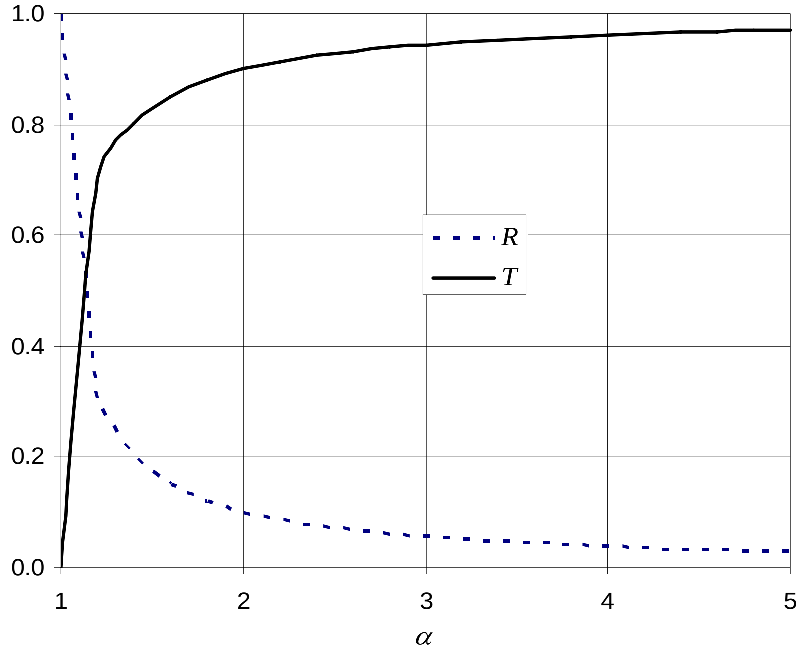
$$T = 1-R = 1-\left(\frac{1-\sqrt{\frac{\alpha-1}{\alpha}}}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)^2$$

A spreadsheet program to plot R and T as functions of α is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	1.0	α
B2	(1−SQRT((A2−1)/A2))/ (1+SQRT((A2−1)/A2))^2	$\left(\frac{1-\sqrt{\frac{\alpha-1}{\alpha}}}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)^2$
C2	1−B2	$1-\left(\frac{1-\sqrt{\frac{\alpha-1}{\alpha}}}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)^2$

	A	B	C
1	α	R	T
2	1.0	1.000	0.000
3	1.2	0.298	0.702
4	1.4	0.198	0.802
5	1.6	0.149	0.851
18	4.2	0.036	0.964
19	4.4	0.034	0.966
20	4.6	0.032	0.968
21	4.8	0.031	0.969
22	5.0	0.029	0.971

The following graph was plotted using the data in the above table:



14 •• Suppose that the potential energy in Problem 13 is equal to zero everywhere in the region $x < 0$ and is equal to $-U_0$ everywhere in the region $x > 0$, where $U_0 > 0$. The wave number for the incident particle is again k_1, and the total energy is $2U_0$. (*a*) What is the wave number for the particle in the region where $x > 0$? (*b*) What is the reflection coefficient *R*? (*c*) What is the transmission coefficient *T*? (*d*) If each of one million particles that are in the region $x < 0$ are traveling with wave number k_1 in the direction of increasing *x* are incident upon the potential energy drop at $x = 0$, how many of these particles are expected to continue along in the direction of increasing *x*? How does this compare with the classical prediction?

Picture the Problem We can use the total energy of the particle in the region $x > 0$ to find k_2. Knowing k_2, we can use $R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ to find *R* and $T = 1 - R$ to determine the transmission coefficient *T*.

(*a*) Using conservation of energy, express the particle in the region $x > 0$:

$$\frac{\hbar^2 k_2^2}{2m} - U_0 = 2U_0 \Rightarrow k_2 = \frac{\sqrt{6mU_0}}{\hbar}$$

From the equation for the total energy of the particle:

$$k_1 = \frac{\sqrt{4mU_0}}{\hbar}$$

Express the ratio of k_2 to k_1 and solve for k_2 to obtain:

$$\frac{k_2}{k_1}=\frac{\frac{\sqrt{6mU_0}}{\hbar}}{\frac{\sqrt{4mU_0}}{\hbar}}=\sqrt{\frac{3}{2}}\Rightarrow k_2=\boxed{\sqrt{\frac{3}{2}}k_1}$$

(b) The reflection coefficient R is given by:

$$R=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}=\frac{\left(1-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{k_2}{k_1}\right)^2}$$

Substitute for k_2/k_1 and evaluate R:

$$R=\frac{\left(1-\sqrt{\frac{3}{2}}\right)^2}{\left(1+\sqrt{\frac{3}{2}}\right)^2}=\boxed{0.0102}$$

(c) Because $R+T=1$:

$$T=1-R=1-0.0102=\boxed{0.990}$$

(d) If we let N_0 represent the number of particles incident upon the potential step, then the number that continue beyond is:

$$N_0T=10^6\times 0.990=\boxed{9.90\times 10^5}$$

Classically, all 10^6 would continue to move past the potential step.

15 •• [SSM] A 10-eV electron (an electron with a kinetic energy of 10 eV) is incident on a potential-energy barrier that has a height equal to 25 eV and a width equal to 1.0 nm. (a) Use Equation 35-29 to calculate the order of magnitude of the probability that the electron will tunnel through the barrier. (b) Repeat your calculation for a width of 0.10 nm.

Picture the Problem The probability that the electron with a given energy will tunnel through the given barrier is given by Equation 35-29.

(a) Equation 35-29 is:

$$T\propto e^{-2\alpha a}$$

where

$$\alpha=\sqrt{\frac{2m(U_0-E)}{\hbar^2}}=\frac{\sqrt{2m(U_0-E)}}{\hbar}$$

Multiply the numerator and denominator of α by c to obtain:

$$\alpha = \frac{\sqrt{2mc^2(U_0 - E)}}{\hbar c}$$

where

$$\hbar c = 1.974\times10^{-13}\ \mathrm{MeV\cdot m}$$

Using $m_e c^2 = 511\,\mathrm{keV}$, evaluate T:

$$T \propto \exp\left\{-2\left(1.0\times10^{-9}\ \mathrm{m}\right)\frac{\sqrt{2(511\,\mathrm{keV})(25\,\mathrm{eV}-10\,\mathrm{eV})}}{1.974\times10^{-13}\ \mathrm{MeV\cdot m}}\right\} = 5.9\times10^{-18} \approx \boxed{10^{-17}}$$

(b) Repeat with a = 0.10 nm:

$$T \propto \exp\left\{-2\left(0.10\times10^{-9}\ \mathrm{m}\right)\frac{\sqrt{2(511\,\mathrm{keV})(25\,\mathrm{eV}-10\,\mathrm{eV})}}{1.974\times10^{-13}\ \mathrm{MeV\cdot m}}\right\} = 1.9\times10^{-2} \approx \boxed{10^{-2}}$$

16 •• Use Equation 35-29 to calculate the order of magnitude of the probability that a proton will tunnel out of a nucleus in one collision with the nuclear barrier if the proton has an energy 6.0 MeV below the top of the potential-energy barrier and the barrier thickness is 1.2×10^{-15} m.

Picture the Problem The probability that a proton will tunnel out of a nucleus in one collision with a nuclear barrier if it has a given energy is given by Equation 35-29.

Equation 35-29 is:

$$T \propto e^{-2\alpha a}$$

where

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

Multiply the numerator and denominator of α by c to obtain:

$$\alpha = \frac{\sqrt{2mc^2(U_0 - E)}}{\hbar c}$$

where

$$\hbar c = 1.974\times10^{-13}\ \mathrm{MeV\cdot m}$$

Using $m_p c^2 = 938\,\mathrm{MeV}$, evaluate T:

$$T \propto \exp\left\{-2\left(1.2\times10^{-15}\ \mathrm{m}\right)\frac{\sqrt{2(938\,\mathrm{MeV})(6.0\,\mathrm{MeV})}}{1.974\times10^{-13}\ \mathrm{MeV\cdot m}}\right\} = 0.275 \approx \boxed{10^{-1}}$$

17 ••• To understand how a small change in α-particle energy can dramatically change the probability of the α particle tunneling from a nucleus, consider an α particle emitted by a uranium nucleus ($Z = 92$). (*a*) Referring to Figure 35-17, calculate the center-to-center distance of closest approach r_1 that α particles that have kinetic energies of 4.0 MeV and 7.0 MeV could make to the uranium nucleus. (*b*) Use the result from Part (*a*) to calculate the relative transmission coefficient $e^{-2\alpha a}$ for the same α particles. (Note: *The actual half-lives of uranium nuclei vary over nine orders of magnitude. Your calculation will show a smaller range than this; however, to find half-life, you must also include the frequency with which the α particle strikes the barrier.*)

Picture the Problems We can find the distance of closest approach by equating the kinetic energy of the alpha particle and the Coulomb potential energy. The probability that the electron with a given energy will tunnel through the given barrier is given by $T \propto e^{-2\alpha a}$ where α is the transmission coefficient and depends on ΔE.

(*a*) The distance of closest approach is related to the kinetic energy E of the alpha particles:

$$E = \frac{k2eZe}{r_1} \Rightarrow r_1 = \frac{2kZe^2}{E}$$

For $E = 4.0$ MeV:

$$r_{1,4.0\,\text{MeV}} = \frac{2\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(92)\left(1.602\times10^{-19}\ \text{C}\right)^2}{4.0\,\text{MeV}\times\dfrac{1.602\times10^{-19}\ \text{J}}{\text{eV}}} = 6.623\times10^{-14}\ \text{m}$$

$$= \boxed{66\,\text{fm}}$$

For $E = 7.0$ MeV:

$$r_{1,7.0\,\text{MeV}} = \frac{2\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(92)\left(1.602\times10^{-19}\ \text{C}\right)^2}{7.0\,\text{MeV}\times\dfrac{1.602\times10^{-19}\ \text{J}}{\text{eV}}} = 3.785\times10^{-14}\ \text{m}$$

$$= \boxed{38\,\text{fm}}$$

(*b*) The transmission coefficient T is given by:

$$T \propto e^{-2\alpha a} \qquad (1)$$

where

$$\alpha = \sqrt{\frac{2m\Delta E}{\hbar^2}} = \frac{\sqrt{2m\Delta E}}{\hbar}$$

Evaluate $\alpha_{4.0\text{ MeV}}$ for ΔE = 4.0 MeV:

$$\alpha_{4.0\,\text{MeV}} = \frac{\sqrt{2\left(4\,\text{u}\times\frac{1.661\times10^{-27}\,\text{kg}}{\text{u}}\right)\left(4.0\,\text{MeV}\times\frac{1.602\times10^{-19}\,\text{J}}{\text{eV}}\right)}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} = 8.747\times10^{14}\,\text{m}^{-1}$$

Evaluate $\alpha_{7.0\text{ MeV}}$ for ΔE = 7.0 MeV:

$$\alpha_{7.0\,\text{MeV}} = \frac{\sqrt{2\left(4\,\text{u}\times\frac{1.661\times10^{-27}\,\text{kg}}{\text{u}}\right)\left(7.0\,\text{MeV}\times\frac{1.602\times10^{-19}\,\text{J}}{\text{eV}}\right)}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} = 1.157\times10^{15}\,\text{m}^{-1}$$

Substitute numerical values in equation (1) and evaluate $T_{4.0\text{ MeV}}$ for $a = r_{1,4.0\,\text{MeV}}$:

$$T_{4.0\,\text{MeV}} \propto \exp\left[-2\left(8.747\times10^{14}\,\text{m}^{-1}\right)\left(6.623\times10^{-14}\,\text{m}\right)\right] \approx \boxed{10^{-51}}$$

Substitute numerical values in equation (1) and evaluate $T_{7.0\text{ MeV}}$ for $a = r_{1,7.0\,\text{MeV}}$:

$$T_{7.0\,\text{MeV}} \propto \exp\left[-2\left(1.157\times10^{15}\,\text{m}^{-1}\right)\left(3.785\times10^{-14}\,\text{m}\right)\right] \approx \boxed{10^{-38}}$$

The Schrödinger Equation in Three Dimensions

18 •• (*a*) A particle is confined to a three-dimensional box that has sides L_1, $L_2 = 2L_1$, and $L_3 = 3L_1$. Give the quantum numbers n_1, n_2 and n_3 that correspond to the ten lowest-energy quantum states of this box. (Hint: *A spreadsheet can be helpful.*) (*b*) What quantum numbers, if any, correspond to degenerate energy levels? (*c*) Give a wave function for the fifth excited state. (There are only five states that have energy levels below the energy level of the fifth excited state.)

Picture the Problem (*a*) The allowed energies of the particle, as a function of the lengths L_1, L_2 and L_3 of the box and the quantum numbers n_1, n_2 and n_3 are given by $E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$. (*b*) Degenerate energy levels correspond to sets of quantum numbers that have the same energy. (*c*) The wave functions are of the form $\psi = A\sin\left(\frac{n_1\pi}{L_1}x\right)\sin\left(\frac{n_2\pi}{2L_1}y\right)\sin\left(\frac{n_3\pi}{3L_1}z\right)$.

(*a*) The energies of the quantum states are given by Equation 35-34:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$$

For a box with sides L_1, $L_2 = 2L_1$, and $L_3 = 3L_1$:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{4L_1^2}+\frac{n_3^2}{9L_1^2}\right)$$
$$= \frac{h^2}{8mL_1^2}\left(n_1^2+\frac{n_2^2}{4}+\frac{n_3^2}{9}\right)$$
$$= \frac{h^2}{288mL_1^2}\left(36n_1^2+9n_2^2+n_3^2\right)$$

The energies in units of $\frac{h^2}{288mL_1^2}$ are listed in the following table (generated using a spreadsheet program):

n_1	1	1	1	1	1	1	1	1	1	1
n_2	1	1	1	1	1	2	2	1	2	2
n_3	1	2	3	4	5	1	2	6	3	4
E	46	49	54	61	70	73	76	81	81	88

(*b*) The quantum numbers that correspond to degenerate energy levels are $\boxed{(1,1,6)}$ and $\boxed{(1,2,3)}$.

(*c*) For the fifth excited state, $(n_1, n_2, n_3) = (1, 2, 1)$ and the wave function is:

$$\psi(1,2,1) = \boxed{A\sin\left(\frac{\pi}{L_1}x\right)\sin\left(\frac{\pi}{L_1}y\right)\sin\left(\frac{\pi}{3L_1}z\right)}$$

19 •• (*a*) A particle is confined to a three-dimensional box that has sides L_1, $L_2 = 2L_1$, and $L_3 = 4L_1$. Give the quantum numbers n_1, n_2 and n_3 that correspond to the ten lowest-energy quantum states of this box. (Hint: *A spreadsheet can be helpful.*) (*b*) What combinations of these quantum numbers, if any, correspond to degenerate energy levels? (*c*) Give the wave function for the fourth excited energy state. (There are only four states that have energy levels below the energy level of the fourth excited state.)

Picture the Problem (*a*) The allowed energies of the particle, as a function of the lengths L_1, L_2 and L_3 of the box and the quantum numbers n_1, n_2 and n_3 are given by $E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$. (*b*) Degenerate energy levels correspond to

sets of quantum numbers that have the same energy. (*c*) The wave functions are of the form $\psi = A\sin\left(\frac{n_1\pi}{L_1}x\right)\sin\left(\frac{n_2\pi}{2L_1}y\right)\sin\left(\frac{n_3\pi}{4L_1}z\right)$.

(*a*) The energies of the quantum states are given by Equation 35-34:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$$

For a box with sides L_1, $L_2 = 2L_1$, and $L_3 = 4L_1$:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{4L_1^2}+\frac{n_3^2}{16L_1^2}\right) = \frac{h^2}{8mL_1^2}\left(n_1^2+\frac{n_2^2}{4}+\frac{n_3^2}{16}\right) = \frac{h^2}{128mL_1^2}\left(16n_1^2+4n_2^2+n_3^2\right)$$

The energies in units of $\frac{h^2}{128mL_1^2}$ are listed in the following table (generated using a spreadsheet program):

n_1	1	1	1	1	1	1	1	1	1	1
n_2	1	1	1	2	1	2	2	1	2	3
n_3	1	2	3	1	4	2	3	5	4	1
E	21	24	29	33	36	36	41	45	48	53

(*b*) Referring to the table, we see that the quantum numbers that correspond to degenerate energy levels are $\boxed{(1,1,4)}$ and $\boxed{(1,2,2)}$.

(*c*) For the fourth excited state, $(n_1, n_2, n_3) = (1, 1, 4)$ and the wave function is:

$$\psi(1,1,4) = \boxed{A\sin\left(\frac{\pi}{L_1}x\right)\sin\left(\frac{\pi}{2L_1}y\right)\sin\left(\frac{\pi}{L_1}z\right)}$$

20 • A particle moves in a potential well given by $U(x, y, z) = 0$ for $-L/2 < x < L/2$, $0 < y < L$, and $0 < z < L$; $U = \infty$ outside these ranges. (*a*) Write an expression for the ground-state wave function for this particle. (*b*) How do the allowed energies compare with those for a well having $U = 0$ for $0 < x < L$, rather than for $-L/2 < x < L/2$? Explain your answers.

Picture the Problem The boundary conditions in the y and z directions are the same as those in Figure 35-1. In the x direction, we'll require the $\psi = 0$ at $-L/2$ and $L/2$.

(a) The boundary conditions in the x direction are:

$$\psi\left(-\tfrac{1}{2}L\right) = \psi\left(\tfrac{1}{2}L\right) = 0$$

The general solution of the time-independent Schrödinger equation is:

$$\psi(x) = A\sin kx + B\cos kx$$

Apply the boundary conditions to obtain:

$$\psi\left(-\tfrac{1}{2}L\right) = -A\sin\left(\frac{kL}{2}\right) + B\cos\left(\frac{kL}{2}\right) = 0$$

and

$$\psi\left(\tfrac{1}{2}L\right) = A\sin\left(\frac{kL}{2}\right) + B\cos\left(\frac{kL}{2}\right) = 0$$

Eliminate the terms in B by subtracting the equations:

$$A\sin\frac{kL}{2} = 0$$

For $A \neq 0$:

$$\frac{kL}{2} = \sin^{-1} 0 = 0, \pi, 2\pi, \ldots$$

or

$$k = \frac{n_1\pi}{L}, n_1 = 0, 2, 4, \ldots$$

Eliminating the terms in A by adding the equations gives:

$$B\sin\frac{kL}{2} = 0$$

For $B \neq 0$:

$$\frac{kL}{2} = \cos^{-1} 0 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

or

$$k = \frac{n_2\pi}{L}, n_2 = 1, 3, 5, \ldots$$

Thus:

$$\psi(x,y,z) = B\cos\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)\sin\left(\frac{n_3\pi}{L}z\right), n_1 = 2n+1$$

and

$$\psi(x,y,z) = A\sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)\sin\left(\frac{n_3\pi}{L}z\right),\ n_1 = 2n$$

The ground-state wave function is:

$$\boxed{\psi(1,1,1) = A\cos\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)\cos\left(\frac{\pi}{L}z\right)}$$

(*b*) The allowed energies are the same as those for a well with $U = 0$ for $0 < x < L$.

21 •• A particle that is constrained to the two-dimensional region defined by $0 \le x \le L$ and $0 \le y \le L$ can move freely throughout that region. (*a*) Find the wave functions that meet these conditions and are solutions of the time-independent Schrödinger equation. (*b*) Find the energies that correspond to the wave functions in Part (*a*). (*c*) Find the quantum numbers of the two lowest states that have the same energy (that are degenerate). (*d*) Find the quantum numbers of the three states that have the same energy.

Picture the Problem (*a*) The wave functions that meet the given conditions and are solutions of the time-independent Schrödinger equation of the form $\psi = A\sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)$. (*b*) The allowed energies of the particle, as a function of the dimensions of the two-dimensional region and the quantum numbers n_1 and n_2 are given by $E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L^2} + \frac{n_2^2}{L^2}\right)$. (*c*) Degenerate energy levels correspond to sets of quantum numbers that have the same energy. These energies can be found by trial-and-error searching or by using a spreadsheet program. (*d*) We must find three different sets of quantum numbers (n_1,n_2) for which the sum of the squares are the same.

(*a*) The wave functions that satisfy the given conditions and are solutions of the time-independent Schrödinger equation are:

$$\psi(x,y) = A\sin k_1 x \sin k_2 y$$

$$= \boxed{A\sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)}$$

where n_1 and n_2 are positive integers.

(*b*) The energies that correspond to the wave functions in Part (*a*) are:

$$E_{n_1,n_2} = \boxed{\frac{h^2}{8mL^2}\left(n_1^2 + n_2^2\right)}$$

(*c*) The quantum numbers of the two lowest states that have the same energy are:

$\boxed{(1,2)}$ and $\boxed{(2,1)}$

(*d*) The energies of three lowest states that have the same energies (in units of $\frac{h^2}{8mL^2}$) are listed in the table to the right:

n_1	n_2	E_{n_1,n_2}
1	7	50
7	1	50
5	5	50

The quantum numbers for the three states are: $\boxed{(1,7)}$, $\boxed{(7,1)}$ and $\boxed{(5,5)}$

The Schrödinger Equation for Two Identical Particles

22 • Show that the two-particle wave function

$$\psi_{12} = A\sin(\pi x_1/L)\sin(2\pi x_2/L),\ 0 < x_1 < L \text{ and } 0 < x_2 < L,$$

(Equation 35-37) is a solution of

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_2^2} + U\psi(x_1,x_2) = E\psi(x_1,x_2)$$

(Equation 35-35), if $U(x_1, x_2) = 0$, and find the energy of the state represented by this wave function.

Picture the Problem We must differentiate Equation 35-37 twice and substitute these derivatives in this equation to show that it is a solution.

With $U = 0$, Equation 35-35 becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_2^2} = E\psi(x_1,x_2) \quad (1)$$

Differentiate Equation 35-7 with respect to x_1:

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial}{\partial x_1}\left[A\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right] = \frac{A\pi}{L}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$$

Compute the second derivative with respect to x_1:

$$\frac{\partial^2\psi}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[\frac{A\pi}{L}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right] = -\frac{A\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$$

Differentiate Equation 35-7 with respect to x_2:

$$\frac{\partial\psi}{\partial x_2}=\frac{\partial}{\partial x_2}\left[A\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right]$$
$$=\frac{2A\pi}{L}\sin\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{2\pi x_2}{L}\right)$$

Compute the second derivative with respect to x_2:

$$\frac{\partial^2\psi}{\partial x_2^2}=\frac{\partial}{\partial x_2}\left[\frac{2A\pi}{L}\sin\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{2\pi x_2}{L}\right)\right]$$
$$=-\frac{4A\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$$

Substitute in equation (1) to obtain:

$$-\frac{\hbar^2}{2m}\left[-\frac{A\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right]-\frac{\hbar^2}{2m}\left[-\frac{4A\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right]$$
$$=EA\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$$

Solving for E yields:

$$E=\frac{5\hbar^2\pi^2}{2mL^2}=\frac{5h^2}{8mL^2}$$

Thus we've shown that Equation 35-37 satisfies Equation 35-35 provided $E=\dfrac{5\hbar^2\pi^2}{2mL^2}=\dfrac{5h^2}{8mL^2}$.

23 • What is the ground-state energy of ten non-interacting bosons in a one-dimensional box of length L?

Picture the Problem Because bosons have symmetric wave functions and do not obey the Pauli exclusion principle, they can occupy the same ground state.

For ten noninteracting bosons in a one-dimensional box:

$$E_{1,10\,\text{bosons}}=10E_1$$

where E_1 is the ground-state energy of a single particle in a one-dimensional box.

The ground-state energy of a single particle in a one-dimensional box of length L is given by Equation 35-14:

$$E_1=\frac{h^2}{8mL^2}$$

Substituting for E_1 and simplifying yields:

$$E_{1,10\text{ bosons}} = 10\left(\frac{h^2}{8mL^2}\right) = \boxed{\frac{5h^2}{4mL^2}}$$

24 •• What is the ground-state energy of seven identical non-interacting fermions in a one-dimensional box of length *L*? (Because the quantum number associated with spin can have two values, each spatial state can be occupied two fermions.)

Picture the Problem For fermions (particles for which the spin quantum number is ½) two particles can occupy the same spatial state.

The ground-state energy of seven fermions is:

$$E_{1,7\text{ fermions}} = E_1\left[2\left(1^2 + 2^2 + 3^2\right) + 4^2\right]$$
$$= 44E_1$$

where E_1 is the ground-state energy of a single particle in a one-dimensional box.

The ground-state energy of a single particle in a one-dimensional box of length *L* is given by Equation 35-14:

$$E_1 = \frac{h^2}{8mL^2}$$

Substituting for E_1 and simplifying yields:

$$E_{1,7\text{ fermions}} = 44\left(\frac{h^2}{8mL^2}\right) = \boxed{\frac{11h^2}{2mL^2}}$$

Orthogonality of Wave Functions

25 •• Show that the ground-state and the first excited state wave functions of the harmonic oscillator are orthogonal; that is, show that $\int \psi_0(x)\,\psi_1(x)dx = 0$.

Picture the Problem We need to show that $\int_{-\infty}^{\infty}\psi_0(x)\psi_1(x)dx = 0$, where $\psi_0(x)$ and $\psi_1(x)$ are given by Equations 35-23 and 35-25, respectively.

Equations 35-23 and 35-25 are:

$$\psi_0(x) = A_0 e^{-ax^2} \qquad 35\text{-}23$$

and

$$\psi_1(x) = A_1 x e^{-ax^2} \qquad 35\text{-}25$$

Note that $\psi_1(x)$ is antisymmetric, whereas $\psi_0(x)$ is symmetric. Because the product of an

$\psi_0(x)\psi_1(x)$ is antisymmetric

antisymmetric function and a symmetric function is antisymmetric:

Because the integral of an antisymmetric function over symmetric limits is zero:

$$\boxed{\int_{-\infty}^{\infty}\psi_0(x)\psi_1(x)dx = 0}$$

26 •• The wave function for the state $n = 2$ of the harmonic oscillator is $\psi_2(x) = A_2\left(2ax^2 - \frac{1}{2}\right)e^{-ax^2}$, where A_2 is the normalization constant and a is a positive constant. Show that the wave functions for the states $n = 1$ and $n = 2$ of the harmonic oscillator are orthogonal.

Picture the Problem We need to show that $\int_{-\infty}^{\infty}\psi_1(x)\psi_2(x)dx = 0$, where $\psi_2(x)$ is given in the problem statement and $\psi_1(x) = A_1xe^{-ax^2}$.

Note that $\psi_1(x)$ is antisymmetric, whereas $\psi_2(x)$ is symmetric. Because the product of a symmetric function and an antisymmetric function is antisymmetric, $\psi_1(x)\psi_2(x)$ is antisymmetric. Because the integral of an antisymmetric function over symmetric limits is zero:

$$\boxed{\int_{-\infty}^{\infty}\psi_1(x)\psi_2(x)dx = 0}$$

27 •• For the wave functions

$$\psi_n(x) = A\sin(n\pi x/L) \quad n = 1,\ 2,\ 3,\ k$$

corresponding to a particle in an infinite square-well potential from 0 to L, show that $\int_0^L \psi_m(x)\,\psi_n(x)dx = 0$ for all positive integers m and n, where $m \neq n$; that is, show that the wave functions are orthogonal.

Picture the Problem We need to show that $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$

Use the trigonometric identity $(\sin a\alpha)(\sin b\alpha) = \frac{1}{2}\{\cos[(a-b)\alpha] - \cos[(a+b)\alpha]\}$

to rewrite the product of the two sine functions as the difference of two cosine functions:

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)=\frac{1}{2}\left\{\cos\left[(n-m)\frac{\pi x}{L}\right]-\cos\left[(n+m)\frac{\pi x}{L}\right]\right\}$$

Substitute for $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ and evaluate $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$:

$$\int_0^L \frac{1}{2}\left\{\cos\left[(n-m)\frac{\pi x}{L}\right]-\cos\left[(n+m)\frac{\pi x}{L}\right]\right\}dx$$

$$=\frac{L}{\pi}\left[\frac{\sin\left[(n-m)\frac{\pi x}{L}\right]}{n-m}-\frac{\sin\left[(n+m)\frac{\pi x}{L}\right]}{n+m}\right]_0^L$$

Because n and m are integers and $n \neq m$, the sine functions vanish at the two limits $x = 0$ and $x = L$. Therefore, $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$ for $n \neq m$.

General Problems

28 •• Consider a particle in an infinite one-dimensional box that has a length L and is centered at the origin. (*a*) What are the values of $\psi_1(0)$ and $\psi_2(0)$? (*b*) What are the values of $\langle x\rangle$ for the $n = 1$ and $n = 2$ states? (*c*) Evaluate $\langle x^2\rangle$ for the $n = 1$ and $n = 2$ states.

Picture the Problem We can use the wave functions $\psi_1(x)$ and $\psi_2(x)$ and the definitions of $\langle x\rangle$ and $\langle x^2\rangle$ to evaluate these quantities and the wave functions at $x = 0$.

(*a*) The wave functions $\psi_1(x)$ and $\psi_2(x)$ are:

$$\psi_m=\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right), m = 2n$$

and

$$\psi_m=\sqrt{\frac{2}{L}}\cos\left(\frac{m\pi}{L}x\right), m = 2n+1$$

where $n = 0, 1, 2, \ldots$

Evaluate these functions at $x = 0$ to obtain:

$$\psi_1(0) = \sqrt{\frac{2}{L}}\sin\left[\frac{\pi}{L}(0)\right] = \boxed{0}$$

and

$$\psi_2(0) = \sqrt{\frac{2}{L}}\cos\left[\frac{\pi}{L}(0)\right] = \boxed{\sqrt{\frac{2}{L}}}$$

(*b*) Because $\left|\psi_m(x)\right|^2$ is an even function of x in all cases, $x\psi_m^2(x)$ is an odd function of x and:

$$\langle x\rangle = \int_{-L/2}^{L/2} x\psi_m^2(x)dx = \boxed{0}$$

(*c*) For $n = 1$:

$$\left\langle x^2\right\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\frac{\pi}{L}xdx$$

From integral tables:

$$\int x^2\sin^2(ax)dx = \frac{x^3}{6} - \left(\frac{x^3}{4a} - \frac{1}{8a^3}\right)\sin(2ax) - \frac{x\cos(2ax)}{4a^2}$$

Use this integral with $a = \pi/L$ to obtain:

$$\begin{aligned}
\left\langle x^2\right\rangle &= \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\frac{\pi}{L}xdx \\
&= \frac{2}{L}\left[\frac{x^3}{6} - \left(\frac{x^3}{4\left(\frac{\pi}{L}\right)} - \frac{1}{8\left(\frac{\pi}{L}\right)^3}\right)\sin\left(2\left(\frac{\pi}{L}\right)x\right) - \frac{x\cos\left(2\left(\frac{\pi}{L}\right)x\right)}{4\left(\frac{\pi}{L}\right)^2}\right]_{-L/2}^{L/2} \\
&= \frac{2}{L}\left(\frac{L^3}{24} - \frac{L^3}{4\pi^2}\cos\pi\right) \\
&= \boxed{\frac{L^2}{12}\left(1 + \frac{6}{\pi^2}\right)}
\end{aligned}$$

For $n = 2$:

$$\left\langle x^2\right\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\left(\frac{2\pi}{L}x\right)dx$$

Use the integral given above with $a = 2\pi/L$ to obtain:

$$\left\langle x^2\right\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2 \sin^2\left(\frac{\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left[\frac{x^3}{6} - \left(\frac{x^3}{4\left(\frac{2\pi}{L}\right)} - \frac{1}{8\left(\frac{2\pi}{L}\right)^3}\right)\sin\left(2\left(\frac{2\pi}{L}\right)x\right) - \frac{x\cos\left(2\left(\frac{2\pi}{L}\right)x\right)}{4\left(\frac{2\pi}{L}\right)^2}\right]_{-L/2}^{L/2}$$

$$= \boxed{\frac{L^2}{12}\left(1 + \frac{3}{2\pi^2}\right)}$$

Remarks: Note that for any value of *m*, $\boldsymbol{\left\langle x^2\right\rangle = \frac{L^2}{12}\left(1 + \frac{6}{m^2\pi^2}\right)}$**.**

29 •• [SSM] Eight identical non-interacting fermions are confined to an infinite two-dimensional square box of side length L. Determine the energies of the three lowest-energy states. (See Problem 22.)

Picture the Problem We can determine the energies of the state by identifying the four lowest quantum states that are occupied in the ground state and computing their combined energies. We can then find the energy difference between the ground state and the first excited state and use this information to find the energy of the excited state.

Each n, m state can accommodate only 2 particles. Therefore, in the ground state of the system of 8 fermions, the four lowest quantum states are occupied. These are:

(1,1), (1,2), (2,1) and (2,2)

Note that the states (1,2) and (2,1) are distinctly different states because the x and y directions are distinguishable.

The energies are quantized to the values given by:

$$E_{n_1,n_2} = 2\left(\frac{h^2}{8mL^2}\right)\left(n_1^2 + n_2^2\right)$$

The energy of the ground state is the sum of the energies of the four lowest quantum states:

$$\begin{aligned}E_0 &= E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}\\ &= 2\left(\frac{h^2}{8mL^2}\right)\left(1^2+1^2\right) + 2\left(\frac{h^2}{8mL^2}\right)\left(1^2+2^2\right) + 2\left(\frac{h^2}{8mL^2}\right)\left(2^2+1^2\right)\\ &\qquad + 2\left(\frac{h^2}{8mL^2}\right)\left(2^2+2^2\right)\\ &= 2\left(\frac{h^2}{8mL^2}\right)(2+5+5+8) = \frac{5h^2}{mL^2}\end{aligned}$$

The next higher state is achieved by taking one fermion from the (2, 2) state and raising it to the next higher unoccupied state. That state is the (1, 3) state. The energy difference between the ground state and this state is:

$$\begin{aligned}\Delta E &= E_{1,3} - E_{2,2}\\ &= \frac{h^2}{8mL^2}\left(1^2+3^2\right) - \frac{h^2}{8mL^2}\left(2^2+2^2\right)\\ &= \frac{h^2}{4mL^2}\end{aligned}$$

Hence, the energies of the degenerate states (1,3) and (3,1) are:

$$\begin{aligned}E_{1,3} = E_{3,1} &= E_0 + \Delta E\\ &= \frac{5h^2}{mL^2} + \frac{h^2}{4mL^2} = \frac{21h^2}{4mL^2}\end{aligned}$$

The three lowest energy levels are therefore:

$$E_0 = \boxed{\frac{5h^2}{mL^2}}$$

and two states of energy

$$E_1 = E_2 = \boxed{\frac{21h^2}{4mL^2}}$$

30 •• A particle is confined to a two-dimensional box defined by the following boundary conditions: $U(x, y) = 0$ for $-L/2 \le x \le L/2$ and $-3L/2 \le y \le 3L/2$, and $U(x, y) = \infty$ outside these ranges. (*a*) Determine the energies of the three lowest-energy states. Are any of these states degenerate? (*b*) Identify the quantum numbers of the two lowest-energy degenerate states and determine the energy of these states.

Picture the Problem (*a*) The energy levels for a two-dimensional box is given by Equation 35-34:

$$E_{n_1,n_2} = \frac{h^2}{8m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}\right)$$

We can simplify the calculation by substituting L for L_1 and $3L$ for L_2..

(*b*) Degenerate energy states have the same energy but different quantum numbers. We can use the dependence of E_{n_1,n_2} on the quantum numbers n_1 and n_2 to identify the quantum numbers of the two lowest-energy degenerate states. Knowing their quantum numbers, we can find the energy of these states.

(*a*) The energies of the bound states are given by:

$$E_{n_1,n_2} = \frac{h^2}{8m}\left(\frac{n_1^2}{L^2} + \frac{n_2^2}{9L^2}\right) = \frac{h^2}{72mL^2}\left(9n_1^2 + n_2^2\right) \tag{1}$$

The three lowest energy states are:

$$E_{1,1} = \frac{h^2}{72mL^2}(9+1) = \boxed{\frac{5h^2}{36mL^2}}$$

$$E_{1,2} = \frac{h^2}{72mL^2}(9+4) = \boxed{\frac{13h^2}{72mL^2}}$$

and

$$E_{1,3} = \frac{h^2}{72mL^2}(9+9) = \boxed{\frac{h^2}{4mL^2}}$$

$$\boxed{\text{None of these states are degenerate.}}$$

(*b*) Let n_1, n_2 be the quantum numbers for one state and let m_1, m_2 be the quantum numbers for a second state. Then the energies of these states are given by equation (1):

$$E_{n_1,n_2} = \frac{h^2}{72mL^2}\left(9n_1^2 + n_2^2\right)$$

and

$$E_{m_1,m_2} = \frac{h^2}{72mL^2}\left(9m_1^2 + m_2^2\right)$$

Degenerate states have equal energies. Equating E_{n_1,m_1} and E_{n_2,m_2} yields:

$$9n_1^2 + n_2^2 = 9m_1^2 + m_2^2$$

We can solve this by a massive computational effort using a spreadsheet or by trial and error using educated guesses. Using educated guesses, we first look for a solution with $n_1 = 1$ and $m_1 = 2$. Substituting, this gives:

$$9 \cdot 1^2 + n_2^2 = 9 \cdot 2^2 + m_2^2$$

or

$$9 + n_2^2 = 36 + m_2^2$$

Notice that 9 and 36 are perfect squares:

$$3^2 + n_2^2 = 6^2 + m_2^2$$

We can see one solution by inspection:

$$n_2 = 6 \text{ and } m_2 = 3$$

Thus, the quantum numbers of interest are:

$$n_1 = 2, m_1 = 3, \text{and } n_2 = 1,\ m_2 = 6$$

The energy of this doubly degenerate state is:

$$E_{2,3} = \frac{h^2}{72mL^2}(36+9) = \boxed{\frac{5h^2}{8mL^2}}$$

31 ••• The classical probability distribution function for a particle in an infinite one-dimensional well of length L is $P = 1/L$. (See Example 34-5.) (*a*) Show that the classical expectation value of x^2 for a particle in an infinite one-dimensional well of length L that is centered at the origin is $L^2/12$. (*b*) Find the quantum expectation value of x^2 for the nth state of a particle in the one-dimensional box and show that it approaches the classical limit $L^2/12$ as n approaches infinity.

Picture the Problem We can use the definition of the classical expectation value (average value) to show that the classical expectation value of x^2 for a particle in a one-dimensional box of length L centered at the origin is $L^2/12$. In (*b*) we'll proceed as in (*a*) using the definition of the quantum expectation value of x^2.

(*a*) The classical expectation value of x^2 for a particle in a one-dimensional box of length L centered at the origin is given by:

$$\langle x^2 \rangle = \frac{1}{\frac{1}{2}L - \left(-\frac{1}{2}L\right)} \int_{-L/2}^{L/2} x^2 dx$$

$$= \frac{1}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{1}{L}\left(\frac{L^3}{12}\right)$$

$$= \boxed{\frac{L^2}{12}}$$

(*b*) For a particle in the *n*th state in a one-dimensional box:

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\left(\frac{n\pi}{L}x\right)dx$$

From integral tables:

$$\int x^2\sin^2(ax)dx = \frac{x^3}{6}-\left(\frac{x^3}{4a}-\frac{1}{8a^3}\right)\sin(2ax)-\frac{x\cos(2ax)}{4a^2}$$

Use this integral with $a = n\pi/L$ to obtain:

$$\begin{aligned}\langle x^2\rangle &= \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\left(\frac{n\pi}{L}x\right)dx\\ &= \frac{2}{L}\left[\frac{x^3}{6}-\left(\frac{x^3}{4\left(\frac{n\pi}{L}\right)}-\frac{1}{8\left(\frac{n\pi}{L}\right)^3}\right)\sin\left(2\left(\frac{n\pi}{L}\right)x\right)-\frac{x\cos\left(2\left(\frac{n\pi}{L}\right)x\right)}{4\left(\frac{n\pi}{L}\right)^2}\right]_{-L/2}^{L/2}\\ &= \boxed{\frac{2}{L}\left(\frac{L^3}{24}-\frac{L^3}{4n^2\pi^2}\cos n\pi\right)}\end{aligned}$$

In the limit $n >> 1$ we have:

$$\langle x^2\rangle = \boxed{\frac{L^2}{12}}$$

32 •• Show that Equation 35-27 and Equation 35-28 imply that the transmission coefficient for particles of energy E incident on a step barrier $U_0 < E$ is given by $T = \dfrac{4k_1k_2}{(k_1+k_2)^2} = \dfrac{4r}{(1+r)^2}$ where $r = k_2/k_1$.

Picture the Problem We can solve Equation 35-28 for T and substitute for R using Equation 35-27. Letting $r = k_2/k_1$ and simplifying will lead to the given result.

Equation 35-28 is:

$$T + R = 1 \Rightarrow T = 1 - R$$

From Equation 35-27:

$$R=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}=\frac{\left(1-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{k_2}{k_1}\right)^2}=\frac{(1-r)^2}{(1+r)^2}$$

where $r = k_2/k_1$

Substitute for R to obtain:

$$T=1-\frac{(1-r)^2}{(1+r)^2}=\frac{(1+r)^2-(1-r)^2}{(1+r)^2}$$

$$=\boxed{\frac{4r}{(1+r)^2}}$$

Substitute for k_2/k_1 for r and simplify to obtain:

$$T=\boxed{\frac{4k_1k_2}{(k_1+k_2)^2}}$$

33 •• (*a*) Show that for the case of a particle of energy E incident on a step barrier $U_0 < E$, the wave numbers k_1 and k_2 are related by $\frac{k_2}{k_1}=r=\sqrt{1-\frac{U_0}{E}}$. (*b*) Use this result to show that $R = (1 - r)^2/(1 + r)^2$.

Picture the Problem We can use the energies in the regions $U = 0$ and $U = U_0$ to express the ratio of the wave numbers k_1 and k_2 in these regions in terms of E and U_0 and the definition of the reflection coefficient R to show that $R=\frac{(1-r)^2}{(1+r)^2}$.

(*a*) In the region $U_0 = 0$:

$$E=\frac{\hbar^2k_1^2}{2m}\Rightarrow k_1=\sqrt{\frac{2mE}{\hbar^2}}$$

In the region $U = U_0$:

$$E-U_0=\frac{\hbar^2k_2^2}{2m}\Rightarrow k_2=\sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

Let r equal the ratio of k_2 to k_1:

$$r=\frac{k_2}{k_1}=\frac{\sqrt{\frac{2m(E-U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}}}=\boxed{\sqrt{1-\frac{U_0}{E}}}$$

(*b*) The reflection coefficient R is given by:

$$R=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

Factor k_1 from the numerator and denominator to obtain:

$$R = \frac{\left(1 - \frac{k_2}{k_1}\right)^2}{\left(1 + \frac{k_2}{k_1}\right)^2}$$

Substitute for k_2/k_1 to obtain:

$$\boxed{R = \frac{(1-r)^2}{(1+r)^2}}$$

34 •• (*a*) Using a **spreadsheet** program or graphing calculator and the results of Problem 32 and Problem 33, graph the transmission coefficient T and reflection coefficient R as a function of incident energy E for values of E ranging from $E = U_0$ to $E = 10.0\ U_0$. (*b*) What limiting values do your graphs indicate?

Picture the Problem

(*a*) From Problem 33 we have:

$$R = \frac{(1-r)^2}{(1+r)^2}, \text{where } r = \sqrt{1 - \frac{U_0}{E}}$$

Because $E = \alpha U_0$, R can be written:

$$r = \sqrt{1 - \frac{1}{\alpha}}$$

From Problem 32 we have:

$$T = \frac{4r}{(1+r)^2}$$

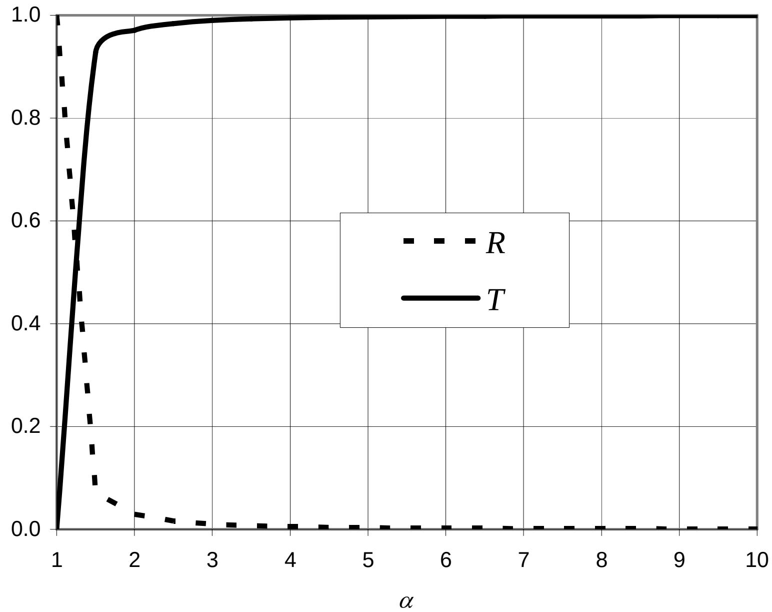
A spreadsheet program to plot R and T as functions of α is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	1.0	α
B2	SQRT(1–1/A2)	$\sqrt{1-\frac{1}{\alpha}}$
C2	(1–B2)^2/(1+B2)^2	$\frac{(1-r)^2}{(1+r)^2}$
D2	4*B2//(1+B2)^2	$\frac{4r}{(1+r)^2}$

	A	B	C	D
1	α	r	R	T
2	1.0	0.000	1.000	0.000
3	1.5	0.577	0.072	0.928
4	2.0	0.707	0.029	0.971

5	2.5	0.775	0.016	0.984
16	8.0	0.935	0.001	0.999
17	8.5	0.939	0.001	0.999
18	9.0	0.943	0.001	0.999
19	9.5	0.946	0.001	0.999
20	10.0	0.949	0.001	0.999

The following graph of R and T as functions of α was plotted using the data in the table:



(b) Referring to the graph, note that, as $\alpha \to \infty$. $T \to 1$ and $R \to 0$. The graph also shows that, as $\alpha \to 1$, $T \to 0$ and $R \to 1$.

35 ••• The wave function for the state $n = 2$ of the harmonic oscillator is $\psi_2(x) = A_2\left(2ax^2 - \frac{1}{2}\right)e^{-ax^2}$, where $a = \frac{1}{2}m\omega_0/h$. Determine the normalization constant A_2.

Picture the Problem We can find the normalization constant by requiring that

$$A_2^2 \int_{-\infty}^{\infty} \left(2ax^2 - \tfrac{1}{2}\right)^2 e^{-2ax^2} dx = 2A_2^2 \int_0^{\infty} \left(2ax^2 - \tfrac{1}{2}\right)^2 e^{-2ax^2} dx = 1.$$

Expand the integrand to obtain:

$$\left(2ax^2 - \tfrac{1}{2}\right)^2 e^{-2ax^2} = \left(4a^2x^4 - 2ax^2 + \tfrac{1}{4}\right)e^{-2ax^2} = 4a^2x^4e^{-2ax^2} - 2ax^2e^{-2ax^2} + \tfrac{1}{4}e^{-2ax^2}$$

Substitute in the integral expression to obtain:

$$2A_2^2\int_0^\infty\left(4a^2x^4e^{-2ax^2}-2ax^2e^{-2ax^2}+\tfrac{1}{4}e^{-2ax^2}\right)dx=1$$

or

$$8a^2A_2^2\int_0^\infty x^4e^{-2ax^2}dx-4aA_2^2\int_0^\infty x^2e^{-2ax^2}dx+\tfrac{1}{2}A_2^2\int_0^\infty e^{-2ax^2}dx=1 \qquad (1)$$

Use the definite integral

$$\int_0^\infty e^{-bx^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{b}} \text{ and } \int_0^\infty x^{2n}e^{-bx^2}dx=\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}b^n}\sqrt{\frac{\pi}{b}}\quad, n\geq 1,$$

(see Table D-5) to integrate equation (1) term by term:

$$8a^2A_2^2\left[\frac{3}{2^3(2a)^2}\sqrt{\frac{\pi}{2a}}\right]-4aA_2^2\left[\frac{1}{2^2(2a)}\sqrt{\frac{\pi}{2a}}\right]+\tfrac{1}{2}A_2^2\left[\frac{1}{2}\sqrt{\frac{\pi}{2a}}\right]=1$$

Simplify to obtain:

$$A_2^2\left[\frac{3}{4}\sqrt{\frac{\pi}{2a}}-\frac{1}{2}\sqrt{\frac{\pi}{2a}}+\frac{1}{4}\sqrt{\frac{\pi}{2a}}\right]=1$$

or

$$A_2^2\left[\frac{1}{2}\sqrt{\frac{\pi}{2a}}\right]=1\Rightarrow A_2=\sqrt{2\sqrt{\frac{2a}{\pi}}}$$

Because $a=\dfrac{m\omega_0}{2\hbar}=\dfrac{m\omega_0\pi}{h}$

$$A_2=\boxed{\sqrt[4]{\frac{8m\omega_0}{h}}}$$

36 ••• Consider the time-independent, one-dimensional Schrödinger equation when the potential function is symmetric about the origin, that is, when $U(x)$ is even.[1] (*a*) Show that if $\psi(x)$ is a solution of the Schrödinger equation and has energy E, then $\psi(-x)$ is also a solution that has the same energy E. Furthermore, $\psi(x)$ and $\psi(-x)$ can differ by only a multiplicative constant. (*b*) Write $\psi(x)=C\psi(-x)$, and show that $C=\pm1$. Note that $C=+1$ means that $\psi(x)$ is an even function of x, and $C=-1$ means that $\psi(x)$ is an odd function of x.

[1] A function $f(x)$ is even if $f(x)=f(-x)$ for all x, and a function $f(x)$ is odd if $f(x)=-f(-x)$ for all x.

Picture the Problem
(a) Let $x = -x$. The second derivative is an even operator, that is, $d^2\psi(-x)/d(-x)^2 = d^2\psi(x)/dx^2$. Therefore, if $U(-x) = U(x)$, the Schrödinger equation for $\psi(-x) = \psi(x)$ and must give the same values for the energy E. If $\psi(-x)$ differs from $\psi(x)$, the ratio $\psi(-x)/\psi(x)$ cannot be a function of x and must be a constant. Hence, $\psi(x) = C\psi(-x)$.

(b) The previous result means that replacing the argument of the wave function by its negative is equivalent to multiplication by C. Thus, if we replace $C\psi(-x)$ argument by its negative, that is, by x, we must multiply by C again. Thus, $\psi(x) = C^2\psi(x)$, $C^2 = 1$, and $C = \pm 1$.

37 ••• [SSM] In this problem, you will derive the ground-state energy of the harmonic oscillator using the precise form of the uncertainty principle, $\Delta x\Delta p_x \geq h/2$, where Δx and Δp_x are defined to be the standard deviations $(\Delta x)^2 = \left\langle\left(x - \langle x\rangle\right)^2\right\rangle$ and $(\Delta p_x)^2 = \left\langle\left(p_x - \langle p_x\rangle\right)^2\right\rangle$. Proceed as follows:
1. Write the total classical energy in terms of the position x and momentum p_x using $U(x) = \frac{1}{2}m\omega_0^2x^2$ and $K = \frac{1}{2}p_x^2/m$.
2. Show that $(\Delta x)^2 = \left\langle\left(x - \langle x\rangle\right)^2\right\rangle = \langle x^2\rangle - \langle x\rangle^2$ and $(\Delta p_x)^2 = \left\langle\left(p_x - \langle p_x\rangle\right)^2\right\rangle = \langle p_x^2\rangle - \langle p_x\rangle^2$. (Hint: See Equations 17-34a and 17-34b.)
3. Use the symmetry of the potential energy function to argue that $\langle x\rangle$ and $\langle p_x\rangle$ must be zero, so that $(\Delta x)^2 = \langle x^2\rangle$ and $(\Delta p_x)^2 = \langle p_x^2\rangle$.
4. Assume that $\Delta x\Delta p_x = h/2$ to eliminate $\langle p_x^2\rangle$ from the average energy $\langle E\rangle = \left\langle\frac{1}{2}p_x^2/m + \frac{1}{2}m\omega_0^2x^2\right\rangle = \frac{1}{2}\langle p_x^2\rangle/m + \frac{1}{2}m\omega_0^2\langle x^2\rangle$ and write $\langle E\rangle$ as $\langle E\rangle = h^2/(8mZ) + \frac{1}{2}m\omega_0^2 Z$, where $Z = \langle x^2\rangle$.
5. Set $d\langle E\rangle/dZ = 0$ to find the value of Z for which $\langle E\rangle$ is a minimum.
6. Show that the minimum energy is given by $\langle E\rangle_{\text{min}} = +\frac{1}{2}h\omega_0$.

Picture the Problem We can follow the step-by-step procedure outlined in the problem statement to show that $\langle E\rangle_{\text{min}} = +\frac{1}{2}h\omega_0$.

1. The total classical energy is:

$$E = K + U(x) = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_0^2x^2$$

Hence the average classical energy is given by:

$$\langle E\rangle = \left\langle \frac{p_x^2}{2m} + \tfrac{1}{2}m\omega_0^2 x^2 \right\rangle \qquad (1)$$

2. Express the standard deviation of Δp_x:

$$\begin{aligned}(\Delta p_x)^2 &= \left\langle \left(p_x - \langle p_x\rangle\right)^2 \right\rangle \\ &= \left\langle p_x^2 - 2p_x\langle p_x\rangle + \langle p_x\rangle^2 \right\rangle \\ &= \left\langle p_x^2\right\rangle - 2\langle p_x\rangle\langle p_x\rangle + \langle p_x\rangle^2 \\ &= \left\langle p_x^2\right\rangle - \langle p_x\rangle^2\end{aligned}$$

Proceeding similarly for the standard deviation of Δx gives:

$$(\Delta x)^2 = \left\langle x^2\right\rangle - \langle x\rangle^2$$

3. From the symmetry of the potential energy function we can conclude that $\langle x\rangle$ and $\langle p_x\rangle$ must be zero. Hence:

$$(\Delta x)^2 = \left\langle x^2\right\rangle$$

and

$$(\Delta p_x)^2 = \left\langle p_x^2\right\rangle$$

4. Rewrite equation (1) in terms of $(\Delta p_x)^2$ to obtain:

$$\begin{aligned}\langle E\rangle &= \frac{\left\langle p_x^2\right\rangle}{2m} + \tfrac{1}{2}m\omega_0^2\left\langle x^2\right\rangle \\ &= \frac{(\Delta p_x)^2}{2m} + \tfrac{1}{2}m\omega_0^2\left\langle x^2\right\rangle\end{aligned}$$

Using the uncertainty principle ($\Delta x\,\Delta p_x = h/2$) to eliminate $\left\langle p_x^2\right\rangle$ gives:

$$\begin{aligned}\langle E\rangle &= \frac{\left(\dfrac{h}{2\Delta x}\right)^2}{2m} + \tfrac{1}{2}m\omega_0^2\left\langle x^2\right\rangle \\ &= \frac{h^2}{8m(\Delta x)^2} + \tfrac{1}{2}m\omega_0^2\left\langle x^2\right\rangle \\ &= \frac{h^2}{8m\left\langle x^2\right\rangle} + \tfrac{1}{2}m\omega_0^2\left\langle x^2\right\rangle\end{aligned}$$

Letting $Z = \left\langle x^2\right\rangle$ yields:

$$\langle E\rangle = \frac{h^2}{8mZ} + \tfrac{1}{2}m\omega_0^2 Z$$

5. Differentiate $\langle E\rangle$ with respect to Z and set this derivative equal to zero:

$$\frac{d\langle E\rangle}{dZ}=\frac{d}{dZ}\left[\frac{h^2}{8mZ}+\tfrac{1}{2}m\omega_0^2 Z\right]$$
$$=-\frac{h^2}{8mZ^2}+\tfrac{1}{2}m\omega_0^2$$
$$=0 \text{ for extrema}$$

Solve for Z to find the value of Z that minimizes $\langle E\rangle$ (see the remark below):

$$Z=\frac{h}{2m\omega_0}$$

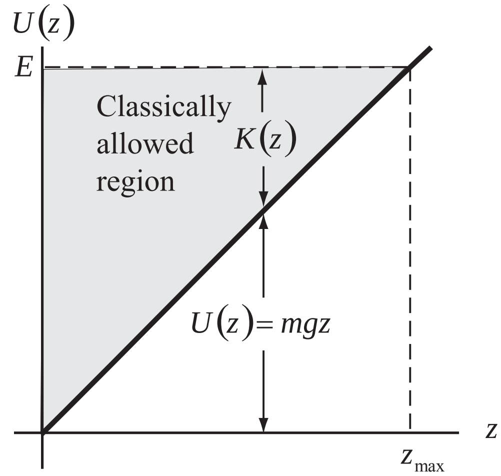
6. Evaluating $\langle E\rangle$ when $Z=\dfrac{h}{2m\omega_0}$ gives:

$$\langle E\rangle_{\text{min}}=\frac{h^2}{8m}\left(\frac{2m\omega_0}{h}\right)+\tfrac{1}{2}m\omega_0^2\left(\frac{h}{2m\omega_0}\right)$$
$$=\boxed{+\tfrac{1}{2}h\omega_0}$$

Remarks: All we've shown is that $Z = h/(2m\omega_0)$ is an extreme value, i.e., either a *maximum* or a *minimum*. To show that $Z = \hbar/(2m\omega_0)$ minimizes <E> we must either 1) show that the second derivative of <E> with respect to Z evaluated at $Z = h/(2m\omega_0)$ is positive, or 2) confirm that the graph of <E> as a function of Z opens upward at $Z = h/(2m\omega_0)$.

38 ••• A particle that has mass m and is near Earth's surface, at which $z = 0$, can be described by the potential energy function $U = mgz$ in the region $z > 0$, and by $U = \infty$ in the region $z < 0$. Sketch a graph of $U(z)$ versus z. For some positive value of total energy E, indicate the classically allowed region on the graph and plot the classical kinetic energy versus z on the graph. The Schrödinger equation for this problem is quite difficult to solve. Using arguments similar to those in Section 35-2 about the concavity of the wave function as given by the Schrödinger equation, sketch the shape of the wave function for the ground state and for the first two excited states.

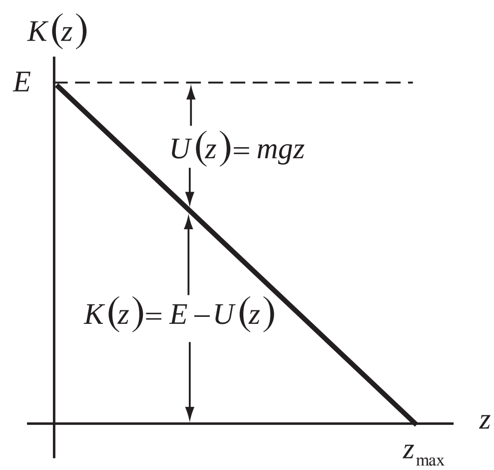
Picture the Problem
The classically allowed region is for $E \ge U(z)$. In the figure to the right, this region extends from $z = 0$ to $z = z_{max}$.



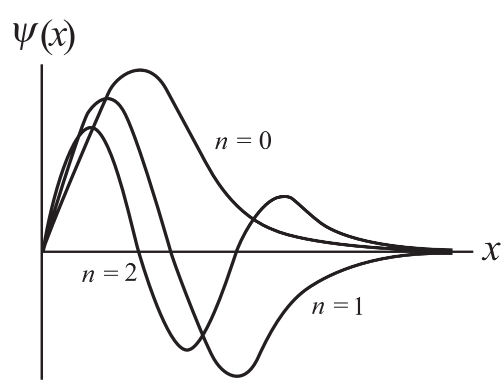
The total energy of the particle is the sum of its kinetic and potential energies:

$$E = K(z) + U(z) \Rightarrow K(z) = E - U(z)$$

A graph of the classical kinetic energy versus z is shown to the right:



A sketch of the wave functions for the lowest three energy states is shown to the right:



Chapter 36
Atoms

Conceptual Problems

1 • For the hydrogen atom, as n increases, does the spacing of adjacent energy levels on an energy-level diagram increase or decrease?

Determine the Concept Examination of Figure 35-4 indicates that as n increases, the spacing of adjacent energy levels decreases**.**

2 • The energy of the ground state of doubly ionized lithium (Z = 3) is _____, where E_0 = 13.6 eV. (a) $-9E_0$, (b) $-3E_0$, (c) $-E_0/3$, (d) $-E_0/9$.

Picture the Problem The energy of an atom of atomic number Z, with exactly one electron in its nth energy state is given by $E_n = -Z^2\dfrac{E_0}{n^2}, n = 1, 2, 3, \ldots$.

Express the energy of an atom of atomic number Z, with exactly one electron, in its nth energy state:

$$E_n = -Z^2\frac{E_0}{n^2}, n = 1, 2, 3, \ldots$$

where E_0 is the atom's ground state energy.

For lithium (Z = 3) in its first excited state (n = 1) this expression becomes:

$$E_2 = -(3)^2\frac{E_0}{1^2} = -9E_0$$

and $\boxed{(a)}$ is correct.

3 • Bohr's quantum condition on electron orbits requires (a) that the orbital angular momentum of the electron about the hydrogen nucleus equals $n\hbar$, where n is an integer, (b) that no more than one electron occupy a given state, (c) that the electrons spiral into the nucleus while radiating electromagnetic waves, (d) that the energies of an electron in a hydrogen atom be equal to nE_0, where E_0 is a constant and n is an integer, (e) none of the above.

Determine the Concept Bohr's postulates are 1) the electron in the hydrogen atom can move only in certain non-radiating, circular orbits called *stationary states*, 2) if E_i and E_f are the initial and final energies of the atom, the frequency f of the emitted radiation during a transition is given by $f = [E_i - E_f]/h$, and 3) the angular momentum of a circular orbit is constrained by $mvr = n\hbar$. $\boxed{(a)}$ is correct.

4 •• According to the Bohr model, if an electron moves to a larger orbit, does the electron's total energy increase or decrease? Does the electron's kinetic energy increase or decrease?

Picture the Problem We can express the kinetic energy of the orbiting electron as well as its total energy as functions of its radius r.

Express the total energy of an orbiting electron:

$$E = K + U$$

Express the orbital kinetic energy of an electron:

$$K = \frac{kZe^2}{2r} \qquad (1)$$

Express the potential energy of an orbiting electron:

$$U = -\frac{kZe^2}{r}$$

Thus, as r increases, E becomes less negative and therefore increases.

Examination of the expression for K makes it clear that if r increases, K decreases.

5 • According to the Bohr model, the kinetic energy of the electron in the ground state of hydrogen is E_0, where $E_0 = 13.6$ eV. The kinetic energy of the electron in the state $n = 2$ is (*a*) $4E_0$, (*b*) $2E_0$, (*c*) $E_0/2$, (*d*) $E_0/4$.

Picture the Problem We can relate the kinetic energy of the electron in the $n = 2$ state to its total energy using $E_2 = K_2 + U_2$.

Express the total energy of the hydrogen atom in its $n = 2$ state:

$$E_2 = K_2 + U_2 = K_2 - 2K_2 = -K_2$$

or

$$K_2 = -E_2$$

Express the energy of hydrogen in its nth energy state:

$$E_n = -Z^2\frac{E_0}{n^2} = -(1)^2\frac{E_0}{n^2} = -\frac{E_0}{n^2}$$

where E_0 is hydrogen's ground state energy and $Z = 1$.

Substitute to obtain:

$$K_n = \frac{E_0}{n^2} \text{ and } K_2 = \frac{E_0}{2^2} = \frac{E_0}{4}$$

$\boxed{(d)}$ is correct.

6 • According to the Bohr model, the radius of the $n = 1$ orbit in the hydrogen atom is $a_0 = 0.053$ nm. What is the radius of the $n = 5$ orbit? (*a*) $25a_0$, (*b*) $5a_0$, (*c*) a_0, (*d*) $a_0/5$, (*e*) $a_0/25$.

Picture the Problem The orbital radius r depends on the $n = 1$ orbital radius a_0, the atomic number Z, and the orbital quantum number n according to $r = n^2a_0/Z$.

The radius of the $n = 5$ orbit is:

$$r_5 = 5^2\left(\frac{a_0}{1}\right) = 25a_0$$

because $Z = 1$ for hydrogen.

$\boxed{(a)}$ is correct.

7 • **[SSM]** For the principal quantum number $n = 4$, how many different values can the orbital quantum number ℓ have? (*a*) 4, (*b*) 3, (*c*) 7, (*d*) 16, (*e*) 25

Determine the Concept We can find the possible values of ℓ by using the constraints on the quantum numbers *n* and ℓ.

The allowed values for the orbital quantum number ℓ for $n = 1, 2, 3$, and 4 are summarized in table shown to the right:

n	ℓ
1	0
2	0, 1
3	0, 1, 2
4	0, 1, 2, 3

From the table it is clear that ℓ can have 4 values. $\boxed{(a)}$ is correct.

8 • For the principal quantum number $n = 4$, how many different combinations of ℓ and m_ℓ can occur? (*a*) 4, (*b*) 3, (*c*) 7, (*d*) 16, (*e*) 25

Picture the Problem We can find the number of different values m_ℓ can have by enumerating the possibilities when the principal quantum number $n = 4$.

The allowed values for the orbital quantum number ℓ and the magnetic quantum number m_ℓ for $n = 4$ are summarized to the right:

$\ell = 0, 1, 2, 3$

and

$m_\ell = -3, -2, -1, 0, 1, 2, 3$

When $\ell = 0$, $m_\ell = 0 \Rightarrow$ 1 combination: (0,0)

When $\ell = 1$, $m_\ell = -1,0,1 \Rightarrow$ 3 combinations: (1, −1), (1,0) and (1,1)

When $\ell = 2$ $m_\ell = -2, -1,0,1,2 \Rightarrow$ 5 combinations: (2, −2), (2,−1), (2,0), (2,1) and (2,2)

When $\ell = 3$ $m_\ell = -3,-2, -1,0,1,2,3 \Rightarrow$ 7 combinations: (3, −3), (3,−2), (3, −1), (3,0), (3,1), (3,2) and (3,3)

From this enumeration we can see that 16 combinations of ℓ and m_ℓ can occur.

$\boxed{(d)}$ is correct.

Remarks: For n = 4 the orbital quantum number ℓ can be 0, 1, 2, or 3, and for each value ℓ there are 2*ℓ + 1 values of the magnetic quantum number. Thus the total number of combinations are (2*3 + 1) + (2*2 + 1) + (2*1 + 1) + (2*0 + 1) = 7 + 5 + 3 + 1 = 16.

9 •• **[SSM]** Why is the energy of the 3s state considerably lower than the energy of the 3p state for sodium, whereas in hydrogen 3s and 3p states have essentially the same energy?

Determine the Concept The s state, with $\ell = 0$, is a ″penetrating″ state in which the probability density near the nucleus is significant. Consequently, the 3s electron in sodium is in a region of low potential energy for a significant portion of the time. In the state $\ell = 1$, the probability density at the nucleus is zero, so the 3p electron of sodium is shielded from the nuclear charge by the 1s electrons. In hydrogen, the 3s and 3p electrons experience the same nuclear potential.

10 • The d state of an electron configuration corresponds to (*a*) $n = 2$, (*b*) $\ell = 3$, (*c*) $\ell = 2$, (*d*) $n = 3$, (*e*) $\ell = 0$.

Picture the Problem We can visualize the relationship between the quantum number ℓ and the electronic configuration as shown in the table below.

	s	p	d	f	g	h
ℓ value	0	1	2	3	4	5

Because the d state corresponds to $\ell = 2$, $\boxed{(c)}$ is correct.

11 •• Why are three quantum numbers inadequate to describe the states of the electrons in atoms that have more than one electron?

Determine the Concept In conformity with the exclusion principle, the total number of electrons that can be accommodated in states of quantum number n is n^2 (see Problem 48). The fact that closed shells correspond to $2n^2$ electrons indicates that there is another quantum number that can have two possible values.

12 •• Group the following six atoms—potassium, calcium, titanium, chromium, manganese, and copper—according to their ground state electron configurations for the $n = 4$ shell.

Picture the Problem We can group these atoms by using Table 36-1 to look for a common outer electronic configuration in the ground states.

The following atoms have an outer $4s^2$ configuration in the ground state: $\boxed{\text{titanium, manganese, and calcium}}$

The following atoms have an outer 4s configuration in the ground state: $\boxed{\text{potassium, chromium, and copper.}}$

Remarks: It is to be expected that atoms of the first group will have similar properties, and, likewise, that atoms of the second group will have similar properties.

13 • What element has the electron configuration (*a*) $1s^22s^22p^63s^23p^3$ and (*b*) $1s^22s^22p^63s^23p^63d^54s^1$?

Picture the Problem We can use the fact that the sum of the exponents in the electronic configuration representation is the atomic number to identify these two elements.

(*a*) Adding the exponents yields a sum of 15. Because this sum is the atomic number, *Z*, the element must be $\boxed{\text{phosphorus.}}$

(*b*) Adding the exponents yields a sum of 24. Because this sum is the atomic number, *Z*, the element must be $\boxed{\text{chromium.}}$

Remarks: Checking the electronic configurations in Table 36-1 further confirms these conclusions.

14 • For the principal quantum number $n = 3$, what are the possible combinations of the quantum numbers ℓ and m_ℓ?

Picture the Problem We can apply the constraints on the quantum numbers ℓ and m_ℓ to find the possible values for each when $n = 3$.

Express the constraints on the quantum numbers n, ℓ, and m_ℓ:

$$n = 1, 2, 3, \ldots,$$
$$\ell = 0, 1, 2, \ldots, n-1,$$
and
$$m_\ell = -\ell, -\ell+1, \ldots, \ell$$

For $n = 3$, the constraints on ℓ limit it to the values:

$$\ell = 0, 1, \text{and } 2$$

m_ℓ can take on the values:

$$m_\ell = -2, -1, 0, 1, 2$$

When $\ell = 0$, $m_\ell = 0 \Rightarrow$ 1 combination: (0,0)

When $\ell = 1$, $m_\ell = -1,0,1 \Rightarrow$ 3 combinations: (1, −1), (1,0) and (1,1)

When $\ell = 2$ $m_\ell = -2, -1,0,1,2 \Rightarrow$ 5 combinations: (2, −2), (2,−1), (2,0), (2,1) and (2,2)

The possible combinations of the quantum numbers ℓ and m_ℓ are (0,0), (1, −1), (1,0), (1,1), (2, −2), (2,−1), (2,0), (2,1) and (2,2).

15 • An electron in the *L* shell means that the electron is represented by (*a*) $\ell = 0$, (*b*) $\ell = 1$, (*c*) $n = 1$, (*d*) $n = 2$, or (*e*) $m_\ell = 2$.

Determine the Concept The correspondence between the letter designations K, L, M, N, O, and P for the shells and the principal quantum number *n* is summarized in the table below.

Shell designation	K	L	M	N	O	P	Q
n	1	2	3	4	5	6	7
ℓ	0	0	0	0	0	0	0
		1	1	1	1	1	1
			2	2	2	2	2
				3	3	3	3
					4	4	4
						5	5
							6

While $n = 2$ for the L shell, ℓ can be either 0 or 1. $\boxed{(d)}$ is correct.

16 •• The Bohr model and the quantum-mechanical model of the hydrogen atom give the same results for the energy levels. Discuss the advantages and disadvantages of each model.

Picture the Problem The strengths and weaknesses of each model are summarized in the following table.

	Bohr Model	Quantum-Mechanical Model
Ease of application	Easy	Difficult
Prediction of stationary state energies	Correct predictions	Correct predictions
Prediction of angular momenta	Predicts incorrect results	Predicts correct results
Spatial distribution of electrons	Predicts incorrect results	Predicts correct probabilistic distribution

17 •• The Sommerfeld–Hosser displacement theorem states that the optical spectrum of any atom is very similar to the spectrum of the singly charged positive ion of the element immediately following it in the periodic table. Discuss why this theorem is accurate.

Determine the Concept The optical spectrum of any atom is due to the configuration of its outer-shell electrons. Ionizing the next atom in the periodic table gives you an ion with the same number of outer-shell electrons, and almost the same nuclear charge. Hence, the spectra should be very similar.

18 • Using the triplet of numbers (n, ℓ, m_ℓ) to represent an electron that has the principal quantum number n, orbital quantum number ℓ, and magnetic quantum number m_ℓ, which of the following transitions is allowed?
(*a*) $(5, 2, 2) \rightarrow (3, 1, -2)$, (*b*) $(2, 1, 0) \rightarrow (3, 0, 0)$, (*c*) $(4, 3, -2) \rightarrow (3, 2, -1)$,
(*d*) $(1, 0, 0) \rightarrow (2, 1, -1)$, (*e*) $(2, 1, 0) \rightarrow (3, 1, 0)$

Determine the Concept An allowed transition must satisfy the selection rules $\Delta m_\ell = 0$ or ± 1 and $\Delta\ell = \pm 1$.

(*a*) $\Delta\ell = -1$ and $\Delta m_\ell = -4 \Rightarrow$ $\boxed{\text{not allowed}}$

(*b*) $\Delta\ell = -1$ and $\Delta m_\ell = 0 \Rightarrow$ $\boxed{\text{allowed}}$

(*c*) $\Delta\ell = -1$ and $\Delta m_\ell = +1 \Rightarrow$ $\boxed{\text{allowed}}$

(*d*) $\Delta\ell = +1$ and $\Delta m_\ell = -1 \Rightarrow$ $\boxed{\text{allowed}}$

(*e*) $\Delta\ell = 0$ and $\Delta m_\ell = 0 \Rightarrow$: $\boxed{\text{not allowed}}$

19 •• [SSM] The Ritz combination principle states that for any atom, one can find different spectral lines λ_1, λ_2, λ_3, and λ_4, so that $1/\lambda_1 + 1/\lambda_2 = 1/\lambda_3 + 1/\lambda_4$. Show why this is true using an energy-level diagram.

Determine the Concept The Ritz combination principle is due to the quantization of energy levels in the atom. We can use the relationship between the wavelength of the emitted photon and the difference in energy levels within the atom that results in the emission of the photon to express each of the wavelengths and then the sum of the reciprocals of the first and second wavelengths and the sum of the reciprocals of the third and fourth wavelengths.

Express the wavelengths of the spectral lines λ_1, λ_2, λ_3, and λ_4 in terms of the corresponding energy transitions:

$$\lambda_1 = \frac{hc}{E_3 - E_2},\ \lambda_2 = \frac{hc}{E_2 - E_0}$$

$$\lambda_3 = \frac{hc}{E_3 - E_1} \text{ and } \lambda_4 = \frac{hc}{E_1 - E_0}$$

Add the reciprocals of λ_1 and λ_2 to obtain:

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{E_3 - E_2}{hc} + \frac{E_2 - E_0}{hc} = \frac{E_3 - E_0}{hc} \quad (1)$$

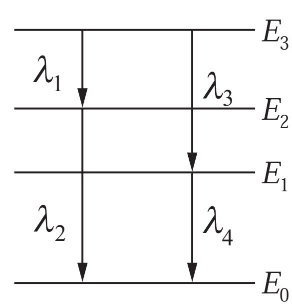
Add the reciprocals of λ_3 and λ_4 to obtain:

$$\frac{1}{\lambda_3} + \frac{1}{\lambda_4} = \frac{E_3 - E_1}{hc} + \frac{E_1 - E_0}{hc} = \frac{E_3 - E_0}{hc} \quad (2)$$

Because the right-hand sides of equations (1) and (2) are equal:

$$\boxed{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{1}{\lambda_3} + \frac{1}{\lambda_4}}$$

One possible set of energy levels is shown to the right:



Estimation and Approximation

20 •• (*a*) We can define a thermal de Broglie wavelength λ_T for an atom in a gas at temperature T as being the de Broglie wavelength for an atom moving at the rms speed appropriate to that temperature. (The average kinetic energy of an

atom is equal to kT, where k is the Boltzman constant. Use this value to calculate the rms speed of the atoms.) Show that $\lambda_T = \sqrt{h^2/3mkT}$, where m is the mass of the atom. (*b*) Cooled atoms can form a Bose *condensate* (a new state of matter) when their thermal de Broglie wavelength becomes larger than the average interatomic spacing. From this criterion, estimate the temperature needed to create a Bose condensate in a gas of ^{85}Rb atoms whose number density is 10^{12} atoms/cm^3.

Picture the Problem We can use the relationship between the kinetic energy of an atom and its momentum, together with the de Broglie equation, to derive the expression for the thermal de Broglie wavelength. In Part (*b*), we can use the definition of the number density of atoms and the result from Part (*a*), with the interatomic spacing set equal to the thermal de Broglie wavelength, to estimate the temperature needed to create a Bose condensate.

(*a*) Express the kinetic energy of an atom in terms of its momentum:

$$K = \frac{p^2}{2m}$$

Use the de Broglie relationship to express the atom's momentum in terms of its de Broglie wavelength:

$$p = \frac{h}{\lambda_T}$$

where λ_T is the thermal de Broglie wavelength.

Substituting for p gives:

$$K = \frac{h^2}{2m\lambda_T^2}$$

The kinetic energy of an atom is also a function of its temperature T:

$$K = \frac{3}{2}kT$$

Equate these expressions for K to obtain:

$$\frac{3}{2}kT = \frac{h^2}{2m\lambda_T^2} \Rightarrow \lambda_T = \boxed{\sqrt{\frac{h^2}{3mkT}}}$$

(*b*) The number density of atoms ρ is given by:

$$\rho = \frac{N}{V}$$

where N is the number of atoms and V is the volume they occupy.

Assume that each atom is arrayed on a cubic lattice of spacing d to obtain:

$V = Nd^3$ and $\rho = \dfrac{N}{Nd^3} = \dfrac{1}{d^3}$

Solving for d yields:

$$d = \rho^{-1/3}$$

Setting $d = \lambda_T$ gives:

$$\rho^{-1/3} = \sqrt{\frac{h^2}{3mkT}} \Rightarrow T = \frac{h^2 \rho^{2/3}}{3mk}$$

Substitute numerical values and evaluate T:

$$T = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2 \left(10^{12}\ \frac{\text{atoms}}{\text{cm}^3} \times \frac{10^6\ \text{cm}^3}{\text{m}^3}\right)^{2/3}}{3(85\,\text{u})\left(1.661\times10^{-27}\ \frac{\text{kg}}{\text{u}}\right)\left(1.381\times10^{-23}\ \frac{\text{J}}{\text{K}}\right)} = \boxed{75\,\text{nK}}$$

21 •• In laser cooling and trapping, a beam of atoms traveling in one direction are slowed by interaction with an intense laser beam in the opposite direction. The photons scatter off the atoms by resonance absorption, a process by which the incident photon is absorbed by the atom, and a short time later a photon of equal energy is emitted in a random direction. The net result of a single such scattering event is a transfer of momentum to the atom in a direction opposite to the motion of the atom, followed by a second transfer of momentum to the atom in a random direction. Thus, during photon absorption the atom loses speed, but during photon emission the change in speed of the atom is, on average, zero (because the directions of the emitted photons are random). An analogy often made to this process is that of slowing down a bowling ball by bouncing ping-pong balls off of it. (*a*) Given that the typical photon energy used in these experiments is about 1 eV, and that the typical kinetic energy of an atom in the beam is the typical kinetic energy of the atoms in a gas that has a temperature of about 500 K (a typical temperature for an oven that produces an atomic beam), estimate the number of photon-atom collisions that are required to bring an atom to rest. (The average kinetic energy of an atom is equal to $\frac{3}{2}kT$, where k is the Boltzmann constant and T is the temperature. Use this to estimate the speed of the atoms.) (*b*) Compare the Part (*a*) result with the number of ping-pong ball–bowling ball collisions that are required to bring the bowling ball to rest. Assume the typical speed of the incident ping-pong balls are all equal to the initial speed of the bowling ball.) (*c*) ^{85}Rb is a type of atom often used during cooling experiments. The wavelength of the light resonant with the cooling transition of these atoms is $\lambda = 780.24$ nm. Estimate the number of photons needed to slow down an ^{85}Rb atom from a typical thermal velocity of 300 m/s to a stop.

Picture the Problem The number of photons needed to stop a ^{85}Rb atom traveling at 300 m/s is the ratio of its momentum to that of a typical photon. In Part (*b*), assume the mass of a bowling ball to be 6.0 kg and the mass of a ping pong ball to be 4.0 g.

(*a*) The number N of photon-atom collisions needed to bring an atom to rest is the ratio of the change in the momentum of the atom as it stops to the momentum brought to the collision by each photon:

$$N = \frac{\Delta p_{\text{atom}}}{p_{\text{photon}}} = \frac{mv}{\frac{E}{c}} = \frac{mvc}{E}$$

where m is the mass of the atom.

The kinetic energy of an atom whose temperature is T is:

$$\tfrac{1}{2}mv^2 = \tfrac{3}{2}kT \Rightarrow v = \sqrt{\frac{3kT}{m}}$$

Substituting for v gives:

$$N = \frac{mc}{E}\sqrt{\frac{3kT}{m}} = \frac{c}{E}\sqrt{3mkT}$$

For an atom whose mass is 50 u:

$$N = \frac{2.998\times10^{8}\ \text{m/s}}{1\,\text{eV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}}\sqrt{3\left(50\,\text{u}\times\frac{1.661\times10^{-27}\ \text{kg}}{\text{u}}\right)\left(1.381\times10^{-23}\ \text{J/K}\right)\left(500\,\text{K}\right)}$$
$$\approx \boxed{10^{5}}$$

(*b*) The number N of ping-pong ball-bowling ball collisions needed to bring the bowling ball to rest is the ratio of the change in the momentum of the bowling ball as it stops to the momentum brought to the collision by each ping-pong ball:

$$N = \frac{\Delta p_{\text{bowling ball}}}{p_{\text{ping-pong ball}}} = \frac{m_{\text{bb}}v_{\text{bb}}}{m_{\text{ppb}}v_{\text{ppb}}}$$

Provided the speeds of the approaching bowling ball and ping-pong ball are approximately the same:

$$N = \frac{\Delta p_{\text{bowling ball}}}{p_{\text{ping-pong ball}}} \approx \frac{m_{\text{bb}}}{m_{\text{ppb}}} \approx \frac{6.0\,\text{kg}}{4.0\,\text{g}} \approx \boxed{10^{3}}$$

(*c*) The number of photons N needed to stop a ^{85}Rb atom is the ratio of the change in the momentum of the atom to the momentum brought to the collision by each photon:

$$N = \frac{\Delta p_{\text{atom}}}{p_{\text{photon}}} = \frac{mv}{\frac{h}{\lambda}} = \frac{mv\lambda}{h}$$

Substitute numerical values and evaluate N:

$$N = \frac{85\left(1.661\times10^{-27}\ \text{kg}\right)\left(300\,\text{m/s}\right)\left(780.24\,\text{nm}\right)}{6.626\times10^{-34}\ \text{J}\cdot\text{s}} = \boxed{5.08\times10^{4}}$$

The Bohr Model of the Hydrogen Atom

22 • The first Bohr radius is given by $a_0 = \hbar^2/\left(mke^2\right) = 0.0529\text{ nm}$ (Equation 36-12). Use the known values of the constants in the equation to show that a_0 is equal to 0.0529 nm.

Picture the Problem The radius of the first Bohr orbit is given by $a_0 = \dfrac{\hbar^2}{mke^2}$.

Equation 36-12 is:

$$a_0 = \frac{\hbar^2}{mke^2}$$

Substitute numerical values and evaluate a_0:

$$a_0 = \frac{\left(1.055\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{\left(9.109\times10^{-31}\text{kg}\right)\left(8.988\times10^{9}\,\dfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\left(1.601\times10^{-19}\text{ C}\right)^2} = 5.30\times10^{-11}\text{ m}$$

$$= \boxed{0.0529\,\text{nm}}$$

23 • The longest wavelength in the Lyman series for the hydrogen atom was calculated in Example 36-2. Find the wavelengths for the transitions (*a*) $n_i = 3$ to $n_f = 1$ and (*b*) $n_i = 4$ to $n_f = 1$.

Picture the Problem We can use the equation relating the wavelength of the radiation emitted during a transition between two energy states to find the wavelengths for the transitions specified in the problem statement.

Express the wavelength of the radiation emitted during an energy transformation from one energy state to another:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{E_i - E_f}$$

provided the energies are expressed in eV. Note that this relationship tells us that the longest wavelength corresponds to the smallest energy difference.

Because $E_f = -13.6$ eV:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{E_i + 13.6\,\text{eV}}$$

Express the energy of the *n*th energy state of the atom:

$$E_n = -\frac{E_0}{n^2} = -\frac{13.6\,\text{eV}}{n^2}$$

Substitute for E_n and simplify to obtain:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{-\frac{13.6\,\text{eV}}{n_i^2}+13.6\,\text{eV}} = \frac{1240\,\text{eV}\cdot\text{nm}}{13.6\,\text{eV}\left(1-\frac{1}{n_i^2}\right)} \quad (1)$$

(a) Evaluate equation (1) for $n_i = 3$:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{13.6\,\text{eV}\left(1-\frac{1}{3^2}\right)} = \boxed{103\,\text{nm}}$$

(b) Evaluate equation (1) for $n_i = 4$:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{13.6\,\text{eV}\left(1-\frac{1}{4^2}\right)} = \boxed{97.3\,\text{nm}}$$

24 • Find the photon energies for the three longest wavelengths in the Balmer series for the hydrogen atom, and calculate the three wavelengths.

Picture the Problem For the Balmer series, $E_f = E_2 = -3.40$ eV. The wavelength associated with each transition is related to the difference in energy between the states by $\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{E_i - E_f}$.

Express the wavelength of the radiation emitted during an energy transformation from one energy state to another:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \quad (1)$$

provided the energies are expressed in eV. Note that this relationship tells us that the longest wavelength corresponds to the smallest energy difference.

Evaluate ΔE for the transition $n = 3$ to $n = 2$:

$$\Delta E_{n\to 2} = E_n - E_2 = \frac{E_0}{n^2} - E_2$$

Because $E_f = E_2 = -3.40$ eV and $E_0 = -13.6$ eV:

$$\Delta E_{n\to 2} = -\frac{13.6\,\text{eV}}{n^2} + 3.40\,\text{eV} \quad (2)$$

Evaluate equation (2) for $n = 3$:

$$\Delta E_{3\to 2} = -\frac{13.6\,\text{eV}}{3^2} + 3.40\,\text{eV} = \boxed{1.89\,\text{eV}}$$

Substitute in equation (1) to obtain:

$$\lambda_{3\to 2}=\frac{1240\,\text{eV}\cdot\text{nm}}{1.89\,\text{eV}}=\boxed{656\,\text{nm}}$$

Evaluate equation (2) for $n = 4$:

$$\Delta E_{4\to 2}=-\frac{13.6\,\text{eV}}{4^2}+3.40\,\text{eV}=\boxed{2.55\,\text{eV}}$$

Substitute in equation (1) to obtain:

$$\lambda_{4\to 2}=\frac{1240\,\text{eV}\cdot\text{nm}}{2.55\,\text{eV}}=\boxed{486\,\text{nm}}$$

Evaluate equation (2) for $n = 5$:

$$\Delta E_{5\to 2}=-\frac{13.6\,\text{eV}}{5^2}+3.40\,\text{eV}=\boxed{2.86\,\text{eV}}$$

Substitute in equation (1) to obtain:

$$\lambda_{5\to 2}=\frac{1240\,\text{eV}\cdot\text{nm}}{2.86\,\text{eV}}=\boxed{434\,\text{nm}}$$

25 •• (*a*) Find the photon energy and wavelength for the series limit (shortest wavelength) in the Paschen series ($n_f = 3$) for the hydrogen atom. (*b*) Calculate the wavelength for the three longest wavelengths in Paschen series.

Picture the Problem We can use Bohr's second postulate to relate the photon energy to its frequency and use $\lambda=\dfrac{1240\,\text{eV}\cdot\text{nm}}{E_\text{i}-E_\text{f}}$ to find the wavelengths of the three longest wavelengths in the Paschen series.

(*a*) Use Bohr's second postulate to express the energy of the photons in the Paschen series:

$$hf=\Delta E=E_\text{i}-E_\text{f} \qquad (1)$$

For the series limit:

$$n_\text{i}=\infty \text{ and } E_\text{i}=0$$

Substitute for n_i and E_i to obtain:

$$\Delta E=-E_\text{f}=-\left(-\frac{E_0}{n_\text{f}^2}\right)=\frac{E_0}{n_\text{f}^2}$$

Evaluate the photon energy for $n_\text{f} = 3$:

$$hf=\frac{13.6\,\text{eV}}{3^2}=\boxed{1.51\,\text{eV}}$$

Express the wavelength of the radiation resulting from an energy transition $\Delta E = hf$:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \qquad (2)$$

provided the energies are expressed in eV.

Evaluate λ_{min} for the transition $n_\text{i} = \infty$ to $n_\text{f} = 3$:

$$\lambda_{\text{min}} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.51\,\text{eV}} = \boxed{821\,\text{nm}}$$

(*b*) For the three longest wavelengths:

$$n_\text{i} = 4, 5, \text{and } 6$$

Equation (1) becomes:

$$\begin{aligned} hf = E_\text{i} - E_\text{f} &= -\frac{E_0}{n_\text{i}^2} - \left(-\frac{E_0}{n_\text{f}^2}\right) \\ &= E_0\left(\frac{1}{n_\text{f}^2} - \frac{1}{n_\text{i}^2}\right) = E_0\left(\frac{1}{9} - \frac{1}{n_\text{i}^2}\right) \end{aligned} \qquad (3)$$

Evaluate equation (3) for $n_\text{i} = 4$:

$$\Delta E_{4\to 3} = (13.6\,\text{eV})\left(\frac{1}{9} - \frac{1}{16}\right) = \boxed{0.661\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 0.661$ eV:

$$\lambda_{4\to 3} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.661\,\text{eV}} = \boxed{1880\,\text{nm}}$$

Evaluate equation (3) for $n_\text{i} = 5$:

$$\Delta E_{5\to 3} = (13.6\,\text{eV})\left(\frac{1}{9} - \frac{1}{25}\right) = \boxed{0.967\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 0.967$ eV:

$$\lambda_{5\to 3} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.967\,\text{eV}} = \boxed{1280\,\text{nm}}$$

Evaluate equation (3) for $n_\text{i} = 6$:

$$\Delta E_{6\to 3} = (13.6\,\text{eV})\left(\frac{1}{9} - \frac{1}{36}\right) = \boxed{1.13\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 1.13$ eV:

$$\lambda_{6\to 3} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.13\,\text{eV}} = \boxed{1100\,\text{nm}}$$

The positions of these lines on a horizontal linear scale are shown below with the wavelengths and transitions indicated.

```
  6→3       5→3                              4→3
---|---------|-------------------------------|-----
1100 nm   1280 nm                         1880 nm
```

26 •• (*a*) Find the photon energy and wavelength for the series limit (shortest wavelength) in the Brackett series ($n_f = 4$) for the hydrogen atom. (*b*) Calculate the wavelength for the three longest wavelengths in Brackett.

Picture the Problem We can use Bohr's second postulate to relate the photon energy to its frequency and use $\lambda = \dfrac{1240\,\text{eV}\cdot\text{nm}}{E_\text{i} - E_\text{f}}$ to find the wavelengths of the three longest wavelengths in the Brackett series.

(*a*) Use Bohr's second postulate to express the energy of the photons in the Paschen series:

$$hf = \Delta E = E_\text{i} - E_\text{f} \quad (1)$$

For the series limit:

$$n_\text{i} = \infty \text{ and } E_\text{i} = 0$$

Substitute to obtain:

$$\Delta E = -E_\text{f} = -\left(-\frac{E_0}{n_\text{f}^2}\right) = \frac{E_0}{n_\text{f}^2}$$

Evaluate the photon energy for $n_f = 4$:

$$hf = \frac{13.6\,\text{eV}}{4^2} = \boxed{0.850\,\text{eV}}$$

Express the wavelength of the radiation resulting from an energy transition $\Delta E = hf$:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \quad (2)$$

provided the energies are expressed in eV.

Evaluate λ_min for the transition $n_i = \infty$ to $n_f = 4$:

$$\lambda_\text{min} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.850\,\text{eV}} = \boxed{1460\,\text{nm}}$$

(*b*) For the three longest wavelengths:

$$n_\text{i} = 5, 6, \text{and } 7$$

Equation (1) becomes:

$$\begin{aligned}\Delta E = E_\text{i} - E_\text{f} &= -\frac{E_0}{n_\text{i}^2} - \left(-\frac{E_0}{n_\text{f}^2}\right) \\ &= E_0\left(\frac{1}{n_\text{f}^2} - \frac{1}{n_\text{i}^2}\right) = E_0\left(\frac{1}{16} - \frac{1}{n_\text{i}^2}\right)\end{aligned} \quad (3)$$

Evaluate equation (3) for $n_\text{i} = 5$:

$$\Delta E_{5\to 4} = (13.6\,\text{eV})\left(\frac{1}{16} - \frac{1}{25}\right) = \boxed{0.306\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 0.306$ eV:

$$\lambda_{5\to 4} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.306\,\text{eV}} = \boxed{4050\,\text{nm}}$$

Evaluate equation (3) for $n_\text{i} = 6$:

$$\Delta E_{6\to 4} = (13.6\,\text{eV})\left(\frac{1}{16} - \frac{1}{36}\right) = \boxed{0.472\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 0.472$ eV:

$$\lambda_{6\to 4} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.472\,\text{eV}} = \boxed{2630\,\text{nm}}$$

Evaluate equation (3) for $n_\text{i} = 7$:

$$\Delta E_{7\to 4} = (13.6\,\text{eV})\left(\frac{1}{16} - \frac{1}{49}\right) = \boxed{0.572\,\text{eV}}$$

Evaluate equation (2) for $\Delta E = 0.572$ eV:

$$\lambda_{7\to 4} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.572\,\text{eV}} = \boxed{2170\,\text{nm}}$$

The positions of these lines on a horizontal linear scale are shown below with the wavelengths and transitions indicated.

```
  7→4   6→4                              5→4
---|---------|-------------------------------------------|-----
2170 nm 2630 nm                                   4050 nm
```

27 ••• [SSM] In the center-of-mass reference frame of a hydrogen atom, the electron and nucleus have momenta that have equal magnitudes p and opposite directions. (a) Using the Bohr model, show that the total kinetic energy of the electron and nucleus can be written $K = p^2/(2\mu)$ where $\mu = m_\text{e}M/(M + m_\text{e})$ is called the reduced mass, m_e is the mass of the electron, and M is the mass of the

nucleus. (*b*) For the equations for the Bohr model of the atom, the motion of the nucleus can be taken into account by replacing the mass of the electron with the reduced mass. Use Equation 36-14 to calculate the Rydberg constant for a hydrogen atom that has a nucleus of mass $M = m_p$. Find the approximate value of the Rydberg constant by letting M go to infinity in the reduced mass formula. To how many figures does this approximate value agree with the actual value? (*c*) Find the percentage correction for the ground-state energy of the hydrogen atom by using the reduced mass in Equation 36-16. *Remark: In general, the reduced mass for a two-body problem with masses m_1 and m_2 is given by*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Picture the Problem We can express the total kinetic energy of the electron-nucleus system as the sum of the kinetic energies of the electron and the nucleus. Rewriting these kinetic energies in terms of the momenta of the electron and nucleus will lead to $K = p^2/2m_r$.

(*a*) Express the total kinetic energy of the electron-nucleus system:

$$K = K_e + K_n$$

Express the kinetic energies of the electron and the nucleus in terms of their momenta:

$$K_e = \frac{p^2}{2m_e} \text{ and } K_n = \frac{p^2}{2M}$$

Substitute and simplify to obtain:

$$K = \frac{p^2}{2m_e} + \frac{p^2}{2M} = \frac{p^2}{2}\left(\frac{1}{m_e} + \frac{1}{M}\right)$$

$$= \frac{p^2}{2}\left(\frac{M + m_e}{m_e M}\right) = \frac{p^2}{2\left(\dfrac{m_e M}{M + m_e}\right)}$$

$$= \boxed{\frac{p^2}{2\mu}}$$

provided we let $\mu = m_e M/(M + m_e)$.

(*b*) From Equation 36-14 we have:

$$R = \frac{m_r k^2 e^4}{4\pi c \hbar^3} = C\left(\frac{m_e}{1 + \dfrac{m_e}{M}}\right) \qquad (1)$$

where

$$C = \frac{k^2 e^4}{4\pi c \hbar^3}$$

Use the Table of Physical Constants at the end of the text to obtain:

$$C = 1.204663 \times 10^{37}\ \mathrm{m^{-1}/kg}$$

For H:

$$R_{\mathrm{H}} = C\left(\frac{m_{\mathrm{e}}}{1+\dfrac{m_{\mathrm{e}}}{m_{\mathrm{p}}}}\right)$$

Substitute numerical values and evaluate R_{H}:

$$\begin{aligned} R_{\mathrm{H}} &= \left(1.204663 \times 10^{37}\ \frac{\mathrm{m^{-1}}}{\mathrm{kg}}\right)\left(\frac{9.10938188 \times 10^{-31}\ \mathrm{kg}}{1+\dfrac{9.10938188 \times 10^{-31}\ \mathrm{kg}}{1.67262158 \times 10^{-27}\ \mathrm{kg}}}\right) \\ &= \boxed{1.096850 \times 10^{7}\ \mathrm{m^{-1}}} \end{aligned}$$

Let $M \to \infty$ in equation (1) to obtain $R_{\mathrm{H,approx}}$:

$$R_{\mathrm{H,approx}} = Cm_{\mathrm{e}}$$

Substitute numerical values and evaluate $R_{\mathrm{H,approx}}$:

$$\begin{aligned} R_{\mathrm{H,approx}} &= \left(1.204663 \times 10^{37}\ \mathrm{m^{-1}/kg}\right)\left(9.10938188 \times 10^{-31}\ \mathrm{kg}\right) \\ &= \boxed{1.097448 \times 10^{7}\ \mathrm{m^{-1}}} \end{aligned}$$

R_{H} and $R_{\mathrm{H,approx}}$ agree to three significant figures.

(*c*) Express the ratio of the kinetic energy K of the electron in its orbit about a stationary nucleus to the kinetic energy of the reduced-mass system K':

$$\begin{aligned} \frac{K}{K'} &= \frac{\dfrac{p^2}{2m_{\mathrm{e}}}}{\dfrac{p^2}{2\mu}} = \frac{\mu}{m_{\mathrm{e}}} = \frac{1}{m_{\mathrm{e}}}\left(\frac{m_{\mathrm{p}}m_{\mathrm{e}}}{m_{\mathrm{p}}+m_{\mathrm{e}}}\right) \\ &= \frac{m_{\mathrm{p}}}{m_{\mathrm{p}}+m_{\mathrm{e}}} = \frac{1}{1+\dfrac{m_{\mathrm{e}}}{m_{\mathrm{p}}}} \end{aligned}$$

Substitute numerical values and evaluate the ratio of the kinetic energies:

$$\frac{K}{K'}=\frac{1}{1+\dfrac{9.10938188\times10^{-31}\text{ kg}}{1.67262158\times10^{-27}\text{ kg}}}=0.999455$$

or

$$K=0.999455K'$$

and the correction factor is the ratio of the masses or $\boxed{0.0545\%}$

Remarks: **The correct energy is slightly less than that calculated neglecting the motion of the nucleus.**

28 •• The Pickering series of the spectrum of He^+ (singly-ionized helium) consists of spectral lines due to transitions to the $n = 4$ state of He^+. Every other line of the Pickering series is very close to a spectral line in the Balmer series for hydrogen transitions to $n = 2$. (*a*) Show that this statement is accurate. (*b*) Calculate the wavelength of the photon during a transition from the $n = 6$ level to the $n = 4$ level of He^+, and show that it corresponds to one of the Balmer series lines.

Picture the Problem We can use Equation 36-15 with $Z = 2$ to explain how it is that every other line of the Pickering series is very close to a line in the Balmer series. We can use the relationship between the energy difference between two quantum states and the wavelength of the photon emitted during a transition from the higher state to the lower state to find the wavelength of the photon corresponding to a transition from the $n = 6$ to the $n = 4$ level of He^+.

(*a*) From Equation 36-15, the energy levels of an atom are given by:

$$E_n=-Z^2\frac{E_0}{n^2}$$

where E_0 is the ground state energy (13.6 eV).

For He^+, $Z = 2$ and:

$$E_n=-4\frac{E_0}{n^2}$$

Because of this, an energy level with even principal quantum number n in He^+ will have the same energy as a level with quantum number $n/2$ in H. Therefore, a transition between levels with principal quantum numbers $2m$ and $2p$ in He^+ will have almost the same energy as a transition between level m and p in H. In particular, transitions from $2m$ to $2p = 4$ in He^+ will have the same energy as transitions from m to $n = 2$ in H (the Balmer series).

(*b*) Transitions between these energy levels result in the emission or absorption of a photon whose wavelength is given by:

$$\lambda = \frac{hc}{E_6 - E_4} \qquad (1)$$

Evaluate E_6 and E_4:

$$E_6 = -4\left(\frac{13.6\,\text{eV}}{6^2}\right) = -1.51\,\text{eV}$$

and

$$E_4 = -4\left(\frac{13.6\,\text{eV}}{4^2}\right) = -3.40\,\text{eV}$$

Substitute for E_6 and E_4 in equation (1) and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{-1.51\,\text{eV} - (-3.40\,\text{eV})} = \boxed{656\,\text{nm}}$$

which is the same as the $n = 3$ to $n = 2$ transition in H.

Quantum Numbers in Spherical Coordinates

29 • For an electron in an atom that has an orbital quantum number $\ell = 1$, find (*a*) the magnitude of the angular momentum *L* and (*b*) the possible values of the magnetic quantum number m_ℓ. (*c*) Draw a vector diagram to scale showing the possible orientations of $\vec{L}$ relative to the +*z* direction.

Picture the Problem We can use the expression relating *L* to ℓ to find the magnitude of the angular momentum and the constraints on the quantum numbers to determine the allowed values for *m*.

(*a*) Express the angular momentum as a function of ℓ:

$$L = \sqrt{\ell(\ell+1)}\hbar$$

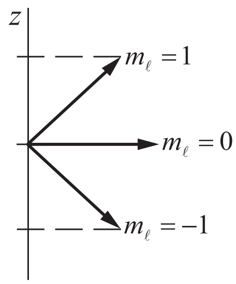
Substitute numerical values and evaluate *L*:

$$L = \sqrt{1(1+1)}\hbar = \sqrt{2}\hbar$$
$$= \sqrt{2}\left(1.055\times10^{-34}\,\text{J}\cdot\text{s}\right)$$
$$= \boxed{1.49\times10^{-34}\,\text{J}\cdot\text{s}}$$

(*b*) Because $m_\ell = -\ell, \ldots, 0, \ldots, \ell$ the allowed values for $\ell = 1$ are:

$$m_\ell = \boxed{-1, 0, +1}$$

(*c*) The vector diagram is shown on the right. Note that because $L_z = m_\ell \hbar$ and $L = \sqrt{2}\hbar$, the vectors for $m_\ell = -1$ and $m_\ell = 1$ must make angles of 45° with the z axis.



30 • For an electron in an atom that has an orbital quantum number $\ell = 3$, find (*a*) the magnitude of the angular momentum L and (*b*) the possible values of m_ℓ. (*c*) Draw a vector diagram to scale showing the possible orientations of $\vec{L}$ relative to the $+z$ direction.

Picture the Problem We can use the expression relating L to ℓ to find the magnitude of the angular momentum and the constraints on the quantum numbers to determine the allowed values for m.

(*a*) Express the angular momentum as a function of ℓ:

$$L = \sqrt{\ell(\ell+1)}\hbar$$

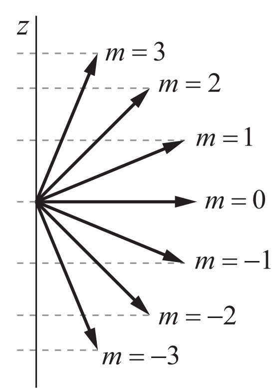
Substitute numerical values and evaluate L:

$$L = \sqrt{3(3+1)}\hbar = 2\sqrt{3}\hbar = 2\sqrt{3}\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right) = \boxed{3.65\times10^{-34}\ \text{J}\cdot\text{s}}$$

(*b*) Because $m_\ell = -\ell, \ldots, 0, \ldots, \ell$ the allowed values for $\ell = 3$ are:

$$m_\ell = \boxed{-3, -2, -1, 0, +1, +2, +3}$$

(*c*) The vector diagram is shown on the right. Note that because $L_z = m_\ell \hbar$ and $L = 2\sqrt{3}\hbar$, the angles between the vectors and the z axis are determined by $\cos\theta_m = m_\ell / 2\sqrt{3}$. Thus $\theta_3 = 30°$, $\theta_2 = 54.7°$, and $\theta_1 = 73.2°$. The spacing between the allowed values of L_z is constant and equal to $\hbar$.



31 • An electron in an atom has principle quantum number $n = 3$. (*a*) What are the possible values of ℓ? (*b*) What are the possible combinations of ℓ and m_ℓ. (*c*) Using the fact that there are two quantum states for each combination of ℓ and m_ℓ because of electron spin, find the total number of electron states for $n = 3$.

Picture the Problem We can find the possible values of ℓ by using the constraints on the quantum numbers. A more analytical solution is to first derive the number of electron states for an arbitrary value of n and then substitute the specific value of n.

(*a*) When $n = 3$: $\ell = \boxed{0,1,2}$

(*b*) For $\ell = 0$: $m_\ell = \boxed{0}$

For $\ell = 1$: $m_\ell = \boxed{-1,0,+1}$

For $\ell = 2$: $m_\ell = \boxed{-2,-1,0,+1,+2}$

(*c*) We can find the total number of electron states by enumerating the possibilities as shown in the table.

n	ℓ	m_ℓ
3	0	0
3	1	−1
3	1	0
3	1	1
3	2	−2
3	2	−1
3	2	0
3	2	1
3	2	2

Note that there are 9 states. Because *N,* the number of electron states is twice the number of m_ℓ states, the number of electron states is $\boxed{18}$.

Alternatively, we can derive an expression for the number of electron states for an arbitrary value of n and then substitute specific values of n.

The number of m_ℓ states for a given n is given by:

$$N_m = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1}(1)$$

Express the sum of all integers from 0 to p:

$$\sum_{\ell=0}^{p}\ell = \tfrac{1}{2}p(p+1)$$

Use this result to evaluate $2\sum_{\ell=0}^{n-1}\ell$:

$$2\sum_{\ell=0}^{n-1}\ell = 2\left[\tfrac{1}{2}(n-1)n\right] = n^2 - n$$

Evaluate the second term to obtain: $\sum_{\ell=0}^{n-1}(1) = n$

Substitute to obtain: $N_m = n^2 - n + n = n^2$

and, because N, the number of electron states is twice the number of m states, the number of electron states is $N = 2n^2$.

Hence, for $n = 3$, the number of electron states is: $N = 2n^2 = 2(3)^2 = \boxed{18}$

32 • In an atom, find the total number of electron states that have (*a*) $n = 4$ and (*b*) $n = 2$. (See Problem 33.)

Picture the Problem While we could find the number of electron states by finding the possible values of ℓ from the constraints on the quantum numbers and then enumerating the states, we'll take a more analytical approach by deriving an expression for the number of electron states for an arbitrary value of n and then substitute specific values of n.

The number of m states for a given n is given by: $N_m = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1}(1)$

Express the sum of all integers from 0 to p: $\sum_{\ell=0}^{p}\ell = \frac{1}{2}p(p+1)$

Use this result to evaluate $2\sum_{\ell=0}^{n-1}\ell$: $2\sum_{\ell=0}^{n-1}\ell = 2\left[\frac{1}{2}(n-1)n\right] = n^2 - n$

Evaluate the second term to obtain: $\sum_{\ell=0}^{n-1}(1) = n$

Substitute to obtain: $N_m = n^2 - n + n = n^2$

and, because N, the number of electron states is twice the number of m states, the number of electron states is $N = 2n^2$.

(*a*) For $n = 4$, the number of electron states is: $N = 2n^2 = 2(4)^2 = \boxed{32}$

(*b*) For $n = 2$, the number of electron states is:

$$N = 2n^2 = 2(2)^2 = \boxed{8}$$

33 •• **[SSM]** Find the minimum value of the angle θ between $\vec{L}$ and the $+z$ direction for an electron in an atom that has (*a*) $\ell = 1$, (*b*) $\ell = 4$, and (*c*) $\ell = 50$.

Picture the Problem The minimum angle between the *z* axis and $\vec{L}$ is the angle between the $\vec{L}$ vector for $m = \ell$ and the *z* axis.

Express the angle θ as a function of L_z and L:

$$\theta = \cos^{-1}\left(\frac{L_z}{L}\right)$$

Relate the *z* component of $\vec{L}$ to m_ℓ and ℓ:

$$L_z = m_\ell \hbar = \ell\hbar$$

Express the angular momentum L:

$$L = \sqrt{\ell(\ell+1)}\hbar$$

Substitute to obtain:

$$\theta = \cos^{-1}\left(\frac{\ell\hbar}{\sqrt{\ell(\ell+1)}\hbar}\right) = \cos^{-1}\left(\sqrt{\frac{\ell}{\ell+1}}\right)$$

(*a*) For $\ell = 1$:

$$\theta = \cos^{-1}\left(\sqrt{\frac{1}{1+1}}\right) = \boxed{45.0^\circ}$$

(*b*) For $\ell = 4$:

$$\theta = \cos^{-1}\left(\sqrt{\frac{4}{4+1}}\right) = \boxed{26.6^\circ}$$

(*c*) For $\ell = 50$:

$$\theta = \cos^{-1}\left(\sqrt{\frac{50}{50+1}}\right) = \boxed{8.05^\circ}$$

34 •• What are the possible values of n and m_ℓ for an electron in an atom that has (*a*) $\ell = 3$, (*b*) $\ell = 4$, and (*c*) $\ell = 0$?

Picture the Problem We can use constraints on the quantum numbers in spherical coordinates to find the possible values of n and m_ℓ for each of the values of ℓ.

The constraints on n, m_ℓ, and ℓ are:

$n = 1, 2, 3, ...$
$\ell = 0, 1, 2, ..., n - 1$
$m_\ell = -\ell, (-\ell + 1), ... 0, 1, 2, ... \ell$

(*a*) For $\ell = 3$: $\boxed{n \geq 4 \text{ and } m_\ell = -3, -2, -1, 0, 1, 2, 3}$

(*b*) For $\ell = 4$: $\boxed{n \geq 5 \text{ and } m_\ell = -4, -3, -2, -1, 0, 1, 2, 3, 4}$

(*c*) For $\ell = 0$: $\boxed{n \geq 1 \text{ and } m_\ell = 0}$

35 •• For an electron in an atom that is in an $\ell = 2$ state, find (*a*) the magnitude of the angular momentum squared L^2, (*b*) the maximum value of L_z^2, and (*c*) the smallest value of $L_x^2 + L_y^2$.

Picture the Problem The magnitude of the orbital angular momentum L of an electron in an atom is related to the orbital quantum number ℓ by $L = \sqrt{\ell(\ell+1)}\hbar$ and the z component of the angular momentum of the electron is given by $L_z = m\hbar$.

(*a*) For the $\ell = 2$ state, the square magnitude of the angular momentum is:

$$L^2 = 2(2+1)\hbar^2 = \boxed{6\hbar^2}$$

(*b*) For the $\ell = 2$ state, the maximum value of L_z^2 is:

$$L_z^2 = 2^2\hbar^2 = \boxed{4\hbar^2}$$

(*c*) The smallest value of $L_x^2 + L_y^2$ is given by:

$$L_x^2 + L_y^2 = L^2 - L_z^2 = 6\hbar^2 - 4\hbar^2 = \boxed{2\hbar^2}$$

Quantum Theory of the Hydrogen Atom

36 • For the ground state of the hydrogen atom, find the values of (*a*) $\psi(r)$ at $r = a_0$, (*b*) $\psi^2(r)$ at $r = a_0$, and (*c*) the radial probability density $P(r)$ at $r = a_0$. Give your answers in terms of a_0.

Picture the Problem We can use $\psi(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$ to evaluate the normalized ground-state wave function and its square at $r = a_0$ and $P(r) = 4\pi r^2|\psi|^2$ to find the radial probability density at the same location.

(*a*) Noting that $Z = 1$ for hydrogen, evaluate $\psi(a_0)$ to obtain:

$$\psi(a_0) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-a_0/a_0}$$

$$= \boxed{\frac{1}{ea_0\sqrt{\pi a_0}}}$$

where e is the base of the natural logarithms.

(*b*) Square $\psi(a_0)$ to obtain:

$$\psi^2(a_0) = \left(\frac{1}{ea_0\sqrt{\pi a_0}}\right)^2 = \boxed{\frac{1}{e^2 a_0^3 \pi}}$$

where e is the base of the natural logarithms.

(*c*) Use the result from Part (*b*) to evaluate $P(a_0)$:

$$P(a_0) = 4\pi a_0^2 |\psi|^2 = 4\pi a_0^2\left(\frac{1}{e^2 a_0^3 \pi}\right)$$

$$= \boxed{\frac{4}{e^2 a_0}}$$

where e is the base of the natural logarithms.

37 • (*a*) If electron spin is not included, how many different wave functions are there corresponding to the first excited energy level $n = 2$ for a hydrogen atom? (*b*) Specify the quantum numbers for each of these wave functions.

Picture the Problem We can use the constraints on n, ℓ and m to determine the number of different wave functions, excluding spin, corresponding to the first excited energy state of hydrogen.

For $n = 2$: $\ell = 0$ or 1

(*a*) For $\ell = 0$, $m_\ell = 0$ and we have: 1 state

For $\ell = 1$, $m_\ell = -1, 0, +1$ and we have: 3 states

Hence, for $n = 2$ we have: $\boxed{4\text{ states}}$

(*b*) The four wave functions are summarized to the right.

n	ℓ	m_ℓ	(n, ℓ, m_ℓ)
2	0	0	(2,0,0)
2	1	−1	(2,1,−1)
2	1	0	(2,1,0)
2	1	1	(2,1,1)

38 •• For the ground state of the hydrogen atom, calculate the probability of finding the electron in the region between r and $r + \Delta r$, where $\Delta r = 0.03a_0$ and (*a*) $r = a_0$ and (*b*) $r = 2a_0$.

Picture the Problem We can use $\psi(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$ to evaluate the normalized ground-state wave function and its square and $P(r) = 4\pi r^2|\psi|^2$ to find the radial probability density at the same location. Because the range Δr is so small, the variation in the radial probability density $P(r)$ can be neglected. The probability of finding the electron in some small range Δr is then $P(r)\,\Delta r$.

Express the probability of finding the electron in the range Δr:

$$\text{Prob} = \int P(r)dr \qquad (1)$$

where $P(r)$ is the radial probability density function.

The radial probability density function is:

$$P(r) = 4\pi r^2|\psi|^2 \qquad (2)$$

Express the normalized ground-state wave function:

$$\psi(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

Evaluate the normalized ground-state wave function evaluated at $r = a_0$ to obtain:

$$\psi(a_0) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-a_0/a_0} = \frac{1}{ea_0\sqrt{\pi a_0}}$$

Square $\psi(a_0)$ to obtain:

$$\psi^2(a_0) = \left(\frac{1}{ea_0\sqrt{\pi a_0}}\right)^2 = \frac{1}{e^2a_0^3\pi}$$

Substitute in equation (2) to obtain:

$$P(a_0) = 4\pi a_0^2|\psi|^2 = 4\pi a_0^2\left(\frac{1}{e^2a_0^3\pi}\right)$$

$$= \frac{4}{e^2a_0}$$

(*a*) Substitute in equation (1) to find the probability of finding the electron in the small range $\Delta r = 0.03a_0$:

$$\text{Prob} = \int P(a_0)dr \approx P(a_0)\Delta r$$
$$= \frac{4}{e^2 a_0}(0.03a_0) = \boxed{0.02}$$

(*b*) Evaluate the normalized ground-state wave function at $r = 2a_0$ to obtain:

$$\psi(2a_0) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-2a_0/a_0}$$
$$= \frac{1}{e^2 a_0 \sqrt{\pi a_0}}$$

Squaring $\psi(2a_0)$ yields:

$$\psi^2(2a_0) = \left(\frac{1}{e^2 a_0 \sqrt{\pi a_0}}\right)^2 = \frac{1}{e^4 a_0^3 \pi}$$

Substitute in equation (2) to obtain:

$$P(2a_0) = 4\pi(2a_0)^2|\psi|^2 = 16\pi a_0^2\left(\frac{1}{e^4 a_0^3 \pi}\right)$$
$$= \frac{16}{e^4 a_0}$$

Substitute in equation (1) to find the probability of finding the electron in some small range $\Delta r = 0.03a_0$:

$$\text{Prob} = \int P(2a_0)dr \approx P(2a_0)\Delta r$$
$$= \frac{16}{e^4 a_0}(0.03a_0)$$
$$= \boxed{0.009}$$

Remarks: There is about a 2% chance of finding the electron in this range at $r = a_0$, but at $r = 2a_0$, the chance is only about 0.9%.

39 •• The value of the constant C_{200} in the equation

$$\psi_{200} = C_{200}\left(2 - \frac{Zr}{a_0}\right)e^{-Zr/(2a_0)} \text{ (Equation 36-35)}$$

is given by $C_{2,0,0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}$. Find the values of (*a*) $\psi(r)$ at $r = a_0$, (*b*) $\psi^2(r)$ at $r = a_0$, and (*c*) the radial probability density $P(r)$ at $r = a_0$ for the state $n = 2$, $\ell = 0$, and $m_l = 0$ of a hydrogen atom. Give your answers in terms of a_0.

Picture the Problem We can use Equation 36-35 and the given expression for C_{200} to evaluate the spherically symmetric wave function ψ for $n = 2$, $\ell = 0$, $m_\ell = 0$, and $Z = 1$ and then use this result to evaluate ψ^2 and $P(r)$ for $r = a_0$.

(*a*) Express the spherically symmetric wave function for $n = 2$, $\ell = 0$, $m_\ell = 0$, and $Z = 1$ (Equation 36-35):

$$\psi_{200} = C_{200}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$$

Evaluate this expression for $r = a_0$:

$$\psi_{200}(a_0) = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2-\frac{a_0}{a_0}\right)e^{-a_0/2a_0} = \frac{1}{4\sqrt{2e\pi}\,a_0^{3/2}} = \boxed{\frac{0.0605}{a_0^{3/2}}}$$

where e is the base of the natural logarithms.

(*b*) Squaring $\psi_{200}(a_0)$ gives:

$$[\psi_{200}(a_0)]^2 = \left(\frac{0.0605}{a_0^{3/2}}\right)^2 = \boxed{\frac{0.00366}{a_0^3}}$$

(*c*) The radial probability density is given by:

$$P(r) = 4\pi r^2\psi^2(r)$$

Substitute for $\psi^2(r)$ and simplify to obtain:

$$P(a_0) = 4\pi a_0^2\left(\frac{0.00366}{a_0^3}\right) = \boxed{\frac{0.0460}{a_0}}$$

40 ••• Show that the radial probability density for the $n = 2$, $\ell = 1$, and $m_\ell = 0$ state of a one-electron atom can be written as $P(r) = A\cos^2\theta\, r^4 e^{-Zr/a_0}$, where A is a constant.

Picture the Problem We can use the definition of the radial probability density and the wave function (Equation 36-36) for the state (2, 1, 0) to obtain the result given in the problem statement.

Using Equation 36-36, express the wave function for the state (2, 1, 0):

$$\psi_{210}(r) = C_{210}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta$$

Square $\psi_{210}(r)$ to obtain:

$$[\psi_{210}(r)]^2 = \left(C_{210}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta\right)^2 = C_{210}^2\frac{Z^2r^2}{a_0^2}e^{-Zr/a_0}\cos^2\theta$$

Express the radial probability density:

$$P(r) = 4\pi r^2\psi^2(r)$$

Substitute for $\psi^2(r)$ and simplify to obtain:

$$P(r) = 4\pi r^2 C_{210}^2\frac{Z^2r^2}{a_0^2}e^{-Zr/a_0}\cos^2\theta = \boxed{Ar^4\cos^2\theta\, e^{-Zr/a_0}}$$

where

$$A == \boxed{\frac{4\pi C_{210}^2 Z^2}{a_0^2}}$$

41 ••• Calculate the probability of finding the electron in the region between r and $r + \Delta r$, where $\Delta r = 0.02a_0$ and (*a*) $r = a_0$ and (*b*) $r = 2a_0$ for the state $n = 2$, $\ell = 0$, and $m_l = 0$ in hydrogen. (See Problem 41 for the value of C_{200}.)

Picture the Problem In this instance, $\int P(r)dr$ extends over a sufficiently narrow interval $\Delta r << 0.02a_0$ that we can neglect the dependence of $P(r)$ on r. Hence, we can set $\int P(r)dr = P(r)\Delta r$ and use the wave function (Equation 36-35) for the state (2, 0, 0) and the expression for C_{200} from Problem 41 to find ψ_{200}, ψ_{200}^2, $P(r)$, and the probability of finding the electron in the range specified at $r = a_0$ and $r = 2a_0$.

(*a*) Express the probability of finding the electron in the range Δr:

$$\text{Prob} = \int P(r)dr \qquad (1)$$

where $P(r)$ is the radial probability density function.

The radial probability density function is:

$$P(r) = 4\pi r^2|\psi|^2 \qquad (2)$$

The normalized wave function for the (2, 0, 0) state of hydrogen is given by Equation 36-35:

$$\psi_{200}(r) = C_{200}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$$

From Problem 41 we have, for hydrogen:

$$C_{200} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}$$

Substitute to obtain:

$$\psi_{200}(r) = C_{200}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$$

$$= \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$$

Evaluate the normalized ground-state wave function at $r = a_0$ to obtain:

$$\psi_{200}(a_0) = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-a_0/2a_0}$$

$$= \frac{1}{4\sqrt{2\pi e}a_0^{3/2}} = \frac{0.0605}{a_0^{3/2}}$$

Square $\psi_{200}(a_0)$ to obtain:

$$[\psi_{200}(a_0)]^2 = \left(\frac{0.0605}{a_0^{3/2}}\right)^2 = \frac{0.00366}{a_0^3}$$

Substitute for $\psi^2(r)$ in equation (2) to obtain:

$$P(a_0) = 4\pi a_0^2\psi^2(a_0) = 4\pi a_0^2\left(\frac{0.00366}{a_0^3}\right)$$

$$= \frac{4\pi(0.00366)}{a_0} = \frac{0.0460}{a_0}$$

Substitute in equation (1) to find the probability of finding the electron in some small range $\Delta r = 0.02\ a_0$:

$$\text{Prob} = \int P(a_0)dr \approx P(a_0)\Delta r$$

$$= \frac{0.0460}{a_0}(0.02a_0)$$

$$= \boxed{9\times10^{-4}}$$

(*b*) Evaluate the normalized ground-state wave function at $r = 2a_0$ to obtain:

$$\psi_{200}(2a_0) = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2 - \frac{2a_0}{a_0}\right)e^{-2a_0/2a_0} = 0$$

Squaring $\psi_{200}(2a_0)$ gives:

$$[\psi_{200}(a_0)]^2 = 0$$

Substitute for $\psi^2(r)$ in equation (2) to obtain:

$$P(2a_0) = 0$$

Substitute in equation (1) to find the probability of finding the electron in some small range $\Delta r = 0.02\ a_0$:

$$\text{Prob} = \int P(2a_0)dr \approx P(2a_0)\Delta r = \boxed{0}$$

42 •• Show that the ground-state hydrogen atom wave function $\psi_{100} = \pi^{-1/2}\left(Z/a_0\right)^{3/2} e^{-Zr/a_0}$ (Equation 36-32) is a solution to Schrödinger's equation in spherical coordinates:

$$\frac{-\hbar^2}{2mr^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)+\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]\right\}+U(r)\psi = E\psi$$

where $U(r) = kZe^2/r$ (Equation 36-25).

Picture the Problem We wish to show that $\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} = Ce^{-Zr/a_0}$ is a solution to $\frac{-\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+U(r)\psi = E\psi$, where $U(r) = -\frac{kZe^2}{r}$. Because the ground state is spherically symmetric, we do not need to consider the angular partial derivatives in Equation 36-32.

The normalized ground-state wave function is:

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} = Ce^{-Zr/a_0}$$

Differentiate this expression with respect to r to obtain:

$$\frac{\partial\psi_{100}}{\partial r} = C\frac{\partial}{\partial r}\left[e^{-Zr/a_0}\right] = -C\frac{Z}{a_0}e^{-Zr/a_0}$$

Multiply both sides of this equation by r^2:

$$r^2\frac{\partial\psi_{100}}{\partial r} = -C\frac{Z}{a_0}r^2e^{-Zr/a_0}$$

Differentiate this expression with respect to r to obtain:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_{100}}{\partial r}\right) = -C\frac{Z}{a_0}\frac{\partial}{\partial r}\left(r^2e^{-Zr/a_0}\right) = \left[-\frac{2Zr}{a_0}+r^2\left(\frac{Z}{a_0}\right)^2\right]Ce^{-Zr/a_0}$$

Substitute in Schrödinger's equation to obtain:

$$\frac{-\hbar^2}{2mr^2}\left[-\frac{2Zr}{a_0}+r^2\left(\frac{Z}{a_0}\right)^2\right]Ce^{-Zr/a_0} - \frac{kZe^2}{r}Ce^{-Zr/a_0} = ECe^{-Zr/a_0}$$

Solving for E yields:

$$E=\frac{-\hbar^2}{2mr^2}\left[-\frac{2Zr}{a_0}+r^2\left(\frac{Z}{a_0}\right)^2\right]-\frac{kZe^2}{r}$$

Because $a_0=\dfrac{\hbar^2}{mke^2}$:

$$E=\frac{-\hbar^2}{2mr^2}\left[-\frac{2mke^2Zr}{\hbar^2}+r^2\left(\frac{Zmke^2}{\hbar^2}\right)^2\right]-\frac{kZe^2}{r}=\frac{kZe^2}{r}-\frac{Z^2k^2e^4m}{2\hbar^2}-\frac{kZe^2}{r}$$

$$=\boxed{-\frac{Z^2k^2e^4m}{2\hbar^2}}$$

Because this is the correct ground state energy, we have shown that Equation 36-32 is a solution to Schrödinger's Equation with the potential energy function given by Equation 36-25.

43 •• Show by dimensional analysis that the expression for the hydrogen atom ground-state energy given by $E_0=\frac{1}{2}mk^2e^4/\hbar^2$ (Equation 36-27) has the dimensions of energy.

Picture the Problem For the purposes of this problem, we'll take unit analysis and dimensional analysis to be interchangeable. Then we can substitute for the units of the physical quantities in Equation 36-27 and simplify the result to show that the expression for the hydrogen atom ground-state energy has the units of energy.

Equation 36-27 is:

$$E_0=\frac{mk^2e^4}{2\hbar^2}$$

Substituting the units of m, k, e, and $\hbar$ and simplifying yields:

$$\frac{\mathrm{kg}\left[\dfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right]^2[\mathrm{C}]^4}{[\mathrm{J\cdot s}]^2}=\frac{\mathrm{kg}\left[\mathrm{N\cdot m^2}\right]^2}{[\mathrm{N\cdot m\cdot s}]^2}=\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}}=\boxed{\mathrm{J}}$$

Because the joule is a unit of energy, we can conclude that the expression for the ground-state energy given by Equation 36-27 has the dimensions of energy.

44 •• By dimensional analysis, show that the expression for the first Bohr radius given by $a_0\,\hbar^2/(mke^2)$ (Equation 36-12) has the dimensions of length.

Picture the Problem For the purposes of this problem, we'll take unit analysis and dimensional analysis to be interchangeable. Then we can substitute for the units of the physical quantities in the expression for the Bohr radius and simplify the result to show that the expression for the first Bohr radius has the units of length.

Equation 36-12 is:

$$a_0 = \frac{\hbar^2}{mke^2}$$

Substituting the units of *m, k, e,* and $\hbar$ and simplifying yields:

$$\frac{[\mathrm{J\cdot s}]^2}{\mathrm{kg}\cdot\dfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\cdot\mathrm{C^2}} = \frac{[\mathrm{N\cdot m\cdot s}]^2}{\mathrm{kg}\cdot\dfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\cdot\mathrm{C^2}} = \frac{\mathrm{N\cdot s^2}}{\mathrm{kg}} = \frac{\mathrm{kg}\cdot\dfrac{\mathrm{m}}{\mathrm{s^2}}\cdot\mathrm{s^2}}{\mathrm{kg}} = \boxed{\mathrm{m}}$$

Because the meter is a unit of length, we can conclude that the expression for the first Bohr radius has the dimensions of length.

45 •• The radial probability distribution function for a one-electron atom in its ground state can be written $P(r) = Cr^2e^{-2Zr/a_0}$, where C is a constant. Show that $P(r)$ has its maximum value at $r = a_0/Z$.

Picture the Problem This is an extreme value problem. We'll begin its solution with the radial probability distribution function, differentiate it with respect to its independent variable *r*, set this derivative equal to zero, and solve for an extreme value for *r*. We can show that this value corresponds to a maximum by evaluating the second derivative of $P(r)$ at the location found from the first derivative.

Differentiate the radial probability distribution function with respect to *r* to obtain:

$$\begin{aligned}\frac{dP(r)}{dr} &= C\frac{d}{dr}\left[r^2e^{-2Zr/a_0}\right]\\ &= C\left[2re^{-2Zr/a_0} - \frac{2Zr^2}{a_0}e^{-2Zr/a_0}\right]\\ &= \frac{2CZr}{a_0}e^{-2Zr/a_0}\left(\frac{a_0}{Z} - r\right)\\ &= 0 \text{ for extrema}\end{aligned}$$

Solve for *r* to obtain:

$$r = \frac{a_0}{Z}$$

To show that this value for r corresponds to a maximum, differentiate $dP(r)/dr$ to obtain:

$$\frac{d^2P(r)}{dr^2} = -\frac{2CZr}{a_0}e^{-2Zr/a_0} + \left(\frac{a_0}{Z} - r\right)\left(-\frac{4CZ^2}{a_0^2} + \frac{2CZ}{a_0}\right)e^{-2Zr/a_0}$$

Evaluate this derivative at $r = a_0/Z$:

$$\left.\frac{d^2P(r)}{dr^2}\right|_{r=\frac{a_0}{Z}} = -2Ce^{-2} < 0$$

because C is a positive constant. Hence, $P(r)$ has its maximum value at

$$r = \boxed{\frac{a_0}{Z}}$$

46 ••• Show that the number of states in the hydrogen atom for a given n is $2n^2$.

Picture the Problem We can double the sum of the number of m states for a given n to show that the number of states in the hydrogen atom for a given n is $2n^2$.

The number of m_ℓ states for a given n is:

$$N_{m_\ell} = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1}(1)$$

The sum of all integers from 0 to p is:

$$\sum_{\ell=0}^{p}\ell = \tfrac{1}{2}p(p+1)$$

and

$$2\sum_{\ell=0}^{n-1}\ell = 2\left(\tfrac{1}{2}\right)(n-1)(n) = n^2 - n$$

The second term is:

$$\sum_{\ell=0}^{n-1}(1) = n$$

Substitute for $2\sum_{\ell=0}^{n-1}\ell$ and $\sum_{\ell=0}^{n-1}(1)$ in the expression for N_m to obtain:

$$N_{m_\ell} = n^2 - n + n = n^2$$

Because N, the number of electron states, is twice the number of m_ℓ states, the number of electron states is:

$$N = 2N_{m_\ell} = \boxed{2n^2}$$

47 ••• Calculate the probability that the electron in the ground state of a hydrogen atom is in the region $0 < r < a_0$.

Picture the Problem The ground state of a hydrogen atom is the state described by $n = 1$, $\ell = 0$, $m_\ell = 0$. We can calculate the probability that the electron in the ground state of the hydrogen atom is in the region $0 < r < a_0$ by evaluating the integral $\int_0^{a_0} 4\pi r^2 \psi_{100}^2(r)dr$.

Express the probability that the electron in the ground state of a hydrogen atom is in the region $0 < r < a_0$:

$$P(r) = \int_0^{a_0} 4\pi r^2 \psi_{100}^2(r)dr$$

Express the ground-state wave function for hydrogen:

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

Square the wave function to obtain:

$$\psi_{100}^2(r) = \frac{1}{\pi a_0^3} e^{-2r/a_0}$$

Substitute for $\psi_{100}^2(r)$ to obtain:

$$P(r) = \int_0^{a_0} 4\pi r^2 \left(\frac{1}{\pi a_0^3} e^{-2r/a_0}\right)dr$$

$$= \frac{4}{a_0^3}\int_0^{a_0} r^2 e^{-2r/a_0}\,dr$$

Use a table of integrals to find:

$$\int x^2 e^{bx} dx = \frac{e^{bx}}{b^3}\left(b^2x^2 - 2bx + 2\right)$$

Use this integral to show that:

$$P(r) = -e^{-2r/a_0}\left(\frac{2r^2}{a_0^2} + \frac{2r}{a_0} + 1\right)\Bigg|_0^{a_0}$$

$$= 1 - 5e^{-2} = \boxed{0.323}$$

The Spin-Orbit Effect and Fine Structure

48 • The potential energy of a magnetic moment in an external magnetic field is given by $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}}$. (*a*) Calculate the difference in energy between the two possible orientations of an electron in a magnetic field $\vec{\boldsymbol{B}} = (1.50\ \text{T})\hat{\boldsymbol{k}}$. (*b*) If these electrons are bombarded with photons of energy equal to this energy difference, ″spin flip″ transitions can be induced. Find the wavelength of the photons needed for such transitions. This phenomenon is called *electron spin resonance*.

Picture the Problem The energy difference between the two possible orientations of an electron in a magnetic field is $2\mu B$ and the wavelength of the photons required to induce a spin-flip transition can be found from $hc/\Delta E$. The magnetic moment μ_B associated with the spin of an electron is 5.79×10^{-5} eV/T.

(*a*) The difference in energy between the two spin orientations is given by:

$$\Delta E = 2\mu B$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = 2\left(5.79\times10^{-5}\ \text{eV/T}\right)\left(1.50\ \text{T}\right) = 1.737\times10^{-4}\ \text{eV} = \boxed{174\ \mu\text{eV}}$$

(*b*) Relate the wavelength of the photon needed to induce such a transition to the energy required:

$$\lambda = \frac{hc}{\Delta E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{1.737\times10^{-4}\ \text{eV}} = 7.139\times10^{6}\,\text{nm} = \boxed{7.14\,\text{mm}}$$

49 • The total angular momentum of a hydrogen atom in a certain excited state has the quantum number $j = \frac{1}{2}$. What can you say about the value of the orbital angular-momentum quantum number ℓ?

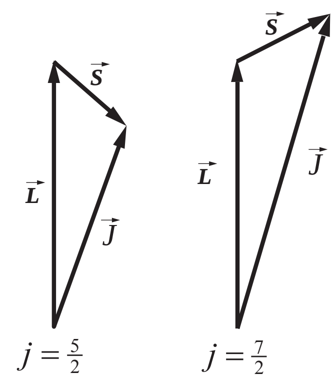
Determine the Concept j and ℓ are constrained according to $j = \ell \pm \frac{1}{2}$. For j = ½, $\ell = \frac{1}{2} \pm \frac{1}{2}$ or $\ell = \boxed{0\,\text{or}\,1}$.

50 • A hydrogen atom is in the state $n = 3$, $\ell = 2$. What are the possible values of j?

Determine the Concept j and ℓ are constrained according to $j = \ell \pm \frac{1}{2}$. For $\ell = 2$, $j = 2 \pm \frac{1}{2}$, or $j = \boxed{\frac{3}{2} \text{ or } \frac{5}{2}}$

51 • Using a scaled vector diagram, show how the orbital angular momentum $\vec{L}$ combines with the spin angular momentum $\vec{S}$ to produce the two possible values of total angular momentum $\vec{J}$ for the $\ell = 3$ state of the hydrogen atom.

Picture the Problem The total angular momentum vector $\vec{J}$ is the sum of the orbital momentum vector $\vec{L}$ and the spin orbital angular momentum vector $\vec{S}$. The quantum number j can be either $\ell + \frac{1}{2}$ or $\ell - \frac{1}{2}$, where $\ell \neq 0$. Hence, j can take on the values 3 + 1/2 = 7/2 and 3 − 1/2 = 5/2. The scaled vector diagrams are shown to the right.



The Periodic Table

52 • The total number of states of a hydrogen atom that has principal quantum number $n = 4$ is (*a*) 4, (*b*) 16, (*c*) 32, (*d*) 36, (*e*) 48.

Determine the Concept The total number of quantum states of hydrogen with quantum number n is $2n^2$. For $n = 4$, we have $2(4)^2 = 32$. $\boxed{(c)}$ is correct.

53 • How many of the eight electrons in an oxygen atom in the ground state are found in a p state? (*a*) 0, (*b*) 2, (*c*) 4, (*d*) 6, (*e*) 8

Determine the Concept From Table 36-1, oxygen's electronic configuration is $1s^22s^22p^4$. Because there are 4 electrons in the p state, $\boxed{(c)}$ is correct.

54 • Write the ground-state electron configuration of (*a*) an atom of carbon and (*b*) an atom of oxygen.

Determine the Concept We can use the atomic numbers of carbon and oxygen to determine the sum of the exponents in their electronic configurations and then use the rules for the filling of the shells to find their electronic configurations.

(*a*) The atomic number Z of carbon is 6. So we must fill the subshells of the electronic configuration until we have placed its 6 electrons. This is accomplished by writing:

$$\boxed{1s^2 2s^2 2p^2}$$

(*b*) The atomic number Z of oxygen is 8. So we must fill the subshells of the electronic configuration until we have placed its 8 electrons. This is accomplished by writing:

$$\boxed{1s^2 2s^2 2p^4}$$

55 • Give the possible values of the z component of the orbital angular momentum of (*a*) a d electron, and (*b*) an f electron.

Determine the Concept We can find the z component of the orbital angular momentum using $L_z = m_\ell \hbar$ and the relationship between the quantum numbers ℓ (which we know from the state of the electrons) and m_ℓ (which is related to ℓ through $m_\ell = -\ell, (-\ell + 1), \ldots, 0, 1, 2, \ldots \ell$).

(*a*) For a d electron $\ell = 2$. For $\ell = 2$, $m_\ell = -2, -1, 0, 1$ or 2. Because $L_z = -m_\ell\hbar, \ldots, m_\ell\hbar$:

$$L_z = \boxed{-2\hbar, -\hbar, 0, \hbar, 2\hbar}$$

(*b*) For an f electron, $\ell = 3$. For $\ell = 3$, $m_\ell = -3, -2, -1, 0, 1, 2, 3$. Because $L_z = -m_\ell\hbar, \ldots, m_\ell\hbar$:

$$L_z = \boxed{-3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar}$$

Optical Spectra and X-Ray Spectra

56 • The optical spectra of atoms that have two electrons in the same highest energy shell are similar, but they are quite different from the spectra of atoms that have just one electron in the highest energy shell because of the interaction of the two electrons. Group the elements according to similar spectra: lithium, beryllium, sodium, magnesium, potassium, calcium, chromium, nickel, cesium, and barium.

Determine the Concept Lithium, sodium, potassium, chromium, and cesium have one outer s electron and hence belong in the same group. Beryllium, magnesium, calcium, nickel, and barium have two outer s electrons and, hence, belong in the same group.

57 • Write down the possible electron configurations for the first excited state of (*a*) a hydrogen atom, (*b*) a sodium atom, and (*c*) a helium atom.

Determine the Concept We can use Table 35-1 to find the electronic configurations for the first excited states of these atoms.

(*a*) For H, E depends only on n and the lowest excited state is: $\boxed{\text{2s or 2p}}$

(*b*) For Na, the 3p state is higher energy than the 3s state and the lowest excited state is: $\boxed{1s^2 2s^2 2p^6 3p}$

(*c*) For He, the lowest excited state has one electron in the 2s state and the lowest excited state is: $\boxed{1s2s}$

58 • Indicate which of the following atoms should have optical spectra similar to a hydrogen atom and which of the following atoms should have optical spectra similar to a helium atom: Li, Ca, Ti, Rb, Hg, Ag, Cd, Ba, Fr, and Ra.

Determine the Concept Atoms with one outer electron have spectra similar to H: Li, Rb, Ag, Fr. Atoms with two outer electrons have spectra similar to He: Ca, Ti, Hg, Cd, Ba, Ra. Therefore, the table should be completed as shown below:

Optical Spectra Similar to Hydrogen	Optical Spectra Similar to Helium
Li, Rb, Ag, Fr	Ca, Ti, Hg, Cd, Ba, Ra

59 • (*a*) Calculate the next two longest wavelengths in the K series (after the K_α line) of molybdenum. (*b*) What is the wavelength of the shortest wavelength in this series?

Picture the Problem When an electron from state n drops into a vacated state in the n = 1 shell, a photon of energy $\Delta E = E_n - E_1$ is emitted. We can find the wavelength of this photon using $\lambda = hc/\Delta E$. The second and third longest wavelengths in the K series correspond to transitions from n = 3 to n = 1 and n = 4 to n = 1 and the shortest wavelength to the transition from $n = \infty$ to n = 1.

Express the wavelength of the emitted photon in terms of the energy transition within the atom:

$$\lambda = \frac{hc}{E_n - E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{E_n - E_1}$$

Express the energy of the nth energy state:

$$E_n = -(Z-1)^2 \frac{E_0}{n^2} \text{ where } n = 1, 2, \ldots$$

Substitute to obtain:

$$\lambda = \frac{hc}{E_n - E_1}$$
$$= \frac{1240\,\text{eV}\cdot\text{nm}}{-(Z-1)^2 \frac{E_0}{n^2} - \left(-(Z-1)^2 \frac{E_0}{1^2}\right)}$$
$$= \frac{1240\,\text{eV}\cdot\text{nm}}{(Z-1)^2 E_0 \left(1 - \frac{1}{n^2}\right)}$$

(*a*) Evaluate this expression with $n = 3$ and $Z = 42$ to obtain:

$$\lambda_3 = \frac{1240\,\text{eV}\cdot\text{nm}}{(42-1)^2(13.6\,\text{eV})\left(1-\frac{1}{3^2}\right)} = \boxed{0.0610\,\text{nm}}$$

Use $n = 4$ and $Z = 42$ to obtain:

$$\lambda_4 = \frac{1240\,\text{eV}\cdot\text{nm}}{(42-1)^2(13.6\,\text{eV})\left(1-\frac{1}{4^2}\right)} = \boxed{0.0578\,\text{nm}}$$

(*b*) The shortest wavelength in the series corresponds to the largest energy difference between the initial and final states. Repeat the calculation in Part (*a*) with $n = \infty$ to obtain:

$$\lambda_\infty = \frac{1240\,\text{eV}\cdot\text{nm}}{(42-1)^2(13.6\,\text{eV})(1-0)} = \boxed{0.0542\,\text{nm}}$$

60 • The wavelength of the K_α line for a certain element is 0.3368 nm. What is the element?

Picture the Problem When an electron from state n drops into the vacated state in the $n = 1$ shell, a photon of energy $E_n - E_1$ is emitted. The wavelength of this photon is $\lambda = \frac{hc}{(Z-1)^2(13.6\,\text{eV})\left(1-\frac{1}{n^2}\right)}$. Hence, if we know the wavelength of the K_α line we can solve for the atomic number of the element and use its value to identify the element.

Express the wavelength of the K$_\alpha$ line as a function of the atomic number of the element:

$$\lambda = \frac{hc}{(Z-1)^2(13.6\,\text{eV})\left(1-\frac{1}{n^2}\right)}$$

Solving for Z gives:

$$Z = 1 + \sqrt{\frac{hc}{\lambda(13.6\,\text{eV})\left(1-\frac{1}{n^2}\right)}}$$

Substitute numerical values and evaluate Z:

$$Z = 1 + \sqrt{\frac{1240\,\text{eV}\cdot\text{nm}}{(0.3368\,\text{nm})(13.6\,\text{eV})\left(1-\frac{1}{2^2}\right)}} = 20$$

The element whose atomic number is 20 is $\boxed{\text{calcium}}$.

61 • Calculate the wavelength of the K_α line in (*a*) a magnesium ($Z = 12$) atom and (*b*) a copper ($Z = 29$) atom.

Picture the Problem The K_α corresponds to a transition from $n = 2$ to $n = 1$. Equation 36-46 relates the atomic number Z to the wavelength of the emitted photon.

From Equation 36-46 we have:

$$\lambda_{K_\alpha} = \frac{hc}{(Z-1)^2 E_0\left(1-\frac{1}{2^2}\right)}$$

(*a*) Substitute numerical values and evaluate λ_{K_α} for magnesium:

$$\lambda_{K_\alpha} = \frac{1240\,\text{eV}\cdot\text{nm}}{(12-1)^2(13.6\,\text{eV})\left(1-\frac{1}{2^2}\right)} = \boxed{1.00\,\text{nm}}$$

(*b*) Substitute numerical values and evaluate λ_{K_α} for copper:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{(29-1)^2(13.6\,\text{eV})\left(1-\frac{1}{2^2}\right)} = \boxed{0.155\,\text{nm}}$$

General Problems

62 • What is the energy of the shortest wavelength photon emitted by the hydrogen atom?

Picture the Problem The energy associated with a transition from an initial state to some final state is given by $\Delta E = E_i - E_f$ and the wavelength λ of a photon emitted in such a transition is given by $\lambda = hc/\Delta E$. Hence, the shortest wavelength corresponds to the largest energy difference.

Express the wavelength of the emitted photon in terms of the energy difference ΔE between the atom's initial and final states:

$$\lambda = \frac{hc}{\Delta E} \Rightarrow \Delta E = \frac{hc}{\lambda}$$

For λ_{min}, ΔE will be the energy required to ionize a hydrogen atom:

$$\Delta E_{max} = \frac{hc}{\lambda_{min}} = \boxed{13.6\,\text{eV}}$$

63 • The wavelength of a spectral line of hydrogen is 97.254 nm. Identify the transition that results in this line, assuming that the transition is to the ground state.

Picture the Problem This spectral line is due to a transition from some initial state n_i to a final state n_f (we're given that the final state is the ground state). The wavelength of the spectral line is related to the difference in energy ΔE between these states according to $\lambda = 1240\,\text{eV}\cdot\text{nm}/\Delta E$ and the energy of the nth state is given (for hydrogen, $Z = 1$) by $E_n = (1^2)(-13.6\text{ eV})/n^2$.

Relate the wavelength of a spectral line to the energy transition within the atom:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \qquad (1)$$

Express the energy difference ΔE in a transition:

$$\Delta E = E_i - E_f = -\frac{Z^2E_0}{n_i^2} + \frac{Z^2E_0}{n_f^2}$$

$$= Z^2E_0\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

For $Z = 1$ and $E_0 = 13.6$ eV:

$$\Delta E = (13.6\,\text{eV})\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Substitute in equation (1) to obtain:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{(13.6\,\text{eV})\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)} = \frac{91.2\,\text{nm}}{\frac{1}{n_f^2} - \frac{1}{n_i^2}}$$

or

$$\frac{1}{n_f^2} - \frac{1}{n_i^2} = \frac{91.2\,\text{nm}}{\lambda}$$

For $\lambda = 97.254$ nm and $n_f = 1$ this expression simplifies to:

$$1 - \frac{1}{n_i^2} = \frac{91.2\,\text{nm}}{97.254\,\text{nm}} = 0.938$$

Solving for n_i gives:

$$n_i = 4$$

The transition that produced the given wavelength was from $\boxed{n_i = 4 \text{ to } n_f = 1}$.

64 •• The wavelength of a spectral line of hydrogen is 1093.8 nm. Identify the transition that results in this line.

Picture the Problem This spectral line is due to a transition from some initial state n_i to a final state n_f. The wavelength of the spectral line is related to the difference in energy ΔE between these states according to $\lambda = 1240\,\text{eV}\cdot\text{nm}/\Delta E$ and the energy of the nth state is given (for hydrogen, $Z = 1$) by $E_n = (1^2)(-13.6\text{ eV})/n^2$.

Relate the wavelength of a spectral line to the energy transition within the atom:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \qquad (1)$$

Express the energy difference ΔE in a transition:

$$\Delta E = E_i - E_f = -\frac{Z^2 E_0}{n_i^2} + \frac{Z^2 E_0}{n_f^2} = Z^2 E_0\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

For $Z = 1$ and $E_0 = 13.6$ eV:

$$\Delta E = (13.6\,\text{eV})\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Substitute in equation (1) to obtain:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{(13.6\,\text{eV})\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)} = \frac{91.2\,\text{nm}}{\frac{1}{n_f^2}-\frac{1}{n_i^2}}$$

or

$$\frac{1}{n_f^2}-\frac{1}{n_i^2} = \frac{91.2\,\text{nm}}{\lambda}$$

Solving for n_f and simplifying gives :

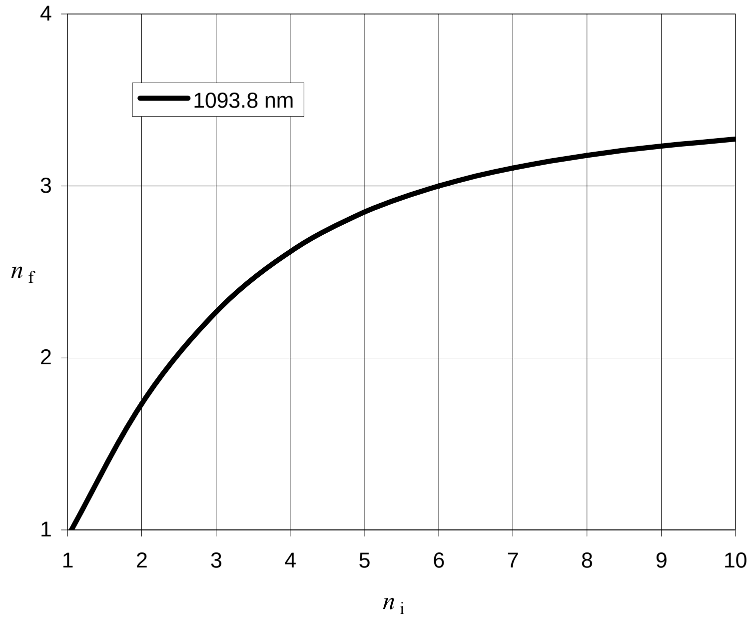
$$n_f = \frac{1}{\sqrt{\frac{91.2\text{ nm}}{\lambda}+\frac{1}{n_i^2}}}$$

Note that n_f and n_i are integers. A spreadsheet program to generate data for a graph of n_f as a function of n_i is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
B2	1093.8	λ
A5	1	n_i
B5	1/SQRT((91.2/$B$2)+(1/A5^2))	$\frac{1}{\sqrt{\frac{91.2\text{ nm}}{\lambda}+\frac{1}{n_i^2}}}$

	A	B	C
1			
2	lambda=	1093.8	nm
3			
4	*n*_i	*n*_f	
5	1	0.96	
6	2	1.73	
7	3	2.27	
8	4	2.62	
9	5	2.85	
10	**6**	**3.00**	
11	7	3.10	
12	8	3.18	
13	9	3.23	
14	10	3.27	

A graph of n_f as a function of n_i follows. Consulting either the spreadsheet table or the graph we can see that the transition that produces the given wavelength is from $\boxed{n_i = 6 \text{ to } n_f = 3}$.



65 •• Spectral lines of the following wavelengths are emitted by a singly ionized helium atom: 164 nm, 230.6 nm, and 541 nm. Identify the transitions that result in these spectral lines.

Picture the Problem These spectral lines are due to transitions in singly ionized helium from some initial state n_i to a final state n_f. The wavelengths of the spectral lines are related to the difference in energy ΔE between these states according to $\lambda = 1240\,\text{eV}\cdot\text{nm}/\Delta E$ and the energy of the nth state is given (for helium, $Z = 2$) by $E_n = (2^2)(-13.6\text{ eV})/n^2$.

Relate the wavelength of a spectral line to the energy transition within the atom:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E} \qquad (1)$$

Express the energy difference ΔE in a transition:

$$\Delta E = E_i - E_f = -\frac{Z^2E_0}{n_i^2} + \frac{Z^2E_0}{n_f^2}$$

$$= Z^2E_0\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

For $Z = 2$ and $E_0 = 13.6$ eV:

$$\Delta E = 2^2(13.6\,\text{eV})\left(\frac{1}{n_\text{f}^2}-\frac{1}{n_\text{i}^2}\right) = (54.4\,\text{eV})\left(\frac{1}{n_\text{f}^2}-\frac{1}{n_\text{i}^2}\right)$$

Substitute in equation (1) to obtain:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{(54.4\,\text{eV})\left(\frac{1}{n_\text{f}^2}-\frac{1}{n_\text{i}^2}\right)} = \frac{22.8\,\text{nm}}{\frac{1}{n_\text{f}^2}-\frac{1}{n_\text{i}^2}}$$

or

$$\frac{1}{n_\text{f}^2}-\frac{1}{n_\text{i}^2} = \frac{22.8\,\text{nm}}{\lambda}$$

Solving for n_f and simplifying gives :

$$n_\text{f} = \frac{1}{\sqrt{\frac{22.8\,\text{nm}}{\lambda}+\frac{1}{n_\text{i}^2}}}$$

Note that n_f and n_i are integers. A spreadsheet program to generate data for a graph of n_f as a function of n_i is shown below. The formulas used to calculate the quantities in the columns are as follows:

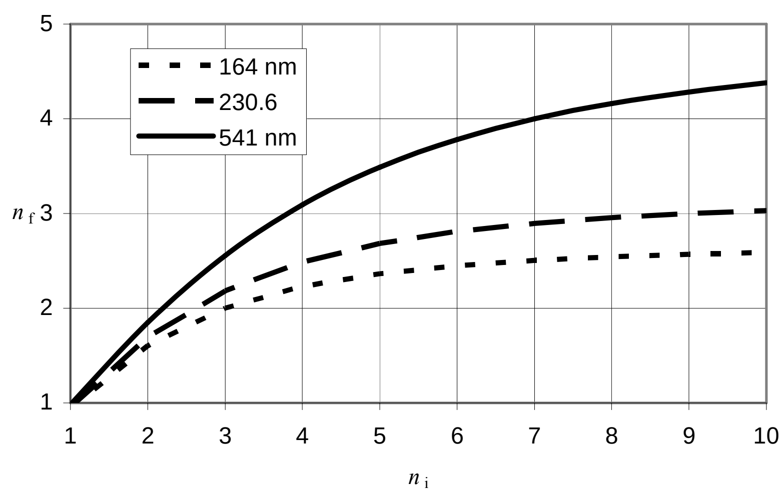
Cell	Content/Formula	Algebraic Form
B2	164	λ
C2	230.6	λ
D2	541	λ
A5	1	n_i
B5	1/SQRT((91.2/$B$2)+(1/A5^2))	$\frac{1}{\sqrt{\frac{22.8\,\text{nm}}{\lambda}+\frac{1}{n_\text{i}^2}}}$
C5	1/SQRT((91.2/$C$2)+(1/A5^2))	$\frac{1}{\sqrt{\frac{22.8\,\text{nm}}{\lambda}+\frac{1}{n_\text{i}^2}}}$
D5	1/SQRT((91.2/$D$2)+(1/A5^2))	$\frac{1}{\sqrt{\frac{22.8\,\text{nm}}{\lambda}+\frac{1}{n_\text{i}^2}}}$

	A	B	C	D	E
1					
2	lambda=	164	230.6	541	nm
3					

4	n_i	n_f	n_f	n_f	
5	1	0.94	0.95	0.98	
6	2	1.60	1.69	1.85	
7	**3**	**2.00**	2.18	2.55	
8	4	2.23	2.49	3.09	
9	5	2.36	2.68	3.49	
10	6	2.45	2.81	3.78	
11	**7**	2.50	2.90	**4.00**	
12	8	2.54	2.96	4.16	
13	**9**	2.57	**3.00**	4.28	
14	10	2.59	3.03	4.38	

A graph of n_f as a function of n_i follows. Consulting either the spreadsheet table or the graph we can see that the transitions corresponding to the given wavelengths are:

λ, nm	n_i	n_f
164	3	2
230.6	9	3
541	7	4



66 •• The combination of physical constants $\alpha = e^2k/\hbar c$, where k is the Coulomb constant, is known as the *fine-structure constant*. It appears in numerous relations in atomic physics. (*a*) Show that α is dimensionless. (*b*) Show that in the Bohr model of the hydrogen atom $v_n = c\alpha/n$, where v_n is the speed of the electron in the state of quantum number n.

Picture the Problem We can show that α is dimensionless by showing that it has no units. In Part (*b*) we can use Bohr's 3rd postulate and the expression for the radii of the Bohr orbits, together with the definition of α, to show that the speed of the electron in a stationary state of quantum number *n* is related to α according to $v_n = c\alpha/n$.

(*a*) Express the units of α:

$$\frac{\mathrm{C}^2\left(\dfrac{\mathrm{N\cdot m^2}}{\mathrm{C}^2}\right)}{(\mathrm{J\cdot s})\dfrac{\mathrm{m}}{\mathrm{s}}}=\frac{\mathrm{N\cdot m^2}}{\mathrm{J\cdot m}}=1$$

Because α is unitless, it is also dimensionless.

(*b*) Apply the quantization of angular momentum postulate to obtain:

$$v_n=\frac{n\hbar}{mr_n}$$

The radii of the Bohr orbits are given by:

$$r_n=n^2\frac{\hbar^2}{mkZe^2}$$

or, because $Z = 1$ for hydrogen,

$$r_n=n^2\frac{\hbar^2}{mke^2}$$

Substitute and simplify to obtain:

$$v_n=\frac{n\hbar}{mn^2\dfrac{\hbar^2}{mke^2}}=\frac{ke^2}{n\hbar}$$

Divide this expression by the definition of α to obtain:

$$\frac{v_n}{\alpha}=\frac{\dfrac{ke^2}{n\hbar}}{\dfrac{e^2k}{\hbar c}}=\frac{c}{n}\Rightarrow v_n=\boxed{\frac{\alpha c}{n}}$$

67 •• The wavelengths of the photons emitted by a potassium atom corresponding to transitions from the $4P_{3/2}$ and $4P_{1/2}$ states to the ground state are 766.41 nm and 769.90 nm. (*a*) Calculate the energies of these photons in electron volts. (*b*) The difference in the energies of these photons equals the difference in energy ΔE between the $4P_{3/2}$ and $4P_{1/2}$ states in potassium. Calculate ΔE. (*c*) Estimate the magnetic field that the 4p electron in potassium experiences.

Picture the Problem Because the energies of the photons emitted by potassium during these transitions are related to their wavelengths through $hf = (1240\ \text{eV}\cdot\text{nm}/\lambda)$ eV where λ is in nm, we can use this relationship to find the energies of the given photons. The difference in energy between these states can be found using its definition and is related to the magnetic field through $\Delta E = 2\mu_B B$.

(*a*) For $\lambda = 766.41$ nm:

$$hf = \frac{1240\,\text{eV}\cdot\text{nm}}{766.41\,\text{nm}} = \boxed{1.6179\,\text{eV}}$$

For $\lambda = 769.90$ nm:

$$hf = \frac{1240\,\text{eV}\cdot\text{nm}}{769.90\,\text{nm}} = \boxed{1.6106\,\text{eV}}$$

(*b*) Using its definition, express the difference in energy between these two states:

$$\Delta E = 1.6179\,\text{eV} - 1.6106\,\text{eV} = \boxed{0.0073\,\text{eV}}$$

(*c*) Relate the energy difference between these states ΔE to the magnetic field B and the quantum unit of magnetic moment (a Bohr magneton) μ_B:

$$\Delta E = 2\mu_B B \Rightarrow B = \frac{\Delta E}{2\mu_B}$$

Substitute numerical values and evaluate B:

$$B = \frac{0.0073\,\text{eV}}{2\left(5.79\times10^{-5}\ \text{eV/T}\right)} = \boxed{63\,\text{T}}$$

Remarks: This magnetic field is about 42 times that of commercial magnetic resonance imagers.

68 •• To observe the characteristic *K* lines of the X-ray spectrum, one of the $n = 1$ electrons must be ejected from the atom. This is generally accomplished by bombarding the target material with electrons of sufficient energy to eject this tightly bound electron. What is the minimum energy required to observe the *K* lines of (*a*) a tungsten atom, (*b*) a molybdenum atom, and (*c*) a copper atom?

Picture the Problem One 1s electron must be released from the atom. Because the electron is shielded from the nuclear charge *Z* by one other 1s electron, the effective charge is $Z - 1$, and the ionization energy for that 1s electron is $E_{min} = (Z - 1)^2 E_0$.

(*a*) For tungsten, $Z = 74$, and:

$$E_{min} = 73^2(13.6\,\text{eV}) = \boxed{72.5\,\text{keV}}$$

(*b*) For molybdenum, $Z = 42$, and:

$$E_{\min} = 41^2(13.6\,\text{eV}) = \boxed{22.9\,\text{keV}}$$

(*c*) For copper, $Z = 29$, and:

$$E_{\min} = 28^2(13.6\,\text{eV}) = \boxed{10.7\,\text{keV}}$$

69 •• **[SSM]** We are often interested in finding the quantity ke^2/r in electron volts when r is given in nanometers. Show that $ke^2 = 1.44$ eV·nm.

Picture the Problem We can show that $ke^2 = 1.44$ eV·nm by solving the equation for the ground state energy of an atom for ke^2.

Express the ground state energy of an atom as a function of k, e, and a_0:

$$E_0 = \frac{ke^2}{2a_0} \Rightarrow ke^2 = 2E_0a_0$$

Substitute numerical values and evaluate ke^2:

$$ke^2 = 2(13.6\,\text{eV})(0.0529\,\text{nm}) = \boxed{1.44\,\text{eV}\cdot\text{nm}}$$

70 •• The *positron* is a particle that has the same mass as the electron and carries a charge equal to $+e$. *Positronium* is a bound state of an electron-positron combination. (*a*) Calculate the energies of the five lowest energy states of positronium using the reduced mass, as given in Problem 29. (*b*) Do transitions between any of the levels found in Part (*a*) fall in the visible range of wavelengths? If so, which transitions are these?

Picture the Problem We can use Problem 29 to express the energy levels of positronium in terms of the reduced mass of the electron-positron system. In Part (*b*) we can find the energies corresponding to 400 nm and 700 nm to decide whether the transitions between any of the levels found in (*a*) fall in the visible range of wavelengths

(*a*) Express the energy of positronium as a function of the quantum number n:

$$E_n = -\frac{m_rk^2e^4}{2\hbar^2}\frac{Z^2}{n^2}$$

From Problem 29 we have:

$$m_r = \frac{m_\text{e}m_\text{pos}}{m_\text{e} + m_\text{pos}}$$

Because $m_\text{e} = m_\text{pos}$:

$$m_r = \frac{m_\text{e}m_\text{e}}{m_\text{e} + m_\text{e}} = \frac{m_\text{e}}{2}$$

Substitute and simplify to obtain:

$$E_n = -\frac{m_\text{e}k^2e^4}{4\hbar^2}\frac{1}{n^2} = -\frac{E_0}{2n^2}$$

Evaluate E_n for $n = 1, 2, 3, 4$, and 5 to obtain:

n	E_n (eV)
1	−6.80
2	−1.70
3	−0.756
4	−0.425
5	−0.272

(*b*) Relate the wavelength of the emitted photons to the energy-level differences:

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\ \text{eV}\cdot\text{nm}}{\Delta E}$$

Solving for ΔE yields:

$$\Delta E = \frac{1240\ \text{eV}\cdot\text{nm}}{\lambda}$$

Evaluate ΔE for $\lambda = 400$ nm and $\lambda = 700$ nm:

$$\Delta E_{400\ \text{nm}} = \frac{1240\ \text{eV}\cdot\text{nm}}{400\ \text{nm}} = 3.10\ \text{eV}$$

and

$$\Delta E_{700\ \text{nm}} = \frac{1240\ \text{eV}\cdot\text{nm}}{700\ \text{nm}} = 1.77\ \text{eV}$$

Because none of the energies in the above table are in the interval 1.77 eV to 3.10 eV, no transitions are in the visible range of wavelengths.

71 • In 1947, Lamb and Retherford showed that there is a very small energy difference between the $2S_{1/2}$ and the $2P_{1/2}$ states of the hydrogen atom. They measured this difference essentially by causing transitions between the two states using very long wavelength electromagnetic radiation. The energy difference (the Lamb shift) is 4.372×10^{-6} eV and is explained by quantum electrodynamics as being due to fluctuations in the energy level of the vacuum. (*a*) What is the frequency of a photon whose energy is equal to the Lamb shift energy? (*b*) What is the wavelength of this photon? In what spectral region does it belong?

Picture the Problem We can use $E = hf$ to find the frequency of the photon and $\lambda = hc/E$ to find its wavelength.

(*a*) The energy of the photon whose energy is equal to the Lamb shift energy is given by:

$$E = hf \Rightarrow f = \frac{E}{h}$$

Substitute numerical values and evaluate *f*:

$$f=\frac{4.372\times10^{-6}\text{ eV}}{4.136\times10^{-15}\text{ eV}\cdot\text{s}}=\boxed{1.06\,\text{GHz}}$$

(*b*) The wavelength of this photon is given by:

$$\lambda=\frac{hc}{E}$$

Substitute numerical values and evaluate λ:

$$\lambda=\frac{1240\,\text{eV}\cdot\text{nm}}{4.372\times10^{-6}\text{ eV}}=\boxed{28.4\,\text{cm}}$$

This wavelength is in the microwave portion of the electromagnetic spectrum.

72 • A Rydberg atom is one in which an electron is in a *very* high excited state ($n \approx 40$ or higher). Such atoms are useful for experiments that probe the transition from quantum-mechanical behavior to classical. Furthermore, these excited states have extremely long lifetimes (i.e., the electron will stay in this high excited state for a very long time). A hydrogen atom is in the $n = 45$ state. (*a*) What is the ionization energy of the atom when it is in this state? (*b*) What is the energy level separation (in electron volts) between this state and the $n = 44$ state? (*c*) What is the wavelength of a photon resonant with this transition between these two states? (*d*) What is the radius of the atom when it is in the $n = 45$ state?

Picture the Problem The ionization energy of the electron is the magnitude of the energy of the atom in the given state. We can use $E = -E_0/n^2$, where E_0 is the ground-state energy, to find the energy levels in the 44th and 45th states and, hence, the energy level separation between the states. The wavelength of a photon resonant with this transition can be found from $\lambda = hc/\Delta E$. We'll approximate the size of the atom in the $n = 45$ state by finding the radius of the outer-shell electron.

(*a*) The energy of the atom in its *n*th state is:

$$E_n=-\frac{E_0}{n^2}$$

The energy of the atom in the $n = 45$ state is:

$$E_{45}=-\frac{13.6\,\text{eV}}{(45)^2}=-6.72\,\text{meV}$$

The ionization energy is the negative of the energy in the state $n = 45$:

$$E_{\text{ionizing}}=-E_{45}=\boxed{6.72\,\text{meV}}$$

(*b*) The energy level separation between the $n = 45$ and $n = 44$ state is:

$$E_{45\to44}=-\left(\frac{13.6\text{ eV}}{(45)^2}-\frac{13.6\text{ eV}}{(44)^2}\right)$$
$$=3.087\times10^{-4}\text{ eV}$$
$$=\boxed{3.09\times10^{-4}\text{ eV}}$$

(*c*) The photon wavelength is:

$$\lambda = \frac{hc}{\Delta E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\ \text{eV}\cdot\text{nm}}{3.087\times10^{-4}\ \text{eV}} = \boxed{4.02\ \text{mm}}$$

(*d*) The radii of the Bohr orbits are given by:

$$r = n^2\frac{a_0}{Z}$$

Substitute numerical values and evaluate the radius of the 45th Bohr orbit:

$$r = (45)^2\frac{0.0529\ \text{nm}}{1} = \boxed{107\ \text{nm}}$$

73 •• The deuteron, the nucleus of deuterium (heavy hydrogen), was first recognized from the spectrum of hydrogen. The deuteron has a mass that is approximately twice the mass of the proton. (*a*) Calculate the Rydberg constant for hydrogen and for deuterium using the reduced mass as given in Problem 29. (*b*) Using the result obtained in Part (*a*), determine the difference between the longest wavelength Balmer line of hydrogen protium and the longest wavelength Balmer line of deuterium.

Picture the Problem We can use the definition of the Rydberg constant and the equation for the reduced mass from Problem 29 to calculate the Rydberg constant for hydrogen and for deuterium. We can find the wavelength difference between the longest wavelength Balmer lines of hydrogen and deuterium by finding the longest wavelengths from the Rydberg-Ritz equation, using the appropriate value for R, and taking their difference.

(*a*) The Rydberg constant is given by:

$$R = \frac{m_r k^2 e^4}{4\pi c\hbar^3} = C\left(\frac{m_e}{1+\dfrac{m_e}{M}}\right)$$

where

$$C = \frac{k^2 e^4}{4\pi c\hbar^3} = 1.204662\times10^{37}\ \text{m}^{-1}/\text{kg}$$

For H:

$$R_H = C\left(\frac{m_e}{1+\dfrac{m_e}{m_p}}\right)$$

Substitute numerical values and evaluate R_H:

$$R_{\mathrm{H}} = \left(1.204662\times10^{37}\ \mathrm{m^{-1}/kg}\right)\times\left(\frac{9.109390\times10^{-31}\ \mathrm{kg}}{1+\dfrac{9.109390\times10^{-31}\ \mathrm{kg}}{1.672623\times10^{-27}\ \mathrm{kg}}}\right) = \boxed{1.096776\times10^{7}\ \mathrm{m^{-1}}}$$

For deuterium:

$$R_{\mathrm{D}} = C\left(\frac{m_{\mathrm{e}}}{1+\dfrac{m_{\mathrm{e}}}{2m_{\mathrm{p}}}}\right)$$

Substitute numerical values and evaluate R_D:

$$R_{\mathrm{D}} = \left(1.204662\times10^{37}\ \mathrm{m^{-1}/kg}\right)\left(\frac{9.109390\times10^{-31}\ \mathrm{kg}}{1+\dfrac{9.109390\times10^{-31}\ \mathrm{kg}}{2\left(1.672623\times10^{-27}\ \mathrm{kg}\right)}}\right) = \boxed{1.097075\times10^{7}\ \mathrm{m^{-1}}}$$

(*b*) Express the wavelength difference between the longest wavelength Balmer lines of hydrogen and deuterium:

$$\Delta\lambda = \lambda_{\text{longest, H}} - \lambda_{\text{longest, D}}$$

Use the Rydberg-Ritz formula to express the reciprocal wavelength:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right)$$

where n_1 and n_2 are integers and $n_1 > n_2$.

Solve for λ to obtain:

$$\lambda = \frac{n_1^2 n_2^2}{R\left(n_1^2-n_2^2\right)}$$

The longest wavelength in the Balmer series corresponds to a transition from $n_1 = 3$ to $n_2 = 2$. Use $R = R_H$ to evaluate $\lambda_{\text{longest, H}}$:

$$\lambda_{\text{longest, H}} = \frac{3^2\left(2^2\right)}{\left(1.096776\times10^{7}\ \mathrm{m^{-1}}\right)\left(3^2-2^2\right)} = 656.470\ \mathrm{nm}$$

Find $\lambda_{\text{longest,D}}$ using $R = R_{\text{D}}$:

$$\lambda_{\text{longest, D}} = \frac{3^2(2^2)}{(1.097075\times10^7\ \text{m}^{-1})(3^2-2^2)} = 656.291\,\text{nm}$$

Substitute numerical values and evaluate $\Delta\lambda$:

$$\Delta\lambda = 656.470\,\text{nm} - 656.291\,\text{nm} = \boxed{0.179\,\text{nm}}$$

74 •• The muonium atom is a hydrogen atom that has the electron replaced by a μ^- particle. The μ^- has a mass 207 times as great as the electron. (*a*) Calculate the energies of the five lowest energy levels of muonium using the reduced mass as given in Problem 29. (*b*) Do transitions between any of the levels found in Part (*a*) fall in the visible range of wavelengths, (for example, between λ = 700 nm and 400 nm)? If so, which transitions are these?

Picture the Problem We can use Problem 29 to express the energy levels of muonium in terms of the reduced mass of the muonium-proton system. In Part (*b*) we can find the energies corresponding to 400 nm and 700 nm to decide whether the transitions between any of the levels found in Part (*a*) fall in the visible range of wavelengths

(*a*) Express the energy of muonium as a function of the quantum number *n*:

$$E_n = -\frac{m_r k^2 e^4}{2\hbar^2}\frac{Z^2}{n^2} \qquad (1)$$

The reduced mass m_r is given by:

$$m_r = \frac{m_{\text{p}} m_{\mu^-}}{m_{\text{p}} + m_{\mu^-}}$$

Because $m_{\mu^-} = 207 m_{\text{e}}$:

$$m_r = \frac{207 m_{\text{p}} m_{\text{e}}}{m_{\text{p}} + 207 m_{\text{e}}} = \frac{207 m_{\text{e}}}{1 + 207\dfrac{m_{\text{e}}}{m_{\text{p}}}}$$

Because $m_{\text{p}} = 1836 m_{\text{e}}$:

$$m_r = \frac{207 m_{\text{e}}}{1 + \dfrac{207}{1836}} = 186 m_{\text{e}}$$

Substitute in equation (1) and simplify to obtain:

$$E_n = -\frac{186 m_{\text{e}} k^2 e^4}{2\hbar^2}\frac{1}{n^2} = -\frac{186 E_0}{n^2}$$

Evaluate E_n for n = 1, 2, 3, 4, and 5 to obtain:

n	E_n
	(keV)
1	−2.53
2	−0.633
3	−0.281
4	−0.158
5	−0.101

(*b*) Relate the wavelength of the emitted photons to the energy-level differences:

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{\Delta E}$$

Solving for ΔE yields:

$$\Delta E = \frac{1240\,\text{eV}\cdot\text{nm}}{\lambda}$$

Evaluate ΔE for λ = 400 nm and λ = 700 nm:

$$\Delta E_{400\,\text{nm}} = \frac{1240\,\text{eV}\cdot\text{nm}}{400\,\text{nm}} = 3.10\,\text{eV}$$

and

$$\Delta E_{700\,\text{nm}} = \frac{1240\,\text{eV}\cdot\text{nm}}{700\,\text{nm}} = 1.77\,\text{eV}$$

Because none of the energies in the table shown above are in the interval 1.77 eV to 3.10 eV, no transitions are in the visible range of wavelengths.

75 •• The triton, a nucleus consisting of a proton and two neutrons, is unstable and has a half-life of approximately 12 years. Tritium is an atom consisting of an electron and a triton. (*a*) Calculate the Rydberg constant of tritium using the reduced mass as given in Problem 29. (*b*) Determine the difference between the longest wavelength of the Balmer lines of tritium and the longest wavelengths of the Balmer lines of deuterium (see Problem 75). In addition, (*c*) determine the difference between the longest wavelength of the Balmer lines of tritium and the longest wavelengths of the Balmer lines of hydrogen protium.

Picture the Problem We can use the definition of the Rydberg constant and the equation for the reduced mass from Problem 29 to calculate the Rydberg constant for hydrogen, tritium, and deuterium. We can find the wavelength difference between the longest wavelength Balmer lines of tritium and deuterium and tritium and hydrogen by finding the longest wavelengths from the Rydberg-Ritz equation, using the appropriate value for R, and taking their difference.

(*a*) From Problem 29 we have:

$$R = \frac{m_r k^2 e^4}{4\pi c\hbar^3} = C\left(\frac{m_e}{1+\frac{m_e}{M}}\right)$$

where

$$C = \frac{k^2 e^4}{4\pi c\hbar^3} = 1.204662\times10^{37}\ \mathrm{m^{-1}/kg}$$

For tritium:

$$R_T = C\left(\frac{m_e}{1+\frac{m_e}{m_p+2m_n}}\right)$$

Substitute numerical values and evaluate R_T:

$$R_T = \left(1.204662\times10^{37}\ \mathrm{m^{-1}/kg}\right)\left(\frac{9.109382\times10^{-31}\ \mathrm{kg}}{1+\frac{9.109382\times10^{-31}\ \mathrm{kg}}{1.672622\times10^{-27}\ \mathrm{kg}+2\left(1.674927\times10^{-27}\ \mathrm{kg}\right)}}\right)$$

$$= \boxed{1.097174\times10^{7}\ \mathrm{m^{-1}}}$$

(*b*) Express the wavelength difference between the longest wavelength Balmer lines of hydrogen and deuterium:

$$\Delta\lambda = \lambda_{\mathrm{longest,\,D}} - \lambda_{\mathrm{longest,\,T}}$$

Use the Rydberg-Ritz formula to express the reciprocal wavelength:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

where n_1 and n_2 are integers and $n_1 > n_2$.

Solve for λ to obtain:

$$\lambda = \frac{n_1^2 n_2^2}{R\left(n_1^2 - n_2^2\right)}$$

The longest wavelength in the Balmer series corresponds to a transition from $n_1 = 3$ to $n_2 = 2$. Use $R = R_T$ to evaluate $\lambda_{\mathrm{longest,\,T}}$:

$$\lambda_{\mathrm{longest,\,T}} = \frac{3^2\left(2^2\right)}{\left(1.097174\times10^{7}\ \mathrm{m^{-1}}\right)\left(3^2 - 2^2\right)}$$

$$= 656.231\,\mathrm{nm}$$

For deuterium:

$$R_D = C\left(\frac{m_e}{1+\frac{m_e}{2m_p}}\right)$$

Substitute numerical values and evaluate R_D:

$$R_D = \left(1.204662\times10^{37}\ \text{m}^{-1}/\text{kg}\right)\left(\frac{9.109382\times10^{-31}\ \text{kg}}{1+\frac{9.109382\times10^{-31}\ \text{kg}}{2\left(1.672623\times10^{-27}\ \text{kg}\right)}}\right)$$
$$= 1.097075\times10^{7}\ \text{m}^{-1}$$

Find $\lambda_{\text{longest,D}}$ using $R = R_D$:

$$\lambda_{\text{longest, D}} = \frac{3^2\left(2^2\right)}{\left(1.097075\times10^{7}\ \text{m}^{-1}\right)\left(3^2-2^2\right)}$$
$$= 656.291\,\text{nm}$$

Substitute numerical values and evaluate $\Delta\lambda$:

$$\Delta\lambda = 656.291\,\text{nm} - 656.231\,\text{nm}$$
$$= \boxed{0.0600\,\text{nm}}$$

(*c*) Proceed similarly to show that for tritium and hydrogen protium:

$$\Delta\lambda = 656.4695\,\text{nm} - 656.2314\,\text{nm}$$
$$= \boxed{0.238\,\text{nm}}$$

Chapter 37
Molecules

Conceptual Problems

1 • [SSM] Would you expect NaCl to be polar or nonpolar?

Determine the Concept Yes. Because the center of charge of the positive Na ion does not coincide with the center of charge for the negative Cl ion, the NaCl molecule has a permanent dipole moment. Hence, it is a polar molecule.

2 • Would you expect N_2 to be polar or nonpolar?

Determine the Concept Because a N_2 molecule has no permanent dipole moment, it is a non-polar molecule.

3 • Does neon naturally occur as Ne or Ne_2? Explain your answer.

Determine the Concept No. Neon naturally occurs as Ne, not Ne_2. Neon is a rare gas atom with a closed shell electron configuration.

4 • What type of bonding mechanism would you expect for atoms of (*a*) HF, (*b*) KBr, (*c*) N_2, (*d*) Ag in solid silver?

Determine the Concept

(*a*) Because an electron is transferred from the H atom to the F atom, the bonding mechanism is ionic.

(*b*) Because an electron is transferred from the K atom to the Br atom, the bonding mechanism is ionic.

(*c*) Because the atoms share two electrons, the bonding mechanism is covalent.

(*d*) Because each valence electron is shared by many atoms, the bonding mechanism is metallic bonding.

5 •• [SSM] The elements on the far right column of the periodic table are sometimes called noble gases, both because they are gasses under a wide range of conditions, and because atoms of these elements almost never react with other atoms to form molecules or ionic compounds. However, atoms of noble gases can react if the resulting molecule is formed in an electronic excited state. An example is ArF. When it is formed in the excited state, it is written ArF* and is called an excimer (for <u>ex</u>cited di<u>mer</u>). Refer to Figure 37-13 and discuss how a

diagram for the electronic, vibrational, and rotation energy levels of ArF and ArF* would look in which the ArF ground state is unstable and the ArF* excited state is stable. *Remark: Excimers are used in certain kinds of lasers*.

Determine the Concept The diagram would consist of a non-bonding ground state with no vibrational or rotational states for ArF (similar to the upper curve in Figure 37-4) but for ArF* there should be a bonding excited state with a definite minimum with respect to inter-nuclear separation and several vibrational states as in the excited state curve of Figure 37-13.

6 • Find other atoms that have the same subshell electron configurations in their two highest energy orbitals as carbon atoms do. Would you expect the same type of hybridization for these orbitals as for carbon?

Determine the Concept Elements similar to carbon in outer shell configurations are silicon, germanium, tin, and lead. We would expect the same hybridization for these as for carbon, and this is indeed the case for silicon and germanium whose crystal structure is the diamond structure. Tin and lead, however, are metallic and here the metallic bond is dominant.

7 • How does the value of the effective force constant calculated for a CO molecule in Example 37-4 compare with the value of the force constant of the suspension springs on a typical automobile, which is about 1.5 kN/m?

Determine the Concept The effective force constant from Example 37-4 is 1.85×10^{3} N/m. This value is about 25% larger than the given value of the force constant of the suspension springs on a typical automobile.

8 • Explain why the moment of inertia of a diatomic molecule increases slightly with increasing angular momentum.

Determine the Concept As the angular momentum increases, the separation between the nuclei also increases (the effective force between the nuclei is similar to that of a stiff spring). Consequently, the moment of inertia also increases.

9 • Why would you expect the separation distance between the two protons to be larger in a H_2^+ ion than in a H_2 molecule?

Determine the Concept For H_2, the concentration of negative charge between the two protons holds the protons together. In a H_2^+ ion, there is only one electron that is shared by the two positive charges such that most of the electronic charge is again between the two protons. However, the negative charge in a H_2^+ ion is not as effective as the larger charge in a H_2 molecule, and the protons should be

farther apart. The experimental values support this argument. For H_2, r_0 = 0.074 nm, while for H_2^+, r_0 = 0.106 nm.

10 • At room temperature an atom typically absorb radiation only from the ground state, whereas a diatomic molecule typically absorbs radiation from many different rotational states. Why?

Determine the Concept The energy of the first excited state of an atom is orders of magnitude greater than kT at ordinary temperatures. Consequently, practically all atoms are in the ground state. By contrast, the energy separation between the ground rotational state and nearby higher rotational states is less than or roughly equal to kT at ordinary temperatures, and so these higher states are thermally excited and occupied.

11 •• The vibrational energy levels of diatomic molecules are described by a single vibrational frequency f that is the frequency of vibration of the two atoms of the molecule along the line through their centers. Would you expect to see one, or more than one, vibrational frequency in molecules that have three or more atoms? Consider in particular a water molecule H_2O (Figure 37-9).

Determine the Concept With more than two atoms in the molecule there will be more than just one frequency of vibration because there are more possible relative motions. In advanced mechanics these are known as normal modes of vibration.

Estimation and Approximation

12 •• The potential energy for a diatomic molecule has a minimum as shown in Figure 37-13. Near this minimum, the graph for the energy as a function of distance between the atoms may be approximated as a parabola, leading to the harmonic oscillator model for the vibrating molecule. An improved approximation is called the anharmonic oscillator and leads to a modification of the expression for the energy $E_\nu = \left(\nu + \frac{1}{2}\right)hf$, where ν = 0, 1, 2, …(Equation 37-18). The modified expression for energy is $E_\nu = \left(\nu + \frac{1}{2}\right)hf - \left(\nu + \frac{1}{2}\right)^2 hf\alpha$, where ν = 0, 1, 2, …. For a O_2 molecule, the constants have the values $f = 4.74 \times 10^{13}\ \text{s}^{-1}$ and $\alpha = 7.6 \times 10^{-3}$. Use this formula to estimate the smallest value of the quantum number ν for which the modified expression differs from the original expression by 10 percent.

Picture the Problem We can estimate the value of the quantum number ν for which the improved formula corrects the original formula by 10 percent by setting the ratio of the correction term to the first term equal to 10 percent and solving for ν.

Express the ratio of the correction term to the first term of the expression for E_ν and simplify to obtain:

$$\frac{\left(\nu+\frac{1}{2}\right)^2 hf\alpha}{\left(\nu+\frac{1}{2}\right)hf}=\left(\nu+\tfrac{1}{2}\right)\alpha$$

For a correction of 10 percent:

$$\left(\nu+\tfrac{1}{2}\right)\alpha=0.10\Rightarrow\nu=\frac{1}{10\alpha}-\frac{1}{2}$$

Substitute numerical values and evaluate ν:

$$\nu=\frac{1}{10\left(7.6\times10^{-3}\right)}-\frac{1}{2}=12.7\approx\boxed{13}$$

13 •• To understand why quantum mechanics is not needed to describe many macroscopic systems, estimate the rotational energy quantum number ℓ and spacing between adjacent energy levels for a baseball (m ~ 300 g, r ~ 3 cm) spinning about its own axis at 20 rev/min. *Hint: Pick* ℓ *so the quantum energy formula* $E_\ell=\ell(\ell+1)\hbar^2/(2I)$ *, where* ℓ = 0, 1, 2, ..., *(Equation 37-12) gives the correct energy for the given system. Then find the energy increase for the next highest energy level.*

Picture the Problem We can solve Equation 37-12 for ℓ and substitute for the moment of inertia and rotational kinetic energy of the baseball to estimate the quantum number ℓ and spacing between adjacent energy levels for a baseball spinning about its own axis.

The rotational energy levels are given by Equation 37-12:

$$E_\ell=\frac{\ell(\ell+1)\hbar^2}{2I}$$

where ℓ = 0, 1, 2, … is the rotational energy quantum number and I is the moment of inertia of the ball.

Solving for $\ell(\ell+1)$ yields:

$$\ell(\ell+1)=\frac{2IE_\ell}{\hbar^2}$$

Factoring ℓ from the parentheses gives:

$$\ell^2\left(1+\frac{1}{\ell}\right)=\frac{2IE_\ell}{\hbar^2}$$

The result of our calculation of ℓ will show that ℓ >> 1. Assuming for the moment that this is the case:

$$\ell^2\approx\frac{2IE_\ell}{\hbar^2}\Rightarrow\ell\approx\sqrt{\frac{2IE_\ell}{\hbar^2}}$$

Because the energy of the ball is rotational kinetic energy:

$$E_\ell=K_{\text{rot}}=\tfrac{1}{2}I\omega^2$$

Substitute for E_ℓ in the expression for ℓ and simplify to obtain:

$$\ell \approx \frac{\sqrt{2I\left(\frac{1}{2}I\omega^2\right)}}{\hbar} = \frac{\sqrt{\left(I^2\omega^2\right)}}{\hbar} = \frac{I\omega}{\hbar}$$

The moment of inertia of a ball about an axis through its diameter is (see Table 9-1):

$$I = \tfrac{2}{5}mr^2$$

Substituting for I gives:

$$\ell \approx \frac{2mr^2\omega}{5\hbar}$$

Substitute numerical values and evaluate ℓ:

$$\ell \approx \frac{2(0.300\,\text{kg})(3\,\text{cm})^2\left(\frac{20\,\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)}{5\left(1.055\times10^{-34}\,\text{J}\cdot\text{s}\right)} \approx \boxed{2\times10^{30}}$$

Set $\ell = 0$ to express the spacing between adjacent energy levels:

$$E_{0\text{r}} = \frac{\hbar^2}{2I} = \frac{5\hbar^2}{4mr^2}$$

Substitute numerical values and evaluate $E_{0\text{r}}$:

$$E_{0\text{r}} = \frac{5\left(1.055\times10^{-34}\,\text{J}\cdot\text{s}\right)^2}{4(300\,\text{g})(3.0\,\text{cm})^2} \approx \boxed{5\times10^{-65}\,\text{J}}$$

Remarks: Note that our value for ℓ justifies our assumption that ℓ >> 1.

14 •• Estimate the quantum number ν and spacing between adjacent energy levels for a 1.0-kg mass attached to spring. The spring has a force constant equal to 1200-N/m and the mass-spring system is vibrating with an amplitude of 3.0 cm. *Hint: Pick ν so that the quantum energy formula* $E_\nu = \left(\nu + \frac{1}{2}\right)hf$, *where* $\nu = 0, 1, 2, \ldots$ *(Equation 37-18) gives the correct energy for the given system. Then find the energy increase for the next highest energy level.*

Picture the Problem We can solve Equation 37-18 for ν and substitute for the frequency of the mass-and-spring oscillator to estimate the quantum number ν and spacing between adjacent energy levels for this system.

The vibrational energy levels are given by Equation 37-18:

$$E_\nu = \left(\nu + \tfrac{1}{2}\right)hf$$

where $\nu = 0, 1, 2, \ldots$

Solve for ν to obtain:

$$\nu = \frac{E_\nu}{hf} - \frac{1}{2}$$

or, because $\nu >> 1$,

$$\nu \approx \frac{E_\nu}{hf}$$

The vibrational energy of the object attached to the spring is:

$$E_\nu = \tfrac{1}{2}kA^2$$

where A is the amplitude of its motion.

Substitute for E_ν in the expression for ν to obtain:

$$\nu = \frac{kA^2}{2hf}$$

The frequency of oscillation f of the mass-and-spring oscillator is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute for f in the expression for ν and simplify to obtain:

$$\nu = \frac{\pi kA^2}{h}\sqrt{\frac{m}{k}} = \frac{\pi A^2}{h}\sqrt{mk}$$

Substitute numerical values and evaluate ν:

$$\nu = \frac{\pi(3.0\,\text{cm})^2}{6.626\times10^{-34}\ \text{J}\cdot\text{s}}\sqrt{(1.0\,\text{kg})(1200\,\text{N/m})} \approx \boxed{1\times10^{32}}$$

Set $\nu = 0$ in Equation 37-18 to express the spacing between adjacent energy levels:

$$E_{0\nu} = \frac{1}{2}hf = \frac{h}{4\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate $E_{0\nu}$:

$$E_{0\nu} = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{4\pi}\sqrt{\frac{1200\,\text{N/m}}{1.0\,\text{kg}}} = \boxed{1.8\times10^{-33}\ \text{J}}$$

Remarks: Note that our value for ν justifies our assumption that $\nu >> 1$.

Molecular Bonding

15 • Calculate the separation of of Na^+ and Cl^- ions for which the potential energy of a single ionic unit (one Na^+ ion and one Cl^- ion) is –1.52 eV.

Picture the Problem The electrostatic potential energy with U_e with ions separated a distance r is given by $U = -ke^2/r$.

Relate the electrostatic potential energy of the ions to their separation:

$$U_e = -\frac{ke^2}{r} \Rightarrow r = -\frac{ke^2}{U_e}$$

Substitute numerical values and evaluate r:

$$r = -\frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{\left(-1.52\,\text{eV}\right)\left(1.602\times10^{-19}\ \frac{\text{J}}{\text{eV}}\right)} = \boxed{0.947\,\text{nm}}$$

16 • The equilibrium separation of the atoms in a HF molecule is 0.0917 nm and the measured electric dipole moment of the molecule is 6.40×10^{-30} C·m. What percentage of the HF bond is ionic?

Picture the Problem The percentage of the bonding that is ionic is given by $100\left(p_{\text{meas}}/p_{100}\right)$.

Express the percentage of the bonding that is ionic:

$$\text{Percent}_{\text{ ionic bonding}} = 100\left(\frac{p_{\text{meas}}}{p_{100}}\right)$$

Express the dipole moment for 100% ionic bonding:

$$p_{100} = er$$

Substituting for p_{100} gives:

$$\text{Percent}_{\text{ ionic bonding}} = 100\left(\frac{p_{\text{meas}}}{er}\right)$$

Substitute numerical values and evaluate the percent ionic bonding:

$$\text{Percent}_{\text{ ionic bonding}} = 100\left[\frac{6.40\times10^{-30}\ \text{C}\cdot\text{m}}{\left(1.602\times10^{-19}\ \text{C}\right)\left(0.0917\,\text{nm}\right)}\right] = \boxed{43.6\%}$$

17 •• The dissociation energy of RbF is 5.12 eV, and the equilibrium separation of RbF is 0.227 nm. The electron affinity of a fluorine atom is –3.40 eV and the ionization energy of rubidium is 4.18 eV. Determine the core-repulsion energy of RbF.

Picture the Problem If we choose the potential energy at infinity to be ΔE, the total potential energy is $U_{tot} = U_e + \Delta E + U_{rep}$, where U_{rep} is the energy of repulsion, which is found by setting the dissociation energy equal to $-U_{tot}$.

Express the total potential energy of the molecule:

$$U_{tot} = U_e + \Delta E + U_{rep}$$

The core-repulsive energy is :

$$U_{rep} = -(\Delta E + U_e + E_d) \qquad (1)$$

Calculate the energy ΔE needed to form Rb^+ and F^- ions from neutral rubidium and fluorine atoms:

$$\Delta E = 4.18\,\text{eV} - 3.40\,\text{eV} = 0.78\,\text{eV}$$

The electrostatic potential energy is given by:

$$U_e = -\frac{ke^2}{r}$$

Substitute numerical values and evaluate U_e:

$$U_e = -\frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(1.602\times10^{-19}\ \text{C})^2}{(0.227\,\text{nm})\left(1.602\times10^{-19}\ \frac{\text{J}}{\text{eV}}\right)} = -6.34\,\text{eV}$$

Substitute numerical values in equation (1) and evaluate U_{rep}:

$$U_{rep} = -(0.78\,\text{eV} - 6.34\,\text{eV} + 5.12\,\text{eV}) = \boxed{0.44\,\text{eV}}$$

18 •• The equilibrium separation of the K^+ and Cl^- ions in KCl is about 0.267 nm. (*a*) Calculate the potential energy of attraction of the ions. Assume that the ions are point charges at this separation. (*b*) The ionization energy of potassium is 4.34 eV and the electron affinity of chlorine is –3.62 eV. Calculate a value for the dissociation energy under the assumption that the energy of repulsion is negligible. (See Figure 37-1) (*c*) The measured dissociation energy is 4.49 eV. What is the energy due to repulsion of the ions at the equilibrium separation?

Picture the Problem The potential energy of attraction of the ions is $U_e = -ke^2/r$. We can find the dissociation energy from the negative of the sum of the potential energy of attraction and the difference between the ionization energy of potassium and the electron affinity of chlorine.

(a) The potential energy of attraction of the ions is given by:

$$U_e = -\frac{ke^2}{r}$$

where $ke^2 = 1.44$ eV·nm

Substitute numerical values and evaluate U_e:

$$U_e = -\frac{1.44\,\text{eV}\cdot\text{nm}}{0.267\,\text{nm}} = \boxed{-5.39\,\text{eV}}$$

(b) Express the total potential energy of the molecule:

$$U_{tot} = U_e + \Delta E + U_{rep}$$

or, neglecting any energy of repulsion,

$$U_{tot} = U_e + \Delta E$$

The dissociation energy is the negative of the total potential energy:

$$E_{d,calc} = -U_{tot} = -(U_e + \Delta E)$$

ΔE is the difference between the ionization energy of potassium and the electron affinity of Cl:

$$\Delta E = 4.34\,\text{eV} - 3.62\,\text{eV} = 0.72\,\text{eV}$$

Substitute numerical values and evaluate $E_{d,calc}$:

$$E_{d,calc} = -(-5.39\,\text{eV} + 0.72\,\text{eV}) = \boxed{4.67\,\text{eV}}$$

(c) The energy due to repulsion of the ions at equilibrium separation is given by:

$$U_{rep} = E_{d,calc} - E_{d,meas}$$

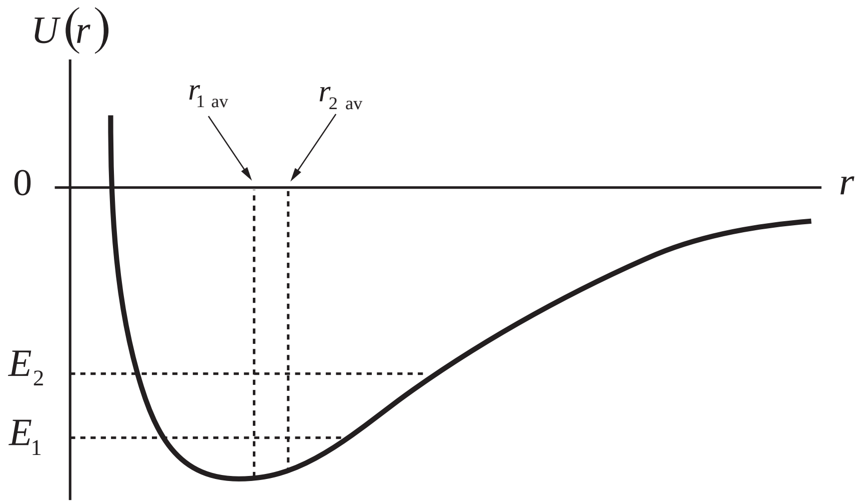
Substitute numerical values and evaluate U_{rep}:

$$U_{rep} = 4.67\,\text{eV} - 4.49\,\text{eV} = \boxed{0.18\,\text{eV}}$$

19 •• Indicate an approximate value for the average value of the separation distance r for two vibrational levels on the potential energy curve for a diatomic molecule (one of the curves in Figure 37-13). Your teacher claims that the increase in r_{av} with increases in vibration energy explains why solids expand when heated. Do you agree? If so, give an argument supporting this claim. If not, give an argument opposing this claim.

Picture the Problem You should agree. The potential energy curve is shown in the following diagram. The turning points for vibrations of energy E_1 and E_2 are at the values of r where the energies equal $U(r)$. The average value of r for the vibrational levels E_1 and E_2 are labeled $r_{1\,av}$ and $r_{2\,av}$. Note that the estimate of $r_{1\,av}$ is midway between $r_{1\,min}$ and $r_{1\,max}$. The potential is like a special spring that has a

greater spring constant for compressions than it has for extensions. The period of a spring-and-mass oscillator is inversely proportional to the square root of the spring constant, so our ″special spring″ spends more time in extension than in compression. As a result, $r_{1\,\text{av}}$ will be greater than the equilibrium radius.



This argument can be extended to explain why $r_{2\,\text{av}}$ is greater than $r_{1\text{av}}$. It is because the ″spring constant″ for extension, which can be estimated by taking the average slope of the potential energy curve in the region to the right of the equilibrium position, is greater for $E = E_2$ than for $E = E_1$. It is also because the ″spring constant″ for compression is greater for $E = E_2$ than for $E = E_1$. It follows that $r_{2\,\text{av}}$ is greater than $r_{1\,\text{av}}$. Because $r_{2,\text{av}}$ is greater than $r_{1\,\text{av}}$, it follows that as the vibrational energy of a diatomic molecule increases, the average separation of the atoms of the molecule increases and, hence, the solid expands with heating.

20 •• Calculate the potential energy of attraction between the Na^+ and Cl^- ions at the equilibrium separation $r_0 = 0.236$ nm. Compare this result with the dissociation energy given in Figure 37-1. What is the energy due to repulsion of the ions at the equilibrium separation?

Picture the Problem We can use $U_\text{e} = -ke^2/r_0$ to calculate the potential energy of attraction between the Na^+ and Cl^- ions at the equilibrium separation $r_0 = 0.236$ nm. We can find the energy due to repulsion of the ions at the equilibrium separation from $U_\text{rep} = -(U_\text{e} + E_\text{d} + \Delta E)$.

The potential energy of attraction between the Na^+ and Cl^- ions at the equilibrium separation r_0 is given by:

$$U_\text{e} = -\frac{ke^2}{r_0}$$

where $ke^2 = 1.44$ eV·nm.

Substitute numerical values and evaluate U_e:

$$U_e = -\frac{1.44\,\text{eV}\cdot\text{nm}}{0.236\,\text{nm}} = \boxed{-6.10\,\text{eV}}$$

From Figure 37-1:

$$E_d = 4.27\,\text{eV}$$

The ratio of the magnitude of the potential energy of attraction to the dissociation energy is:

$$\frac{|U_e|}{E_d} = \frac{6.10\,\text{eV}}{4.27\,\text{eV}} = \boxed{1.43}$$

U_{rep} is related to U_e, E_d, and ΔE according to:

$$U_{rep} = -(U_e + E_d + \Delta E)$$

From Figure 37-1:

$$\Delta E = 1.52\,\text{eV}$$

Substitute numerical values and evaluate U_{rep}:

$$U_{rep} = -(-6.10\,\text{eV} + 4.27\,\text{eV} + 1.52\,\text{eV}) = \boxed{0.31\text{eV}}$$

21 •• The equilibrium separation of the K^+ and F^- ions in KF is about 0.217 nm. (*a*) Calculate the potential energy of attraction of these ions. Assume that the ions are point charges at this separation. (*b*) The ionization energy of potassium is 4.34 eV and the electron affinity of fluorine is –3.40 eV. Find the dissociation energy by neglecting any energy of repulsion. (*c*) The measured dissociation energy is 5.07 eV. Calculate the energy due to repulsion of the ions at the equilibrium separation.

Picture the Problem The potential energy of attraction of the ions is $U_e = -ke^2/r$. We can find the dissociation energy from the negative of the sum of the potential energy of attraction and the difference between the ionization energy of potassium and the electron affinity of fluorine.

(*a*) The potential energy of attraction between the K^+ and F^- ions at the equilibrium separation r_0 is given by:

$$U_e = -\frac{ke^2}{r_0}$$

where ke^2 = 1.44 eV·nm.

Substitute numerical values and evaluate U_e:

$$U_e = -\frac{1.44\,\text{eV}\cdot\text{nm}}{0.217\,\text{nm}} = \boxed{-6.64\,\text{eV}}$$

(*b*) Express the total potential energy of the molecule:

$$U_{\text{tot}} = U_{\text{e}} + \Delta E + U_{\text{rep}}$$

or, neglecting any energy of repulsion,

$$U_{\text{tot}} = U_{\text{e}} + \Delta E$$

The dissociation energy is the negative of the total potential energy:

$$E_{\text{d,calc}} = -U_{\text{tot}} = -\left(U_{\text{e}} + \Delta E\right)$$

ΔE is the difference between the ionization energy of potassium and the electron affinity of fluorine:

$$\Delta E = 4.34\,\text{eV} - 3.40\,\text{eV} = 0.94\,\text{eV}$$

Substitute numerical values and evaluate $E_{\text{d,calc}}$:

$$E_{\text{d,calc}} = -\left(-6.64\,\text{eV} + 0.94\,\text{eV}\right) = \boxed{5.70\,\text{eV}}$$

(*c*) The energy due to repulsion of the ions at equilibrium separation is given by:

$$U_{\text{rep}} = E_{\text{d,calc}} - E_{\text{d,meas}}$$

Substitute numerical values and evaluate U_{rep}:

$$U_{\text{rep}} = 5.70\,\text{eV} - 5.07\,\text{eV} = \boxed{0.63\,\text{eV}}$$

Energy Levels of Spectra of Diatomic Molecules

22 • The characteristic rotational energy $E_{0\text{r}}$ for the rotation of a N_2 molecule is 2.48×10^{-4} eV. Using this value, find the separation distance of the 2 nitrogen atoms.

Picture the Problem We can relate the characteristic rotational energy $E_{0\text{r}}$ to the moment of inertia of the molecule and model the moment of inertia of the N_2 molecule as two point objects separated by a distance r.

The characteristic rotational energy of a molecule is given by:

$$E_{0r} = \frac{\hbar^2}{2I}$$

The moment of inertia of the molecule is given by:

$$I = 2M_{\text{N}}\left(\frac{r}{2}\right)^2 = \tfrac{1}{2}M_{\text{N}}r^2$$

Substitute for I to obtain:

$$E_{0r} = \frac{\hbar^2}{2\left(\frac{1}{2}M_{\text{N}}r^2\right)} = \frac{\hbar^2}{M_{\text{N}}r^2} = \frac{\hbar^2}{14m_{\text{p}}r^2}$$

Solving for r gives:

$$r = \hbar\sqrt{\frac{1}{14E_{0r}m_{\mathrm{p}}}}$$

Substitute numerical values and evaluate r:

$$r = \left(1.055\times10^{-34}\ \mathrm{J\cdot s}\right)\sqrt{\frac{1}{14\left(2.48\times10^{-4}\ \mathrm{eV}\right)\left(1.602\times10^{-19}\ \mathrm{J/eV}\right)\left(1.673\times10^{-27}\ \mathrm{kg}\right)}}$$
$$= \boxed{0.109\,\mathrm{nm}}$$

23 • [SSM] The separation of the two oxygen atoms in a molecule of O_2 is actually slightly greater than the 0.100 nm used in Example 37-3. Furthermore, the characteristic energy of rotation E_{0r} for O_2 is 1.78×10^{-4} eV rather than the result obtained in that example. Use this value to calculate the separation distance of the two oxygen atoms.

Picture the Problem We can relate the characteristic rotational energy E_{0r} to the moment of inertia of the molecule and model the moment of inertia of the O_2 molecule as two point objects separated by a distance r.

The characteristic rotational energy of a molecule is given by:

$$E_{0r} = \frac{\hbar^2}{2I}$$

The moment of inertia of the molecule is given by:

$$I = 2M_{\mathrm{O}}\left(\frac{r}{2}\right)^2 = \tfrac{1}{2}M_{\mathrm{O}}r^2$$

Substitute for I to obtain:

$$E_{0r} = \frac{\hbar^2}{2\left(\frac{1}{2}M_{\mathrm{O}}r^2\right)} = \frac{\hbar^2}{M_{\mathrm{O}}r^2} = \frac{\hbar^2}{16m_{\mathrm{p}}r^2}$$

Solving for r yields:

$$r = \frac{\hbar}{4}\sqrt{\frac{1}{E_{0r}m_{\mathrm{p}}}}$$

Substitute numerical values and evaluate r:

$$r = \left(\frac{1.055\times10^{-34}\ \mathrm{J\cdot s}}{4}\right)\sqrt{\frac{1}{\left(1.78\times10^{-4}\ \mathrm{eV}\right)\left(1.602\times10^{-19}\ \mathrm{J/eV}\right)\left(1.673\times10^{-27}\ \mathrm{kg}\right)}}$$
$$= \boxed{0.121\,\mathrm{nm}}$$

24 •• Show that the reduced mass of a diatomic molecule is always smaller than the mass of the molecule. Calculate the reduced mass for (*a*) H_2, (*b*) N_2, (*c*) CO, and (*d*) HCl. Express your answers in unified atomic mass units.

Picture the Problem We can use the definition of the reduced mass to show that the reduced mass is smaller than either mass in a diatomic molecule.

Express the reduced mass of a two-body system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Divide the numerator and denominator of this expression by m_2 to obtain:

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} \qquad (1)$$

Divide the numerator and denominator of this expression by m_1 to obtain:

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} \qquad (2)$$

Because the denominator is greater than 1 in equations (1) and (2):

$$\boxed{\mu < m_1} \text{ and } \boxed{\mu < m_2}$$

(*a*) For H_2, $m_1 = m_2 = 1$ u:

$$\mu_{H_2} = \frac{(1\,\text{u})(1\,\text{u})}{1\,\text{u} + 1\,\text{u}} = \boxed{0.500\,\text{u}}$$

(*b*) For N_2, $m_1 = m_2 = 14$ u:

$$\mu_{N_2} = \frac{(14\,\text{u})(14\,\text{u})}{14\,\text{u} + 14\,\text{u}} = \boxed{7.00\,\text{u}}$$

(*c*) For CO, $m_1 = 12$ u and $m_2 = 16$ u:

$$\mu_{CO} = \frac{(12\,\text{u})(16\,\text{u})}{12\,\text{u} + 16\,\text{u}} = \boxed{6.86\,\text{u}}$$

(*d*) For HCl, $m_1 = 1$ u and $m_2 = 35.5$ u:

$$\mu_{HCl} = \frac{(1\,\text{u})(35.5\,\text{u})}{1\,\text{u} + 35.5\,\text{u}} = \boxed{0.973\,\text{u}}$$

25 •• A CO molecule has a binding energy of approximately 11 eV. Find the vibrational quantum number ν that corresponds to 11 eV. (If a CO molecule actually had this much vibrational energy, it would ″shake″ apart.)

Picture the Problem We can solve Equation 37-18 for ν and substitute for the frequency of the CO molecule (see Example 37-4) and its binding energy to estimate the quantum number ν.

The vibrational energy levels are given by Equation 37-18:

$$E_\nu = \left(\nu + \tfrac{1}{2}\right)hf \Rightarrow \nu = \frac{E_\nu}{hf} - \frac{1}{2}$$

where $\nu = 0, 1, 2, \ldots, k$

Substitute numerical values and evaluate ν:

$$\nu = \frac{11\,\text{eV} \times \dfrac{1.602\times10^{-19}\ \text{J}}{\text{eV}}}{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(6.42\times10^{13}\ \text{Hz}\right)} - \frac{1}{2}$$

$$\approx \boxed{41}$$

26 •• Derive the equation $I = \mu r_0^2$ (Equation 37-14) for the moment of inertia in terms of the reduced mass of a diatomic molecule.

Picture the Problem Let the origin of coordinates be at the point mass m_1 and point mass m_2 be at a distance r_0 from the origin. We can express the moment of inertia of a diatomic molecule with respect to its center of mass using the definitions of the center of mass and the moment of inertia of point particles.

Express the moment of inertia of a diatomic molecule:

$$I = m_1 r_1^2 + m_2 r_2^2 \quad (1)$$

The r coordinate of the center of mass is:

$$r_{\text{CM}} = \frac{m_2}{m_1 + m_2} r_0$$

The distances of m_1 and m_2 from the center of mass are:

$$r_1 = r_{\text{CM}}$$

and

$$r_2 = r_0 - r_{\text{CM}} = r_0 - \frac{m_2}{m_1 + m_2} r_0$$

$$= \frac{m_1}{m_1 + m_2} r_0$$

Substitute for r_1 and r_2 in equation (1) to obtain:

$$I = m_1 \left(\frac{m_2}{m_1 + m_2} r_0\right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} r_0\right)^2$$

Simplifying this expression leads to:

$$I = \frac{m_1 m_2}{m_1 + m_2} r_0^2 \Rightarrow \boxed{I = \mu r_0^2} \quad \text{37-14}$$

where $\boxed{\mu = \dfrac{m_1 m_2}{m_1 + m_2}}$

27 •• [SSM] The equilibrium separation between the atoms of a LiH molecule is 0.16 nm. Determine the energy separation between the $\ell = 3$ and $\ell = 2$ rotational levels of this diatomic molecule.

Picture the Problem We can use the expression for the rotational energy levels of the diatomic molecule to express the energy separation ΔE between the $\ell = 3$ and $\ell = 2$ rotational levels and model the moment of inertia of the LiH molecule as two point objects separated by a distance r_0.

The energy separation between the $\ell = 3$ and $\ell = 2$ rotational levels of this diatomic molecule is given by:

$$\Delta E = E_{\ell=3} - E_{\ell=2}$$

Express the rotational energy levels $E_{\ell=3}$ and $E_{\ell=2}$ in terms of E_{0r}:

$$E_{\ell=3} = 3(3+1)E_{0r} = 12E_{0r}$$

and

$$E_{\ell=2} = 2(2+1)E_{0r} = 6E_{0r}$$

Substitute for $E_{\ell=3}$ and $E_{\ell=2}$ to obtain:

$$\Delta E = 12E_{0r} - 6E_{0r} = 6E_{0r}$$

The characteristic rotational energy of a molecule is given by $E_{0r} = \dfrac{\hbar^2}{2I}$. Hence:

$$\Delta E = 6\left(\frac{\hbar^2}{2I}\right) = \frac{3\hbar^2}{I}$$

The moment of inertia of the molecule is given by:

$$I = \mu r_0^2$$

where μ is the reduced mass of the molecule.

Substituting for I yields:

$$\Delta E = \frac{3\hbar^2}{\mu r_0^2} = \frac{3\hbar^2}{\dfrac{m_{\mathrm{Li}} m_{\mathrm{H}}}{m_{\mathrm{Li}} + m_{\mathrm{H}}} r_0^2} = \frac{3\hbar^2 (m_{\mathrm{Li}} + m_{\mathrm{H}})}{m_{\mathrm{Li}} m_{\mathrm{H}}\, r_0^2}$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \frac{3(1.055\times10^{-34}\ \mathrm{J\cdot s})^2(6.94\,\mathrm{u} + 1\,\mathrm{u})}{(6.94\,\mathrm{u})(1\,\mathrm{u})(0.16\,\mathrm{nm})^2(1.602\times10^{-19}\ \mathrm{J/eV})(1.661\times10^{-27}\ \mathrm{kg/u})} = \boxed{5.6\,\mathrm{meV}}$$

28 •• The equilibrium separation of the K^+ and Cl^- ions in KCl is about 0.267 nm. Use this value together with the reduced mass of KCl to calculate the characteristic rotational energy E_{0r} (Equation 37-13) of KCl.

Picture the Problem We can relate the characteristic rotational energy E_{0r} to the moment of inertia of the molecule and model the moment of inertia of the KCl molecule as two point objects of reduced mass μ separated by a distance r.

The characteristic rotational energy of a molecule is given by:

$$E_{0r} = \frac{\hbar^2}{2I}$$

Express the moment of inertia of the molecule:

$$I = \mu r_0^2$$

where $\mu = \dfrac{m_K m_{Cl}}{m_K + m_{Cl}}$

Substituting for I and simplifying gives:

$$E_{0r} = \frac{\hbar^2}{2\mu r_0^2} = \frac{\hbar^2 (m_K + m_{Cl})}{2m_k m_{Cl} r_0^2}$$

Substitute numerical values and evaluate E_{0r}:

$$\begin{aligned} E_{0r} &= \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2 (39.1\,\text{u} + 35.5\,\text{u})}{2(39.1\text{u})(35.5\,\text{u})(0.267\,\text{nm})^2 (1.661\times10^{-27}\ \text{kg/u})} \\ &= 2.53\times10^{-24}\ \text{J}\times\frac{1}{1.602\times10^{-19}\ \text{J/eV}} = \boxed{0.0158\,\text{meV}} \end{aligned}$$

29 •• The central frequency for the absorption band of HCl shown in Figure 37-14 is at 8.66×10^{13} Hz, and the absorption peaks to either side of the central frequency are separated by about 6×10^{11} Hz. Use this information to find (*a*) the lowest (zero-point) vibrational energy for HCl, (*b*) the moment of inertia of HCl, and (*c*) the equilibrium separation of the two atoms.

Picture the Problem We can use the expression for the vibrational energies of a molecule to find the lowest vibrational energy. Because the difference in the vibrational energy levels depends on both Δf and the moment of inertia I of the molecule, we can relate these quantities and solve for I. Finally, we can use $I = \mu r^2$, with μ representing the reduced mass of the molecule, to find the equilibrium separation of the atoms.

(*a*) The vibrational energy levels are given by:

$$E_\nu = \left(\nu + \tfrac{1}{2}\right)hf, \quad \nu = 0,1,2,\ldots$$

The lowest vibrational energy corresponds to $\nu = 0$:

$$E_0 = \tfrac{1}{2}hf$$

Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(8.66\times10^{13}\ \text{Hz}\right) = 2.87\times10^{-20}\ \text{J}\times\frac{1}{1.602\times10^{-19}\ \text{J/eV}}$$

$$= \boxed{0.179\,\text{eV}}$$

(*b*) For $\Delta\ell = \pm1$:

$$\Delta E_\ell = \frac{\ell\hbar^2}{I} = \ell h\Delta f$$

Solving for I and simplifying gives:

$$I = \frac{\hbar^2}{h\Delta f} = \frac{h^2}{4\pi^2 h\Delta f} = \frac{h}{4\pi^2\Delta f}$$

Substitute numerical values and evaluate I:

$$I = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{4\pi^2\left(6\times10^{11}\ \text{Hz}\right)}$$

$$= 2.80\times10^{-47}\ \text{kg}\cdot\text{m}^2$$

$$\approx \boxed{3\times10^{-47}\ \text{kg}\cdot\text{m}^2}$$

(*c*) The moment of inertia of a HCL molecule is given by:

$$I = \mu r^2$$

Replace μ by the reduced mass of a HCl molecule and r by r_0 to obtain:

$$I = \frac{m_\text{H}m_\text{Cl}}{m_\text{H}+m_\text{Cl}}r_0^2 \Rightarrow r_0 = \sqrt{\frac{m_\text{H}+m_\text{Cl}}{m_\text{H}m_\text{Cl}}I}$$

Substitute numerical values and evaluate r_0:

$$r_0 = \sqrt{\left[\frac{1\text{u}+34.453\text{u}}{(1\text{u})(34.453\text{u})}\right]\frac{\left(2.80\times10^{-47}\ \text{kg}\cdot\text{m}^2\right)}{1.661\times10^{-27}\ \text{kg/u}}} = \boxed{0.1\text{nm}}$$

30 •• Calculate the effective force constant for HCl from its reduced mass and from the fundamental vibrational frequency obtained from Figure 37-14.

Picture the Problem Let the numeral 1 refer to the H^+ and the numeral 2 to the Cl^- ion. For a two-mass and spring system on which no external forces are acting, the center of mass must remain fixed. We can use this condition to express the net force acting on either the H^+ or Cl^- ion. Because this force is a linear restoring

force, we can conclude that the motion of the object whose mass is m_1 will be simple harmonic with an angular frequency given by $\omega = \sqrt{k_{\text{eff}}/m_1}$.

If the particle whose mass is m_1 moves a distance r_1 from (or toward) the center of mass, then the particle whose mass is m_2 must move a distance:

$\Delta r_2 = \frac{m_1}{m_2}\Delta r_1$ from (or toward) the center of mass.

Express the force exerted by the spring:

$$F = -k\Delta r = -k(\Delta r_1 + \Delta r_2)$$

Substitute for Δr_2 to obtain:

$$F = -k\left(\Delta r_1 + \frac{m_1}{m_2}\Delta r_1\right) = -k\left(\frac{m_1 + m_2}{m_2}\right)\Delta r_1$$

A displacement Δr_1 of m_1 results in a restoring force:

$$F = -k\left(\frac{m_1 + m_2}{m_2}\right)\Delta r_1 = -k_{\text{eff}}\Delta r_1$$

where $k_{\text{eff}} = k\left(\frac{m_1 + m_2}{m_2}\right)$

Because this is a linear restoring force, we know that the motion will be simple harmonic with the angular frequency given by:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_1}} \Rightarrow 2\pi f = \sqrt{\frac{k_{\text{eff}}}{m_1}}$$

Substitute for k_{eff} and simplify to obtain:

$$2\pi f = \sqrt{k\left(\frac{m_1 + m_2}{m_1 m_2}\right)}$$

or, because $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_{\text{H}} m_{\text{Cl}}}{m_{\text{H}} + m_{\text{Cl}}}$ is the reduced mass of the two-particle system, $2\pi f = \sqrt{\frac{k}{\mu}}$

Solving for k yields:

$$k = 4\pi^2 f^2 \mu = 4\pi^2 f^2 \frac{m_{\text{H}} m_{\text{Cl}}}{m_{\text{H}} + m_{\text{Cl}}}$$

Substitute numerical values and evaluate k:

$$k = \frac{4\pi^2\left(8.66\times10^{13}\text{ Hz}\right)^2(1\text{u})(35.453\text{u})\left(1.661\times10^{-27}\text{ kg/u}\right)}{(1\text{u}+35.453\text{u})} = \boxed{478\,\text{N/m}}$$

31 •• [SSM] The equilibrium separation between the atoms of a CO molecule is 0.113 nm. For a molecule, such as CO, that has a permanent electric dipole moment, radiative transitions obeying the selection rule $\Delta\ell = \pm1$ between two rotational energy levels of the same vibrational level are allowed. (That is, the selection rule $\Delta\nu = \pm1$ does not hold.) (*a*) Find the moment of inertia of CO and calculate the characteristic rotational energy E_{0r} (in eV). (*b*) Make an energy-level diagram for the rotational levels from $\ell = 0$ to $\ell = 5$ for some vibrational level. Label the energies in electron volts, starting with $E = 0$ for $\ell = 0$. Indicate on your diagram the transitions that obey $\Delta\ell = -1$, and calculate the energies of the photons emitted. (*c*) Find the wavelength of the photons emitted during each transition in (*b*). In what region of the electromagnetic spectrum are these photons?

Picture the Problem We can find the reduced mass of CO and the moment of inertia of a CO molecule from their definitions. The energy level diagram for the rotational levels from $\ell = 0$ to $\ell = 5$ can be found using $\Delta E_{\ell,\ell-1} = 2\ell E_{0r}$. Finally, we can find the wavelength of the photons emitted for each transition using $\lambda_{\ell,\ell-1} = \dfrac{hc}{\Delta E_{\ell,\ell-1}} = \dfrac{hc}{2\ell\Delta E_{0r}}$.

(*a*) Express the moment of inertia of CO:

$$I = \mu r_0^2$$

where μ is the reduced mass of the CO molecule.

The reduced mass of the CO molecule is given by:

$$\mu = \frac{m_C m_O}{m_C + m_O}$$

Substituting for μ gives:

$$I = \frac{m_C m_O r_0^2}{m_C + m_O}$$

Substitute numerical values (r_0 = 0.113 nm) and evaluate I:

$$I = \left(\frac{(12\,\text{u})(16\,\text{u})}{12\,\text{u}+16\,\text{u}}\right)(1.661\times10^{-27}\ \text{kg/u})(0.113\,\text{nm})^2 = 1.454\times10^{-46}\ \text{kg}\cdot\text{m}^2$$
$$= \boxed{1.45\times10^{-46}\ \text{kg}\cdot\text{m}^2}$$

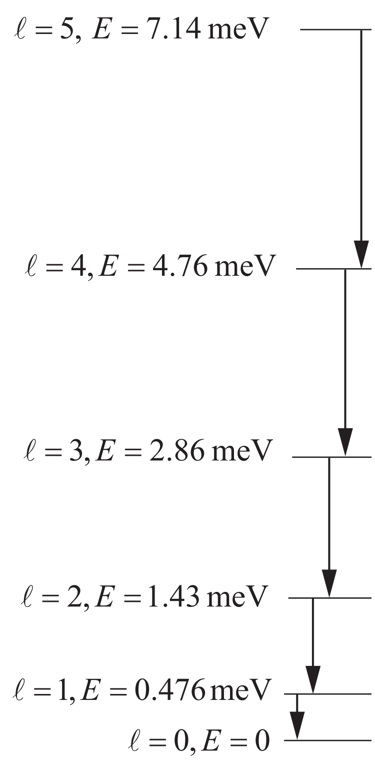
The characteristic rotational energy E_{0r} is given by:

$$E_{0r} = \frac{\hbar^2}{2I}$$

Substitute numerical values and evaluate E_{0r}:

$$E_{0r} = \frac{(6.58\times10^{-16}\ \text{eV}\cdot\text{s})^2(1.602\times10^{-19}\ \text{J/eV})}{2(1.454\times10^{-46}\ \text{kg}\cdot\text{m}^2)} = 0.2385\,\text{meV} = \boxed{0.239\,\text{meV}}$$

(*b*) The energy level diagram is shown to the right. Note that $\Delta E_{\ell,\ell-1}$, the energy difference between adjacent levels for $\Delta\ell = -1$, is $\Delta E_{\ell,\ell-1} = 2\ell E_{0r}$.



The energies of the photons emitted in each transition are given by $\Delta E_{\ell,\ell-1}$. Hence:

$$\Delta E_{5,4} = 7.14\,\text{meV} - 4.76\,\text{meV} = \boxed{2.38\,\text{meV}}$$

$$\Delta E_{4,3} = 4.76\,\text{meV} - 2.86\,\text{meV} = \boxed{1.90\,\text{meV}}$$

$$\Delta E_{3,2} = 2.86\,\text{meV} - 1.43\,\text{meV} = \boxed{1.43\,\text{meV}}$$

$$\Delta E_{2,1} = 1.43\,\text{meV} - 0.476\,\text{meV} = \boxed{1.25\,\text{meV}}$$

$$\Delta E_{1,0} = 0.476\,\text{meV} - 0 = \boxed{0.476\,\text{meV}}$$

(*c*) Express the energy difference $\Delta E_{\ell.\ell-1}$ between energy levels in terms of the frequency of the emitted radiation:

$$\Delta E_{\ell,\ell-1} = hf_{\ell,\ell-1}$$

Because $c = f_{\ell,\ell-1}\lambda_{\ell,\ell-1}$:

$$\lambda_{\ell,\ell-1} = \frac{hc}{\Delta E_{\ell,\ell-1}} = \frac{hc}{2\ell\Delta E_{0r}}$$

Substitute numerical values to obtain:

$$\lambda_{\ell,\ell-1} = \frac{(4.136\times10^{-15}\ \text{eV}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{2\ell(0.2385\,\text{meV})} = \frac{2600\,\mu\text{m}}{\ell}$$

For ℓ = 1:

$$\lambda_{1,0} = \frac{2600\,\mu\text{m}}{1} = \boxed{2600\,\mu\text{m}}$$

For ℓ = 2:

$$\lambda_{2,1} = \frac{2600\,\mu\text{m}}{2} = \boxed{1300\,\mu\text{m}}$$

For ℓ = 3:

$$\lambda_{3,2} = \frac{2600\,\mu\text{m}}{3} = \boxed{867\,\mu\text{m}}$$

For ℓ = 4:

$$\lambda_{4,3} = \frac{2600\,\mu\text{m}}{4} = \boxed{650\,\mu\text{m}}$$

For $\ell = 5$:

$$\lambda_{5,4} = \frac{2600\,\mu\text{m}}{5} = \boxed{520\,\mu\text{m}}$$

These wavelengths fall in the microwave region of the electromagnetic spectrum.

32 •• Two objects, one of mass m_1 and the other of mass m_2, are connected to opposite ends of a spring of force constant k_F. The objects are released from rest with the spring compressed. (*a*) Show that when the spring is extended and the object of mass m_1 is a distance Δr_1 from its equilibrium position in the center-of-mass reference frame, the force exerted by the spring is given by $F = -k_F(m_1/\mu)\Delta r_1$, where μ is the reduced mass. (*b*) Show that the frequency of oscillation f is related to k and μ by $2\pi f = \sqrt{k_F/\mu}$.

Picture the Problem For a two-mass and spring system on which no external forces are acting, the center of mass must remain fixed. We can use this condition to express the net force acting on either object. Because this force is a linear restoring force, we can conclude that the motion of the object whose mass is m_1 will be simple harmonic with an angular frequency given by $\omega = \sqrt{k_{\text{eff}}/m_1}$. Substitution for k_{eff} will lead us to the result given in (*b*).

(*a*) If the particle whose mass is m_1 moves a distance Δr_1 from (or toward) the center of mass, then the particle whose mass is m_2 must move a distance:

$\Delta r_2 = \dfrac{m_1}{m_2}\Delta r_1$ from (or toward) the center of mass.

Express the force exerted by the spring:

$$F = -k_F \Delta r = -k_F(\Delta r_1 + \Delta r_2)$$

Substitute for Δr_2 and simplify to obtain:

$$F = -k_F\left(\Delta r_1 + \frac{m_1}{m_2}\Delta r_1\right)$$

$$= -k_F\left(\frac{m_1 + m_2}{m_2}\right)\Delta r_1$$

$$= -k_F\left(\frac{m_1}{\dfrac{m_1 m_2}{m_1 + m_2}}\right)\Delta r_1 = \boxed{-k_F\left(\frac{m_1}{\mu}\right)\Delta r_1}$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the spring and mass oscillator.

(*b*) A displacement Δr_1 of m_1 results in a restoring force given by:

$$F = -k_F\left(\frac{m_1 + m_2}{m_2}\right)\Delta r_1 = -k_{eff}\,\Delta r_1$$

where $k_{eff} = k_F\left(\frac{m_1 + m_2}{m_2}\right)$

Because this is a linear restoring force, we know that the motion will be simple harmonic with an angular frequency given by:

$$\omega = \sqrt{\frac{k_{eff}}{m_1}} \Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_{eff}}{m_1}}$$

Substitute for k_{eff} and simplify to obtain:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{k_F\left(\frac{m_1 + m_2}{m_1 m_2}\right)}$$

or, because $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the two-particle system,

$$2\pi f = \boxed{\sqrt{\frac{k_F}{\mu}}}\,.$$

33 ••• Calculate the reduced masses for $H^{35}Cl$ and $H^{37}Cl$ molecules and the fractional difference $\Delta\mu/\mu$. Show that the mixture of isotopes in HCl leads to a fractional difference in the frequency of a transition from one rotational state to another given by $\Delta f/f = -\Delta\mu/\mu$. Compute $\Delta f/f$ and compare your result with Figure 37-17.

Picture the Problem We can use the definition of the reduced mass to find the reduced mass for the $H^{35}Cl$ and $H^{37}Cl$ molecules and the fractional difference $\Delta\mu/\mu$. Because the rotational frequency is proportional to $1/I$, where I is the moment of inertia of the system, and I is proportional to μ, we can obtain an expression for f as a function of μ that we differentiate implicitly to show that $\Delta f/f = -\Delta\mu/\mu$.

For $H^{35}Cl$:

$$\mu_{H^{35}Cl} = \frac{(35\,u)(1\,u)}{35\,u + 1\,u} = \frac{35}{36}\,u = \boxed{0.972\,u}$$

For $H^{37}Cl$:

$$\mu_{H^{37}Cl} = \frac{(37\,u)(1\,u)}{37\,u + 1\,u} = \frac{37}{38}\,u = \boxed{0.974\,u}$$

The fractional difference is:

$$\frac{\Delta\mu}{\mu}=\frac{\frac{37}{38}\text{u}-\frac{35}{36}\text{u}}{\frac{1}{2}\left(\frac{35}{36}\text{u}+\frac{37}{38}\text{u}\right)}=\frac{\frac{36\times37-35\times38}{36\times38}\text{u}}{\frac{35\times38+36\times37}{2(36)(38)}\text{u}}=\boxed{0.00150}$$

The rotational frequency is proportional to I^{-1}, where I is the moment of inertia of the system. Because I is proportional to μ:

$$f=\frac{C}{\mu}$$

and

$$df=-C\mu^{-2}d\mu$$

Divide df by f to obtain:

$$\frac{df}{f}=-\frac{d\mu}{\mu}\text{ and }\frac{\Delta f}{f}\approx-\frac{\Delta\mu}{\mu}$$

From Figure 37-17:

$$\Delta f\approx0.01\times10^{13}\text{ Hz}=1\times10^{11}\text{ Hz}$$

For $f = 8.40\times10^{13}$ Hz:

$$\frac{\Delta f}{f}\approx\frac{1\times10^{11}\text{ Hz}}{8.40\times10^{13}\text{ Hz}}=\boxed{0.001}$$

This result is in fair agreement (about 20% difference) with the calculated result. Note that Δf is difficult to determine precisely from Figure 37-17.

General Problems

34 • Show that when one atom of a diatomic molecule is much more massive than the other the reduced mass is approximately equal to the mass of the lighter atom.

Picture the Problem We can use the definition of the reduced mass to show that when one atom in a diatomic molecule is much more massive than the other the reduced mass is approximately equal to the mass of the lighter atom.

Express the reduced mass of a two-body system:

$$\mu=\frac{m_1m_2}{m_1+m_2}$$

Divide the numerator and denominator of this expression by m_2 to obtain:

$$\mu=\frac{m_1}{1+\frac{m_1}{m_2}}$$

If $m_2 >> m_1$, then: $\frac{m_1}{m_2} << 1$ and $\boxed{\mu \approx m_1}$

35 •• The equilibrium separation between the nuclei of a CO molecule is 0.113 nm. Determine the energy difference between the $\ell = 2$ and $\ell = 1$ rotational energy levels of this molecule.

Picture the Problem The rotational energy levels are given by $E = \frac{\ell(\ell+1)\hbar^2}{2I}, \ell = 0, 1, 2, ...$

Express the energy difference between these rotational energy levels:

$$\Delta E_{1,2} = E_2 - E_1$$

Express E_2 and E_1:

$$E_2 = \frac{2(2+1)\hbar^2}{2I} = \frac{3\hbar^2}{I}$$

and

$$E_1 = \frac{1(1+1)\hbar^2}{2I} = \frac{\hbar^2}{I}$$

Substitute to obtain:

$$\Delta E_{1,2} = \frac{3\hbar^2}{I} - \frac{\hbar^2}{I} = \frac{2\hbar^2}{I}$$

The moment of inertia of the molecule is given by:

$$I = \mu r_0^2$$

where μ is the reduced mass of the molecule.

Substituting for I yields:

$$\Delta E_{1,2} = \frac{2\hbar^2}{\mu r_0^2} = \frac{2\hbar^2}{\frac{m_C m_O}{m_C + m_O} r_0^2} = \frac{2\hbar^2 (m_C + m_O)}{m_C m_O r_0^2}$$

Substitute numerical values and evaluate $\Delta E_{1,2}$:

$$\Delta E_{1,2} = \frac{2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2 (12\,\text{u} + 16\,\text{u})}{(16\,\text{u})(12\,\text{u})(0.113\,\text{nm})^2 (1.661\times10^{-27}\ \text{kg/u})(1.602\times10^{-19}\ \text{J/eV})} = \boxed{0.955\ \text{meV}}$$

36 •• The effective force constant for a HF molecule is 970 N/m. Find the frequency of vibration for this molecule.

Picture the Problem We can use the result of Problem 32 to find the frequency of vibration of the HF molecule.

In Problem 32 it was established that:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$ where μ is the reduced mass of the HF molecule.

The reduced mass of the HF molecule is given by:

$$\mu = \frac{m_H m_F}{m_H + m_F}$$

Substitute for μ and simplify to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\frac{m_H m_F}{m_H + m_F}}} = \frac{1}{2\pi}\sqrt{\frac{k(m_H + m_F)}{m_H m_F}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{(970\,\text{N/m})(1\,\text{u} + 19\,\text{u})}{(1\,\text{u})(19\,\text{u})(1.661\times10^{-27}\,\text{kg/u})}} = \boxed{1.25\times10^{14}\,\text{Hz}}$$

37 •• The frequency of vibration of a NO molecule is 5.63×10^{13} Hz. Find the effective force constant for NO.

Picture the Problem We can use the result of Problem 32 to find the effective force constant for NO.

In Problem 32 it was established that:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} \Rightarrow k = 4\pi^2 f^2 \mu$$ where μ is the reduced mass of a NO molecule.

The reduced mass of the NO molecule is given by:

$$\mu = \frac{m_N m_O}{m_N + m_O}$$

Substituting for μ gives:

$$k = \frac{4\pi^2 f^2 m_N m_O}{m_N + m_O}$$

Substitute numerical values and evaluate k:

$$k=\frac{4\pi^2\left(5.63\times10^{13}\ \mathrm{s}^{-1}\right)^2(14\,\mathrm{u})(16\,\mathrm{u})\left(1.661\times10^{-27}\ \mathrm{kg/u}\right)}{14\,\mathrm{u}+16\,\mathrm{u}}=\boxed{1.55\,\mathrm{kN/m}}$$

38 •• The effective force constant of the hydrogen bond in a H_2 molecule is 580 N/m. Obtain the energies of the four lowest vibrational levels of the H_2, HD, and D_2 molecules, where H is protium and D is deuterium, and find the wavelengths of photons resulting from transitions between adjacent vibrational levels for each of these molecules.

Picture the Problem We can use the expression for the vibrational energy levels of a molecule and the expression for the frequency of oscillation from Problem 32 to find the four lowest vibrational levels of the given molecules.

The vibrational energy levels are given by:

$$E_\nu=\left(\nu+\tfrac{1}{2}\right)hf,\ \text{where}\ \nu=0,1,2,\ldots$$

In Problem 32 we showed that the frequency of oscillation is:

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{\mu}}\ \text{ where }\ \mu=\frac{m_1m_2}{m_1+m_2}$$

Substitute for f and μ to obtain:

$$E_\nu=\frac{\left(\nu+\frac{1}{2}\right)h}{2\pi}\sqrt{\frac{(m_1+m_2)k}{m_1m_2}}=\frac{\left(\nu+\frac{1}{2}\right)h}{2\pi}\sqrt{k}\sqrt{\frac{(m_1+m_2)}{m_1m_2}}$$

Substitute numerical values to obtain:

$$E_\nu=\frac{\left(\nu+\frac{1}{2}\right)\left(4.136\times10^{-15}\ \mathrm{eV\cdot s}\right)}{2\pi}\sqrt{\frac{580\,\mathrm{N/m}}{1.661\times10^{-27}\ \mathrm{kg/u}}}\sqrt{\frac{(m_1+m_2)}{m_1m_2}}=\left(\nu+\tfrac{1}{2}\right)(0.389\,\mathrm{eV\cdot u})\sqrt{\frac{(m_1+m_2)}{m_1m_2}}$$

Substitute for m_1 and m_2 and evaluate E_0 for H_2:

$$E_0=\tfrac{1}{2}(0.389\,\mathrm{eV\cdot u})\sqrt{\frac{(1\,\mathrm{u}+1\,\mathrm{u})}{(1\,\mathrm{u})(1\,\mathrm{u})}}=0.275\,\mathrm{eV}$$

Proceed similarly to complete the table to the right:

	H_2	HD	D_2
	(eV)	(eV)	(eV)
0	$\boxed{0.275}$	$\boxed{0.238}$	$\boxed{0.195}$
1	$\boxed{0.825}$	$\boxed{0.715}$	$\boxed{0.584}$
2	$\boxed{1.375}$	$\boxed{1.191}$	$\boxed{0.973}$
3	$\boxed{1.925}$	$\boxed{1.667}$	$\boxed{1.362}$

The energies of the photons resulting from transitions between adjacent vibrational levels of these molecules are given by:

$$\Delta E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E}$$

Evaluate $\lambda(H_2)$:

$$\lambda(\mathrm{H}_2) = \frac{1240\,\mathrm{eV\cdot nm}}{0.550\,\mathrm{eV}} = \boxed{2.25\,\mu\mathrm{m}}$$

Evaluate $\lambda(\mathrm{HD})$:

$$\lambda(\mathrm{HD}) = \frac{1240\,\mathrm{eV\cdot nm}}{0.477\,\mathrm{eV}} = \boxed{2.60\,\mu\mathrm{m}}$$

Evaluate $\lambda(D_2)$:

$$\lambda(\mathrm{D}_2) = \frac{1240\,\mathrm{eV\cdot nm}}{0.389\,\mathrm{eV}} = \boxed{3.19\,\mu\mathrm{m}}$$

39 •• The potential energy between two atoms in a molecule separated by a distance r can often be described rather well by the Lenard–Jones (or 6-12) potential function, which can be written as

$$U = U_0\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right]$$

where U_0 and a are constants. Find the equilibrium separation r_0 in terms of a. *Hint: At the equilibrium separation the potential energy is a minimum.* Find U_{min}, the value of U when $r = r_0$. Use Figure 37-4 to obtain numerical values of r_0 and U_0 for a H_2 molecule, and express your answers in nanometers and electron volts.

Picture the Problem We can set the derivative of the potential energy function equal to zero to find the value of r for which it is either a maximum or a minimum. Examination of the second derivative of this function at the value for r obtained from setting the first derivative equal to zero will establish whether the function is a relative maximum or relative minimum at this point.

Differentiate the potential energy function with respect to r:

$$\frac{dU}{dr}=\frac{d}{dr}\left\{U_0\left[\left(\frac{a}{r}\right)^{12}-2\left(\frac{a}{r}\right)^{6}\right]\right\}$$
$$=-\frac{U_0}{r}\left[12\left(\frac{a}{r}\right)^{11}-12\left(\frac{a}{r}\right)^{5}\right]$$

Set the derivative equal to zero for extrema:

$$\frac{dU}{dr}=-\frac{U_0}{r_0}\left[12\left(\frac{a}{r_0}\right)^{11}-12\left(\frac{a}{r_0}\right)^{5}\right]=0$$

Solve for r_0 to obtain a candidate for r that minimizes the Lenard-Jones potential:

$$r_0=\boxed{a}$$

To show that $r_0 = a$ corresponds to a minimum, differentiate U a second time to obtain:

$$\frac{d^2U}{dr^2}=\frac{d}{dr}\left\{-\frac{U_0}{r}\left[12\left(\frac{a}{r}\right)^{11}-12\left(\frac{a}{r}\right)^{5}\right]\right\}$$
$$=\frac{U_0}{r^2}\left[132\left(\frac{a}{r}\right)^{10}-60\left(\frac{a}{r}\right)^{4}\right]$$

Evaluate this second derivative of the potential at $r_0 = a$:

$$\left.\frac{d^2U}{dr^2}\right|_{r=a}=\frac{U_0}{a^2}[132-60]=\frac{72U_0}{a^2}>0$$

Therefore, we can conclude that $r_0 = a$ minimizes the potential function.

Evaluate $U_{\min}$:

$$U_{\min}=U(a)=U_0\left[\left(\frac{a}{a}\right)^{12}-2\left(\frac{a}{a}\right)^{6}\right]$$
$$=\boxed{-U_0}$$

From Figure 37-4:

$$r_0=\boxed{0.074\,\text{nm}}$$

and

$$U_0=\boxed{4.52\,\text{eV}}$$

40 •• In this problem, you are to determine how the van der Waals force between a polar molecule and a nonpolar molecule depends on the distance between the molecules. Let the polar molecule be at the origin and let its dipole moment be in the +x direction. In addition, let the nonpolar molecule be on the x axis a distance x away. (a) How does the electric field strength due to an electric

dipole vary with the distance from the dipole in a given direction? (*b*) Use (1) that the potential energy *U* of an electric dipole of dipole moment $\vec{\boldsymbol{p}}$ in an electric field $\vec{\boldsymbol{E}}$ can be expressed as $U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}$, and (2) that the induced dipole moment $\vec{\boldsymbol{p}}'$ of the nonpolar molecule is in the direction of $\vec{\boldsymbol{E}}$, and that p' is proportional to *E*, to determine how the potential energy of interaction of the two molecules depends on the separation distance *x*. (*c*) Using $F_x = -dU/dx$, determine how the force between the two molecules depends on distance.

Picture the Problem We can use Equation 21-10 to establish the dependence of *E* on *x* and the dependence of an induced dipole on the field that induces it to establish the dependence of *p* and *U* on *x*.

(*a*) In terms of the dipole moment, the electric field on the axis of the dipole at a point a great distance $|x|$ away has the magnitude (see Equation 21-10):

$$E = \frac{2kp}{|x|^3} \Rightarrow E \propto \boxed{\frac{1}{|x|^3}}$$

(*b*) Because the induced dipole moment is proportional to the field that induces it:

$$p \propto \boxed{\frac{1}{x^3}} \text{ and } U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}} \propto \boxed{\frac{1}{x^6}}$$

(*c*) Differentiate *U* with respect to *x* to obtain:

$$F_x = -\frac{dU}{dx} \propto \boxed{\frac{1}{x^7}}$$

41 •• Find the dependence of the force on the separation distance between the two polar molecules described in Problem 40.

Picture the Problem For the case of two polar molecules, *p* does not depend on the field *E*. The dependence of *F* on the separation distance *x* can be found from *dU*/*dx*.

Because *p* does not depend on the electric field in which the polar molecules find themselves:

$$U \propto \frac{1}{x^3}$$

Differentiate *U* with respect to *x* to obtain:

$$F_x = -\frac{dU}{dx} \propto \boxed{\frac{1}{x^4}}$$

42 •• Use the infrared absorption spectrum of HCl in Figure 37-17 to obtain (*a*) the characteristic rotational energy E_{0r} (in eV) and (*b*) find vibrational frequency f and the vibrational energy hf (in eV).

Picture the Problem (*a*) Except for a gap of $4E_{0r}/h$ at the vibrational frequency f, the absorption spectrum contains frequencies equally spaced at $2E_{0r}/h$. We can measure this spacing of the absorption peaks to obtain E_{0r}. (*b*) We can use the expression for the vibrational and rotational energies of a molecule, in conjunction with Figure 37-17 to find f and hf.

(*a*) Letting Δf represent the frequency gap, we have:

$$\Delta f = \frac{2E_{0r}}{h} \Rightarrow E_{0r} = \tfrac{1}{2}h\Delta f$$

Estimating from Figure 37-17, the frequency gap is approximately:

$$\Delta f = 7.0\times10^{11}\text{ Hz}$$

where the peaks at 8.40×10^{13} Hz and 8.61×10^{13} Hz were used.

Substitute numerical values and evaluate E_{0r}:

$$E_{0r} = \tfrac{1}{2}\left(4.136\times10^{-15}\text{ eV}\cdot\text{s}\right)\left(7.0\times10^{11}\text{ Hz}\right) = \boxed{1.4\text{ meV}}$$

(*b*) The vibrational energy levels are given by:

$$E_\nu = \left(\nu+\tfrac{1}{2}\right)hf,\quad \nu = 0,1,2,\ldots$$

The lowest vibrational energy corresponds to $\nu = 0$:

$$E_0 = \tfrac{1}{2}hf \Rightarrow hf = 2E_0 \qquad (1)$$

Determine f from Figure 37-17:

$$f = \boxed{8.66\times10^{13}\text{ Hz}}$$

Substitute for f and h and evaluate E_0:

$$E_0 = \tfrac{1}{2}\left(4.136\times10^{-15}\text{ eV}\cdot\text{s}\right)\left(8.66\times10^{13}\text{ Hz}\right) = 0.179\text{ eV}$$

Substitute in equation (1) and evaluate hf:

$$hf = 2\left(0.179\,\text{eV}\right) = \boxed{0.358\,\text{eV}}$$

43 • The dissociation energy is sometimes expressed in kilocalories per mole (kcal/mol). (*a*) Find the relation between the units eV/molecule and kcal/mol. (*b*) Find the dissociation energy of NaCl in kcal/mol.

Picture the Problem We can use conversion factors to convert eV/molecule into kcal/mol.

(*a*)

$$1\frac{\text{eV}}{\text{molecule}}=1\frac{\text{eV}}{\text{molecule}}\times\frac{1\,\text{kcal}}{4184\,\text{J}}\times\frac{6.022\times10^{23}\text{ molecules}}{\text{mole}}\times\frac{1.602\times10^{-19}\text{ J}}{\text{eV}}$$
$$=\boxed{23.0\,\text{kcal/mol}}$$

(*b*) The dissociation energy of NaCl, in eV/molecule, is 4.27 eV/molecule. Using the conversion factor in (*a*), express this energy in kcal/mol:

$$\frac{4.27\,\text{eV}}{\text{molecule}}\times\frac{23.0\,\text{kcal}}{\text{mol}}=\boxed{98.2\,\text{kcal/mol}}$$

Chapter 38
Solids

Conceptual Problems

1 • In the classical model of conduction, the electron loses energy on average during a collision because it loses the drift velocity it had acquired since the last collision. Where does this energy appear?

Determine the Concept The energy lost by the electrons in collision with the ions of the crystal lattice appears as Joule heat (I^2R).

2 • A metal is a good conductor because the valence energy band for electrons is (*a*) empty, (*b*) partly filled, (*c*) filled, but there is only a small gap to a higher empty band, (*d*) completely filled, (*e*) None of the above.

Determine the Concept If the valence band is only partially full, there are many available empty energy states in the band, and the electrons in the band can easily be raised to a higher energy state by an electric field. $\boxed{(b)}$ is correct.

3 • Thomas refuses to believe that a potential difference can be created simply by bringing two different metals into contact with each other. John talks him into making a small wager and is about to win the bet. (*a*) Which two metals from Table 38-2 would demonstrate his point most effectively? (*b*) What is the value of that contact potential?

Picture the Problem The contact potential is given by $V_{\text{contact}} = \dfrac{\phi_1 - \phi_2}{e}$, where ϕ_1 and ϕ_2 are the work functions of the two different metals in contact with each other.

(*a*) Express the contact potential in terms of the work functions of the metals:

$$V_{\text{contact}} = \frac{\phi_1 - \phi_2}{e}$$

Examining Table 38-2, we see that the greatest difference between the work functions will occur when potassium and nickel are joined.

(*b*) Substitute numerical values and evaluate V_{contact}:

$$V_{\text{contact}} = \frac{5.2\,\text{eV} - 2.1\,\text{eV}}{e} = \boxed{3.1\,\text{V}}$$

4 • (*a*) In Problem 3, which choices of different metals would make the least impressive demonstration? (*b*) What is the value of that contact potential?

Picture the Problem The contact potential is given by $V_{\text{contact}} = \frac{\phi_1 - \phi_2}{e}$, where ϕ_1 and ϕ_2 are the work functions of the two different metals in contact with each other.

(*a*) Express the contact potential in terms of the work functions of the metals:

$$V_{\text{contact}} = \frac{\phi_1 - \phi_2}{e}$$

Examining Table 38-2, we see that the least difference between the work functions will occur when silver and gold are joined.

(*b*) Substitute numerical values and evaluate V_{contact}:

$$V_{\text{contact}} = \frac{4.8\,\text{eV} - 4.7\,\text{eV}}{e} = \boxed{0.1\,\text{V}}$$

5 • [SSM] When a sample of pure copper is cooled from 300 K to 4 K, its resistivity decreases more than the resistivity of a sample of brass when it is cooled through the same temperature difference. Why?

Determine the Concept The resistivity of brass at 4 K is almost entirely due to the ″residual resistance,″ the resistance due to impurities and other imperfections of the crystal lattice. In brass, the zinc ions act as impurities in copper. In pure copper, the resistivity at 4 K is due to its residual resistance, which is very low if the copper is very pure.

6 • Insulators are poor conductors of electricity because (*a*) there is a small energy gap between the filled valence band and the next higher band where electrons can exist, (*b*) there is a large energy gap between the completely filled valence band and the next higher band where electrons can exist, (*c*) the valence band has a few vacancies for electrons, (*d*) the valence band is only partly filled, (*e*) None of the above.

Determine the Concept Insulators are poor conductors of electricity because there is a large energy gap between the filled valence band and the next higher band where electrons can exist. $\boxed{(b)}$ is correct.

7 • [SSM] How does the sign of the change in the resistivity of a sample of copper compare with the sign of the change of the resistivity of a sample of silicon when the temperatures of both samples increase?

Determine the Concept The resistivity of copper increases with increasing temperature; the resistivity of (pure) silicon decreases with increasing temperature because the number of charge carriers increases.

8 • True or false:

(*a*) Solids that are good electrical conductors are usually good heat conductors.
(*b*) At $T = 0$, an intrinsic semiconductor is an insulator.
(*c*) The Fermi energy is the average energy of an electron in a solid.
(*d*) At $T = 0$, the value of the Fermi factor can be either 1 or 0.
(*e*) Semiconductors conduct current in one direction only.
(*f*) The classical free-electron theory adequately explains the heat capacity of metals.
(*g*) The contact potential between two metals is proportional to the difference in the work functions of the two metals.

(*a*) True

(*b*) True

(*c*) False. The Fermi energy is the energy of the last filled (or half-filled) level at $T = 0$.

(*d*) True

(*e*) False

(*f*) False. The classical free-electron theory predicts heat capacities for metals that are not observed experimentally.

(*g*) True

9 • Which of the following elements are most likely to act as acceptor impurities in germanium? (*a*) bromine (*b*) gallium (*c*) silicon (*d*) phosphorus (*e*) magnesium

Determine the Concept Because a gallium atom can accept electrons from the valence band of germanium to complete its four covalent bonds, $\boxed{(b)}$ is correct.

10 • Which of the following elements are most likely to serve as donor impurities in germanium? (*a*) bromine (*b*) gallium (*c*) silicon (*d*) phosphorus (*e*) magnesium

Determine the Concept Because phosphorus has 3 electrons that it can donate to the conduction band of germanium without leaving holes in the valence band, $\boxed{(d)}$ is correct.

11 • An electron hole is created when a photon is absorbed by a semiconductor. How does this hole enable the semiconductor to conduct electricity?

Determine the Concept The excited electron is in the conduction band and can conduct electricity. The positive hole left in the valence band moves through the band and also contributes to the current.

12 • Examine the positions of phosphorus, boron, thallium, and antimony in Table 36-1. (*a*) Which of these elements can be used to dope silicon to create an *n*-type semiconductor? (*b*) Which of these elements can be used to dope silicon to create a *p*-type semiconductor?

Determine the Concept
(a) Phosphorus and antimony will make n-type semiconductors since each has one more valence electron than silicon.

(b) Boron and thallium will make p-type semiconductors since each has one less valence electron than silicon.

13 • When photons of visible light strike the *p*-type semiconductor in a *pn* junction solar cell, (*a*) only free electrons are created, (*b*) only positive holes are created, (*c*) both electrons and holes are created, (*d*) protons are created, (*e*) None of the above.

Determine the Concept Because a *pn* junction solar cell has donor impurities on one side and acceptor impurities on the other, both electrons and holes are created when photons of visible light strike the *p*-type semiconductor cell. $\boxed{(c)}$ is correct.

Estimation and Approximation

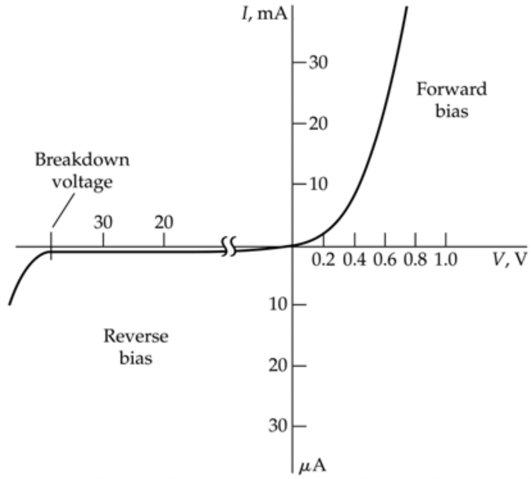
14 • The ratio of the resistivity of the most resistive (least conductive) material to that of the least resistive material (excluding superconductors) is approximately 10^{24}. You can develop a feeling for how remarkable this range is by considering what the ratio is of the largest values to smallest values of other material properties. Choose any three properties of materials, and using tables in this book or some other resource, calculate the ratio of the largest instance of the property to the smallest instance of that property (other than zero) and rank these in decreasing order. Can you find any other property that shows a range as large as that of electrical resistivity?

Picture the Problem We can use the list of tables on the inside back covers of volumes 1 and 2 to find tables of material properties. A representative sample is included in the following table in which all the units are SI:

Table	Material property	Largest value	Smallest value	Ratio (order of magnitude)
13-1	Mass density	22.5×10^{3} (Osmium)	0.08994 (Hydrogen)	10^5
20-3	Thermal conductivity	429 (Ag)	0.026 (air)	10^4
20-1	Thermal expansion	51×10^{-6} (ice)	10^{-6} (invar)	10^2
12-8	Tensile Strength	520 (steel)	2 (concrete)	10^2
12-8	Young's modulus	200 (steel)	9 (bone)	10
18-1	Heat capacity	4.18 (water)	0.900 (Al)	1

15 • A device is said to be "ohmic" if a graph of current versus applied voltage is a straight line through the origin. The resistance R of the device is the reciprocal of the slope of this line. A *pn* junction is an example of a nonohmic device, as may be seen from Figure 38-21. For nonohmic devices, it is sometimes convenient to define the *differential resistance* as the reciprocal of the slope of the I versus V curve. Using the curve in Figure 38-21, estimate the differential resistance of the *pn* junction at applied voltages of –20 V, +0.2 V, +0.4 V, +0.6 V, and +0.8 V.

Picture the Problem Figure 38-21 is reproduced below. We can draw tangent lines at each of the voltages and estimate the slope. The differential resistance is the reciprocal of the slope.



V (V)	1/slope (Ω)
–20	$\boxed{\infty}$
+0.2	$\boxed{40}$
+0.4	$\boxed{20}$
+0.6	$\boxed{10}$
+0.8	$\boxed{5}$

Remarks: Note that, because of the difficulty in determining the slopes, these results are only approximations.

The Structure of Solids

16 • Calculate the center-to-center separation distance r_0 between the K^+ and the Cl^- ions in KCl. Do this by assuming that each ion occupies a cubic volume of side r_0. The molar mass of KCl is 74.55 g/mol and its density is 1.984 g/cm^3.

Picture the Problem We can use the geometry of the ion to relate the volume per mole to the length of its side r_0 and the definition of density to express the volume per mole in terms of its molar mass and density.

Because the cube length/ion is r_0, the volume/mole is given by:

$$V_{\text{mol}} = 2N_{\text{A}}r_0^3 \Rightarrow r_0 = \sqrt[3]{\frac{V_{\text{mol}}}{2N_{\text{A}}}}$$

The volume/mole is given by:

$$V_{\text{mol}} = \frac{M}{\rho}$$

where M is the molar mass of KCl.

Substitute for V_{mol} in the expression for r_0 to obtain:

$$r_0 = \sqrt[3]{\frac{M}{2\rho N_{\text{A}}}}$$

Substitute numerical values and evaluate r_0:

$$r_0 = \sqrt[3]{\frac{74.55\,\text{g/mol}}{2(1.984\,\text{g/cm}^3)(6.022\times10^{23}\ \text{particles/mol})}} = \boxed{0.315\,\text{nm}}$$

17 • The center-to-center separation distance between the Li^+ and Cl^- ions in LiCl is 0.257 nm. Use this value and the molar mass of LiCl (42.4 g/mol) to compute the density of LiCl.

Picture the Problem We can use the definition of density and the geometry of the ions to compute the density of LiCl.

The density of LiCl is given by:

$$\rho = \frac{M_{\text{unit cell}}}{V_{\text{unit cell}}}$$

Express the volume of the unit cell:

$$V_{\text{unit cell}} = (2r_0)^3$$

Because the unit cell has four molecules, its mass is given by:

$$M_{\text{unit cell}} = \frac{4M}{N_{\text{A}}}$$

Substitute for $V_{\text{unit cell}}$ and $M_{\text{unit cell}}$ to obtain:

$$\rho = \frac{\frac{4M}{N_{\text{A}}}}{(2r_0)^3} = \frac{4M}{N_{\text{A}}(2r_0)^3}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{4(42.4\,\text{g/mol})}{(6.022\times10^{23}\ \text{particles/mol})\left[2(0.257\times10^{-9}\ \text{m})\right]^3} = \boxed{2.07\,\text{g/cm}^3}$$

18 • Find the value of n in Equation 38-6 that gives the measured dissociation energy of 741 kJ/mol for LiCl, which has the same structure as NaCl and for which r_0 = 0.257 nm.

Picture the Problem We can solve Equation 38-6 for n and then substitute numerical values to evaluate n.

Equation 38-6 is:

$$U(r_0) = -\alpha\frac{ke^2}{r_0}\left(1-\frac{1}{n}\right)$$

and

$$|U(r_0)| = \alpha\frac{ke^2}{r_0}\left(1-\frac{1}{n}\right)$$

Solve for *n* to obtain:

$$n = \frac{1}{1-\dfrac{|U(r_0)|r_0}{\alpha ke^2}}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{1}{1 - \dfrac{(741\,\text{kJ/mol})\left(\dfrac{1\text{eV/ion pair}}{96.47\,\text{kJ/mol}}\right)(0.257\,\text{nm})}{(1.7476)(1.44\,\text{eV}\cdot\text{nm})}} = \boxed{4.64}$$

19 •• (*a*) Use Equation 38-6 and calculate $U(r_0)$ for calcium oxide, CaO, where $r_0 = 0.208$ nm. Assume $n = 8$. (*b*) If *n* increase from 8 to 10, what is the fractional change in $U(r_0)$?

Picture the Problem We can substitute numerical values in Equation 38-6 to evaluate $U(r_o)$ for $n = 8$ and $n = 10$.

(*a*) Equation 38-6 is:

$$U(r_0) = -\alpha\frac{ke^2}{r_0}\left(1-\frac{1}{n}\right)$$

Substitute numerical values and evaluate $U(r_0)$:

$$U(r_o) = -\frac{(1.7476)(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.602\times10^{-19}\,\text{C})^2}{(0.208\times10^{-9}\,\text{m})\left(\dfrac{1.602\times10^{-19}\,\text{J}}{1\text{eV}}\right)}\left(1-\frac{1}{8}\right) = \boxed{-10.6\,\text{eV}}$$

(*b*) The fractional change is given by:

$$\frac{\Delta U(r_0)}{U(r_0)} = \frac{U_{n=10} - U_{n=8}}{U_{n=8}} = \frac{U_{n=10}}{U_{n=8}} - 1$$

Substitute numerical values and evaluate $U(r_0)$ for n = 10:

$$U_{n=10}(r_o) = -\frac{(1.7476)\left(8.988\times10^9\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left(1.602\times10^{-19}\ \mathrm{C}\right)^2}{\left(0.208\times10^{-9}\ \mathrm{m}\right)\left(\frac{1.602\times10^{-19}\ \mathrm{J}}{1\mathrm{eV}}\right)}\left(1-\frac{1}{10}\right) = -10.9\,\mathrm{eV}$$

Substitute numerical values and evaluate the fractional change in $U(r_0)$:

$$\frac{\Delta U(r_0)}{U(r_0)} = \frac{-10.9\,\mathrm{eV}}{-10.6\,\mathrm{eV}} - 1 = \boxed{2.83\%}$$

A Microscopic Picture of Conduction

20 • A measure of the density of the free electrons in a metal is the distance r_s, which is defined as the radius of the sphere whose volume equals the volume per conduction electron. (*a*) Show that $r_s = [3/(4\pi n)]^{1/3}$, where n is the free-electron number density. (*b*) Calculate r_s for copper in nanometers.

Picture the Problem We can use the expression for the volume occupied by one electron to show that $r_s = \left(\frac{3}{4\pi n}\right)^{1/3}$.

(*a*) The volume occupied by one electron is:

$$\frac{1}{n} = \frac{4}{3}\pi r_s^3 \Rightarrow r_s = \boxed{\left(\frac{3}{4\pi n}\right)^{1/3}}$$

(*b*) From Table 38-1:

$$n_{\mathrm{Cu}} = 8.47\times10^{28}\ \mathrm{m^{-3}}$$

Substitute numerical values and evaluate r_s for copper:

$$r_s = \sqrt[3]{\frac{3}{4\pi\left(8.47\times10^{28}\ \mathrm{m^{-3}}\right)}} = \boxed{0.141\mathrm{nm}}$$

21 • (*a*) Given a mean free path λ = 0.400 nm and a mean speed $v_{av} = 1.17\times10^5$ m/s for the charge flow in copper at a temperature of 300 K, calculate the classical value for the resistivity ρ of copper. (*b*) The classical model suggests that the mean free path is temperature independent and that v_{av} depends on temperature. According to this model, what would ρ be at 100 K?

Picture the Problem (*a*) We can use the expression for the resistivity of the copper in terms of v_{av} and λ to find the classical value for the resistivity ρ of copper. In Part (*b*) we can use $v_{av} = \sqrt{\frac{3kT}{m_e}}$ to relate the average speed to the temperature.

(*a*) In terms of the mean free path and the mean speed, the resistivity is:

$$\rho = \frac{m_e v_{av}}{n_e e^2 \lambda}$$

Substitute numerical values and evaluate ρ (see Table 38-1 for the free-electron number density of copper):

$$\rho = \frac{(9.109\times10^{-31}\text{ kg})(1.17\times10^{5}\text{ m/s})}{(8.47\times10^{28}\text{ m}^{-3})(1.602\times10^{-19}\text{ C})^2(0.400\,\text{nm})} = \boxed{0.123\,\mu\Omega\cdot\text{m}}$$

(*b*) Relate the average speed of the electrons to the temperature:

$$v_{av} = \sqrt{\frac{3kT}{m_e}}$$

Substitute for v_{av} in the expression for ρ to obtain:

$$\rho = \frac{m_e}{n_e e^2 \lambda}\sqrt{\frac{3kT}{m_e}} = \frac{1}{n_e e^2 \lambda}\sqrt{3m_e kT}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{\sqrt{3(9.109\times10^{-31}\text{ kg})(1.381\times10^{-23}\text{ J/K})(100\,\text{K})}}{(8.47\times10^{28}\text{ m}^{-3})(1.602\times10^{-19}\text{ C})^2(0.400\,\text{nm})} = \boxed{70.7\,\text{n}\Omega\cdot\text{m}}$$

Free Electrons in a Solid

22 •• Silicon has a molar mass of 28.09 g/mol and a density of 2.41×10^3 kg/m^3. Each atom of silicon has four valence electrons and the Fermi energy of the material is 4.88 eV. (*a*) Given that the electron mean free path at room temperature is l = 27.0 nm, estimate the resistivity. (*b*) The accepted value for the resistivity of silicon is 640 Ω·m (at room temperature). How does this accepted value compare to the value calculated in Part (*a*)?

Picture the Problem We can use Equation 38-14 to estimate the resistivity of silicon.

(*a*) From Equation 38-14:

$$\rho = \frac{m_e v_{av}}{n_e e^2 \lambda}$$

The speed of the electrons is given by:

$$v_{av} = u_F = \sqrt{\frac{2E_F}{m_e}}$$

Substituting for v_{av} yields:

$$\rho = \frac{m_e\sqrt{\dfrac{2E_F}{m_e}}}{n_e e^2 \lambda} = \frac{\sqrt{2m_e E_F}}{n_e e^2 \lambda}$$

The electron density of Si is given by:

$$n_e = MN_A N_{atom}$$

where N_{atom} is the number of electrons per atom.

Substitute for n_e to obtain:

$$\rho = \frac{\sqrt{2m_e E_F}}{MN_A N_{atom} e^2 \lambda}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{\sqrt{2\left(9.109\times10^{-31}\,\text{kg}\right)\left(4.88\,\text{eV}\right)\left(\dfrac{1.602\times10^{-19}\,\text{J}}{1\,\text{eV}}\right)}}{\left(2.41\times10^{3}\,\dfrac{\text{kg}}{\text{m}^3}\right)\left(\dfrac{6.022\times10^{23}\,\text{atoms}}{0.02809\,\text{kg}}\right)\left(\dfrac{4\,\text{e}}{\text{atom}}\right)\left(1.602\times10^{-19}\,\text{C}\right)^2\left(27.0\times10^{-9}\,\text{m}\right)}$$

$$= \boxed{8.3\,\text{n}\Omega\cdot\text{m}}$$

(*b*) The accepted resistivity of 640 Ω·m is much greater than the calculated value. We assume that valence electrons will produce conduction in the material. Silicon is a semiconductor and a gap between the valence band and conduction band exists. Only electrons with sufficient energies will be found in the conduction band.

23 • Calculate the number density of free electrons in (*a*) Ag (ρ = 10.5 g/cm^3) and (*b*) Au (ρ = 19.3 g/cm^3), assuming one free electron per atom, and compare your results with the values listed in Table 38-1.

Picture the Problem The number density of free electrons is given by $n = \rho N_A / M$, where N_A is Avogadro's number, ρ is the density of the element, and M is its molar mass.

Relate the number density of free electrons to the density ρ and molar mass M of the element:

$$\frac{n}{N_A} = \frac{\rho}{M} \Rightarrow n = \frac{\rho N_A}{M}$$

(*a*) For Ag:

$$n_{\text{Ag}} = \frac{\left(10.5\,\text{g/cm}^3\right)\left(6.022\times10^{23}\text{ electrons/mol}\right)}{107.87\,\text{g/mol}} = \boxed{5.86\times10^{22}\text{ electrons/cm}^3}$$

(*b*) For Au:

$$n_{\text{Ag}} = \frac{\left(19.3\,\text{g/cm}^3\right)\left(6.022\times10^{23}\text{ electrons/mol}\right)}{196.97\,\text{g/mol}} = \boxed{5.90\times10^{22}\text{ electrons/cm}^3}$$

Both these results agree with the values in Table 38-1.

24 • The density of aluminum is 2.7 g/cm^3. How many free electrons are present per aluminum atom?

Picture the Problem The number of free electrons per atom n_e is given by $n_e = nM/\rho N_A$, where N_A is Avogadro's number, ρ is the density of the element, M is its molar mass, and n is the free electron number density for the element.

The number of free electrons per atom is given by:

$$n_e = \frac{nM}{\rho N_A}$$

Substitute numerical values and evaluate n_e:

$$n_e = \frac{\left(18.1\times10^{22}\text{ electrons/cm}^3\right)\left(26.98\,\text{g/mol}\right)}{\left(2.7\,\text{g/cm}^3\right)\left(6.022\times10^{23}\text{ electrons/mol}\right)} = \boxed{3.0}$$

25 • The density of tin is 7.3 g/cm^3. How many free electrons are present per tin atom?

Picture the Problem The number of free electrons per atom n_e is given by $n_e = nM/\rho N_A$, where N_A is Avogadro's number, ρ is the density of the element, M is its molar mass, and n is the free electron number density for the element.

The number of free electrons per atom is given by:

$$n_e = \frac{nM}{\rho N_A}$$

Substitute numerical values and evaluate n_e:

$$n_e = \frac{\left(14.8\times10^{22}\ \text{electrons/cm}^3\right)\left(118.69\ \text{g/mol}\right)}{\left(7.3\ \text{g/cm}^3\right)\left(6.022\times10^{23}\ \text{electrons/mol}\right)} = \boxed{4.0}$$

26 • Calculate the Fermi temperature for (*a*) Mg, (*b*) Mn, and (*c*) Zn.

Picture the Problem The Fermi temperature T_F is defined by $kT_F = E_F$, where E_F is the Fermi energy. See Table 38-1 for Fermi energies at $T = 0$.

The Fermi temperature is given by:

$$T_F = \frac{E_F}{k}$$

(*a*) For Mg:

$$T_F = \frac{7.11\ \text{eV}}{8.617\times10^{-5}\ \text{eV/K}} = \boxed{8.25\times10^{4}\ \text{K}}$$

(*b*) For Mn:

$$T_F = \frac{11.0\ \text{eV}}{8.617\times10^{-5}\ \text{eV/K}} = \boxed{1.28\times10^{5}\ \text{K}}$$

(*c*) For Zn:

$$T_F = \frac{9.46\ \text{eV}}{8.617\times10^{-5}\ \text{eV/K}} = \boxed{1.10\times10^{5}\ \text{K}}$$

27 • What is the speed of a conduction electron whose energy is equal to the Fermi energy E_F for (*a*) Na, (*b*) Au, and (*c*) Sn?

Picture the Problem We can solve the expression for the Fermi energy for the speed of a conduction electron.

Express the Fermi energy in terms of the Fermi speed of a conduction electron:

$$E_F = \tfrac{1}{2} m_e u_F^2 \Rightarrow u_F = \sqrt{\frac{2E_F}{m_e}}$$

(*a*) Substitute numerical values (see Table 38-1 for E_F) and evaluate u_F for Na:

$$u_F = \sqrt{\frac{2\left(3.24\ \text{eV}\right)\left(1.602\times10^{-19}\ \text{J/eV}\right)}{9.109\times10^{-31}\ \text{kg}}} = \boxed{1.07\times10^{6}\ \text{m/s}}$$

(*b*) Substitute numerical values and evaluate u_F for Au:

$$u_F = \sqrt{\frac{2(5.53\,\text{eV})(1.602\times10^{-19}\ \text{J/eV})}{9.109\times10^{-31}\ \text{kg}}} = \boxed{1.39\times10^{6}\ \text{m/s}}$$

(*c*) Substitute numerical values and evaluate u_F for Sn:

$$u_F = \sqrt{\frac{2(10.2\,\text{eV})(1.602\times10^{-19}\ \text{J/eV})}{9.109\times10^{-31}\ \text{kg}}} = \boxed{1.89\times10^{6}\ \text{m/s}}$$

28 • Calculate the Fermi energy for (*a*) Al, (*b*) K, and (*c*) Sn using the number densities given in Table 38-1.

Picture the Problem The Fermi energy at $T = 0$ depends on the number of electrons per unit volume (the number density) according to $E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(N/V)^{2/3}$.

The Fermi energy at $T = 0$ is given by:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)\left(\frac{N}{V}\right)^{2/3}$$

(*a*) For Al, $N/V = 18.1\times10^{22}$ electrons/cm^3 (see Table 38-1) and:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(18.1\times10^{22}\ \text{electrons/cm}^3)^{2/3} = \boxed{11.7\,\text{eV}}$$

(*b*) For K, $N/V = 1.40 \times 10^{22}$ electrons/cm^3 and:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(1.40\times10^{22}\ \text{electrons/cm}^3)^{2/3} = \boxed{2.12\,\text{eV}}$$

(*c*) For Sn, $N/V = 14.8\times10^{22}$ electrons/cm^3 and:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(14.8\times10^{22}\ \text{electrons/cm}^3)^{2/3} = \boxed{10.2\,\text{eV}}$$

29 • Find the average energy of the conduction electrons at $T = 0$ in (*a*) copper and (*b*) lithium.

Picture the Problem The average energy of electrons in a Fermi gas at $T = 0$ is three-fifths of the Fermi energy.

The average energy at $T = 0$ is given by:

$$E_{av} = \tfrac{3}{5}E_F$$

(*a*) For copper, $E_F = 7.04$ eV (see Table 38-1) and:

$$E_{av} = \tfrac{3}{5}(7.04\,\text{eV}) = \boxed{4.22\,\text{eV}}$$

(*b*) For lithium, $E_F = 4.75$ eV and:

$$E_{av} = \tfrac{3}{5}(4.75\,\text{eV}) = \boxed{2.85\,\text{eV}}$$

30 • Calculate (*a*) the Fermi temperature and (*b*) the Fermi energy at $T = 0$ for iron.

Picture the Problem The Fermi energy at $T = 0$ is given by $E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(N/V)^{3/2}$, where N/V is the free-electron number density and the Fermi temperature is related to the Fermi energy according to $kT_F = E_F$.

(*a*) The Fermi temperature for iron is given by:

$$T_F = \frac{E_F}{k}$$

Substitute numerical values (see Table 38-1) and evaluate T_F:

$$T_F = \frac{11.2\,\text{eV}}{8.617\times10^{-5}\,\text{eV/K}} = \boxed{1.30\times10^{5}\,\text{K}}$$

(*b*) The Fermi energy at $T = 0$ is given by:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)\left(\frac{N}{V}\right)^{3/2}$$

Substitute numerical values (see Table 38-1) and evaluate E_F:

$$E_F = (0.365\,\text{eV}\cdot\text{nm}^2)(17.0\times10^{22}\,\text{electrons/cm}^3)^{2/3} = \boxed{11.2\,\text{eV}}$$

31 •• **[SSM]** (*a*) Assuming that each gold atom in a sample of gold metal contributes one free electron, calculate the free-electron density in gold knowing that its atomic mass is 196.97 g/mol and its density is 19.3×10^3 kg/m^3. (*b*) If the Fermi speed for gold is 1.39×10^6 m/s, what is the Fermi energy in electron volts? (*c*) By what factor is the Fermi energy higher than the kT energy at room temperature? (*d*) Explain the difference between the Fermi energy and the kT energy.

Picture the Problem We can use $n_e = \rho V = \dfrac{\rho N_A N_{atom}}{m}$, where N_{atom} is the number of electrons per atom, to calculate the electron density of gold. The Fermi energy is given by $E_F = \tfrac{1}{2}m_e u_F^2$.

(*a*) The electron density of gold is given by:

$$n_e = \rho V = \frac{\rho N_A N_{atom}}{m}$$

Substitute numerical values and evaluate n_e:

$$n_e = \frac{\left(19.3\times10^3\ \frac{\text{kg}}{\text{m}^3}\right)\left(6.022\times10^{23}\,\text{atoms}\right)\left(\frac{1\text{e}}{1\,\text{atom}}\right)}{0.197\ \text{kg}} = \boxed{5.90\times10^{28}\ \text{e/m}^3}$$

(*b*) The Fermi energy is given by:

$$E_F = \tfrac{1}{2} m_e u_F^2$$

Substitute numerical values and evaluate E_F:

$$E_F = \frac{1}{2}\left(9.109\times10^{-31}\,\text{kg}\right)\left(1.39\times10^6\,\text{m/s}\right)^2\left(\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right) = \boxed{5.50\,\text{eV}}$$

(*c*) The factor by which the Fermi energy is higher than the kT energy at room temperature is:

$$f = \frac{E_F}{kT}$$

At room temperature $kT = 0.026$ eV. Substitute numerical values and evaluate f:

$$f = \frac{5.50\ \text{eV}}{0.026\ \text{eV}} = \boxed{212}$$

(*d*) The ratio E_F/kT is equal to 212 $T = 300$ K. The Fermi energy is the energy of the most energetic conduction electron when the crystal is at absolute zero. Because no two conduction electrons can occupy the same state, the Fermi energy is quite high compared with kT. The kT energy is the energy the average conduction electron would have when the crystal is at temperature T if the electrons did not obey the exclusion principle.

32 •• The bulk modulus B of a material can be defined by $B = -V\,\partial P/\partial V$. (*a*) Use the monatomic ideal-gas relation $PV = \tfrac{2}{3} N E_{av}$, where E_{av} is the average kinetic energy, Equation 38-22 and Equation 38-23 to show that $P = \tfrac{2}{5} N E_F / V = C V^{-5/3}$, where C is a constant independent of V. (*b*) Show that the bulk modulus of the free electrons in a solid metal is therefore $B = \tfrac{5}{3} P = \tfrac{2}{3} N E_F / V$. (*c*) Compute the bulk modulus in newtons per square meter for the free electrons in a sample of copper and compare your result with the measured value of 140×10^9 N/m^2.

Picture the Problem We can follow the procedure given in the problem statement to show that $P = \dfrac{2NE_F}{5V} = CV^{-5/3}$ and $B = \dfrac{5}{3}P = \dfrac{2NE_F}{3V}$.

(a) We're given that, for a monatomic ideal-gas:

$$PV = \tfrac{2}{3}NE_{\text{av}}$$

Because $E_{\text{av}} = \tfrac{3}{5}E_{\text{F}}$:

$$P = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{3}{5}E_{\text{F}}\right) = \frac{2}{5}\left(\frac{N}{V}\right)E_{\text{F}} \qquad (1)$$

The Fermi energy is given by:

$$E_{\text{F}} = \frac{h^2}{8m_{\text{e}}}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m_{\text{e}}}\left(\frac{3N}{\pi}\right)^{2/3} V^{-2/3}$$

Substitute to obtain:

$$P = \frac{2}{5}\left(\frac{N}{V}\right)\frac{h^2}{8m_{\text{e}}}\left(\frac{3N}{\pi}\right)^{2/3} V^{-2/3}$$

$$= \frac{N^{5/3}h^2}{20m_{\text{e}}}\left(\frac{3}{\pi}\right)^{2/3} V^{-5/3} = \boxed{CV^{-5/3}}$$

where $C = \dfrac{N^{5/3}h^2}{20m_{\text{e}}}\left(\dfrac{3}{\pi}\right)^{2/3}$ is a constant.

(b) The bulk modulus is given by:

$$B = -V\frac{dP}{dV} = -V\frac{d}{dV}\left[CV^{-5/3}\right]$$

$$= -CV\left(-\frac{5}{3}V^{-8/3}\right) = \frac{5}{3}CV^{-5/3}$$

$$= \frac{5}{3}P$$

Substitute for P from equation (1) to obtain:

$$B = \frac{5}{3}\left[\frac{2}{5}\left(\frac{N}{V}\right)E_{\text{F}}\right] = \boxed{\tfrac{2}{3}NE_{\text{F}}/V}$$

(c) Substitute numerical values and evaluate B for copper:

$$B = \frac{2}{3}\left(8.47\times10^{22}\text{ electrons/cm}^3\right)\left(7.04\,\text{eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right) = \boxed{63.6\text{ GN/m}^2}$$

From Table 13-2:

$$B_{\text{Cu}} = 140\text{ GN/m}^2$$

Divide the calculated value for B by the value from Table 13-2 to obtain:

$$\frac{B}{B_{\text{Cu}}} = \frac{63.6\,\text{GN/m}^2}{140\,\text{GN/m}^2} = 0.454$$

or

$$B = \boxed{0.454B_{\text{Cu}}}$$

33 •• [SSM] The pressure of a monatomic ideal gas is related to the average kinetic energy of the gas particles by $PV = \frac{2}{3}NE_{av}$, where N is the number of particles and E_{av} is the average kinetic energy. Use this information to calculate the pressure of the free electrons in a sample of copper in newtons per square meter, and compare your result with atmospheric pressure, which is about 10^5 N/m^2. (Note: The units are most easily handled by using the conversion factors 1 N/m^2 = 1 J/m^3 and 1 eV = 1.602×10^{-19} J.)

Picture the Problem We can solve $PV = \frac{2}{3}NE_{av}$ for P and substitute for E_{av} in order to express P in terms of N/V and E_F.

Solve $PV = \frac{2}{3}NE_{av}$ for P:

$$P = \frac{2}{3}\left(\frac{N}{V}\right)E_{av}$$

Because $E_{av} = \frac{3}{5}E_F$:

$$P = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{3}{5}E_F\right) = \frac{2}{5}\left(\frac{N}{V}\right)E_F$$

Substitute numerical values (see Table 38-1) and evaluate P:

$$P = \frac{2}{5}\left(8.47\times10^{22}\ \text{electrons/cm}^3\right)\left(7.04\,\text{eV}\right)\left(1.602\times10^{-19}\ \text{J/eV}\right)$$

$$= \boxed{3.82\times10^{10}\ \text{N/m}^2} = 3.82\times10^{10}\ \text{N/m}^2 \times \frac{1\,\text{atm}}{101.325\times10^3\ \text{N/m}^2}$$

$$= \boxed{3.77\times10^5\,\text{atm}}$$

34 • Calculate the contact potential between (*a*) Ag and Cu, (*b*) Ag and Ni, and (*c*) Ca and Cu.

Picture the Problem The contact potential is given by $V_{contact} = \frac{\phi_1 - \phi_2}{e}$, where ϕ_1 and ϕ_2 are the work functions of the two different metals in contact with each other.

The contact potential is given by:

$$V_{contact} = \frac{\phi_1 - \phi_2}{e}$$

(*a*) For Ag and Cu (see Table 38-1):

$$V_{contact} = \frac{4.7\,\text{eV} - 4.1\,\text{eV}}{e} = \boxed{0.6\,\text{V}}$$

(*b*) For Ag and Ni:

$$V_{contact} = \frac{5.2\,\text{eV} - 4.7\,\text{eV}}{e} = \boxed{0.5\,\text{V}}$$

(*c*) For Ca and Cu (see Table 38-1):

$$V_{\text{contact}} = \frac{4.1\,\text{eV} - 3.2\,\text{eV}}{e} = \boxed{0.9\,\text{V}}$$

Heat Capacity Due to Electrons in a Metal

35 •• [SSM] Gold has a Fermi energy of 5.53 eV. Determine the molar specific heat at constant volume at room temperature for gold.

Picture the Problem We can use Equation 38-29 to find the molar specific heat of gold at constant volume and room temperature.

The molar specific heat is given by Equation 38-29:

$$c'_{\text{V}} = \frac{\pi^2 RT}{2T_{\text{F}}}$$

The Fermi energy is given by:

$$E_{\text{F}} = kT_{\text{F}} \Rightarrow T_{\text{F}} = \frac{E_{\text{F}}}{k}$$

Substitute for T_{F} to obtain:

$$c'_{\text{V}} = \frac{\pi^2 RkT}{2E_{\text{F}}}$$

Substitute numerical values and evaluate c'_{V}:

$$c'_{\text{V}} = \frac{\pi^2 (8.314\,\text{J/mol K})(1.381\times10^{-23}\,\text{J/mol})\left(\dfrac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right)(300\,\text{K})}{2(5.53\,\text{eV})}$$

$$= \boxed{0.192\,\text{J/mol}\cdot\text{K}}$$

Remarks: The value 0.192 J/mol K is for a mole of gold atoms. Because each gold atom contributes one electron to the metal, a mole of gold corresponds to a mole of electrons.

Quantum Theory of Electrical Conduction

36 • The resistivities and Fermi speeds of Na, Au, and Sn at T = 273 K are 4.2 $\mu\Omega$·cm, 2.04 $\mu\Omega$·cm and 10.6 $\mu\Omega$·cm, and 1.07×10^6 m/s, 1.39×10^6 m/s and 1.89×10^6 m/s respectively. Use these values to find the mean free paths for the conduction electrons in these elements.

Picture the Problem We can solve the equation giving the resistivity of a conductor in terms of the mean free path and the mean speed for the mean free path and use the Fermi speeds as the mean speeds.

In terms of the mean free path and the mean speed, the resistivity is:

$$\rho = \frac{m_e v_{av}}{ne^2\lambda} \Rightarrow \lambda = \frac{m_e v_{av}}{ne^2\rho}$$

Using the Fermi speeds as the average speeds, substitute numerical values and evaluate the mean free path of Na:

$$\lambda_{Na} = \frac{(9.109\times10^{-31}\ \text{kg})(1.07\times10^{6}\ \text{m/s})}{(2.65\times10^{22}\ \text{electrons/cm}^3)(1.602\times10^{-19}\ \text{C})^2(4.2\ \mu\Omega\cdot\text{cm})} = \boxed{34\,\text{nm}}$$

Proceed as above for Au:

$$\lambda_{Au} = \frac{(9.109\times10^{-31}\ \text{kg})(1.39\times10^{6}\ \text{m/s})}{(5.90\times10^{22}\ \text{electrons/cm}^3)(1.602\times10^{-19}\ \text{C})^2(2.04\ \mu\Omega\cdot\text{cm})} = \boxed{41.1\,\text{nm}}$$

Proceed as above for Sn:

$$\lambda_{Sn} = \frac{(9.109\times10^{-31}\ \text{kg})(1.89\times10^{6}\ \text{m/s})}{(14.8\times10^{22}\ \text{electrons/cm}^3)(1.602\times10^{-19}\ \text{C})^2(10.6\ \mu\Omega\cdot\text{cm})} = \boxed{4.29\,\text{nm}}$$

37 •• [SSM] The resistivity of pure copper increases by approximately $1.0 \times 10^{-8}\ \Omega\cdot\text{m}$ with the addition of 1.0 percent (by number of atoms) of an impurity distributed throughout the metal. The mean free path λ depends on both the impurity and the oscillations of the lattice ions according to the equation $1/\lambda = 1/\lambda_t + 1/\lambda_i$, where λ_t is the mean free path associated with the thermal vibrations of the ions and λ_i is the mean free path associated with the impurities. (*a*) Estimate λ_i using Equation 38-14 and the data given in Table 38-1. (*b*) If *r* is the effective radius of an impurity lattice ion seen by an electron, the scattering cross section is πr^2. Estimate this area, using the fact that *r* is related to λ_i by Equation 38-16.

Picture the Problem We can solve the resistivity equation for the mean free path and then substitute the Fermi speed for the average speed to express the mean free path as a function of the Fermi energy.

(*a*) In terms of the mean free path and the mean speed, the resistivity is:

$$\rho_i = \frac{m_e v_{av}}{ne^2\lambda_i} = \frac{m_e u_F}{ne^2\lambda_i} \Rightarrow \lambda_i = \frac{m_e u_F}{ne^2\rho_i}$$

Express the Fermi speed u_F in terms of the Fermi energy E_F:

$$u_F = \sqrt{\frac{2E_F}{m_e}}$$

Substitute for u_F and simplify to obtain:

$$\lambda_i = \frac{\sqrt{2m_e E_F}}{ne^2\rho_i}$$

Substitute numerical values (see Table 38-1) and evaluate λ_i:

$$\lambda_i = \frac{\sqrt{2\left(9.109\times10^{-31}\text{ kg}\right)\left(7.04\,\text{eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)}}{\left(8.47\times10^{28}\text{ electrons/m}^3\right)\left(1.602\times10^{-19}\text{ C}\right)^2\left(1.0\times10^{-8}\ \Omega\cdot\text{m}\right)} = 66.1\,\text{nm}$$
$$= \boxed{66\,\text{nm}}$$

(*b*) From Equation 38-16 we have:

$$\lambda = \frac{1}{n\pi r^2} \Rightarrow \pi r^2 = \frac{1}{n\lambda}$$

Substitute numerical values and evaluate πr^2:

$$\pi r^2 = \frac{1}{\left(8.47\times10^{28}\text{ m}^{-3}\right)\left(66.1\text{nm}\right)}$$
$$= \boxed{1.8\times10^{-4}\text{ nm}^2}$$

Band Theory of Solids

38 • Electromagnetic radiation is incident on the surface of a semiconductor. The maximum wavelength of this light that is required if electrons are to cross the energy gap between the valence band and the conduction band is 380.0 nm. What is the energy gap, in electron volts, for this semiconductor?

Picture the Problem We can use $E_g = hc/\lambda$ to calculate the energy gap for this semiconductor.

The lowest photon energy to increase conductivity is given by:

$$E_g = \frac{hc}{\lambda}$$

Substitute numerical values and evaluate E_g:

$$E_g = \frac{\left(6.626\times10^{-34}\,\text{J}\cdot\text{s}\right)\left(2.998\times10^{8}\,\text{m/s}\right)}{\left(380.0\times10^{-9}\,\text{m}\right)\left(1.602\times10^{-19}\,\text{J/eV}\right)}$$
$$= \boxed{3.26\,\text{eV}}$$

39 • **[SSM]** An electron occupies the highest energy level of the valence band in a silicon sample. What is the maximum photon wavelength that will excite the electron across the energy gap if the gap is 1.14 eV?

Picture the Problem We can relate the maximum photon wavelength to the energy gag using $\Delta E = hf = hc/\lambda$.

Express the energy gap as a function of the wavelength of the photon:

$$\Delta E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{1.14\,\text{eV}} = \boxed{1.09\,\mu\text{m}}$$

40 • An electron occupies the highest energy level of the valence band in a germanium sample. What is the maximum photon wavelength that will excite the electron into the conduction band? In germanium, the energy gap between the valence and conduction bands is 0.74 eV.

Picture the Problem We can relate the maximum photon wavelength to the energy gag using $\Delta E = hf = hc/\lambda$.

Express the energy gap as a function of the wavelength of the photon:

$$\Delta E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{0.74\,\text{eV}} = \boxed{1.7\,\mu\text{m}}$$

41 • An electron occupies the highest energy level of the valence band in a diamond sample. What is the maximum photon wavelength that will excite the electron into the conduction band? In diamond, the energy gap between the valence and conduction bands is 7.0 eV.

Picture the Problem We can relate the maximum photon wavelength to the energy gag using $\Delta E = hf = hc/\lambda$.

Express the energy gap as a function of the wavelength of the photon:

$$\Delta E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{7.0\,\text{eV}} = \boxed{180\,\text{nm}}$$

42 •• A photon of wavelength 3.35 μm has just enough energy to raise an electron from the valence band to the conduction band in a lead sulfide sample. (*a*) Find the energy gap between these bands in lead sulfide. (*b*) Find the temperature T for which kT equals this energy gap.

Picture the Problem We can use $\Delta E = hf = hc/\lambda$ to find the energy gap between these bands and $T = E_g/k$ to find the temperature for which kT equals this energy gap.

(*a*) Express the energy gap as a function of the wavelength of the photon:

$$\Delta E = \frac{hc}{\lambda}$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \frac{1240\,\text{eV}\cdot\text{nm}}{3.35\,\mu\text{m}} = 0.3701\,\text{eV} = \boxed{0.370\,\text{eV}}$$

(*b*) The temperature is related to the energy gap E_g according to:

$$T = \frac{E_g}{k}$$

Substitute numerical values and evaluate T:

$$T = \frac{0.3701\,\text{eV}}{8.617\times10^{-5}\,\text{eV/K}} = \boxed{4.29\times10^{3}\,\text{K}}$$

Semiconductors

43 • The donor energy levels in an *n*-type semiconductor are 0.0100 eV below the conduction band. Find the temperature for which kT = 0.0100 eV.

Picture the Problem We can use $\Delta E = kT$ to find the temperature for which kT = 0.0100 eV

Express the temperature T in terms of the energy gap ΔE:

$$T = \frac{\Delta E}{k}$$

Substitute numerical values and evaluate T:

$$T = \frac{0.0100\,\text{eV}}{1.381\times10^{-23}\,\text{J/K}\times\dfrac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}} = \boxed{116\,\text{K}}$$

44 •• When a thin slab of semiconducting material is illuminated with monochromatic electromagnetic radiation, most of the radiation is transmitted through the slab if the wavelength is greater than 1.85 mm. For wavelengths less than 1.85 mm, most of the incident radiation is absorbed. Determine the energy gap of this semiconductor.

Picture the Problem We can use $E = hf$ to find the energy gap of this semiconductor.

The energy gap of the semiconductor is given by:

$$E_g = hf = \frac{hc}{\lambda} \text{ where } hc = 1240 \text{ eV}\cdot\text{nm}$$

Substitute numerical values and evaluate E_g:

$$E_g = \frac{1240\,\text{eV}\cdot\text{nm}}{1.85\,\mu\text{m}} = \boxed{0.670\,\text{eV}}$$

45 •• The relative binding of the extra electron in the arsenic atom that replaces an atom in silicon or germanium can be understood from a calculation of the first Bohr radius of this electron in these materials. Four of arsenic's valence electrons form covalent bonds, so the fifth electron sees a center of attraction with a charge of $+e$. This model is a modified hydrogen atom. In the Bohr model of the hydrogen atom, the electron moves in free space at a radius a_0 given by $a_0 = 4\pi \in_0 \hbar^2/(m_e e^2)$ (Equation 36-12). When an electron moves in a crystal, we can approximate the effect of the other atoms by replacing $\in_0$ with $\kappa \in_0$ and m_e with an effective mass for the electron. For silicon, κ is 12 and the effective mass is approximately $0.2m_e$. For germanium, κ is 16 and the effective mass is approximately $0.1m_e$. Estimate the Bohr radii for the valence electron as it orbits the impurity arsenic atom in silicon and germanium.

Picture the Problem We can make the indicated substitutions in the expression for a_0 (= 0.0529 nm) to obtain an expression for the Bohr radii for the outer electron as it orbits the impurity arsenic atom in silicon and germanium.

Make the indicated substitutions in the expression for a_0 to obtain:

$$a_B = \frac{\kappa \in_0 h^2}{\pi\, m_{eff} e^2} = \frac{\kappa \in_0 m_e h^2}{\pi\, m_e m_{eff} e^2} = \frac{\kappa m_e}{m_{eff}} \frac{\in_0 h^2}{\pi\, m_e e^2} = \frac{\kappa m_e}{m_{eff}} a_0$$

For silicon:

$$a_{B,Si} = \frac{12m_e}{0.2m_e}(0.0529\,\text{nm}) = \boxed{3\,\text{nm}}$$

For germanium:

$$a_{B,Ge} = \frac{16m_e}{0.1m_e}(0.0529\,\text{nm}) = \boxed{8\,\text{nm}}$$

46 •• The ground-state energy of the hydrogen atom is given by $E_1 = -m_e e^4 / \left(8 \in_0^2 h^2\right)$ (Equation 36-15 and 36-16 where $4\pi\in_0$ is substituted for k^{-1}). Modify this equation using information in Problem 45 by replacing $\in_0$ with $\kappa\in_0$ and m_e with an effective mass for the electron to estimate the binding energy of the extra electron of an impurity arsenic atom in (*a*) silicon and (*b*) germanium.

Picture the Problem We can make the same substitutions we made in Problem 45 in the expression for E_1 (= 13.6 eV) to obtain an expression that we can use to estimate the binding energy of the extra electron of an impurity arsenic atom in silicon and germanium.

Make the indicated substitutions in the expression for E_1 to obtain:

$$E_{\text{binding}} = -\frac{e^4 m_{\text{eff}}}{8(\kappa \in_0)^2 h^2} = -\frac{e^4 m_e m_{\text{eff}}}{8 m_e \kappa^2 \in_0^2 h^2}$$
$$= -\frac{m_{\text{eff}}}{m_e \kappa^2 \in_0^2} \frac{e^4 m_e}{8 \in_0^2 h^2}$$
$$= -\frac{m_{\text{eff}}}{m_e \kappa^2 \in_0^2} E_1$$

(*a*) For silicon:

$$E_{\text{binding}} = -\frac{0.2 m_e}{m_e (12)^2}(13.6\,\text{eV})$$
$$= \boxed{-0.02\,\text{eV}}$$

(*b*) For germanium:

$$E_{\text{binding}} = -\frac{0.1 m_e}{m_e (16)^2}(13.6\,\text{eV})$$
$$= \boxed{-0.05\,\text{eV}}$$

47 •• A doped *n*-type silicon sample has 1.00×10^{16} electrons per cubic centimeter in the conduction band and has a resistivity of 5.00×10^{23} Ω·m at 300 K. Find the mean free path of the electrons. Use the effective mass of $0.2m_e$ for the mass of the electrons. (See Problem 45.) Compare this mean free path with that of conduction electrons in copper at 300 K.

Picture the Problem We can use the expression for the resistivity ρ of the sample as a function of the mean free path λ of the conduction electrons in conjunction with the expression for the average speed v_{av} of the electrons to derive an expression that we can use to calculate the mean free path of the electrons.

Express the resistivity ρ of the sample as a function of the mean free path λ of the conduction electrons:

$$\rho = \frac{m_e v_{av}}{n_e e^2 \lambda} \Rightarrow \lambda = \frac{m_e v_{av}}{n_e e^2 \rho} \qquad (1)$$

The average speed v_{av} of the electrons is given by:

$$v_{av} \approx v_{rms} = \sqrt{\frac{3kT}{m_e}}$$

Substitute for v_{av} in the expression for λ and simplify to obtain:

$$\lambda = \frac{m_e}{n_e e^2 \rho}\sqrt{\frac{3kT}{m_e}} = \frac{\sqrt{3km_eT}}{n_e e^2 \rho}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{\sqrt{3(1.381\times10^{-23}\text{ J/K})(0.2)(9.109\times10^{-31}\text{ kg})(300\text{ K})}}{(10^{16}\text{ cm}^{-3})(1.602\times10^{-19}\text{ C})^2(5.00\times10^{-3}\ \Omega\cdot\text{m})} = \boxed{37.1\text{ nm}}$$

The number density of electrons n_e is related to the mass density ρ_m, Avogadro's number N_A, and the molar mass M.:

$$n_e = \frac{\rho_m N_A}{M}$$

Substitute numerical values (For copper, ρ = 8.93 g/cm^3 and M = 63.5 g/mol.) and evaluate n_e:

$$n_e = \frac{(8.93\text{ g/cm}^3)(6.022\times10^{23}\text{ electrons/mol})}{63.5\text{ g/mol}} = 8.47\times10^{28}\text{ electrons/m}^3$$

Using equation (1), evaluate λ_{Cu} (see Table 25-1 for the resistivity of copper and Example 38-5 for u_F):

$$\lambda_{Cu} = \frac{(9.109\times10^{-31}\text{ kg})(1.57\times10^{6}\text{ m/s})}{(8.47\times10^{28}\text{ electrons/m}^3)(1.602\times10^{-19}\text{ C})^2(1.70\times10^{-8}\ \Omega\cdot\text{m})} = \boxed{38.7\text{ nm}}$$

The mean free paths agree to within about 4%.

48 ••• In the Hall effect, the Hall coefficient R_H is the proportionality constant between the transverse electric field and the product of the applied magnetic field and the current density. That is, $E_y = R_H B_z J_x$, where the current density, the transverse electric field and the applied magnetic field are in the +x, +y and +z directions, respectively. (The Hall effect is presented in Chapter 26.) The measured Hall coefficient of a doped silicon sample is 0.0400 V·m/(A·T) at room temperature. If all the doping impurities have contributed to the total number of charge carriers of the sample, find (a) the type of impurity (donor or acceptor) used to dope the sample and (b) the concentration of these impurities.

Picture the Problem We can use the definition of the Hall electric field in the slab, the expression for the Hall voltage across it, and the definition of current density to show that the Hall coefficient is given by 1/(*nq*). We can use the expression for the Hall coefficient to determine the type of impurity and the concentration of these impurities.

(*a*) The Hall coefficient is:

$$R=\frac{E_y}{J_xB_z}$$

where J_x is *x* component of the current density in the material, B_z is the *z* component of the magnetic field, and E_y is the *y* component resulting Hall electric field.

Using its definition, express the Hall electric field in the slab:

$$E_y=\frac{V_\mathrm{H}}{w}$$

The current density in the slab is:

$$J_x=\frac{I}{wt}=nqv_\mathrm{d}$$

Substitute for E_y and J_x and simplify to obtain:

$$R=\frac{\frac{V_\mathrm{H}}{w}}{nqv_\mathrm{d}B_z}=\frac{V_\mathrm{H}}{nqv_\mathrm{d}wB_z}$$

Express the Hall voltage in terms of v_d, *B*, and *w*:

$$V_\mathrm{H}=v_\mathrm{d}B_zw$$

Substitute for V_H and simplify to obtain:

$$R=\frac{v_\mathrm{d}B_zw}{nqv_\mathrm{d}wB_z}=\frac{1}{nq} \qquad (1)$$

Because $R > 0$, $q > 0$ and conduction is by holes and the sample contains acceptor impurities.

(*b*) Solve equation (1) for *n*:

$$n=\frac{1}{Rq}$$

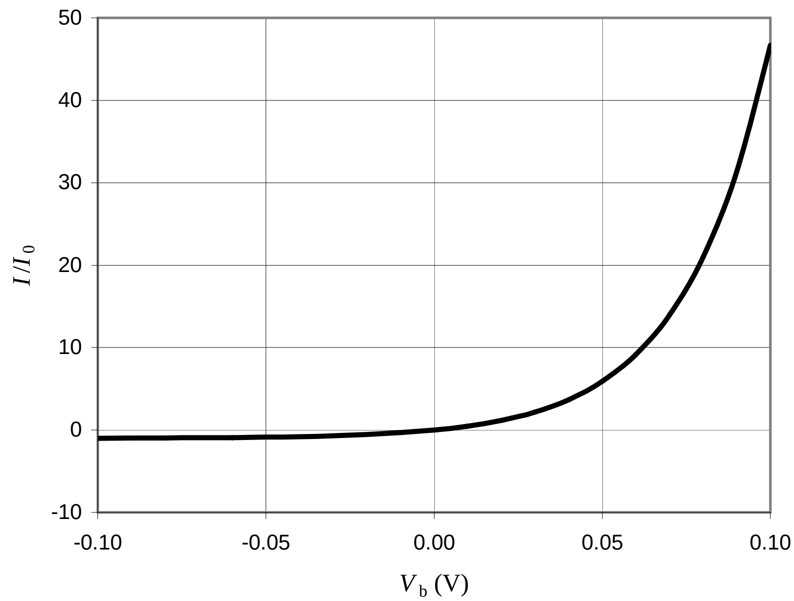
Substitute numerical values and evaluate *n*:

$$n=\frac{1}{(0.0400\,\mathrm{V\cdot m/A\cdot T})(1.602\times10^{-19}\,\mathrm{C})}=\boxed{1.56\times10^{20}\ \mathrm{m}^{-3}}$$

Semiconductor Junctions and Devices

49 •• Simple theory for the current versus the bias voltage across a *pn* junction yields the equation $I = I_0\left(e^{eV_b/kT} - 1\right)$. Sketch I versus V_b for both positive and negative values of V_b using this equation.

Picture the Problem The following graph of I/I_0 versus V_b was plotted using a spreadsheet program.



50 • The base current in an *npn* transistor circuit is 25.0 mA. If 88.0 percent of the electrons entering the base from the emitter reach the collector, what is the base current?

Picture the Problem The base current is the difference between the emitter current and the plate current.

The base current I_B is given by: $$I_B = I - I_C$$

We're given that: $$I_C = 0.880I = 25.0\,\text{mA}$$

Solving for I gives: $$I = \frac{25.0\text{ mA}}{0.880} = 28.4\,\text{mA}$$

Substitute numerical values for I and I_C and evaluate I_B: $$I_B = 28.4\,\text{mA} - 25.0\,\text{mA} = \boxed{3.4\,\text{mA}}$$

51 •• [SSM] In Figure 38-28 for the *pnp*-transistor amplifier, suppose $R_b = 2.00\ \text{k}\Omega$ and $R_L = 10.0\ \text{k}\Omega$. Suppose further that a 10.0-μA ac base current i_b generates a 0.500-mA ac collector current i_c. What is the voltage gain of the amplifier?

Picture the Problem We can use its definition to compute the voltage gain of the amplifier.

The voltage gain of the amplifier is given by:

$$\text{Voltage gain} = \frac{i_c R_L}{i_b R_b}$$

Substitute numerical values and evaluate the voltage gain:

$$\text{Voltage gain} = \frac{(0.500\,\text{mA})(10.0\,\text{k}\Omega)}{(10.0\,\mu\text{A})(2.00\,\text{k}\Omega)} = \boxed{250}$$

52 •• Germanium can be used to measure the energy of incident photons. Consider a 660-keV gamma ray emitted from ^{137}Cs. (*a*) Given that the band gap in germanium is 0.72 eV, how many electron–hole pairs can be generated as this gamma ray travels through germanium? (*b*) The number *N* of pairs in Part (*a*) will have statistical fluctuations given by $\pm\sqrt{N}$. What then is the energy resolution of this detector in this photon energy region?

Picture the Problem The number of electron-hole pairs *N* is related to the energy *E* of the incident beam and the energy gap E_g.

(*a*) The number of electron-hole pairs *N* is given by:

$$N = \frac{E}{E_g}$$

Substitute numerical values and evaluate *N*:

$$N = \frac{660\,\text{keV}}{0.72\,\text{eV}} = 9.17\times10^5 = \boxed{9.2\times10^5}$$

(*b*) The energy resolution of the detector is given by:

$$\frac{\Delta E}{E} = \frac{\Delta N}{N}$$

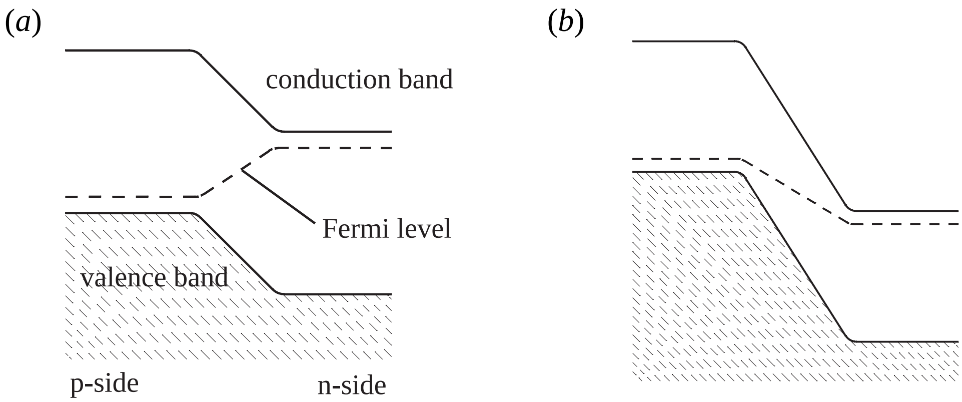
For $\Delta N = 1$ and $N = \sqrt{9.17\times10^5}$:

$$\frac{\Delta E}{E} = \frac{1}{\sqrt{9.17\times10^5}} = \boxed{0.10\%}$$

53 •• Make a sketch showing the valence and conduction band edges and Fermi energy of a *pn*-junction diode when biased (*a*) in the forward direction and (*b*) in the reverse direction.

Picture the Problem The nearly full valence band is shown shaded in the

diagram to the right. The Fermi level is shown by the dashed line.



54 •• A good silicon diode has the current–voltage characteristic given by $I = I_0\left(e^{eV_b/kT} - 1\right)$. Let kT = 0.025 eV (room temperature) and the saturation current I_0 = 1.0 nA. (*a*) Show that for small reverse-bias voltages, the resistance is 25 MΩ. *Hint: Do a Taylor-series expansion of the exponential function about* V_b = 0, *or use the expansion for* e^x *found in Table M-4 of the Math Tutorial.* (*b*) Find the dc resistance for a reverse bias of 0.50 V. (*c*) Find the resistance V/I for a 0.50-V forward bias. What is the current in this case? (*d*) Calculate the ac resistance dV/dI for a 0.50-V forward bias.

Picture the Problem We can use Ohm's law and the expression for the current given in the problem statement to find the resistance for small reverse-and-forward bias voltages.

(*a*) Use Ohm's law to express the resistance:

$$R = \frac{V_b}{I} \quad (1)$$

We're given that:

$$I = I_0\left(e^{eV_b/kT} - 1\right) \quad (2)$$

For $eV_b << kT$:

$$e^{eV_b/kT} - 1 \approx 1 + \frac{eV_b}{kT} - 1 = \frac{eV_b}{kT}$$

Substitute for $e^{eV_b/kT} - 1$ to obtain:

$$I = I_0\frac{eV_b}{kT}$$

Substitute for I in equation (1) and simplify:

$$R = \frac{V_b}{I_0\dfrac{eV_b}{kT}} = \frac{kT}{eI_0}$$

Substitute numerical values and evaluate R:

$$R = \frac{(0.025\,\mathrm{eV})(1.602\times10^{-19}\ \mathrm{J/eV})}{(1.602\times10^{-19}\ \mathrm{C})(1.0\,\mathrm{nA})} = \boxed{25\,\mathrm{M}\Omega}$$

(b) Substitute equation (2) in equation (1) to obtain:

$$R = \frac{V_\mathrm{b}}{I_0\left(e^{eV_\mathrm{b}/kT}-1\right)} \qquad (3)$$

Evaluate $\frac{eV_\mathrm{b}}{kT}$ for $V_\mathrm{b} = -\,0.50$ V:

$$\frac{eV_\mathrm{b}}{kT} = \frac{(1.602\times10^{-19}\ \mathrm{C})(-\,0.50\,\mathrm{V})}{(0.025\,\mathrm{eV})(1.602\times10^{-19}\ \mathrm{J/eV})} = -19.8$$

Evaluate equation (3) for $V_\mathrm{b} = -\,0.50$ V:

$$R = \frac{-\,0.50\,\mathrm{V}}{(1.0\,\mathrm{nA})\left(e^{-19.8}-1\right)} = \boxed{5.0\times10^{8}\ \Omega}$$

(c) The resistance V/I is given by:

$$R = \frac{V}{I} = \frac{V}{I_0\left(e^{eV_\mathrm{b}/kT}-1\right)} \qquad (4)$$

Evaluate $\frac{eV_\mathrm{b}}{kT}$ for $V_\mathrm{b} = +\,0.50$ V:

$$\frac{eV_\mathrm{b}}{kT} = \frac{(1.602\times10^{-19}\ \mathrm{C})(0.50\,\mathrm{V})}{(1.381\times10^{-23}\ \mathrm{J/K})(293\,\mathrm{K})} = 19.8$$

Substituting numerical values in equation (4) gives:

$$R = \frac{0.50\,\mathrm{V}}{(1.0\,\mathrm{nA})\left(e^{19.8}-1\right)} = \boxed{1.3\Omega}$$

From equation (1):

$$I = \frac{0.50\,\mathrm{V}}{1.3\Omega} = \boxed{0.38\,\mathrm{A}}$$

(d) Evaluate $R_\mathrm{ac} = dV/dI$ to obtain:

$$R_\mathrm{ac} = \frac{dV}{dI} = \left(\frac{dI}{dV}\right)^{-1} = \left\{\frac{d}{dV}\left[I_0\left(e^{eV_\mathrm{b}/kT}-1\right)\right]\right\}^{-1} = \left\{\frac{eI_0}{kT}e^{eV_\mathrm{b}/kT}\right\}^{-1} = \frac{kT}{eI_0}e^{-eV_\mathrm{b}/kT}$$

Substitute numerical values and evaluate R_ac:

$$R_\mathrm{ac} = (25\,\mathrm{M}\Omega)e^{-19.8} = \boxed{63\,\mathrm{m}\Omega}$$

55 •• A differential long slab of silicon of thickness t = 1.0 mm and width w = 1.0 cm is placed in a magnetic field B = 0.40 T. The slab is in the xy plane with the length of the slab parallel with the x axis, and the magnetic field points in the +z direction. When a current of 0.20 A exists in the sample in the +x direction, a potential difference of 5.0 mV develops across the width of the sample and the electric field in the sample points in the +y direction. Determine the semiconductor type (n or p) and the concentration of charge carriers. (The Hall effect is presented in Chapter 26.)

Picture the Problem We can use the Hall-effect equation to find the concentration of charge carriers in the slab of silicon. We can determine the semiconductor type by determining the directions of the magnetic and electric fields.

Use the expression for the Hall-effect voltage to relate the concentration of charge carriers n to the Hall voltage V_H:

$$V_H = \frac{IB}{nte} \Rightarrow n = \frac{IB}{teV_H}$$

Substitute numerical values and evaluate n:

$$n = \frac{(0.20\,\text{A})(0.40\,\text{T})}{(1.0\,\text{mm})(1.602\times10^{-19}\,\text{C})(5.0\,\text{mV})} = \boxed{1.0\times10^{23}\,\text{m}^{-3}}$$

Referring to Figure 26-28, note that $\vec{B}$ points out of the page. $\vec{E}$ is in the +y direction. Therefore, the charge carriers are holes and the semiconductor is p-type.

The BCS Theory

56 • (a) Use Equation 38-37 to calculate the superconducting energy gap for tin, and compare your result with the measured value of 6.00×10^{-4} eV. (b) Use the measured value to calculate the minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in tin (T_c = 3.72 K) at T = 0.

Picture the Problem We can calculate E_g using $E_g = 3.5kT_c$ and find the minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in tin at T = 0 using $\lambda = hc/E_g$.

(a) From Equation 38-37 we have:

$$E_g = 3.5kT_c$$

Substitute numerical values and evaluate E_g:

$$E_g = 3.5(8.617\times10^{-5}\,\text{eV/K})(3.72\,\text{K}) = \boxed{1.12\,\text{meV}}$$

Express the ratio of E_g to $E_{g,measured}$:

$$\frac{E_g}{E_{g,measured}} = \frac{1.12\,meV}{6.0\times10^{-4}\,eV} = 1.87$$

or

$$E_g \approx \boxed{2E_{g,measured}}$$

(*b*) The minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in tin at $T = 0$ is given by:

$$\lambda = \frac{hc}{E_g}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,eV\cdot nm}{6.00\times10^{-4}\,eV} = \boxed{2.07\,mm}$$

57 • **[SSM]** (*a*) Use Equation 38-37 to calculate the superconducting energy gap for lead, and compare your result with the measured value of 2.73×10^{-3} eV. (*b*) Use the measured value to calculate the minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in lead (T_c = 7.19 K) at $T = 0$.

Picture the Problem We can calculate E_g using $E_g = 3.5kT_c$ and find the minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in lead at $T = 0$ using $\lambda = hc/E_g$.

(*a*) From Equation 38-37 we have:

$$E_g = 3.5kT_c$$

Substitute numerical values and evaluate E_g:

$$E_g = 3.5\left(8.62\times10^{-5}\,eV/K\right)\left(7.19\,K\right) = \boxed{2.17\,meV}$$

Express the ratio of E_g to $E_{g,measured}$:

$$\frac{E_g}{E_{g,measured}} = \frac{2.17\,meV}{2.73\times10^{-3}\,eV} = 0.795$$

or

$$E_g \approx \boxed{0.8E_{g,measured}}$$

(*b*) The minimum value of the wavelength of a photon that has sufficient energy to break up Cooper pairs in lead at $T = 0$ is given by:

$$\lambda = \frac{hc}{E_g}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{2.73\times10^{-3}\,\text{eV}} = \boxed{0.454\,\text{mm}}$$

The Fermi-Dirac Distribution

58 •• The number of electrons in the conduction band of an insulator or intrinsic semiconductor is governed chiefly by the Fermi factor. Because the valence band in these materials is nearly filled and the conduction band is nearly empty, the Fermi energy E_F is generally midway between the top of the valence band and the bottom of the conduction band; that is, it is at $E_g/2$, where E_g is the band gap between the two bands and the energy is measured from the top of the valence band. (*a*) In silicon, $E_g \approx 1.0$ eV. Show that in this case the Fermi factor for electrons at the bottom of the conduction band is given by $\exp(-E_g/2kT)$ and evaluate this factor. Discuss the significance of this result if there are 10^{22} valence electrons per cubic centimeter and the probability of finding an electron in the conduction band is given by the Fermi factor. (*b*) Repeat the calculation in Part (*a*) for an insulator with a band gap of 6.0 eV.

Picture the Problem We can evaluate the Fermi factor at the bottom of the conduction band for T near room temperature to show that this factor is given by $\exp(-E_g/2kT)$.

(*a*) At the bottom of the conduction band:

$e^{(E-E_F)/kT} = e^{E_g/2kT} >>> 1$ for T near room temperature.

Neglecting the 1 in the denominator of the Fermi function yields:

$$f\left(\tfrac{1}{2}E_g\right) = \frac{1}{e^{E_g/2kT}} = \boxed{e^{-E_g/2kT}}$$

Substitute numerical values and evaluate $f\left(\tfrac{1}{2}E_g\right)$ for $T = 300$ K:

$$f\left(\tfrac{1}{2}E_g\right) = \exp\left[\frac{-1.0\,\text{eV}}{2\left(8.62\times10^{-5}\,\text{eV/K}\right)\left(300\,\text{K}\right)}\right] = \boxed{4.0\times10^{-9}}$$

Given that low a probability of finding an electron in a state near the bottom of the conduction band, the exclusion principle has no significant impact on the distribution function. With 10^{22} valence electrons per cm^3, the number of electrons in the conduction band will be about 4×10^{13} per cm^3.

(*b*) Evaluate $f\left(\frac{1}{2}E_g\right)$ for $T = 300$ K and $E_g = 6$ eV:

$$f\left(\tfrac{1}{2}E_g\right) = \exp\left[\frac{-6.0\,\text{eV}}{2\left(8.62\times10^{-5}\ \text{eV/K}\right)\left(300\,\text{K}\right)}\right] = \boxed{4.2\times10^{-51}}$$

The probability of finding even one electron in the conduction band is negligibly small (approximately 4×10^{-51}).

59 •• Approximately how many energy states that have energies between 2.00 eV and 2.20 eV are available to electrons in a cube of silver measuring 1.00 mm on a side?

Picture the Problem The number of energy states per unit volume per unit energy interval N is given by $N \approx g(E)\Delta E$, where N is only approximate, because ΔE is not infinitesimal and $g(E) = \frac{8\sqrt{2}\pi m_e^{3/2}V}{h^3}E^{1/2}$ is the density of states.

The number of states N is the product of the density of states and the energy interval:

$$N \approx g(E)\Delta E \qquad (1)$$

The density of states is given by:

$$g(E) = \frac{8\sqrt{2}\pi m_e^{3/2}V}{h^3}E^{1/2}$$

Substitute numerical values and evaluate $g(E)$:

$$g(E) = \frac{8\sqrt{2}\pi\left(9.109\times10^{-31}\ \text{kg}\right)^{3/2}\left(1.00\times10^{-3}\ \text{m}\right)^3}{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^3}\left(2.10\,\text{eV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)^{1/2}$$
$$= 6.15\times10^{37}\ \text{J}^{-1}$$

Substitute numerical values in equation (1) and evaluate N:

$$N \approx \left(6.15\times10^{37}\ \text{J}^{-1}\right)\left(2.20\,\text{eV} - 2.00\,\text{eV}\right)\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}} \approx \boxed{2.0\times10^{18}}$$

60 •• Show that at $E = E_F$, the expression for the Fermi factor (Equation 38-24) is equal to 0.5.

Picture the Problem Evaluating $f(E_F)$ at $E = E_F$ will show that $f(E) = 0.5$.

The Fermi factor is:

$$f(E) = \frac{1}{e^{(E-E_F)/kT} + 1}$$

Evaluate $f(E_F)$:

$$f(E_F) = \frac{1}{e^{(E_F-E_F)/kT} + 1} = \frac{1}{1+1} = \boxed{0.5}$$

61 •• **[SSM]** (*a*) Using the equation $E_F = \left[h^2/(8m_e)\right]\left[3N/(\pi V)\right]^{2/3}$ (Equation 38-22*a*), calculate the Fermi energy for silver. (*b*) Determine the average kinetic energy of a free electron and (*c*) find the Fermi speed for silver.

Picture the Problem Equation 38-22*a* expresses the dependence of the Fermi energy E_F on the number density of free electrons. Once we've determined the Fermi energy for silver, we can find the average kinetic energy of a free electron from the Fermi energy for silver and then use the average kinetic energy of a free electron to find the Fermi speed for silver.

(*a*) From Equation 38-22*a* we have:

$$E_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3}$$

Use Table 27-1 to find the free-electron number density N/V for silver:

$$\frac{N}{V} = 5.86\text{x}10^{22}\ \frac{\text{electrons}}{\text{cm}^3} = 5.86\text{x}10^{28}\ \frac{\text{electrons}}{\text{m}^3}$$

Substitute numerical values and evaluate E_F:

$$E_F = \frac{\left(6.626\times10^{-34}\,\text{J}\cdot\text{s}\right)^2}{8\left(9.109\times10^{-31}\,\text{kg}\right)}\left[\frac{3\left(5.86\times10^{28}\,\text{electrons/m}^3\right)}{\pi}\right]^{2/3}\left(\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}\right) = \boxed{5.51\,\text{eV}}$$

(*b*) The average kinetic energy of a free electron is given by:

$$E_{av} = \frac{3}{5}E_F$$

Substitute numerical values and evaluate E_{av}:

$$E_{av} = \frac{3}{5}(5.51\,\text{eV}) = \boxed{3.31\,\text{eV}}$$

(*c*) Express the Fermi energy in terms of the Fermi speed of the electrons:

$$E_F = \tfrac{1}{2}m_e u_F^2 \Rightarrow u_F = \sqrt{\frac{2E_F}{m_e}}$$

Substitute numerical values and evaluate u_F:

$$u_F = \sqrt{\frac{2(3.31\,\text{eV})}{9.109\times10^{-31}\,\text{kg}}\left(\frac{1.602\times10^{-19}\,\text{J}}{1\,\text{eV}}\right)} = \boxed{1.08\times10^{6}\ \text{m/s}}$$

62 •• What is the difference between the energies at which the Fermi factor is 0.9 and 0.1 at 300 K in (*a*) copper, (*b*) potassium, and (*c*) aluminum.

Picture the Problem We can find the difference between the energies at which the Fermi factor has the given values by solving the expression for Fermi factor for *E* and then deriving an expression for Δ*E*.

(*a*) The Fermi factor is:

$$f(E) = \frac{1}{e^{(E-E_F)/kT}+1}$$

Solving for *E* gives:

$$E = E_F + kT\ln\left(\frac{1}{f(E)}-1\right)$$

The difference between the energies is given by:

$$\begin{aligned}
\Delta E &= E(0.1) - E(0.9) \\
&= E_F + \frac{(1.381\times10^{-23}\ \text{J/K})(300\,\text{K})}{1.602\times10^{-19}\ \text{J/eV}}\ln\left(\frac{1}{0.1}-1\right) \\
&\quad - \left\{E_F + \frac{(1.381\times10^{-23}\ \text{J/K})(300\,\text{K})}{1.602\times10^{-19}\ \text{J/eV}}\ln\left(\frac{1}{0.9}-1\right)\right\} \\
&= \frac{(1.381\times10^{-23}\ \text{J/K})(300\,\text{K})}{1.602\times10^{-19}\ \text{J/eV}}\left[\ln\left(\frac{1}{0.1}-1\right) - \ln\left(\frac{1}{0.9}-1\right)\right] \\
&= \boxed{0.114\,\text{eV}}
\end{aligned}$$

(*b*) and (*c*) Because Δ*E* is independent of E_F, Δ*E* is the same as in (*a*).

63 •• **[SSM]** What is the probability that a conduction electron in silver will have a kinetic energy of 4.90 eV at *T* = 300 K?

Picture the Problem The probability that a conduction electron will have a given kinetic energy is given by the Fermi factor.

The Fermi factor is:

$$f(E)=\frac{1}{e^{(E-E_\mathrm{F})/kT}+1}$$

Because E_F – 4.90 eV >> 300k:

$$f(4.90\,\mathrm{eV})=\frac{1}{0+1}=\boxed{1}$$

64 •• Show that $g(E)=\frac{3}{2}NE_\mathrm{F}^{-3/2}E^{1/2}$ (Equation 38-43) follows from Equation 38-41 for $g(E)$, and from Equation 38-22a for E_F.

Picture the Problem We can solve Equation 38-22a for V and substitute in Equation 38-41 to show that $g(E)=\frac{3}{2}NE_\mathrm{F}^{-3/2}E^{1/2}$.

From Equation 38-22a we have:

$$E_\mathrm{F}=\frac{h^2}{8m_\mathrm{e}}\left(\frac{3N}{\pi V}\right)^{2/3}$$

Solving for V yields:

$$V=\frac{3N}{\pi}\left(\frac{h^2}{8m_\mathrm{e}E_\mathrm{F}}\right)^{3/2}$$

The density $g(E)$ is given by Equation 38-41:

$$g(E)=\frac{8\pi\sqrt{2}m_\mathrm{e}^{3/2}V}{h^3}E^{1/2}$$

Substitute for V and simplify to obtain:

$$g(E)=\frac{8\pi\sqrt{2}m_\mathrm{e}^{3/2}}{h^3}\left[\frac{3N}{\pi}\left(\frac{h^2}{8m_\mathrm{e}E_\mathrm{F}}\right)^{3/2}\right]E^{1/2}$$
$$=\boxed{\frac{3}{2}NE_\mathrm{F}^{-3/2}E^{1/2}}$$

65 •• Carry out the integration $E_\mathrm{av}=(1/N)\int_0^{E_\mathrm{F}}Eg(E)dE$ to show that the average energy at T = 0 is $\frac{3}{5}E_\mathrm{F}$.

Picture the Problem We can use the expression for $g(E)$ from Problem 64 to show that the average energy at T = 0 is $\frac{3}{5}E_\mathrm{F}$.

From Problem 64 we have:

$$g(E)=\frac{3}{2}NE_\mathrm{F}^{-3/2}E^{1/2}$$

Substitute in the expression for E_{av} and simplify to obtain:

$$E_{av}=\frac{1}{N}\int_0^{E_F}Eg(E)dE$$
$$=\frac{1}{N}\int_0^{E_F}E\left(\frac{3N}{2}E_F^{-3/2}E^{1/2}\right)dE$$
$$=\frac{3}{2}E_F^{-3/2}\int_0^{E_F}E^{3/2}dE$$

Integrating the expression for E_{av} gives:

$$E_{av}=\frac{3}{2}E_F^{-3/2}\int_0^{E_F}E^{3/2}dE$$
$$=\left(\tfrac{3}{2}E_F^{-3/2}\right)\left(\tfrac{2}{5}E_F^{5/2}\right)$$
$$=\boxed{\tfrac{3}{5}E_F}$$

66 •• The density of the electron states in a metal can be written $g(E) = AE^{1/2}$, where A is a constant and E is measured from the bottom of the conduction band. (*a*) Show that the total number of states is $\tfrac{2}{3}AE_F^{3/2}$. (*b*) Approximately what fraction of the conduction electrons are within kT of the Fermi energy? (*c*) Evaluate this fraction for copper at T = 300 K.

Picture the Problem We can integrate $g(E)$ from 0 to E_F to show that the total number of states is $\tfrac{2}{3}AE_F^{3/2}$.

(*a*) Integrate $g(E)$ from 0 to E_F:

$$N=\int_0^{E_F}AE^{1/2}dE=\boxed{\tfrac{2}{3}AE_F^{3/2}}$$

(*b*) Express the fraction of N within kT of E_F:

$$\frac{kTg(E_F)}{N}=\frac{kTAE_F^{1/2}}{\tfrac{2}{3}AE_F^{3/2}}=\boxed{\frac{3kT}{2E_F}}$$

(*c*) Substitute numerical values and evaluate the expression obtained in (*b*) for copper:

$$\frac{3kT}{2E_F}=\frac{3\left(8.62\times10^{-5}\ \text{eV/K}\right)\left(300\ \text{K}\right)}{2\left(7.04\ \text{eV}\right)}$$
$$=\boxed{5.51\times10^{-3}}$$

67 •• What is the probability that a conduction electron in silver will have a kinetic energy of 5.49 eV at T = 300 K?

Picture the Problem The probability that a conduction electron in metal is the Fermi factor.

The Fermi factor is:

$$f(E)=\frac{1}{e^{(E-E_\mathrm{F})/kT}+1}$$

Calculate the dimensionless exponent in the Fermi factor:

$$\frac{E-E_\mathrm{F}}{kT}=\frac{5.49\,\mathrm{eV}-5.50\,\mathrm{eV}}{(8.617\times10^{-5}\ \mathrm{eV/K})(300\,\mathrm{K})}$$
$$=-0.387$$

Use this result to calculate the Fermi factor:

$$f(5.49\,\mathrm{eV})=\frac{1}{e^{-0.387}+1}=\boxed{0.60}$$

68 •• Using the expression $g(E)=\left(8\sqrt{2}\pi m_\mathrm{e}^{3/2}V/h^3\right)E^{1/2}$ (Equation38-41) for the density of states, estimate the fraction of the conduction electrons in copper that can absorb energy from collisions with the vibrating lattice ions at (*a*) 77 K and (*b*) 300 K.

Picture the Problem We can integrate the density-of-states function, Equation 38-41, to find the number of occupied states *N*. The fraction of these states that are within *kT* of E_F can then be found from the ratio of $kTg(E_\mathrm{F})$ to *N*.

The density of states function is:

$$g(E)=AE^{1/2}\text{ where }A=\frac{8\pi\sqrt{2}m_\mathrm{e}^{3/2}V}{h^3}$$

Integrate $g(E)$ from 0 to E_F to find the total number of occupied states:

$$N=\int_0^{E_\mathrm{F}}AE^{1/2}dE=\tfrac{2}{3}AE_{E_\mathrm{F}}^{3/2}$$

Express the fraction of *N* within *kT* of E_F:

$$\frac{kTg(E_\mathrm{F})}{N}=\frac{kTAE_\mathrm{F}^{1/2}}{\frac{2}{3}AE_\mathrm{F}^{3/2}}=\frac{3kT}{2E_\mathrm{F}}$$

(*a*) Substitute numerical values and evaluate the expression obtained above for copper at T = 77 K:

$$\frac{3kT}{2E_\mathrm{F}}=\frac{3(8.617\times10^{-5}\ \mathrm{eV/K})(77\,\mathrm{K})}{2(7.04\,\mathrm{eV})}$$
$$=\boxed{1.4\times10^{-3}}$$

(*b*) At T = 300 K:

$$\frac{3kT}{2E_\mathrm{F}}=\frac{3(8.617\times10^{-5}\ \mathrm{eV/K})(300\,\mathrm{K})}{2(7.04\,\mathrm{eV})}$$
$$=\boxed{5.51\times10^{-3}}$$

69 •• In an intrinsic semiconductor, the Fermi energy is about midway between the top of the valence band and the bottom of the conduction band. In germanium, the forbidden energy band has a width of 0.70 eV. Show that at room

temperature the distribution function of electrons in the conduction band is given by the Maxwell–Boltzmann distribution function.

Picture the Problem The distribution function of electrons in the conduction band is given by $n(E)=g(E)f(E)$ where $f(E)$ is the Fermi factor and $g(E)$ is the density of states in terms of E_F.

Express the number of electrons n with energy E:

$$n(E)=g(E)f(E) \qquad (1)$$

where

$$g(E)=\frac{3N}{2}E_{E_F}^{-3/2}E^{1/2}$$

and

$$f(E)=\frac{1}{e^{(E-E_F)/kT}+1}$$

The dimensionless exponent in the Fermi factor is:

$$\frac{E-E_F}{kT}=\frac{E-\frac{1}{2}E_g}{kT}>>1$$

and

$$\exp\left(\frac{E-\frac{1}{2}E_g}{kT}\right)>>1$$

Hence:

$$f(E)=\frac{1}{e^{\left(E-\frac{1}{2}E_g\right)/kT}}\approx e^{E_g/2kT}e^{-E/kT}$$

Substitute in equation (1) and simplify to obtain:

$$n(E)=\boxed{\left(\tfrac{3}{2}NE_F^{-3/2}e^{E_g/2kT}\right)E^{1/2}e^{-E/kT}}$$

There is an additional temperature dependence that arises from the fact that E_F depends on T. At room temperature,

$$\exp[(E-E_g/2)/kT]\geq\exp(0.35\text{ eV}/0.0259\text{ eV})=7.4\times10^5,$$

so the approximation leading to the Boltzmann distribution is justified.

70 ••• The root-mean-square (rms) value of a variable is obtained by calculating the average value of the square of that variable and then taking the square root of the result. Use this procedure to determine the rms energy of a Fermi distribution. Express your result in terms of E_F and compare it to the average energy. Why do E_{av} and E_{rms} differ?

Picture the Problem We can follow the procedure outlined in the problem statement to determine the rms energy of a Fermi distribution.

Express the E_{rms} in terms of $g(E)$:

$$E_{rms} = \left(\frac{1}{N}\int_0^{E_F} g(E)E^2 dE\right)^{1/2}$$

The density of states $g(E)$ is given by:

$$g(E) = \frac{3N}{2}E_F^{-3/2}E^{1/2}$$

Substitute to obtain:

$$E_{rms} = \left(\frac{1}{2NE_F^{3/2}}\int_0^{E_F} E^{3/2} dE\right)^{1/2}$$

Evaluate the integral and simplify to obtain:

$$E_{rms} = \sqrt{\frac{3}{7}}E_F = \boxed{0.655E_F}$$

$E_{rms} > E_{av}$ because the process of averaging the square of the energy weights larger energies more heavily.

General Problems

71 • The density of potassium is 0.851 g/cm^3. How many free electrons are there per potassium atom in a crystal of potassium?

Picture the Problem The number of free electrons per atom n_e is given by $n_e = nM/\rho N_A$ where N_A is Avogadro's number, ρ is the density of the element, M is its molar mass, and n is the free electron number density for the element.

The number of electrons per atom is given by:

$$n_e = \frac{nM}{\rho N_A}$$

From Table 38-1:

$$n = 14.0\ \text{electrons/nm}^3$$
$$= 1.40\times10^{22}\ \text{electrons/cm}^3$$

Substitute numerical values and evaluate n_e:

$$n_e = \frac{(1.40\times10^{22}\ \text{electrons/cm}^3)(39.098\,\text{g}/\text{mol})}{(0.851\text{g/cm}^3)(6.022\times10^{23}\ \text{electrons/mol})} = \boxed{1.07}$$

Remarks: Our result of 1.07 free electrons per potassium atom means that, for every 100 potassium atoms, there are 107 free electrons.

72 • Calculate the number density of free electrons for (*a*) magnesium, which has a density of 1.74 g/cm^3, and (*b*) zinc, which has a density of 7.14 g/cm^3. For these calculations assume there are two free electrons per atom, and compare your results with the values listed in Table 38-1.

Picture the Problem The number of free electrons per atom n_e is related to the number density of free electrons n by $n_e = nM/\rho N_A$, where N_A is Avogadro's number, ρ is the density of the element, and M is its molar mass.

The number of electrons per atom is given by:

$$n_e = \frac{nM}{\rho N_A}$$

Solving for n gives:

$$n = \frac{n_e \rho N_A}{M}$$

(*a*) Substitute numerical values and evaluate n for Mg:

$$n_{Mg} = \frac{(2)(1.74\,\mathrm{g/cm^3})(6.022\times10^{23}\,\mathrm{electrons/mol})}{24.31\,\mathrm{g/mol}} = \boxed{8.62\times10^{22}\,\mathrm{electrons/cm^3}}$$

(*b*) Substitute numerical values and evaluate n for Zn:

$$n_{Zn} = \frac{(2)(7.14\,\mathrm{g/cm^3})(6.022\times10^{23}\,\mathrm{electrons/mol})}{65.38\,\mathrm{g/mol}} = \boxed{1.32\times10^{23}\,\mathrm{electrons/cm^3}}$$

Both results agree with the values in Table 38-1 to within 1%.

73 •• Estimate the fraction of free electrons in copper that are in energy states above the Fermi energy at (*a*) 300 K (about room temperature) and (*b*) at 1000 K.

Picture the Problem We can integrate $g(E)$ from 0 to E_F to show that the total number of states is $\frac{2}{3}AE_F^{3/2}$ and then use this result to find the fraction of the free electrons that are above the Fermi energy at the given temperatures.

Integrate $g(E)$ from 0 to E_F:

$$N = \int_0^{E_F} AE^{1/2}dE = \frac{2}{3}AE_{E_F}^{3/2}$$

Express the fraction of N within kT of E_F:

$$\frac{kTg(E_F)}{N}=\frac{kTAE_F^{1/2}}{\frac{2}{3}AE_F^{3/2}}=\frac{3kT}{2E_F}$$

(*a*) Substitute numerical values and evaluate this fraction for copper at 300 K:

$$\frac{3kT}{2E_F}=\frac{3(8.617\times10^{-5}\text{ eV/K})(300\text{ K})}{2(7.04\text{ eV})}=\boxed{5.51\times10^{-3}}$$

(*b*) Evaluate the same fraction at 1000 K:

$$\frac{3kT}{2E_F}=\frac{3(8.617\times10^{-5}\text{ eV/K})(1000\text{ K})}{2(7.04\text{ eV})}=\boxed{1.84\times10^{-2}}$$

74 •• A certain free-electron energy state of manganese has a 10.0 percent chance of being occupied when the temperature of the manganese is T = 1300 K. What is the energy of this state?

Picture the Problem The Fermi factor gives the probability of an energy state being occupied as a function of the energy of the state E, the Fermi energy E_F for the particular material, and the temperature T.

The Fermi factor is:

$$f(E)=\frac{1}{e^{(E-E_F)/kT}+1}$$

For 10 percent probability:

$$0.1=\frac{1}{e^{(E-E_F)/kT}+1}\Rightarrow e^{(E-E_F)/kT}=9$$

Take the natural logarithm of both sides of $e^{(E-E_F)/kT}=9$ to obtain:

$$\frac{E-E_F}{kT}=\ln(9)\Rightarrow E=E_F+kT\ln(9)$$

From Table 37-1, $E_{F,\,Mn}$ = 11.0 eV. Substitute numerical values and evaluate E:

$$E=11.0\text{ eV}+(1.381\times10^{-23}\text{ J/K})(1300\text{ K})\left(\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}}\right)\ln 9=\boxed{11.2\text{ eV}}$$

75 •• The semiconducting compound CdSe is widely used for light-emitting diodes (LEDs). The energy gap in CdSe is 1.80 eV. What is the frequency of the light emitted by a CdSe LED?

Picture the Problem The energy gap for the semiconductor is related to the wavelength of the emitted light according to $E_g=hc/\lambda$.

The frequency of the emitted light is related to its wavelength:

$$f = \frac{c}{\lambda}$$

Express the energy gap E_g in terms of the wavelength λ of the emitted light:

$$E_g = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_g}$$

Substituting for λ and simplifying yields:

$$f = \frac{c}{\frac{hc}{E_g}} = \frac{cE_g}{hc}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{(2.998\times10^8\ \text{m/s})(1.80\ \text{eV})}{1240\ \text{eV}\cdot\text{nm}} = \boxed{4.35\times10^{14}\ \text{Hz}}$$

Remarks: This frequency is in the red portion of the visible spectrum.

76 ••• A 2.00-cm^2 wafer of pure silicon is irradiated with electromagnetic radiation having a wavelength of 775 nm. The intensity of the radiation is 4.00 W/m^2 and every photon that strikes the sample is absorbed and creates an electron–hole pair. (*a*) How many electron–hole pairs are produced in one second? (*b*) If the number of electron–hole pairs in the sample is 6.25×10^{11} in the steady state, at what rate do the electron–hole pairs recombine? (*c*) If every recombination event results in the radiation of one photon, at what rate is energy radiated by the sample?

Picture the Problem The rate of production of electron-hole pairs is the ratio of the incident energy to the energy required to produce an electron-hole pair.

(*a*)The number of electron-hole pairs *N* produced in one second is:

$$N = \frac{IA}{\frac{hc}{\lambda}} = \frac{IA\lambda}{hc}$$

Substitute numerical values and evaluate *N*:

$$N = \frac{(4.00\,\text{W/m}^2)(2.00\times10^{-4}\ \text{m}^2)(775\,\text{nm})}{(1240\,\text{eV}\cdot\text{nm})(1.602\times10^{-19}\ \text{J/eV})} = \boxed{3.12\times10^{15}\ \text{s}^{-1}}$$

(*b*) In the steady state, the rate of recombination equals the rate of generation. Therefore:

$$N = \boxed{3.12\times10^{15}\ \text{s}^{-1}}$$

(*c*) The power radiated equals the power absorbed:

$$P_{\text{rad}} = IA$$

Substitute numerical values and evaluate P_{rad}:

$$P_{\text{rad}} = \left(4.00\,\text{W/m}^2\right)\left(2.00\times10^{-4}\,\text{m}^2\right) = \boxed{0.800\,\text{mJ/s}}$$

Chapter 39
Relativity

Conceptual Problems

1 • [SSM] The approximate total energy of a particle of mass m moving at speed $u << c$ is (a) $mc^2 + \frac{1}{2}mu^2$, (b) $\frac{1}{2}mu^2$, (c) cmu, (d) mc^2, (e) $\frac{1}{2}cmu$.

Determine the Concept The total relativistic energy E of a particle is defined to be the sum of its kinetic and rest energies.

The sum of the kinetic and rest energies of a particle is given by:

$$E = K + mc^2 = \tfrac{1}{2}mu^2 + mc^2$$

and $\boxed{(a)}$ is correct.

2 • A set of twins work in an office building. One twin works on the top floor and the other twin works in the basement. Considering general relativity, which twin will age more quickly? (a) They will age at the same rate. (b) The twin who works on the top floor will age more quickly. (c) The twin who works in the basement will age more quickly. (d) It depends on the speed of the office building. (e) None of the above.

Determine the Concept The gravitational field of Earth is slightly greater in the basement of the office building than it is at the top floor. Because clocks run more slowly in regions of low gravitational potential, clocks in the basement will run more slowly than clocks on the top floor. Hence, the twin who works on the top floor will age more quickly. $\boxed{(b)}$ is correct.

3 • True or false:

(a) The speed of light is the same in all reference frames.
(b) The time interval between two events is never shorter than the proper time interval between the two events.
(c) Absolute motion can be determined by means of length contraction.
(d) The light-year is a unit of distance.
(e) Simultaneous events must occur at the same place.
(f) If two events are not simultaneous in one frame, they cannot be simultaneous in any other frame.
(g) The mass of a system that consists of two particles that are tightly bound together by attractive forces is less than the sum of the masses of the individual particles when separated.

(a) True

(*b*) True

(*c*) False. The shortening of the length of an object in the direction in which it is moving is independent of the velocity of the frame of reference from which it is observed.

(*d*) True

(*e*) False. Consider two explosions equidistant, but in opposite directions, from an observer in the observer's frame of reference.

(*f*) False. Whether events appear to be simultaneous depends on the motion of the observer.

(*g*) True

4 • An observer sees a system moving past her that consists of a mass oscillating on the end of a spring and measures the period T of the oscillations. A second observer, who is moving with the mass–spring system, also measures its period. The second observer will find a period that is (*a*) equal to T, (*b*) less than T, (*c*) greater than T, (*d*) either (*a*) or (*b*) depending on whether the system was approaching or receding from the first observer, (*e*) Not enough information is given to answer the question.

Determine the Concept Because the clock is moving with respect to the first observer, a time interval will be longer for this observer than for the observer moving with the spring-and-mass oscillator. Hence, the observer moving with the system will measure a period that is less than T. $\boxed{(b)}$ is correct.

5 • The Lorentz transformation for y and z is the same as the classical result: $y = y'$ and $z = z'$. Yet the relativistic velocity transformation does not give the classical result $u_y = u'_y$ and $u_z = u'_z$. Explain why this result occurs.

Determine the Concept Although $\Delta y = \Delta y'$, $\Delta t \neq \Delta t'$. Consequently, $u_y = \Delta y/\Delta t' \neq \Delta y'/\Delta t' = u_y'$.

Estimation and Approximation

6 •• The Sun radiates energy at the rate of approximately 4×10^{26} W. Assume that this energy is produced by a reaction whose net result is the fusion of four protons to form a single ^{4}He nucleus and the release of 25 MeV of energy that is radiated into space. Calculate the Sun's loss of mass per day.

Picture the Problem We can calculate the Sun's loss of mass per day from the number of reactions per second and the loss of mass per reaction.

Express the rate at which the Sun loses mass:

$$\frac{\Delta M}{\Delta t} = N\Delta m \Rightarrow \Delta M = N\Delta m\Delta t$$

where N is the number of reactions per second and Δm is the loss of mass per reaction.

The number of reactions per second, N is given by:

$$N = \frac{P}{E/\text{reaction}}$$

Substitute for N to obtain:

$$\Delta M = \frac{P}{E/\text{reaction}}\Delta m\Delta t$$

The loss of mass per reaction Δm is:

$$\Delta m = \frac{E/\text{reaction}}{c^2}$$

Substituting for Δm and simplifying yields:

$$\Delta M = \left(\frac{P}{E/\text{reaction}}\right)\left(\frac{E/\text{reaction}}{c^2}\right)\Delta t = \frac{P}{c^2}\Delta t$$

Substitute numerical values and evaluate ΔM:

$$\Delta M = \frac{4\times10^{26}\ \text{W}}{\left(2.998\times10^{8}\ \text{m/s}\right)^2}\left(1\,\text{d}\times\frac{86.4\,\text{ks}}{\text{d}}\right) \approx \boxed{4\times10^{14}\ \text{kg}}$$

7 •• **[SSM]** The most distant galaxies that can be seen by the Hubble telescope are moving away from us and have a redshift parameter of about $z = 5$. (The redshift parameter z is defined as $(f - f')/f'$, where f is the frequency measured in the rest frame of the emitter, and f' is the frequency measured in the rest frame of the receiver.) (*a*) What is the speed of these galaxies relative to us (expressed as a fraction of the speed of light)? (*b*) *Hubble's law* states that the recession speed is given by the expression $v = Hx$, where v is the speed of recession, x is the distance, and H, the Hubble constant, is equal to 75 km/s/Mpc, where 1 pc = 3.26 c·y. (The abbreviation for parsec is pc.) Estimate the distance of such a galaxy from us using the information given.

Picture the Problem (*a*) We can use the definition of the redshift parameter and the relativistic Doppler shift equation to show that, for light that is Doppler-shifted with respect to an observer, $v = c\left(\frac{u^2 - 1}{u^2 + 1}\right)$, where $u = z + 1$, and to find the ratio of v to c. In Part (*b*) we can solve Hubble's law for x and substitute our result from Part (*a*) to estimate the distance to the galaxy.

(*a*) The red-shift parameter is defined to be:

$$z = \frac{f_0 - f'}{f'}$$

The relativistic Doppler shift for recession is given by:

$$f' = f_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

Substitute for f' and simplify to obtain:

$$z = \frac{f_0 - f_0\sqrt{\frac{1 - v/c}{1 + v/c}}}{f_0\sqrt{\frac{1 - v/c}{1 + v/c}}} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1$$

Letting $u = z + 1$ and simplifying yields:

$$u = z + 1 = \sqrt{\frac{1 + v/c}{1 - v/c}} \Rightarrow \frac{v}{c} = \frac{u^2 - 1}{u^2 + 1}$$

Substitute for u to express v/c as a function of z:

$$\frac{v}{c} = \frac{(z + 1)^2 - 1}{(z + 1)^2 + 1}$$

Substituting the numerical value of z and evaluating v/c gives:

$$\frac{v}{c} = \frac{(5 + 1)^2 - 1}{(5 + 1)^2 + 1} = \boxed{0.946}$$

(*b*) Solving Hubble's law for x yields:

$$x = \frac{v}{H}$$

Substitute numerical values and evaluate x:

$$x = \frac{0.946c}{H} = \frac{0.946c}{75\frac{\text{km/s}}{\text{Mpc}}} \times \frac{3.26 \times 10^6\ c \cdot \text{y}}{\text{Mpc}}$$

$$= \boxed{12.3\,\text{G}c \cdot \text{y}}$$

Time Dilation and Length Contraction

8 • The proper mean lifetime of a muon is 2.2 μs. Muons in a beam are traveling through a laboratory at 0.95*c*. (*a*) What is their mean lifetime as measured in the laboratory? (*b*) How far do they travel, on average, before they decay?

Picture the Problem We can find the mean lifetime of a muon as measured in the laboratory using $t' = \gamma t$ where $\gamma = 1\Big/\sqrt{1-(v/c)^2}$ and t is the proper mean lifetime of the muon. The distance L that the muon travels is the product of its speed and its mean lifetime in the laboratory.

(*a*) The mean lifetime of the muon, as measured in the laboratory, is given by:

$$t' = \frac{t}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate t':

$$t' = \frac{2.2\ \mu\text{s}}{\sqrt{1-\left(\frac{0.95c}{c}\right)^2}} = 7.046\ \mu\text{s} = \boxed{7.0\ \mu\text{s}}$$

(*b*) The distance L that the muon travels is related to its mean lifetime in the laboratory:

$$L = vt' = 0.95ct'$$

Substitute numerical values and evaluate L:

$$L = 0.95\left(2.998\times10^8\ \text{m/s}\right)\left(7.046\ \mu\text{s}\right) = \boxed{2.0\,\text{km}}$$

9 •• In the Stanford linear collider, small bundles of electrons and positrons are fired at each other. In the laboratory's frame of reference, each bundle is approximately 1.0 cm long and 10 μm in diameter. In the collision region, each particle has an energy of 50 GeV, and the electrons and the positrons are moving in opposite directions. (*a*) How long and how wide is each bundle in its own reference frame? (*b*) What must be the minimum proper length of the accelerator for a bundle to have both its ends simultaneously in the accelerator in its own reference frame? (The actual proper length of the accelerator is less than 1000 m.) (*c*) What is the length of a positron bundle in a reference frame that moves with the electron bundle?

Picture the Problem The proper length L_p of the beam is its length as measured in a reference frame in which it is not moving. The proper length is related to its length in the frame in which it is measured by $L_p = \gamma L$.

(*a*) Relate the proper length L_p of the beam to its length L in the laboratory frame of reference:

$$L_p = \gamma L$$

The energy of the beam also depends on γ:

$$E = \gamma mc^2$$

Solve for and evaluate γ:

$$\gamma = \frac{E}{mc^2} = \frac{50\,\text{GeV}}{0.511\,\text{MeV}} = 9.785\times10^4$$

Substitute numerical values and evaluate L_p:

$$L_p = \left(9.785\times10^4\right)(1.0\,\text{cm}) = 978.5\,\text{m} = \boxed{0.98\,\text{km}}$$

and the width w of the beam is unchanged.

(*b*) Express the length of the accelerator in the electron beam's frame of reference:

$$L_{\text{acc}} = \frac{L_{\text{acc,p}}}{\gamma}$$

Set $L_{\text{acc}} = L_p$:

$$L_p = \frac{L_{\text{acc,p}}}{\gamma} \Rightarrow L_{\text{acc,p}} = \gamma L_p$$

Substitute numerical values and evaluate $L_{\text{acc,p}}$:

$$L_{\text{acc,p}} = \left(9.785\times10^4\right)(978.5\,\text{m}) = \boxed{9.6\times10^7\ \text{m}}$$

(*c*) The length of the positron bundle in the electron's frame of reference is:

$$L_{\text{pos}} = \frac{L}{\gamma}$$

Substitute numerical values and evaluate L_{pos}:

$$L_{\text{pos}} = \frac{1.0\,\text{cm}}{9.785\times10^4} = \boxed{0.10\,\mu\text{m}}$$

10 • Use the binomial expansion equation

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \approx 1 + nx, \qquad x << 1$$

to derive the following results for the case when v is much less than c, and use the results when applicable in the following problems:

(a) $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

(b) $\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$

(c) $\gamma - 1 \approx 1 - \frac{1}{\gamma} \approx \frac{1}{2}\frac{v^2}{c^2}$

Picture the Problem We can use the definition of γ and the binomial expansion of $(1 + x)^n$ to show that each of these relationships holds provided $v << c$.

(a) Express the gamma factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Expand the radical factor binomially to obtain:

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \text{higher order terms}$$

For $v << c$:

$$\gamma \approx \boxed{1 + \frac{1}{2}\frac{v^2}{c^2}}$$

(b) Express the reciprocal of γ:

$$\frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}$$

Expand the radical binomially to obtain:

$$\frac{1}{\gamma} = \left(1-\frac{v^2}{c^2}\right)^{1/2}$$

$$= 1 + \left(\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \text{higher order terms}$$

For $v << c$:

$$\frac{1}{\gamma} \approx \boxed{1 - \frac{1}{2}\frac{v^2}{c^2}}$$

(c) Express the gamma factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Subtract one from both sides of the equation to obtain:

$$\gamma - 1 = \left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1$$

Expand the radical binomially to obtain:

$$\gamma - 1 = 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) - 1 + \text{higher order terms}$$

For $v << c$:

$$\gamma - 1 \approx \boxed{\frac{1}{2}\frac{v^2}{c^2}}$$

11 •• Star A and Star B are at rest relative to Earth. Star A is 27 c·y from Earth, and as viewed from Earth, Star B is located beyond (behind) Star A. (a) A spaceship is making a trip from Earth to Star A at a speed such that the trip from Earth to Star A takes 12 y according to clocks on the spaceship. At what speed, relative to Earth, must the spaceship travel? (Assume that the times for the accelerations are very short compared to the overall trip time.) (b) Upon reaching Star A, the spaceship speeds up and departs for Star B at a speed such that the gamma factor, γ, is twice that of Part (a). The trip from Star A to Star B takes 5.0 y (spaceship's time). How far, in c·y, is Star B from Star A in the rest frame of Earth and the two stars? (c) Upon reaching Star B, the spaceship departs for Earth at the same speed as in Part (b). It takes it 10 y (spaceship's time) to return to Earth. If you were born on Earth the day the ship left Earth and you remain on Earth, how old are you on the day the ship returns to Earth?

Picture the Problem We can use the time-dilation relationship to find the speed of the spacecraft. The distance to the second star is the product of the new gamma factor, the speed of the spacecraft, and the elapsed time. Finally, the time that has elapsed on Earth (your age) is the sum of the elapsed times for the three legs of the journey.

(a) From the point of view of an observer on Earth, the elapsed time for the trip will be:

$$\Delta t = \frac{L}{v}$$

From the point of view of an observer on the spaceship, the elapsed time for the trip will be:

$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{L}{\gamma v}$$

Substitute for γ to obtain:

$$\Delta t' = \frac{L}{v}\sqrt{1-\frac{v^2}{c^2}} \Rightarrow v = \frac{Lc}{\sqrt{L^2 + c^2(\Delta t')^2}}$$

Substitute numerical values and evaluate v:

$$v = \frac{(27\,c\cdot\text{y})c}{\sqrt{(27\,c\cdot\text{y})^2 + c^2(12\,\text{y})^2}} = 0.914c$$
$$= \boxed{0.91c}$$

Note that from the point of view of an Earth observer, this part of the trip has taken 27 $c\cdot$y/0.914c = 29.5 y.

(b) The distance the ship travels, from the point of view of an Earth observer, in 5 y is:

$$\Delta L' = 2\gamma\Delta L = 2\gamma v\Delta t$$

where γ is the gamma factor for the first part of the trip.

The gamma factor in Part (a) is:

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{0.914c}{c}\right)^2}}$$
$$= 2.46$$

Substitute numerical values and evaluate $\Delta L'$:

$$\Delta L' = 2(2.46)(0.914c)(5.0\,\text{y})$$
$$= 22.5\,c\cdot\text{y} = \boxed{22\,c\cdot\text{y}}$$

(c) The elapsed time Δt on Earth (your age) is the sum of the times for the spacecraft to travel to the star 27 $c\cdot$y away, to the second star, and to return home from the second star:

$$\Delta t = 29.5\,\text{y} + 22.5\,\text{y} + \Delta t_{\substack{\text{returning}\\ \text{home}}}$$

The elapsed time on Earth while the spacecraft is returning to Earth is:

$$\Delta t_{\substack{\text{returning}\\ \text{home}}} = 2\gamma\Delta t_{\text{ship's time}}$$
$$= 2(2.46)(10\,\text{y})$$
$$= 49.2\,\text{y}$$

Substitute for $\Delta t_{\substack{\text{returning}\\ \text{home}}}$ and evaluate Δt:

$$\Delta t = 29.5\,\text{y} + 22.5\,\text{y} + 49.2\,\text{y}$$
$$= \boxed{101\,\text{y}}$$

12 • A spaceship travels to a star 35 $c\cdot$y away at a speed of 2.7×10^8 m/s. How long does the spaceship take to get to the star (a) as measured on Earth and (b) as measured by a passenger on the spaceship?

Picture the Problem We can use $\Delta t = L/v$, where L is the distance to the star and v is the speed of the spaceship to find the time Δt for the trip as measured on Earth. The travel time as measured by a passenger on the spaceship can be found using $\Delta t' = \Delta t/\gamma$.

(*a*) The travel time as measured on Earth is the ratio of the distance traveled L to speed of the spaceship:

$$\Delta t = \frac{L}{v}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{35c\cdot\text{y}}{2.7\times10^{8}\ \text{m/s}} = \frac{35\,\text{y}}{\dfrac{2.7\times10^{8}\ \text{m/s}}{c}}$$

$$= \frac{35\,\text{y}}{0.9} = 38.9\,\text{y} = \boxed{39\,\text{y}}$$

(*a*) The travel time as measured by a passenger on the spaceship is given by:

$$\Delta t' = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^{2}}$$

Substitute numerical values and evaluate $\Delta t'$:

$$\Delta t' = (38.9\,\text{y})\sqrt{1-(0.90)^{2}} = \boxed{17\,\text{y}}$$

13 •• [SSM] Unobtainium (Un) is an unstable particle that decays into normalium (Nr) and standardium (St) particles. (*a*) An accelerator produces a beam of Un that travels to a detector located 100 m away from the accelerator. The particles travel with a velocity of $v = 0.866c$. How long do the particles take (in the laboratory frame) to get to the detector? (*b*) By the time the particles get to the detector, half of the particles have decayed. What is the half-life of Un? (*Note:* half-life as it would be measured in a frame moving with the particles) (*c*) A new detector is going to be used, which is located 1000 m away from the accelerator. How fast should the particles be moving if half of the particles are to make it to the new detector?

Picture the Problem The time required for the particles to reach the detector, as measured in the laboratory frame of reference is the ratio of the distance they must travel to their speed. The half life of the particles is the trip time as measured in a frame traveling with the particles. We can find the speed at which the particles must move if they are to reach the more distant detector by equating their half life to the ratio of the distance to the detector in the particle's frame of reference to their speed.

(*a*) The time required to reach the detector is the ratio of the distance to the detector and the speed with which the particles are traveling:

$$\Delta t = \frac{\Delta x}{v} = \frac{\Delta x}{0.866c}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{100\,\text{m}}{0.866\left(2.998\times10^{8}\,\text{m/s}\right)} = \boxed{0.385\,\mu\text{s}}$$

(*b*) The half life is the trip time as measured in a frame traveling with the particles:

$$\Delta t' = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate $\Delta t'$:

$$\Delta t' = \left(0.385\,\mu\text{s}\right)\sqrt{1-\left(\frac{0.866c}{c}\right)^2} = \boxed{0.193\,\mu\text{s}}$$

(*c*) In order for half the particles to reach the detector:

$$\Delta t' = \frac{\Delta x'}{\gamma v} = \frac{\Delta x'\sqrt{1-\left(\frac{v}{c}\right)^2}}{v}$$

where $\Delta x'$ is the distance to the new detector.

Rewrite this expression to obtain:

$$\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{\Delta x'}{\Delta t'}$$

Squaring both sides of the equation yields:

$$\frac{v^2}{1-\left(\frac{v}{c}\right)^2} = \left(\frac{\Delta x'}{\Delta t'}\right)^2$$

Substitute numerical values for $\Delta x'$ and $\Delta t'$ and simplify to obtain:

$$\frac{v^2}{1-\left(\frac{v}{c}\right)^2} = \left(\frac{1000\,\text{m}}{0.193\,\mu\text{s}}\right)^2 = \left(17.3c\right)^2$$

Divide both sides of the equation by c^2 to obtain:

$$\frac{\frac{v^2}{c^2}}{1-\left(\frac{v}{c}\right)^2}=(17.3)^2$$

Solving this equation for v^2/c^2 gives:

$$\frac{v^2}{c^2}=\frac{(17.3)^2}{1+(17.3)^2}=0.9967$$

Finally, solving for v yields:

$$v=\boxed{0.998c}$$

14 •• A clock on Spaceship A measures the time interval between two events, both of which occur at the location of the clock. You are on Spaceship B. According to your careful measurements, the time interval between the two events is 1.00 percent longer than that measured by the two clocks on Spaceship A. How fast is Spaceship A moving relative to Spaceship B. (*Hint: Use one or more of the results of Problem 10.*)

Picture the Problem We can express the fractional difference in your time-interval measurements as a function of γ and solve the resulting equation for the relative speed of the two spaceships.

Express the fractional difference in the time-interval measurements of the two observers:

$$\frac{\Delta t-\Delta t'}{\Delta t}=1-\frac{\Delta t'}{\Delta t}=0.0100$$

Since $\Delta t'/\Delta t=1/\gamma$:

$$\frac{\Delta t-\Delta t'}{\Delta t}=1-\frac{1}{\gamma}=0.0100$$

From Problem 10(*b*) we have:

$$\frac{1}{\gamma}\approx 1-\frac{1}{2}\frac{v^2}{c^2}$$

Substitute to obtain:

$$1-\frac{1}{\gamma}\approx 1-\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)=0.0100$$

or

$$\frac{1}{2}\frac{v^2}{c^2}=0.0100$$

Solve for v to obtain:

$$v=\sqrt{0.0200}c=0.141c$$

$$=\boxed{4.23\times10^7\ \text{m/s}}$$

15 •• If a plane flies at a speed of 2000 km/h, how long must the plane fly before its clock loses 1.00 s because of time dilation? (*Hint: Use one or more of the results of Problem 10.*)

Picture the Problem We can use the time dilation equation to relate the time lost by the clock to the speed of the plane and the time it must fly.

Express the time δt lost by the clock:

$$\delta t = \Delta t - \Delta t_{\text{p}} = \Delta t - \frac{\Delta t}{\gamma} = \Delta t\left(1 - \frac{1}{\gamma}\right)$$

Because $V << c$, we can use Part (*b*) of Problem 10:

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{V^2}{c^2}$$

Substitute to obtain:

$$\delta t = \Delta t\left[1 - \left(1 - \frac{1}{2}\frac{V^2}{c^2}\right)\right] = \frac{1}{2}\frac{V^2}{c^2}\Delta t$$

Solving for Δt gives:

$$\Delta t = \frac{2\delta t\, c^2}{V^2}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(1.00\,\text{s})(2.998\times10^8\ \text{m/s})^2}{(2000\,\text{km/h}\times1\,\text{h}/3600\,\text{s})^2}$$

$$= 5.824\times10^{11}\ \text{s}\times\frac{1\,\text{y}}{31.56\,\text{Ms}}$$

$$= \boxed{1.85\times10^4\ \text{y}}$$

The Lorentz Transformation, Clock Synchronization, and Simultaneity

16 •• Show that when $v << c$ the relativistic transformation equations for x, t, and u_x reduce to the classical transformation equations.

Picture the Problem We can use the inverse Lorentz transformations and the result of Problem 10(*c*) to show that when $u << c$ the transformation equations for x, t, and u reduce to the Galilean equations.

The inverse transformation for x is:

$$x' = \gamma(x - vt)$$

From Problem 10(*c*):

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Substitute for γ and expand to obtain:

$$x' = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)(x - vt)$$
$$= x - vt + \frac{1}{2}\frac{v^2}{c^2}x - \frac{1}{2}\frac{v^3}{c^2}t$$

When $v << c$:

$$\boxed{x' \approx x - vt}$$

The inverse transformation for t is:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Substitute for γ and expand to obtain:

$$t' = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\left(t - \frac{vx}{c^2}\right)$$
$$= t - \frac{vx}{c^2} + \frac{1}{2}\frac{v^2}{c^2}t - \frac{1}{2}\frac{v^3}{c^4}x$$

When $v << c$:

$$\boxed{t' \approx t}$$

The inverse velocity transformation for motion in the x direction is:

$$u_x' = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

When $v << c$:

$$\boxed{u_x' \approx u_x - v}$$

17 •• [SSM] A spaceship of proper length $L_p = 400$ m moves past a transmitting station at a speed of $0.760c$. (The transmitting station broadcasts signals that travel at the speed of light.) A clock is attached to the nose of the spaceship and a second clock is attached to the transmitting station. The instant that the nose of the spaceship passes the transmitter, the clock attached to the transmitter and the clock attached to the nose of the spaceship are set equal to zero. The instant that the tail of the spaceship passes the transmitter a signal is sent by the transmitter that is subsequently detected by a receiver in the nose of the spaceship. (*a*) When, according to the clock attached to the nose of spaceship, is the signal sent? (*b*) When, according to the clocks attached to the nose of spaceship, is the signal received? (*c*) When, according to the clock attached to the transmitter, is the signal received by the spaceship? (*d*) According to an observer that works at the transmitting station, how far from the transmitter is the nose of the spaceship when the signal is received?

Picture the Problem Let S be the reference frame of the spaceship and S' be that of Earth (transmitter station). Let event A be the emission of the light pulse and event B the reception of the light pulse at the nose of the spaceship. In (*a*) and (*c*)

we can use the classical distance, rate, and time relationship and in (*b*) and (*d*) we can apply the inverse Lorentz transformations.

(*a*) In both *S* and *S*′ the pulse travels at the speed *c*. Thus:

$$t_A = \frac{L_p}{v} = \frac{400\,\text{m}}{0.760\text{c}} = \boxed{1.76\,\mu\text{s}}$$

(*b*) The inverse time transformation is:

$$t_B' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where

$$\gamma = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\dfrac{(0.760c)^2}{c^2}}} = 1.54$$

Substitute numerical values and evaluate t_B':

$$t_B' = (1.54)\left(3.09\,\mu\text{s} - \frac{(-0.760c)(400\,\text{m})}{c^2}\right)$$

$$= (1.54)\left(3.09\,\mu\text{s} - \frac{(-0.760)(400\,\text{m})}{2.998\times10^8\ \text{m/s}}\right)$$

$$= \boxed{6.32\,\mu\text{s}}$$

(*c*) The elapsed time, according to the clock on the ship is:

$$t_B = t_{\substack{\text{pulse to travel}\\ \text{length of ship}}} + t_A$$

Find the time of travel of the pulse to the nose of the ship:

$$t_{\substack{\text{pulse to travel}\\ \text{length of ship}}} = \frac{400\,\text{m}}{2.998\times10^8\ \text{m/s}} = 1.33\,\mu\text{s}$$

Substitute numerical values and evaluate t_B:

$$t_B = 1.33\,\mu\text{s} + 1.76\,\mu\text{s} = \boxed{3.09\,\mu\text{s}}$$

(*d*) The inverse transformation for *x* is:

$$x' = \gamma(x - vt)$$

Substitute numerical values and evaluate x':

$$x' = (1.54)\left[400\,\text{m} - (-0.760)(2.998\times10^8\ \text{m/s})(3.09\times10^{-6}\ \text{s})\right] = \boxed{1.70\,\text{km}}$$

18 •• In frame S, event B occurs 2.0 μs after event A, and event A occurs at the origin whereas event B occurs on the x axis at x = 1.5 km. How fast and in what direction must an observer be traveling along the x axis so that events A and B occur simultaneously? Is it possible for event B to precede event A for some observer?

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to find the required speed of the observer.

Use Equation 39-12 to obtain:

$$\Delta t' = t_B' - t_A' = \gamma\left[(t_B - t_A) - \frac{v}{c^2}(x_B - x_A)\right] = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

where $\Delta t = t_B - t_A$ and $\Delta x = x_B - x_A$.

Setting $\Delta t'$ equal to zero gives:

$$\Delta t - \frac{v\Delta x}{c^2} = 0 \Rightarrow v = \frac{c\Delta t}{\Delta x}c$$

Substitute numerical values and evaluate v:

$$v = \frac{(2.998\times10^8\ \text{m/s})(2.0\ \mu\text{s})}{1.5\,\text{km}}c = \boxed{0.40\text{c}}$$

Because $\Delta t = t_B - t_A = 2.0\ \mu$s :

$\boxed{\text{The observer is moving in the } +x \text{ direction.}}$

If B occurs before A, then $\Delta t' < 0$:

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) < 0$$

and

$$\Delta t - \frac{v\Delta x}{c^2} < 0$$

Solving for v yields:

$$v > \frac{c^2\Delta t}{\Delta x} = 0.40c$$

Yes. Event B will precede event A for some observer (t'_B will be less than t'_A) provided $v > 0.40c$.

19 •• Observers in reference frame S see an explosion located on the x axis at x_1 = 480 m. A second explosion occurs, 5.0 μs later, at x_2 = 1200 m. In

reference frame S', which is moving along the x axis in the $+x$ direction at speed v, the two explosions occur at the same point in space. What is the separation in time between the two explosions as measured in S'?

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to express the separation in time $\Delta t'$ between the two explosions as measured in S' as a function of the speed v of the observer and Equation 39-11, the inverse position transformation equation, to find the speed of the observer.

Use Equation 39-12 to express the separation in time between the two explosions as measured in S':

$$\Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right]$$

$$= \frac{\Delta t - \dfrac{v}{c^2}\Delta x}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}} \qquad (1)$$

From Equation 39-11:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

Because the explosions occur at the same point in space, $\Delta x' = 0$:

$$0 = \gamma(\Delta x - v\Delta t) \Rightarrow v = \frac{\Delta x}{\Delta t}$$

Substitute numerical values and evaluate v:

$$v = \frac{1200\,\text{m} - 480\,\text{m}}{5.0\,\mu\text{s}} = 1.44\times10^8\ \text{m/s}$$

Substitute numerical values in equation (1) and evaluate $\Delta t'$:

$$\Delta t' = \frac{5.0\,\mu\text{s} - \dfrac{1.44\times10^8\ \text{m/s}}{\left(2.998\times10^8\ \text{m/s}\right)^2}(1200\,\text{m} - 480\,\text{m})}{\sqrt{1-\left(\dfrac{1.44\times10^8\ \text{m/s}}{2.998\times10^8\ \text{m/s}}\right)^2}} = \boxed{4.4\,\mu\text{s}}$$

20 ••• In reference frame S, events 1 and 2 are separated by a distance $D = x_2 - x_1$ and a time $T = t_2 - t_1$. (*a*) Use the Lorentz transformation to show that in frame S', which is moving along the x axis with speed v relative to S, the time separation is $t_2' - t_1' = \gamma\left(T - vD/c^2\right)$. (*b*) Show that the events can be simultaneous in frame S' only if D is greater than cT. (*c*) If one of the events is the *cause* of the other, the separation D must be less than cT, because D/c is the smallest time that a signal can take to travel from x_1 to x_2 in frame S. Show that if D is less than cT, t_2' is greater than t_1' in all reference frames. This shows that if the cause precedes the effect in one frame, it must precede it in all reference

frames. (*d*) Suppose that a signal could be sent with speed $c' > c$ so that in frame S the cause precedes the effect by the time $T = D/c'$. Show that there is then a reference frame moving with speed v less than c in which the effect precedes the cause.

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to establish the results called for in this problem.

(*a*) Use Equation 39-12 to obtain:

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right] = \boxed{\gamma\left(T - \frac{vD}{c^2}\right)}$$

where $T = t_2 - t_1$ and $D = x_2 - x_1$.

(*b*) Events 1 and 2 are simultaneous in S' if:

$$t_2' = t_1'$$

or

$$T - \frac{vD}{c^2} = 0 \Rightarrow D = \frac{c^2T}{v}$$

Because $v \leq c$:

$$D \geq \boxed{cT}$$

(*c*) If $D < cT$, then $t_2' > t_1'$ and the events are not simultaneous in S'.

(*d*) If $D = c'T > cT$, then:

$$T - \frac{vD}{c^2} = T\left[1 - \frac{v}{c}\frac{c'}{c}\right] = t_2' - t_1'$$

In this case, $t_2' > t_1'$ could be negative; i.e., t_2' could be less than t_1', or the effect could precede the cause.

21 ••• A rocket that has a proper length of 700 m is moving to the right at a speed of 0.900*c*. It has two clocks—one in the nose and one in the tail—that have been synchronized in the frame of the rocket. A clock on the ground and the clock in the nose of the rocket both read zero as they pass by each other. (*a*) At the instant the clock on the ground reads zero, what does the clock in the tail of the rocket read according to observers on the ground? When the clock in the tail of the rocket passes the clock on the ground, and (*b*) what does the clock in the tail read according to observers on the ground, (*c*) what does the clock in the nose read according to observers on the ground, and (*d*) what does the clock in the nose read according to observers on the rocket? (*e*) At the instant the clock in the nose of the rocket reads 1.00 h, a light signal is sent from the nose of the rocket to an observer standing by the clock on the ground. What does the clock on the ground read when the observer on the ground receives this signal? (*f*) When the observer

on the ground receives the signal, he immediately sends a return signal to the nose of the rocket. What is the reading of the clock in the nose of the rocket when this signal is received at the nose of the rocket?

Picture the Problem Let S be the ground reference frame, S' the reference frame of the rocket, and $v = 0.900c$ be the speed of the rocket relative to S. Denote the tail and nose of the rocket by T and N, respectively. The initial conditions in S' are $t_N' = 0$, $x_N' = 0$, and $x_T' = -L' = -700\,\text{m}$.

(*a*) The reading of the tail clock is given by:

$$t_T' = \gamma\left(t_T - \frac{vx_T}{c^2}\right) = -\frac{\gamma v x_T}{c^2}$$

because $t_T = 0$

We can find x_T using the length contraction equation:

$$x_T = -\frac{L'}{\gamma}$$

Substitute for x_T to obtain:

$$t_T' = \frac{vL'}{c^2}$$

Substitute numerical values and evaluate t_T':

$$t_T' = \frac{(0.900)(700\,\text{m})}{2.998\times10^8\ \text{m/s}} = 2.101\,\mu\text{s}$$

$$= \boxed{2.10\,\mu\text{s}}$$

(*b*) The time for the rocket to move a distance L' is given by:

$$t_T' = \frac{L'}{v} = \frac{L'}{0.9c}$$

Substitute numerical values and evaluate t_T':

$$t_T' = \frac{700\,\text{m}}{(0.900)(2.998\times10^8\ \text{m/s})}$$

$$= 2.594\,\mu\text{s} = \boxed{2.59\,\mu\text{s}}$$

(*c*) As seen by an observer on the ground:

$$t_N = \Delta t' = 2.59\,\mu\text{s} - 2.10\,\mu\text{s}$$

$$= \boxed{0.49\,\mu\text{s}}$$

(*d*) Because the clocks are synchronized in S':

$$t_N' = t_T' = \boxed{2.59\,\mu\text{s}}$$

(*e*) The time the signal is received on the ground is the sum of the time when the signal is sent and the time for it to travel to the ground:

$$t_{\text{rec}} = \Delta t + \Delta t_{\text{travel}}$$

Find Δt, the time the signal is sent:

$$\Delta t = \gamma \Delta t_{\text{p}} = \frac{1.00\,\text{h}}{\sqrt{1-\frac{(0.900c)^2}{c^2}}} = 2.294\,\text{h}$$

Find Δt_{travel}, the time for the signal to travel to the ground:

$$\Delta t_{\text{travel}} = \frac{\Delta x}{c} = \frac{(2.294\,\text{h})(0.900c)}{c} = 2.065\,\text{h}$$

Substitute for Δt and Δt_{travel} and evaluate t_{rec}:

$$t_{\text{rec}} = 2.294\,\text{h} + 2.065\,\text{h} = 4.359\,\text{h} = \boxed{4.36\,\text{h}}$$

(*f*) Find Δx when the signal is sent:

$$\Delta x = (4.359\,\text{h})(0.900c) = 3.923\,c\cdot\text{h}$$

In *S,* the signal arrives at $0.1c$ relative to the rocket. The time required for the signal to travel to the rocket is:

$$\Delta t = \frac{\Delta x}{0.100c} = \frac{3.923\,c\cdot\text{h}}{0.100c} = 39.23\,\text{h}$$

Find the time when the signal reaches the rocket:

$$t = 39.23\ \text{h} + 3.923\ \text{h} = 43.15\ \text{h}$$

Finally, use the time dilation equation to find $t_N{}'$:

$$t_N{}' = \frac{t}{\gamma} = (43.15\,\text{h})\sqrt{1-\frac{(0.900c)^2}{c^2}} = \boxed{18.8\,\text{h}}$$

The Velocity Transformation

22 •• A spaceship, at rest in a certain reference frame *S*, is given a speed increase of $0.50c$ (call this increase boost 1). Relative to its new rest frame, the spaceship is given a further $0.50c$ increase 10 seconds later (as measured in its new rest frame; call this increase boost 2). This process is continued indefinitely, at 10-s intervals, as measured in the rest frame of the spaceship. (Assume that the boosts take a very short time compared to 10 s.) (*a*) Using a **spreadsheet** program, calculate and graph the speed of the spaceship in reference frame *S* as a

function of the boost number for boost 1 to boost 10. (*b*) Graph the gamma factor the same manner. (*c*) How many boosts does it take until the speed of the ship in *S* is greater than 0.999*c*? (*d*) How far does the spaceship move between boost 1 and boost 6, as measured in reference frame *S*? What is the average speed of the spaceship between boost 1 and boost 6, as measured in *S*?

Picture the Problem We'll let the speed (in *S*) of the spaceship after the i^th boost be v_i and derive an expression for the ratio of v to c after the spaceship's $(i + 1)$th boost as a function of the number of boosts *N*. We can use the definition of γ, in terms of v/c to plot γ as a function of *N*.

(*a*) and (*b*) The speed of the spaceship after the $(i + 1)$th boost is given by relativistic velocity addition equation:

$$v_{i+1}=\frac{v_i+0.50c}{1+\frac{(0.50c)v_i}{c^2}}$$

Factor c from both the numerator and denominator to obtain:

$$v_{i+1}=\frac{\frac{v_i}{c}+0.50}{1+(0.50)\frac{v_i}{c}}$$

γ_i is given by:

$$\gamma_i=\frac{1}{\sqrt{1-\left(\frac{v_i}{c}\right)}}$$

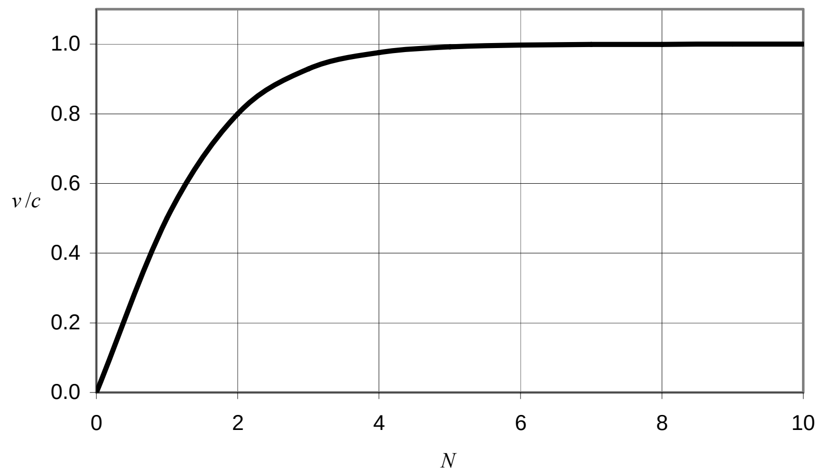
A spreadsheet program to calculate v/c and γ as functions of the number of boosts *N* is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A3	0	N
B2	0	v_0
B3	(B2+0.5)/(1+0.5*B2)	v_{i+1}
C1	1/(1−B2^2)^0.5	γ

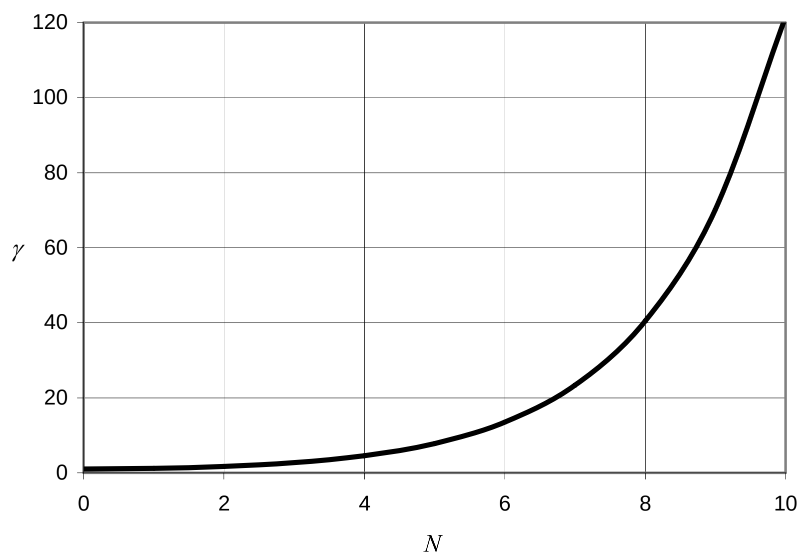
	A	B	C
1	N	v/c	γ
2	0	0.000	1.00
3	1	0.500	1.15
4	2	0.800	1.67
5	3	0.929	2.69
6	4	0.976	4.56
7	5	0.992	7.83
8	6	0.997	13.52
9	7	0.999	23.39

10	8	1.000	40.51
11	9	1.000	70.15
12	10	1.000	121.50

A graph of v/c as a function of N follows:



A graph of γ as a function of N follows:



(c) Examination of the spreadsheet or of the graph of v/c as a function of N indicates that, after 8 boosts, the speed of the spaceship is greater than $0.999c$.

(*d*) After 5 boosts, the spaceship has traveled a distance Δx, measured in the Earth frame of reference (S), given by:

$$\begin{aligned}\Delta x &= \Delta x_{1\to2} + \Delta x_{2\to3} + \Delta x_{3\to4} + \Delta x_{4\to5} + \Delta x_{5\to6} \\ &= (0.5c)(10\,\text{s})\gamma_{1\to2} + (0.8c)(10\,\text{s})\gamma_{2\to3} + (0.929c)(10\,\text{s})\gamma_{3\to4} \\ &\quad + (0.976c)(10\,\text{s})\gamma_{4\to5} + (0.992c)(10\,\text{s})\gamma_{5\to6} \\ &= (0.5c)(10\,\text{s})(1.15) + (0.8c)(10\,\text{s})(1.67) + (0.929c)(10\,\text{s})(2.69) \\ &\quad + (0.976c)(10\,\text{s})(4.56) + (0.992c)(10\,\text{s})(7.83) \\ &= \boxed{166c\cdot\text{s}}\end{aligned}$$

The average speed of the spaceship, between boost 1 and boost 5, as measured in S is given by:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t}$$

where Δt is the travel time as measured in the Earth frame of reference.

Express Δt as the sum of the times the spaceship travels during each 10-s interval following a boost in its speed:

$$\begin{aligned}\Delta t &= \Delta t_{1\to2} + \Delta t_{2\to3} + \Delta t_{3\to4} + \Delta t_{4\to5} + \Delta t_{5\to6} \\ &= (10\,\text{s})\gamma_{1\to2} + (10\,\text{s})\gamma_{2\to3} + (10\,\text{s})\gamma_{3\to4} + (10\,\text{s})\gamma_{4\to5} + (10\,\text{s})\gamma_{5\to6} \\ &= (10\,\text{s})(\gamma_{1\to2} + \gamma_{2\to3} + \gamma_{3\to4} + \gamma_{4\to5} + \gamma_{5\to6})\end{aligned}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = (10\,\text{s})(1.15 + 1.67 + 2.69 + 4.56 + 7.83) = 179\,\text{s}$$

Substitute for Δx and Δt and evaluate v_{av}:

$$v_{\text{av}} = \frac{166c\cdot\text{s}}{179\,\text{s}} = \boxed{0.927c}$$

Remarks: This result seems to be reasonable. Relativistic time dilation implies that the spacecraft will be spending larger amounts of time at high speed (as seen in reference frame *S*).

The Relativistic Doppler Shift

23 • Light is emitted by a sodium sample that is moving toward Earth with speed v. The wavelength of the light is 589 nm in the rest frame of the sample. The wavelength measured in the frame of Earth is 547 nm. Find v.

Picture the Problem We can substitute, using $c = f\lambda$, in the relativistic Doppler shift equation and solve for the speed of the source.

Using the expression for the relativistic Doppler shift, express f' as a function of v:

$$f' = f\sqrt{\frac{1+v/c}{1-v/c}}$$

Substitute using $c = f\lambda$ and simplify to obtain:

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \frac{\lambda}{\lambda'} = \sqrt{\frac{1+v/c}{1-v/c}}$$

or

$$\left(\frac{\lambda}{\lambda'}\right)^2 = \frac{1+v/c}{1-v/c}$$

Solving for v gives:

$$v = \left[\frac{\left(\frac{\lambda}{\lambda'}\right)^2 - 1}{\left(\frac{\lambda}{\lambda'}\right)^2 + 1}\right]c$$

Substitute numerical values and evaluate v:

$$v = \left[\frac{\left(\frac{589\,\text{nm}}{547\,\text{nm}}\right)^2 - 1}{\left(\frac{589\,\text{nm}}{547\,\text{nm}}\right)^2 + 1}\right]\left(2.998\times10^8\ \text{m/s}\right) = \boxed{2.22\times10^7\ \text{m/s}}$$

24 • A distant galaxy is moving away from us at a speed of 1.85×10^7 m/s. Calculate the fractional redshift $(\lambda' - \lambda_0)/\lambda_0$ that we observe the light from this galaxy to have.

Picture the Problem We can use the relativistic Doppler shift, when the source and the receiver are receding, to relate the frequencies of the two wavelengths and $c = f\lambda$ to express the ratio of the wavelengths as a function of the speed of the galaxy.

When the source and receiver are moving away from each other, the relativistic Doppler shift is given by:

$$f' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,f_0$$

Use the relationship between the wavelength and frequency to obtain:

$$\frac{c}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\frac{c}{\lambda_0} \Rightarrow \frac{\lambda_0}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

Solving for λ'/λ_0 yields:

$$\frac{\lambda'}{\lambda_0}=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Express the fractional redshift:

$$\frac{\lambda'-\lambda_0}{\lambda_0}=\frac{\lambda'}{\lambda_0}-1=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}-1$$

Substitute numerical values and evaluate $(\lambda'-\lambda_0)/\lambda_0$:

$$\frac{\lambda'-\lambda_0}{\lambda_0}=\sqrt{\frac{1+\frac{1.85\times10^7\text{ m/s}}{2.998\times10^8\text{ m/s}}}{1-\frac{1.85\times10^7\text{ m/s}}{2.998\times10^8\text{ m/s}}}}-1$$

$$=\boxed{0.0637}$$

25 •• Derive $f'=f_0\sqrt{1-\left(v^2/c^2\right)}\Big/\left[1-\left(v/c\right)\right]$ (Equation **Error! Reference source not found.**a) for the frequency received by an observer moving with speed v toward a stationary source of electromagnetic waves.

Picture the Problem We can begin the derivation by expressing the number of waves received by the observer, in the rest frame of the source, in a time interval Δt. We can then relate this time interval to the time interval in the rest frame of the observer to complete the derivation of Equation 39-16a.

Express the number of waves n encountered by the observer, in the rest frame of the source, in a time interval Δt_s:

$$n=\frac{(c+v)\Delta t_s}{\lambda}=\frac{(c+v)f_o\Delta t_s}{c}$$
$$=f_o\left(1+\frac{v}{c}\right)\Delta t_s$$

This time interval in the rest frame of the observer is given by:

$$\Delta t_o=\frac{\Delta t_s}{\gamma}$$

Express the frequency received by the observer and simplify to obtain:

$$f'=\frac{n}{\Delta t_o}=\gamma\left(1+\frac{v}{c}\right)f_o=\frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}f_o=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}f_o=\boxed{\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}f_o}$$

26 • Show that if v is much less than c, the Doppler shift is given approximately by $\Delta f/f \approx \pm v/c$.

Picture the Problem We can use the expression for the relativistic Doppler shift to show that, to a good approximation, $\Delta f/f \approx \pm v/c$.

Express the fractional Doppler shift in terms of f and f_0:

$$\frac{\Delta f}{f_0} = \frac{f - f_0}{f_0} = \frac{f}{f_0} - 1$$

When the source and receiver are approaching each other, the relativistic Doppler shift is given by:

$$f = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\, f_0 \Rightarrow \frac{f}{f_0} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Substitute in the expression for $\Delta f/f_0$ to obtain:

$$\frac{\Delta f}{f_0} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1$$

$$= \left(1+\frac{v}{c}\right)^{1/2}\left(1-\frac{v}{c}\right)^{-1/2} - 1$$

Keeping just the lowest order terms in v/c, expand binomially to obtain:

$$\frac{\Delta f}{f_0} = \left(1+\frac{1}{2}\frac{v}{c}\right)\left(1+\frac{1}{2}\frac{v}{c}\right) - 1$$

$$\approx 1 + \frac{v}{c} - 1 = \boxed{\frac{v}{c}}$$

The sign depends on whether the source and receiver are approaching or receding. Here we have assumed that they are approaching.

27 •• **[SSM]** A clock is placed in a satellite that orbits Earth with an orbital period of 90 min. By what time interval will this clock differ from an identical clock on Earth after 1.0 y? (Assume that special relativity applies and neglect general relativity.)

Picture the Problem Due to its motion, the orbiting clock will run more slowly than the Earth-bound clock. We can use Kepler's third law to find the radius of the satellite's orbit in terms of its period, the definition of speed to find the orbital speed of the satellite from the radius of its orbit, and the time dilation equation to find the difference δ in the readings of the two clocks.

Express the time δ lost by the clock:

$$\delta = \Delta t - \Delta t_{\mathrm{p}} = \Delta t - \frac{\Delta t}{\gamma} = \Delta t\left(1 - \frac{1}{\gamma}\right)$$

Because $v << c$, we can use Part (b) of Problem 10:

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$$

Substitute to obtain:

$$\delta = \Delta t\left[1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\right] = \frac{1}{2}\frac{v^2}{c^2}\Delta t \qquad (1)$$

Express the square of the speed of the satellite in its orbit:

$$v^2 = \left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 r^2}{T^2} \qquad (2)$$

where T is its period and r is the radius of its (assumed) circular orbit.

Use Kepler's third law to relate the period of the satellite to the radius of its orbit about Earth:

$$T^2 = \frac{4\pi^2}{GM_{\mathrm{E}}} r^3 = \frac{4\pi^2}{gR_{\mathrm{E}}^2} r^3 \Rightarrow r = \sqrt[3]{\frac{gR_{\mathrm{E}}^2 T^2}{4\pi^2}}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\frac{(9.81\,\mathrm{m/s^2})(6370\,\mathrm{km})^2(90\,\mathrm{min} \times 60\,\mathrm{s/min})^2}{4\pi^2}} = 6.65 \times 10^6\,\mathrm{m}$$

Substitute numerical values in equation (2) and evaluate v^2:

$$v^2 = \frac{4\pi^2(6.65 \times 10^6\,\mathrm{m})^2}{(90\,\mathrm{min} \times 60\,\mathrm{s/min})^2} = 5.99 \times 10^7\,\mathrm{m^2/s^2}$$

Finally, substitute for v^2 in equation (1) and evaluate δ:

$$\delta = \frac{1}{2}\frac{(5.99 \times 10^7\,\mathrm{m^2/s^2})(1.0\,\mathrm{y} \times 31.56\,\mathrm{Ms/y})}{(2.998 \times 10^8\,\mathrm{m/s})^2} = \boxed{11\,\mathrm{ms}}$$

28 •• For light that is Doppler-shifted with respect to an observer, we define the redshift parameter $z = (f - f')/f'$, where f is the frequency of the light measured in the rest frame of the emitter, and f' is the frequency measured in the rest frame of the receiver. If the emitter is moving directly away from the receiver, show that the relative velocity between the emitter and the receiver is $v = c(u^2 - 1)/(u^2 + 1)$, where $u = z + 1$.

Picture the Problem We can use the definition of the redshift parameter and the relativistic Doppler shift equation to show that $v = c\left(\dfrac{u^2-1}{u^2+1}\right)$, where $u = z + 1$.

The red-shift parameter is defined to be:

$$z = \frac{f_0 - f'}{f'}$$

The relativistic Doppler shift for recessional motion is given by:

$$f' = f_0\sqrt{\frac{1-v/c}{1+v/c}}$$

Substitute for f' and simplify to obtain:

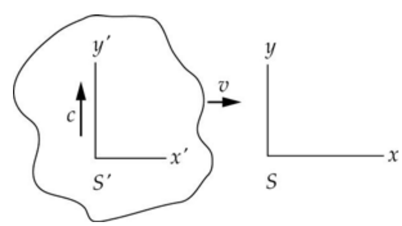
$$z = \frac{f_0 - f_0\sqrt{\dfrac{1-v/c}{1+v/c}}}{f_0\sqrt{\dfrac{1-v/c}{1+v/c}}} = \sqrt{\frac{1+v/c}{1-v/c}} - 1$$

Letting $u = z + 1$:

$$u = z + 1 = \sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow v = \boxed{c\left(\frac{u^2-1}{u^2+1}\right)}$$

29 • A light beam moves along the y' axis with speed c in frame S', which is moving to the in the $+x$ direction with speed v relative to frame S. (*a*) Find the x and y components of the velocity of the light beam in frame S. (*b*) Show that, according to the velocity transformation equations, the magnitude of the velocity of the light beam in S is c.

Picture the Problem We can use the velocity transformation equations for the x and y directions to express the x and y components of the velocity of the light beam in frame S.



(*a*) The x and y components of the velocity of the light beam in frame S are:

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}} \text{ and } u_y = \frac{u_y'}{\gamma\left(1 + \dfrac{vu_x'}{c^2}\right)}$$

Because $u_x' = 0$:

$$u_x = \boxed{v} \text{ and } u_y = \boxed{\frac{c}{\gamma}}$$

(*b*) The magnitude of the velocity of the light beam in *S* is given by:

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{v^2 + \frac{c^2}{\gamma^2}}$$

$$= \sqrt{v^2 + \left(1 - \frac{v^2}{c^2}\right)c^2} = \boxed{c}$$

30 •• A spaceship is moving east at speed 0.90c relative to Earth. A second spaceship is moving west at speed 0.90c relative to Earth. What is the speed of one spaceship relative to the other spaceship?

Picture the Problem Let S be the Earth reference frame and S' be that of the ship traveling East (+x direction). Then in the reference frame S', the velocity of S is directed West, i.e., $v = -u_x$. We can apply the inverse velocity transformation equation to express u'_x in terms of u_x.

Apply the inverse velocity transformation equation to obtain:

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

Substituting for v gives:

$$u_x' = \frac{u_x + u_x}{1 + \frac{u_x^2}{c^2}} = \frac{2u_x}{1 + \frac{u_x^2}{c^2}}$$

Because $u_x = 0.90c$:

$$u_x' = \frac{2(0.90c)}{1 + \frac{(0.90c)^2}{c^2}} = \boxed{0.99c}$$

31 •• **[SSM]** A particle moves with speed 0.800c in the $+x''$ direction along the x'' axis of frame S'', which moves with the same speed and in the same direction along the x' axis relative to frame S'. Frame S' moves with the same speed and in the same direction along the x axis relative to frame S. (*a*) Find the speed of the particle relative to frame S'. (*b*) Find the speed of the particle relative to frame S.

Picture the Problem We can apply the inverse velocity transformation equation to express the speed of the particle relative to both frames of reference.

(*a*) Express u_x' in terms of u_x'':

$$u_x' = \frac{u_x'' + v}{1 + \frac{vu_x''}{c^2}}$$

where v of S', relative to S'', is 0.800c.

Substitute numerical values and evaluate u_x':

$$u_x' = \frac{0.800c + 0.800c}{1 + \frac{(0.800c)^2}{c^2}} = \frac{1.60c}{1.64} = \boxed{0.976c}$$

(*b*) Express u_x in terms of u_x':

$$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}}$$ where v, the speed of S, relative to S', is $0.800c$.

Substitute numerical values and evaluate u_x:

$$u_x = \frac{0.976c + 0.800c}{1 + \frac{(0.800c)(0.976c)}{c^2}} = \frac{1.776c}{1.781} = \boxed{0.997c}$$

Relativistic Momentum and Relativistic Energy

32 • A proton that has a rest energy equal to 938 MeV has a total energy of 2200 MeV. (*a*) What is its speed? (*b*) What is its momentum?

Picture the Problem We can use the relation for the total energy, momentum, and rest energy to find the momentum of the proton and Equation 39-25 to relate the speed of the proton to its energy and momentum.

Relate the energy of the proton to its momentum:

$$E^2 = p^2c^2 + (mc^2)^2$$

(*b*) Solving for p gives:

$$p = \sqrt{\frac{E^2 - (mc^2)^2}{c^2}}$$

Substitute numerical values and evaluate p:

$$p = \frac{\sqrt{(2200\,\text{MeV})^2 - (938\,\text{MeV})^2}}{c} = \boxed{1.99\frac{\text{GeV}}{c}}$$

(*a*) From Equation 39-25 we have:

$$\frac{v}{c} = \frac{pc}{E} \Rightarrow v = \frac{pc}{E}c$$

Substitute numerical values and evaluate v:

$$v = \frac{1.99\,\text{GeV}}{2200\,\text{MeV}}c = \boxed{0.905c}$$

33 • If the kinetic energy of a particle equals twice its rest energy, what percentage error is made by using $p = mu$ for the magnitude of its momentum?

Picture the Problem We can use $E^2 = p^2c^2 + (mc^2)^2$ (Equation R-17) to find the relativistic momentum of the particle in terms of γ and the fact that the kinetic energy of the particle equals twice its rest energy to find the error made in using mv for the momentum of the particle.

Express the error e in using $p' = mv$ for the momentum of the particle:

$$e = \frac{p - p'}{p} = 1 - \frac{p'}{p} \qquad (1)$$

The relationship between the total energy E, momentum p, and rest energy mc^2 of the particle is:

$$E^2 = p^2c^2 + (mc^2)^2 \qquad \text{R-17}$$

Solve for p and simplify to obtain:

$$p = \sqrt{\frac{E^2}{c^2} - m^2c^2} = \sqrt{m^2c^2\left(\frac{E^2}{m^2c^4} - 1\right)}$$

Because $E = \gamma mc^2$:

$$p = mc\sqrt{\gamma^2 - 1}$$

Substitute for p and p' in equation (1) to obtain:

$$e = 1 - \frac{mu}{mc\sqrt{\gamma^2 - 1}} = 1 - \frac{u}{c\sqrt{\gamma^2 - 1}} \qquad (2)$$

From the definition of γ:

$$\frac{u}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

Eliminate v/c in equation (2) to obtain:

$$e = 1 - \frac{1}{\sqrt{\gamma^2 - 1}}\sqrt{1 - \frac{1}{\gamma^2}} = 1 - \frac{1}{\gamma} \qquad (3)$$

The kinetic energy of the particle is related to its rest energy:

$$K = (\gamma - 1)mc^2 \Rightarrow \gamma = 1 + \frac{K}{mc^2}$$

Because the kinetic energy of the particle is twice its rest energy:

$$\gamma = 1 + \frac{2mc^2}{mc^2} = 3$$

Substitute for γ in equation (3) and evaluate e:

$$e = 1 - \frac{1}{3} = 0.667 = \boxed{66.7\%}$$

34 •• In a certain reference frame, a particle has momentum of 6.00 MeV/c and total energy of 8.00 MeV. (a) Determine the mass of the particle. (b) What is the total energy of the particle in a reference frame in which its momentum is 4.00 MeV/c? (c) What is the relative speed of the two reference frames?

Picture the Problem (a) The mass of the particle is given by $m = E_0/c^2$. (b) We can use Equation R-17 [$E^2 = p^2c^2 + \left(mc^2\right)^2$] to find the total energy of the particle in a reference frame in which its momentum is 4.00 MeV/c. (c) We can apply the inverse velocity transformation equation to find the relative speeds of the two reference frames.

(a) The mass of the particle is given by:

$$m = \frac{E_0}{c^2}$$

Substituting numerical values yields:

$$m = \frac{(8.00\,\text{MeV})\left(1.602\times10^{-19}\,\text{J/eV}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2} = \boxed{1.43\times10^{-29}\ \text{kg}}$$

(b) Solving equation R-17 for E gives:

$$E = \sqrt{p^2c^2 + \left(mc^2\right)^2} = \sqrt{p^2c^2 + E_0^2} \quad (1)$$

We can use Equation R-17 to find the energy of the particle in the reference frame in which it has momentum of 6.00 MeV/c and total energy of 8.00 MeV:

$$E^2 = p^2c^2 + E_0^2 \Rightarrow E_0 = \sqrt{E^2 - p^2c^2}$$

Substitute numerical values and evaluate E_0:

$$E_0 = \sqrt{(8.00\,\text{MeV})^2 - (6.00\,\text{MeV}/c)^2c^2} = \sqrt{(8.00\,\text{MeV})^2 - (6.00\,\text{MeV})^2} = 5.292\,\text{MeV}$$

Substitute numerical values in equation (1) and evaluate the total energy E of the particle:

$$E = \sqrt{(4.00\,\text{MeV}/c)^2c^2 + (5.292\,\text{MeV})^2} = \boxed{6.63\,\text{MeV}}$$

(c) The inverse speed transformation is:

$$u_b = \frac{u_a - v}{1 - \dfrac{vu_a}{c^2}}$$

where the subscripts refer to the speeds in Parts (a) and (b) of the problem.

Solving for v gives:

$$v=\frac{u_a-u_b}{1-\frac{u_a u_b}{c^2}} \qquad (1)$$

Relate the relativistic energy of the particle in (*a*) to its speed:

$$E=\frac{E_0}{\sqrt{1-\frac{u_a^2}{c^2}}}\Rightarrow u_a=c\sqrt{1-\left(\frac{E_0}{E}\right)^2}$$

Substitute numerical values and evaluate u_a:

$$u_a=c\sqrt{1-\left(\frac{5.292\,\text{MeV}}{8.00\,\text{MeV}}\right)^2}=0.7499c$$

Relate the relativistic momentum of the particle in (*b*) to its speed:

$$p=\frac{m_0u_b}{\sqrt{1-\frac{u_b^2}{c^2}}}\Rightarrow u_b=\frac{pc}{\sqrt{p^2+m_0c^2}}$$

Substitute numerical values and evaluate u_b:

$$u_b=\frac{(4.00\,\text{MeV}/c)c}{\sqrt{(4.00\,\text{MeV}/c)^2+(5.292\,\text{MeV}/c)^2}}=0.6030c$$

Substitute in equation (1) and evaluate v:

$$v=\frac{0.7499c-0.6030c}{1-\frac{(0.7499c)(0.6030c)}{c^2}}=\boxed{0.268c}$$

35 •• Show that $d\left(\frac{m_0u}{\sqrt{1-u^2/c^2}}\right)=m_0\left(1-\frac{u^2}{c^2}\right)^{-3/2}du$. *Note*: This relationship was used to derive the relativistically correct expression for kinetic energy (Equation 39-22).

Picture the Problem We can use the rule for the derivative of a quotient to establish the result given in the problem statement.

Use the expression for the derivative of a quotient to obtain:

$$\frac{d}{du}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)=\frac{\sqrt{1-\frac{u^2}{c^2}}m+\frac{mu^2}{c^2}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}{1-\frac{u^2}{c^2}}$$

Multiply the numerator and denominator of the right-hand side of this expression by $\sqrt{1-\frac{u^2}{c^2}}$ and simplify to obtain:

$$\frac{d}{du}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)=\frac{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}m+\frac{mu^2}{c^2}\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}}{\left(1-\frac{u^2}{c^2}\right)\sqrt{1-\frac{u^2}{c^2}}}=\frac{\left(1-\frac{u^2}{c^2}\right)m+\frac{mu^2}{c^2}}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

$$=m\left(1-\frac{u^2}{c^2}\right)^{-3/2}$$

and

$$d\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)=\boxed{m\left(1-\frac{u^2}{c^2}\right)^{-3/2}du}$$

36 •• The K^0 particle has a mass of 497.7 MeV/c^2. It decays into a π^- and π^+, each having mass 139.6 MeV/c^2. Following the decay of a K^0, one of the pions is at rest in the laboratory. Determine the kinetic energy of the other pion after the decay and of the K^0 prior to the decay.

Picture the Problem We will first consider the decay process in the center of mass reference frame and then transform to the laboratory reference frame in which one of the pions is at rest.

Apply energy conservation in the center of mass frame of reference to obtain:

$$m_{K_0}c^2=2m_{\pi_0}\gamma c^2\Rightarrow\gamma=\frac{m_{K_0}}{2m_{\pi_0}}$$

Substitute numerical values and evaluate γ:

$$\gamma=\frac{497.7\,\text{MeV}/c^2}{2\left(139.6\,\text{MeV}/c^2\right)}=1.783$$

Because one of the pions is at rest in the laboratory frame, γ = 1.78 for the transformation to the laboratory frame. The kinetic energy of the K^0 particle is:

$$\begin{aligned}K_{K^0}&=(\gamma-1)E\\&=(1.783-1)(497.7\,\text{MeV})\\&=388.2\,\text{MeV}\\&=\boxed{388\,\text{MeV}}\end{aligned}$$

The total initial energy in the laboratory frame is:

$$E = 497.7\,\text{MeV} + 388.2\,\text{MeV} = 885.9\,\text{MeV}$$

Express the energy of the other pion:

$$E_\pi = E - 2m_{0\pi}c^2$$

Substitute numerical values and evaluate E_π:

$$E_\pi = 885.9\,\text{MeV} - 2(139.6\,\text{MeV}) = \boxed{607\,\text{MeV}}$$

37 •• [SSM] In reference frame *S'*, two protons, each moving at 0.500*c*, approach each other head-on. (*a*) Calculate the total kinetic energy of the two protons in frame *S*′. (*b*) Calculate the total kinetic energy of the protons as seen in reference frame *S*, which is moving with one of the protons.

Picture the Problem The total kinetic energy of the two protons in Part (*a*) is the sum of their kinetic energies and is given by $K = 2(\gamma - 1)E_0$. Part (*b*) differs from Part (*a*) in that we need to find the speed of the moving proton relative to frame *S*.

(*a*) The total kinetic energy of the protons in frame *S*′ is given by:

$$K = 2(\gamma - 1)E_0$$

Substitute for γ and E_0 and evaluate *K*:

$$K = 2\left(\frac{1}{\sqrt{1 - \frac{(0.500c)^2}{c^2}}} - 1\right)(938.28\,\text{MeV}) = \boxed{290\,\text{MeV}}$$

(*b*) The kinetic energy of the moving proton in frame *S* is given by:

$$K = (\gamma - 1)E_0 \qquad (1)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{uv}{c^2}}}$$

Express the speed *u* of the proton in frame *S*:

$$u = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$

Substitute numerical values and evaluate *u*:

$$u = \frac{0.500c + 0.500c}{1 + \frac{(0.500c)(0.500c)}{c^2}} = 0.800c$$

Evaluate γ:

$$\gamma = \frac{1}{\sqrt{1-\frac{(0.800c)(0.800c)}{c^2}}} = 1.67$$

Substitute numerical values in equation (1) and evaluate K:

$$K = (1.67-1)(938.28\,\text{MeV}) = \boxed{629\,\text{MeV}}$$

38 •• An antiproton $\overline{\text{p}}$ has the same mass m as a proton p. The antiproton is created during the reaction $\text{p} + \text{p} \to \text{p} + \text{p} + \text{p} + \overline{\text{p}}$. During an experiment, protons at rest in the laboratory are bombarded with protons of kinetic energy K_L, which must be great enough so that an amount of kinetic energy equal to $2mc^2$ can be converted into the rest energy of the two particles. In the frame of the laboratory, the total kinetic energy cannot be converted into rest energy because of conservation of momentum. However, in the zero-momentum reference frame in which the two initial protons are moving toward each other with equal speed u, the total kinetic energy can be converted into rest energy. (*a*) Find the speed of each proton u so that the total kinetic energy in the zero-momentum frame is $2mc^2$. (*b*) Transform to the laboratory's frame in which one proton is at rest, and find the speed u' of the other proton. (*c*) Show that the kinetic energy of the moving proton in the laboratory's frame is $K_\text{L} = 6mc^2$.

Picture the Problem (*a*) We can find the speed of each proton by equating their total relativistic kinetic energy to $2mc^2$. In Part (*b*) we can use the inverse velocity transformation with $V = u$ and $u_x = -u$ to find u_x'. In Part (*c*) we'll need to evaluate γ' for the kinetic energy transformation $K_\text{L} = (\gamma'-1)E_0$.

(*a*) Set the relativistic kinetic energy of the protons equal to $2mc^2$ to obtain:

$$2(\gamma-1)E_0 = 2mc^2 \Rightarrow \gamma = 2$$

Substitute for γ:

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 2 \Rightarrow u = \frac{\sqrt{3}}{2}c = \boxed{0.866c}$$

(*b*) Use the inverse velocity transformation with $v = u$ and $u_x = -u$ to find u_x':

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} = \frac{-\frac{\sqrt{3}}{2}c - \frac{\sqrt{3}}{2}c}{1 - \frac{\left(-\frac{\sqrt{3}}{2}c\right)\left(\frac{\sqrt{3}}{2}\mathrm{c}\right)}{c^2}}$$

$$= \frac{-4\sqrt{3}c}{7} = \boxed{-0.990c}$$

(*c*) The kinetic energy of the moving proton in the laboratory's frame is given by:

$$K_\mathrm{L} = (\gamma' - 1)E_0$$

where

$$\gamma' = \frac{1}{\sqrt{1 - \frac{(u_x')^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\left(-\frac{4\sqrt{3}}{7}c\right)^2}{c^2}}} = 7$$

Substitute for γ' and E_0 and evaluate K_L:

$$K_\mathrm{L} = (7-1)mc^2 = \boxed{6mc^2}$$

39 **•••** A particle of mass 1.0 MeV/c^2 and kinetic energy 2.0 MeV collides with a stationary particle of mass 2.0 MeV/c^2. After the collision, the particles stick together. Find (*a*) the speed of the first particle before the collision, (*b*) the total energy of the first particle before the collision, (*c*) the initial total momentum of the system, (*d*) the total kinetic energy after the collision, and (*e*) the mass of the system after the collision.

Picture the Problem Parts (*a*) and (*b*) The initial speed of the particle can be found from its total energy and its total energy found using $E = K + E_0 = \gamma E_0$. (*c*) We can solve $E^2 = p^2c^2 + (mc^2)^2$ for the initial momentum of the system. In Parts (*d*) and (*e*) we can use conservation of energy and conservation of momentum to find the total kinetic energy after the collision and the mass of the system after the collision.

(*a*) Express the total energy of the particle:

$$E = K + E_0 = \gamma E_0$$

Because the kinetic energy of the particle is twice its energy:

$$2E_0 + E_0 = \gamma E_0 \text{ and } \gamma = 3$$

Solving $\gamma = \dfrac{1}{\sqrt{1-(u/c)^2}}$ for u yields:

$$u = c\sqrt{1-\frac{1}{\gamma^2}}$$

Substituting the numerical value of γ and evaluating u gives:

$$u = c\sqrt{1-\frac{1}{3^2}} = c\sqrt{\frac{8}{9}} = \boxed{0.943c}$$

(*b*) The total energy of the particle is:

$$E = K + E_0 = \gamma E_0 = 3E_0$$

Substitute for E_0 and evaluate E:

$$E = 3(1.0\,\text{MeV}) = \boxed{3.0\,\text{MeV}}$$

(*c*) The initial momentum of the incoming particle is related to its energy and mass according to:

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

Solving for p yields:

$$p = \frac{1}{c}\sqrt{E^2 - \left(mc^2\right)^2}$$

Substitute for E and mc^2 and simplify to obtain:

$$p = \frac{1}{c}\sqrt{(3E_0)^2 - (E_0)^2} = \frac{\sqrt{8}E_0}{c}$$

Substitute for E_0 and evaluate p:

$$p = \frac{\sqrt{8}(1.0\,\text{MeV})}{c} = \boxed{2.8\,\text{MeV}/c}$$

(*d*) and (*e*) From conservation of energy we have:

$$E_\text{f} = E_\text{i} = 5.0\,\text{MeV}$$

From conservation of momentum we have:

$$p_\text{f} = p_\text{i}$$

The final momentum of the system is related to its energy and mass according to:

$$E_\text{f}^2 = p_\text{f}^2c^2 + E_\text{f0}^2 \Rightarrow E_\text{f0} = \sqrt{E_\text{f}^2 - p_\text{f}^2c^2}$$

Substitute numerical values and evaluate E_f0:

$$E_\text{f0} = \sqrt{(5.0\,\text{MeV})^2 - (2.83\,\text{MeV}/c)^2c^2} = 4.12\,\text{MeV}$$

Because $E_\text{f0} = m_\text{f0}c^2$:

$$m_\text{f0} = \frac{E_\text{f0}}{c^2} = \boxed{4.1\,\text{MeV}/c^2}$$

The total kinetic energy after the collision is given by:

$$K_f = E_f - E_{f0} = 5.0\,\text{MeV} - 4.122\,\text{MeV} = \boxed{0.9\,\text{MeV}}$$

General Relativity

40 •• Light traveling in the direction of increasing gravitational potential undergoes a frequency redshift. Calculate the shift in wavelength if a beam of light of wavelength $\lambda = 632.8$ nm is sent up a vertical shaft of height $L = 100$ m.

Picture the Problem Let m represent the mass equivalent of a photon. We can equate the change in the gravitational potential energy of a photon as it rises a distance L in the gravitational field to $h\Delta f$ and then express the wavelength shift in terms of the frequency shift.

The speed of the photons in the light beam are related to their frequency and wavelength:

$$c = f\lambda \Rightarrow f = \frac{c}{\lambda}$$

Differentiate this expression with respect to λ to obtain:

$$\frac{df}{d\lambda} = -c\lambda^{-2} = -\frac{c}{\lambda^2}$$

Approximate $df/d\lambda$ by $\Delta f/\Delta\lambda$ and solve for Δf:

$$\Delta f = -\frac{c}{\lambda^2}\Delta\lambda$$

Divide both sides of this equation by f to obtain:

$$\frac{\Delta f}{f} = \frac{-\dfrac{c}{\lambda^2}\Delta\lambda}{\dfrac{c}{\lambda}} = -\frac{\Delta\lambda}{\lambda}$$

Solving for $\Delta\lambda$ gives:

$$\Delta\lambda = -\lambda\frac{\Delta f}{f} \qquad (1)$$

The change in the energy of the photon as it rises a distance L in a gravitational field is given by:

$$\Delta E = \Delta U = mgL$$

Because $\Delta E = h\Delta f$:

$$h\Delta f = mgL \qquad (2)$$

Letting m represent the mass equivalent of the photon:

$$E = mc^2 = hf \qquad (3)$$

Divide equation (2) by equation (3) to obtain:

$$\frac{h\Delta f}{hf}=\frac{mgL}{mc^2}\Rightarrow\frac{\Delta f}{f}=\frac{gL}{c^2}$$

Substitute for $\Delta f/f$ in equation (1):

$$\Delta\lambda=-\frac{gL\lambda}{c^2}$$

Substitute numerical values and evaluate $\Delta\lambda$:

$$\Delta\lambda=-\frac{(9.81\,\text{m/s}^2)(100\,\text{m})(632.8\,\text{nm})}{(2.998\times10^8\,\text{m/s})^2}=\boxed{-6.90\times10^{-12}\,\text{nm}}$$

41 •• Let us revisit a problem from Chapter 3: Two cannons are pointed directly toward each other, as shown in Figure 39-17. When fired, the cannonballs will follow the trajectories shown. Point *P* is the point where the trajectories cross each other. Ignore any effects due to air resistance. Using the principle of equivalence, show that if the cannons are fired simultaneously (in the rest frame of the cannons), the cannonballs will hit each other at point *P*.

Picture the Problem In a freely falling reference frame, both cannonballs travel along straight lines, so they must hit each other, as they were pointed at each other when they were fired. On the other hand, in Earth's homogeneous gravitational field both balls will have the same vertical acceleration and so will hit each other at *P*. This shows that a homogeneous gravitational field is completely equivalent to a uniformly accelerated reference frame.

42 ••• A horizontal turntable rotates with angular speed ω. There is a clock at the center of the turntable and an identical clock mounted on a turntable a distance r from the center. In an inertial reference frame in which the clock at the center is at rest, the clock at distance r is moving with speed $u = r\omega$. (*a*) Show that from time dilation according to special relativity, the time between ticks, Δt_0 for the clock at rest and Δt_R for the moving clock, are related by

$$\frac{\Delta t_r-\Delta t_0}{\Delta t_0}=-\frac{r^2\omega^2}{2c^2}\qquad r\omega = c$$

(*b*) In a reference frame rotating with the table, both clocks are at rest. Show that the clock at distance r experiences a pseudoforce $F_r = mr\omega^2$ in this rotating frame and that this is equivalent to a difference in gravitational potential between r and the origin of $\phi_r-\phi_0=-\frac{1}{2}r^2\omega^2$. Use the difference in gravitational potential given in Part (*b*) to show that in this frame the difference in time intervals is the same as in the inertial frame.

Picture the Problem Consider the turntable to be a giant hollow cylinder in space that is spinning about its axis. Someone on the inside surface of the cylinder would experience a centripetal acceleration caused by the normal force of the surface pushing them toward the rotation axis. Alternatively, they can consider

that they are not accelerating but a gravitational field $\vec{\boldsymbol{g}} = \omega^2 r\hat{\boldsymbol{r}}$ is pushing them away from the axis ($\hat{\boldsymbol{r}}$ is away from the axis). This is the principle of equivalence. From this perspective, up is toward the axis and the points closer to the axis are at the higher gravitational potential. (Just like the electric field points in the direction of decreasing electric potential, the gravitational field points in the direction of decreasing gravitational potential.)

(*a*) From the time dilation equation we have:

$$\Delta t_{\mathrm{r}} = \frac{\Delta t_0}{\gamma} \text{ and } \frac{\Delta t_{\mathrm{r}} - \Delta t_0}{\Delta t_0} = \frac{1}{\gamma} - 1$$

Because $r\omega/c << 1$:

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{u^2}{c^2} = 1 - \frac{r^2\omega^2}{2c^2}$$

Substitute for $\frac{1}{\gamma}$ and simplify to obtain:

$$\frac{\Delta t_{\mathrm{r}} - \Delta t_0}{\Delta t_0} = 1 - \frac{r^2\omega^2}{2c^2} - 1 = \boxed{-\frac{r^2\omega^2}{2c^2}}$$

(*b*) The pseudoforce is given by:

$$F_{\mathrm{P}} = -ma$$

where a is the acceleration of the non-inertial reference frame.

In this case a is the centripetal acceleration:

$$a = -r\omega^2 \Rightarrow F_{\mathrm{P}} = F_r = \boxed{mr\omega^2}$$

To relate this problem to Equation 39-31, point 2 is a distance r from the axis and point 1 is on the axis. The term in parentheses on the right hand side of Equation 39-31 is $\phi_2 - \phi_1$, which translates to $\phi_{\mathrm{r}} - \phi_0$. Because ϕ_{r} is at a lower potential than ϕ_0, this term is negative. Hence:

$$\phi_r - \phi_0 = -\int_0^r \vec{\boldsymbol{g}} \cdot d\vec{\ell} = -\int_0^r \omega^2 r\,\hat{\boldsymbol{r}} \cdot d\vec{\ell} = -\int_0^r \omega^2 r\,dr = \boxed{-\tfrac{1}{2}r^2\omega^2}$$

From Equation 39-31:

$$\frac{\Delta t_{\mathrm{r}} - \Delta t_0}{\Delta t_0} = \frac{1}{c^2}(\phi_r - \phi_0) = \frac{1}{c^2}\left(-\tfrac{1}{2}r^2\omega^2\right)$$

$$= \boxed{-\frac{r^2\omega^2}{2c^2}}$$

General Problems

43 • How fast must a muon travel so that its mean lifetime is 46 μs if its mean lifetime at rest is 2.2 μs?

Picture the Problem We can use the definition of γ and the time dilation equation to find the speed of the muon.

(a) From the definition of γ we have:

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} \Rightarrow \frac{u}{c} = \sqrt{1-\frac{1}{\gamma^2}}$$

Relate the mean lifetime of the muon to its proper lifetime:

$$\Delta t = \gamma \Delta t_p \Rightarrow \gamma = \frac{\Delta t}{\Delta t_p}$$

Substitute in the expression for u/c to obtain:

$$\frac{u}{c} = \sqrt{1-\left(\frac{\Delta t_p}{\Delta t}\right)^2}$$

Substitute numerical values and evaluate u/c:

$$\frac{u}{c} = \sqrt{1-\left(\frac{2.2\ \mu\text{s}}{46\ \mu\text{s}}\right)^2} = 0.999$$

or

$$u = \boxed{0.999c}$$

44 • A distant galaxy is moving away from Earth with a speed that results in each wavelength received on Earth being shifted so that $\lambda' = 2\lambda_0$. Find the speed of the galaxy relative to Earth.

Picture the Problem We can use the relativistic Doppler shift, when the source and the receiver are receding, to relate the frequencies of the two wavelengths and $c = f\lambda$ to express the ratio of the wavelengths as a function of the speed of the galaxy.

When the source and receiver are moving away from each other, the relativistic Doppler shift is given by:

$$f' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,f_0$$

Use the relationship between the wavelength and frequency to obtain:

$$\frac{c}{\lambda'}=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\frac{c}{\lambda_0}\Rightarrow\frac{\lambda_0}{\lambda'}=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

Because $\lambda' = 2\lambda_0$:

$$\frac{\lambda_0}{2\lambda_0}=\frac{1}{2}=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\Rightarrow v=\boxed{0.600c}$$

45 •• **[SSM]** Frames S and S' are moving relative to each other along the x and x' axes (which superpose). Observers at rest in the two frames set their clocks to $t = 0$ when the two origins coincide. In frame S, event 1 occurs at $x_1 = 1.0\ c\cdot\text{y}$ and $t_1 = 1.00$ y and event 2 occurs at $x_2 = 2.0\ c\cdot\text{y}$ and $t_2 = 0.50$ y. These events occur simultaneously in frame S'. (*a*) Find the magnitude and direction of the velocity of S' relative to S. (*b*) At what time do both these events occur as measured in S'?

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to relate the elapsed times and separations of the events in the two systems to the velocity of S' relative to S. We can use this same relationship in Part (*b*) to find the time at which these events occur as measured in S'.

(*a*) Use Equation 39-12 to obtain:

$$\Delta t' = t_2' - t_1' = \gamma\left[(t_2-t_1)-\frac{v}{c^2}(x_2-x_1)\right]$$
$$=\gamma\left[\Delta t-\frac{v}{c^2}\Delta x\right]$$

Because the events occur simultaneously in frame S', $\Delta t' = 0$ and:

$$0=\Delta t-\frac{v}{c^2}\Delta x\Rightarrow v=\frac{c^2\Delta t}{\Delta x}$$

Substitute for Δt and Δx and evaluate v:

$$v=\frac{c^2(0.50\,\text{y}-1.00\,\text{y})}{2.0\,c\cdot\text{y}-1.00\,c\cdot\text{y}}=\boxed{-0.50c}$$

Because $\Delta t = t_2 - t_1 = -0.50\,\text{y}$:

$$\boxed{S'\text{ moves in the }-x\text{ direction.}}$$

(*b*) Use the inverse time transformation to obtain:

$$t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) = \frac{t_2 - \dfrac{vx_2}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

Substitute numerical values and evaluate t_2' and t_1':

$$t_2' = t_1' = \frac{0.50\,\text{y} - \dfrac{(-0.50c)(2.0c\cdot\text{y})}{c^2}}{\sqrt{1-\dfrac{(-0.50c)^2}{c^2}}}$$

$$= \boxed{1.7\,\text{y}}$$

46 •• An interstellar spaceship travels from Earth to a star system 12 light-years away (as measured in Earth's frame). The trip takes 15 y as measured by clocks on the spaceship. (*a*) What is the speed of the spaceship relative to Earth? (*b*) When the spaceship arrives, it sends an electromagnetic signal to Earth. How long after the spaceship leaves Earth will observers on Earth receive the signal?

Picture the Problem We can use the relationship between distance, speed, and time and the length contraction relationship to find the speed of the ship relative to Earth. The elapsed time between the departure of the spaceship and the receipt of the signal at Earth is the sum of the travel time to the distant star system and the time it takes the signal to return to Earth.

(*a*) Express the travel time as measured on the spaceship:

$$\Delta t' = \frac{L'}{u} = \frac{L}{\gamma u} \Rightarrow \gamma u = \frac{L}{\Delta t'}$$

Substitute numerical values and evaluate γu:

$$\gamma u = \frac{12c\cdot\text{y}}{15\,\text{y}} = 0.80c$$

or

$$\frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} = \frac{0.80c}{u} \Rightarrow u = \boxed{0.63c}$$

(*b*) The elapsed time T before Earth receives the signal is the sum of the travel time to the distant star system and the time it takes the signal to return:

$$T = \frac{L}{u} + \frac{L}{c}$$

Substitute numerical values and evaluate T:

$$T = \frac{12c\cdot\text{y}}{0.625c} + \frac{12c\cdot\text{y}}{c} = \boxed{31\,\text{y}}$$

47 •• The neutral pion π^0 has a rest mass of 135.0 MeV/c^2. This particle can be created in a proton–proton collision: p + p $\rightarrow$ p + p + π^0. Determine the threshold kinetic energy for the creation of a π^0 in a collision of a moving and stationary proton. (See Problem 38.)

Picture the Problem We can use conservation of energy to find γ in the CM frame of reference and then use the definition of γ to find the speed u of the projectile proton. We can then use the velocity transformation equation to find the speed and kinetic energy of this proton in the laboratory frame of reference.

Use conservation of energy to find γ in the CM frame of reference:

$$\gamma E_\text{i} = E_\text{f} \Rightarrow \gamma = \frac{E_\text{f}}{E_\text{i}}$$

E_i and E_f are:

$$E_\text{i} = 938\,\text{Mev} + 938\,\text{MeV} = 1876\,\text{MeV}$$

and

$$E_\text{f} = 938\,\text{Mev} + 938\,\text{MeV} + 135.0\,\text{MeV} = 2011\,\text{MeV}$$

Substitute E_i and E_f and evaluate γ:

$$\gamma = \frac{2011\,\text{MeV}}{1876\,\text{MeV}} = 1.072$$

Express γ as a function of the speed u of the projectile proton:

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow u = c\sqrt{1-\frac{1}{\gamma^2}}$$

Substitute for γ and evaluate u:

$$u = c\sqrt{1-\frac{1}{(1.072)^2}} = 0.360c$$

Transform to the laboratory frame and find u':

$$u' = \frac{u-v}{1-\frac{vu}{c^2}} = \frac{0.360c-(-0.360c)}{1-\frac{(0.360c)(-0.360\text{c})}{c^2}} = 0.637c$$

The kinetic energy of the moving proton in the laboratory's frame is given by:

$$K_{\mathrm{L}} = (\gamma_{\mathrm{L}} - 1)E_0$$

where

$$\gamma_{\mathrm{L}} = \frac{1}{\sqrt{1 - \frac{(u')^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.637c)^2}{c^2}}} = 1.297$$

Substitute for γ_{L} and E_0 and evaluate K_{L}:

$$K_{\mathrm{L}} = (1.297 - 1)(938\,\mathrm{MeV}) = \boxed{281\,\mathrm{MeV}}$$

Remarks: In Problem 51 we show that the threshold kinetic energy of the projectile is given by $K_{\mathrm{th}} = \dfrac{\left(\sum m_{\mathrm{in}} + \sum m_{\mathrm{fin}}\right)\left(\sum m_{\mathrm{fin}} - \sum m_{\mathrm{in}}\right)c^2}{2m_{\mathrm{target}}}$.

48 •• A rocket that has a proper length of 1000 m moves away from a space station and in the $+x$ direction at $0.60c$ relative to an observer on the station. An astronaut stands at the rear of the rocket and fires a dart toward the front of the rocket at $0.80c$ relative to the rocket. How long does it take the dart to reach the front of the rocket (*a*) as measured in the frame of the rocket, (*b*) as measured in the frame of the space station, and (*c*) as measured in the frame of the dart?

Picture the Problem Let S be the frame of the space station and let S′ be the frame of the rocket. Then $v = +0.60c$. In addition, let event A be the dart leaving the gun, event B be the dart arriving at the front of the ship, and L be the rest length of the ship. Then $x'_{\mathrm{A}} = 0$, $x'_{\mathrm{B}} = L$, and $t'_{\mathrm{A}} = 0$. Finally, let u_x and u'_x be the speed of the dart in reference frames S and S′ respectively.

(*a*) In the reference frame of the rocket:

$$t'_{\mathrm{B}} - t'_{\mathrm{A}} = \frac{L}{u'_x} = \frac{L}{0.80c}$$

or, because $t'_{\mathrm{A}} = 0$,

$$t'_{\mathrm{B}} = \frac{L}{0.80c}$$

Substitute numerical values and evaluate t'_{B}:

$$t'_{\mathrm{B}} = \frac{1000\,\mathrm{m}}{(0.80)(2.998\times10^8\ \mathrm{m/s})} = \boxed{4.2\,\mu\mathrm{s}}$$

(*b*) In the reference frame of the space station:

$$t'_B - t'_A = \frac{\dfrac{L}{u'_x} + \dfrac{v(x'_B - x'_B)}{c^2}}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}$$

or, because $t'_A = 0$, $x'_A = 0$, and $x'_B = L$,

$$t'_B = \frac{\dfrac{L}{u'_x} + \dfrac{vL}{c^2}}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}} = \frac{\left(\dfrac{1}{u'_x} + \dfrac{v}{c^2}\right)L}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}$$

Substituting numerical values gives:

$$t'_B = \frac{\left(\dfrac{1}{0.80c} + \dfrac{0.60c}{c^2}\right)(1000\text{ m})}{\sqrt{1-\left(\dfrac{0.60c}{c}\right)^2}}$$

$$= \frac{2.313\text{ km}}{c} = \boxed{7.7\,\mu\text{s}}$$

<u>Alternate solution to Part (*b*)</u>

In the reference frame of the dart the speed of the rocket is 0.80c and the length of the ship is:

$$L' = L\sqrt{1 - \frac{(u'_x)^2}{c^2}}$$

or, because $u'_x = 0.80c$,

$$L' = L\sqrt{1 - \frac{(0.80c)^2}{c^2}} = 0.60L$$

The time between events A and B (the proper time) is:

$$t'_B - t'_A = \frac{L'}{u'_x}$$

Substituting for t'_A, L' and u'_x gives:

$$t'_B = \frac{0.60L}{0.80c} = 0.75\frac{L}{c}$$

The speed of the dart in the frame of reference of the space station is given by Equation 39-18*a*:

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}}$$

Substituting numerical values and simplifying yields:

$$u_x = \frac{0.60c + 0.80c}{1 + \dfrac{(0.80c)(0.60c)}{c^2}} = 0.9459c$$

The time between the events in S is related to the proper time between the events by the time dilation formula:

$$t_B - t_A = \frac{t'_B - t'_A}{\sqrt{1-\left(\frac{u_x}{c}\right)^2}}$$

Substituting for L, t_A, t'_B, and u_x gives:

$$t_B = \frac{0.75\frac{(1000\text{ m})}{c}}{\sqrt{1-\left(\frac{0.9459c}{c}\right)^2}} = \frac{2.313\text{ km}}{c}$$

$$= \boxed{7.7\ \mu\text{s}}$$

(*c*) In the bullet's frame of reference, the elapsed time is given by:

$$t'_B - t'_A = \frac{L'}{u'_x}$$

or, because $t'_A = 0$,

$$t'_B = \frac{L'}{u'_x} \qquad (1)$$

The length L' of the rocket in the dart's frame of reference is given by:

$$L' = \frac{L}{\gamma} = L\sqrt{1-\frac{(u'_x)^2}{c^2}}$$

Substitute for L' in equation (1) to obtain:

$$t'_B = \frac{L\sqrt{1-\frac{(u'_x)^2}{c^2}}}{u'_x}$$

Substituting numerical values gives:

$$t'_B = \frac{(1000\text{ m})\sqrt{1-\frac{(0.80c)^2}{c^2}}}{0.80c} = \frac{750\text{ m}}{\text{c}}$$

$$= \boxed{2.5\ \mu\text{s}}$$

49 •• **[SSM]** Using a simple thought experiment, Einstein showed that there is mass associated with electromagnetic radiation. Consider a box of length *L* and mass *M* resting on a frictionless surface. Attached to the left wall of the box is a light source that emits a directed pulse of radiation of energy *E*, which is completely absorbed at the right wall of the box. According to classical electromagnetic theory, this radiation carries momentum of magnitude $p = E/c$ (Equation 32-13). The box recoils when the pulse is emitted by the light source. (*a*) Find the recoil velocity of the box so that momentum is conserved when the light is emitted. (Because *p* is small and *M* is large, you may use classical

mechanics.) (*b*) When the light is absorbed at the right wall of the box the box stops, so the total momentum of the system remains zero. If we neglect the very small velocity of the box, the time it takes for the radiation to travel across the box is $\Delta t = L/c$. Find the distance moved by the box in this time. (*c*) Show that if the center of mass of the system is to remain at the same place, the radiation must carry mass $m = E/c^2$.

Picture the Problem We can use conservation of energy to express the recoil velocity of the box and the relationship between distance, speed, and time to find the distance traveled by the box in time $\Delta t = L/c$. Equating the initial and final locations of the center of mass will allow us to show that the radiation must carry mass $m = E/c^2$.

(*a*) Apply conservation of momentum to obtain:

$$\frac{E}{c} + Mv = p_{\mathrm{i}} = 0 \Rightarrow v = \boxed{-\frac{E}{Mc}}$$

(*b*) The distance traveled by the box in time $\Delta t = L/c$ is:

$$d = v\Delta t = \frac{vL}{c}$$

Substitute for v from (*a*) to obtain:

$$d = \frac{L}{c}\left(-\frac{E}{Mc}\right) = \boxed{-\frac{LE}{Mc^2}}$$

(*c*) Let $x = 0$ be at the center of the box and let the mass of the photon be m. Then initially the center of mass is at:

$$x_{\mathrm{CM}} = \frac{-\frac{1}{2}mL}{M+m}$$

When the photon is absorbed at the other end of the box, the center of mass is at:

$$x_{\mathrm{CM}} = \frac{\left[\frac{-MEL}{Mc^2} + m\left(\frac{1}{2}L - \frac{EL}{Mc^2}\right)\right]}{M+m}$$

Because no external forces act on the system, these expressions for x_{CM} must be equal:

$$\frac{-\frac{1}{2}mL}{M+m} = \frac{\left[\frac{-MEL}{Mc^2} + m\left(\frac{1}{2}L - \frac{EL}{Mc^2}\right)\right]}{M+m}$$

Solving for m yields:

$$m = \frac{E}{c^2\left(1 - \frac{E}{Mc^2}\right)}$$

Because Mc^2 is of the order of 10^{16} J and $E = hf$ is of the order of 1 J for reasonable values of f, $E/Mc^2 << 1$ and:

$$m = \boxed{\frac{E}{c^2}}$$

50 ••• Using the relativistic conservation of momentum and energy and the relation between energy and momentum for a photon $E = pc$, prove that a free electron (an electron not bound to an atomic nucleus) cannot absorb or emit a photon.

Picture the Problem Without loss of generality, we'll consider the absorption case. We'll assume that the electron is initially at rest and that it travels with a speed v after it absorbs the photon. Applying the conservation of energy and the conservation of momentum will lead us to an absurd conclusion that, in turn, will force us to abandon our initial assumption that an electron can absorb a photon. Such an argument is known as a *reductio ad absurdum* argument.

When the electron absorbs a photon, the conservation of relativistic momentum requires that its momentum become:

$$p = \gamma mv$$

From the conservation of energy:

$$mc^2 + pc = \gamma mc^2 \Rightarrow p = (\gamma - 1)mc$$

Equate these expression for p to obtain:

$$\gamma mv = (\gamma - 1)mc \Rightarrow v = \left(\frac{\gamma - 1}{\gamma}\right)c \quad (1)$$

Squaring both sides of the equation gives:

$$v^2 = \left(\frac{\gamma - 1}{\gamma}\right)^2 c^2 = \frac{\gamma^2 - 2\gamma + 1}{\gamma^2}c^2 \quad (2)$$

From the definition of γ:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \frac{c^2}{c^2 - v^2}$$

Solve for v^2 to obtain:

$$v^2 = \frac{\gamma^2 - 1}{\gamma^2}c^2$$

Substituting for v^2 in equation (2) and simplifying gives:

$$\frac{\gamma^2 - 1}{\gamma^2}c^2 = \frac{\gamma^2 - 2\gamma + 1}{\gamma^2}c^2$$

or

$$-1 = -2\gamma + 1 \Rightarrow \gamma = 1$$

Substitute for γ in equation (1) and evaluate v:

$$v = \left(\frac{1-1}{1}\right)c = 0$$

Our assumption that an electron can absorb a photon has led to the contradictory conclusion that its speed after the absorption is zero. Hence we must conclude that the electron cannot absorb a photon.

51 ••• [SSM] When a moving particle that has a kinetic energy greater than the threshold kinetic energy K_{th} strikes a stationary target particle, one or more particles may be created in the inelastic collision. Show that the threshold kinetic energy of the moving particle is given by

$$K_{th} = \frac{(\Sigma m_{in} + \Sigma m_{fin})(\Sigma m_{fin} + \Sigma m_{in})c^2}{2m_{target}}$$

Here Σm_{in} is the sum of the masses of the particles prior to the collision, Σm_{fin} is the sum of the masses of the particles following the collision, and m_{target} is the mass of the target particle. Use this expression to determine the threshold kinetic energy of protons incident on a stationary proton target for the production of a proton–antiproton pair; compare your result with the result of Problem 38.

Picture the Problem Let m_i denote the mass of the incident (projectile) particle. Then $\sum m_{in} = m_i + m_{target}$ and we can use this expression to determine the threshold kinetic energy of protons incident on a stationary proton target for the production of a proton–antiproton pair.

Consider the situation in the center of mass reference frame. At threshold we have:

$$E^2 - p^2c^2 = \sum m_{fin}c^2$$

Note that this is a relativistically invariant expression.

In the laboratory frame, the target is at rest so:

$$E_{target} = E_t = E_{t,0}$$

We can, therefore, write:

$$(E_i + E_{t,0})^2 - p_i^2c^2 = \left(\sum m_{fin}c^2\right)^2$$

For the incident particle:

$$E_i^2 - p_i^2c^2 = E_{i,0}^2$$

and

$$E_i = E_{i,0} + K_{th}$$

where K_{th} is the threshold kinetic energy of the incident particle in the laboratory frame.

Express K_{th} in terms of the rest energies:

$$\left(E_{\text{t,0}} + E_{\text{i.0}}\right)^2 + 2K_{\text{th}}E_{\text{t,0}} = \left(\sum m_{\text{fin}}c^2\right)^2$$

where

$$E_{\text{t,0}} + E_{\text{i.0}} = \sum m_{\text{fin}}c^2$$

and

$$E_{\text{t,0}} = m_{\text{target}}c^2$$

Substitute to obtain:

$$\left(\sum m_{\text{fin}}c^2\right)^2 + 2K_{\text{th}}m_{\text{target}}c^2 = \left(\sum m_{\text{fin}}c^2\right)^2$$

Solving for K_{th} gives:

$$K_{\text{th}} = \boxed{\frac{\left(\sum m_{\text{in}} + \sum m_{\text{fin}}\right)\left(\sum m_{\text{fin}} - \sum m_{\text{in}}\right)c^2}{2m_{\text{target}}}}$$

For the creation of a proton-antiproton pair in a proton-proton collision:

$$\sum m_{\text{in}} = 2m_{\text{p}}\ ,\ \sum m_{\text{fin}} = 4m_{\text{p}} \text{ and } m_{\text{target}} = m_{\text{p}}$$

Substituting for the sums and simplifying yields:

$$K_{\text{th}} = \frac{\left(2m_{\text{p}} + 4m_{\text{p}}\right)\left(4m_{\text{p}} - 2m_{\text{p}}\right)c^2}{2m_{\text{p}}} = \frac{\left(6m_{\text{p}}\right)\left(2m_{\text{p}}\right)c^2}{2m_{\text{p}}} = \boxed{6m_{\text{p}}c^2}$$

in agreement with Problem 38.

52 ••• A particle of mass *M* decays into two identical particles, each of mass *m*, where *m* = 0.30*M*. Prior to the decay, the particle of mass *M* has a total energy of $4.0Mc^2$ in the laboratory reference frame. The velocities of the decay products are along the direction of motion of *M*. Find the velocities of the decay products in the laboratory reference frame.

Picture the Problem We'll solve the problem for the general case of a particle of rest mass *M* decaying into two identical particles each of rest mass *m*.

In the center of mass reference frame:

$$Mc^2 = 2mc^2 = 2\gamma mc^2$$

Solve for *u*/*c* to obtain:

$$\frac{u}{c} = \sqrt{1 - \left(\frac{2m}{M}\right)^2}$$

where *u* is the speed of each of the decay particles in the CM frame.

Next we determine the speed v of the laboratory frame relative to the CM frame. The energy of the particle of rest mass M is:

$$\gamma_{\mathrm{CM}}Mc^2 \text{ where } \gamma_{\mathrm{CM}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{and } \frac{v}{c} = \beta_{\mathrm{CM}} = \sqrt{1-\frac{1}{\gamma_{\mathrm{CM}}^2}}$$

Use Equation 39-18a to express u_{lab}, the speeds of the decay products in the laboratory reference frame:

$$u_{\mathrm{lab}} = \frac{\beta_{\mathrm{CM}} \pm \frac{u}{c}}{1 \pm \beta_{\mathrm{CM}}\frac{u}{c}}c$$

where ± refers to the fact that one of the decay particles will travel in the direction of M, and the other in the direction opposite to that of M.

In this problem we have:

$$\gamma_{\mathrm{CM}} = 4.0\,,\ \beta_{\mathrm{CM}} = 0.968\,,\ \frac{2m_0}{M_0} = 0.600\,,$$

$$\text{and } \frac{u}{c} = 0.800$$

Substituting numerical values gives:

$$u_{\mathrm{lab}} = \frac{0.968 + 0.800}{1+(0.968)(0.800)}c = \boxed{0.996c}$$

and

$$u_{\mathrm{lab}} = \frac{0.968 - 0.800}{1-(0.968)(0.800)}c = \boxed{0.745c}$$

53 ••• A rod of proper length L_{p} makes an angle θ with the x axis in frame S. Show that the angle θ' made with the x' axis in frame S', which is moving in the $+x$ direction with speed v, is given by $\tan\theta' = \gamma\tan\theta$ and that the length of the stick in S' is $L' = L_{\mathrm{p}}\left(\gamma^{-2}\cos^2\theta + \sin^2\theta\right)^{1/2}$.

Picture the Problem We can write the components of the stick in its reference frame and then apply the Lorentz length contraction equation to obtain the given result.

In its reference frame, the stick has x and y components:

$$L_{\mathrm{p}x} = L_{\mathrm{p}}\cos\theta \text{ and } L_{\mathrm{p}y} = L_{\mathrm{p}}\sin\theta$$

Only $L_{\mathrm{p}x}$ is Lorentz contracted:

$$L_x' = \frac{L_{\mathrm{p}x}}{\gamma}$$

Hence, the length in the reference frame S' is:

$$L' = \left[(L_x')^2 + (L_y')^2\right]^{1/2} = \boxed{L_\mathrm{p}\left(\frac{\cos^2\theta}{\gamma^2} + \sin^2\theta\right)^{1/2}}$$

The angle that L' makes with the x' axis is given by:

$$\tan\theta' = \frac{L_y'}{L_x'} = \frac{\sin\theta}{\dfrac{\cos\theta}{\gamma}} = \boxed{\gamma\tan\theta}$$

54 ••• Show that if a particle moves at an angle θ with the x axis with speed u in frame S, it moves at an angle θ' with the x' axis in S' given by

$$\tan\theta' = \gamma^{-1}\sin\theta/(\cos\theta - v/u)$$

Picture the Problem We can express the tangent of the angle u' makes with the x' axis and then use the velocity transformation equations to obtain the given result.

Express the tangent of the angle u' makes with the x' axis:

$$\tan\theta' = \frac{u_y'}{u_x'}$$

Substitute for u_y' and u_x':

$$\tan\theta' = \frac{\dfrac{u_y}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}}{\dfrac{u_x - v}{1 - \dfrac{vu_x}{c^2}}} = \frac{u_y}{\gamma(u_x - v)}$$

Substitute for u_y and u_x and simplify to obtain:

$$\tan\theta' = \frac{u\sin\theta}{\gamma(u\cos\theta - v)} = \boxed{\frac{\gamma^{-1}\sin\theta}{\cos\theta - v/u}}$$

55 ••• **[SSM]** For the special case of a particle moving with speed u along the y axis in frame S, show that its momentum and energy in frame S' are related to its momentum and energy in S by the transformation equations

$$p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right) \qquad p_y' = p_y \qquad p_z' = p_z \qquad \frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{vp_x}{c}\right).$$

Compare these equations with the Lorentz transformation equations for x', y', z', and t'. Notice that that the quantities p_x, p_y, p_z and E/c transform in the same way as do x, y, z, and ct.

Picture the Problem We can use the expressions for $\vec{p}$ and E in S together with the relations we wish to verify and the inverse velocity transformation equations to establish the condition $u'^2 = (u_x')^2 + (u_y')^2 + (u_z')^2 = v^2 + \frac{u^2}{\gamma^2}$ and then use this result to verify the given expressions for p_x', p_y', p_z' and E'/c.

In any inertial frame the momentum and energy are given by:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \text{ and } E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

where $\vec{u}$ is the velocity of the particle and u is its speed.

The components of $\vec{p}$ in S are:

$$p_x = \frac{mu_x}{\sqrt{1-\frac{u^2}{c^2}}},\ p_y = \frac{mu_y}{\sqrt{1-\frac{u^2}{c^2}}}, \text{ and}$$

$$p_z = \frac{mu_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

Because $u_x = u_z = 0$ and $u_y = u$:

$$p_x = p_z = 0 \text{ and } p_y = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}$$

Substituting zeros for p_x and p_z in the relations we are trying to show yields:

$$p_x' = \gamma\left(0 - \frac{vE}{c^2}\right) = -\gamma\frac{vE}{c^2},\ p_y' = p_y,$$

$p_z' = 0$, and

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - 0\right) = \gamma\frac{E}{c}$$

In S' the momentum components are:

$$p_x' = \frac{mu_x'}{\sqrt{1-\frac{u'^2}{c^2}}},\ p_y' = \frac{mu_y'}{\sqrt{1-\frac{u'^2}{c^2}}}, \text{ and}$$

$$p_z' = \frac{mu_z'}{\sqrt{1-\frac{u'^2}{c^2}}}$$

The inverse velocity transformations are:

$$u_x' = \frac{u_x - v}{\sqrt{1-\frac{vu_x}{c^2}}},\ u_y' = \frac{u_y}{\sqrt{1-\frac{vu_y}{c^2}}},\ \text{and}$$

$$u_z' = \frac{u_z}{\sqrt{1-\frac{vu_z}{c^2}}}$$

Substitute $u_x = u_z = 0$ and $u_y = u$ to obtain:

$$u_x' = -v,\ u_y' = \gamma u,\ \text{and}\ u_z' = 0$$

Thus:

$$u'^2 = (u_x')^2 + (u_y')^2 + (u_z')^2$$
$$= v^2 + \frac{u^2}{\gamma^2}$$

First, we verify that $p_z' = p_z = 0$:

$$p_z' = \frac{m(0)}{\sqrt{1-\frac{u'^2}{c^2}}} = p_z = \boxed{0}$$

Next, we verify that $p_y' = p_y$:

$$p_y' = \frac{mu_y'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{mu}{\gamma\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}} = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\sqrt{1-\frac{u^2}{c^2}}}{\gamma\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}}$$
$$= \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\sqrt{\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = p_y\sqrt{\frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}}$$
$$= \boxed{p_y}$$

Next, we verify that $p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right)$:

$$p_x' = \frac{mu_x'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{-mv}{\gamma\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}} = -\frac{\gamma v}{c^2}\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}}$$

$$= -\frac{\gamma v}{c^2}E\sqrt{\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = -\frac{\gamma v}{c^2}E\sqrt{\frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}}$$

$$= \boxed{-\frac{\gamma v}{c^2}E}$$

Finally, we verify that $\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{vp_x}{c}\right) = \gamma\frac{E}{c}$, or $E' = \gamma E$:

$$E' = \frac{mc^2}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{\gamma mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma E\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}}$$

$$= \gamma E\sqrt{\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = \gamma E\sqrt{\frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}}$$

$$= \boxed{\gamma E}$$

The *x, y, z,* and *t* transformation equations are:

$x' = \gamma(x - vt)$, $y' = y$, $z' = z$
and
$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

The $x, y, z,$ and ct transformation equations are:

$$x' = \gamma\left(x - \frac{v}{c}ct\right),\ y' = y,\ z' = z$$

and

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

The $p_x, p_y, p_z,$ and E/c transformation equations are:

$$p_x' = \gamma\left(p_x - \frac{v}{c}\frac{E}{c}\right),\ p_y' = p_y,\ p_z' = p_z$$

and

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{v}{c}p_x\right)$$

Note that the transformation equations for $x, y, z,$ and ct and the transformation equations for $p_x, p_y, p_z,$ and E/c are identical.

56 ••• The equation for the spherical wavefront of a light pulse that begins at the origin at time $t = 0$ is $x^2 + y^2 + z^2 - (ct)^2 = 0$. Frame S' moves with velocity v along the x axis. Using the Lorentz transformation, show that such a light pulse also has a spherical wavefront in frame S' by showing that $x'^2 + y'^2 + z'^2 - (ct')^2 = 0$.

Picture the Problem The Lorentz transformation was derived on the basis of the postulate that the speed of light is c in any inertial reference frame. Thus, if the clocks in S and S' are synchronized at $t = t' = 0$, then it follows from the Einstein postulate that $r^2 = c^2t^2$ and $r'^2 = c^2t'^2$ or $r^2 - c^2t^2 = 0 = r'^2 - c^2t'^2$. In other words, the quantity $s^2 = r^2 - c^2t^2 = 0$ is a relativistic invariant, which can also be written as $x^2 + y^2 + z^2 - c^2t^2 = 0$.

Using the Lorentz transformation equations for x, y, z, and t we have:

$$x'^2 + y'^2 + z'^2 - (ct')^2 = \gamma^2(x^2 - 2vxt + v^2t^2) + y^2 + z^2 - \gamma^2(c^2t^2 - 2vxt + v^2x^2/c^2)$$

The terms linear in x cancel and the terms in x^2 combine to give:

$$\gamma^2x^2(1 - v^2/c^2) = x^2$$

The coefficients of the terms in $(ct)^2$ give:

$$\gamma^2(v^2/c^2 - 1) = -1$$

Thus, $r^2 - c^2t^2 = r'^2 - c^2t'^2$ as required by the Einstein postulate.

57 ••• In Problem 56, you showed that the quantity $x^2 + y^2 + z^2 - (ct)^2$ has the same value (0) in both S and S'. A quantity that has the same value in all inertial frames is called a *Lorentz invariant*. From the results of Problem 55, the quantity $p_x^2 + p_y^2 + p_z^2 - E^2/c^2$ must also be a Lorentz invariant. Show that this quantity has the value $-m^2c^2$ in both the S and S' reference frames.

Picture the Problem We'll use Equation 39-27 to show that this quantity has the value $-mc^2$ in both the S and S' reference frames.

From Equation 39-27, the relationship between total energy E, momentum p, and rest energy mc^2 is:

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

or

$$p^2c^2 - E^2 = -\left(mc^2\right)^2$$

Divide both sides of this equation by c^2 to obtain:

$$p^2 - \left(\frac{E}{c}\right)^2 = -(mc)^2 \qquad (1)$$

We can relate p to p_x, p_y, and p_z:

$$p^2 = p_x^2 + p_y^2 + p_z^2$$

Substitute for p^2 in equation (1) to obtain:

$$p_x^2 + p_y^2 + p_z^2 - \left(\frac{E}{c}\right)^2 = -m^2c^2$$

Because m is the mass of the particle in its rest frame, it is constant. Hence:

$p^2 - \left(\dfrac{E}{c}\right)^2$ must be a relativistic invariant.

Also, in Problem 55 we saw that the components of p and the quantity E/c transform like the components of r and the quantity ct. In Problem 56 we demonstrated that $r^2 - (ct)^2$ is a relativistic invariant. Consequently, $p^2 - (E/c)^2$ must also be relativistically invariant.

58 ••• A long rod that is parallel to the x axis is released from rest. Subsequently, it is in free fall with an acceleration of magnitude g in the $-y$ direction. An observer in a rocket ship moving with speed v parallel to the x axis passes by. Using the Lorentz transformations, show that the observer on the rocket ship will measure the rod to be bent into a parabolic shape. Is the parabola concave upward or concave downward?

Picture the Problem We can use the inverse Lorentz transformation for time to show that the observer will conclude that the rod is bent into a parabolic shape.

In frame S where the rod is not moving along the x axis, the height of the rod at time t is:

$$y(t) = -\tfrac{1}{2}gt^2$$

The inverse Lorentz time transformation is:

$$t = \gamma\left(t' + \frac{vx}{c^2}\right)$$

Express $y'(t)$ in the moving frame of reference:

$$y'(t) = -\tfrac{1}{2}g\gamma\left(t' + \frac{vx}{c^2}\right)^2$$

Evaluate $y'(t)$ at $t' = 0$ to obtain:

$$y'(t) = -\frac{g\gamma v^2}{2c^2}x^2$$

Because $y'(t) = -\frac{g\gamma v^2}{2c^2}x^2$ is the equation of a parabola, we've shown that the moving observer will conclude that the rod is bent into a parabolic shape. Because the coefficient of x^2 is negative, the parabola is concave downward.

59 •• Show that if u_x' and v in Equation 39-18*a* are both less than c, then u_x is less than c. (*Hint*: Let $u_x' = (1-\varepsilon_1)c$ and $v = (1-\varepsilon_2)c$, where ε_1 and ε_2 are small positive numbers that are less than 1.)

Picture the Problem We can make the substitutions given in the hint in Equation 39-18*a* and simplify the resulting expression to show that $u_x < c$.

Equation 39-18*a* gives the x direction relativistic velocity transformation:

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}} \quad \text{or} \quad \frac{u_x}{c} = \frac{u_x' + v}{c + \frac{vu_x'}{c}}$$

Make the substitutions given in the hint to obtain:

$$\frac{u_x}{c} = \frac{(1-\varepsilon_1)c + (1-\varepsilon_2)c}{c + \frac{(1-\varepsilon_2)c(1-\varepsilon_1)c}{c}}$$
$$= \frac{2-(\varepsilon_1+\varepsilon_2)}{1+(1-\varepsilon_2)(1-\varepsilon_1)}$$
$$= \frac{2-(\varepsilon_1+\varepsilon_2)}{2-(\varepsilon_1+\varepsilon_2)+\varepsilon_1\varepsilon_2}$$

Because ε_1 and ε_2 are small positive numbers that are less than 1, the numerator is less than the denominator and:

$$\frac{u_x}{c} < 1 \Rightarrow u_x < \boxed{c}$$

60 ••• In reference frame S the acceleration of a particle is $\vec{a} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}$. Derive expressions for the acceleration components a_x', a_y', and a_z' of the particle in reference frame S' that is moving relative to S in the x direction with velocity v.

Picture the Problem We can evaluate the differentials of Equations 39-19*a, b,* and c and 39-12 and express their ratio to obtain expressions for a_x', a_y', and a_z'.

From Equation 39-19*a* we have:

$$du_x' = d\left(\frac{u_x - v}{1 - \frac{vu_x}{c^2}}\right) = \frac{\left(1 - \frac{vu_x}{c^2}\right)du_x + (u_x - v)\left(\frac{v}{c^2}\right)du_x}{\left(1 - \frac{vu_x}{c^2}\right)^2} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{vu_x}{c^2}\right)^2}du_x$$

From Equation 39-12:

$$dt' = \gamma d\left(t - \frac{vx}{c^2}\right) = \gamma dt - \frac{\gamma v}{c^2}dx = \frac{\gamma v}{c^2}\frac{dx}{dt}dt = \gamma\left(1 - \frac{vu_x}{c^2}\right)dt$$

Divide du_x' by dt' to obtain:

$$a_x' = \frac{du_x'}{dt'} = \frac{\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{vu_x}{c^2}\right)^2}du_x}{\gamma\left(1 - \frac{vu_x}{c^2}\right)dt} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\gamma\left(1 - \frac{vu_x}{c^2}\right)^3}\frac{du_x}{dt} = \boxed{\frac{1}{\gamma^3\delta^3}a_x} \text{ where } \delta = 1 - \frac{vu_x}{c^2}$$

Proceeding in exactly the same manner, one obtains:

$$a_y' = \boxed{\frac{1}{\gamma^2\delta^2}a_y + \frac{vu_y}{\gamma^3\delta^3c^2}a_x}$$

and an identical expression for a_z' with z replacing y.

Chapter 40
Nuclear Physics

Conceptual Problems

1 • Isotopes of nitrogen, iron and tin have stable isotopes ^{14}N, ^{56}Fe and ^{118}Sn. Give the symbols for two other isotopes of (*a*) nitrogen, (*b*) iron, and (*c*) tin.

Determine the Concept Two or more nuclides with the same atomic number Z but different *N* and *A* numbers are called isotopes.

(*a*) Two other isotopes of ^{14}N are: ^{15}N, ^{16}N

(*b*) Two other isotopes of ^{56}Fe are: ^{54}Fe, ^{55}Fe

(*c*) Two other isotopes of ^{118}Sn are: ^{117}Sn, ^{119}Sn

2 • Why is the decay chain $A = 4n + 1$ not found in nature?

Determine the Concept The parent of that series, ^{237}Np, the longest-lived member of this decay series, has a half-life of 2×10^6 y that is much shorter than the age of Earth. There is no naturally occurring Np remaining on Earth.

3 • A decay by α emission is often followed by β decay. When this occurs, it is by β^- and not β^+ decay. Why?

Determine the Concept Generally, β-decay leaves the daughter nucleus neutron rich, i.e., above the line of stability. The daughter nucleus therefore tends to decay via β^- emission which converts a nuclear neutron to a proton.

4 • The half-life of ^{14}C is much less than the age of the universe, yet ^{14}C is found in nature. Why?

Determine the Concept ^{14}C is found on Earth because it is constantly being formed by cosmic rays in the upper atmosphere in the reaction $^{14}N + n \rightarrow {}^{14}C + {}^{1}H$.

5 • What effect would a long-term variation in cosmic-ray activity have on the accuracy of ^{14}C dating?

Determine the Concept It would make the dating unreliable because the current concentration of ^{14}C is not equal to that at some earlier time.

6 • Why does an element that has $Z = 130$ not exist?

Determine the Concept An element with such a high Z value (the Coulomb-repulsion force is proportional to Z^2) would either fission spontaneously or decay almost immediately by α emission (see Figure 40-3).

7 • Why is a moderator needed in an ordinary nuclear fission reactor?

Determine the Concept The probability for neutron capture by the fissionable nucleus is large only for slow (thermal) neutrons. The neutrons emitted in the fission process are fast (high energy) neutrons and must be slowed to thermal neutrons before they are likely to be captured by another fissionable nucleus.

8 • Explain why water is more effective than lead in slowing down fast neutrons.

Determine the Concept The process of ″slowing down″ involves the sharing of energy of a fast neutron and another nucleus in an elastic collision. The fast particle will lose maximum energy in such a collision if the target particle is of the same mass as the incident particle. Hence, neutron-proton collisions are most effective in slowing down neutrons. However, ordinary water cannot be used as a moderator because protons will capture the slow neutrons and form deuterons.

9 • The stable isotope of sodium is ^{23}Na. What kind of beta decay would you expect of (*a*) ^{22}Na and (*b*) ^{24}Na?

Determine the Concept Beta decay occurs in nuclei that have too many or too few neutrons for stability. In β decay, A remains the same while Z either increases by 1 (β^- decay) or decreases by 1 (β^+ decay).

(*a*) The reaction is:

$$^{22}_{11}\text{Na} \to {}^{22}_{10}\text{Ne} + {}^{0}_{+1}\beta \Rightarrow \boxed{\beta^+}\ \text{decay}$$

(*b*) The reaction is:

$$^{24}_{11}\text{Na} \to {}^{24}_{12}\text{Mg} + {}^{0}_{-1}\beta \Rightarrow \boxed{\beta^-}\ \text{decay}$$

10 • What is the advantage of a breeder reactor over an ordinary reactor? What are the disadvantages of a breeder reactor?

Advantages	Disadvantages
The reactor uses ^{238}U, which, by neutron capture and subsequent decays, produces ^{239}Pu. Thus plutonium isotope fissions by fast neutron capture. Thus, the breeder reactor uses the plentiful uranium isotope and does not need a moderator to slow the neutrons needed for fission.	The fraction of delayed neutrons emitted in the fission of ^{239}Pu is very small. Consequently, control of the fission reaction is very difficult, and the safety hazards are more severe than for the ordinary reactor that uses ^{235}U as fuel.

11 • True or false:

(*a*) In a breeder reactor, fuel can be produced as fast as it is consumed.
(*b*) The atomic nucleus is composed of protons, neutrons, and electrons.
(*c*) The mass of a ^{2}H nucleus is less than the mass of a ^{1}H nucleus plus the mass of a neutron.
(*d*) After two half-lives, all the radioactive nuclei in a given sample have decayed.

(*a*) True (given an unlimited supply of ^{238}U).

(*b*) False. The nucleus does not contain electrons.

(*c*) True.

(*d*) False. After two half-lives, three-fourths of the radioactive nuclei in a given sample have decayed.

12 • Why is it that extreme changes in the temperature or the pressure of a radioactive sample have little or no effect on the radioactivity?

Determine the Concept Exponential time dependence is characteristic of all radioactivity and indicates that radioactive decay is a statistical process. Because each nucleus is well shielded from others by the atomic electrons, pressure and temperature changes have little or no effect on the rate of radioactive decay or other nuclear properties.

Estimation and Approximation

13 • We found in Chapter 25 that the ratio of the resistivity of the most insulating material to the resistivity of the least resistive material (excluding superconductors) is approximately 10^{22}. Few properties of materials show such a wide range of values. Using information in the textbook or other resources, find the ratio of largest to smallest for some nuclear properties of matter. Some examples might be the range of mass densities found in an atom, the half-life of radioactive nuclei, or the range of nuclear masses.

Picture the Problem There is no table of half lives in the text although the information is mentioned in the alpha particle discussion for alpha decay (about 15 orders of magnitude). Mass density in an atom ranges roughly as the cube of the radius of an atom to that of the nucleus, also about 15 orders of magnitude. Nuclear masses only range 2 orders of magnitude.

Material property	Ratio (order of magnitude)
Mass density	10^{15}
Half life	10^{15}
Nuclear masses	2

14 •• According to the United States Department of Energy, the U.S. population consumes approximately 10^{20} joules of energy each year. Estimate the mass (in kilograms) of (*a*) uranium that would be needed to produce this much energy using nuclear fission and (*b*) deuterium and tritium that would be needed to produce this much energy using nuclear fusion.

Picture the Problem The mass of ^{235}U required is given by $m_{235} = (N/N_A)M_{235}$, where M_{235} is the molecular mass of ^{235}U and N is the number of fissions required to produce 10^{20} J. The mass of deuterium and tritium required can be found similarly.

(*a*) Relate the mass of ^{235}U required to the number of fissions N required:

$$m_{235} = \frac{N}{N_A} M_{235}$$

where M_{235} is the molecular mass of ^{235}U.

Because $N = \dfrac{E_{\text{annual}}}{E_{\text{per fission}}}$:

$$m_{235} = \frac{E_{\text{annual}} M_{235}}{N_A E_{\text{per fission}}}$$

Substitute numerical values and evaluate m_{235}:

$$m_{235} = \frac{(1\times10^{20}\ \text{J})(235\,\text{g/mol})}{\left(6.022\times10^{23}\,\frac{\text{nuclei}}{\text{mol}}\right)\left(200\,\text{MeV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)} \approx \boxed{5\times10^{6}\ \text{kg}}$$

(*b*) Relate the mass of ^{2}H and ^{3}H required to the number of fusions N required:

$$m_{^2\text{H}+^3\text{H}} = \frac{N}{N_\text{A}} M_{^2\text{H}+^3\text{H}}$$

where $M_{^2\text{H}+^3\text{H}}$ is the molecular mass of ^{2}H + ^{3}H.

Because $N = \dfrac{E_\text{annual}}{E_\text{per fission}}$:

$$m_{^2\text{H}+^3\text{H}} = \frac{E_\text{annual} M_{^2\text{H}+^3\text{H}}}{N_\text{A} E_\text{per fission}}$$

Substitute numerical values and evaluate $m_{^2\text{H}+^3\text{H}}$:

$$m_{^2\text{H}+^3\text{H}} = \frac{(1\times10^{20}\ \text{J})(5\,\text{g/mol})}{\left(6.022\times10^{23}\,\frac{\text{nuclei}}{\text{mol}}\right)\left(18\,\text{MeV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}\right)} \approx \boxed{3\times10^{6}\ \text{kg}}$$

Properties of Nuclei

15 • [SSM] Calculate the binding energy and the binding energy per nucleon from the masses given in Table 40-1 for (*a*) ^{12}C, (*b*) ^{56}Fe, and (*c*) ^{238}U.

Picture the Problem To find the binding energy of a nucleus we add the mass of its neutrons to the mass of its protons and then subtract the mass of the nucleus and multiply by c^2. The binding energy per nucleon is the ratio of the binding energy to the mass number of the nucleus.

(*a*) For ^{12}C, Z = 6 and N = 6. Add the mass of the neutrons to that of the protons:

$$6m_\text{p} + 6m_\text{n} = 6\times1.007\,825\,\text{u} + 6\times1.008\,665\,\text{u} = 12.098\,940\,\text{u}$$

Subtract the mass of ^{12}C from this result:

$$(6m_\text{p} + 6m_\text{n}) - m_{^{12}\text{C}} = 12.098\,940\,\text{u} - 12\,\text{u} = 0.098\,940\,\text{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_b = (\Delta m)c^2 = (0.098\ 940\,\mathrm{u})c^2\left(\frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}}\right) = \boxed{92.2\,\mathrm{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_b}{A} = \dfrac{92.2\,\mathrm{MeV}}{12} = \boxed{7.68\,\mathrm{MeV}}$

(*b*) For ^{56}Fe, Z = 26 and N = 30. Add the mass of the neutrons to that of the protons:

$$26m_p + 30m_n = 26\times 1.007\ 825\,\mathrm{u} + 30\times 1.008\ 665\,\mathrm{u} = 56.463\ 400\,\mathrm{u}$$

Subtract the mass of ^{56}Fe from this result:

$$(26m_p + 30m_n) - m_{^{12}\mathrm{C}} = 56.463\ 400\,\mathrm{u} - 55.934\ 942\,\mathrm{u} = 0.528\ 458\,\mathrm{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_b = (\Delta m)c^2 = (0.528\ 458\,\mathrm{u})c^2\left(\frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}}\right) = \boxed{492\,\mathrm{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_b}{A} = \dfrac{492\,\mathrm{MeV}}{56} = \boxed{8.79\,\mathrm{MeV}}$

(*c*) For ^{238}U, Z = 92 and N = 146. Add the mass of the neutrons to that of the protons:

$$92m_p + 146m_n = 92\times 1.007\ 825\,\mathrm{u} + 146\times 1.008\ 665\,\mathrm{u} = 239.984\ 990\,\mathrm{u}$$

Subtract the mass of ^{238}U from this result:

$$(92m_p + 146m_n) - m_{^{238}\mathrm{U}} = 239.984\ 990\,\mathrm{u} - 238.050\ 783\,\mathrm{u} = 1.934\ 207\,\mathrm{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_b = (\Delta m)c^2 = (1.934\ 207\,\mathrm{u})c^2\left(\frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}}\right) = \boxed{1802\,\mathrm{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_b}{A} = \dfrac{1802\,\mathrm{MeV}}{238} = \boxed{7.57\,\mathrm{MeV}}$

16 • Calculate the binding energy and the binding energy per nucleon from the masses given in Table 40-1 for (*a*) ^{6}Li, (*b*) ^{39}K, and (*c*) ^{208}Pb.

Picture the Problem To find the binding energy of a nucleus we add the mass of its neutrons to the mass of its protons and then subtract the mass of the nucleus and multiply by c^2. The binding energy per nucleon is the ratio of the binding energy to the mass number of the nucleus.

(*a*) For ^{6}Li, Z = 3 and N = 3. Add the mass of the neutrons to that of the protons:

$$3m_\mathrm{p} + 3m_\mathrm{n} = 3\times 1.007\ 825\,\mathrm{u} + 3\times 1.008\ 665\,\mathrm{u} = 6.049\ 470\,\mathrm{u}$$

Subtract the mass of ^{6}Li from this result:

$$(3m_\mathrm{p} + 3m_\mathrm{n}) - m_{^6\mathrm{L}i} = 6.049\ 470\,\mathrm{u} - 6.015\,122\,\mathrm{u} = 0.034\ 348\,\mathrm{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_\mathrm{b} = (\Delta m)c^2 = (0.034\ 348\,\mathrm{u})c^2\left(\frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}}\right) = \boxed{32.0\,\mathrm{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_\mathrm{b}}{A} = \dfrac{32.0\,\mathrm{MeV}}{6} = \boxed{5.33\,\mathrm{MeV}}$

(*b*) For ^{39}K, Z = 19 and N = 20. Add the mass of the neutrons to that of the protons:

$$19m_\mathrm{p} + 20m_\mathrm{n} = 19\times 1.007\ 825\,\mathrm{u} + 20\times 1.008\ 665\,\mathrm{u} = 39.321\,975\,\mathrm{u}$$

Subtract the mass of ^{39}K from this result:

$$(19m_\mathrm{p} + 20m_\mathrm{n}) - m_{^{39}\mathrm{K}} = 39.321\,975\,\mathrm{u} - 38.963\ 707\,\mathrm{u} = 0.358\ 268\,\mathrm{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_\mathrm{b} = (\Delta m)c^2 = (0.358\ 268\,\mathrm{u})c^2\left(\frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}}\right) = \boxed{334\,\mathrm{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_\mathrm{b}}{A} = \dfrac{334\,\mathrm{MeV}}{39} = \boxed{8.56\,\mathrm{MeV}}$

(*c*) For ^{208}Pb, Z = 82 and N = 126. Add the mass of the neutrons to that of the protons:

$$82m_{\text{p}} + 126m_{\text{n}} = 82 \times 1.007\ 825\,\text{u} + 126 \times 1.008\ 665\,\text{u} = 209.733\ 440\,\text{u}$$

Subtract the mass of ^{208}Pb from this result:

$$\left(82m_{\text{p}} + 126m_{\text{n}}\right) - m_{^{208}\text{Pb}} = 209.733\ 440\,\text{u} - 207.976\ 636\,\text{u} = 1.756\ 804\,\text{u}$$

Multiply the mass difference by c^2 and convert to MeV:

$$E_{\text{b}} = (\Delta m)c^2 = (1.756\ 804\,\text{u})c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right) = \boxed{1636\,\text{MeV}}$$

and the binding energy per nucleon is $\dfrac{E_{\text{b}}}{A} = \dfrac{1636\,\text{MeV}}{208} = \boxed{7.87\,\text{MeV}}$

17 • Use the radius formula $R = R_0 A^{1/3}$ (Equation 40-1), where R_0 = 1.2 fm, to compute the radii of the following nuclei: (*a*) ^{16}O, (*b*) ^{56}Fe, and (*c*) ^{197}Au.

Picture the Problem The nuclear radius is given by $R = R_0 A^{1/3}$ where R_0 = 1.2 fm.

(*a*) The radius of ^{16}O is: $R_{^{16}\text{O}} = (1.2\,\text{fm})(16)^{1/3} = \boxed{3.0\,\text{fm}}$

(*b*) The radius of ^{56}Fe is: $R_{^{56}\text{Fe}} = (1.2\,\text{fm})(56)^{1/3} = \boxed{4.6\,\text{fm}}$

(*c*) The radius of ^{197}Au is: $R_{^{197}\text{Au}} = (1.2\,\text{fm})(197)^{1/3} = \boxed{7.0\,\text{fm}}$

18 • During a fission process, a ^{239}Pu nucleus splits into two nuclei whose mass number ratio is 3 to 1. Calculate the radii of the nuclei formed during this process.

Picture the Problem The nuclear radius is given by $R = R_0 A^{1/3}$ where R_0 = 1.2 fm.

The radii of the daughter nuclei are given by:

$$R = R_0 A^{1/3} \qquad (1)$$

Because the ratio of the mass numbers of the daughter nuclei is 3 to 1:

$$A_1 = \frac{3}{4}(239) \text{ and } A_2 = \frac{1}{4}(239)$$

Substitute in equation (1) to obtain:

$$R_1 = (1.2\,\text{fm})\left(\frac{3\times 239}{4}\right)^{1/3} = \boxed{6.8\,\text{fm}}$$

and

$$R_2 = (1.2\,\text{fm})\left(\frac{239}{4}\right)^{1/3} = \boxed{4.7\,\text{fm}}$$

19 •• [SSM] The neutron, when isolated from an atomic nucleus, decays into a proton, an electron, and an antineutrino as follows: $\text{n} \to {}^1\text{H} + \text{e}^- + \bar{\nu}$. The thermal energy of a neutron is of the order of *kT*, where *k* is the Boltzmann constant. (*a*) In both joules and electron volts, calculate the energy of a thermal neutron at 25ºC. (*b*) What is the speed of this thermal neutron? (*c*) A beam of monoenergetic thermal neutrons is produced at 25ºC and has an intensity *I*. After traveling 1350 km, the beam has an intensity of $\frac{1}{2}I$. Using this information, estimate the half-life of the neutron. Express your answer in minutes.

Picture the Problem The speed of the neutrons can be found from their thermal energy. The time taken to reduce the intensity of the beam by one-half, from *I* to *I*/2, is the half-life of the neutron. Because the beam is monoenergetic, the neutrons all travel at the same speed.

(*a*) The thermal energy of the neutron is:

$$\begin{aligned} E_{\text{thermal}} &= kT \\ &= (1.381\times 10^{-23}\ \text{J/K})(25+273)\text{K} \\ &= \boxed{4.11\times 10^{-21}\ \text{J}} \\ &= 4.11\times 10^{-21}\ \text{J}\times\frac{1\text{eV}}{1.602\times 10^{-19}\ \text{J}} \\ &= \boxed{25.7\,\text{meV}} \end{aligned}$$

(*b*) Equate E_{thermal} and the kinetic energy of the neutron to obtain:

$$E_{\text{thermal}} = \tfrac{1}{2}m_\text{n}v^2 \Rightarrow v = \sqrt{\frac{2E_{\text{thermal}}}{m_\text{n}}}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{\frac{2(4.11\times 10^{-21}\ \text{J})}{1.673\times 10^{-27}\ \text{kg}}} = \boxed{2.22\,\text{km/s}}$$

(*c*) Relate the half-life, $t_{1/2}$, to the speed of the neutrons in the beam:

$$t_{1/2} = \frac{x}{v}$$

Substitute numerical values and evaluate $t_{1/2}$:

$$t_{1/2} = \frac{1350\text{ km}}{2.22\text{ km/s}} = 608\text{ s} \times \frac{1\text{min}}{60\text{ s}} = \boxed{10.1\text{min}}$$

20 • Use the radius formula $R = R_0 A^{1/3}$ (Equation 40-1), where R_0 = 1.2 fm, for the radius of a spherical nucleus to calculate the density of nuclear matter. Express your answer in grams per cubic centimeter.

Picture the Problem We can use the definition of density, the equation for the volume of a sphere, and the given approximation to calculate the density of nuclear matter in grams per cubic centimeter.

Express the density of a spherical nucleus:

$$\rho = \frac{m}{V}$$

The approximate mass is:

$$m = \left(1.66 \times 10^{-27}\text{ kg}\right)A$$

The volume of the nucleus is given by:

$$V = \tfrac{4}{3}\pi\left(R_0 A^{1/3}\right)^3 = \tfrac{4}{3}\pi R_0^3 A$$

Substitute for m and V to obtain:

$$\rho = \frac{\left(1.66 \times 10^{-27}\text{ kg}\right)A}{\tfrac{4}{3}\pi R_0^3 A} = \frac{3\left(1.66 \times 10^{-27}\text{ kg}\right)}{4\pi R_0^3}$$

Substitute numerical values and evaluate ρ:

$$\rho = \frac{3\left(1.66 \times 10^{-27}\text{ kg}\right)}{4\pi\left(1.2\text{ fm}\right)^3} = 2.29 \times 10^{17}\text{ kg/m}^3 = \boxed{2.3 \times 10^{14}\text{ g/cm}^3}$$

21 •• **[SSM]** In 1920, 12 years before the discovery of the neutron, Ernest Rutherford argued that proton–electron pairs might exist in the confines of the nucleus in order to explain the mass number, A, being greater than the nuclear charge, Z. He also used this argument to account for the source of beta particles in radioactive decay. Rutherford's scattering experiments in 1910 showed that the nucleus had a diameter of approximately 10 fm. Using this nuclear diameter, the uncertainty principle, and that beta particles have an energy range of 0.02 MeV to 3.40 MeV, show why the hypothetical electrons cannot be confined to a region occupied by the nucleus.

Picture the Problem The Heisenberg uncertainty principle relates the uncertainty in position, Δx, to the uncertainty in momentum, Δp, by $\Delta x\Delta p \geq \frac{1}{2}\hbar$.

Solve the Heisenberg equation for Δp:

$$\Delta p \approx \frac{\hbar}{2\Delta x}$$

Assuming that electrons are contained within the nucleus, the uncertainty in their momenta must be:

$$\Delta p \approx \frac{1.05\times10^{-34}\ \mathrm{J\cdot s}}{2\left(10\times10^{-15}\ \mathrm{m}\right)} = 5.25\times10^{-21}\ \mathrm{kg\cdot m/s}$$

The kinetic energy of the electron is given by:

$$K = pc$$

Substitute numerical values and evaluate K for a nuclear electron:

$$K = \left(5.25\times10^{-21}\ \mathrm{kg\cdot m/s}\right)\left(2.998\times10^{8}\ \mathrm{m/s}\right) = 1.58\times10^{-12}\ \mathrm{J}\times\frac{1\mathrm{eV}}{1.602\times10^{-19}\ \mathrm{J}}$$

$$\approx 10\ \mathrm{MeV}$$

This result contradicts experimental observations that show that the energy of electrons in unstable atoms is of the order of 1 to 1000 eV. Hence the premise on which it is based (that electrons are contained within the nucleus) must be false.

22 •• Consider the following fission process: ${}^{236}_{92}\mathrm{U} + \mathrm{n} \rightarrow {}^{95}_{37}\mathrm{Rb} + {}^{137}_{55}\mathrm{Cs} + 4\mathrm{n}$. Determine the electrostatic potential energy, in MeV, of the reaction products when the surfaces of the ${}^{95}\mathrm{Rb}$ nucleus and the ${}^{137}\mathrm{Cs}$ nucleus are just touching immediately after being formed during the fission process.

Picture the Problem The separation of the nuclei when they are just touching is the sum of their radii. The radius of a nucleus whose atomic number is A is given by $R = R_0A^{1/3}$, where R_0 = 1.2 fm.

The electrostatic potential energy of the system is given by:

$$U = k\frac{q_1q_2}{R} = k\frac{\left(Z_{\mathrm{Rb}}e\right)\left(Z_{\mathrm{Cs}}e\right)}{R_{\mathrm{Rb}} + R_{\mathrm{Cs}}}$$

where R is the distance from the center of the ${}^{95}\mathrm{Rb}$ nucleus to the center of the ${}^{137}\mathrm{Cs}$ nucleus.

The radii of the nuclei are:

$$R_{\mathrm{Rb}} = R_{\mathrm{o}}A_{\mathrm{Rb}}^{1/3} \text{ and } R_{\mathrm{Cs}} = R_{\mathrm{o}}A_{\mathrm{Cs}}^{1/3}$$

Substitute for R_{Rb} and R_{Cs} and simplify to obtain:

$$U = ke^2 \frac{(Z_{Rb})(Z_{Cs})}{R_o A_{Rb}^{1/3} + R_o A_{Cs}^{1/3}} = \frac{ke^2}{R_o} \frac{(Z_{Rb})(Z_{Cs})}{\left(A_{Rb}^{1/3} + A_{Cs}^{1/3}\right)}$$

Substitute numerical values and evaluate U:

$$U = \frac{\left(8.988\times10^9\ \text{Nm}^2/\text{C}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{\left(1.2\times10^{-15}\ \text{m}\right)} \frac{(37)(55)}{\left(95^{1/3} + 137^{1/3}\right)}$$
$$= 4.026\times10^{-11}\ \text{J}\times\frac{1\text{eV}}{1.602\times10^{-19}\ \text{J}} = \boxed{0.25\ \text{GeV}}$$

Radioactivity

23 • Homer enters the visitors’ chambers, and his Geiger beeper sounds. He shuts off the beeper, removes the device from his shoulder patch, and holds it near the only new object in the room—an orb that is to be presented as a gift from the visiting Cartesians. Pushing a button marked ″monitor, ″ Homer reads that the device is reading a counting rate of 4000 counts/s above the background counting rate. After 10 min, the counting rate has dropped to 1000 counts/s above the background rate. (*a*) What is the half-life of the source? (*b*) How high will the counting rate be counting above the background counting rate 20 min after the monitoring device was switched on?

Picture the Problem The counting rate, as a function of the number of half-lives *n*, is given by $R = \left(\frac{1}{2}\right)^n R_0$, where R_0 = 4000 counts/s.

(*a*) The counting rate after *n* half-lives is:

$$R = \left(\tfrac{1}{2}\right)^n R_0 \Rightarrow n = \frac{\ln(R/R_0)}{\ln\left(\frac{1}{2}\right)}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{\ln\left(\dfrac{1000\,\text{counts/s}}{4000\,\text{counts/s}}\right)}{\ln\left(\frac{1}{2}\right)} = 2$$

Because there are two half-lives in 10 min:

$$t_{1/2} = \boxed{5\,\text{min}}$$

(*b*) At the end of 4 half-lives:

$$R = \left(\tfrac{1}{2}\right)^4 (4000\,\text{counts/s}) = \boxed{250\,\text{Bq}}$$

24 • A certain source gives 2000 counts/s at time $t = 0$. Its half-life is 2.0 min. How many counts per second will it give after (*a*) 4.0 min, (*b*) 6.0 min, and (*c*) 8.0 min?

Picture the Problem The counting rate, as a function of the number of half-lives n, is given by $R = \left(\frac{1}{2}\right)^n R_0$.

(*a*) When t = 4.0 min, two half-lives will have passed and n = 2:

$$R = \left(\tfrac{1}{2}\right)^2 (2000\,\text{counts/s}) = \boxed{500\,\text{Bq}}$$

(*b*) When t = 6.0 min, three half-lives will have passed and n = 3:

$$R = \left(\tfrac{1}{2}\right)^3 (2000\,\text{counts/s}) = \boxed{250\,\text{Bq}}$$

(*c*) When t = 8.0 min, four half-lives will have passed and n = 4:

$$R = \left(\tfrac{1}{2}\right)^4 (2000\,\text{counts/s}) = \boxed{125\,\text{Bq}}$$

25 • The counting rate from a radioactive source is 8000 counts/s at time t = 0, and 10 min later the rate is 1000 counts/s. (*a*) What is the half-life? (*b*) What is the decay constant? (*c*) What is the counting rate after 20 min?

Picture the Problem The counting rate, as a function of the number of half-lives n, is given by $R = \left(\frac{1}{2}\right)^n R_0$ and the decay constant λ is related to the half-life by $t_{1/2} = \ln(2)/\lambda$.

(*a*) Relate the counting rate at time t = 10 min to the counting rate at t = 0:

$$R_{10\,\text{min}} = \left(\tfrac{1}{2}\right)^n R_0 \Rightarrow n = \frac{\ln\left(R_{10\,\text{min}}/R_0\right)}{\ln\left(\frac{1}{2}\right)}$$

Substitute numerical values and evaluate n:

$$n = \frac{\ln\left(\dfrac{1000\,\text{counts/s}}{8000\,\text{counts/s}}\right)}{\ln\left(\frac{1}{2}\right)} = 3$$

Therefore, 3 half-lives have passed in 10 min:

$$3t_{1/2} = 10\,\text{min} \Rightarrow t_{1/2} = \boxed{200\,\text{s}}$$

(*b*) The decay constant λ is related to the half-life by:

$$t_{1/2} = \frac{\ln(2)}{\lambda} \Rightarrow \lambda = \frac{\ln(2)}{t_{1/2}}$$

Substitute for $t_{1/2}$ and evaluate λ:

$$\lambda = \frac{\ln(2)}{200\,\text{s}} = \boxed{3.5\times10^{-3}\,\text{s}^{-1}}$$

(*c*) Six half-lives will have passed in 20 min:

$$R_{20\,\text{min}} = \left(\tfrac{1}{2}\right)^6 (8000\,\text{Bq}) = \boxed{125\,\text{Bq}}$$

26 • The half-life of radium is 1620 y. Calculate the number of disintegrations per second of 1.00 g of radium and show that the disintegration rate is approximately 1.0 Ci.

Picture the Problem We can use $R = \lambda N_0 e^{-\lambda t}$ to show that the disintegration rate is approximately 1.0 Ci.

The decay rate is given by:

$$R = \lambda N_0 e^{-\lambda t}$$

where N_0 is the number of nuclei at $t = 0$.

The decay constant λ is related to the half-life by:

$$t_{1/2} = \frac{\ln(2)}{\lambda} \Rightarrow \lambda = \frac{\ln(2)}{t_{1/2}}$$

Substitute for $t_{1/2}$ and evaluate λ:

$$\lambda = \frac{\ln(2)}{1620\,\text{y}} = 4.28\times10^{-4}\ \text{y}^{-1}\times\frac{1\,\text{y}}{31.56\,\text{Ms}}$$
$$= 1.356\times10^{-11}\ \text{s}^{-1}$$

The number of radioactive nuclei N_0 at $t = 0$ can be found from Avogadro's number N_A, the mass m_{Ra} of the sample, and the molar mass M of the sample:

$$\frac{N_0}{N_A} = \frac{m_{Ra}}{M} \Rightarrow N_0 = N_A\frac{m_{Ra}}{M}$$

Substitute numerical values and evaluate N_0:

$$N_0 = \left(6.022\times10^{23}\,\text{nuclei/mol}\right)\left(\frac{1.00\ \text{g}}{226\ \text{g/mol}}\right) = 2.664\times10^{21}\ \text{nuclei}$$

Substitute numerical values for λ and N_0 and evaluate R:

$$R = \left(1.356\times10^{-11}\ \text{s}^{-1}\right)\left(2.664\times10^{21}\right)e^{-\left(1.356\times10^{-11}\ \text{s}^{-1}\right)(0)} = 3.61\times10^{10}\ \text{s}^{-1}$$
$$\approx 3.7\times10^{10}\ \text{s}^{-1} = \boxed{1.0\ \text{Ci}}$$

27 • A radioactive piece of silver foil ($t_{1/2}$ = 2.4 min) is placed near a Geiger counter and 1000 counts/s are observed at time t = 0. (*a*) What is the counting rate at t = 2.4 min and at t = 4.8 min? (*b*) If the counting efficiency is 20 percent, how many radioactive silver nuclei are there at time t = 0? At time t = 2.4 min? (*c*) At what time will the counting rate be about 30 counts/s?

Picture the Problem We can use $R=\left(\frac{1}{2}\right)^n R_0$ to relate the counting rate R to the number of half-lives n that have passed since $t = 0$. The detection efficiency depends on the probability that a radioactive decay particle will enter the detector and the probability that upon entering the detector it will produce a count. If the efficiency is 20 percent, the decay rate must be 5 times the counting rate.

(*a*) When $t = 2.4$ min, $n = 1$ and:

$$R_{2.4\,\text{min}}=\left(\tfrac{1}{2}\right)^1(1000\,\text{counts/s})=\boxed{500\,\text{Bq}}$$

When $t = 4.8$ min, $n = 2$ and:

$$R_{4.8\,\text{min}}=\left(\tfrac{1}{2}\right)^2(1000\,\text{counts/s})=\boxed{250\,\text{Bq}}$$

(*b*) The number of radioactive nuclei is related to the decay rate R, and the decay constant λ:

$$R=\lambda N\Rightarrow N=\frac{R}{\lambda}\qquad(1)$$

The decay constant is related to the half-life:

$$\lambda=\frac{0.693}{t_{1/2}}=\frac{0.693}{2.4\,\text{min}}=\frac{0.693}{144\,\text{s}}=4.813\times10^{-3}\,\text{s}^{-1}$$

Calculate the decay rate at $t = 0$ from the counting rate:

$$R_0=5\times1000\,\text{counts/s}=5000\,\text{s}^{-1}$$

Substitute in equation (1) and evaluate N_0 at $t = 0$:

$$N_0=\frac{R_0}{\lambda}=\frac{5000\,\text{s}^{-1}}{4.813\times10^{-3}\,\text{s}^{-1}}=\boxed{1.0\times10^6}$$

Calculate the decay rate at $t = 2.4$ min from the counting rate:

$$R_{2.4\,\text{min}}=5\times500\,\text{counts/s}=2500\,\text{s}^{-1}$$

Substitute in equation (1) and evaluate $N_{2.4\text{ min}}$:

$$N_{2.4\,\text{min}}=\frac{R_{2.4\,\text{min}}}{\lambda}=\frac{2500\,\text{s}^{-1}}{4.813\times10^{-3}\,\text{s}^{-1}}=\boxed{5.2\times10^5}$$

(*c*) The time at which the counting rate will be about 30 counts/s is the product of the number of half-lives that will have passed and the half-life:

$$t=nt_{1/2}\qquad(2)$$

The counting rate R after n half-lives is related to the counting rate at $t = 0$ by:

$$R = \left(\tfrac{1}{2}\right)^n R_0 \Rightarrow n = \frac{\ln(R/R_0)}{\ln\left(\tfrac{1}{2}\right)}$$

Substitute numerical values and evaluate n:

$$n = \frac{\ln(30\,\text{counts/s}/1000\,\text{counts/s})}{\ln\left(\tfrac{1}{2}\right)} = 5.059$$

Substitute numerical values for n and $t_{1/2}$ in equation (2) and evaluate t:

$$t = (5.059)(2.4\,\text{min}) = \boxed{12\text{ min}}$$

28 • Use Table 40-1 to calculate the energy release, in MeV, for the α decay of (*a*) ^{226}Ra and (*b*) ^{242}Pu.

Picture the Problem Knowing each of these reactions, we can use Table 40-1 to find the differences in the masses of the nuclei and then convert this difference into the energy released in each reaction.

(*a*) Write the reaction:

$$^{226}\text{Ra} \rightarrow {}^{222}\text{Rn} + {}^{4}\text{He}$$

Use Table 40-1 to find ΔE:

$$\Delta E = (226.025\ 403\,\text{u} - 222.017\ 571\,\text{u} - 4.002\ 603\,\text{u})c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

$$= \boxed{4.87\,\text{MeV}}$$

(*b*) Write the reaction:

$$^{242}\text{Pu} \rightarrow {}^{238}\text{U} + {}^{4}\text{He}$$

Use Table 40-1 to find ΔE:

$$\Delta E = (242.058\ 737\,\text{u} - 238.050\ 783\,\text{u} - 4.002\ 603\,\text{u})c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

$$= \boxed{4.98\,\text{MeV}}$$

29 •• **[SSM]** Plutonium is very toxic to the human body. Once it enters the body it collects primarily in the bones, although it also can be found in other organs. Red blood cells are synthesized within the marrow of the bones. The isotope ^{239}Pu is an alpha emitter that has a half-life of 24 360 years. Because alpha particles are an ionizing radiation, the blood-making ability of the marrow is, in time, destroyed by the presence of ^{239}Pu. In addition, many kinds of cancers will

also develop in the surrounding tissues because of the ionizing effects of the alpha particles. (*a*) If a person accidentally ingested 2.0 μg of ^{239}Pu and all of it is absorbed by the bones of the person, how many alpha particles are produced per second within the body of the person? (*b*) When, in years, will the activity be 1000 alpha particles per second?

Picture the Problem Each ^{239}Pu nucleus emits an alpha particle whose activity, *A*, depends on the decay constant of ^{239}Pu and on the number *N* of nuclei present in the ingested ^{239}Pu. We can find the decay constant from the half-life and the number of nuclei present from the mass ingested and the atomic mass of ^{239}Pu. Finally, we can use the dependence of the activity on time to find the time at which the activity will be 1000 alpha particles per second.

(*a*) The activity of the nuclei present in the ingested ^{239}Pu is given by:

$$R = \lambda N \qquad (1)$$

Find the constant for the decay of ^{239}Pu:

$$\lambda = \frac{\ln(2)}{t_{1/2}} = \frac{0.693}{(24360\,\text{y})(31.56\,\text{Ms/y})} = 9.016\times10^{-13}\,\text{s}^{-1}$$

Express the number of nuclei *N* present in the mass m_{Pu} of ^{239}Pu ingested:

$$\frac{N}{N_{\text{A}}} = \frac{m_{\text{Pu}}}{M_{\text{Pu}}} \Rightarrow N = m_{\text{Pu}}\frac{N_{\text{A}}}{M_{\text{Pu}}}$$

where M_{Pu} is the atomic mass of ^{239}Pu.

Substitute numerical values and evaluate *N*:

$$N = (2.0\,\mu\text{g})\left(\frac{6.022\times10^{23}\,\text{nuclei/mol}}{239\,\text{g/mol}}\right) = 5.039\times10^{15}\,\text{nuclei}$$

Substitute numerical values in equation (1) and evaluate *A*:

$$R = (9.016\times10^{-13}\,\text{s}^{-1})(5.039\times10^{15}\,\alpha) = 4.543\times10^{3}\,\alpha/\text{s} = \boxed{4.5\times10^{3}\,\alpha/\text{s}}$$

(*b*) The activity varies with time according to:

$$R = R_0 e^{-\lambda t} \Rightarrow t = \frac{\ln\left(\dfrac{R}{R_0}\right)}{-\lambda}$$

Substitute numerical values and evaluate *t*:

$$t = \frac{\ln\left(\dfrac{1000\,\alpha/\text{s}}{4.543\times10^{3}\,\alpha/\text{s}}\right)}{-(9.016\times10^{-13}\,\text{s}^{-1})\left(\dfrac{31.56\,\text{Ms}}{\text{y}}\right)} = \boxed{5.3\times10^{4}\,\text{y}}$$

30 •• Consider an alpha-emitting parent nucleus ${}^{A}_{Z}X$ initially at rest. The nucleus decays into a daughter nucleus Y and an alpha particle as follows: ${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\alpha + Q$. (*a*) Show that the alpha particle has a kinetic energy of $(A - 4)Q/A$. (*b*) Show that the kinetic energy of the recoiling daughter nucleus is given by $K_Y = 4Q/A$.

Picture the Problem We can use conservation of energy and conservation of linear momentum to relate the momenta and kinetic energies of the nuclei to the decay's Q value.

(*a*) Express the kinetic energies of the alpha particle and daughter nucleus:

$$K_\alpha = \tfrac{1}{2}m_\alpha v_\alpha^2 = \frac{p_\alpha^2}{2m_\alpha} \qquad (1)$$

and

$$K_Y = \tfrac{1}{2}m_Y v_Y^2 = \frac{p_Y^2}{2m_Y} \qquad (2)$$

Solve equations (1) and (2) for p_α^2 and p_D^2:

$p_\alpha^2 = 2m_\alpha K_\alpha$ and $p_Y^2 = 2m_Y K_Y$

From the conservation of linear momentum we have:

$\vec{p}_i = \vec{p}_f$

or, because the parent is initially at rest, $0 = p_\alpha - p_Y$ and $p_\alpha = p_Y$.

Because the momenta are equal:

$$2m_Y K_Y = 2m_\alpha K_\alpha$$

Solving for K_Y and simplifying gives:

$$K_Y = \frac{m_\alpha}{m_Y}K_\alpha = \frac{4}{A-4}K_\alpha$$

Because the daughter nucleus and the alpha particle share the Q-value:

$$Q = K_Y + K_\alpha = \frac{4}{A-4}K_\alpha + K_\alpha = \left(\frac{4}{A-4}+1\right)K_\alpha = \left(\frac{A}{A-4}\right)K_\alpha$$

Solve for K_α to obtain:

$$K_\alpha = \boxed{\left(\frac{A-4}{A}\right)Q}$$

(*b*) Substituting for K_α in the expression for Q yields:

$$Q = K_Y + \left(\frac{A-4}{A}\right)Q$$

Solving for K_Y gives:

$$K_Y = Q - \left(\frac{A-4}{A}\right)Q = \boxed{\frac{4Q}{A}}$$

31 • [SSM] The fissile material ^{239}Pu is an alpha emitter. Write the reaction that describes ^{239}Pu undergoing alpha decay. Given that ^{239}Pu, ^{235}U, and an alpha particle have respective masses of 239.052 156 u, 235.043 923 u, and 4.002 603 u, use the relations appearing in Problem 30 to calculate the kinetic energies of the alpha particle and the recoiling daughter nucleus.

Picture the Problem We can write the equation of the decay process by using the fact that the post-decay sum of the *Z* and *A* numbers must equal the pre-decay values of the parent nucleus. The *Q* value in the equations from Problem 30 is given by $Q = -(\Delta m)c^2$.

^{239}Pu undergoes alpha decay according to:

$$\boxed{{}^{239}_{94}\text{Pu} \to {}^{235}_{92}\text{U} + {}^{4}_{2}\alpha + Q}$$

The *Q* value, in MeV, for the decay is given by:

$$Q = \left[\left(m_{\text{Pu}}\right) - \left(m_{\text{U}} + m_\alpha\right)\right]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

Substitute numerical values and evaluate *Q*:

$$Q = \left[(239.052\,156\,\text{u}) - (235.043\,923\,\text{u} + 4.002\,603\,\text{u})\right]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$
$$= \boxed{5.24\,\text{MeV}}$$

From Problem 30, the kinetic energy of the alpha particle is given by:

$$K_\alpha = \left(\frac{A-4}{A}\right)Q$$

Substitute numerical values and evaluate K_α:

$$K_\alpha = \left(\frac{239-4}{239}\right)(5.24\,\text{MeV})$$
$$= \boxed{5.15\,\text{MeV}}$$

From Problem 30, the kinetic energy of ^{235}U is given by:

$$K_{\text{U}} = \frac{4Q}{A}$$

Substitute numerical values and evaluate $K_{^{235}\text{U}}$:

$$K_{^{235}\text{U}} = \frac{4(5.24\,\text{MeV})}{235} = \boxed{89.2\,\text{keV}}$$

32 • Through a friend in the security department at the museum, Angela obtains a sample of a wooden tool handle that contains 175 g of carbon. The decay rate of the ^{14}C in the sample is 8.1 Bq. How long ago was the wood in the handle last alive?

Picture the Problem We can find the age of the sample using $R_n = \left(\frac{1}{2}\right)^n R_0$ to find n and then applying $t = nt_{1/2}$.

Express the age of the bone in terms of the half-life of ^{14}C and the number n of half-lives that have elapsed:

$$t = nt_{1/2} \qquad (1)$$

The decay rate R_n after n half-lives is related to the counting rate R_0 at $t = 0$ by:

$$R_n = \left(\tfrac{1}{2}\right)^n R_0 \Rightarrow n = \frac{\ln(R/R_0)}{\ln\left(\frac{1}{2}\right)}$$

Because there are 15.0 decays per minute per gram of carbon in a living organism:

$$R_0 = 15.0\,\frac{\text{decays}}{\text{min}\cdot\text{g}}\times\frac{1\,\text{min}}{60\,\text{s}}\times 175\,\text{g} = 43.75\,\text{Bq}$$

Substitute numerical values for R and R_0 and evaluate n:

$$n = \frac{\ln\left(\dfrac{8.1\,\text{Bq}}{43.75\,\text{Bq}}\right)}{\ln\left(\frac{1}{2}\right)} = 2.433$$

Substitute numerical values in equation (1) and evaluate t:

$$t = (2.433)(5730\,\text{y}) \approx \boxed{14{,}000\,\text{y}}$$

33 • A sample of a radioactive isotope is found to have an activity of 115.0 Bq immediately after it is pulled from the reactor that formed the isotope. Its activity 2 h 15 min later is measured to be 85.2 Bq. (*a*) Calculate the decay constant and the half-life of the sample. (*b*) How many radioactive nuclei were there in the sample initially?

Picture the Problem We can solve $R = R_0 e^{-\lambda t}$ for λ to find the decay constant of the sample and use $t_{1/2} = \dfrac{\ln 2}{\lambda}$ to find its half-life. The number of radioactive nuclei in the sample initially can be found from $R_0 = \lambda N_0$.

(*a*) The decay rate is given by:

$$R = R_0 e^{-\lambda t} \Rightarrow \lambda = \frac{\ln\left(\dfrac{R}{R_0}\right)}{-t}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{\ln\left(\frac{85.2\,\text{Bq}}{115.0\,\text{Bq}}\right)}{-2.25\,\text{h}} = \boxed{0.133\,\text{h}^{-1}}$$

The half-life is related to the decay constant:

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{\ln(2)}{0.133\,\text{h}^{-1}} = \boxed{5.20\,\text{h}}$$

(b) The number N_0 of radioactive nuclei in the sample initially is related to the decay constant λ and the initial decay rate R_0:

$$R_0 = \lambda N_0 \Rightarrow N_0 = \frac{R_0}{\lambda}$$

Substitute numerical values and evaluate N_0:

$$N_0 = \frac{115\,\text{Bq}}{0.133\,\text{h}^{-1} \times \frac{1\,\text{h}}{3600\,\text{s}}} = \boxed{3.11\times10^{6}}$$

34 •• A 1.00-mg sample of substance that has an atomic mass of 59.934 u and emits β particles has an activity of 1.131 Ci. Find the decay constant for this substance in reciprocal seconds and find the half-life in years.

Picture the Problem The decay constant λ can be found from the decay rate R and the number of radioactive nuclei N at the moment of interest and the half-life, in turn, can be found from the decay constant.

The decay rate R is related to the decay constant λ and the number of radioactive nuclei N at the moment of interest:

$$R = \lambda N \Rightarrow \lambda = \frac{R}{N} \qquad (1)$$

The number of radioactive nuclei N at the moment of interest can be found from Avogadro's number, the mass m of the sample, and the molar mass M of the sample:

$$\frac{N}{N_A} = \frac{m}{M} \Rightarrow N = N_A\frac{m}{M}$$

Substitute numerical values and evaluate N:

$$N = \left(6.022\times10^{23}\ \text{nuclei/mol}\right)\frac{1.00\ \text{mg}}{59.934\,\text{g/mol}} = 1.004\times10^{19}\ \text{nuclei}$$

Substitute numerical values in equation (1) and evaluate λ:

$$\lambda = \frac{1.131\,\text{Ci} \times \dfrac{3.7\times10^{10}\ \text{Bq}}{\text{Ci}}}{1.004\times10^{19}\ \text{nuclei}} = \boxed{4.17\times10^{-9}\ \text{s}^{-1}}$$

Find the half-life from the decay constant:

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{\ln(2)}{4.17\times10^{-9}\ \text{s}^{-1}} = 1.67\times10^{8}\ \text{s}\times\frac{1\,\text{y}}{31.56\,\text{Ms}} = \boxed{5.27\,\text{y}}$$

35 •• [SSM] Radiation has been used for a long time in medical therapy to control the development and growth of cancer cells. Cobalt-60, a gamma emitter that emits photons that have energies of 1.17 MeV and 1.33 MeV, is used to irradiate and destroy deep-rooted cancers. Small needles made of ^{60}Co of a specified activity are encased in gold and used as body implants in tumors for time periods that are related to tumor size, tumor cell reproductive rate, and the activity of the needle. (*a*) A 1.00 μg sample of ^{60}Co, that has a half-life of 5.27 y, that is used to irradiate a small internal tumor with gamma rays, is prepared in the cyclotron of a medical center. Determine the activity of the sample in curies. (*b*) What will the activity of the sample be 1.75 y from now?

Picture the Problem We can use $R_0 = \lambda N$ to find the initial activity of the sample and $R = R_0 e^{-\lambda t}$ to find the activity of the sample after 1.75 y.

(*a*) The initial activity of the sample is the product of the decay constant λ for ^{60}Co and the number of atoms N of ^{60}Co initially present in the sample:

$$R_0 = \lambda N$$

Express N in terms of the mass m of the sample, the molar mass M of ^{60}Co, and Avogadro's number N_A:

$$\frac{N}{N_A} = \frac{m}{M} \Rightarrow N = \frac{m}{M}N_A$$

Substituting for N yields:

$$R_0 = \frac{\lambda m N_A}{M}$$

The decay constant is given by:

$$\lambda = \frac{\ln(2)}{t_{1/2}}$$

Substitute for λ to obtain:

$$R_0 = \frac{\ln(2) m N_A}{t_{1/2} M}$$

Substitute numerical values and evaluate R_0:

$$R_0 = \frac{\ln(2)\left(1.00\times10^{-6}\,\text{g}\right)\left(6.022\times10^{23}\,\frac{\text{nuclei}}{\text{mol}}\right)}{\left(5.27\,\text{y}\times\frac{31.56\,\text{Ms}}{\text{y}}\right)\left(60\,\frac{\text{g}}{\text{mol}}\right)} = 4.183\times10^{7}\,\text{s}^{-1}\times\frac{1\,\text{Ci}}{3.7\times10^{10}\,\text{s}^{-1}}$$

$$= \boxed{1.13\,\text{mCi}}$$

(b) The activity varies with time according to:

$$R = R_0 e^{-\lambda t} = R_0 e^{-\left(\frac{0.693t}{5.27\text{y}}\right)}$$

Evaluate R at t = 1.75 y:

$$R = (1.13\,\text{mCi})\, e^{-\left(\frac{0.693\times1.75\text{y}}{5.27\text{y}}\right)} = \boxed{0.898\,\text{mCi}}$$

36 •• (a) Show that if the decay rate is R_0 at time t = 0 and R_1 at some later time $t = t_1$, the decay constant is given by $\lambda = t_1^{-1}\ln(R_0/R_1)$ and the half-life is given by $t_{1/2} = t_1 \ln(2)/\ln(R_0/R_1)$. ($b$) Use these results to find the decay constant and the half-life if the decay rate is 1200 Bq at t = 0 and 800 Bq at t_1 = 60.0 s.

Picture the Problem We can solve Equation 40-7 for λ to show that $\lambda = t_1^{-1}\ln(R_0/R_1)$.

(a) The half-life as a function of the decay constant λ is given by:

$$t_{1/2} = \frac{\ln(2)}{\lambda} \qquad (1)$$

From Equation 40-7 we have:

$$R_1 = R_0 e^{-\lambda t_1} \Rightarrow \lambda = \boxed{t_1^{-1}\ln(R_0/R_1)}$$

Substituting for λ in equation (1) and simplifying gives:

$$t_{1/2} = \frac{\ln(2)}{t_1^{-1}\ln(R_0/R_1)} = \boxed{\frac{t_1\ln(2)}{\ln(R_0/R_1)}}$$

(b) Substitute numerical values for t, R_1, and R_0 and evaluate λ:

$$\lambda = \frac{1}{60.0\,\text{s}}\ln\left(\frac{1200\,\text{Bq}}{800\,\text{Bq}}\right) = \boxed{0.00676\,\text{s}^{-1}}$$

Use the decay constant to find the half-life:

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{\ln(2)}{0.00676\,\text{s}^{-1}} = \boxed{103\,\text{s}}$$

39 •• A wooden casket is thought to be 18 000 years old. How much carbon would have to be recovered from this object to yield a ^{14}C counting rate of no less than 5 counts/min with a detection efficiency of 20 percent?

Picture the Problem Because the detection efficiency is 20 percent, the decay rate must be five times the counting rate. The required mass is given by M = (5 counts/min)/R, where R is the current counting rate per gram of carbon. We can use the assumed age of the casket to find the number of half-lives that have elapsed and $R = \left(\frac{1}{2}\right)^n R_0$ to find the current counting rate per gram of ^{14}C.

The mass of carbon required is:

$$M = \frac{5(5\,\text{counts/min})}{R} \qquad (1)$$

Because there were about 15.0 decays per minute per gram of the living wood, the counting rate per gram is:

$$R = \left(\tfrac{1}{2}\right)^n R_0 = \left(\tfrac{1}{2}\right)^n (15\,\text{counts/min}\cdot\text{g})$$

We can find n from the assumed age of the casket and the half-life of ^{14}C:

$$n = \frac{18{,}000\,\text{y}}{5730\,\text{y}} = 3.141$$

Substitute for n and evaluate R:

$$R = \left(\tfrac{1}{2}\right)^{3.141}(15\,\text{counts/min}\cdot\text{g})$$
$$= 1.70\,\text{counts/min}\cdot\text{g}$$

Substitute for R in equation (1) and evaluate M:

$$M = \frac{5(5\,\text{counts/min})}{(1.70\,\text{counts/min}\cdot\text{g})} \approx \boxed{15\,\text{g}}$$

38 •• A sample of radioactive material is initially found to have an activity of 115.0 decays/min. After 4 d 5 h, its activity is measured to be 73.5 decays/min. (*a*) Calculate the half-life of the material. (*b*) How long (from the initial time) will it take for the sample to reach an activity of 10.0 decays/min?

Picture the Problem We can use the decay rate equation $R = R_0 e^{-\lambda t}$ and the expression relating the half-life of a source to its decay constant to find the half-life of the sample. Solving the decay-rate equation for t will yield the time at which the activity level drops to any given value.

(*a*) The half-life of the material is given by:

$$t_{1/2} = \frac{\ln(2)}{\lambda} \qquad (1)$$

The decay rate is given by:

$$R = R_0 e^{-\lambda t} \;\Rightarrow\; \lambda = \frac{\ln(R_0/R)}{t} \qquad (2)$$

Substitute for λ in equation (1) and simplify to obtain:

$$t_{1/2}=\frac{\ln(2)}{\frac{\ln(R_0/R)}{t}}=\frac{\ln(2)}{\ln(R_0/R)}t$$

Substitute numerical values and evaluate $t_{1/2}$:

$$t_{1/2}=\frac{\ln(2)}{\ln\left(\frac{115\,\text{decays/min}}{73.5\,\text{decays/min}}\right)}(101\,\text{h})=\boxed{156\,\text{h}}$$

(b) Solving equation (2) for t yields:

$$t=\frac{\ln(R/R_0)}{-\lambda}$$

Express λ in terms of $t_{1/2}$:

$$\lambda=\frac{\ln(2)}{t_{1/2}}$$

Substitute for λ in the expression for t to obtain:

$$t=\frac{\ln(R/R_0)}{-\ln(2)}t_{1/2}$$

Substitute numerical values and evaluate t:

$$t=\frac{\ln\left(\frac{10\,\text{decays/min}}{115\,\text{decays/min}}\right)}{-\ln(2)}(156\,\text{h})=550\,\text{h}\times\frac{1\,\text{d}}{24\,\text{h}}=\boxed{22.9\,\text{d}}$$

39 •• The rubidium isotope ^{87}Rb is a β^- emitter that has a half-life of 4.9×10^{10} y. It decays into ^{87}Sr. This nuclear decay is used to determine the age of rocks and fossils. Rocks containing the fossils of early animals have a ratio of ^{87}Sr to ^{87}Rb of 0.0100. Assuming that there was no ^{87}Sr present when the rocks were formed, calculate the age of these fossils.

Picture the Problem We can use the decay rate equation $R=R_0e^{-\lambda t}$ and the expression relating the half-life of a source to its decay constant to find the age of the fossils.

The decay rate is given by:

$$R=R_0e^{-\lambda t}\Rightarrow t=\frac{\ln(R/R_0)}{-\lambda}$$

Express λ in terms of $t_{1/2}$:

$$\lambda=\frac{\ln(2)}{t_{1/2}}$$

Substitute for λ in the expression for t to obtain:

$$t = \frac{\ln(R/R_0)}{-\ln(2)} t_{1/2}$$

or, because the activity at any time is proportional to the number of radioactive nuclei present,

$$t = \frac{\ln(N_{\text{Rb}}/N_{0,\text{Rb}})}{-\ln(2)} t_{1/2} \qquad (1)$$

The number of ^{87}Sr nuclei present in the rocks is given by:

$$N_{\text{Sr}} = N_{0,\text{Rb}} - N_{\text{Rb}} \Rightarrow N_{0,\text{Rb}} = N_{\text{Sr}} + N_{\text{Rb}}$$

We're given that:

$$N_{\text{Sr}} = 0.01 N_{\text{Rb}} \Rightarrow \frac{N_{\text{Sr}}}{N_{\text{Rb}}} = 0.01$$

Express the ratio of $N_{0,\text{Rb}}$ to N_{Rb}:

$$\frac{N_{0,\text{Rb}}}{N_{\text{Rb}}} = \frac{N_{\text{Sr}} + N_{\text{Rb}}}{N_{\text{Rb}}} = \frac{N_{\text{Sr}}}{N_{\text{Rb}}} + 1$$
$$= 0.01 + 1 = 1.01$$

Substitute numerical values in equation (1) and evaluate the age of the fossils:

$$t = \frac{\ln\left(\frac{1}{1.01}\right)}{-\ln(2)} (4.9\times10^{10}\text{ y}) \approx \boxed{7.0\times10^{8}\text{ y}}$$

40 ••• Consider a single nucleus of a radioactive isotope that has a decay rate equal to λ. The nucleus has not decayed at t = 0. The probability that the nucleus will decay between time t and time $t + dt$ is equal to $\lambda e^{-\lambda t} dt$. (*a*) Show that this statement is consistent with the fact that the probability is 1 that the nucleus will decay between t = 0 and $t = \infty$. (*b*) Show that the expected lifetime of the nucleus is equal to $1/\lambda$. *Hint: The expected lifetime is equal to* $\int_0^\infty t\lambda e^{-\lambda t} dt$ *divided by* $\int_0^\infty \lambda e^{-\lambda t} dt$. (*c*) A sample of material contains a significant number of these radioactive nuclei at time t = 0. What will have been the mean lifetime of these nuclei some time later, after they have all decayed?

Picture the Problem (*a*) Integrating the probability function between 0 and ∞ will show that its value is 1. (*b*) We can evaluate this integral by changing variables to obtain a form that we can find in a table of integrals.

(*a*) The probability that the nucleus will decay between t = 0 and $t = \infty$ is given by:

$$P = \int_0^\infty \lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-\lambda t} dt$$

Integrating yields:

$$P = \lambda \left.\frac{e^{-\lambda t}}{-\lambda}\right]_0^\infty = -e^{-\lambda t}\Big]_0^\infty = \boxed{1}$$

(*b*) The expected lifetime of the nucleus is given by:

$$\tau = \frac{\int_0^\infty t\lambda e^{-\lambda t}dt}{\int_0^\infty \lambda e^{-\lambda t}dt}$$

or, because $\int_0^\infty \lambda e^{-\lambda t}dt = 1$,

$$\tau = \int_0^\infty t\lambda e^{-\lambda t}dt$$

Change variables by letting $x = \lambda t$. Then:

$$dx = \lambda dt\,,\ dt = \frac{dx}{\lambda}\text{, and } t = \frac{x}{\lambda}$$

Substitute to obtain:

$$\tau = \int_0^\infty t\lambda e^{-\lambda t}dt = \int_0^\infty \frac{x}{\lambda}\lambda e^{-x}\frac{dx}{\lambda} = \frac{1}{\lambda}\int_0^\infty xe^{-x}dx$$

From integral tables:

$$\int_0^\infty xe^{-x}dx = 1$$

Substitute in the expression for τ to obtain:

$$\tau = \boxed{1/\lambda}$$

(*c*) The mean lifetime after all the nuclei have decayed is $\boxed{1/\lambda}$, the same as the expected lifetime.

Nuclear Reactions

41 • Using Table 40-1, find the Q values for the following reactions: (*a*) $^1H + {}^3H \rightarrow {}^3He + n + Q$ and (*b*) $^2H + {}^2H \rightarrow {}^3He + n + Q$.

Picture the Problem We can use $Q = -(\Delta m)c^2$ to find the Q values for these reactions.

(*a*) Find the mass of each atom from Table 40-1:

$$m_{^1\text{H}} = 1.007\ 825\ \text{u}$$
$$m_{^3\text{H}} = 3.016\ 049\ \text{u}$$
$$m_{^3\text{He}} = 3.016\ 029\ \text{u}$$
$$m_{\text{n}} = 1.008\ 665\,\text{u}$$

Calculate the initial mass m_i of the incoming particles:

$$m_{\text{i}} = 1.007\ 825\ \text{u} + 3.016\ 049\ \text{u} = 4.023\,874\ \text{u}$$

Calculate the final mass m_f:

$$m_{\text{f}} = 3.016\ 029\ \text{u} + 1.008\ 665\ \text{u} = 4.024\ 694\ \text{u}$$

Calculate the increase in mass:

$$\Delta m = m_{\text{f}} - m_{\text{i}} = 4.024\ 694\ \text{u} - 4.023\,874\ \text{u} = 0.000\ 820\ \text{u}$$

Calculate the Q value:

$$Q = -(\Delta m)c^2 = -(0.000\ 820\,\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{u}\right) = \boxed{-0.764\,\text{MeV}}$$

(*b*) Proceed as in (*a*) to obtain:

$$Q = (0.003\ 510\,\text{u})\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right)c^2 = \boxed{3.27\,\text{MeV}}$$

Remarks: Because *Q* < 0 for the first reaction, it is endothermic. Because *Q* > 0 for the second reaction, it is exothermic.

42 • Using Table 40-1, find the Q values for the following reactions: (*a*) $^2\text{H} + {}^2\text{H} \rightarrow {}^3\text{H} + {}^1\text{H} + Q$, (*b*) $^2\text{H} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^1\text{H} + Q$, and (*c*) $^6\text{Li} + \text{n} \rightarrow {}^3\text{H} + {}^4\text{He} + \text{Q}$.

Picture the Problem We can use $Q = -(\Delta m)c^2$ to find the Q values for these reactions.

(*a*) Find the mass of each atom from Table 40-1:

$$m_{^2\text{H}} = 2.014\,102\ \text{u}$$
$$m_{^3\text{H}} = 3.016\ 049\ \text{u}$$
$$m_{^1\text{H}} = 1.007\ 825\ \text{u}$$

Calculate the initial mass m_i of the incoming particles:

$$m_i = 2(2.014\,102\text{ u}) = 4.028\,204\text{ u}$$

Calculate the final mass m_f:

$$m_f = 3.016\,049\text{ u} + 1.007\,825\text{ u} = 4.023\,874\text{ u}$$

Calculate the increase in mass:

$$\Delta m = m_f - m_i = 4.023\,874\text{ u} - 4.028\,204\text{ u} = -0.004\,330\text{ u}$$

Calculate the Q value:

$$Q = -(\Delta m)c^2 = -(-0.004\,330\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{u}\right) = \boxed{4.03\,\text{MeV}}$$

(b) Proceed as in (a) to obtain:

$$Q = -(\Delta m)c^2 = -(-0.019\,703\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right) = \boxed{18.4\,\text{MeV}}$$

(c) Proceed as in (a) to obtain:

$$Q = -(\Delta m)c^2 = -(-0.005\,135\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right) = \boxed{4.78\,\text{MeV}}$$

43 •• [SSM] (a) Use the values 14.003 242 u and 14.003 074 u for the atomic masses of ^{14}C and ^{14}N, respectively, to calculate the Q value (in MeV) for the β-decay reaction $^{14}_{6}\text{C} \rightarrow {}^{14}_{7}\text{N} + \text{e}^- + \bar{\nu}_e$. ($b$) Explain why you should not add the mass of the electron to that of atomic ^{14}N for the calculation in (a).

Picture the Problem We can use $Q = -(\Delta m)c^2$ to find the Q values for this reaction.

(a) The masses of the atoms are:

$$m_{^{14}\text{C}} = 14.003\,242\,\text{u}$$
$$m_{^{14}\text{N}} = 14.003\,074\,\text{u}$$

Calculate the increase in mass:

$$\Delta m = m_f - m_i = 14.003\,074\,\text{u} - 14.003\,242\,\text{u} = -0.000\,168\,\text{u}$$

Calculate the Q value:

$$Q = -(\Delta m)c^2 = -(-0.000\,168\,\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right) = \boxed{0.156\,\text{MeV}}$$

(b) The masses given are for atoms, not nuclei, so for nuclear masses the masses are too large by the atomic number multiplied by the mass of an electron. For the given nuclear reaction, the mass of the carbon atom is too large by $6m_e$ and the mass of the nitrogen atom is too large by $7m_e$. Subtracting $6m_e$ from both sides of the reaction equation leaves an extra electron mass on the right. Not including the mass of the beta particle (electron) is mathematically equivalent to explicitly subtracting $1m_e$ from the right side of the equation.

44 •• (a) Use the values 13.005 738 u and 13.003 354 u for the atomic masses of ${}^{13}_{7}\text{N}$ and ${}^{13}_{6}\text{C}$, respectively, to calculate the Q value (in MeV) for the β-decay reaction ${}^{13}_{7}\text{N} \rightarrow {}^{13}_{6}\text{C} + e^+ + \nu_e$. ($b$) Explain why you need to add twice the mass of an electron to the mass of ${}^{13}_{6}\text{C}$ during the calculation of the Q value for the reaction in (a).

Picture the Problem We can use $Q = -(\Delta m)c^2$ to find the Q values for this reaction.

(a) The masses of the atoms are:

$$m_{^{13}\text{N}} = 13.005\,738\,\text{u}$$

and

$$m_{^{13}\text{C}} = 13.003\,354\,\text{u}$$

For β^+ decay:

$$Q = (m_i - m_f - 2m_e)c^2 = (m_i - m_f)c^2 - 2m_e c^2$$

Calculate $m_i - m_f$:

$$m_i - m_f = 13.005\,738\,\text{u} - 13.003\,354\,\text{u} = 0.002\,384\,\text{u}$$

Calculate the Q value:

$$Q = (0.002\ 384\,\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{u}\right) - 2(0.511\ \text{MeV}) = \boxed{1.20\,\text{MeV}}$$

(b) The atomic masses include the masses of the electrons of the neutral atoms. In this reaction the initial atom has 7 electrons and the final atom has 6 electrons. Moreover, in addition to one electron not included in the atomic masses, a positron of mass equal to that of an electron is created. Consequently, one must add the rest energies of two electrons to the rest energy of the daughter atomic mass when calculating Q.

Fission and Fusion

45 • Assuming an average energy of 200 MeV per fission, calculate the number of fissions per second needed for a 500-MW reactor.

Picture the Problem The power output of the reactor is the product of the number of fissions per second and energy liberated per fission.

Express the required number N of fissions per second in terms of the power output P and the energy released per fission $E_{\text{per fission}}$:

$$N = \frac{P}{E_{\text{per fission}}}$$

Substitute numerical values and evaluate N:

$$N = \frac{500\,\text{MW}}{200\,\text{MeV}} = \frac{5.00\times10^8\,\frac{\text{J}}{\text{s}}\times\frac{1\text{eV}}{1.602\times10^{-19}\ \text{J}}}{200\,\text{MeV}} = \boxed{1.56\times10^{19}\,\text{s}^{-1}}$$

46 • If the reproduction factor in a reactor is 1.1, find the number of generations needed for the power level to (a) double, (b) increase by a factor of 10, and (c) increase by a factor of 100. Find the time needed in each case if (d) there are no delayed neutrons, so that the time between generations is 1.0 ms, and (e) there are delayed neutrons that make the average time between generations 100 ms.

Picture the Problem If k = 1.1, the reaction rate after N generations is 1.1^N. We can find the number of generations by setting 1.1^N equal, in turn, to 2, 10, and 100 and solving for N. The time to increase by a given factor is the number of generations N needed to increase by that factor times the generation time.

(*a*) Set 1.1^N equal to 2 and solve for N:

$$(1.1)^N = 2 \Rightarrow N\ln(1.1) = \ln(2)$$

and $N = \dfrac{\ln(2)}{\ln(1.1)} = \boxed{7.3}$

(*b*) Set 1.1^N equal to 10 and solve for N:

$$(1.1)^N = 10 \Rightarrow N\ln(1.1) = \ln(10)$$

and $N = \dfrac{\ln(10)}{\ln(1.1)} = \boxed{24}$

(*c*) Set 1.1^N equal to 100 and solve for N:

$$(1.1)^N = 100 \Rightarrow N\ln(1.1) = \ln(100)$$

and $N = \dfrac{\ln(100)}{\ln(1.1)} = \boxed{48}$

(*d*) Multiply the number of generations by the generation time:

$$t_2 = Nt_1 = (7.27)(1.0\,\text{ms}) = \boxed{7.3\,\text{ms}}$$

$$t_{10} = Nt_1 = (24.2)(1.0\,\text{ms}) = \boxed{24\,\text{ms}}$$

$$t_{100} = Nt_1 = (48.3)(1.0\,\text{ms}) = \boxed{48\,\text{ms}}$$

(*e*) Multiply the number of generations by the generation time:

$$t_2 = Nt_1 = (7.27)(100\,\text{ms}) = \boxed{0.73\,\text{s}}$$

$$t_{10} = Nt_1 = (24.2)(100\,\text{ms}) = \boxed{2.4\,\text{s}}$$

$$t_{100} = Nt_1 = (48.3)(100\,\text{ms}) = \boxed{4.8\,\text{s}}$$

47 •• [SSM] Consider the following fission reaction:
${}^{235}_{92}\text{U} + \text{n} \to {}^{95}_{42}\text{Mo} + {}^{139}_{57}\text{La} + 2\text{n} + Q$. The masses of the neutron, ^{235}U, ^{95}Mo, and ^{139}La are 1.008 665 u, 235.043 923 u, 94.905 842 u, and 138.906 348 u, respectively. Calculate the Q value, in MeV, for this fission reaction.

Picture the Problem We can use $Q = -(\Delta m)c^2$, where $\Delta m = m_\text{f} - m_\text{i}$, to calculate the Q value.

The Q value, in MeV, is given by:

$$Q = -(\Delta m)c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

Calculate the change in mass Δm:

$$\begin{aligned}\Delta m &= m_f - m_i \\ &= 94.905\,842\text{ u} + 138.906\,348\text{ u} + 2(1.008\,665\text{ u}) \\ &\qquad - (235.043\,923\text{ u} + 1.008\,665\text{ u}) \\ &= -0.223\,068\text{ u}\end{aligned}$$

Substitute for Δm and evaluate Q:

$$Q = -(-0.223\,068\text{ u})c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right) = \boxed{208\text{ MeV}}$$

48 •• In 1989, researchers claimed to have achieved fusion in an electrochemical cell at room temperature. Their now thoroughly discredited claim was that a power output of 4.00 W was produced by deuterium fusion reactions in the palladium electrode of their apparatus. The two most likely reactions are

$$^{2}\text{H} + {}^{2}\text{H} \rightarrow {}^{3}\text{He} + \text{n} + 3.27\text{ MeV}$$

and

$$^{2}\text{H} + {}^{2}\text{H} \rightarrow {}^{3}\text{H} + {}^{1}\text{H} + 4.03\text{ MeV}$$

Of the deuterium nuclei that participated in these reactions, assume half of the deuterium nuclei participated in the first reaction and the other half participated in the second reaction. How many neutrons per second would we expect to be emitted in the generation of 4.00 W of power?

Picture the Problem We can find the number of neutrons per second in the generation of 4 W of power from the number of reactions per second.

The number of neutrons emitted per second is:

$$N_n = \tfrac{1}{2}N$$

where N is the number of reactions per second.

The number of reactions per second is:

$$N = 2\left(\frac{4.00\frac{\text{J}}{\text{s}} \times \frac{1\text{eV}}{1.602\times10^{-19}\text{ J}}}{3.27\,\text{MeV} + 4.03\,\text{MeV}}\right)$$

$$= 6.841\times10^{12}\text{ s}^{-1}$$

Substitute for N and evaluate N_n:

$$N_n = \tfrac{1}{2}\left(6.841\times10^{12}\ \text{s}^{-1}\right) = \boxed{3.42\times10^{12}\ \text{neutrons/s}}$$

49 •• A fusion reactor that uses only deuterium for fuel would have the two reactions in Problem 50 taking place in the reactor. The ^{3}H produced in the second reaction reacts immediately with another ^{2}H in the reaction

$$^3\text{H} + {}^2\text{H} \rightarrow {}^4\text{He} + \text{n} + 17.6\ \text{MeV}$$

The ratio of ^{2}H to ^{1}H atoms in naturally occurring hydrogen is 1.5×10^{-4}. How much energy would be produced from 4.0 L of water if all of the ^{2}H nuclei undergo fusion?

Picture the Problem We can use the energy released in the reactions of Problem 50, together with the 17.6 MeV released in the reaction described in this problem, to find the energy released using 5 ^{2}H nuclei. Finding the number of D atoms in 4.0 L of H_2O, we can then find the energy produced if all of the ^{2}H nuclei undergo fusion.

Find the energy released using 5 ^{2}H nuclei:

$$Q = 3.27\,\text{MeV} + 4.03\,\text{MeV} + 17.6\,\text{MeV} = 24.9\,\text{MeV}$$

The number of H atoms in 4.0 L of H_2O is:

$$N_H = 2\left(\frac{m_{H_2O}}{18\,\text{g/mol}}\right)N_A$$

Substitute numerical values and evaluate N_H:

$$N_H = 2\left(\frac{4.0\,\text{kg}}{18\,\text{g/mol}}\right)\left(6.022\times10^{23}\ \text{atoms/mol}\right) = 2.676\times10^{26}$$

The number of D atoms in 4.0 L of H_2O is:

$$N_D = \left(1.5\times10^{-4}\right)N_H = \left(1.5\times10^{-4}\right)\left(2.676\times10^{26}\right) = 4.01\times10^{22}$$

The energy produced is given by:

$$E = \frac{N_D}{5}Q$$

Substitute numerical values and evaluate E:

$$E = \frac{4.01\times10^{22}}{5}(24.9\,\text{MeV})$$
$$= 1.997\times10^{23}\ \text{MeV}\times\frac{1.602\times10^{-19}\ \text{J}}{\text{eV}}$$
$$= \boxed{3.2\times10^{10}\ \text{J}}$$

50 ••• The fusion reaction between ^{2}H and ^{3}H is

$$^{3}\text{H} + {}^{2}\text{H} \rightarrow {}^{4}\text{He} + \text{n} + 17.6\ \text{MeV}$$

Using the conservation of momentum and the given Q value, find the final energies of both the ^{4}He nucleus and the neutron, assuming the initial kinetic energy of the system is 1.00 MeV and the initial momentum of the system is zero.

Picture the Problem We can use the conservation of momentum and the given Q value to find the final energies of both the ^{4}He nucleus and the neutron, assuming that the initial momentum of the system is zero.

Apply conservation of energy to obtain:

$$18.6\,\text{MeV} = \tfrac{1}{2}m_{\text{He}}v_{\text{He}}^2 + \tfrac{1}{2}m_{\text{n}}v_{\text{n}}^2 = K_{\text{He}} + K_{\text{n}} \quad (1)$$

Apply conservation of momentum to obtain:

$$m_{\text{He}}v_{\text{He}} + m_{\text{n}}v_{\text{n}} = 0 \quad (2)$$

Solve equation (2) for v_{He}:

$$v_{\text{He}} = -\frac{m_{\text{n}}v_{\text{n}}}{m_{\text{He}}} \Rightarrow v_{\text{He}}^2 = \left(\frac{m_{\text{n}}}{m_{\text{He}}}\right)^2 v_{\text{n}}^2$$

Substitute for v_{He}^2 in equation (1):

$$18.6\,\text{MeV} = \tfrac{1}{2}m_{\text{He}}\left(\frac{m_{\text{n}}}{m_{\text{He}}}\right)^2 v_{\text{n}}^2 + \tfrac{1}{2}m_{\text{n}}v_{\text{n}}^2$$

or

$$18.6\,\text{MeV} = \tfrac{1}{2}m_{\text{n}}v_{\text{n}}^2\left(1+\frac{m_{\text{n}}}{m_{\text{He}}}\right) = K_{\text{n}}\left(1+\frac{m_{\text{n}}}{m_{\text{He}}}\right)$$

Solving for K_{n} yields:

$$K_{\text{n}} = \frac{18.6\,\text{MeV}}{1+\dfrac{m_{\text{n}}}{m_{\text{He}}}}$$

Substitute numerical values for m_n and m_{He} and evaluate K_n:

$$K_n = \frac{18.6\,\text{MeV}}{1+\dfrac{1.008665\,\text{u}}{4.002603\,\text{u}}} = 14.86\,\text{MeV} = \boxed{14.9\,\text{MeV}}$$

Use equation (1) to find K_{He}:

$$K_{He} = 18.6\,\text{MeV} - K_n = 18.6\,\text{MeV} - 14.86\,\text{MeV} = \boxed{3.7\,\text{MeV}}$$

51 ••• Energy is generated in the Sun and other stars by fusion. One of the fusion cycles, the proton–proton cycle, consists of the following reactions:

$$^1\text{H} + {}^1\text{H} \to {}^2\text{H} + e^+ + \nu_e$$
$$^1\text{H} + {}^2\text{H} \to {}^3\text{He} + \gamma$$

followed by

$$^1\text{H} + {}^3\text{He} \to {}^4\text{He} + e^+ + \nu_e$$

(*a*) Show that the net effect of these reactions is

$$4\,{}^1\text{H} \to {}^4\text{He} + 2e^+ + 2\nu_e + \gamma$$

(*b*) Show that 24.7 MeV is released during this cycle (not counting the additional energy of 1.02 MeV that is released when each positron meets an electron and the two annihilate). (*c*) The Sun radiates energy at the rate of approximately 4.0×10^{26} W. Assuming that this is due to the conversion of four protons into helium, γ-rays, and neutrinos, which releases 26.7 MeV, what is the rate of proton consumption in the Sun? How long will the Sun last if it continues to radiate at its present level? (Assume that protons constitute about half of the total mass, 2.0×10^{30} kg, of the Sun.)

Picture the Problem Adding the three reactions will yield their net effect. We can use $(\Delta m)c^2$ to find the rest energy released in the cycle and find the rate of proton consumption from the ratio of the Sun's power output to the released per proton in fusion.

(*a*) Add the three reactions to obtain:

$$^1\text{H} + {}^1\text{H} + {}^1\text{H} + {}^2\text{H} + {}^1\text{H} + {}^3\text{He} \to {}^2\text{H} + \beta^+ + \nu_e + {}^3\text{He} + \gamma + {}^4\text{He} + \beta^+ + \nu_e$$

Simplify to obtain:

$$\boxed{4\,{}^1\text{H} \to {}^4\text{He} + 2\beta^+ + 2\nu_e + \gamma}$$

(*b*) Express the rest energy released in this cycle:

$$(\Delta m)c^2 = \left(4m_{\mathrm{p}} - m_\alpha - 4m_{\mathrm{e}}\right)c^2$$

Use Table 40-1 to find the masses of the participants in the reaction and evaluate $(\Delta m)c^2$:

$$(\Delta m)c^2 = [4(1.007\ 825\ \mathrm{u}) - 4.002\ 603\ \mathrm{u}]c^2 \times \frac{931.5\,\mathrm{MeV}/c^2}{\mathrm{u}} - 4(0.511\,\mathrm{MeV}) = \boxed{24.7\,\mathrm{MeV}}$$

(*c*) Express the rate *R* of proton consumption:

$$R = \frac{P}{E} \qquad (1)$$

where *E* is the energy released per proton in fusion.

Find *N*, the number of protons in the Sun:

$$N = \frac{\frac{1}{2}m_{\mathrm{sun}}}{m_{\mathrm{p}}} = \frac{\frac{1}{2}\left(1.99\times10^{30}\ \mathrm{kg}\right)}{1.673\times10^{-27}\ \mathrm{kg}} = 5.96\times10^{56}$$

where we have assumed that protons constitute about half of the total mass of the Sun.

The energy released per proton in fusion is:

$$E = \tfrac{1}{4}(26.7\,\mathrm{MeV}) = 6.675\,\mathrm{MeV} = 6.675\,\mathrm{MeV}\times\frac{1.602\times10^{-19}\ \mathrm{J}}{\mathrm{eV}} = 1.07\times10^{-12}\ \mathrm{J}$$

Substitute numerical values in equation (1) and evaluate *R*:

$$R = \frac{4.0\times10^{26}\ \mathrm{W}}{1.07\times10^{-12}\ \mathrm{J}} = 3.74\times10^{38}\ \mathrm{s}^{-1} \approx \boxed{3.7\times10^{38}\ \mathrm{s}^{-1}}$$

The time *T* for the consumption of all protons is:

$$T = \frac{N}{R} = \frac{5.96\times10^{56}}{3.74\times10^{38}\ \mathrm{s}^{-1}} = 1.59\times10^{18}\ \mathrm{s}\times\frac{1\,\mathrm{y}}{31.56\,\mathrm{Ms}} \approx \boxed{5.0\times10^{10}\ \mathrm{y}}$$

General Problems

52 • (*a*) Show that ke^2 = 1.44 MeV·fm, where k is the Coulomb constant and e is the magnitude of the electron charge. (*b*) Show that hc = 1240 MeV·fm.

Picture the Problem We can use the values of k, e, h, and c and the appropriate conversion factors to show that ke^2 = 1.44 MeV·fm and hc = 1240 MeV·fm

(*a*) Evaluate ke^2 to obtain:

$$\begin{aligned} ke^2 &= \left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\left(1.602\times10^{-19}\ \mathrm{C}\right)^2 = 2.307\times10^{-28}\ \mathrm{N\cdot m^2} \\ &= 2.307\times10^{-28}\ \mathrm{J\cdot m}\times\frac{1\,\mathrm{eV}}{1.602\times10^{-19}\ \mathrm{J}} = 1.44\times10^{-9}\ \mathrm{eV\cdot m} \\ &= 1.44\times10^{-9}\ \mathrm{eV\cdot m}\times\frac{1\,\mathrm{fm}}{10^{-15}\ \mathrm{m}}\times\frac{1\,\mathrm{MeV}}{10^{6}\ \mathrm{eV}} = \boxed{1.44\,\mathrm{MeV\cdot fm}} \end{aligned}$$

(*b*) Evaluate hc to obtain:

$$\begin{aligned} hc &= \left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)\left(2.998\times10^{8}\ \mathrm{m/s}\right) = 1.99\times10^{-25}\ \mathrm{J\cdot m} \\ &= 1.99\times10^{-25}\ \mathrm{J\cdot m}\times\frac{1\,\mathrm{eV}}{1.602\times10^{-19}\ \mathrm{J}} = 1240\times10^{-9}\ \mathrm{eV\cdot m} \\ &= 1240\times10^{-9}\ \mathrm{eV\cdot m}\times\frac{1\,\mathrm{fm}}{10^{-15}\ \mathrm{m}}\times\frac{1\,\mathrm{MeV}}{10^{6}\ \mathrm{eV}} = \boxed{1240\,\mathrm{MeV\cdot fm}} \end{aligned}$$

53 • **[SSM]** The counting rate from a radioactive source is 6400 counts/s. The half-life of the source is 10 s. Make a plot of the counting rate as a function of time for times up to 1 min. What is the decay constant for this source?

Picture the Problem We can use the given information regarding the half-life of the source to find its decay constant. We can then plot a graph of the counting rate as a function of time.

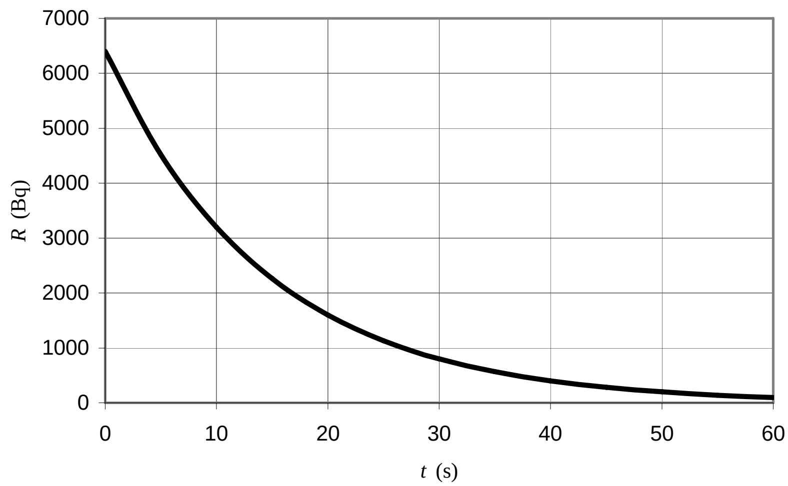
The decay constant is related to the half-life of the source:

$$\lambda = \frac{\ln(2)}{t_{1/2}} = \frac{\ln(2)}{10\,\mathrm{s}} = 0.0693\,\mathrm{s}^{-1} = \boxed{0.069\,\mathrm{s}^{-1}}$$

The activity of the source is given by:

$$R = R_0 e^{-\lambda t} = \left(6400\,\mathrm{Bq}\right)e^{-\left(0.0693\,\mathrm{s}^{-1}\right)t}$$

The following graph of $R = (6400\,\text{Bq})e^{-(0.0693\,\text{s}^{-1})t}$ was plotted using a spreadsheet program.



54 • Find the energy needed to remove a neutron from (*a*) ^{4}He and (*b*) ^{7}Li.

Picture the Problem The energy needed to remove a neutron is given by $Q = (\Delta m)c^2$ where Δm is the difference between the sum of the masses of the reaction products and the mass of the target nucleus.

(*a*) The reaction is: $^4\text{He} + Q = {}^3\text{He} + \text{n}$

The masses are (see Table 40-1):

$$m_{^4\text{He}} = 4.002\,603\ \text{u}$$
$$m_{^3\text{He}} = 3.016\,029\ \text{u}$$
$$m_\text{n} = 1.008\,665\ \text{u}$$

Calculate the final mass:

$$m_\text{f} = 3.016\,029\ \text{u} + 1.008\,665\ \text{u} = 4.024\,694\ \text{u}$$

Calculate the increase in mass:

$$\Delta m = m_\text{f} - m_\text{i} = 4.024\,694\ \text{u} - 4.002\,603\ \text{u} = 0.022\,091\ \text{u}$$

Calculate the energy Q needed to remove a neutron from ^{4}He:

$$Q = (\Delta m)c^2$$
$$= (0.022\ 091\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right)$$
$$= \boxed{20.6\,\text{MeV}}$$

(*b*) The reaction is:

$$^7\text{Li} + Q = {}^6\text{Li} + \text{n}$$

The masses are (see Table 40-1):

$$m_{^7\text{Li}} = 7.016\ 004\ \text{u}$$
$$m_{^6\text{Li}} = 6.015\,122\ \text{u}$$
$$m_\text{n} = 1.008\ 665\,\text{u}$$

Calculate the final mass:

$$m_\text{f} = 6.015\,122\ \text{u} + 1.008\ 665\ \text{u}$$
$$= 7.023\ 787\ \text{u}$$

Calculate the increase in mass:

$$\Delta m = m_\text{f} - m_\text{i}$$
$$= 7.023\ 787\ \text{u} - 7.016\ 004\ \text{u}$$
$$= 0.007\ 783\ \text{u}$$

Calculate the energy Q needed to remove a neutron from ^{4}He:

$$Q = (\Delta m)c^2$$
$$= (0.007\ 783\text{u})c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$
$$= \boxed{7.25\,\text{MeV}}$$

55 • The isotope ^{14}C decays according to $^{14}\text{C} \rightarrow {}^{14}\text{N} + \text{e}^- + \overline{\nu}_\text{e}$. The atomic mass of ^{14}N is 14.003 074 u. Determine the maximum kinetic energy of the electron. (Neglect recoil of the nitrogen atom.)

Picture the Problem The maximum kinetic energy of the electron is given by $K_\text{max} = Q = \left(m_{^{14}\text{C}} - m_{^{14}\text{N}}\right)c^2$.

The maximum kinetic energy of the electron is the Q value for the reaction:

$$K_\text{max} = Q = \left(m_{^{14}\text{C}} - m_{^{14}\text{N}}\right)c^2$$

Find the mass of each atom from Table 40-1:

$$m_{^{14}\text{C}} = 14.003\ 242\ \text{u}$$
$$m_{^{14}\text{N}} = 14.003\,074\ \text{u}$$

Calculate $\Delta m = m_{^{14}\text{C}} - m_{^{14}\text{N}}$:

$$\Delta m = 14.003\,242\text{ u} - 14.003\,074\text{ u} = 0.000\,168\,\text{u}$$

Calculate the maximum kinetic energy of the electron:

$$Q = (\Delta m)c^2 = (0.000\,168\,\text{u})c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right) = \boxed{156\text{ keV}}$$

56 • The density of a neutron star is the same as the density of a nucleus. If our Sun were to collapse to a neutron star, what would be the radius of that object?

Picture the Problem We can use the definition density to find the radius of the neutron star.

Relate the mass of the neutron star to the mass of the sun *M*, the volume *V* of the star and the nuclear density ρ:

$$M = \rho V = \tfrac{4}{3}\pi\rho R^3 \Rightarrow R = \sqrt[3]{\frac{3M}{4\pi\rho}}$$

where *R* is the radius of the star.

In Problem 20 it was established that:

$$\rho = 1.174 \times 10^{17}\text{ kg/m}^3$$

Substitute numerical values and evaluate *R*:

$$R = \sqrt[3]{\frac{3(1.99\times10^{30}\text{ kg})}{4\pi(1.174\times10^{17}\text{ kg/m}^3)}} = \boxed{15.9\,\text{km}}$$

57 •• **[SSM]** Show that the ^{109}Ag nucleus is stable and does not undergo alpha decay, ${}^{109}_{47}\text{Ag} \to {}^{4}_{2}\text{He} + {}^{105}_{45}\text{Rh} + Q$. The mass of the ^{109}Ag nucleus is 108.904 756 u, and the products of the decay are 4.002 603 u and 104.905 250 u, respectively.

Picture the Problem We can show that ^{109}Ag is stable against alpha decay by demonstrating that its *Q* value is negative.

The *Q* value for this reaction is:

$$Q = -[(m_{\text{Rh}} + m_{\alpha}) - m_{\text{Ag}}]c^2\left(931.5\,\frac{\text{MeV}/c^2}{\text{u}}\right)$$

Substitute numerical values and evaluate Q:

$$Q = -\left[\left(4.002\ 603\ \text{u} + 104.905\ 250\ \text{u}\right) - 108.904\ 756\ \text{u}\right]c^2\left(931.5\frac{\text{MeV}/c^2}{\text{u}}\right)$$

$$= \boxed{-2.88\ \text{MeV}}$$

Remarks: Alpha decay occurs spontaneously and the Q value will equal the sum of the kinetic energies of the alpha particle and the recoiling daughter nucleus, $Q = K_\alpha + K_\text{D}$. Kinetic energy cannot be negative; hence, alpha decay cannot occur unless the mass of the parent nucleus is greater than the sum of the masses of the alpha particle and daughter nucleus, $m_\text{P} > m_\alpha + m_\text{D}$. Alpha decay cannot take place unless the total rest mass decreases.

58 •• Gamma rays can be used to induce photofission (fission triggered by the absorption of a photon) in nuclei. Calculate the threshold photon wavelength for the following nuclear reaction: $^2\text{H} + \gamma \rightarrow {}^1\text{H} + \text{n}$. Use Table 40-1 for the masses of the interacting particles.

Picture the Problem We can use $E_\text{threshold} = hf_\text{threshold} = hc/\lambda_\text{threshold}$, where $E_\text{threshold}$ is the binding energy of the deuteron, to find the threshold wavelength for the given nuclear reaction.

Express the threshold energy of the photon:

$$E_\text{threshold} = hf_\text{threshold} = \frac{hc}{\lambda_\text{threshold}}$$

Solve for the threshold wavelength:

$$\lambda_\text{threshold} = \frac{hc}{E_\text{threshold}} \qquad (1)$$

The threshold energy equals the binding energy of the deuteron:

$$E_\text{threshold} = E_\text{B} = \left[m_\text{D} - \left(m_\text{p} + m_\text{n}\right)\right]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

Substitute numerical values and evaluate E_th, the energy that must be added to the deuteron that will cause it to fission:

$$E_\text{threshold} = \left[2.014\,102\ \text{u} - \left(1.007\ 825\ \text{u} + 1.008\ 665\ \text{u}\right)\right]\left(931.5\,\text{MeV/u}\right)$$

$$= -2.22\ \text{MeV} \times \frac{1.602\times10^{-19}\ \text{J}}{\text{eV}} = -3.55\times10^{-13}\ \text{J}$$

Substitute numerical values in equation (1) and evaluate $\lambda_{\text{threshold}}$:

$$\lambda_{\text{threshold}} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{3.55\times10^{-13}\ \text{J}} = 5.60\times10^{-13}\ \text{m} = \boxed{0.560\ \text{pm}}$$

59 • The relative abundance of ^{40}K (potassium 40) is 1.2×10^{-4}. The isotope ^{40}K has a molar mass of 40.0 g/mol, is radioactive and has a half-life of 1.3×10^{9} y. Potassium is an essential element of every living cell. In the human body the mass of potassium constitutes approximately 0.36 percent of the total mass. Determine the activity of this radioactive source in a student whose mass is 60 kg.

Picture the Problem The activity of a radioactive source is the product of the number of radioactive nuclei present and their decay constant.

The activity of the isotope ^{40}K in the student is:

$$R = N_{40}\lambda = \frac{N_{40}\ln(2)}{t_{1/2}} \qquad (1)$$

Find N, the number of K nuclei in the student:

$$\frac{N}{N_{\text{A}}} = \frac{0.0036m}{M} \Rightarrow N = \frac{0.0036mN_{\text{A}}}{M}$$

where m is the mass of the student and M is the atomic mass of K.

Substitute numerical values and evaluate N:

$$N = \frac{(0.0036)(60\ \text{kg})(6.022\times10^{23}\ \text{nuclei/mol})}{39.098\ \text{g/mol}} = 3.326\times10^{24}$$

The number N_{40} of ^{40}K nuclei in the student is the product of the relative abundance and the number of K nuclei in the student:

$$\begin{aligned} N_{40} &= \text{Relative abundance} \times N \\ &= (1.2\times10^{-4})(3.326\times10^{24}) \\ &= 3.991\times10^{20} \end{aligned}$$

Substitute numerical values in equation (1) and evaluate R:

$$R = \frac{(3.991\times10^{20})\ln(2)}{1.3\times10^{9}\ \text{y}\times\dfrac{31.56\ \text{Ms}}{\text{y}}} = \boxed{6.7\times10^{3}\ \text{Bq}}$$

60 •• When a positron makes contact with an electron, the electron–positron pair annihilate by way of the reaction $\beta^+ + \beta^- \to 2\gamma$. Calculate the minimum total energy, in MeV, of the two photons created when a positron–electron pair annihilate.

Picture the Problem We can find the energy released in the reaction $\beta^+ + \beta^- \to Q$ by recognizing that a total of 2 electron masses are converted into energy in this annihilation.

The energy released when a positron-electron pair annihilate is given by:

$$Q = E = 2m_e c^2$$

Substitute numerical values and evaluate Q:

$$Q = 2\left(9.109\times10^{-31}\,\text{kg}\right)\left(2.998\times10^{8}\,\text{m/s}\right)^2 = 1.64\times10^{-13}\,\text{J}\times\frac{1\,\text{eV}}{1.602\times10^{-19}\,\text{J}}$$

$$= \boxed{1.02\,\text{MeV}}$$

61 •• The isotope ^{24}Na is a β emitter and has a half-life of 15 h. A saline solution containing this radioactive isotope has an activity of 600 kBq and is injected into the bloodstream of a patient. Ten hours later, the activity of 1 mL of blood from this individual yields a counting rate of 12 counts/s at a counting efficiency of 20 percent. Determine the volume of blood in this patient.

Picture the Problem We can use the fact that, after n half-lives, the decay rate of the ^{24}Na isotope is $R = \left(\frac{1}{2}\right)^n R_0$, where R_0 is its decay rate at $t = 0$.

The counting rate after n half-lives is related to the initial counting rate:

$$R = \left(\tfrac{1}{2}\right)^n R_0$$

Divide both sides of the equation by the volume V of blood in the patient:

$$\frac{R}{V} = \left(\tfrac{1}{2}\right)^n \frac{R_0}{V}$$

We're given that $n = 2/3$, $R_0 = 600$ kBq, and, after n half-lives, the decay rate per unit volume is 60 Bq/mL:

$$60\,\text{Bq/mL} = \left(\tfrac{1}{2}\right)^{2/3} \frac{600\,\text{kBq}}{V}$$

Solving for V yields:

$$V = \left(\tfrac{1}{2}\right)^{2/3}\left(\frac{600\,\text{kBq}}{60\,\text{Bq/mL}}\right) = 6.30\times10^3\,\text{mL}$$

$$= \boxed{6.3\,\text{L}}$$

62 •• (*a*) Determine the distance of closest approach of an 8.0-MeV α particle in a head-on collision with a stationary nucleus of ^{197}Au and with a stationary nucleus of ^{10}B, neglecting the recoil of the struck nuclei. (*b*) Repeat the calculation taking into account the recoil of the struck nuclei.

Picture the Problem We can solve this problem in the center of mass reference frame for the general case of an α particle in a head-on collision with a stationary nucleus of atomic mass M u and then substitute data for a stationary nucleus of ^{197}Au and a nucleus of ^{10}B.

In the CM frame, the kinetic energy is:

$$K_{\text{CM}} = \frac{K_{\text{lab}}}{1+\dfrac{m_\alpha}{M}} = \frac{K_{\text{lab}}}{1+\dfrac{4\,\text{u}}{M}}$$

At the point of closest approach:

$$K_{\text{CM}} = \frac{kq_1q_2}{R_{\min}} = \frac{k(2e)(Ze)}{R_{\min}} = \frac{ke^2\,2Z}{R_{\min}}$$

or, because $ke^2 = 1.44$ MeV·fm,

$$K_{\text{CM}} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2Z)}{R_{\min}}$$

Solve for $R_{\min}$ to obtain:

$$R_{\min} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2Z)}{K_{\text{CM}}} \qquad (1)$$

(*a*) Neglecting the recoil of the target nucleus is equivalent to replacing K_{CM} by K_{lab}. Evaluate equation (1) for ^{197}Au:

$$R_{\min} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2\times 79)}{8.0\,\text{MeV}} = \boxed{28\,\text{fm}}$$

Evaluate equation (1) for ^{10}B:

$$R_{\min} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2\times 5)}{8.0\,\text{MeV}} = \boxed{1.8\,\text{fm}}$$

(*b*) Find K_{CM} for the ^{197}Au nucleus:

$$K_{\text{CM}} = \frac{8.0\,\text{MeV}}{1+\dfrac{4\,\text{u}}{197\,\text{u}}} = 7.841\,\text{MeV}$$

Substitute numerical values in equation (1) and evaluate R_{min}:

$$R_{\text{min}} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2\times 79)}{7.841\,\text{MeV}} = \boxed{29\,\text{fm}}$$

Note that this result is about 2% greater that R_{min} calculated ignoring recoil.

Find K_{CM} for the ^{10}B nucleus:

$$K_{\text{CM}} = \frac{8.0\,\text{MeV}}{1+\dfrac{4\,\text{u}}{10\,\text{u}}} = 5.714\,\text{MeV}$$

Substitute numerical values in equation (1) and evaluate R_{min}:

$$R_{\text{min}} = \frac{(1.44\,\text{MeV}\cdot\text{fm})(2\times 5)}{5.714\,\text{MeV}} = \boxed{2.5\,\text{fm}}$$

Note that this result is about 40% greater that R_{min} calculated ignoring recoil.

63 •• Twelve nucleons are in a one-dimensional infinite square well of length L = 3.0 fm. (*a*) Using the approximation that the mass of a nucleon is 1.0 u, find the lowest energy of a nucleon in the well. Express your answer in MeV. What is the ground-state energy of the system of 12 nucleons in the well if (*b*) all the nucleons are neutrons so that there can be no more than 2 in each spatial state and (*c*) 6 of the nucleons are neutrons and 6 are protons so that there can be as many as 4 nucleons in each spatial state? (Neglect the energy of Coulomb repulsion of the protons.)

Picture the Problem The allowed energy levels in a one-dimensional infinite square well are given by $E_n = n^2\left(\dfrac{h^2}{8mL^2}\right)$.

(*a*) The lowest energy of a nucleon of mass 1.0 u in the well corresponds to n = 1:

$$E_1 = \frac{(6.626\times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(1.0\,\text{u})(1.661\times 10^{-27}\ \text{kg/u})(3.0\,\text{fm})^2} = 3.678\times 10^{-12}\ \text{J}\times\frac{1\text{eV}}{1.602\times 10^{-19}\ \text{J}} = \boxed{23\,\text{MeV}}$$

(*b*) Because neutrons are fermions, there can be no more than two per state:

$$E = 2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6) = 2(E_1 + 2^2 E_1 + 3^2 E_1 + 4^2 E_1 + 5^2 E_1 + 6^2 E_1)$$
$$= 182 E_1 = 182(23\,\text{MeV}) = \boxed{4.2\,\text{GeV}}$$

(*c*) Find E for 4 protons and 4 neutrons:

$$E = 4(E_1 + E_2 + E_3) = 4(E_1 + 2^2 E_1 + 3^2 E_1)$$
$$= 56 E_1 = 56(23\,\text{MeV})$$
$$= \boxed{1.3\,\text{GeV}}$$

64 •• The helium nucleus or α particle is a very tightly bound system. Nuclei with $N = Z = 2n$, where n is an integer (for example, ^{12}C, ^{16}O, ^{20}Ne, and ^{24}Mg), can be modeled as agglomerates of α particles. (*a*) Use this model to estimate the binding energy of a pair of α particles from the atomic masses of ^{4}He and ^{16}O. Assume that the four α particles in ^{16}O form a regular tetrahedron that has one α particle at each vertex. (*b*) From the result obtained in Part (*a*) determine, on the basis of this model, the binding energy of ^{12}C and compare your result with the result obtained from the atomic mass of ^{12}C.

Picture the Problem We can apply BE = $(\Delta m)c^2$ to the model to find the binding energies and the binding energies/bond.

(*a*) Find the binding energy BE for this model:

$$\text{BE} = \left(4m_\alpha - m_{^{16}\text{O}}\right)c^2$$
$$= [4(4.002\ 603\ \text{u}) - 15.994\ 915\ \text{u}]c^2$$
$$= (0.015\ 497\ \text{u})c^2$$

Because there is a bond between adjacent α particles (4) and a bond between diagonally opposite α particles (2), there are 6 bonds for the regular tetrahedron:

$$\frac{\text{BE}}{\text{bond}} = \tfrac{1}{6}\text{BE} = \tfrac{1}{6}(0.015\ 497\,\text{u})c^2 = \tfrac{1}{6}(0.015\ 497\,\text{u})c^2\left(\frac{931.5\,\text{MeV/c}^2}{\text{u}}\right)$$
$$= \boxed{2.406\,\text{MeV}}$$

(*b*) ^{12}C has 3 pairwise α particle bonds. Find the total BE for ^{12}C with this model:

$$\text{BE}\left(^{12}\text{C}\right) = 3 \times \text{BE}\left(^{4}\text{He}\right) + 3(2.406\,\text{MeV})$$

Calculate BE(^{4}He):

$$\begin{aligned}\text{BE}\left({}^4\text{He}\right) &= \left[2\left(m_\text{p} + m_\text{n}\right) - m_{{}^4\text{He}}\right]c^2 \\ &= \left[2\left(1.007\ 825\,\text{u} + 1.008\ 665\,\text{u}\right) - 4.002\ 603\,\text{u}\right]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right) \\ &= 28.30\,\text{MeV}\end{aligned}$$

Substitute numerical values and evaluate $\text{BE}\left({}^{12}\text{C}\right)$:

$$\text{BE}\left({}^{12}\text{C}\right) = 3\left(28.30\,\text{MeV}\right) + 3\left(2.406\,\text{MeV}\right) = \boxed{92.1\,\text{MeV}}$$

Use values in Table 40-1 to find $\text{BE}\left({}^{12}\text{C}\right)$:

$$\begin{aligned}\text{BE}\left({}^{12}\text{C}\right) &= \left[6\left(m_\text{p} + m_\text{n}\right) - m_{{}^{12}\text{C}}\right]c^2 \\ &= \left[6\left(1.007\ 825\,\text{u} + 1.008\ 665\,\text{u}\right) - 12.000\ 000\,\text{u}\right]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right) \\ &= 92.2\,\text{MeV}\end{aligned}$$

Note that this result is in good agreement with the model.

65 •• Nuclei of a radioactive isotope that has a decay constant of λ are produced in an accelerator at a constant rate R_p. The number of radioactive nuclei N then obeys the equation $dN/dt = R_\text{p} - \lambda N$. (*a*) If N is zero at $t = 0$, sketch N versus t for this situation. (*b*) The isotope ^{62}Cu is produced at a rate of 100 per second by placing ordinary copper (^{63}Cu) in a beam of high-energy photons. The reaction is

$$\gamma + {}^{63}\text{Cu} \rightarrow {}^{62}\text{Cu} + \text{n}$$

The isotope ^{62}Cu decays by β decay and has a half-life of 10 min. After a time long enough so that $dN/dt \approx 0$, how many ^{62}Cu nuclei are there?

Picture the Problem We can separate the variables in the differential equation $dN/dt = R_\text{p} - \lambda N$ and integrate to express N as a function of t. When $dN/dt \approx 0$, $R_\text{p} - \lambda N_\infty = 0$, a condition we can use to find N_∞.

(*a*) Separate the variables in the differential equation to obtain:

$$\frac{dN}{R_\text{p} - \lambda N} = dt$$

Integrate the left side of the equation from 0 to N and the right side from 0 to t to obtain:

$$\int_0^N \frac{dN'}{R_\text{p} - \lambda N'} = \int_0^t dt'$$

Let $u = R_p - \lambda N'$. Then: $du = -\lambda dN'$

and

$$\int_0^N \frac{dN'}{R_p - \lambda N'} = -\frac{1}{\lambda}\int_{\ell_1}^{\ell_2} \frac{du}{u} = -\frac{1}{\lambda}\ln u\Big|_{\ell_1}^{\ell_2}$$

$$= -\frac{1}{\lambda}\ln(R_p - \lambda N')\Big|_0^N$$

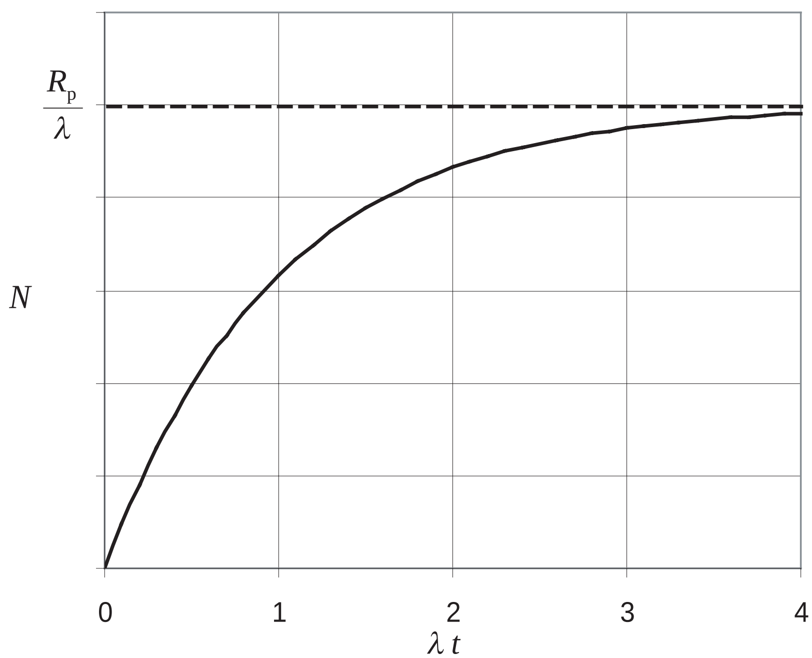
$$= -\frac{1}{\lambda}\ln(R_p - \lambda N) + \frac{1}{\lambda}\ln(R_p)$$

$$= \frac{1}{\lambda}\ln\left(\frac{R_p}{R_p - \lambda N}\right)$$

Because $\int_0^t dt' = t$:

$$\frac{1}{\lambda}\ln\left(\frac{R_p}{R_p - \lambda N}\right) = t \Rightarrow N = \frac{R_p}{\lambda}\left(1 - e^{-\lambda t}\right)$$

The following graph of $N(t) = (R_p/\lambda)(1 - e^{-\lambda t})$ was plotted using a spreadsheet program. Note that $N(t)$ approaches R_p/λ in the same manner that the charge on a capacitor approaches the value CV.



(*b*) When $dN/dt = 0$:

$$R_p - \lambda N_\infty = 0 \Rightarrow N_\infty = \frac{R_p}{\lambda}$$

The decay constant is:

$$\lambda = \frac{\ln(2)}{t_{1/2}}$$

Substitute for λ to obtain:

$$N_\infty = \frac{R_p}{\ln(2)} t_{1/2}$$

Substitute numerical values and evaluate N_∞:

$$N_\infty = \frac{100\,\text{s}^{-1}}{\ln(2)}\left(10\,\text{min} \times \frac{60\,\text{s}}{\text{min}}\right) = \boxed{8.7 \times 10^4}$$

66 •• The total energy consumed in the United States in 1 y is approximately 7.0×10^{19} J. How many kilograms of ^{235}U would be needed to provide this amount of energy if we assume that 200 MeV of energy is released by each fissioning uranium nucleus, that all of the uranium atoms undergo fission, and that all of the energy-conversion mechanisms used are 100 percent efficient?

Picture the Problem The mass of ^{235}U required is given by $m_{235} = \dfrac{N}{N_A} M_{235}$, where M_{235} is the molecular mass of ^{235}U and N is the number of fissions required to produce $7.0{\times}10^{19}$ J.

Relate the mass of ^{235}U required to the number of fissions N required:

$$m_{235} = \frac{N}{N_A} M_{235}$$

where M_{235} is the molecular mass of ^{235}U.

Because $N = \dfrac{E_{\text{annual}}}{E_{\text{per fission}}}$:

$$m_{235} = \frac{E_{\text{annual}} M_{235}}{N_A E_{\text{per fission}}}$$

Substitute numerical values in equation (1) and evaluate m_{235}:

$$m_{235} = \frac{(7.0 \times 10^{19}\,\text{J})(235\,\text{g/mol})}{\left(6.022 \times 10^{23}\,\dfrac{\text{nuclei}}{\text{mol}}\right)\left(200\,\text{MeV} \times \dfrac{1.602 \times 10^{-19}\,\text{J}}{\text{eV}}\right)} = \boxed{8.5 \times 10^5\,\text{kg}}$$

67 •• (*a*) Find the wavelength of a particle in the ground state of a one-dimensional infinite square well of length L = 2.00 fm. (*b*) Find the momentum in units of MeV/c for a particle that has this wavelength. (*c*) Show that the total energy of an electron that has this wavelength is approximately $E \approx pc$. (*d*) What is the kinetic energy of an electron in the ground state of this well? This

calculation shows that if an electron were confined in a region of space as small as a nucleus, it would have a very large kinetic energy.

Picture the Problem In the ground state of a one-dimensional infinite square well of length L the wavelength of a particle is $2L$. We can use de Broglie's equation to find p for the particle and the relationship $E^2 = E_0^2 + p^2c^2$ with $E_0 << pc$ to show that $E \approx pc$.

(a) In the ground state of a one-dimensional infinite square well of length L:

$$\lambda = 2L = 2(2.00\,\text{fm}) = \boxed{4.00\,\text{fm}}$$

(b) Use de Broglie's relation to obtain:

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c}$$

Substitute numerical values and evaluate p:

$$p = \frac{1240\,\text{eV}\cdot\text{nm}}{(4.00\,\text{fm})c} = \boxed{310\,\text{MeV}/c}$$

(c) Relate the total energy of the electron to its rest energy and momentum:

$$E^2 = E_0^2 + p^2c^2 = p^2c^2\left(1+\frac{E_0^2}{p^2c^2}\right)$$

Because $E_0 << pc$:

$$E^2 \approx p^2c^2 \Rightarrow E \approx \boxed{pc}$$

(d) The kinetic energy of an electron in the ground state of this well is given by:

$$K = E - E_0 \approx E = pc$$
$$= (310\,\text{MeV}/c)c = \boxed{310\,\text{MeV}}$$

68 •• If ^{12}C, ^{11}B, and ^{1}H have respective masses of 12.000 000 u, 11.009 306 u, and 1.007 825 u, determine the minimum energy, Q, in MeV, required to remove a proton from a ^{12}C nucleus.

Picture the Problem When a single proton is removed from a ^{12}C nucleus, a ^{11}B nucleus remains and we can use $Q = \Delta mc^2$ to determine the minimum energy required to remove a proton.

The nuclear reaction is:

$${}^{12}_{6}\text{C} + Q \rightarrow {}^{11}_{5}\text{B} + {}^{1}_{1}\text{H}$$

The minimum energy Q required is:

$$Q = \left(m_{^{11}\text{B}} + m_{^{1}\text{H}} - m_{^{12}\text{C}}\right)c^2$$

Substitute numerical values and evaluate Q:

$$Q = [(11.009\,306\,\text{u} + 1.007\,825\,\text{u}) - 12.000\,000\,\text{u}]c^2\left(\frac{931.5\,\text{MeV}/c^2}{\text{u}}\right)$$

$$= \boxed{16.0\,\text{MeV}}$$

69 ••• [SSM] Assume that a neutron decays into a proton and an electron without the emission of a neutrino. The kinetic energy shared by the proton and the electron is then 0.782 MeV. In the rest frame of the neutron, the total momentum is zero, so the momentum of the proton must be equal and opposite the momentum of the electron. This determines the ratio of the kinetic energies of the two particles, but because the electron is relativistic, the exact calculation of these relative kinetic energies is somewhat challenging. (a) Assume that the kinetic energy of the electron is 0.782 MeV and calculate the momentum p of the electron in units of MeV/c. *Hint: Use* $E^2 = p^2c^2 + (mc^2)^2$ *(Equation 39-27).* (b) Using your result from Part (a), calculate the kinetic energy $p^2/2m_p$ of the proton. (c) Because the total kinetic energy of the electron and the proton is 0.782 MeV, the calculation in Part (b) gives a correction to the assumption that the kinetic energy of the electron is 0.782 MeV. What percentage of 0.782 MeV is this correction?

Picture the Problem The momentum of the electron is related to its total energy through $E^2 = p^2c^2 + E_0^2$ and its total relativistic energy E is the sum of its kinetic and rest energies.

(a) Relate the total energy of the electron to its momentum and rest energy:

$$E^2 = p^2c^2 + E_0^2 \qquad (1)$$

The total relativistic energy E of the electron is the sum of its kinetic energy and its rest energy:

$$E = K + E_0$$

Substitute for E in equation (1) to obtain:

$$(K + E_0)^2 = p^2c^2 + E_0^2$$

Solving for p gives:

$$p = \frac{\sqrt{K(K + 2E_0)}}{c}$$

Substitute numerical values and evaluate p:

$$p=\frac{\sqrt{(0.782\,\text{MeV})(0.782\,\text{MeV}+2\times0.511\,\text{MeV})}}{c}=1.188\,\text{MeV}/c=\boxed{1.19\,\text{MeV}/c}$$

(b) Because $p_{\text{p}} = -p_{\text{e}}$:

$$K_{\text{p}}=\frac{p_{\text{p}}^2}{2m_{\text{p}}}$$

Substitute numerical values (see Table 7-1 for the rest energy of a proton) and evaluate K_{p}:

$$K_{\text{p}}=\frac{(1.188\,\text{MeV}/c)^2}{2(938.28\,\text{MeV}/c^2)}=\boxed{752\,\text{eV}}$$

(c) The percent correction is:

$$\frac{K_{\text{p}}}{K}=\frac{752\,\text{eV}}{0.782\,\text{MeV}}=\boxed{0.0962\%}$$

70 ••• In the laboratory reference frame, a neutron of mass m moving with speed v_{L} makes an elastic head-on collision with a nucleus of mass M that is at rest. (a) Show that the speed of the center of mass in the lab frame is $V = mv_{\text{L}}/(m + M)$. ($b$) What is the speed of the nucleus in the center-of-mass frame before the collision and after the collision? (c) What is the speed of the nucleus in the laboratory frame after the collision? (d) Show that the energy of the nucleus after the collision in the laboratory frame is

$$\frac{1}{2}M(2V)^2=\frac{4mM}{(m+M)^2}\left(\frac{1}{2}mv_{\text{L}}^2\right)$$

(e) Show that the fraction of the energy lost by the neutron in this elastic collision is

$$\frac{-\Delta E}{E}=\frac{4mM}{(m+M)^2}=\frac{4(m/M)}{(1+m/M)^2}$$

Picture the Problem Conservation of momentum and conservation of energy allow us to find the final speeds of the neutron and nucleus.

(a) Apply conservation of momentum to the collision to obtain:

$$(m+M)V=mv_{\text{L}}\Rightarrow V=\boxed{\left(\frac{m}{m+M}\right)v_{\text{L}}}$$

(b) In the CM frame, $V_{M\text{i}} = V$ and so:

$$V_{M\text{i}}=\boxed{V}$$

In the CM frame, $V_f = -V_i$ and so:

$$V_{Mf} = \boxed{-V}$$

(*c*) Use conservation of momentum to obtain one relation for the final velocities:

$$mv_L = mv_f + MV_{Mf} \qquad (1)$$

The equality of the initial and final kinetic energies provides a second equation relating the two final velocities. This is implemented by equating the speeds of recession and approach:

$$V_{Mf} - v_f = -(V_{Mi} - v_L) = 0 + v_L$$

and so

$$v_f = V_{Mf} - v_L$$

To eliminate v_f, substitute for mv_L in equation (1) :

$$mv_L = m(V_{Mf} - v_L) + MV_{Mf}$$

Solving for V_{Mf} yields:

$$V_{Mf} = \boxed{\left(\frac{2m}{m+M}\right)v_L}$$

(*d*) The kinetic energy of the nucleus after the collision in the laboratory frame is:

$$K_M = \tfrac{1}{2}MV_{Mf}^2$$

Substitute for V_{Mf} and simplify to obtain:

$$K_{Mf} = \tfrac{1}{2}M\left(\frac{2m}{m+M}v_L\right)^2 = \boxed{\frac{4mM}{(m+M)^2}\left(\tfrac{1}{2}mv_L^2\right)}$$

(*e*) The fraction of the energy lost by the neutron in the elastic collision is given by:

$$\frac{-\Delta E}{E} = \frac{-(K_{mf} - K_{mi})}{\frac{1}{2}mv_L^2} = \frac{-\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_L^2\right)}{\frac{1}{2}mv_L^2} = \frac{-\frac{1}{2}m\left(v_f^2 - v_L^2\right)}{\frac{1}{2}mv_L^2}$$

Because $v_f = V_{Mf} - v_L$:

$$\frac{-\Delta E}{E} = \frac{-\frac{1}{2}m\left[(V_{Mf} - v_L)^2 - v_L^2\right]}{\frac{1}{2}mv_L^2}$$

Substituting for $V_{M\mathrm{f}}$ from Part (c) gives:

$$\frac{-\Delta E}{E}=\frac{-\frac{1}{2}m\left[\left(\left(\frac{2m}{m+M}\right)v_{\mathrm{L}}-v_{\mathrm{L}}\right)^2-v_{\mathrm{L}}^2\right]}{\frac{1}{2}mv_{\mathrm{L}}^2}=\frac{-\frac{1}{2}mv_{\mathrm{L}}^2\left[\left(\left(\frac{2m}{m+M}\right)-1\right)^2-1\right]}{\frac{1}{2}mv_{\mathrm{L}}^2}$$

$$=-\left[\left(\frac{2m}{m+M}\right)-1\right]^2-1=-\left[\left(\frac{2m}{m+M}\right)-1\right]^2-1=-\left[\frac{2m-m-M}{m+M}\right]^2-1$$

Further simplification yields:

$$\frac{-\Delta E}{E}=\frac{4mM}{(M+m)^2}=\frac{4mM}{M^2\left(1+\frac{m}{M}\right)^2}=\boxed{\frac{4m/M}{(1+m/M)^2}}$$

71 ••• (a) Use the result from Part (e) of Problem 72 to show that after N head-on collisions of a neutron with carbon nuclei at rest, the energy of the neutron is approximately $(0.714)^N E_0$, where E_0 is its original energy.
(b) How many head-on collisions are required to reduce the energy of the neutron from 2.0 MeV to 0.020 eV, assuming stationary carbon nuclei?

Picture the Problem We can use the result of Problem 70, Part (e), to find the fraction $f=E_{\mathrm{f}}/E_0$ of its initial energy lost per collision and then use this result to show that, after N collisions, $E=(0.714)^N E_0$.

(a) Determine $f=E_{\mathrm{f}}/E_0$ per collision:

$$f=\frac{E_0-\Delta E}{E_0}=1-\frac{\Delta E}{E_0}$$

From Problem 70, Part (e):

$$\frac{\Delta E}{E_0}=\frac{4m}{M\left(1+\frac{m}{M}\right)^2}$$

Substitute for $\Delta E/E_0$ in the expression for f to obtain:

$$f=1-\frac{4m}{M\left(1+\frac{m}{M}\right)^2}$$

Substitute numerical values and evaluate f:

$$f = 1 - \frac{4(1.008665\,\text{u})}{12.000000\,\text{u}\left(1 + \frac{1.008665\,\text{u}}{12.000000\,\text{u}}\right)^2} = 0.714$$

After N collisions:

$$E_{fN} = f^N E_0 = \boxed{(0.714)^N E_0} \quad (1)$$

(b) Solve equation (1) for N:

$$N = \frac{\ln\left(\frac{E_{fN}}{E_0}\right)}{\ln(0.714)}$$

Substitute numerical values and evaluate N:

$$N = \frac{\ln\left(\frac{0.020\,\text{eV}}{2.0\,\text{MeV}}\right)}{\ln(0.714)} \approx 55$$

55 head-on collisions are required to reduce the energy of the neutron from 2.0 MeV to 0.020 eV.

72 ••• On the average, a neutron loses 63 percent of its energy in a collision with a hydrogen atom and 11 percent of its energy in a collision with a carbon atom. Calculate the number of collisions needed to reduce the energy of a neutron from 2.0 MeV to 0.020 eV if the neutron collides with (a) hydrogen atoms and (b) carbon atoms.

Picture the Problem We can use the result of Problem 70, Part (e), to find the fraction $f = E_\text{f}/E_0$ of its initial energy lost per collision. Note the difference between the energy loss per collision specified here and that of the preceding problem. In the preceding problem it was assumed that all collisions are head-on collisions.

(a) Determine $f = E_\text{f}/E_0$ per collision:

$$f = \frac{E_0 - \Delta E}{E_0} = 1 - \frac{\Delta E}{E_0}$$

In a collision with a hydrogen atom:

$$\frac{\Delta E}{E_0} = 0.63 \Rightarrow f = 0.37$$

After N collisions:

$$E_{fN} = f^N E_0 = (0.37)^N E_0 \quad (1)$$

Solve equation (1) for N:

$$N = \frac{\ln\left(\frac{E_{fN}}{E_0}\right)}{\ln(0.37)} \quad (2)$$

Substitute numerical values and evaluate N:

$$N = \frac{\ln\left(\frac{0.020\,\text{eV}}{2.0\,\text{MeV}}\right)}{\ln(0.37)} \approx 19$$

19 head-on collisions with an atom of hydrogen are required to reduce the energy of the neutron from 2.0 MeV to 0.020 eV.

(*b*) In a collision with a carbon atom:

$$\frac{\Delta E}{E_0} = 0.11 \Rightarrow f = 0.89$$

Equation (2) becomes:

$$N = \frac{\ln\left(\frac{E_{fN}}{E_0}\right)}{\ln(0.89)}$$

Substitute numerical values and evaluate N:

$$N = \frac{\ln\left(\frac{0.020\,\text{eV}}{2.0\,\text{MeV}}\right)}{\ln(0.89)} \approx 160$$

160 head-on collisions with an atom of carbon are required to reduce the energy of the neutron from 2.0 MeV to 0.020 eV.

73 ••• [SSM] Frequently, the daughter nucleus of a radioactive parent nucleus is itself radioactive. Suppose the parent nucleus, designated by P, has a decay constant λ_P; while the daughter nucleus, designated by D, has a decay constant λ_D. The number of daughter nuclei N_D are then given by the solution to the differential equation

$$dN_D/dt = \lambda_P N_P - \lambda_D N_D$$

where N_P is the number of parent nuclei. (*a*) Justify this differential equation. (*b*) Show that the solution for this equation is

$$N_D(t) = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right)$$

where N_{P0} is the number of parent nuclei present at $t = 0$ when there are no daughter nuclei. (*c*) Show that the expression for N_D in Part (*b*) gives $N_D(t) > 0$

whether $\lambda_P > \lambda_D$ or $\lambda_D > \lambda_P$. (*d*) Make a plot of $N_P(t)$ and $N_D(t)$ as a function of time when $\tau_D = 3\tau_P$, where τ_D and τ_P are the mean lifetimes of the daughter and parent nuclei, respectively.

Picture the Problem We can differentiate $N_D(t) = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right)$ with respect to t to show that it is the solution to the differential equation $dN_D/dt = \lambda_P N_P - \lambda_D N_D$.

(*a*) The rate of change of N_D is the rate of generation of D nuclei minus the rate of decay of D nuclei. The generation rate is equal to the decay rate of P nuclei, which equals $\lambda_P N_P$. The decay rate of D nuclei is $\lambda_D N_D$.

(*b*) We're given that:

$$\frac{dN_D}{dt} = \lambda_P N_P - \lambda_D N_D \qquad (1)$$

$$N_D(t) = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right) \qquad (2)$$

$$N_P = N_{P0}e^{-\lambda_P t} \qquad (3)$$

Differentiate equation (2) with respect to t to obtain:

$$\frac{d}{dt}\left[N_D(t)\right] = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\frac{d}{dt}\left[\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right)\right] = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left[-\lambda_P e^{-\lambda_P t} + \lambda_D e^{-\lambda_D t}\right]$$

Substitute this derivative in equation (1) to obtain:

$$\frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left[-\lambda_P e^{-\lambda_P t} + \lambda_D e^{-\lambda_D t}\right] = \lambda_P N_{P0}e^{-\lambda_P t} - \lambda_D\left[\frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right)\right]$$

Multiply both sides by $\frac{\lambda_D - \lambda_P}{\lambda_D\lambda_P}$ and simplify to obtain:

$$\begin{aligned}
\frac{N_{P0}}{\lambda_D}\left[-\lambda_P e^{-\lambda_P t} + \lambda_D e^{-\lambda_D t}\right] &= \frac{\lambda_D - \lambda_P}{\lambda_D}N_{P0}e^{-\lambda_P t} - N_{P0}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right) \\
&= N_{P0}e^{-\lambda_P t} - \frac{N_{P0}\lambda_P}{\lambda_D}e^{-\lambda_P t} - N_{P0}e^{-\lambda_P t} + N_{P0}e^{-\lambda_D t} \\
&= -\frac{N_{P0}\lambda_P}{\lambda_D}e^{-\lambda_P t} + N_{P0}e^{-\lambda_D t} \\
&= \frac{N_{P0}}{\lambda_D}\left[-\lambda_P e^{-\lambda_P t} + \lambda_D e^{-\lambda_D t}\right]
\end{aligned}$$

Because this is an identity, it confirms that equation (2) is the solution to equation (1).

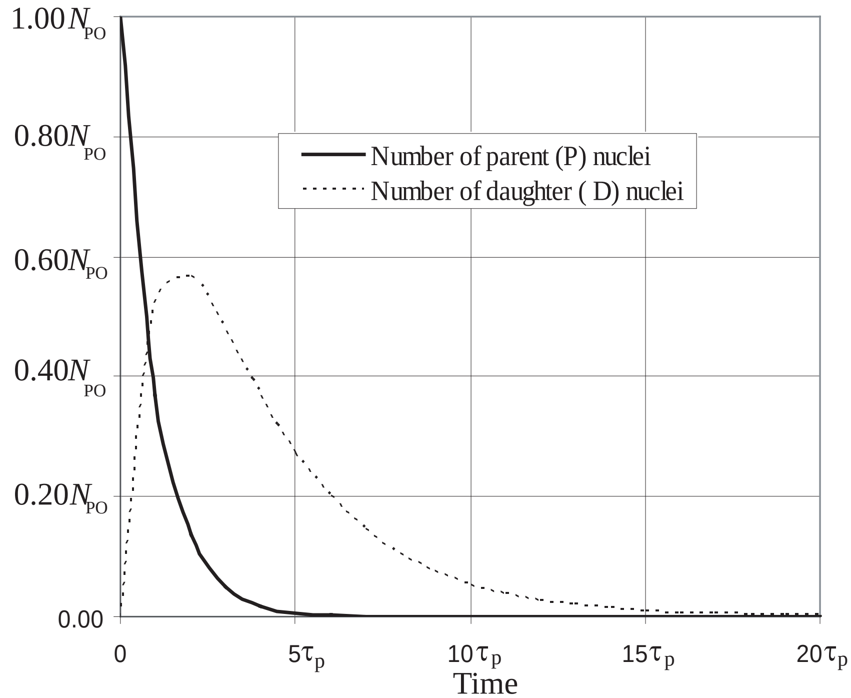
(*c*) If $\lambda_P > \lambda_D$ the denominator and expression in parentheses are both negative for $t > 0$. If $\lambda_P < \lambda_D$ the denominator and expression in parentheses are both positive for $t > 0$.

(*d*) A spreadsheet program to calculate N_P and N_D as functions of time is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A4	0	t
A5	A4+1	$t + 1$
B4	EXP(−A4)	$N_{P0}e^{-\lambda_P t}$
C4	−1.5*(EXP(−A4) −EXP(−(1/3)*A4))	$\frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right)$

	A	B	C
1			
2			
3	*Time*	N_P	N_D
4	0	1	0
5	1	0.367879	0.522978
6	2	0.135335	0.567123
7	3	0.049787	0.477139
22	18	1.52E−08	0.003718
23	19	5.6E−09	0.002664
24	20	2.06E−09	0.001909

A graph of N_P and N_D as functions of time follows:



74 ••• Suppose isotope A decays to isotope B and has a decay constant λ_A, and isotope B in turn decays and has a decay constant λ_B. Suppose a sample contains, at $t = 0$, only isotope A nuclei. Derive an expression for the time at which the number of isotope B nuclei will be a maximum. (See Problem 75.)

Picture the Problem We can express the time at which the number of isotope B nuclei will be a maximum by setting dN_B/dt equal to zero and solving for t.

From Problem 75 we have:

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B = 0 \text{ for extrema}$$

Replace $\lambda_A N_A$ by $\lambda_A N_{A0} e^{-\lambda_A t}$ and N_B by $\dfrac{N_{A0}\lambda_A}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$:

$$\lambda_A N_{A0} e^{-\lambda_A t} - \lambda_B \frac{N_{A0}\lambda_A}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) = 0$$

Simplify to obtain:

$$e^{-\lambda_A t} - \frac{\lambda_B}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) = 0$$

$$(\lambda_B - \lambda_A)e^{-\lambda_A t} - \lambda_B\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) = 0$$

Remove the parentheses and combine like terms to obtain:

$$\lambda_A e^{-\lambda_A t} = \lambda_B e^{-\lambda_B t} \Rightarrow t = \boxed{\frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}}$$

Remarks: Note that all we've shown is that an *extreme value* exists at $t = \frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}$. To show that this value for *t* maximizes N_B, we need to either 1) examine the second derivative at this value for *t*, or 2) plot a graph of N_B as a function of time (see Problem 77) .

75 ••• An example of the situation discussed in Problem 75 is the radioactive isotope ^{229}Th, an α emitter that has a half-life of 7300 y. Its daughter, ^{225}Ra, is a β emitter that has a half-life of 14.8 d. In this instance, as in many instances, the half-life of the parent is much longer than the half-life of the daughter. Using the expression given in Problem 75, Part (*b*), starting with a sample of pure ^{229}Th containing N_{P0} nuclei, show that the number, N_D, of ^{225}Ra nuclei will, after several years, be given by

$$N_D = \frac{\lambda_P}{\lambda_D} N_{P0}$$

where N_P is number of ^{229}Th nuclei. The number of daughter nuclei are said to be in *secular equilibrium*.

Picture the Problem We can show that, provided $\tau_P >> \tau_D$, $e^{-\lambda_P t} - e^{-\lambda_D t} \approx 1$ and $\frac{\lambda_P}{\lambda_D - \lambda_P} \approx \frac{\lambda_P}{\lambda_D}$ and, hence, that $N_D = \frac{\lambda_P}{\lambda_D} N_{P0}$.

We have, from Problem 75 (*b*):

$$N_D(t) = \frac{N_{P0}\lambda_P}{\lambda_D - \lambda_P}\left(e^{-\lambda_P t} - e^{-\lambda_D t}\right) \quad (1)$$

Because $\tau_P >> \tau_D$:

$$\lambda_P << \lambda_D$$

When several years have passed, because $\lambda_P t << 1$:

$$e^{-\lambda_P t} - e^{-\lambda_D t} \approx 1 \quad (2)$$

Also, when $\lambda_P << \lambda_D$:

$$\frac{\lambda_P}{\lambda_D - \lambda_P} \approx \frac{\lambda_P}{\lambda_D} \quad (3)$$

Substitute (2) and (3) in (1) to obtain:

$$N_D(t) = \boxed{\frac{\lambda_P}{\lambda_D} N_{P0}}$$

Chapter 41
Elementary Particles and the Beginning of the Universe

Conceptual Problems

1 • How are baryons and mesons similar? How are they different?

Similarities	Differences
Baryons and mesons are hadrons, i.e., they participate in the strong interaction. Both are composed of quarks.	Baryons consist of three quarks and are fermions. Mesons consist of two quarks and are bosons. Baryons have baryon number +1 or −1. Mesons have baryon number 0.

2 • The muon and the pion have nearly the same masses. How do these particles differ?

Determine the Concept The muon is a lepton. It is a spin-½ particle and is a fermion. It does not participate in strong interactions. It appears to be an elementary particle like the electron. The pion is a meson. Its spin is 0 and it is a boson. It does participate in strong interactions and is composed of quarks.

3 • **[SSM]** How can you tell whether a decay proceeds by the strong interaction or the weak interaction?

Determine the Concept A decay process involving the strong interaction has a very short lifetime ($\sim 10^{-23}$ s), whereas decay processes that proceed by the weak interaction have lifetimes of order 10^{-10} s.

4 • True or false:

(*a*) All baryons are hadrons.
(*b*) All hadrons are baryons.

(*a*) True

(*b*) False. There are two kinds of hadrons-baryons, which have spin $\frac{1}{2}$ (or $\frac{3}{2}$, $\frac{5}{2}$ and so on), and mesons, which have zero or integral spin.

5 • True or false: All mesons are spin-$\frac{1}{2}$ particles.

False. Mesons have zero or integral spins.

6 • A particle made of exactly two quarks is (*a*) a meson, (*b*) a baryon, (*c*) a lepton, (*d*) either a meson or a baryon, but definitely not a lepton?

Determine the Concept A meson has 2 quarks, a baryon has 3 quarks. Hence $\boxed{(a)}$ is correct.

7 • Have any quark–antiquark combinations whose electric charge is not an integral multiplied by the fundamental charge *e* been observed?

Determine the Concept No; from Table 41-2 it is evident that any quark-antiquark combination always results in an integral or zero charge.

8 • True or false:

(*a*) A lepton is a combination of three quarks.
(*b*) The typical times for decays by the weak interaction are orders of magnitude longer than the typical times for decays by the strong interaction.
(*c*) The muon and the pion are both mesons.

(*a*) False. Leptons are not made up of quarks.

(*b*) True

(*c*) False. The muon is lepton, not a meson.

9 • True or false:

(*a*) Electrons interact with protons by the strong interaction.
(*b*) Strangeness is not conserved in reactions involving the weak interaction.
(*c*) Neutrons have zero charm.

(*a*) False. Electrons interact with protons by the electromagnetic interaction.

(*b*) True

(*c*) True

Estimation and Approximation

10 •• Grand unification theories predict that the proton has a long but finite lifetime. Current experiments based on detecting the decay of protons in water infer that this lifetime is at least 10^{32} years. Assume 10^{32} years is, in fact, the mean lifetime of the proton. Estimate the expected time between proton-decays

that occur in the water of a filled Olympic-size swimming pool. An Olympic-size swimming pool is 100 m × 25 m × 2.0 m. Give your answer in days.

Picture the Problem Assuming that the lifetime of a proton is 10^{32} y, one proton out of every 10^{32} protons should decay every year on average. Hence, we can estimate the expected time between proton-decays that occur in the water of a filled Olympic-size swimming pool by determining the number of protons N in the pool and dividing 10^{32} y by this number.

The mean time between disintegrations is the ratio of the lifetime of the protons to the number of protons N in the pool:

$$\Delta t_{\text{mean}} = \frac{10^{32}\ \text{y}}{N} \qquad (1)$$

The number of protons N in the pool is related to the mass of water in the pool M_{water}, the molar mass of water $m_{\text{molar, water}}$, and the number of protons per molecule n:

$$\frac{N}{M_{\text{water}}} = \frac{nN_{\text{A}}}{m_{\text{molar, water}}}$$

Solve for N to obtain:

$$N = \frac{nN_{\text{A}}M_{\text{water}}}{m_{\text{molar, water}}}$$

Because the mass of the water is the product of its density and the volume of the pool:

$$N = \frac{nN_{\text{A}}\rho_{\text{water}}V_{\text{pool}}}{m_{\text{molar, water}}}$$

Substituting for N in equation (1) yields:

$$\Delta t_{\text{mean}} = \frac{10^{32}\ \text{y}}{\dfrac{nN_{\text{A}}\rho_{\text{water}}V_{\text{pool}}}{m_{\text{molar, water}}}} = \frac{\left(10^{32}\ \text{y}\right)m_{\text{molar, water}}}{nN_{\text{A}}\rho_{\text{water}}V_{\text{pool}}}$$

Because each molecule of water has 10 protons:

$$n = 10\,\frac{\text{protons}}{\text{molecule}}$$

Substitute numerical values and evaluate Δt_{mean}:

$$\Delta t_{\text{mean}} = \frac{\left(10^{32}\ \text{y}\right)\left(18\,\frac{\text{g}}{\text{mol}}\times\frac{1\,\text{kg}}{10^{3}\ \text{g}}\right)}{\left(10\,\frac{\text{protons}}{\text{molecule}}\right)\left(6.02\times10^{23}\,\frac{\text{molecules}}{\text{mol}}\right)\left(10^{3}\,\frac{\text{kg}}{\text{m}^{3}}\right)\left(100\,\text{m}\right)\left(25\,\text{m}\right)\left(2.0\,\text{m}\right)}$$

$$= 0.0598\,\text{y} = 0.0598\,\text{y}\times\frac{365.24\,\text{d}}{\text{y}} \approx \boxed{22\,\text{d}}$$

11 •• Table 41-6 lists some properties of the four fundamental interactions. To better understand the significance of this table, confirm the ratio of the numerical entries in the second and fourth column of the last row of the table by estimating the ratio of the electromagnetic force to the gravitational force between two protons of a nucleus.

Picture the Problem We can use $F_{\text{em}} = kq_{\text{proton}}^2 / r_{\text{nucleus}}^2$ and $F_{\text{grav}} = Gm_{\text{proton}}^2 / r_{\text{nucleus}}^2$ to estimate the ratio of the electromagnetic and gravitational forces between two protons located in a nucleus.

The electromagnetic force between two protons located in a nucleus is given by:

$$F_{\text{em}} = \frac{kq_{\text{proton}}^2}{r_{\text{nucleus}}^2}$$

The gravitational force between these same protons is given by:

$$F_{\text{grav}} = \frac{Gm_{\text{proton}}^2}{r_{\text{nucleus}}^2}$$

Divide F_{em} by F_{grav} to obtain:

$$\frac{F_{\text{em}}}{F_{\text{grav}}} = \frac{\dfrac{kq_{\text{proton}}^2}{r_{\text{nucleus}}^2}}{\dfrac{Gm_{\text{proton}}^2}{r_{\text{nucleus}}^2}} = \frac{kq_{\text{proton}}^2}{Gm_{\text{proton}}^2}$$

Substitute numerical values and evaluate $F_{\text{em}}/F_{\text{grav}}$:

$$\frac{F_{\text{em}}}{F_{\text{grav}}} = \frac{\left(8.99\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(1.60\times10^{-19}\ \text{C}\right)^2}{\left(6.67\times10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\left(1.67\times10^{-27}\ \text{kg}\right)^2} = \boxed{1.24\times10^{36}}$$

Spin and Antiparticles

12 • Two pions at rest annihilate according to the reaction $\pi^+ + \pi^- \to \gamma + \gamma$. (*a*) Why must the energies of the two γ-rays be equal? (*b*) Find the energy of each γ-ray. (*c*) Find the wavelength of each γ-ray.

Picture the Problem We can use both conservation of energy and momentum to explain why the energies of the two γ-rays must be equal. We can find the energy of each γ-ray in Table 41-1 and find their wavelengths using $\lambda = hc/E$.

(*a*) The initial momentum is zero; therefore, the final momentum must be zero. The momentum of the photon is E/c. To conserve both momentum and energy the two photons must have the same momentum magnitude. Hence they must have the same energy.

(*b*) From Table 41-1:

$$E_\gamma = \boxed{139.6\,\text{MeV}}$$

(*c*) The wavelength of each γ ray is given by:

$$\lambda = \frac{hc}{E} = \frac{1240\,\text{MeV}\cdot\text{fm}}{E}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{1240\,\text{MeV}\cdot\text{fm}}{139.6\,\text{MeV}} = \boxed{8.88\,\text{fm}}$$

13 • Find the minimum energy of the photon needed for the following pair-production reactions: (*a*) $\gamma \to \pi^+ + \pi^-$, (*b*) $\gamma \to \text{p} + \text{p}^-$, and (*c*) $\gamma \to \mu^- + \mu^+$.

Picture the Problem In each case, the required energy is given by $E = 2mc^2$ where m is mass of each particle produced in the pair-production reaction. These masses can be found in Tables 41-1 and 41-3.

(*a*) For $\gamma \to \pi^+ + \pi^-$:

$$E = 2m_\pi c^2 = 2\left(139.6\,\text{MeV}/c^2\right)c^2 = \boxed{279.2\,\text{MeV}}$$

(*b*) For $\gamma \to \text{p} + \text{p}^-$:

$$E = 2m_\text{p} c^2 = 2\left(938.3\,\text{MeV}/c^2\right)c^2 = \boxed{1877\,\text{MeV}}$$

(*c*) For $\gamma \to \mu^- + \mu^+$:

$$E = 2m_\mu c^2 = 2\left(105.659\,\text{MeV}/c^2\right)c^2 = \boxed{211.3\,\text{MeV}}$$

The Conservation Laws

14 • State which of the following decays or reactions violate one or more of the conservation laws, and give the law or laws violated in each case:
a) $p^+ \to n + e^+ + \overline{\nu}_e$, (*b*) $n \to p^+ + \pi^-$, (*c*) $e^+ + e^- \to \gamma$, (*d*) $p + p^- \to \gamma + \gamma$, and (*e*) $\overline{\nu}_e + p \to n + e^+$.

Picture the Problem We need to check for conservation of energy, charge, baryon number, and lepton number.

(*a*) Energy conservation:

Because $m_p < m_n$, energy conservation is violated.

Charge conservation:
$+e \to 0 + e + 0 = +e$

Because the net charge is $+e$ before and after the decay, charge is conserved.

Baryon number:
$+1 \to +1 + 0 + 0 = +1$

Because B is +1 before and after the decay, baryon number is conserved.

Lepton number; electrons:
$0 \to 0 + 0 + 0 = 0$

Because $L_e = 0$ before and after the decay, the lepton number for electrons is conserved.

This process is not allowed because it violates conservation of energy.

(*b*) Energy conservation:

Because $m_n < m_p + m_{\pi^-}$, energy conservation is violated.

Charge conservation:
$0 \to +e + (-e) = 0$

Because the net charge is 0 before and after the decay, charge is conserved.

Baryon number:
$+1 \to +1 + 0 = +1$

Because $B = +1$ before and after the decay, baryon number is conserved.

Lepton number; electrons:
$0 \to 0 + 0 = 0$

Because $L = 0$ before and after the decay, lepton number is conserved.

Because energy is not conserved, this decay is not allowed.

(*c*) Momentum conservation is violated. Two (or more) γ rays must be emitted to conserve momentum.

(*d*) Energy conservation:	Energy is conserved.
Charge conservation: $+1 + (-1) \to 0 + 0 \ = 0$	Because the net charge is zero before and after the decay, charge is conserved.
Baryon number: $+1 + (-1) \to 0 + 0 \ = 0$	Because $B = 0$ before and after the decay, baryon number is conserved.
Lepton number; electrons: $0 \to 0 + 0 + 0 = 0$	Because $L_e = 0$ before and after the decay, the lepton number for electrons is conserved.

Because none of the conservation laws are violated, this is an allowed process.

(*e*) Energy conservation:	Because $m_p < m_n + m_{e^+}$, energy is conserved.
Charge conservation: $0 + 1 \to 0 + 1 \ = 1$	Because the net charge is one before and after the decay, charge is conserved.
Baryon number: $0 + 1 \to +1 + 0 \ = +1$	Because $B = +1$ before and after the decay, baryon number is conserved.
Lepton number; electrons: $-1 + 0 \to 0 + (-1) = -1$	Because $L_e = -1$ before and after the decay, the lepton number for electrons is conserved.

Because none of the conservation laws are violated, this is an allowed process.

15 • Determine the change in strangeness in each reaction that follows, and state whether the each decay can proceed by the strong interaction, by the weak interaction, or not at all: (*a*) $\Omega^- \to \Xi^0 + \pi^-$, (*b*) $\Xi^0 \to p + \pi^- + \pi^0$, and (*c*) $\Lambda^0 \to p + \pi^-$.

Picture the Problem The decay will occur via the strong interaction if strangeness is conserved. If $\Delta S = \pm 1$, it will occur via the weak interaction. If S changes by more than 1, the decay will not occur.

(*a*) List the strangeness of Ω^-, Ξ^0, and π^-:

Ω^-: $S = -3$
Ξ^0: $S = -2$
π^-: $S = 0$

Determine ΔS:

$$\Delta S = -2 - (-3) = \boxed{+1}$$

Because $\Delta S = +1$, the reaction can proceed via the weak interaction.

(*b*) List the strangeness of Ξ^0, p, π^-, and π^0:

Ξ^0: $S = -2$
p: $S = 0$
π^-: $S = 0$
π^0: $S = 0$

Determine ΔS:

$$\Delta S = 0 - (-2) = \boxed{+2}$$

Because $\Delta S = +2$, the reaction is not allowed.

(*c*) List the strangeness of Λ^0, p^+, and π^-:

Λ^0: $S = -1$
p^+: $S = 0$
π^-: $S = 0$

Determine ΔS:

$$\Delta S = 0 - (-1) = \boxed{+1}$$

Because $\Delta S = +1$, the reaction can proceed via the weak interaction.

16 • Determine the change in strangeness for each decay, and state whether each decay can proceed by the strong interaction, by the weak interaction, or not at all: (*a*) $\Omega^- \to \Lambda^0 + K^-$ and (*b*) $\Xi^0 \to p + \pi^-$.

Picture the Problem The decay will occur via the strong interaction if strangeness is conserved. If $\Delta S = \pm 1$, it will occur via the weak interaction. If S changes by more than 1, the decay will not occur.

(*a*) List the strangeness of Ω^-, Λ^0, and K^-:

Ω^-: $S = -3$
Λ^0: $S = -1$
K^-: $S = -1$

Determine ΔS:

$$\Delta S = -1 - 1 - (-3) = \boxed{+1}$$

Because $\Delta S = +1$, the reaction can proceed via the weak interaction.

(*b*) List the strangeness of Ξ^0, p, and π^-:

Ξ^0: $S = -2$
p: $S = 0$
π^-: $S = 0$

Determine ΔS:

$$\Delta S = 0-(-2)=\boxed{+2}$$

Because $\Delta S = +2$, the reaction is not allowed.

17 • Determine the change in strangeness for each decay, and state whether each decay can proceed by the strong interaction, by the weak interaction, or not at all: (*a*) $\Omega^- \to \Lambda^0 + \overline{\nu}_e + e^-$ and (*b*) $\Sigma^+ \to p + \pi^0$.

Picture the Problem The decay will occur via the strong interaction if strangeness is conserved. If $\Delta S = \pm 1$, it will occur via the weak interaction. If *S* changes by more than 1, the decay will not occur.

(*a*) List the strangeness of Ω^-, Λ^0, $\overline{\nu}_e$, and e^-:

Ω^-: $S = -3$
Λ^0: $S = -1$
$\overline{\nu}_e$: $S = 0$
e^-: $S = 0$

Determine ΔS:

$$\Delta S = -1-(-3)=\boxed{+2}$$

Because $\Delta S = +2$, the reaction is not allowed.

(*b*) List the strangeness of Σ^+, p, and π^0:

Σ^+: $S = -1$
p: $S = 0$
π^0: $S = 0$

Determine ΔS:

$$\Delta S = 0-(-1)=\boxed{+1}$$

Because $\Delta S = +1$, the reaction can proceed via the weak interaction.

18 • (*a*) Which of the following decays of the τ particle is possible?

$$\tau \to \mu^- + \overline{\nu}_\mu + \nu_\tau$$
$$\tau \to \mu^- + \nu_\mu + \overline{\nu}_\tau$$

(*b*) Explain why the other decay is not possible. (*c*) Calculate the kinetic energy of the decay products for the decay that is possible.

Picture the Problem We can decide whether the given decays of the τ particle are possible by determining whether energy conservation is satisfied and whether conservation of both the τ and μ lepton numbers is satisfied.

(a) The decay $\tau \to \mu^- + \overline{\nu}_\mu + \nu_\tau$ is allowed. The decay satisfies energy conservation and conservation of both the τ and μ lepton numbers.

(b) The decay $\tau \to \mu^- + \nu_\mu + \overline{\nu}_\tau$ is not allowed. The decay scheme does not conserve τ and μ lepton numbers.

(c) The total kinetic for the decay $\tau \to \mu^- + \overline{\nu}_\mu + \nu_\tau$ is:

$$K_{\text{tot}} = m_\tau c^2 - m_\mu c^2$$

From Table 41-3 we have:

$$m_\tau = 1784\,\text{MeV}/c^2$$

and

$$m_\mu = 105.659\,\text{MeV}/c^2$$

Substitute numerical values and evaluate K_{tot}:

$$K_{\text{tot}} = \left(1784\,\text{MeV}/c^2\right)c^2 - \left(106\,\text{MeV}/c^2\right)c^2$$
$$= \boxed{1678\,\text{MeV}}$$

Remarks: Note that the kinetic energy of the individual decay products cannot be determined from the decay scheme alone.

19 •• **[SSM]** Using **Error! Reference source not found.** and the laws of conservation of charge number, baryon number, strangeness, and spin, identify the unknown particle, symbolized by (?), in each of the following reactions: (a) p + π^- $\to$ Σ^0 + (?),
(b) p + p $\to$ π^+ + n + K^+ + (?), and (c) p + K^- $\to$ Ξ^- +(?)

Picture the Problem We can systematically determine Q, B, S, and s for each reaction and then use these values to identify the unknown particles.

(a) For the strong reaction:

$$\text{p} + \pi^- \to \Sigma^0 + (?)$$

Charge number:

$$+1 - 1 = 0 + Q \Rightarrow Q = 0$$

Baryon number:

$$+1 + 0 = +1 + B \Rightarrow B = 0$$

Strangeness:

$$0 + 0 = -1 + S \Rightarrow S = +1$$

Spin:

$$+\tfrac{1}{2} + 0 = +\tfrac{1}{2} + s \Rightarrow s = 0$$

These properties indicate that the particle is the kaon $\boxed{K^0}$.

(*b*) For the strong reaction: $p + p \rightarrow \pi^+ + n + K^+ + (?)$

Charge number: $+1 + 1 = +1 + 0 + 1 + Q \Rightarrow Q = 0$

Baryon number: $+1 + 1 = 0 + 1 + 0 + B \Rightarrow B = +1$

Strangeness: $0 + 0 = 0 + 0 + 1 + S \Rightarrow S = -1$

Spin: $+\frac{1}{2} + \frac{1}{2} = 0 + \frac{1}{2} + 0 + s \Rightarrow s = +\frac{1}{2}$

These properties indicate that the particle is either the $\boxed{\Sigma^0}$ or the $\boxed{\Lambda^0}$ baryon.

(*c*) For the strong reaction: $p + \overline{K}^- \rightarrow \Xi^- + (?)$

Charge number: $+1 - 1 = -1 + Q \Rightarrow Q = +1$

Baryon number: $+1 + 0 = +1 + B \Rightarrow B = 0$

Strangeness: $0 - 1 = -2 + S \Rightarrow S = +1$

Spin: $+\frac{1}{2} + 0 = +\frac{1}{2} + s \Rightarrow s = 0$

These properties indicate that the particle is the kaon $\boxed{K^+}$.

20 •• Test the following decays for violation of the conservation of energy, electric charge, baryon number, and lepton number: (*a*) $n \rightarrow \pi^+ + \pi^- + \mu^+ + \mu^-$ and (*b*) $\pi^0 \rightarrow e^+ + e^- + \gamma$. Assume that linear momentum and angular momentum are conserved. State which conservation laws (if any) are violated in each decay.

Picture the Problem A decay process is allowed if energy, charge, baryon number, and lepton number are conserved.

(*a*) Energy conservation: Because $m_n > 2m_\pi + 2m_\mu$, energy conservation is not violated.

Charge conservation:
$0 \rightarrow +1 + (-1) + 1 + (-1) = 0$

Because the total charge is 0 before and after the decay, charge is conserved.

Baryon number:
$1 \rightarrow 0 + 0 + 0 + 0 = 0$

Because baryon number changes from +1 to 0, conservation of baryon number is violated.

Lepton number:
$0 \to 0 + 0 + 1 + (-1) = 0$

Because $L_{\mu} = 0$ before and after the decay, the lepton number for muons is conserved.

Not allowed. The decay violates conservation of baryon number.

(*b*) Energy conservation:

Because $m_{\pi} > 2m_{\mathrm{e}}$, energy conservation is not violated.

Charge conservation:
$0 \to +1 + (-1) + 0 = 0$

Because the total charge is 0 before and after the decay, charge is conserved.

Baryon number:
$0 \to 0 + 0 + 0 + 0 = 0$

Because $B = 0$ before and after the decay, the baryon number is conserved.

Lepton number:
$0 \to +1 + (-1) + 0 = 0$

Because $L_{\mathrm{e}} = 0$ before and after the decay, the lepton number for electrons is conserved.

Allowed. The decay satisfies all the conservation laws.

Quarks

21 • Find the baryon number, charge, and strangeness for the following quark combinations and identify the hadron: (*a*) *uud*, (*b*) *udd*, (*c*) *uus*, (*d*) *dds*, (*e*) *uss*, and (*f*) *dss*.

Picture the Problem For each quark combination we can determine the baryon number *B*, the charge *Q*, and the strangeness *S* (from Table 41-2) and then use Table 41-1 to find a match and complete the following table.

	Combination	*B*	*Q*	*S*	hadron
(*a*)	*uud*	1	+1	0	p^+
(*b*)	*udd*	1	0	0	n
(*c*)	*uus*	1	+1	−1	Σ^+
(*d*)	*dds*	1	−1	−1	Σ^-
(*e*)	*uss*	1	0	−2	Ξ^0
(*f*)	*dss*	1	−1	−2	Ξ^-

22 • Find the baryon number, charge, and strangeness for the following quark combinations: (*a*) $u\overline{d}$, (*b*) $\overline{u}d$, (*c*) $u\overline{s}$, and (*d*) $\overline{u}s$.

Picture the Problem For each quark combination we can determine the baryon number *B*, the charge *Q*, and the strangeness *S* (from Table 41-2) and then use Table 41-1 to find a match and complete the following table.

	Combination	B	Q	S	hadron
(*a*)	$u\overline{d}$	0	+1	0	π^+
(*b*)	$\overline{u}d$	0	−1	0	π^-
(*c*)	$u\overline{s}$	0	+1	+1	K^+
(*d*)	$\overline{u}s$	0	−1	−1	K^-

23 • The Δ^{++} particle is a baryon that decays by the strong interaction. Its strangeness, charm, topness, and bottomness are all zero. What combination of quarks gives a particle that has these properties?

Determine the Concept From Table 41-2 we see that to satisfy the conditions of charge = +2 and zero strangeness, charm, topness, and bottomness, the quark combination must be *uuu*.

24 • Find a possible combination of quarks that gives the correct values for electric charge, baryon number, and strangeness for (*a*) K^+ and (*b*) K^0.

Picture the Problem Because K^+ and K^0 are mesons, they consist of a quark and an antiquark. We can use Tables 41-1 and 41-2 to find combinations of quarks with the correct values for electric charge, baryon number, and strangeness for these particles.

(*a*) For K^+ we need:

$$Q = +1$$
$$B = 0$$
$$S = +1$$

A combination of quarks with these properties is $\boxed{u\overline{s}.}$

(*b*) For K^0 we need:

$$Q = 0$$
$$B = 0$$
$$S = +1$$

A combination of quarks with these properties is $\boxed{d\overline{s}.}$

25 • The D^+ meson has zero strangeness, but it has charm of +1. (*a*) What is a possible quark combination that will give the correct properties for this particle? (*b*) Repeat Part (*a*) for the D^- meson, which is the antiparticle of the D^+ meson.

Determine the Concept Because D^+ and D^- are mesons, they consist of a quark and an antiquark.

(*a*) $B = 0$, so we must look for a combination of quark and antiquark. Because it has charm of +1, one of the quarks must be *c*. Because the charge is +*e*, the antiquark must be $\overline{d}$. The possible combination for D^+ is $\boxed{c\overline{d}.}$.

(*b*) Because D^- is the antiparticle of D^+, the quark combination is $\boxed{\overline{c}d.}$

26 • Find a possible combination of quarks that gives the correct values for electric charge, baryon number, and strangeness for (*a*) K^- (the K^- is the antiparticle of the K^+) and (*b*) $\overline{K}^0$.

Picture the Problem We can use Tables 41-1 and 41-2 to find combinations of quarks with the correct values for electric charge, baryon number, and strangeness for these particles. Because K^- and $\overline{K}^0$ are mesons, they consist of a quark and an antiquark.

(*a*) For K^- we need:

$$Q = -1$$
$$B = 0$$
$$S = -1$$

A combination of quarks with these properties is $\boxed{\overline{u}s.}$

(*b*) For $\overline{K}^0$ we need:

$$Q = 0$$
$$B = 0$$
$$S = -1$$

A combination of quarks with these properties is $\boxed{\overline{d}s.}$

Remarks: An alternative solution could take advantage of our results in Problem 24 for the antiparticles K^+ and K^0.

27 •• **[SSM]** Find a possible quark combination for the following particles: (*a*) Λ^0, (*b*) p^-, and (*c*) Σ^-.

Picture the Problem Because Λ^0, p^-, and Σ^- are baryons, they are made up of three quarks. We can use Tables 41-1 and 41-2 to find combinations of quarks with the correct values for electric charge, baryon number, and strangeness for these particles.

(*a*) For Λ^0 we need:

$$Q = 0$$
$$B = +1$$
$$S = -1$$

The quark combination that satisfies these conditions is $\boxed{uds.}$

(*b*) For p^- we need:

$$Q = -1$$
$$B = -1$$
$$S = +1$$

The quark combination that satisfies these conditions is $\boxed{\bar{u}\bar{u}\bar{d}.}$

(*c*) For Σ^- we need:

$$Q = -1$$
$$B = +1$$
$$S = -1$$

The quark combination that satisfies these conditions is $\boxed{dds.}$

28 •• Find a possible quark combination for the following particles: (*a*) $\bar{n}$, (*b*) Ξ^0, and (*c*) Σ^+.

Picture the Problem Because $\bar{n}$, Ξ^0, and Σ^+ are baryons, they are made up of three quarks. We can use Tables 41-1 and 41-2 to find combinations of quarks with the correct values for electric charge, baryon number, and strangeness for these particles.

(*a*) For $\bar{n}$ we need:

$$Q = 0$$
$$B = -1$$
$$S = 0$$

The quark combination that satisfies these conditions is $\boxed{\bar{u}\bar{d}\bar{d}.}$

(*b*) For Ξ^0 we need:

$$Q = 0$$
$$B = +1$$
$$S = -2$$

The quark combination that satisfies these conditions is $\boxed{uss.}$

(*c*) For Σ^+ we need:

$$Q = +1$$
$$B = +1$$
$$S = -1$$

The quark combination that satisfies these conditions is $\boxed{uus.}$

29 •• Find a possible quark combination for the following particles: (*a*) Ω^- and (*b*) Ξ^-.

Picture the Problem Because Ω^- and Ξ^- are baryons, they are made up of three quarks. We can use Tables 41-1 and 41-2 to find combinations of quarks with the correct values for electric charge, baryon number, and strangeness for these particles.

(*a*) For Ω^- we need:

$$Q = -1$$
$$B = +1$$
$$S = -3$$

The quark combination that satisfies these conditions is $\boxed{sss.}$

(*b*) For Ξ^- we need:

$$Q = -1$$
$$B = +1$$
$$S = -2$$

The quark combination that satisfies these conditions is $\boxed{ssd.}$

30 •• State the properties of the particles made up of the following quarks: (*a*) *ddd*, (*b*) $u\bar{c}$, (*c*) $u\bar{b}$, and (*d*) $\bar{s}\bar{s}\bar{s}$.

Picture the Problem We can use Table 41-2 to identify the properties of the particles made up of the given quarks.

	Particle	Q	B	S	Charm	Topness	Bottomness
(*a*)	*ddd*	−1	+1	0	0	0	0
(*b*)	$u\bar{c}$	0	0	0	−1	0	0
(*c*)	$u\bar{b}$	+1	0	0	0	0	−1
(*d*)	$\bar{s}\bar{s}\bar{s}$	+1	−1	+3	0	0	0

The Evolution of the Universe

31 • [SSM] A galaxy is receding from Earth at 2.5 percent the speed of light. Estimate the distance from Earth to this galaxy.

Picture the Problem We can use Hubble's law to find the distance from Earth to this galaxy.

The recessional velocity of galaxy is related to its distance by Hubble's law:

$$v = Hr \Rightarrow r = \frac{v}{H}$$

Substitute numerical values and evaluate *r*:

$$r = \frac{(0.025)c}{\dfrac{23\text{ km/s}}{10^6\, c\cdot\text{y}}} = \frac{(0.025)(2.998\times10^5\text{ km/s})}{\dfrac{23\text{ km/s}}{10^6\, c\cdot\text{y}}} = \boxed{3.3\times10^8\, c\cdot\text{y}}$$

32 • Estimate the speed of a galaxy that is 12×10^9 c·y away from us.

Picture the Problem We can use Hubble's law to find the speed of the galaxy.

The recessional velocity of the galaxy is related to its distance by Hubble's law:

$$v = Hr$$

Substitute numerical values and evaluate *v*:

$$v = \left(\frac{23\text{ km/s}}{10^6\, c\cdot\text{y}}\right)(12\times10^9\, c\cdot\text{y})\left(\frac{c}{2.998\times10^5\text{ km/s}}\right) = \boxed{0.92c}$$

33 •• The Doppler frequency shift for a light from a source that is receding from a stationary receiver is given by $f' = f_0\sqrt{(1-\beta)/(1+\beta)}$, where $\beta = v/c$ (Equation 39-16*b*). Show that the Doppler wavelength shift for light is $\lambda' = \lambda_0\sqrt{(1+\beta)/(1-\beta)}$.

Picture the Problem We can substitute for f' and f_0, using $v = f\lambda$, in Equation 39-16*b* to show that the relativistic wavelength shift is $\lambda' = \lambda_0\sqrt{\dfrac{1+\beta}{1-\beta}}$.

From Equation 39-16*b*:

$$f' = f_0\sqrt{\frac{1-\beta}{1+\beta}}$$

Express f' and f_0 in terms of λ' and λ_0:

$$f' = \frac{c}{\lambda'} \text{ and } f_0 = \frac{c}{\lambda_0}$$

Substitute for f' and f_0 to obtain:

$$\frac{c}{\lambda'} = \frac{c}{\lambda_0}\sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \lambda' = \boxed{\lambda_0\sqrt{\frac{1+\beta}{1-\beta}}}$$

34 •• The red line in the spectrum of atomic hydrogen is frequently referred to as the Hα line, and it has a wavelength of 656.3 nm. Using Hubble's law and the Doppler equation for light from Problem 33, determine the wavelength of the Hα line in the spectrum emitted from galaxies at distances of (*a*) 5.00×10^6 *c*·y, (*b*) 500×10^6 *c*·y, and (*c*) 5.00×10^9 *c*·y from Earth.

Picture the Problem Using Hubble's law, we can rewrite the equation from Problem 33 as $\lambda' = \lambda_0\sqrt{\frac{c+Hr}{c-Hr}}$.

From Problem 33 we have:

$$\lambda' = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}} = \lambda_0\sqrt{\frac{1+v/c}{1-v/c}}$$

Use Hubble's law to relate v to r:

$$v = Hr$$

Substitute for v and simplify to obtain:

$$\lambda' = \lambda_0\sqrt{\frac{1+Hr/c}{1-Hr/c}} = \lambda_0\sqrt{\frac{c+Hr}{c-Hr}}$$

(*a*) For $r = 5.00 \times 10^6$ *c*·y:

$$\lambda' = 656.3\,\text{nm}\sqrt{\frac{2.998\times10^5\text{ km/s}+\left(\frac{23\text{ km/s}}{10^6 c\cdot\text{y}}\right)\left(5.00\times10^6 c\cdot\text{y}\right)}{2.998\times10^5\text{ km/s}-\left(\frac{23\text{ km/s}}{10^6 c\cdot\text{y}}\right)\left(5.00\times10^6 c\cdot\text{y}\right)}} = \boxed{6.6\times10^{-7}\text{ m}}$$

(*b*) For $r = 500 \times 10^6\ c{\cdot}\text{y}$:

$$\lambda' = 656.3\,\text{nm}\sqrt{\frac{2.998\times10^5\ \text{km/s} + \left(\dfrac{23\ \text{km/s}}{10^6\,c\cdot\text{y}}\right)\left(500\times10^6\,c\cdot\text{y}\right)}{2.998\times10^5\ \text{km/s} - \left(\dfrac{23\ \text{km/s}}{10^6\,c\cdot\text{y}}\right)\left(500\times10^6\,c\cdot\text{y}\right)}} = \boxed{6.8\times10^{-7}\ \text{m}}$$

(*c*) For $r = 5.00 \times 10^9\ c{\cdot}\text{y}$:

$$\lambda' = 656.3\,\text{nm}\sqrt{\frac{2.998\times10^5\ \text{km/s} + \left(\dfrac{23\ \text{km/s}}{10^6\,c\cdot\text{y}}\right)\left(5.00\times10^9\,c\cdot\text{y}\right)}{2.998\times10^5\ \text{km/s} - \left(\dfrac{23\ \text{km/s}}{10^6\,c\cdot\text{y}}\right)\left(5.00\times10^9\,c\cdot\text{y}\right)}} = \boxed{9.8\times10^{-7}\ \text{m}}$$

General Problems

35 • (*a*) What conditions are necessary for a hadron and its antiparticle to be identical? (*b*) Find the quark combination of both the hadron and its antiparticle for both the π^0 and the Ξ^0 particles. (*c*) Of the π^0 and the Ξ^0 particles, which, if any, is its own antiparticle?

Determine the Concept

(*a*) Baryon number and lepton numbers are conserved quantities. A particle and its antiparticle must have baryon numbers that add to zero and lepton numbers that add to zero. Thus, for a particle and its antiparticle to be identical, its baryon number and all three of its lepton numbers must equal zero. This means it cannot be a lepton or a baryon, so it must be a meson. A particle and its antiparticle have the complementary quark content. That is, if each quark in a particle is replaced by its antiquark, then the resulting entity is the antiparticle of the particle.

(*b*) The quark combination for the π^0 is a linear combination of $u\overline{u}$ and $d\overline{d}$ and the quark combination for the $\overline{\pi}^0$ is a linear combination of $\overline{u}u$ and $\overline{d}d$. The quark combination for the Ξ^0 is uss and that of the $\overline{\Xi}^0$ is $\overline{u}\overline{s}\overline{s}$.

(*c*) The π^0 is a meson with quark content of a linear combination of $u\overline{u}$ and $d\overline{d}$, so the π^0 is its own antiparticle. The Ξ^0 is a baryon. As is explained in the answer to Part (*a*), a baryon cannot be its own antiparticle.

36 •• The red line in the spectrum of atomic hydrogen is frequently referred to as the Hα line, and it has a wavelength of 656.3 nm. Light from a distant galaxy shows a redshift of the Hα line of hydrogen to a wavelength of 1458 nm.

(*a*) What is the recessional velocity of the galaxy? (*b*) Estimate the distance to this galaxy.

Picture the Problem We can solve the equation derived in Problem 33 for the recessional velocity of the galaxy and then use Hubble's equation to estimate the distance to the galaxy.

(*a*) From Problem 33 we have:

$$\lambda' = \lambda_0\sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \frac{1+v/c}{1-v/c} = \left(\frac{\lambda'}{\lambda_0}\right)^2$$

Substitute numerical values for λ' and λ_0:

$$\frac{1+v/c}{1-v/c} = \left(\frac{1458\text{ nm}}{656.3\text{ nm}}\right)^2 = 4.935$$

Simplify to obtain:

$$4.953\left(1-\frac{v}{c}\right) = 1+\frac{v}{c}$$

and

$$5.953\frac{v}{c} = 3.953$$

Solving for v gives:

$$v = 0.664c = 0.664\left(2.998\times10^8\text{ m/s}\right)$$
$$= 1.99\times10^8\text{ m/s} = \boxed{1.99\times10^5\text{ km/s}}$$

(*b*) From the Hubble equation we have:

$$r = \frac{v}{H}$$

Substitute numerical values and evaluate r:

$$r = \frac{1.99\times10^5\text{ km/s}}{\dfrac{23\text{ km/s}}{10^6\, c\cdot\text{y}}} = \boxed{8.65\times10^9\, c\cdot\text{y}}$$

37 •• [SSM] (*a*) In terms of the quark model, show that the reaction $\pi^0 \to \gamma+\gamma$ does not violate any conservation laws. (*b*) Which conservation law is violated by the reaction $\pi^0 \to \gamma$?

Picture the Problem The π^{o} particle is composed of two quarks, u and $\bar{\text{u}}$. Hence, the reaction $\pi^{\text{o}} \to \gamma+\gamma$ is equivalent to $\text{u}\bar{\text{u}} \to \gamma+\gamma$.

(*a*) The u and $\bar{\text{u}}$ annihilate – resulting in the photons.

(*b*) Two or more photons are required to conserve linear momentum.

38 •• Test the following decays for violation of the conservation of energy, electric charge, baryon number, and lepton number: (*a*) $\Lambda^0 \rightarrow \mathrm{p} + \pi^-$, (*b*) $\Sigma^- \rightarrow \mathrm{n} + \mathrm{p}^-$, and (*c*) $\mu^- \rightarrow \mathrm{e}^- + \overline{\nu}_\mathrm{e} + \nu_\mu$. Assume that linear momentum and angular momentum are conserved. State which conservation laws (if any) are violated in each decay.

Picture the Problem A decay process is allowed if energy, charge, baryon number, and lepton numbers are conserved.

(*a*) Energy conservation:

Because $m_\Lambda > m_\mathrm{p} + m_\pi$, energy conservation is not violated.

Charge conservation:
$0 \rightarrow 1 - 1 = 0$

Because the total charge is 0 before and after the decay, charge conservation is not violated.

Baryon number:
$1 \rightarrow 1 + 0 = 1$

Because there is no change in baryon number, baryon number is conserved.

Lepton number:
$0 \rightarrow 0 + 0 = 0$

Because lepton number is 0 before and after the decay, lepton number is conserved.

The decay is allowed.

(*b*) Energy conservation:

Because $m_\Sigma < m_\mathrm{n} + \mathrm{m}_\mathrm{p}$, energy is not conserved.

Charge conservation:
$-1 \rightarrow 0 - 1 = -1$

Because the total charge does not change, charge is conserved.

Baryon number:
$+1 \rightarrow 1 - 1 = 0$

Because B changes from +1 to 0, baryon number is not conserved.

Lepton number:
$0 \rightarrow 0 + 0 = 0$

Because L is 0 before and after the decay, the lepton number is conserved.

Not allowed. The decay violates both energy conservation and baryon number.

(*c*) Energy conservation:

Energy is conserved.

Charge conservation: $-1 \to -1 + 0 + 0 = -1$

Because the total charge does not change, charge is conserved.

Baryon number: $0 \to 0 + 0 + 0 = 0$

Because B does not change, baryon number is conserved.

Lepton number: $1 \to 1 - 1 + 1 = 1$

Because L does not change, lepton number is conserved.

The decay satisfies all conservation laws and is allowed.

Remarks: The decay in Part (*c*) is the decay process for the muon μ^- (see Example 41-2).

39 •• Consider the following high-energy particle reaction: $p + p \to \Lambda^0 + K^0 + p + (?)$, where (?) represents an unknown particle. During this reaction, stationary protons are bombarded with a beam of high-energy protons. (*a*) Use the laws of conservation of charge number, baryon number, strangeness (**Error! Reference source not found.**), and spin to determine the unknown particle. (*b*) Calculate the Q value for the reaction. (*c*) The threshold kinetic energy K_{th} for this reaction is given by $K_{th} = -\frac{1}{2}Q\left(m_p + m_p + M_1 + M_2 + M_3 + M_4\right)/m_p$, where M_1, M_2, M_3 and M_4 are the masses of the reaction products. Find K_{th}.

Picture the Problem We can systematically determine Q, B, S, and s for the reaction and then use these values to identify the unknown particle. The Q value for the reaction is given by $Q = -(\Delta m)c^2$ and the expression for the threshold energy for the reaction is given in the problem statement.

(*a*) For the strong reaction: $p + p \to \Lambda^o + K^o + p + (?)$

Charge number: $+1 + 1 = 0 + 0 + 1 + Q \Rightarrow Q = +1$

baryon number: $+1 + 1 = +1 + 0 + 1 + B \Rightarrow B = 0$

strangeness: $0 + 0 = -1 + 1 + 0 + S \Rightarrow S = 0$

spin: $+\frac{1}{2} + \frac{1}{2} = +\frac{1}{2} + 0 + \frac{1}{2} + s \Rightarrow s = 0$

These properties indicate that the unknown particle is a pion $\boxed{\pi^+}$.

(*b*) The reaction is: $p + p \to \Lambda^o + K^o + p + \pi^+$

The Q-value for the reaction is:

$$Q = \left[\left(m_{\mathrm{p}} + m_{\mathrm{p}}\right) - \left(M_{\Lambda^0} + M_{\mathrm{K}^0} + M_{\mathrm{p}} + M_{\pi^+}\right)\right]c^2$$

Use Table 41-1 to find the mass-energy values:

$$Q = \left[(938.3 + 938.3) - (1116 + 497.7 + 938.3 + 139.6)\right]\mathrm{MeV} = \boxed{-815\,\mathrm{MeV}}$$

Because $Q < 0$, the reaction is endothermic.

(c) The threshold energy for this reaction is:

$$K_{\mathrm{th}} = -\frac{Q}{2m_{\mathrm{p}}}\left(m_{\mathrm{p}} + m_{\mathrm{p}} + M_{\Lambda^0} + M_{\mathrm{K}^0} + M_{\mathrm{p}} + M_{\pi^+}\right)$$

Using Table 41-1 to find the mass-energy values, substitute numerical values and evaluate K_{th}:

$$K_{\mathrm{th}} = -\frac{-815\,\mathrm{MeV}}{2(938.3\,\mathrm{MeV})}(938.3 + 938.3 + 1116 + 497.7 + 938.3 + 139.6)\mathrm{MeV}$$
$$= 1984\,\mathrm{MeV} = \boxed{1.98\ \mathrm{GeV}}$$

40 ••• In this problem, you will calculate the difference in the time of arrival of two neutrinos of different energy from a supernova that is 170 000 light-years away. Let the energies of the neutrinos be E_1 = 20 MeV and E_2 = 5 MeV, and assume that the mass of a neutrino is 2.0 eV/c^2. Because the total energies of the neutrinos is so much greater than their rest energies, the neutrinos have speeds that are very nearly equal to c and energies that are approximately $E \approx pc$. (a) If t_1 and t_2 are the times that the neutrinos with speeds u_1 and u_2 take to travel a distance x , show that $\Delta t = t_2 - t_1 = x\left(u_1 - u_2\right)/u_1 u_2 \approx \left(x\,\Delta u\right)/c^2$. (b) The speed of a neutrino of mass m and total energy E can be found from $E = mc^2/\left[1 - \left(u^2/c^2\right)\right]^{1/2}$ (Equation 39-24). Show that when $E >> mc^2$, the speed ratio u/c is given approximately by $u/c \approx 1 - \frac{1}{2}\left(mc^2/E\right)^2$. (c) Use the results from Part (a) and Part (b) to calculate $u_1 - u_2$ for the energies and mass given, and calculate Δt from the result from Part (a) for x = 170 000 c·y. (d) Repeat the calculation in Part (c) using 20 eV/c^2 for the neutrino mass.

Picture the Problem The solution strategy is outlined in the problem statement.

(a) Express $\Delta t = t_2 - t_1$ in terms of u_2 and u_1:

$$\Delta t = t_2 - t_1 = \frac{x}{u_2} - \frac{x}{u_1} = \frac{x\left(u_1 - u_2\right)}{u_1 u_2}$$

Noting that $u_1 u_2 \approx c^2$, we have:

$$\Delta t \approx \boxed{\frac{x\Delta u}{c^2}} \qquad (1)$$

where $\Delta u = u_1 - u_2$

(b) From Equation 39-25 we have:

$$\frac{u}{c} = \sqrt{1-\left(\frac{mc^2}{E}\right)^2} = \left(1-\left(\frac{mc^2}{E}\right)^2\right)^{1/2}$$

Expand binomially to obtain:

$$\frac{u}{c} = \boxed{1-\tfrac{1}{2}\left(\frac{mc^2}{E}\right)^2}$$

(c) Express $u_1 - u_2$ in terms of E_1, E_2, and mc^2:

$$u_1 - u_2 = \tfrac{1}{2}c\left(mc^2\right)^2\left(\frac{1}{E_2^2}-\frac{1}{E_1^2}\right) = \frac{c\left(mc^2\right)^2\left(E_1^2-E_2^2\right)}{2E_1^2E_2^2}$$

Substitute numerical values and evaluate Δu:

$$\Delta u = \frac{c\left(2.0\frac{\text{eV}}{c^2}c^2\right)^2\left[(20\,\text{MeV})^2-(5.0\,\text{MeV})^2\right]}{2(20\,\text{MeV})^2(5.0\,\text{MeV})^2} = \boxed{7.5\times10^{-14}\,c}$$

Use equation (1) to evaluate Δt:

$$\Delta t \approx \frac{\left(1.70\times10^5\,c\cdot\text{y}\right)\left(7.5\times10^{-14}\,c\right)}{c^2} = 1.275\times10^{-8}\,\text{y}\times\frac{31.56\,\text{Ms}}{\text{y}} = \boxed{0.40\,\text{s}}$$

(d) Using $mc^2 = 20$ eV for the rest energy of a neutrino:

$$\Delta u = \frac{c\left(20\frac{\text{eV}}{c^2}c^2\right)^2\left[(20\,\text{MeV})^2-(5.0\,\text{MeV})^2\right]}{2(20\,\text{MeV})^2(5.0\,\text{MeV})^2} = \boxed{7.5\times10^{-12}\,c}$$

Use equation (1) to evaluate Δt:

$$\Delta t \approx \frac{\left(1.70\times10^{5}\,c\cdot\text{y}\right)\left(7.5\times10^{-12}\,c\right)}{c^{2}}$$
$$=1.275\times10^{-6}\ \text{y}\times\frac{31.56\,\text{Ms}}{\text{y}}$$
$$=\boxed{40\,\text{s}}$$

Remarks: Note that the spread in the arrival time for neutrinos from a supernova can be used to estimate the mass of a neutrino.

41 ••• A Λ^0 at rest decays by the reaction $\Lambda^0 \rightarrow \text{p} + \pi^-$. (*a*) Calculate the total kinetic energy of the decay products. (*b*) Find the ratio of the kinetic energy of the pion to the kinetic energy of the proton. (*c*) Find the kinetic energies of the proton and the pion for this decay.

Picture the Problem We can use the masses of the parent and daughters to find the total kinetic energy of the decay products under the assumption that the Λ^0 is initially at rest. Application of conservation of energy and the definition of kinetic energy will yield the ratio of the kinetic energy of the pion to the kinetic energy of the proton. Finally, we can use our results in Parts (*a*) and (*b*) to find the kinetic energies of the proton and the pion for this decay.

(*a*) The total kinetic energy of the decay products is given by:

$$K_{\text{tot}}=\left(m_\Lambda-m_\text{p}-m_\pi\right)c^2$$

Substitute numerical values (see Table 41-1) and evaluate K_{tot}:

$$K_{\text{tot}}=\left(1116\frac{\text{MeV}}{c^2}-938.3\frac{\text{MeV}}{c^2}-139.6\frac{\text{MeV}}{c^2}\right)c^2=\boxed{38\,\text{MeV}}$$

(*b*) The ratio of the kinetic energies is given by:

$$\frac{K_\pi}{K_\text{p}}=\frac{\frac{1}{2}m_\pi v_\pi^2}{\frac{1}{2}m_\text{p}v_\text{p}^2}=\frac{m_\pi v_\pi^2}{m_\text{p}v_\text{p}^2}$$

Use conservation of momentum (nonrelativistic) to obtain:

$$m_\pi v_\pi=m_\text{p}v_\text{p}\Rightarrow\frac{v_\pi}{v_\text{p}}=\frac{m_\text{p}}{m_\pi}$$

Substitute for the ratio of the speeds to obtain:

$$\frac{K_\pi}{K_\text{p}}=\frac{m_\pi}{m_\text{p}}\left(\frac{m_\text{p}}{m_\pi}\right)^2=\frac{m_\text{p}}{m_\pi}$$

Substitute numerical values and evaluate the ratio of the kinetic energies:

$$\frac{K_\pi}{K_p} = \frac{938.3\frac{\text{MeV}}{c^2}}{139.6\frac{\text{MeV}}{c^2}} = \boxed{6.72}$$

(*c*) Express the total kinetic energy in terms of K_π and K_p:

$$K_p + K_\pi = K_{tot} \qquad (1)$$

Use our results in (*a*) and (*b*) to obtain:

$$K_p + 6.72K_p = 38\,\text{MeV}$$

Solving for K_p gives:

$$K_p = \boxed{5\,\text{MeV}}$$

Substitute in equation (1) to obtain:

$$K_\pi = K_{tot} - K_p = \boxed{33\,\text{MeV}}$$

42 ••• A Σ^0 particle at rest decays by the reaction $\Sigma^0 \rightarrow \Lambda^0 + \gamma$. (*a*) What is the total energy (total energy includes rest energy) of the decay products? (*b*) Assuming that the kinetic energy of the Λ^0 is negligible compared with the energy of the photon, calculate the approximate momentum of the photon. (*c*) Use your result from Part (*b*) to calculate the kinetic energy of the Λ^0. (*d*) Use your result from Part (*c*) to obtain a better estimate of the momentum and the energy of the photon.

Picture the Problem The total energy of the decay products is the rest energy of the Σ^0 particle. We can find the momentum of the photon from its energy and use the conservation of momentum to calculate the kinetic energy of the Λ^0.

(*a*) The total energy of the decay products is given by:

$$E = (m_\Sigma)c^2$$

Substitute numerical values (see Table 41-1) and evaluate K_{tot}:

$$E = \left(1193\frac{\text{MeV}}{c^2}\right)c^2 = \boxed{1193\,\text{MeV}}$$

(*b*) The momentum of the photon is given by:

$$p_\gamma = \frac{E_\gamma}{c} = \frac{E - m_\Lambda c^2}{c}$$

Substitute numerical values and evaluate p_γ:

$$p_\gamma = \frac{1193\,\text{MeV} - \left(1116\frac{\text{MeV}}{c^2}\right)c^2}{c} = \boxed{77\frac{\text{MeV}}{c}}$$

(c) The kinetic energy of the Λ^0 is given by:

$$K_\Lambda = \frac{p_\Lambda^2}{2m_\Lambda}$$

or, because $p_\Lambda = p_\gamma$,

$$K_\Lambda = \frac{p_\gamma^2}{2m_\Lambda}$$

Substitute numerical values and evaluate K_Λ:

$$K_\Lambda = \frac{\left(77\,\frac{\text{MeV}}{c}\right)^2}{2\left(1116\,\frac{\text{MeV}}{c^2}\right)} = \boxed{2.7\,\text{MeV}}$$

(d) A better estimate of the energy of the photon is:

$$E_\gamma = E - m_\Lambda c^2 - K_\Lambda$$

Substitute numerical values and evaluate E_γ:

$$E_\gamma = 1193\,\text{MeV} - \left(1116\,\frac{\text{MeV}}{c^2}\right)c^2 - 2.7\,\text{MeV} = \boxed{74\,\text{MeV}}$$

The improved estimate of the momentum of the photon is:

$$p_\gamma = \frac{E_\lambda}{c} = \boxed{74\,\frac{\text{MeV}}{c}}$$